

GUARANTEED MINIMUM MATURITY BENEFITS IN A SELF-EXCITING STOCHASTIC MORTALITY MODEL: PRICING, ESTIMATION AND CALIBRATION.

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ABSTRACT. The Guaranteed Minimum Maturity Benefit is quite a popular feature embedded in several unit-linked policies offered by insurance companies. The value of this benefit depends on several processes assumed to describe both the mortality and the financial dynamics, typically represented by interest rates and by the fund associated to the unit-linked policy. A large literature is devoted to the valuation of GMMB for different mortality models, in particular when the mortality dynamics is described by affine models of diffusion type. In the present paper we assume a self-exciting behaviour for the mortality dynamics, described by a Hawkes-type process with exponential kernel. This allows us to keep both the Markov and affine features, but introduces jumps with a stochastic intensity. These types of dynamics, exhibiting a jump clustering property, are quite convenient in order to describe mortality in some critical situations, like epidemics, where contagion phenomena make the probability of jump arrivals higher whenever a jump occurs. By assuming diffusion type dynamics for both the fund and the interest rate, and introducing all the possible correlations among the diffusion processes necessary in order to get realistic dynamics, we take advantage of the affine features of the model proposed and compute in a semi-explicit form the value of a GMMB. Lastly, we calibrate our and perform an empirical analysis to determine the effects of excess mortality on GMMB prices as well as a death benefit portfolio.

Key words and phrases: Guaranteed Minimum Maturity Benefit, Self-exciting Processes, Jump Clustering, Hawkes processes, Stochastic Mortality.

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1. INTRODUCTION

Product development in insurance markets worldwide has changed dramatically in the past few decades. Policyholders are becoming more aware of investment opportunities outside the insurance sector; they want to benefit simultaneously from both equity investments and mortality protection. In order to meet this dual demand for insurance and investment, insurers issue various types of equity-linked insurance contracts. Examples include segregated fund contracts in Canada, unit-linked insurance in the United Kingdom, and variable annuities and equity-indexed annuities in the United States; see Hardy [19] for a comprehensive survey. Equity-linked insurance is a contract in which the policyholder's fund level depends on the performance of a stock market indicator. Insurers offer guarantees embedded in equity-linked contracts to provide downside protection.

Such guarantees transfer the risks from policyholders to insurers, making the contracts more attractive. There are several classifications of guaranteed minimum benefits (GMB). A guaranteed minimum maturity benefit (GMMB) is a guarantee that provides the policyholder with a minimum benefit on maturity date. A guaranteed minimum death benefit (GMDB) is a guarantee that pays out a minimum benefit upon death during the term of the contract. A guaranteed minimum accumulation benefit (GMAB) allows the policyholder to renew the contract at a guaranteed level. A guaranteed minimum income benefit (GMIB) gives the policyholder the right to convert the account value at maturity to an annuity at a guaranteed rate.

Various frameworks for GMB valuation have been considered in the literature. The focus of the present paper is on GMMB. The pricing of GMBs entails the modeling of risk factors such as interest rate, mortality rate and the underlying stock index. The simplest approach to such valuation, under a lognormal assumption on the stock/fund dynamics, is provided in Hardy [12]. In Fung, Ignatieva, and Sherris [17] and Da Fonseca and Ziveyi [9], independence between mortality rates and financial markets is assumed. Peng, Leung, and Kwok [25], Dai, Kwok, and Zong [10], Chen, Vetzal, and Forsyth [6], and Marshall, Hardy, and Saunders [23] ignore mortality effects, whereas Fung, Ignatieva, and Sherris [17] set the mortality rate as constant. Some authors endorsed geometric Brownian motion (GBM) as a suitable model for the underlying fund dynamics, like Brennan and Schwartz [2], Bauer, King, and Russ [1], Piscopo and Haberman [26], Feng and Vecer [15]. Mamon, Xiong and Zhao [22] evaluate the GMMB under a regime-switching modelling framework.

The modelling framework we assume in the present paper describes the mortality dynamics by introducing a Hawkes process. In several dramatic situations like epidemics, mortality exhibits jumps which appear in clusters. Hawkes processes were introduced long ago in order to describe phenomena with jump clustering features like earthquakes. Hawkes processes have been extensively used in financial context in order to describe various assets dynamics [13], [3], [27], [18], to model credit risk [14] and to estimate financial risk [4]. In general Hawkes processes are not Markov, but in a particular case they exhibit both Markov and affine properties, and this is the case in which the kernel is of exponential type (short memory case). In this last case the two-dimensional process describing the jump intensity and the counting process is Markov and affine. These very appealing properties allow us to explicitly compute many relevant quantities including some interesting actuarial contracts, like GMMBs. In this paper we present how this explicit computation can be performed. We assume GBM dynamics for the fund/stock and diffusion type dynamics for the short rates, but we include significant correlations among all the relevant processes. We remark that Guy Nijke Leunga and Hainaut [24] considered Hawkes type dynamics in valuation of a GMMB, but the Hawkes process in that paper describes the fund/stock instead of the mortality dynamics. Chiang et al. [7] show how to develop accurate models of COVID-19 transmission by using Hawkes processes with spatial-temporal covariates. They model the conditional intensity of new COVID-19 cases and deaths in the U.S. at the county level, estimating the dynamic

reproduction number of the virus within an EM algorithm through a regression on Google mobility indices and demographic covariates in the maximization step.

In next section we offer a short overview on Hawkes processes. Thereafter, in Section 3 we describe our modelling framework and introduce the most relevant notation in use throughout the paper. In Section 4 we compute the value of a zero-coupon bond in the present model setting; this value will be used later in order to compute the GMMB by a suitable *change of numéraire* technique. Similarly, in Section 5 we compute the value of a pure endowment in the same model setting, while in Section 6 we shall present our computation of the Guaranteed Minimum Maturity Benefit. In Section 7 and Section 8 we present a concrete application of our results, by illustrating an estimation methodology for our model and some graphical illustration of the main achievements in some specific cases. In Section 9 we collect some concluding remarks. The proofs of the main propositions in Sections 5 and 6 can be found in the Appendix.

2. HAWKES PROCESSES

In the present section we offer a gentle survey on Hawkes processes necessary for the reader to easily follow later sections. The interested reader may consult [20].

Definition 2.1 (Hawkes process). *Let $N = \{N_t, t \geq 0\}$ be a counting process satisfying*

$$\mathbb{P}[N_{t+h} - N_t = m | \mathcal{F}_t^N] = \begin{cases} 1 - \lambda_t h + o(h), & m = 0 \\ \lambda_t h + o(h), & m = 1 \\ o(h), & m > 1 \end{cases}.$$

Here, $h > 0$ and λ is a stochastic process of the form

$$\lambda_t = \bar{\lambda} + \int_0^t \mu_{t-u} dL_u,$$

for some $\bar{\lambda} > 0$, function $\mu : (0, \infty) \rightarrow [0, \infty)$, and L satisfying

$$L_t = \sum_{i=1}^{N_t} J_i, \quad t \geq 0, \tag{2.1}$$

where $J = \{J_i\}_{i \geq 1}$ is a sequence of i.i.d. random variables. $\bar{\lambda}$ and $\mu : (0, \infty) \rightarrow [0, \infty)$ are called the background intensity and excitation function, respectively.

To avoid the trivial case we assume $\mu \neq 0$, and additionally require that J and N are independent. Such a process N is then called a Hawkes process.

Remark 2.2. The usual definition of a Hawkes process involves $\lambda_t = \bar{\lambda} + \int_0^t \mu_{t-u} dN_u$ rather than dL , making this a generalised version of the standard Hawkes process. Allowing the jumps of λ to depend on $\{J_i\}_{i \geq 1}$ has two advantages. Firstly, it means that larger jumps in L has a larger effect on λ . Additionally, if a small jump in L remains undetected, as might occur during the calibration procedure in Section 7, this will have a minimal effect on estimates of λ .

Throughout this paper the common law of the jumps $\{J_i\}_{i \geq 1}$ will be denoted by ν and their moment generating function (MGF) by ψ . We also assume that the excitation function decays exponentially, i.e. $\mu_t = \alpha e^{-\beta t}$, $t \geq 0$ for fixed parameters $\alpha, \beta > 0$. In this configuration λ satisfies the following SDE

$$d\lambda_t = \beta(\bar{\lambda} - \lambda_t)dt + \alpha dL_t, \quad t \geq 0. \quad (2.2)$$

Setting some initial condition $\lambda_t|_{t=0} = \lambda_0 > 0$, it turns out by Itô's formula for semimartingales that

$$\lambda_t = e^{-\beta t}(\lambda_0 - \bar{\lambda}) + \bar{\lambda} + \alpha \int_0^t e^{\beta(t-u)} dL_u, \quad t \geq 0. \quad (2.3)$$

In general, neither L nor (λ, L) are Markov processes. However, for the specific choice of exponentially decaying kernel $\mu_t = \alpha e^{-\beta t}$, $t \geq 0$, it is easy to verify that (λ, L) is a Markov process, see [20, Section 3.4]. This allows for the modelling of non-Markovian features such as clustering effects and embed the framework into a Markovian environment.

3. MODELLING FRAMEWORK

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space equipped with a filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t \in [0, T]}$ with $T \in \mathbb{R}$, $T > 0$ some maturity. Here $\mathcal{A} = \mathcal{F}_T$. This probability space carries all the necessary stochastic processes in our modelling framework and $\mathcal{F}$ is simply the $\mathbb{P}$ -augmented joint natural filtration.

We will consider three independent Brownian motions $Z^i = \{Z_t^i, t \geq 0\}$, $i = 1, 2, 3$ and the jump process $L = \{L_t, t \in [0, T]\}$ defined in (2.1), where $N = \{N_t, t \geq 0\}$ is a Hawkes process with stochastic intensity $\lambda = \{\lambda_t, t \geq 0\}$ given satisfying (2.2).

3.1. Interest rate model. The short-rate is modelled by $r = \{r_t, t \geq 0\}$ a stochastic process with dynamics

$$dr_t = a(b - r_t)dt + \tau dW_t^1, \quad r_0 \in \mathbb{R}, \quad t \geq 0, \quad (3.1)$$

where $a, b, \tau \in \mathbb{R}$, $\tau > 0$ and W^1 is a copy of Z^1 .

Remark 3.1. *Although empirical evidence suggests existence of negative interest rates during the last 10 years, standard economic theories assume positive interest rates. A CIR model for short rate can be introduced without substantially changing the procedure outlined in the present paper: by assuming a suitable risk premium form, a structure-preserving measure change can be introduced, under which the dynamics of the force of mortality, short rate, and fund are described by the same stochastic differential equations with different parameters. The computation of the pure endowment can be then performed in the same way outlined for the Vasicek model.*

3.2. Model for the mortality. We denote by μ_t the force of mortality at time t for a specified generation and assume that it evolves as

$$d\mu_t = m(\bar{\mu}_t - \mu_t)dt + \zeta_t dW_t^2 + \eta_t dL_t, \quad (3.2)$$

$$d\lambda_t = \beta(\bar{\lambda} - \lambda_t)dt + \alpha dL_t, \quad (3.3)$$

where $m \in \mathbb{R}$, $m > 0$ and W^2 is such that $[W^1, W^2]_t = \rho_{12}t$. Here, $\bar{\mu}, \zeta$ and η are deterministic continuous functions of time. The parameter m is assumed to be positive so that μ is reverting to a mortality trend closely related to $\bar{\mu}$. The process μ should not be confused with the force of mortality for a given age x and year t , i.e. $\mu(x, t)$, but it is related. When computing probabilities we look at $\mu(x + t, t)$ where x is the age of the individual at initial time and t is the number of years since the initial time. Here, we model $\mu_t = \mu(x + t, t)$ for a given initial age x as in (3.2).

Remark 3.2. *In analogy with the Remark 3.1, it is possible to notice that μ may assume negative values in contrast with any reasonable modelling assumptions. We shall find, however, that, according to the estimates performed in the following sections, the probability of negative values is negligible.*

3.3. Model for the fund. The value of the fund is modelled by a stochastic process $S = \{S_t, t \geq 0\}$ with dynamics given by

$$\frac{dS_t}{S_t} = cdt + \sigma dW_t^3, \quad S_0 > 0, \quad t \geq 0, \quad (3.4)$$

where $c, \sigma \in \mathbb{R}$, $\sigma > 0$ are the trend and volatility and the noise W^3 is such that $[W^1, W^3]_t = \rho_{13}t$ and $[W^2, W^3]_t = \rho_{23}t$.

3.4. Correlation structure and short-hand notations. The correlated noises $\vec{W} = (W^1, W^2, W^3)$ are a linear transformation of $\vec{Z} = (Z^1, Z^2, Z^3)$ via the Cholesky decomposition of the covariance matrix of $\vec{W}$, that is $\vec{W} = \Lambda \vec{Z}$ where Λ is the Cholesky decomposition of $\{\mathbb{E}[W_i^j W_i^k]\}_{i,j,k=1,2,3}$. Actually,

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ \rho_{12} & \sqrt{1 - \rho_{12}^2} & 0 \\ \rho_{13} & \frac{\rho_{23} - \rho_{12}\rho_{13}}{\sqrt{1 - \rho_{12}^2}} & \sqrt{1 - \rho_{13}^2 - \frac{(\rho_{23} - \rho_{12}\rho_{13})^2}{1 - \rho_{12}^2}} \end{pmatrix}.$$

We will denote by $\{\rho_{ij}^*\}_{i,j=1,2,3}$ the entries of Λ which will serve as short-hand notation for correlation parameters involved in measure changes later on. Hence,

$$W^i = \sum_j \rho_{ij}^* Z^j. \quad (3.5)$$

For the calculations we will extensively use the following formulas which are hereby introduced to ease the reading.

