

A blow – up result for the semilinear Moore – Gibson – Thompson equation with nonlinearity of derivative type in the conservative case

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Abstract. In this paper, we study the blow – up of solutions to the semilinear Moore – Gibson – Thompson (MGT) equation with nonlinearity of derivative type $|u_t|^p$ in the conservative case. We apply an iteration method in order to study both the subcritical case and the critical case. Hence, we obtain a blow – up result for the semilinear MGT equation (under suitable assumptions for initial data) when the exponent p for the nonlinear term satisfies $1 < p \leq (n+1)/(n-1)$ for $n \geq 2$ and $p > 1$ for $n = 1$. In particular, we find the same blow – up range for p as in the corresponding semilinear wave equation with nonlinearity of derivative type.

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1. Introduction

Over the last years, the Moore – Gibson – Thompson (MGT) equation (cf. [26, 41]), a linearization of a model for the wave propagation in viscous thermally relaxing fluids, has been studied by several authors (see, for example, [9, 16, 15, 24, 14, 23, 4, 7, 22, 8, 21, 33, 3, 6, 34]).

This model is realized through the third order hyperbolic equation

$$\tau u_{ttt} + u_{tt} - c^2 \Delta u - b \Delta u_t = 0. \quad (1.1)$$

In the physical context of acoustic waves, the unknown function $u = u(t, x)$ denotes the scalar acoustic velocity, c denotes the speed of sound and τ denotes the thermal relaxation. Besides, the coefficient $b = \beta c^2$ is related to the diffusivity of the sound with $\beta \in (0, \tau]$. Let us point out that there is the transition from a linear model that can be described in the case of bounded domains with an exponentially stable strongly continuous semigroup in the case $0 < \beta < \tau$ to the limit case $\beta = \tau$, where the exponential stability of a semigroup is lost and it holds the conservation of a suitable defined energy (see [16, 24]). For this reason, we will refer to the limit case $\beta = \tau$ as to the *conservative case* throughout this paper.

We consider the semilinear Cauchy problem for MGT equation in the conservative case with nonlinearity of derivative type, namely,

$$\begin{cases} \beta u_{ttt} + u_{tt} - \Delta u - \beta \Delta u_t = |u_t|^p, & x \in \mathbb{R}^n, t > 0, \\ (u, u_t, u_{tt})(0, x) = \varepsilon(u_0, u_1, u_2)(x), & x \in \mathbb{R}^n, \end{cases} \quad (1.2)$$

where $p > 1$ and ε is a positive parameter describing the size of initial data. Note that, for the sake of simplicity, we normalized the speed of the sound by putting $c^2 = 1$. We are interested in investigating the blow – up in finite time of local (in time) solutions under suitable sign assumptions

for the Cauchy data regardless of their size. Let us underline that, while the MGT equation has been widely investigated in the case of bounded domains via semigroups theory, very few results concerning nonlinear Cauchy problems for MGT equation are available up to the knowledge of the authors. In [35], the semilinear Cauchy problem with nonlinearity $\partial_t (k(u_t)^2 + |\nabla u|^2)$ is considered in the *dissipative case* $0 < \tau < \beta$, where k is a suitable constant. More precisely, a global existence result for small data solutions is proved providing that initial data are sufficiently regular and satisfy certain integral relations (cf. [35, Theorem 5.1]). Moreover, a blow – up result for the conservative case with power nonlinearity can be found in [5].

Let us provide some results which are related to our model (1.2). By choosing $\beta = 0$, we find that (1.2) corresponds formally to the semilinear wave equation

$$\begin{cases} u_{tt} - \Delta u = |u_t|^p, & x \in \mathbb{R}^n, t > 0, \\ (u, u_t)(0, x) = \varepsilon(u_0, u_1)(x), & x \in \mathbb{R}^n, \end{cases} \quad (1.3)$$

where $p > 1$. According to [13, 38, 25, 37, 36, 1, 10, 42, 45, 11] the critical exponent of (1.3) is the so – called *Glassey exponent* $p_{\text{Gla}}(n) \doteq (n+1)/(n-1)$. Moreover, the sharp behavior of the lifespan $T(\varepsilon)$ of local (in time) solutions to (1.3) with respect to a sufficiently small parameter $\varepsilon > 0$ is given by

$$T(\varepsilon) \approx \begin{cases} C\varepsilon^{-\left(\frac{1}{p-1} - \frac{n-1}{2}\right)^{-1}} & \text{if } 1 < p < p_{\text{Gla}}(n), \\ \exp(C\varepsilon^{-(p-1)}) & \text{if } p = p_{\text{Gla}}(n). \end{cases}$$

The main result of this paper consists of a blow – up result for (1.2) when the power of the nonlinear term is in the sub – Glassey range (including the case $p = p_{\text{Gla}}(n)$).