For $k, h \in \mathbb{R}$ we define the function

$$A_k(h) \triangleq \frac{1}{k} [1 - e^{-kh}], \quad (3.6)$$

Furthermore, for $k_1, k_2 \in \mathbb{R}$ and $h_1, h_2 \in \mathbb{R}$ we have the following relations

$$\begin{aligned} \int_0^h e^{-ks} ds &= A_k(h), \quad \lim_{k \rightarrow 0} A_k(h) = h, \\ \int_0^{h_1} A_k(h_2 + s) ds &= \frac{1}{k} [h_1 - e^{-kh_2} A_k(h_1)], \\ \int_0^h e^{-k_1 s} A_{k_2}(s) ds &= \frac{1}{k_2} [A_{k_1}(h) - A_{k_1+k_2}(h)], \\ \int_0^{h_1} e^{-k_1 s} A_{k_2}(h_2 + s) ds &= \frac{1}{k_2} [A_{k_1}(h_1) - e^{-k_2 h_2} A_{k_1+k_2}(h_1)]. \end{aligned} \quad (3.7)$$

4. ZERO-COUPON BOND AND FORWARD MEASURE

The fair value of a zero T -coupon bond at time t , denoted by $P(t, T)$ is well-known to be

$$P(t, T) = \mathbb{E}_{\mathbb{Q}} \left[e^{-\int_t^T r_s ds} \middle| \mathcal{F}_t \right],$$

where $\mathbb{Q}$ is an equivalent martingale measure such that T -bond prices are $\mathbb{Q}$ -martingales.

We consider, for some $\gamma \in \mathbb{R}$, the following parametrization for the Girsanov transformation of our noise,

$$\begin{aligned} dZ_t^{1,\mathbb{Q}} &= dZ_t^1 + \gamma dt \\ dZ_t^{2,\mathbb{Q}} &= dZ_t^2 \\ dZ_t^{3,\mathbb{Q}} &= dZ_t^3 + \frac{c - r_t - \sigma \gamma \rho_{31}^*}{\sigma \rho_{33}^*} dt, \end{aligned} \quad (4.1)$$

where $Z^{i,\mathbb{Q}}$ is a Brownian motion under $\mathbb{Q}$ for $i \in \{1, 2, 3\}$. Note that this implies that

$$d \left(e^{-\int_0^t r_s ds} S_t \right) = \sigma \left(e^{-\int_0^t r_s ds} S_t \right) dW_t^{3,\mathbb{Q}},$$

where $dW_t^{3,\mathbb{Q}} = \rho_{31}^* dZ_t^{1,\mathbb{Q}} + \rho_{32}^* dZ_t^{2,\mathbb{Q}} + \rho_{33}^* dZ_t^{3,\mathbb{Q}}$, making the discounted asset price a martingale under $\mathbb{Q}$. Furthermore, we get the dynamics of r under $\mathbb{Q}$ given by

$$dr_t = a \left[b - \frac{\gamma \tau}{a} - r_t \right] dt + \tau dW_t^{1,\mathbb{Q}}, \quad t \in [0, T], \quad (4.2)$$

where $dW_t^{1,\mathbb{Q}} = dZ_t^{1,\mathbb{Q}}$.

Hence, the price of the T -bond at time $t \leq T$ is given by

$$P(t, T) = e^{-A_a(T-t)r_t + D_a(T-t)}, \quad t \in [0, T], \quad (4.3)$$

where

$$D_a(h) \triangleq \left(\frac{\tau^2}{2a^2} - b + \frac{\gamma\tau}{a} \right) [h - A_a(h)] + \frac{\tau^2}{4a} A_a(h)^2$$

and A_a is given as in (3.6).

5. VALUATION OF PURE ENDOWMENT

A pure endowment is a contract that pays out one monetary unit to the holder in case they survive until the maturity of the bond.

In the case of deterministic mortality we have that the survival probability that a life aged $x + t$ at time t lives for a further $T - t$ years is given by

$${}_{T-t}p_{x+t} = e^{-\int_t^T \mu_u du}.$$

However, if μ is stochastic, the survival probability is a random variable and the price of a pure endowment at time $t \in [0, T]$ is given by

$$P_\mu(t, T) \triangleq \mathbb{E}_\mathbb{Q} \left[e^{-\int_t^T r_s ds} e^{-\int_t^T \mu_s ds} \middle| \mathcal{F}_t \right].$$

In what follows, we will make use of forward measure, for a more detailed explanation of such measures and related concepts we refer to [16].

Observe that $P(t, T) = P_0(t, T)$ where $P_0(t, T)$ denotes the price at time t of a longevity T -bond with $\mu \equiv 0$. Now, since $P(0, T)e^{\int_0^T r_s ds} > 0$, $\mathbb{P}$ -a.s. and

$$\mathbb{E}_\mathbb{Q} \left[\frac{1}{P(0, T)e^{\int_0^T r_s ds}} \right] = \mathbb{E}_\mathbb{Q} \left[\frac{P(T, T)}{P(0, T)e^{\int_0^T r_s ds}} \right] = 1$$

we can define a new measure $\mathbb{Q}^T \sim \mathbb{Q}$ (T -forward measure) on $\mathcal{F}_T$ by

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}} \Big|_{\mathcal{F}_T} = \frac{1}{P(0, T)e^{\int_0^T r_s ds}}.$$

Then for every $t \in [0, T]$ we have

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}} \Big|_{\mathcal{F}_t} = \mathbb{E}_\mathbb{Q} \left[\frac{d\mathbb{Q}^T}{d\mathbb{Q}} \middle| \mathcal{F}_t \right] = \frac{P(t, T)}{P(0, T)e^{\int_0^t r_s ds}}.$$

Bayes' rule implies that

$$P_\mu(t, T) = \mathbb{E}_\mathbb{Q} \left[e^{-\int_t^T r_s ds} \middle| \mathcal{F}_t \right] \mathbb{E}_{\mathbb{Q}^T} \left[e^{-\int_t^T \mu_s ds} \middle| \mathcal{F}_t \right] = P(t, T)S(t, T), \quad (5.1)$$

where $S(t, T) \triangleq \mathbb{E}_{\mathbb{Q}^T} \left[e^{-\int_t^T \mu_s ds} \middle| \mathcal{F}_t \right]$. The factor $P(t, T)$ is already derived in (4.3), so it remains to find $S(t, T)$. In order to do so, we need the dynamics of μ under $\mathbb{Q}^T$.

We know that $dW_t^{1,\mathbb{Q}^T} = \tau A_a(T-t)dt + dW_t^{1,\mathbb{Q}}$. Moreover, we know that $dW_t^{1,\mathbb{Q}} = \gamma dt + dW_t^1$. Hence,

$$dW_t^{1,\mathbb{Q}^T} = [\tau A_a(T-t) + \gamma] dt + dW_t^1 = [\tau A_a(T-t) + \gamma] dt + dZ_t^1.$$

The other Brownian motions remain unchanged, that is

$$dZ_t^{2,\mathbb{Q}^T} = dZ_t^2, \quad dZ_t^{3,\mathbb{Q}^T} = dZ_t^3.$$

The dynamics of μ under $\mathbb{Q}^T$ become

$$\begin{aligned} d\mu_t &= m(\bar{\mu}_t - \mu_t)dt + \zeta_t dW_t^2 + \eta_t dL_t \\ &= m(\bar{\mu}_t - \mu_t)dt + \zeta_t \left[\rho_{12} dZ_t^1 + \sqrt{1 - \rho_{12}^2} dZ_t^2 \right] + \eta_t dL_t \\ &= m(\bar{\mu}_t - \mu_t)dt + \zeta_t \left[\rho_{12} dW_t^1 + \sqrt{1 - \rho_{12}^2} dZ_t^2 \right] + \eta_t dL_t \\ &= m(\bar{\mu}_t - \mu_t)dt + \zeta_t \left[\rho_{12} \left(dW_t^{1,\mathbb{Q}^T} - (\tau A_a(T-t) + \gamma) dt \right) + \sqrt{1 - \rho_{12}^2} dZ_t^2 \right] + \eta_t dL_t \\ &= [m(\bar{\mu}_t - \mu_t) - \rho_{12}\zeta_t (\tau A_a(T-t) + \gamma)] dt + \zeta_t dW_t^{2,\mathbb{Q}^T} + \eta_t dL_t, \end{aligned}$$

where $dW_t^{2,\mathbb{Q}^T} = \rho_{12} dW_t^{1,\mathbb{Q}^T} + \sqrt{1 - \rho_{12}^2} dZ_t^{2,\mathbb{Q}^T}$.

As a result,

$$d\mu_t = m[\tilde{\mu}_t - \mu_t] dt + \zeta_t dW_t^{2,\mathbb{Q}^T} + \eta_t dL_t, \quad \tilde{\mu}_t \triangleq \bar{\mu}_t - \rho_{12} \frac{\zeta_t}{m} (\tau A_a(T-t) + \gamma). \quad (5.2)$$

In the next proposition we shall derive the affine structure in the exponent of the conditional Laplace transform of $(\mu_t, \lambda_t, M_t, \Lambda_t)$, where we define

$$M_t \triangleq \int_0^t \mu_s ds, \quad \Lambda_t \triangleq \int_0^t \lambda_s ds, \quad t \in [0, T]. \quad (5.3)$$

Proposition 5.1. *Let (μ_t, λ_t) , $t \in [0, T]$ be the Markov process given in (5.2) where λ is the intensity process of the Hawkes process L in (2.3). Further let M_t, Λ_t be as defined in (5.3). The moment generating function of $(\mu_T, \lambda_T, M_T, \Lambda_T)$ under $\mathbb{Q}^T$ conditioned on $\mathcal{F}_t$, $t \in [0, T]$, is then given by*

$$\mathbb{E}_{\mathbb{Q}^T} \left[e^{u_1 \mu_T + u_2 \lambda_T + u_3 M_T + u_4 \Lambda_T} \middle| \mathcal{F}_t \right] = e^{F_T(t) + G_T(t) \mu_t + H_T(t) \lambda_t + u_3 M_t + u_4 \Lambda_t},$$

whenever it is finite. Here

$$G_T(t) = u_1 e^{-m((T-t))} + u_3 A_m(T-t),$$

with A_m given in (3.6) and where F_T and H_T satisfy the following system of ODE's

$$\begin{cases} F'_T = \Theta_1(u_1, u_3, t, H_T), & F_T(T) = 0, \\ H'_T = \Theta_2(u_1, u_3, u_4, t, H_T), & H_T(T) = u_2, \end{cases} \quad (5.4)$$

where

$$\begin{aligned}\Theta_1(u_1, u_3, t, x) &= -m\tilde{\mu}_t G_T - G_T^2 \frac{\zeta_t^2}{2} - \beta \lambda x, \\ \Theta_2(u_1, u_3, u_4, t, x) &= \beta x - [\psi(\eta_t G_T + \alpha x) - 1] - u_4.\end{aligned}$$

Proof. The proof can be found in Appendix A.1. $\square$

Remark 5.2. Note that the functions F_T , G_T and H_T depend on u_1 and u_2 , but for better readability we dropped the dependence in the proof.

Remark 5.3. Note that if $m \rightarrow 0$ in equation (A.2) then $G_T(t)$ becomes $u_1 + u_3(T-t)$ which corresponds to the case with constant drift as in [3, Proposition 2].

Corollary 5.4. The price of a pure endowment is thus given by

$$P_\mu(t, T) = e^{-A_a(T-t)r_t + D_a(T-t)} e^{-A_m(T-t)\mu_t + F_T(t) + H_T(t)\lambda_t},$$

where F_T and H_T are obtained as solutions of the ODE (5.4) with $u_2 = 0$.

Proof. This is a consequence of (5.1) and Proposition 5.1 taking $u_1 = u_2 = u_4 = 0$ and $u_3 = -1$. $\square$

Remark 5.5. Note that in Corollary 5.4 we have an impact of the intensity process λ , which implies that the intensity of self-excitement of μ indeed has an influence on the price of the bond.

6. GUARANTEED MINIMUM MATURITY BENEFIT

A Guaranteed Minimum Maturity Benefit is a contract where the insured invests her premiums in a fund, say S , and collects the maximum value of the fund or a minimum guaranteed benefit, say G , at the end of the contract. That is a contract with payoff $\max\{G, S_T\}$. Observe that

$$\max\{G, S_T\} = S_T + (G - S_T)^+,$$

where $(x)^+$ denotes the positive part of a real number $x \in \mathbb{R}$. The second term above corresponds to a put option on S with maturity T and strike price G , which corresponds to the insurer's liability.

Pricing such contract when both mortality and interest rate are stochastic is more involved than the case of a pure endowment. We shall use the same idea as for the pure endowment.

Let $\tilde{\mathbb{Q}}^T$ be the (forward) measure defined by the Radon-Nikodym derivative

$$\frac{d\tilde{\mathbb{Q}}^T}{d\mathbb{Q}} \Big|_{\mathcal{F}_T} = \frac{e^{-\int_0^T r_s ds} e^{-\int_0^T \mu_s ds} P_\mu(T, T)}{P_\mu(0, T)}.$$

Then for every $t \in [0, T]$ we have

$$\frac{d\tilde{Q}^T}{dQ} \Big|_{\mathcal{F}_t} = \mathbb{E} \left[\frac{d\tilde{Q}^T}{dQ} \Big| \mathcal{F}_t \right] = \frac{e^{-\int_0^t r_s ds} e^{-\int_0^t \mu_s ds} P_\mu(t, T)}{P_\mu(0, T)}.$$

Using Bayes' rule for conditional expectations we arrive at

$$P_{\text{GMMB}}(t, T) = \mathbb{E}_{\mathbb{Q}} \left[e^{-\int_t^T r_s ds} e^{-\int_t^T \mu_s ds} (G - S_T)^+ \Big| \mathcal{F}_t \right] = P_\mu(t, T) \mathbb{E}_{\tilde{\mathbb{Q}}^T} [(G - S_T)^+ | \mathcal{F}_t]. \quad (6.1)$$

For the factor P_μ we have already derived a semi-analytic solution in terms of solutions of ODE's in Corollary 5.4. For the remaining factor, we need to derive the dynamics of S under $\tilde{\mathbb{Q}}^T$.

First, observe that

$$\begin{aligned} dW_t^{1, \tilde{\mathbb{Q}}^T} &= [\tau A_a(T-t) + \rho_{12} \zeta_t A_m(T-t)] dt + dW_t^{1, \mathbb{Q}} \\ dZ_t^{2, \tilde{\mathbb{Q}}^T} &= \sqrt{1 - \rho_{12}^2} \zeta_t A_m(T-t) dt + dZ_t^{2, \mathbb{Q}} \\ dZ_t^{3, \tilde{\mathbb{Q}}^T} &= dZ_t^{3, \mathbb{Q}} \end{aligned} \quad (6.2)$$

where recall that the function A_k , $k \in \mathbb{R}$ is defined in (3.6).

Observe that

$$W_t^{3, \mathbb{Q}} = \rho_{31}^* W_t^{1, \mathbb{Q}} + \rho_{32}^* Z_t^{2, \mathbb{Q}} + \rho_{33}^* Z_t^{3, \mathbb{Q}}.$$

Hence,

$$\begin{aligned} dW_t^{3, \mathbb{Q}} &= \rho_{31}^* \left[dW_t^{1, \tilde{\mathbb{Q}}^T} - [\tau A_a(T-t) + \rho_{12} \zeta_t A_m(T-t)] dt \right] \\ &\quad + \rho_{32}^* \left[dZ_t^{2, \tilde{\mathbb{Q}}^T} - \sqrt{1 - \rho_{12}^2} \zeta_t A_m(T-t) dt \right] \\ &\quad + \rho_{33}^* dZ_t^{3, \tilde{\mathbb{Q}}^T}. \end{aligned}$$

Therefore we have

$$\begin{aligned} dW_t^{3, \mathbb{Q}} &= \\ &= \left[\rho_{31}^* [\tau A_a(T-t) + \rho_{12} \zeta_t A_m(T-t)] + \rho_{32}^* \sqrt{1 - \rho_{12}^2} \zeta_t A_m(T-t) \right] dt + dW_t^{3, \tilde{\mathbb{Q}}^T}, \end{aligned}$$

where $dW_t^{3, \tilde{\mathbb{Q}}^T} = \rho_{31}^* dW_t^{1, \tilde{\mathbb{Q}}^T} + \rho_{32}^* dZ_t^{2, \tilde{\mathbb{Q}}^T} + \rho_{33}^* dZ_t^{3, \tilde{\mathbb{Q}}^T}$.

Notice that $\rho_{31}^* = \rho_{13}$ and that $\rho_{31}^* \rho_{12} + \rho_{32}^* \sqrt{1 - \rho_{12}^2} = \rho_{23}$. As a result,

$$dW_t^{3, \mathbb{Q}} = -[\rho_{13} \tau A_a(T-t) + \rho_{23} \zeta_t A_m(T-t)] dt + dW_t^{3, \tilde{\mathbb{Q}}^T}. \quad (6.3)$$

The following theorem gives a semi-closed formula for the price of a guaranteed minimum maturity benefit under stochastic interest rate and self-exciting mortality.

Theorem 6.1 (Price of guaranteed minimum maturity benefit). *The price at time $t \in [0, T]$ of a guaranteed minimum maturity benefit is given by*

$$P_{GMMB}(t, T) = P_\mu(t, T) \left[G\Phi(d(t, T)) - S_t e^{A_a(T-t)r_t + g(t, T) - h(t, T) + \frac{1}{2}\mathcal{V}(t, T)^2} \Phi(d(t, T) - \mathcal{V}(t, T)) \right],$$

where P_μ is the price of a pure endowment and

$$g(t, T) = \left(b_{\mathbb{Q}} - \frac{\tau^2}{a^2} \right) (T - t) + \left(-b_{\mathbb{Q}} + \frac{2\tau^2}{a^2} \right) A_a(T - t) - \frac{\tau^2}{a^2} A_{2a}(T - t) + \int_t^T \rho_{12} \zeta_u A_m(T - u) A_a(T - u) du,$$

$$h(t, T) = -\frac{\sigma \rho_{13} \tau}{a} A_a(T - t) + \left(\frac{\sigma \rho_{13} \tau}{a} + \frac{\sigma^2}{2} \right) (T - t) + \rho_{23} \sigma \int_t^T \zeta_s A_m(T - s) ds,$$

and

$$\mathcal{V}(t, T) \triangleq \left(\frac{\tau^2}{a^2} + \frac{2\tau \rho_{13} \sigma}{a} + \sigma^2 \right) (T - t) - \left(\frac{2\tau^2}{a^2} + \frac{2\tau \rho_{13} \sigma}{a} \right) A_a(T - t) + \frac{\tau^2}{a^2} A_{2a}(T - t).$$

Here, Φ denotes the distribution function of a standard normally distributed random variable and $d(t, T)$ is the quantile defined as

$$d(t, T) \triangleq \frac{\log\left(\frac{G}{S_t}\right) - (A_a(T - t)r_t + g(t, T) - h(t, T))}{\mathcal{V}(t, T)}.$$

Proof. The proof can be found in Appendix A.2. □

The following corollary provides a similar formula for the value of the total payout of a GMMB, given by

$$P_{GMMB}^{\text{total}}(t, T) = \mathbb{E}_{\mathbb{Q}} \left[e^{-\int_t^T r_s ds} e^{-\int_t^T \mu_s ds} \max\{G, S_T\} | \mathcal{F}_t \right],$$

and proved using the same method as Theorem 6.1.

Corollary 6.2. *The value at time $t \in [0, T]$ of the total payout of a guaranteed minimum maturity benefit is given by*

$$P_{GMMB}^{\text{total}}(t, T) = P_\mu(t, T) \left[G\Phi(d(t, T)) - S_t e^{A_a(T-t)r_t + g(t, T) - h(t, T) + \frac{1}{2}\mathcal{V}(t, T)^2} \left(\Phi(d(t, T) - \mathcal{V}(t, T)) - 1 \right) \right],$$

where $P_\mu(t, T)$ denotes the price of a pure endowment given in Corollary 5.4.

7. ESTIMATION AND CALIBRATION

In order to calibrate the parameters of our models we proceed in 3 steps. Firstly, we calibrate the mortality model to age-grouped data. We then translate these parameters to the setting of an ageing cohort as described in Section 7.2. Lastly we calibrate our financial models.

7.1. Mortality Data Description. The data used will be Spanish mortalities fetched from the human mortality database. We mainly consider weekly mortalities from the period 2012 – 2021, in total $N = 522$ individual points, for both men and women in the age-groups 65 – 74, 75 – 84, and 85+. These weekly data are fetched from the human mortality database’s short-term mortality fluctuations data series.

Since our goal is to calibrate our model to that of ageing individual, we will supplement our weekly age-grouped data with yearly mortalities of the 1-year cohorts born 1926 – 1935 for ages 75 – 84.

Additionally, while the following calibration is presented for Spanish data only, it has been performed for several countries, such as USA, Finland, Poland, Sweden, and Norway. While the parameters vary between countries we generally see comparable values and similar behaviour across the different data-sets. Despite this, we note that for countries or age-groups with a very small population and mortality, the weekly data become too noisy for the algorithm to function properly.

Remark 7.1. *For weekly Spanish mortality, the available data series of short-term mortality fluctuations extends back to the year 2000. However, for the purpose of calibration, we limit our consideration to a 10-year period. This simplifies trend computations as it suffices to consider linear trends which may not have the desired accuracy for longer time periods. In the case where linear trends are not sufficient we need to employ more complicated families of trends which may not be as numerically stable in the presence of excess mortality, which act akin to outliers in the data.*

7.2. Connecting Cohorts and Age-Groups. Consider the mortality of a specific chosen cohort, μ following the dynamics of (3.2). The first preliminary fact we will need is that if we set $\bar{\mu}_t = T_t + T'_t/m$ for some differentiable trend T . We may define $M = \mu - T$ and get

$$\begin{aligned} dM_t &= d(\mu_t - T_t) \\ &= m(\bar{\mu}_t - \mu_t)dt - T'_t dt + \zeta_t dW_t + \eta_t dL_t \\ &= m(\bar{\mu}_t - T'_t/m - T_t - (\mu_t - T_t))dt + \zeta_t dW_t + \eta_t dL_t \\ &= -mM_t dt + \zeta_t dW_t + \eta_t dL_t, \end{aligned}$$

making the de-trended mortality M an Ornstein-Uhlenbeck process with reversion level 0. We can further decompose the mortality by defining

$$\mathcal{H}_t \triangleq \int_{-\infty}^t e^{-m(t-u)} \eta_u dL_u, \quad \mathcal{G}_t \triangleq \int_{-\infty}^t e^{-m(t-u)} \zeta_u dW_u, \quad (7.1)$$

Additionally, we will consider $T_t = Y_t + S_t$, where Y represents the long-term yearly trend and S is the seasonal effect. As a consequence we have

$$\mu_t = Y_t + S_t + \mathcal{H}_t + \mathcal{G}_t$$

We will see in Section 8 that it is reasonable to model η , ζ , and S as proportional to Y in the sense that

$$\eta_t = Y_t \hat{\eta}, \quad \zeta_t = Y_t \hat{\zeta}, \quad S_t^C = Y_t \hat{S}_t, \quad (7.2)$$

for some constants $\hat{\eta}, \hat{\zeta}$, and a periodic function $\hat{S}$ with a period of one year.

Furthermore, we assume a Gompertz model for Y , i.e. that $Y_t = \exp(A_Y + tB_Y)$ for some constants A_Y, B_Y . This allows us to decompose μ as

$$\mu_t = Y_t \left(1 + \hat{S}_t + \hat{\mathcal{H}}_t + \hat{\mathcal{G}}_t \right),$$

where

$$\hat{\mathcal{H}}_t \triangleq \int_{-\infty}^t e^{-(m+B_Y)(t-u)} \hat{\eta} dL_u, \quad \hat{\mathcal{G}}_t \triangleq \int_{-\infty}^t e^{-(m+B_Y)(t-u)} \hat{\zeta} dW_u. \quad (7.3)$$

This decomposition will allow us to examine the mortality of an individual cohort by means of age-grouped data. We then consider an age-group, A , consisting of all cohorts with ages $x \in [A^-, A^+]$ at time t and assume that our chosen cohort belongs to A . Denote the mortality of the cohort aged x at time t by $\mu(x, t)$ and, to avoid confusion, denote the yearly trend of $\mu(x, t)$ by $Y(x, t)$ and the yearly trend of chosen cohort by $Y^C(t)$.

We assume that all cohorts in A at time t follow (3.2) and have the same parameters as our chosen cohort, with the exception of A_Y which is assumed to be constant, but may vary across cohorts. Lastly, assume that mortality rates are driven by the same noise. As a result, the mortality, $\mu(x, t)$ of any cohort will admit the decomposition $\mu(x, t) = Y(t, x) (1 + \hat{S}_t + \hat{\mathcal{H}}_t + \hat{\mathcal{G}}_t)$, where only Y depends on x .

If we further denote by $\mathcal{A}_t$ the measure on $[A^-, A^+]$ defined by

$$\mathcal{A}_t(I) = \frac{\text{number of individuals with age in } I \text{ at time } t}{\text{number of individuals with age in } [A^-, A^+] \text{ at time } t},$$

we can compute the average mortality across the age-group, μ^A , as

$$\begin{aligned} \mu_t^A &= \int_{A^-}^{A^+} \mu(x, t) d\mathcal{A}_t(x) \\ &= (1 + \hat{S}_t + \hat{\mathcal{H}}_t + \hat{\mathcal{G}}_t) \int_{A^-}^{A^+} Y(x, t) d\mathcal{A}_t(x) \\ &= (1 + \hat{S}_t + \hat{\mathcal{H}}_t + \hat{\mathcal{G}}_t) Y_t^A, \end{aligned}$$

where we define Y_t^A , the average yearly trend across the age-group, by $Y_t^A = \int_{A^-}^{A^+} Y(x, t) d\mathcal{A}_t(x)$.

One may show that μ^A follows the dynamics of (3.2), and in order to avoid confusion we shall here denote its parameters by a superscript A and the parameters of the chosen cohort by a superscript C . We will further note that Y^A turns out to

be almost constant in time, which allows us simplify and linearise our decomposition by considering

$$\mu_t^A \approx Y_t^A + \bar{Y}(\widehat{S}_t + \widehat{\mathcal{H}}_t + \widehat{\mathcal{G}}_t),$$

where $\bar{Y}$, to be defined in (7.5), is the average of Y^A over the time period of our data set. Under this approximation we see that $M^A = \mu^A - Y^A - \bar{Y}\widehat{S}$ follows the dynamics

$$dM_t^A = -m^A M_t^A dt + \zeta^A dW_t^2 + \eta^A dL_t,$$

where $\zeta^A = \bar{Y}\widehat{\zeta}$ and $\eta^A = \bar{Y}\widehat{\eta}$ are constants. This allows us to calibrate the parameters of μ^A along with the trend Y^C of our chosen cohort in order to recover all our cohort parameters by

$$\eta_t^C = Y_t \eta_t^A / \bar{Y}, \quad \zeta_t^C = Y_t \zeta_t^A / \bar{Y}, \quad S_t^C = Y_t S_t^A / \bar{Y}, \quad m^C = m^A - B_Y. \quad (7.4)$$

Remark 7.2. *To leverage the weekly data effectively, we make the assumption of a high degree of parameter homogeneity across cohorts within A . This implies that we can not explicitly consider cohort-specific effects beyond what is captured by the yearly trend Y^C .*

7.3. Calibration to Age-Grouped Data. In line with section Section 7.2, we assume that the age-grouped mortality, μ^A , follows (3.2) with η^A and ζ^A constant. To avoid obtrusive notation we will omit the superscript A for the parameters. The age-group calibration can then be split into several parts as illustrated in Figure 1.

In the first part we compute a trend, T , followed by the parameters m , ζ , and $\bar{\mu}$, which allows us to determine a list of time points, J , that are considered to be jumps in L . Using J and m we compute a set J' which are time points that are sufficiently unaffected by the jumps and use J' to compute a new trend T . We then compute m , ζ , and $\bar{\mu}$ based on T and J . These computations are iterated until convergence.

After this is done we can extract the increments of our noise processes W and L , followed by an estimation of the remaining parameters, α , β , and $\bar{\lambda}$. We can compute α and β for a given $\bar{\lambda}$ based on a method of moments and $\bar{\lambda}$ based on a given α and β based on maximum likelihood. As such we start with a guess of $\bar{\lambda}$ and iteratively compute (α, β) and $\bar{\lambda}$ until convergence.