In order to prove this result, we are going to apply an iterative argument for a suitable time – dependent functional, which depends on a local (in time) solution to (1.2). For the choice of the functional we follow [18] whereas concerning the iteration procedure we use some key ideas from [5], where a technique to deal with an unbounded exponential multiplier in the iteration frame is developed. This approach is based on the idea of slicing the interval of integration and it has been introduced by Takamura and coauthors in the study of critical cases for wave models (see [2, 39, 40, 43] for example). Recently, many papers have been devoted to the study of blow – up results for semilinear second order hyperbolic models with the aid of a time dependent multipliers. The first paper in this direction is [17] followed then by [18, 19, 28, 29, 30, 12]. In these papers, the time – dependent multiplier is bounded by positive constants from above and from below and it is used to study semilinear damped wave models with time – dependent coefficients for the damping terms in the scattering producing case. On the other hand, the case of unbounded time – dependent multipliers is considered for semilinear wave models with scale – invariant damping and mass terms in [20, 18, 31, 27, 32].

Before stating the main result of this paper, let us introduce a suitable notion of energy solutions to the Cauchy problem (1.2).

Definition 1.1. Let $(u_0, u_1, u_2) \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. We say that u is an energy solution of (1.2) on $[0, T)$ if

$u \in \mathcal{C}([0, T), H^2(\mathbb{R}^n)) \cap \mathcal{C}^1([0, T), H^1(\mathbb{R}^n)) \cap \mathcal{C}^2([0, T), L^2(\mathbb{R}^n))$ such that $u_t \in L^p_{\text{loc}}([0, T) \times \mathbb{R}^n)$ satisfies $u(0, \cdot) = \varepsilon u_0$ in $H^2(\mathbb{R}^n)$ and the integral relation

$$\begin{aligned} & \beta \int_{\mathbb{R}^n} u_{tt}(t, x) \phi(t, x) dx + \int_{\mathbb{R}^n} u_t(t, x) \phi(t, x) dx - \beta \varepsilon \int_{\mathbb{R}^n} u_2(x) \phi(0, x) dx - \varepsilon \int_{\mathbb{R}^n} u_1(x) \phi(0, x) dx \\ & + \beta \int_0^t \int_{\mathbb{R}^n} (\nabla u_t(s, x) \cdot \nabla \phi(s, x) - u_{tt}(s, x) \phi_t(s, x)) dx ds \\ & + \int_0^t \int_{\mathbb{R}^n} (\nabla u(s, x) \cdot \nabla \phi(s, x) - u_t(s, x) \phi_t(s, x)) dx ds \\ & = \int_0^t \int_{\mathbb{R}^n} |u_t(s, x)|^p \phi(s, x) dx ds \end{aligned} \quad (1.4)$$

for any $\phi \in \mathcal{C}_0^\infty([0, T] \times \mathbb{R}^n)$ and any $t \in (0, T)$.