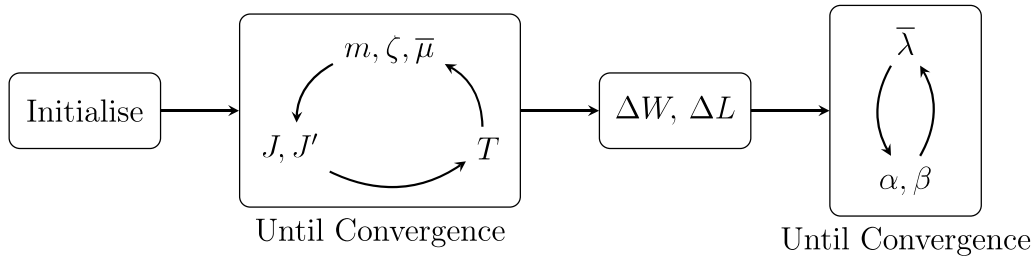


FIGURE 1. Outline of Calibration of Age-grouped Parameters

7.3.1. *Separating Jumps and the Trend.* We first recall the decomposition $M = \mu - T = \mathcal{H} + \mathcal{G}$ where $\mathcal{H}$ and $\mathcal{G}$ are defined in (7.3) and $T_t + T'_t = \bar{\mu}_t$. We see that during periods where $\mathcal{H}_t \approx 0$ we have $\mathbb{E}[\mu_t] \approx \mathbb{E}[\mathcal{G}_t] + T_t = T_t$ which allows us to compute the trend T by considering only such periods.

Furthermore, we again consider the decomposition

$$T(t) = S(t) + Y(t).$$

Here, the yearly trend, Y , of the age-group, is taken to be an affine function in time, i.e. $Y_t = a_Y + b_Y t$ for some constants $a_Y, b_Y \in \mathbb{R}$. In all cases it turns out that b_Y is very small, implying that Y is nearly constant for each age group, as such we will use

$$\bar{Y} = \sum_{i=1}^N \frac{Y_{t_n}}{N}, \quad (7.5)$$

as a reference value. The seasonal effect, S , is taken as a nearly general function, but assumed to be continuously differentiable and periodic with a period of one year. In order to properly capture the exact shape of the seasonality we will opt for a natural smoothing spline for calibration.

However, after the smoothing spline is used to identify the jumps, we want a simpler representation in order to clearly describe how the shape changes across groups, and particularly as a function of age. The exact shape of this trend is essential for the identification of jumps, but once this is completed a basic trigonometric function is sufficient to describe the mortality's change across seasons, along with simplifying certain computations. By regression we will then calibrate the parameters, A_S and B_S , and set

$$S_t = A_S \cos(B_S + 2\pi t).$$

The last preliminary results we will need are moment formulas for $\mathcal{G}$. We note that our data points are computed from the estimated weekly death probabilities and are thus approximately weekly averages of the underlying continuous mortality. To properly handle this we let $\mathcal{M}_n$ denote the residuals of the average mortalities for the n 'th week after removing the trend, i.e.

$$\mathcal{M}_n \triangleq \int_{(n-1)\Delta t}^{n\Delta t} \frac{\mu_u - T_u}{\Delta t} du = \int_{(n-1)\Delta t}^{n\Delta t} \frac{\mathcal{H}_u}{\Delta t} du + \int_{(n-1)\Delta t}^{n\Delta t} \frac{\mathcal{G}_u}{\Delta t} du.$$

Finally, we denote the weekly averages of the components as

$$\tilde{\mathcal{H}}_n \triangleq \int_{(n-1)\Delta t}^{n\Delta t} \frac{\mathcal{H}_u}{\Delta t} du, \quad \tilde{\mathcal{G}}_n \triangleq \int_{(n-1)\Delta t}^{n\Delta t} \frac{\mathcal{G}_u}{\Delta t} du. \quad (7.6)$$

Proposition 7.3. *Let $\mathcal{G}$ be defined as in equation (7.6), we then have that*

$$\begin{aligned} \tilde{\mathcal{G}}_n &= e^{-m\Delta t} \tilde{\mathcal{G}}_{n-1} + \frac{\zeta}{\Delta t} \int_{(n-1)\Delta t}^{n\Delta t} A_m(n\Delta t - s) dW_s \\ &\quad + \frac{\zeta e^{-m\Delta t}}{\Delta t} \int_{(n-2)\Delta t}^{(n-1)\Delta t} A_{-m}(s - (n-2)\Delta t) dW_s. \end{aligned}$$

We also have the following moment formulas

$$\begin{aligned}\mathbb{E} [\tilde{\mathcal{G}}_n \tilde{\mathcal{G}}_{n+1}] &= \zeta^2 \frac{A_m^2(\Delta t) e^{-m\Delta t} + 2A_m(\Delta t)(A_m(\Delta t) - A_{2m}(\Delta t))}{2m(\Delta t)^2}, \\ V_{\tilde{\mathcal{G}}} &\triangleq \mathbb{E} [\tilde{\mathcal{G}}_n^2] = \zeta^2 \frac{mA_m^2(\Delta t) + 2\Delta t - 4A_m(\Delta t) + 2A_{2m}(\Delta t)}{2(m\Delta t)^2}, \\ \rho &\triangleq \frac{\mathbb{E} [\tilde{\mathcal{G}}_n \tilde{\mathcal{G}}_{n+1}]}{V_{\tilde{\mathcal{G}}}} = m \frac{A_m^2(\Delta t) e^{-m\Delta t} + 2A_m(\Delta t)(A_m(\Delta t) - A_{2m}(\Delta t))}{mA_m^2(\Delta t) + 2\Delta t - 4A_m(\Delta t) + 2A_{2m}(\Delta t)}.\end{aligned}$$

Proof. This is a consequence of Fubini's theorem and Itô's isometry. $\square$

The parameter ρ is monotone in m implying we can compute ρ by the empirical first-order autocorrelation and then invert this formula as our estimate of m . This also allows us to compute ζ by the empirical variance which will provide a method of moments for these parameters.

With these results we can proceed by estimating the trend along with the parameters of $\mathcal{G}$. In order to compute the trend we aim to fit it to time periods where the impact of the Hawkes process is negligible. However, this requires knowledge of where the jump spots are located. In order to isolate areas of Hawkes activity and fit the residuals we implement a modified version of the straight-forward peaks over threshold (POT) method used for similar purposes in e.g. [5]. This modified method is performed iteratively and done separately for age- and sex-groups.

For any of these groups we first ignore $\mathcal{H}$ and fit the yearly and seasonal trend to the entire dataset. We then calibrate the parameters of $\mathcal{G}$ from the residuals using a method of moments from Proposition 7.3.

The next step is to locate jumps, for this we standardise the data by removing the effects of mean reversion in order to isolate the noise. For the Gaussian part we note that Proposition 7.3 means that the standardised increments, $\tilde{\mathcal{G}}_n - e^{-m\Delta t} \tilde{\mathcal{G}}_{n-1}$, should be normally distributed unless jumps are present. As for the Hawkes part of the noise we note that if jumps occur only at end-points of the week we have

$$\tilde{\mathcal{H}}_n - e^{-m\Delta t} \tilde{\mathcal{H}}_{n-1} = \frac{A_m(\Delta t)}{\Delta t} (L_{(n-1)\Delta t} - L_{(n-1)\Delta t^-}).$$

Considering these two facts, if we approximate jumps as occurring only at the end of weeks, we can observe that $\mathcal{M}_n - e^{m\Delta t} \mathcal{M}_{n-1}$ follows a Gaussian distribution, unless a jump occurs within the period $[(n-1)\Delta t, n\Delta t)$. We can then determine whether or not a week contains a jump by the POT method.

The POT determines the highest positive increments to be jumps until the remaining increments have a skewness of zero, thereby matching that of a normal distribution. This will then determine our list of jump spots.

Remark 7.4. *Note that we here likely ignore several small jumps which have sizes comparable to those of the Gaussian noise. However, since the effect of jumps on both μ and λ are proportional to their size the effect of ignoring the smallest jumps should be small.*

Furthermore, we do not take into account negative jumps. In the considered data we observe mostly positive deviations of the mortality from the norm, while any large negative changes occur only after a preceding positive jump. As such any big negative jump is adequately handled by mean reversion.

We can use our previous estimate of m to compute the number of weeks it takes for the impact of a jump to decay by at least 95%. The lowest whole number satisfying this is $\lceil -\log(5\%)/m\Delta t \rceil$. With this we consider that number of weeks following a detected jump to be affected by $\mathcal{H}$ and exclude these points for the next trend estimation.

We then repeat this process: a new trend is estimated using points unaffected by jumps, and new parameters for $\mathcal{G}$ are computed using all increments of $\mathcal{M}$ that are not considered to be jumps. This then yields a new list of affected time points that replace the previous list. This process is iterated until the list of affected points match the previous list, at which point the process terminates.

Remark 7.5. *The initial trend is calibrated without the assumption of jumps which cause it to consistently overestimate the actual trend, thereby artificially lowering the skew of the increments. In order to keep this bias from terminating the algorithm early we force the first POT procedure to keep going until the top 5% of increments are counted as jumps. The final set of jump spots has empirically been found not to change if this value is changed as long as it is big enough to overcome the bias from the initial trend. For the majority of groups this initial kick-start has no effect on the final results.*

7.3.2. Noise Increment Extraction. With this we can extract our estimates of

$$\Delta W_n \triangleq W_{n\Delta t} - W_{(n-1)\Delta t}$$

and

$$\Delta L_n \triangleq L_{n\Delta t^-} - L_{(n-1)\Delta t^-} = L_{(n-1)\Delta t} - L_{(n-1)\Delta t^-},$$

the increments of our Gaussian and Hawkes noise processes.

The Brownian motion can be evaluated for all increments, except for those affected by jumps, with the formula from Proposition 7.3

$$\begin{aligned} \Delta \widetilde{W}_n = \mathcal{M}_n - e^{-m\Delta t} \mathcal{M}_{n-1} &= \frac{\zeta}{\Delta t} \int_{(n-1)\Delta t}^{n\Delta t} A_m(n\Delta t - s) dW_s \\ &+ \frac{\zeta e^{-m\Delta t}}{\Delta t} \int_{(n-2)\Delta t}^{(n-1)\Delta t} A_{-m}(s - (n-2)\Delta t) dW_s. \end{aligned} \quad (7.7)$$

Note that $\Delta \widetilde{W}_n$ is not exactly equal to ΔW_n , but this discrepancy will be corrected in subsection Section 7.7, where we compute the correlation factors.

In the spots where a jump occurs we assume that the effect of the Hawkes process significantly overshadows the Gaussian noise, as such we ignore the contribution of $\mathcal{G}$ on the increment and set

$$\Delta L_n = \frac{\mathcal{M}_n - e^{-m\Delta t} \mathcal{M}_{n-1}}{A_m(\Delta t)/\Delta t}.$$

Remark 7.6. *In order to extract the increments of the compound Hawkes process we would in theory need to know exactly how big the detected jump is, which is something we can only approximate. If a jump of size J occurs at time s inside the week $[0, \Delta t]$ we would measure a discrepancy of*

$$\int_s^{\Delta t} \frac{J e^{-m(u-s)}}{\Delta t} du = J A_m(\Delta t - s) / \Delta t.$$

This means that depending on when the jump occurred the measured discrepancy changes. We can compute an average scaling factor by assuming a distribution on s and computing $\mathbb{E}[A_m(\Delta t - s) / \Delta t]$. This means we may alternatively estimate our jumps by

$$\Delta L_n = \frac{\mathcal{M}_n - \rho \mathcal{M}_{n-1}}{\mathbb{E}[A_m(\Delta t - s) / \Delta t]},$$

for any increment determined to be a jump. This method has some advantages for low frequency measurements, but cannot fully extract the individual jumps, it merely scales the results in order to achieve the correct average.

7.3.3. Hawkes Process Parameters. With the jumps estimated we can move on to estimating the parameters of the Hawkes process. It turns out that a pure maximum likelihood method is too unstable for all parameters except $\bar{\lambda}$ due to the data-set's resolution and dense clustering of jumps. To avoid these issues we compute α and β by a method of moments jointly with estimating $\bar{\lambda}$ by maximum likelihood. This allows us to utilise a conservative number of the empirical moments while keeping the stable ML estimator of $\bar{\lambda}$.

Remark 7.7. *The parameters α , η and the scale of the jumps can not be uniquely determined and carries one degree of freedom. In order to extract the jumps we initially set $\eta = \eta^{\text{pre}} = 1$ and then compute $\alpha = \alpha^{\text{pre}}$ along with the jumps. Since the cohort model assumes that α^C is constant while η^C varies in time we expect η to change from group to group, we therefore set $\eta = 1/\alpha^{\text{pre}}$, $\alpha = 1$, and scale the measured jumps accordingly.*

Proposition 7.8. *Let $\mathcal{H}$ and $\tilde{\mathcal{H}}$ be defined as in equation (7.1) and equation (7.6) and denote $\Delta L_t \triangleq L_{t+\Delta t} - L_t$, $\lambda_\infty \triangleq \mathbb{E}[\lambda_t]$. If $\mathbb{E}[J]\alpha/\beta < 1$ we then have the following equalities, furthermore, these equations do not depend on time t and week number n .*

$$\begin{aligned} \mathbb{E}[\tilde{\mathcal{H}}_n] &= \frac{\eta \lambda_\infty \mathbb{E}[J]}{m} \\ \mathbb{E}[\tilde{\mathcal{H}}_n^2] &= \frac{2\mathbb{E}[\mathcal{H}_t^2]}{m^2 \Delta t^2} - \frac{2\mathbb{E}[\mathcal{H}_{t+\Delta t} \mathcal{H}_t]}{m^2 \Delta t^2} + \frac{\eta^2 \mathbb{E}[\Delta L_t^2]}{m^2 \Delta t^2} \\ &\quad - \frac{2\eta \mathbb{E}[\Delta L_t \mathcal{H}_{t+\Delta t}]}{m^2 \Delta t^2} + \frac{2\eta \mathbb{E}[\Delta L_t \mathcal{H}_t]}{m^2 \Delta t^2}, \end{aligned}$$

where the other moments appearing in the formulas are on the following form.

$$\begin{aligned}
 \lambda_\infty &= \frac{\bar{\lambda}}{1 - \mathbb{E}[J]\alpha/\beta} \\
 \mathbb{E}[\lambda_t^2] &= \lambda_\infty^2 + \frac{\alpha^2 \mathbb{E}[J^2] \lambda_\infty}{2(\beta - \alpha \mathbb{E}[J])} \\
 \mathbb{E}[\mathcal{H}_t \lambda_t] &= \frac{\lambda_\infty \alpha \eta \mathbb{E}[J^2] + \mathbb{E}[\lambda_t^2] \eta \mathbb{E}[J] + \mathbb{E}[\mathcal{H}_t] \bar{\lambda}_t \beta}{m + \beta - \alpha \mathbb{E}[J]} \\
 \mathbb{E}[\mathcal{H}_t^2] &= \frac{2\eta \mathbb{E}[J] \mathbb{E}[\mathcal{H}_t \lambda_t] + \lambda_\infty \eta^2 \mathbb{E}[J^2]}{2m} \\
 \mathbb{E}[\mathcal{H}_{t+\Delta t} \mathcal{H}_t] &= e^{-m\Delta t} \mathbb{E}[\mathcal{H}_t^2] + e^{-m\Delta t} A_{\beta - \alpha \mathbb{E}[J] - m}(\Delta t) \eta \mathbb{E}[J] \mathbb{E}[\mathcal{H}_t \lambda_t] \\
 &\quad + (A_m(\Delta t) - e^{-m\Delta t} A_{\beta - \alpha \mathbb{E}[J] - m}(\Delta t)) \eta \mathbb{E}[J] \lambda_\infty \mathbb{E}[\mathcal{H}_t] \\
 \mathbb{E}[\Delta L_t \mathcal{H}_t] &= A_{\beta - \alpha \mathbb{E}[J]}(\Delta t) \mathbb{E}[J] \mathbb{E}[\mathcal{H}_t \lambda_t] \\
 &\quad + (\Delta t - A_{\beta - \alpha \mathbb{E}[J]}(\Delta t)) \mathbb{E}[J] \mathbb{E}[\mathcal{H}_t] \lambda_\infty \\
 \mathbb{E}[\Delta L_t \mathcal{H}_{t+\Delta t}] &= \mathbb{E}[J] A_m(\Delta t) \mathbb{E}[\mathcal{H}_t \lambda_t] + \eta \mathbb{E}[J^2] A_m(\Delta t) \lambda_\infty \\
 &\quad + \frac{\eta \mathbb{E}[J]^2 \lambda_\infty^2 (e^{-m\Delta t} - 1 + m\Delta t)}{m^2} \\
 &\quad + \eta \mathbb{E}[J] \frac{(\mathbb{E}[\lambda_t^2] \mathbb{E}[J] + \lambda_\infty \alpha \mathbb{E}[J^2] - \mathbb{E}[J] \lambda_\infty^2) A_m(\Delta t) e^{-m\Delta t}}{\beta - \alpha \mathbb{E}[J]} \\
 &\quad - \eta \mathbb{E}[J] \frac{(\mathbb{E}[\lambda_t^2] \mathbb{E}[J] + \lambda_\infty \alpha \mathbb{E}[J^2] - \mathbb{E}[J] \lambda_\infty^2) A_{\beta - \alpha \mathbb{E}[J] - m}(\Delta t) e^{-m\Delta t}}{\beta - \alpha \mathbb{E}[J]} \\
 \mathbb{E}[\Delta L_t^2] &= \mathbb{E}[J^2] \lambda_\infty \Delta t + \mathbb{E}[J]^2 \lambda_\infty^2 \Delta t^2 \\
 &\quad + 2\mathbb{E}[J] \left(\frac{(\mathbb{E}[\lambda_t^2] \mathbb{E}[J] + \lambda_\infty \alpha \mathbb{E}[J^2] - \mathbb{E}[J] \lambda_\infty^2)}{\beta - \alpha \mathbb{E}[J]} \right) (t - A_{\beta - \alpha \mathbb{E}[J]}(\Delta t)).
 \end{aligned}$$

Proof. In order to compute these we note that by the Ornstein-Uhlenbeck structure of $\mathcal{H}$ we have for $t = (n-1)\Delta t$ that

$$\mathcal{H}_{t+\Delta t} - \mathcal{H}_t = -m \int_t^{t+\Delta t} \mathcal{H}_s ds + \eta \int_t^{t+\Delta t} dL_s = -m\Delta t \widetilde{\mathcal{H}}_n + \eta(L_{t+\Delta t} - L_t)$$

implying

$$\widetilde{\mathcal{H}}_n = \frac{(\eta(L_{t+\Delta t} - L_t)) - (\mathcal{H}_{t+\Delta t} - \mathcal{H}_t)}{m\Delta t},$$

which means we can compute the moments of $\widetilde{\mathcal{H}}$ by the moments of increments of $\mathcal{H}$ and L . The generator, $\mathcal{A}$, of $(\lambda, \mathcal{H}, L)$ is defined for a function $f : (s, \lambda, \mathcal{H}, L) \mapsto$

$f(s, \lambda, \mathcal{H}, L)$ by

$$\begin{aligned} \mathcal{A}(f) = & \partial_s f + [\beta(\bar{\lambda} - \lambda)] \partial_\lambda f + -m\mathcal{H}\partial_{\mathcal{H}}f \\ & + \lambda \int_0^\infty [f(s, \lambda + \alpha z, \mathcal{H} + \eta z, L + z) - f(s, \lambda, \mathcal{H}, L)] \nu(dz). \end{aligned}$$

We will consider functions on the form $f(s, \mathcal{H}, \lambda, L) = g(s)\mathcal{H}^i\lambda^jL^k$ for some non-negative integers i, j, k , and for some bounded g . Dynkin's formula yields

$$\frac{d}{dt} \mathbb{E} [f(t, \lambda_t, \mathcal{H}_t, L_t) | \mathcal{F}_s] = \mathbb{E} [\mathcal{A}(f)(t, \lambda_t, \mathcal{H}_t, L_t) | \mathcal{F}_s],$$

which allows us to compute conditional expectations by means of the resulting differential equations. Such methods for computing moments of Hawkes-related processes are well established and covered in e.g. [8] or [11]. We can then calculate the unconditional expectations from

$$\mathbb{E} [f(t, \lambda, \mathcal{H}_t, L_t)] = \lim_{s \rightarrow -\infty} \mathbb{E} [f(t, \lambda, \mathcal{H}_t, L_t) | \mathcal{F}_s],$$

which yield the relevant formulas. $\square$

Proposition 7.9. *Let α, β be fixed and assume that jumps only occur at time points $t = n\Delta t$ for $n \in \mathbb{N}, n \leq N - 1$, where N is the size of our dataset (implying $N - 1$ increments). The maximum likelihood estimate of $\bar{\lambda}$ is approximately given by the unique solution to*

$$N - 1 - |\mathcal{N}| = \sum_{n \in \mathcal{N}} \frac{1}{e^{\Delta t \bar{\lambda} + a_n} - 1},$$

where $\mathcal{N}$ is the set of numbers, n , such that week n has a detected jump, $|\mathcal{N}|$ is the size of this set, and

$$a_n = \frac{1 - e^{-\beta \Delta t}}{\beta \Delta t} \sum_{i=0}^{n-2} e^{-(n-2-i)\beta \Delta t} \eta \Delta L_{i\Delta t},$$

with $\Delta L_{i\Delta t} = L_{(i+1)\Delta t} - L_{i\Delta t}$.

Proof. If the intensity at the beginning of the period $[(n-1)\Delta t, n\Delta t]$ is λ_n then the cumulative intensity across the period under the approximating assumption of no jumps within the period will be $\Delta t \bar{\lambda} + (\lambda_n - \bar{\lambda})(1 - e^{-\beta \Delta t})/\beta$. This means that, given our detected jumps, we will have a cumulative intensity over the period $[(n-1)\Delta t, n\Delta t]$ given by

$$\Delta t \bar{\lambda} + \frac{1 - e^{-\beta \Delta t}}{\beta} \int_0^{(n-1)\Delta t} e^{-(\Delta t(n-1)-s)\beta} \eta dL_s$$

If we make the approximation of all jumps within a week occurring at the end of the period this expression equals the feasible value of

$$\Delta t \bar{\lambda} + \frac{1 - e^{-\beta \Delta t}}{\beta} \sum_{i=0}^{n-2} e^{-(n-2-i)\beta \Delta t} \eta \Delta L_{i\Delta t} = \Delta t \bar{\lambda} + a_n.$$

We can then compute the likelihood of week n having no detected jumps by the value $\exp(-\Delta t \bar{\lambda} - a_n)$. Conversely, the likelihood of there being one or more detected jumps in week n must be $1 - \exp(-\Delta t \bar{\lambda} - a_n)$. The log-likelihood of our observed jumps then becomes

$$\sum_{n \in \mathcal{N}^c} (-\Delta t \bar{\lambda} - a_n) + \sum_{n \in \mathcal{N}} \log(1 - e^{-\Delta t \bar{\lambda} - a_n}).$$

Taking the derivative with respect to $\bar{\lambda}$ and setting it to zero gives us

$$0 = \sum_{n \in \mathcal{N}^c} (-\Delta t) + \sum_{n \in \mathcal{N}} \frac{\Delta t e^{-\Delta t \bar{\lambda} - a_n}}{1 - e^{-\Delta t \bar{\lambda} - a_n}},$$

by noting that $|\mathcal{N}^c| = N - 1 - |\mathcal{N}|$ we get

$$N - 1 - |\mathcal{N}| = \sum_{n \in \mathcal{N}} \frac{1}{e^{\Delta t \bar{\lambda} + a_n} - 1}.$$

Which is the desired result, furthermore, since the right hand side of this equation is monotone in $\bar{\lambda}$ for $\bar{\lambda} \geq 0$ it must have a unique solution with no other local minima. $\square$

Using these results we can carry out the last part of the calibration. We note that for the de-trended mortalities we have

$$\mathbb{E}[\mathcal{M}_n] = \mathbb{E}[\tilde{\mathcal{H}}_n],$$

and

$$\mathbb{E}[\mathcal{M}_n^2] = \mathbb{E}[\tilde{\mathcal{H}}_n^2] + \mathbb{E}[\tilde{\mathcal{G}}_n^2] = \mathbb{E}[\tilde{\mathcal{H}}_n^2] + V_{\tilde{\mathcal{G}}}.$$

This allow us to rephrase the moments of $\mathcal{M}$ through the moments of $\tilde{\mathcal{H}}$.

With the formulas from Proposition 7.8 we note that the formula for $\mathbb{E}[\mathcal{H}]$ implies that

$$\frac{\alpha}{\beta} = \left(1 - \frac{\mathbb{E}[J] \bar{\lambda} \eta}{m \mathbb{E}[\mathcal{H}_n]}\right) / E[J],$$

which gives us a value for α/β by plugging in the empirical mean of $\mathcal{M}$. Under this constraint the formula for $\mathbb{E}[\tilde{\mathcal{H}}^2]$ is monotone in β , verified empirically, allowing us to compute α and β by means of the empirical second moment of $\mathcal{M}$.

This gives α and β for a specific $\bar{\lambda}$, conversely, given α and β , we can compute $\bar{\lambda}$ by maximum likelihood via Proposition 7.9.

Finally we combine these estimations by finding the value for $\bar{\lambda}$ implying values of α and β with the original value of $\bar{\lambda}$ as the solution of the likelihood maximization. This value is found by iteratively computing $\bar{\lambda}$ and (α, β) until convergence and has been determined empirically to be resistant to initial conditions.

Remark 7.10. *It is worth mentioning that while there always exists a maximum likelihood estimate of $\bar{\lambda}$ for a given (α, β) the converse is not necessarily true. There are cases where no (α, β) gives the correct moments. While such cases did not*

prevent proper calibrations in this article, it is possible that other datasets do not allow joint calibration of $(\bar{\lambda}, \alpha, \beta)$ by using the method presented. If this is the case, then other methods must be employed, such as considering higher order moments or similar techniques.

7.3.4. Results for Group-Based Parameters. After implementing this method we get the following parameter values, we also here include $\mathbb{E}[J]$ and the average yearly trend, $\bar{Y}$ as useful reference values.

Ages	Sex	β	$\bar{\lambda}$	η	m	ζ	A_S	B_S	$\bar{Y}$	$\mathbb{E}[J]$
65-74	M	5.5	1.1	2.3e-3	61	3.2e-4	1.6e-3	-0.50	1.8e-2	3.7
	F	7.5	1.2	1.2e-3	100	2.3e-4	6.2e-4	-0.45	7.8e-3	4.6
75-84	M	11	1.6	2.8e-3	51	6.8e-4	5.6e-3	-0.66	4.9e-2	7.2
	F	18	2.8	1.1e-3	52	4.5e-4	3.3e-3	-0.65	2.8e-2	9.8
85+	M	12	2.4	8.5e-3	61	2.5e-3	1.8e-2	-0.54	1.5e-1	7.9
	F	15	2.5	5.6e-3	45	1.7e-3	1.4e-2	-0.64	1.2e-1	8.7

TABLE 1. Calibrated Parameters for different Age- and Sex-Groups

We additionally include two plots, namely Figures 2 and 3, in order to visually represent the behaviour of the data. These plots show μ , λ , and T for females aged 85+ and males 75-84 respectively. The most prominent feature is of course the COVID pandemic, where μ exceeds more than the twice the value of T and λ reaches nearly 1/week in both cases. Our model also captures other significant phenomena as jumps, such as severe winters and heatwaves. Notably, Figure 2 shows significantly more excess mortality due to winters than Figure 3, as well as clearly showing the heat wave of 2015, which was the worst on record for Spain at the time. In fact, the effect of non-pandemic-related factors relative to COVID increases in age for both sexes.

An important thing to note is the presence of white noise in the measurements, which decreases the empirical autocorrelation of the mortality and therefore leads to higher mean reversion speeds. The consequences of this can be seen for some of the age-groups in Table 1. This noise is especially prevalent in the younger age groups where mortality is lower, such as women aged 65-74, but is also affected by the size of small groups, such as in the group of males over 85. Consequently, these groups have higher estimated mean reversion speeds than other groups.