Applying further steps of integration by parts in (1.4), we get

$$\begin{aligned}
 & \beta \int_{\mathbb{R}^n} (u_{tt}(t, x)\phi(t, x) - u_t(t, x)\phi_t(t, x) - u(t, x)\Delta\phi(t, x) + u(t, x)\phi_{tt}(t, x)) dx \\
 & + \int_{\mathbb{R}^n} (u_t(t, x)\phi(t, x) - u(t, x)\phi_t(t, x)) dx \\
 & - \beta\varepsilon \int_{\mathbb{R}^n} (u_2(x)\phi(0, x) - u_1(x)\phi_t(0, x) - u_0(x)\Delta\phi(0, x) + u_0(x)\phi_{tt}(0, x)) dx \\
 & - \varepsilon \int_{\mathbb{R}^n} (u_1(x)\phi(0, x) - u_0(x)\phi_t(0, x)) dx \\
 & + \int_0^t \int_{\mathbb{R}^n} u(s, x) (-\beta\phi_{ttt}(s, x) + \phi_{tt}(s, x) - \Delta\phi(s, x) + \beta\Delta\phi_t(s, x)) dx ds \\
 & = \int_0^t \int_{\mathbb{R}^n} |u_t(s, x)|^p \phi(s, x) dx ds.
 \end{aligned}$$

Letting $t \rightarrow T$, we find that u fulfills the definition of weak solution to (1.2).

We now state our main result.

Theorem 1.1. *Let us consider $p > 1$ such that*

$$\begin{cases} p < \infty & \text{if } n = 1, \\ p \leq p_{\text{Gla}}(n) & \text{if } n \geq 2. \end{cases}$$

Let $(u_0, u_1, u_2) \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ be nonnegative and compactly supported functions with supports contained in B_R for some $R > 0$ such that u_1 or u_2 is not identically zero.

Let u be the energy solution to the Cauchy problem (1.2) with lifespan $T(\varepsilon)$ satisfying

$$\text{supp } u(t, \cdot) \subset B_{t+R} \quad \text{for any } t \in (0, T).$$

Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(u_0, u_1, u_2, n, p, R, \beta)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the solution u blows up in finite time. Furthermore, the upper bound estimate for the lifespan

$$T(\varepsilon) \leq \begin{cases} C\varepsilon^{-\left(\frac{1}{p-1} - \frac{n-1}{2}\right)^{-1}} & \text{if } 1 < p < p_{\text{Gla}}(n), \\ \exp(C\varepsilon^{-(p-1)}) & \text{if } p = p_{\text{Gla}}(n), \end{cases}$$

holds, where $C > 0$ is an independent of ε constant.

Remark 1.1. *We point out that the solution to the linear Cauchy problem for MGT equation*

$$\begin{cases} \beta u_{ttt} + u_{tt} - \Delta u - \beta \Delta u_t = F(t, x), & x \in \mathbb{R}^n, t > 0, \\ (u, u_t, u_{tt})(0, x) = \varepsilon(u_0, u_1, u_2)(x), & x \in \mathbb{R}^n, \end{cases} \quad (1.5)$$

fulfills the inhomogeneous wave equation

$$\begin{cases} u_{tt} - \Delta u = \varepsilon e^{-t/\beta}(u_2(x) - \Delta u_0(x)) + 1/\beta \int_0^t e^{(\tau-t)/\beta} F(\tau, x) d\tau, & x \in \mathbb{R}^n, t > 0, \\ (u, u_t)(0, x) = \varepsilon(u_0, u_1)(x), & x \in \mathbb{R}^n. \end{cases} \quad (1.6)$$

Thus, we claim that $\text{supp } u(t, \cdot) \subset B_{t+R}$, if we assume for some $R > 0$ that $\text{supp } u_j \subset B_R$ for any $j = 0, 1, 2$ and $\text{supp } F(t, \cdot) \subset B_{t+R}$ for any $t \geq 0$. Indeed, the source term

$$f(t, x) = \varepsilon e^{-t/\beta}(u_2(x) - \Delta u_0(x)) + 1/\beta \int_0^t e^{(\tau-t)/\beta} F(\tau, x) d\tau$$

in (1.6) has support contained in the forward cone $\{(t, x) : |x| \leq t + R\}$ under these assumptions and we can use the property of finite speed of propagation for the classical wave equation. Therefore, the support condition in Theorem 1.1 for a local in time solution to (1.2) is meaningful.

Notation: We give some notations to be used in this paper. We write $f \lesssim g$ when there exists a positive constant C such that $f \leq Cg$. Moreover, we write $g \lesssim f \lesssim g$ by $f \approx g$. B_R denotes the ball around the origin with radius R in \mathbb{R}^n . Finally, as in the introduction, $p_{\text{Gla}}(n)$ denotes the Glassey exponent.