According to our model we should still expect the quantities m , β , $\bar{\lambda}$, and $\mathbb{E}[J]$ to remain close to constant for a fixed sex, and while they maintain the same order of magnitude, there is still significant variation, possibly caused by numerical instability or imperfections in the model. In particular we note that both β and $\mathbb{E}[J]$ increases with age indicating that the jumps are more heavily clustered for older groups. There is also a noticeable increase in the fraction $\mathbb{E}[\mathcal{H}]/\bar{Y}$ as the groups get older, showing that older generations are more exposed to the causes of excess mortality observed in this time period relative to the base level.

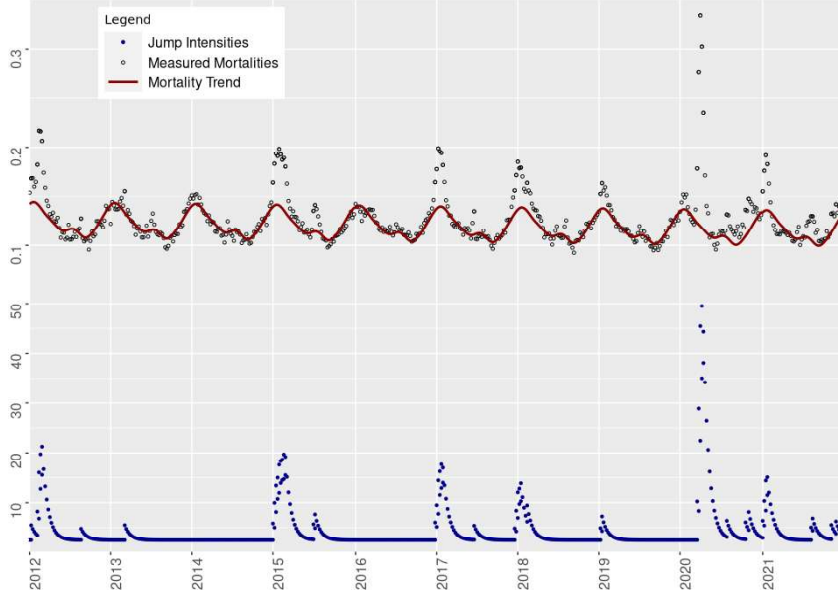


FIGURE 2. Yearly Mortality and Jump Intensity for Females aged 85+

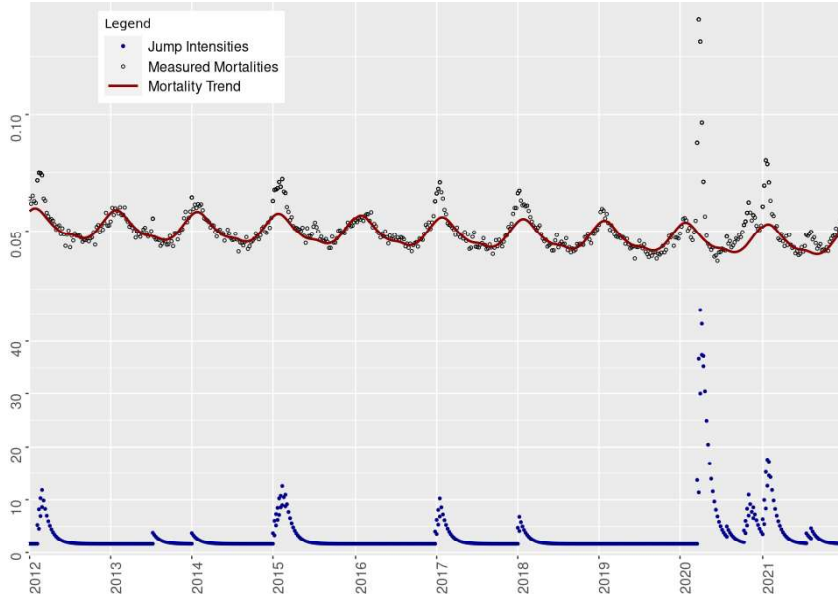


FIGURE 3. Yearly Mortality and Jump Intensity for Males aged 75-84

For the parameters the model expects to change, i.e. (η, ζ, A_S) , we note that all of these values are very closely proportional to $\bar{Y}$. In fact it covers over 97% of the parameters variability across ages in all cases. These results then justify our assumption of proportionality used in Section 7.2 to connect the age-grouped parameters and those for a specific ageing cohort.

Remark 7.11. *One theoretical weakness of our mortality model is the possibility of negative values, however, the likelihood of such an event is fortunately extremely low. To get a sense of how common negative mortality is, given the parameters in Table 1, we consider a lower bound on the average time it would take for μ to hit 0. By ignoring the Hawkes noise and setting the trend to its minimal value, $T_{\min} = \bar{Y} - A_S$ we get a Gaussian Ornstein-Uhlenbeck process. This has an explicit average hitting time given by*

$$\frac{1}{2m} \sum_{i=1}^{\infty} \frac{(\sqrt{2}T_{\min}/\zeta)^i}{i!} \Gamma\left(\frac{i}{2}\right) \approx \frac{\zeta\sqrt{2\pi}}{mT_{\min}} \exp\left(\frac{T_{\min}^2}{2\zeta^2}\right),$$

which serves as a lower bound on the true average hitting time of 0. Among all age groups, the lowest value is observed for women aged 65-74, which is approximately 3e208 years. This implies that while negative mortalities are theoretically possible, they are practically implausible.

7.4. Calibration to Cohort Data. In order to price insurance contracts we will need to translate our model for the mortality of a static group to that of an ageing individual. We will here consider an insured woman aged 75 at the beginning of 2022, i.e. when $t = 0$. This means that for the next 10 years, this individual will belong to the female 75-84 age group. To avoid confusion, we will again denote the parameters for the chosen cohort by superscript C , and the parameters calibrated for the insured's age-group by superscript A .

In line with Section 7.2, we decompose the trend of the insured's cohort by $T^C = S^C + Y^C$, and consider a simple Gompertz model for Y , i.e. $Y_t^C = e^{A_Y + B_Y t}$ for some constants A_Y, B_Y . Furthermore, we also recall (7.4), which will allow us to compute

$$\begin{aligned} \eta_t^C &= \eta^A Y_t^C / \bar{Y}, \\ \zeta_t^C &= \zeta^A Y_t^C / \bar{Y}, \\ S_t^C &= A_S^A \cos(2\pi t + B_S^A) Y_t^C / \bar{Y}, \\ m^C &= m^A - B_Y. \end{aligned}$$

In order to calibrate A_Y and B_Y we first note the following:

$$\begin{aligned} \mathbb{E}[\mu_t] &= Y_t^C + \int_{-\infty}^t \mathbb{E}[J] \lambda_{\infty} \eta_s^C e^{-m^C(t-s)} ds \\ &= Y_t^C + \int_{-\infty}^t \mathbb{E}[J] \lambda_{\infty} \eta_t^C e^{-(B_Y + m^C)(t-s)} ds \\ &= Y_t^C + \frac{\mathbb{E}[J] \lambda_{\infty} \eta_t^C}{m^C + B_Y} \\ &= Y_t^C \left(1 + \frac{\mathbb{E}[J] \lambda_{\infty} \eta^A}{(m^C + B_Y) \bar{Y}} \right). \end{aligned}$$

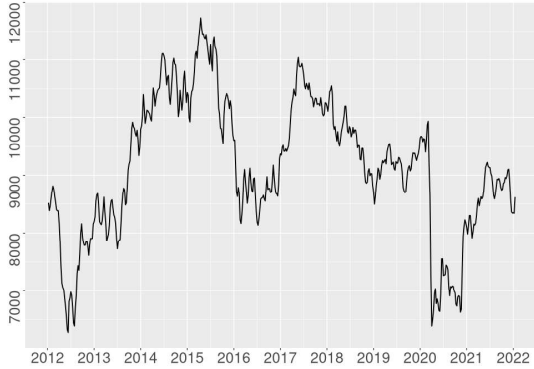
Based on this, and our assumption of Y being log-linear, we see that

$$\mathbb{E}[\mu_t] = \exp(A'_Y + B_Y t), \quad A'_Y = A_Y + \log\left(1 + \frac{\mathbb{E}[J]\lambda_\infty \eta^A}{(m^C + B_Y)\bar{Y}}\right).$$

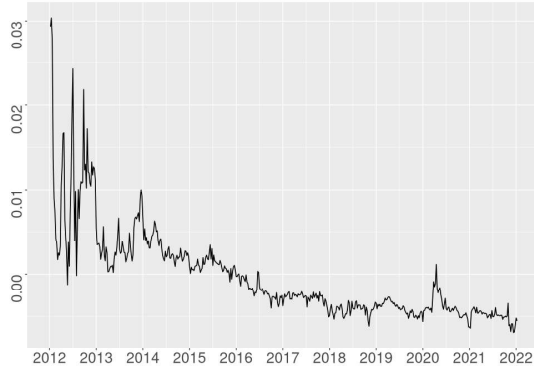
We can then calibrate A'_Y and B_Y on yearly average female mortality over the age period $75 - 84$ for all yearly cohorts born 1927-1936 using linear regression on the log-scale. By noting that the regression parameters are approximately linear as functions of birth year we project the parameter values to the cohort in question, i.e. birth year 1947. We then finally compute A_Y based on our estimate of A'_Y . For the considered cohort we got $A_Y \approx -4.63$, $B_Y \approx 0.160$.

7.5. Financial Data Description. For the financial data we gathered weekly data for the Spanish bond market for fifteen different maturities, ranging from three months to thirty years, over the time period 2012 – 2021. We also collected weekly data for the IBEX35, the main stock index for Spain over the same period which serves as a stand-in for our stock.

7.6. Calibration of Financial Models. Since we assume a simple Black-Scholes model for our asset, plotted in Figure 4A, the log-returns of the index have a mean of $(c - \sigma^2/2)\Delta t$ and a variance of $\sigma^2\Delta t$, and therefore provide estimates of (c, σ) by standard methods. This also readily gives us an estimate of the underlying Brownian motion W^3 by the scaled log-returns. In the case of the IBEX35 we found a yearly standard deviation of $\sigma = 0.16$ and a statistically insignificant drift of $c = 0.014$.



(A) Path of the IBEX35



(B) Path of the Filtered Spanish Short-Rate

For the short-rate calibration we only need the correlation factors and current forward curve, which is already available. In line with this we choose a calibration that most accurately depicts the empirical forward curves.

We recall the formula for the forward curve implied by a set of parameters $(a, b^{\mathbb{Q}}, \tau)$, initial rate r_t and a time to maturity h .

$$P(t, h) = e^{-A_a(h)r_t + D_a(h)}$$

where

$$D_a(h) \triangleq \left(\frac{\tau^2}{2a^2} - b^{\mathbb{Q}}\right)[h - A_a(h)] + \frac{\tau^2}{2a^2}A_a^2(h)$$

For every week we find the parameters which minimise the square error between the theoretical and empirical forward curve. This yields different estimates for the parameters for every week, but also a filtered estimate of the path of r , which is plotted in Figure 4B. For the parameters $(a, b^{\mathbb{Q}}, \tau)$ we take the estimated values for the last week, which corresponds to the beginning of 2022.

When it comes to the extraction of W^1 , as well as estimation of b and γ , we calibrate a standard Gaussian Ornstein-Uhlenbeck process to our estimated r . This yields an estimate of the reversion level b , which gives us γ by $b^{\mathbb{Q}} = b - \gamma\tau/a$. Lastly, the normalised residuals of $r_{t_{i+1}} - e^{-a\Delta t}(r_{t_i} - b)$ give our estimate of the path of W^1 .

Remark 7.12. *It is worth remarking that there appears to be a negative trend in the Spanish short rate, visible in Figure 4B. This trend is not accounted for by the constant reversion level of the Ornstein-Uhlenbeck model and as such results in a drift in the estimated noise.*

After implementing this method we get the following parameters.

a	b	$b^{\mathbb{Q}}$	τ	γ
0.31	-2.6e-4	5.4e-2	8.2e-2	-0.20

7.7. Correlation Factors. By combining our previous estimations of the driving Brownian motions we can compute the correlation factors $\rho_{ij}, i, j = 1, 2, 3$. Note that while ρ_{13} can be computed in a standard fashion we can only compute ρ_{12} and ρ_{23} on the points that are not affected by the Hawkes process.

However, as previously noted, we do not have a direct approximation of W^2 and must instead rely on $\Delta\widetilde{W}_n$ from equation (7.7). One can compute, by Itô's isometry, that for $i = 1, 3$ we have the correlation formula

$$\text{cor}\left[\Delta\widetilde{W}, \Delta W^i\right] = \rho_{2i} \frac{\Delta t - A_m(\Delta t)}{\sqrt{K\Delta t}}, \quad (7.8)$$

where

$$K = \Delta t + A_{2m}(\Delta t) - 2A_m(\Delta t) + e^{-m\Delta t} \left(\Delta t + A_{-2m}(\Delta t) - 2A_{-m}(\Delta t) \right).$$

This allows us to compute ρ_{2i} by substituting the empirical correlation of $\Delta\widetilde{W}$ and ΔW^i in equation (7.8) and inverting the formula.

With all our correlations computable we present the correlations based on the group of females aged 75 - 84.

ρ_{12}	ρ_{23}	ρ_{13}
-13.7%	6.1%	-8.5%

These correlations are all relatively small and only ρ_{12} is statistically significant, but we still see a possible indication that periods of high mortality could effect the economic processes we are modelling. It is worth noting that across all groups we

observe $\rho_{12} < 0$, but due to the highly irregular behaviour of the Spanish bond-market for the considered time period, it is hard to argue for the generality of this phenomenon. We here refer in particular to the effects of the trend mentioned in Remark 7.12, which turns out to significantly affect the correlations. By manually removing the trend in W^1 , we instead obtained $\rho_{12} = -9.7\%$ and $\rho_{13} = -7.7\%$. This highlights the uncertainty of the estimates derived from assuming an Ornstein-Uhlenbeck structure for the short rates. Despite this, it is worth remarking that $\rho_{12} < 0$ holds across all groups even after correcting for the trend.

A more interesting connection, and a subject for further research, is the effect of jumps in mortality on interest rates and the stock market. For example, in our data set, the IBEX35 has a sharp drop coinciding with the start of COVID, matching the biggest jumps in mortality. These types of connections between mortality and financial processes have been studied in a recent paper by Tang et al. [21].

8. EMPIRICAL RESULTS FOR CONTRACT PRICES

With all our parameters set we can move on to computing values of different contracts, starting with the value of our GMMB. To keep the description brief we will still focus on females aged 75 at the beginning of 2022 ($t = 0$). To highlight the potential impact of the Hawkes noise we consider several different scenarios where we vary the initial value of λ and μ , as well as comparing the results to a purely deterministic model:

- Case 0: $\mu_t = \mathbb{E}[\mu_t]$
- Case 1: $\lambda_0 = \bar{\lambda}, \mu_0 = T_0$
- Case 2: $\lambda_0 = \bar{\lambda} + \mathbb{E}[J]\alpha, \mu_0 = T_0 + \mathbb{E}[J]\eta_0$
- Case 3: $\lambda_0 = \bar{\lambda} + \max\{J_i\}\alpha, \mu_0 = T_0 + \max\{J_i\}\eta_0$

Case 0 represents the deterministic model without noise and serves as a baseline of comparison. Case 1 represents the situation where no recent jumps have occurred, while in case 2 and 3 we have recent jumps of respectively average and maximally measured size. It is worth noting that the maximal case corresponds to the initial conditions at the start of the COVID pandemic. For these different scenarios we compute the yields of GMMB's by $\log(P_{\text{GMMB}}^{\text{total}}(0, h)) / h$ for $h \in (0, 5]$ using Corollary 6.2. This requires the parameters calibrated in Section 7 along with the MGF of the jumps, which is estimated by the empirical MGF based on the jumps from the female group of age 75 - 84.

As we see from Figure 5, there is minimal differences between case 0 and case 1. Case 1 has initial conditions $\lambda_0 = \bar{\lambda} < \mathbb{E}[\lambda]$ and $\mu_0 = T_0 < \mathbb{E}[\mu]$, resulting in the initial yield being lower than in case 0. Moreover, the average jump of case 2 has a more moderate impact, initially increasing the yield by roughly 0.2%, but vanishing almost completely over time. Predictably, the effects of case 3 magnifies this effect, more than tripling the initial yield with an increase of 1.3%. Even after a full year the effects of the initial maximal jump produces an increase of approximately 0.1% in terms of yield over the deterministic model, a relative increase of more than 27%. The main conclusion we can draw from these graphs is shown most clearly in Figure 6 where regardless of initial conditions the impact is eventually absorbed

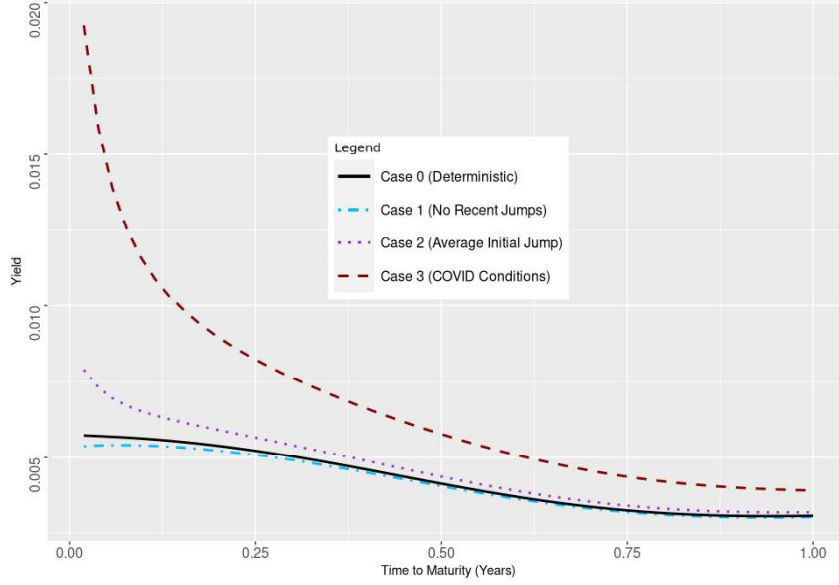


FIGURE 5. Graph of case 0-3 for 1 year

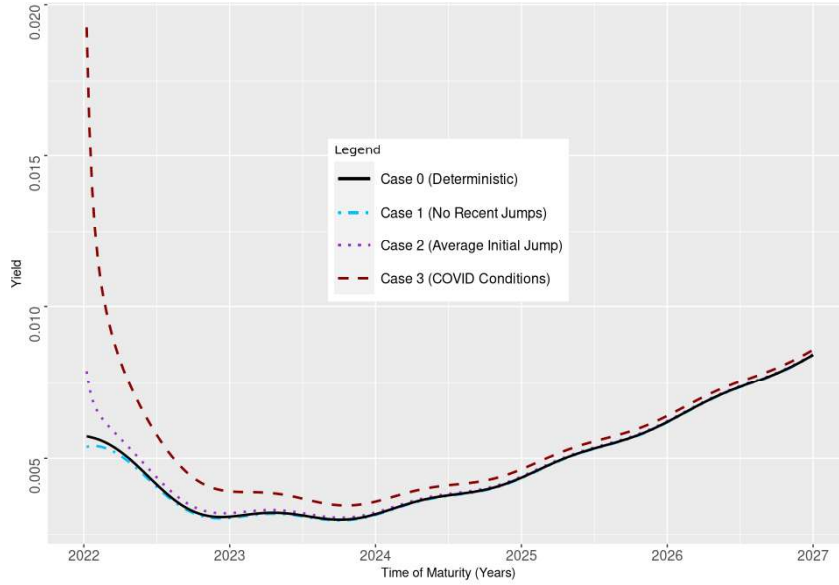


FIGURE 6. Graph of case 0-3 yields for 5 years

by the deterministic average effect. In fact, even case 3 only increases the yield by roughly 0.016% over the deterministic case. This means that even when accounting for shocks in mortality the effect on long-term contracts is negligible.

Remark 8.1. We note that by Corollary 6.2, the value of the total payout of a GMMB, P_{GMMB}^{total} , is proportional to the endowment price P_μ , and their ratio does not depend on μ_0 or λ_0 . This means that while we here show results for P_{GMMB}^{total} we would observe the same phenomena for P_μ .

One of the main properties of our model is the ability to incorporate extreme excess mortality. However, in the case of a GMMB, a high mortality corresponds to low probabilities of the insurance company paying out anything at all. As such we here also examine a portfolio of death benefit contracts, which may cost an insurer significantly more in periods of excess mortality.

For this portfolio we consider 100 insured individuals, all female and aged 75 at the beginning of 2022 ($t = 0$). The death benefit contract pays out 1 unit of currency in the event of the insured's death within a period of 1 year. If we let $\{t_i\}_{i=1}^{100}$ be the death times of the insured individuals the current value of the insurer's liabilities becomes $L = \sum_{t_i \leq 1} \exp(-\int_0^{t_i} r_s ds)$.

By simulating r and μ for case 0-3 we can compute the density of L for the different cases, as summarised in Table 2. Due to the approximation $\exp(-\int_0^{t_i} r_s ds) \approx 1$, the distribution of L is almost discrete, closely approximating the density of the number of deaths within the year. Since the L is concentrated close to $\mathbb{N}$, we focus on tail probabilities for intermediate points. Additionally, the values presented here are relative to case 0, where a value of e.g. +1% corresponds to 101% of the value for case 0.

Statistic	Case 0	Case 1	Case 2	Case 3
$\mathbb{E}[L]$	1.08	-0.2%	+1.0%	+7.1%
$\text{sd}(L)$	1.03	-0.0%	+0.6%	+3.7%
$\mathbb{P}(L > 0.5)$	6.68e-01	-0.2%	+0.5%	+3.6%
$\mathbb{P}(L > 1.5)$	3.00e-01	-0.4%	+1.3%	+9.3%
$\mathbb{P}(L > 2.5)$	9.80e-02	-0.3%	+2.5%	+16.5%
$\mathbb{P}(L > 3.5)$	2.47e-02	-0.3%	+4.0%	+24.9%
$\mathbb{P}(L > 4.5)$	5.03e-03	-0.2%	+5.8%	+35.7%
$\mathbb{P}(L > 5.5)$	8.50e-04	+2.9%	+7.9%	+48.8%
$\mathbb{P}(L > 6.5)$	1.24e-04	+3.6%	+13.0%	+64.6%

TABLE 2. Estimated Statistics for L

In Table 2 we observe many of the same effects shown in Figures 5 and 6, such as the average liabilities increasing by up to 7%. This is expected, as higher initial mortality and jump intensity result in a greater number of deaths and, consequently, more death benefit payouts. It is worth noting that these effects are in line with our observations for the GMMB values as the average liability remains almost unchanged across all cases except case 3, which mirrors the effects observed for the yield of values for a GMMB with one year time to maturity.

Additionally, we see an impact for tail probabilities, which increases the likelihood of extreme events. This effect is more pronounced for higher values, thereby indicating that the density of L has slightly thicker tails under our model. Case 3 naturally exhibits the most significant increase in tail probabilities, but this effect is

also clearly present in case 2, which shows that even moderate shocks can meaningfully increase the risk of extreme events. Interestingly, this effect can also be seen in case 1, albeit at more modest levels. While case 1 has no initial shock, consequently also a slightly lower average, we still note an increase in probabilities for extreme events. As such we see that even in calm periods our model still predict slightly heavier tails due to the risk of future excess mortality.

9. FINAL REMARKS AND CONCLUSIONS

The presented method allows for calibrating the model and yields reasonable and realistic parameters. In order to ensure that the method is coherent it was tested in a simulation experiment, and while the method performed adequately, there were still significant variance in parameter estimates caused by the natural volatility of the Hawkes process. In other words, the variance of empirical moments of $\mathcal{H}$ outweigh any statistical errors inherit in the method itself.

Furthermore, the model still omits several observed phenomena, namely seasonal jump intensity and interdependence of jumps. As is visible from Figure 3 and Figure 2 the jump clusters tend to appear more frequently during the winter months, which is not reflected in the model. Jump sizes are also heavily dependent, we see this most clearly during the initial COVID period where the 3 largest jumps occur in succession, undermining the assumption of independent jumps.

There are also several different approaches one could take in model calibration. As it is the case that most of the jumps observed are caused by external factors such as extreme temperatures, a model without self-excitation would likely be more appropriate. Such a model could be applied while still letting the Hawkes process model the underlying behaviour of the COVID period. Both this and our approach, however, puts a lot of trust in how this 10-year period represents long-run behaviour. The self-exciting pattern of mortality during COVID is undoubtedly over-represented considering most other epidemic and pandemic situations are both rarer and shorter than this data set would indicate. Due to this effect many parameters, but especially the base jump intensity $\bar{\lambda}$, are heavily skewed.

Nevertheless, this model provides a way of considering how different mortality shocks might impact long- and short-term pricing and risk-management of different insurance product. In particular, we have provided several semi-explicit pricing formulas as well as methods for computing more. Additionally we have made several empirical observations based on this model by examining long- and short-term effects of shocks in the mortality. This allowed us to demonstrate that the long-term effects of excess mortality appears to be close to negligible, at least in the case of GMMB's. Additionally we examined the distribution of a short-term death benefit portfolio and found that large shocks, such as those present at the beginning of the COVID pandemic, greatly impact both average and extreme values. Additionally we have demonstrated that the effect on tail probabilities is still present even when the initial conditions are of low to moderate value.

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REFERENCES

- [1] BAUER, D., KLING, A., AND RUSS, J. A universal pricing framework for guaranteed minimum benefits in variable annuities1. *ASTIN Bulletin: The Journal of the IAA* 38, 2 (2008), 621–651.
- [2] BRENNAN, M. J., AND SCHWARTZ, E. S. The pricing of equity-linked life insurance policies with an asset value guarantee. *Journal of Financial Economics* 3, 3 (1976), 195–213.
- [3] BRIGNONE, R., AND SGARRA, C. Asian options pricing in hawkes-type jump-diffusion models. *Annals of Finance* 16, 1 (2020), 101–119.
- [4] CHAVEZ-DEMOULIN*, V., DAVISON, A. C., AND MCNEIL, A. J. Estimating value-at-risk: a point process approach. *Quantitative Finance* 5, 2 (2005), 227–234.
- [5] CHAVEZ-DEMOULIN, V., AND MCGILL, J. High-frequency financial data modeling using hawkes processes. *Journal of Banking & Finance* 36, 12 (2012), 3415–3426.
- [6] CHEN, Z., VETZAL, K., AND FORSYTH, P. A. The effect of modelling parameters on the value of gmwb guarantees. *Insurance: Mathematics and Economics* 43, 1 (2008), 165–173.
- [7] CHIANG, W.-H., LIU, X., AND MOHLER, G. Hawkes process modeling of covid-19 with mobility leading indicators and spatial covariates. *International journal of forecasting* 38, 2 (2022), 505–520.
- [8] CUI, L., HAWKES, A., AND YI, H. An elementary derivation of moments of hawkes processes. *Advances in Applied Probability* 52, 1 (2020), 102–137.
- [9] DA FONSECA, J., AND ZIVEYI, J. Valuing variable annuity guarantees on multiple assets. *Scandinavian Actuarial Journal* 2017, 3 (2017), 209–230.
- [10] DAI, M., KUEN KWOK, Y., AND ZONG, J. Guaranteed minimum withdrawal benefit in variable annuities. *Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics* 18, 4 (2008), 595–611.
- [11] DASSIOS, A., AND ZHAO, H. A dynamic contagion process. *Advances in applied probability* 43, 3 (2011), 814–846.
- [12] DICKSON, D. C., HARDY, M. R., AND WATERS, H. R. Actuarial mathematics for life contingent risks, david cm dickson, mary r. hardy, howard r. waters, cambridge university press, 2013, 616pp.(hardback),£ 50. isbn: 9781107044074. *Annals of Actuarial Science* 8, 2 (2014), 482–482.
- [13] EMBRECHTS, P., LINIGER, T., AND LIN, L. Multivariate hawkes processes: an application to financial data. *Journal of Applied Probability* 48, A (2011), 367–378.
- [14] ERRAIS, E., GIESECKE, K., AND GOLDBERG, L. R. Affine point processes and portfolio credit risk. *SIAM Journal on Financial Mathematics* 1, 1 (2010), 642–665.
- [15] FENG, R., AND VECER, J. Risk based capital for guaranteed minimum withdrawal benefit. *Quantitative Finance* 17, 3 (2017), 471–478.
- [16] FILIPOVIC, D. *Term-Structure Models. A Graduate Course*. Springer, 2009.
- [17] FUNG, M. C., IGNATIEVA, K., AND SHERRIS, M. Systematic mortality risk: An analysis of guaranteed lifetime withdrawal benefits in variable annuities. *Insurance: Mathematics and Economics* 58 (2014), 103–115.
- [18] GONZATO, L., AND SGARRA, C. Self-exciting jumps in the oil market: Bayesian estimation and dynamic hedging. *Energy Economics* 99 (2021), 105279.
- [19] HARDY, M. *Investment guarantees: modeling and risk management for equity-linked life insurance*, vol. 168. John Wiley & Sons, 2003.

- [20] LAUB, P. J., LEE, Y., AND TAIMRE, T. *The elements of hawkes processes*. Springer, 2021.
- [21] LI, H., LIU, H., TANG, Q., AND YUAN, Z. Pricing extreme mortality risk amid the covid-19 pandemic. *Available at SSRN 3899660* (2021).
- [22] MAMON, R., XIONG, H., AND ZHAO, Y. The valuation of a guaranteed minimum maturity benefit under a regime-switching framework. *North American Actuarial Journal* 25, 3 (2021), 334–359.
- [23] MARSHALL, C., HARDY, M., AND SAUNDERS, D. Valuation of a guaranteed minimum income benefit. *North American Actuarial Journal* 14, 1 (2010), 38–58.
- [24] NJIKE LEUNGA, C. G., AND HAINAUT, D. Valuation of annuity guarantees under a self-exciting switching jump model. *Methodology and Computing in Applied Probability* 24, 2 (2022), 963–990.
- [25] PENG, J., LEUNG, K. S., AND KWOK, Y. K. Pricing guaranteed minimum withdrawal benefits under stochastic interest rates. *Quantitative Finance* 12, 6 (2012), 933–941.
- [26] PISCOPO, G., AND HABERMAN, S. The valuation of guaranteed lifelong withdrawal benefit options in variable annuity contracts and the impact of mortality risk. *North American Actuarial Journal* 15, 1 (2011), 59–76.
- [27] RAMBALDI, M., PENNESI, P., AND LILLO, F. Modeling foreign exchange market activity around macroeconomic news: Hawkes-process approach. *Physical Review E* 91, 1 (2015), 012819.

APPENDIX A. TECHNICAL PROOFS

A.1. Proof of Proposition 5.1.

Proof. Let $f(t, \mu_t, \lambda_t, M_t, \Lambda_t) \triangleq \mathbb{E}_{\mathbb{Q}^T} \left[e^{u_1 \mu_T + u_2 \lambda_T + u_3 M_T + u_4 \Lambda_T} \middle| \mathcal{F}_t \right]$. Itô's lemma, and the fact that f is a $\mathbb{Q}^T$ -martingale by our assumption of finite expectation of the moment generating function, implies that

$$\begin{aligned}
 & \partial_t f + \partial_\mu f [m(\tilde{\mu}_t - \mu)] + \partial_{\mu\mu} f \frac{\zeta_t^2}{2} + \partial_\lambda f [\beta(\lambda - \lambda)] \\
 & + \lambda \int_0^\infty [f(t, \mu + \eta_t z, \lambda + \alpha z, M, \Lambda) - f(t, \mu, \lambda, M, \Lambda)] \nu(dz) + \partial_M f \mu + \partial_\Lambda f \lambda = 0.
 \end{aligned} \tag{A.1}$$

Assume that f is of the form

$$f(t, \mu, \lambda, M, \Lambda) = e^{F_T(t) + G_T(t)\mu + H_T(t)\lambda + I_T(t)M + J_T(t)\Lambda}.$$

The partial derivatives of f are given by

$$\begin{aligned}
 \partial_t f &= [F'_T + G'_T \mu + H'_T \lambda + I'_T M + J'_T \Lambda] f, \\
 \partial_\mu f &= G_T f, \quad \partial_{\mu\mu} f = G_T^2 f, \\
 \partial_\lambda f &= H_T f, \\
 \partial_M f &= I_T f, \\
 \partial_\Lambda f &= J_T f.
 \end{aligned}$$

Now, observe that

$$f(t, \mu + \eta_t z, \lambda + \alpha z, M, \Lambda) - f(t, \mu, \lambda, M, \Lambda) = f \left[e^{G_T(t)\eta_t z + H_T(t)\alpha z} - 1 \right].$$

On the other hand,

$$\int_0^\infty \left[e^{G_T(t)\eta_t z + H_T(t)\alpha z} - 1 \right] \nu(dz) = \psi(G_T(t)\eta_t + H_T(t)\alpha) - 1,$$

for every t such that the above integral is finite and where ψ is the moment generating function of the i.i.d. J_i 's.

Altogether,

$$\int_0^\infty [f(t, \mu + \eta_t z, \lambda + \alpha z, M, \Lambda) - f(t, \mu, \lambda, M, \Lambda)] \nu(dz) = f[\psi(\eta_t G_T(t) + \alpha H_T(t)) - 1].$$

Inserting all the expressions into equation (A.1) and cancelling f leads to

$$\begin{aligned} & \left[F'_T + G'_T \mu + H'_T \lambda + I'_T M + J'_T \Lambda \right] + G_T [m(\tilde{\mu}_t - \mu)] + G_T^2 \frac{\zeta_t^2}{2} \\ & + H_T [\beta(\lambda - \lambda)] + \lambda [\psi(G_T \eta_t + H_T \alpha) - 1] + I_T \mu + J_T \lambda = 0. \end{aligned}$$

We can immediately see from the terms in M and Λ that $I_T = u_3$ and $J_T = u_4$. Now, look at the terms in μ . We obtain

$$G'_T - mG_T + u_3 = 0$$

with terminal condition $G_T(T) = u_1$. A particular solution of this equation is trivially $\frac{u_3}{m}$. The homogeneous solution is $Ke^{-m(T-t)}$ for a constant $K \in \mathbb{R}$. Imposing final conditions we obtain

$$G_T(t) = u_1 e^{-m(T-t)} + \frac{u_3}{m} [1 - e^{-m(T-t)}], \quad t \in [0, T]. \quad (\text{A.2})$$

Grouping terms and equating each term to zero we have

$$\begin{aligned} & F'_T + m\tilde{\mu}_t G_T + \frac{\zeta_t^2}{2} G_T^2 + \beta \lambda H_T = 0 \\ & \lambda \left[H'_T - \beta H_T + \psi \left(u_1 \eta_t e^{-m(T-t)} + \frac{u_3 \eta_t}{m} [1 - e^{-m(T-t)}] + \alpha H_T \right) - 1 + u_4 \right] = 0, \end{aligned}$$

which gives the desired ODE's. $\square$

A.2. Proof of Theorem 6.1.

Proof. Due to expression (6.1) it remains to find

$$\mathbb{E}_{\tilde{\mathbb{Q}}^T} [(G - S_T)^+ | \mathcal{F}_t].$$

This requires finding the dynamics of S under the forward measure $\tilde{\mathbb{Q}}^T$. This is easily achieved using (6.3) in connection with the $\mathbb{Q}$ -dynamics of S which are given by

$$\frac{dS_t}{S_t} = r_t dt + \sigma dW_t^{3, \mathbb{Q}}.$$

Then we see that the $\tilde{\mathbb{Q}}^T$ -dynamics of S are given by

$$\frac{dS_t}{S_t} = [r_t - \rho_{13}\sigma\tau A_a(T-t) - \rho_{23}\sigma\zeta_t A_m(T-t)] dt + \sigma dW_t^{3, \tilde{\mathbb{Q}}^T}.$$

Integrating on $[t, T]$ we obtain the following representation

$$\log\left(\frac{S_T}{S_t}\right) = \int_t^T r_s ds - h(t, T) + \int_t^T \sigma dW_s^{3, \tilde{\mathbb{Q}}^T}, \quad (\text{A.3})$$

where h is the deterministic function defined by

$$h(t, T) \triangleq \rho_{13}\sigma\tau \int_t^T A_a(T-s) ds + \rho_{23}\sigma \int_t^T \zeta_s A_m(T-s) ds + \frac{\sigma^2}{2}(T-t).$$

It is easy to see using (3.7) that

$$h(t, T) = -\frac{\sigma\rho_{13}\tau}{a} A_a(T-t) + \left(\frac{\sigma\rho_{13}\tau}{a} + \frac{\sigma^2}{2}\right) (T-t) + \rho_{23}\sigma \int_t^T \zeta_s A_m(T-s) ds.$$

Using the relations in (6.2) we can recast $\int_t^T r_s ds$ under the forward measure $\tilde{\mathbb{Q}}^T$. To this end, we express r in equation (4.2) as

$$r_s = e^{-a(s-t)} r_t + ab_{\mathbb{Q}} A_a(s-t) + \tau \int_t^s e^{-a(s-u)} dW_u^{1, \mathbb{Q}}, \quad s \geq t,$$

where $b_{\mathbb{Q}} \triangleq b - \frac{\gamma\tau}{a}$. In virtue of the relations (6.2) we get

$$\begin{aligned} r_s &= e^{-a(s-t)} r_t + ab_{\mathbb{Q}} A_a(s-t) \\ &\quad - \tau \int_t^s e^{-a(s-u)} [\tau A_a(T-u) + \rho_{12}\zeta_t A_m(T-u)] du \\ &\quad + \tau \int_t^s e^{-a(s-u)} dW_u^{1, \tilde{\mathbb{Q}}^T}, \end{aligned}$$

which yields

$$\int_t^T r_s dt = A_a(T-t)r_t + g(t, T) + \tau \int_t^T A_a(T-s) dW_s^{1, \tilde{\mathbb{Q}}^T}, \quad (\text{A.4})$$

where g is the deterministic function defined by

$$g(t, T) = \int_t^T \left[ab_{\mathbb{Q}} A_a(s-t) - \tau \int_t^s e^{-a(s-u)} [\tau A_a(T-u) + \rho_{12} \zeta_u A_m(T-u)] du \right] ds.$$

Using the change of variables $v = s - u$ and relations in (3.7) one can show that

$$\begin{aligned} g(t, T) &= \left(b_{\mathbb{Q}} - \frac{\tau^2}{a^2} \right) (T-t) + \left(-b_{\mathbb{Q}} + \frac{2\tau^2}{a^2} \right) A_a(T-t) \\ &\quad - \frac{\tau^2}{a^2} A_{2a}(T-t) + \int_t^T \rho_{12} \zeta_u A_m(T-u) A_a(T-u) du. \end{aligned}$$

Combining equations (A.3) and (A.4) we get

$$\begin{aligned} \log \left(\frac{S_T}{S_t} \right) &= A_a(T-t)r_t + g(t, T) - h(t, T) \\ &\quad + \int_t^T [\tau A_a(T-s) + \rho_{31}^* \sigma] dW_s^{1, \tilde{\mathbb{Q}}^T} \\ &\quad + \int_t^T \rho_{32}^* \sigma dZ_s^{2, \tilde{\mathbb{Q}}^T} \\ &\quad + \int_t^T \rho_{33}^* \sigma dZ_s^{3, \tilde{\mathbb{Q}}^T}. \end{aligned}$$

This implies that S_T/S_t , conditional on $\mathcal{F}_t$, is log-normally distributed under $\tilde{\mathbb{Q}}^T$ with log-mean $A_a(T-t)r_t + g(t, T) - h(t, T)$ and log-variance given below. For these computations we note that $\rho_{31}^* = \rho_{13}$, $\rho_{31}^{*2} + \rho_{32}^{*2} + \rho_{33}^{*2} = 1$ and lastly that $W_s^{1, \tilde{\mathbb{Q}}^T}$, $Z_s^{2, \tilde{\mathbb{Q}}^T}$ and $Z_s^{3, \tilde{\mathbb{Q}}^T}$ are independent which yields

$$\begin{aligned}
 \mathcal{V}(t, T)^2 &\triangleq \mathbb{V}ar \left[\log \left(\frac{S_T}{S_t} \right) \middle| \mathcal{F}_t \right] \\
 &= \int_t^T [\tau A_a(T-s) + \rho_{31}^* \sigma]^2 ds + \int_t^T (\rho_{32}^* \sigma)^2 ds + \int_t^T (\rho_{33}^* \sigma)^2 ds \\
 &= \int_t^T \tau^2 A_a(T-s)^2 ds + \int_t^T 2\tau \rho_{31}^* \sigma A_a(T-s) ds + \sigma^2 \int_t^T (\rho_{31}^{*2} + \rho_{32}^{*2} + \rho_{33}^{*2}) ds \\
 &= \frac{\tau^2}{a^2} [T-t - 2A_a(T-t) + A_{2a}(T-t)] + \frac{2\tau \rho_{13} \sigma}{a} [T-t - A_a(T-t)] + \sigma^2 (T-t) \\
 &= \left(\frac{\tau^2}{a^2} + \frac{2\tau \rho_{13} \sigma}{a} + \sigma^2 \right) (T-t) - \left(\frac{2\tau^2}{a^2} + \frac{2\tau \rho_{13} \sigma}{a} \right) A_a(T-t) + \frac{\tau^2}{a^2} A_{2a}(T-t).
 \end{aligned}$$

Note that we have the following equality in law

$$S_T = S_t e^{A_a(T-t)r_t + g(t,T) - h(t,T) + \mathcal{V}(t,T)Z},$$

where Z is a standard normal random variable under $\tilde{\mathbb{Q}}^T$.

The proof then follows by standard calculations of the conditional expectation

$$\mathbb{E}_{\tilde{\mathbb{Q}}^T}[(G - S_t e^{A_a(T-t)r_t + g(t,T) - h(t,T) + \mathcal{V}(t,T)Z})_+ | \mathcal{F}_t].$$

□

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