2. Blow – up result in the subcritical case

2.1. Iteration frame

Let us consider the eigenfunction Φ of the Laplace operator on the whole space

$$\begin{aligned} \Phi(x) &\doteq e^x + e^{-x} && \text{if } n = 1, \\ \Phi(x) &\doteq \int_{\mathbb{S}^{n-1}} e^{x \cdot \omega} d\sigma_\omega && \text{if } n \geq 2, \end{aligned}$$

for any $x \in \mathbb{R}^n$. This function has been employed in the study of blow – up results for the semilinear wave model in the critical case in [44]. The function Φ is positive and smooth and satisfies the following remarkable properties:

$$\Delta \Phi = \Phi, \tag{2.7}$$

$$\Phi(x) \sim |x|^{-\frac{n-1}{2}} e^x \quad \text{as } |x| \rightarrow \infty. \tag{2.8}$$

Hence, we define the function with separate variables $\Psi = \Psi(t, x) \doteq e^{-t} \Phi(x)$. Therefore, Ψ is a solution of the adjoint equation to the homogeneous linear MGT equation, namely,

$$-\beta \partial_t^3 \Psi + \partial_t^2 \Psi - \Delta \Psi + \beta \Delta \partial_t \Psi = 0. \tag{2.9}$$

By using the asymptotic behavior of Ψ (cf. [18, Equation (3.5)]), it follows that there exists a constant $C_1 = C_1(n, R) > 0$ such that

$$\int_{B_{t+R}} \Psi(t, x) dx \leq C_1 (t+R)^{(n-1)/2} \quad \text{for any } t \geq 0. \tag{2.10}$$

Moreover, modifying slightly the proof of Theorem 3.1 in [5] one can prove the existence of local in time energy solutions with support contained in the forward cone $\{(t, x) \in [0, T] \times \mathbb{R}^n : |x| \leq t+R\}$ for any $p > 1$ such that $p \leq n/(n-2)$ when $n \geq 3$, if $(u_0, u_1, u_2) \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ are compactly supported functions with supports contained in B_R for some $R > 0$.

Since u is supported in a forward cone, we may apply the definition of energy solution even though the test function is not compactly supported. So, applying the definition of energy solution with Ψ as test function in (1.4), we get for any $t \in (0, T)$

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^n} |u_t(s, x)|^p \Psi(s, x) dx ds &= \beta \int_{\mathbb{R}^n} u_{tt}(t, x) \Psi(t, x) dx + \int_{\mathbb{R}^n} u_t(t, x) \Psi(t, x) dx \\ &\quad - \beta \varepsilon \int_{\mathbb{R}^n} u_2(x) \Phi(x) dx - \varepsilon \int_{\mathbb{R}^n} u_1(x) \Phi(x) dx \\ &\quad + \beta \int_0^t \int_{\mathbb{R}^n} (\nabla u_t(s, x) \cdot \nabla \Psi(s, x) - u_{tt}(s, x) \Psi_t(s, x)) dx ds \\ &\quad + \int_0^t \int_{\mathbb{R}^n} (\nabla u(s, x) \cdot \nabla \Psi(s, x) - u_t(s, x) \Psi_t(s, x)) dx ds. \end{aligned} \tag{2.11}$$

Consequently, performing integration by parts in (2.11) and employing the properties of Ψ , we find

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^n} |u_t(s, x)|^p \Psi(s, x) dx ds &= \int_{\mathbb{R}^n} (\beta u_{tt}(t, x) + (\beta + 1) u_t(t, x) + u(t, x)) \Psi(t, x) dx \\ &\quad - \varepsilon \int_{\mathbb{R}^n} (\beta u_2(x) + (\beta + 1) u_1(x) + u_0(x)) \Phi(x) dx. \end{aligned} \tag{2.12}$$

Let us introduce

$$I_\beta[u_0, u_1, u_2] \doteq \int_{\mathbb{R}^n} (\beta u_2(x) + (\beta + 1)u_1(x) + u_0(x)) \Phi(x) dx,$$

$$F_1(t) \doteq \int_{\mathbb{R}^n} u_t(t, x) \Psi(t, x) dx.$$

The functional F_1 will play a central role in the iteration argument, as it is the time – dependent quantity that blows up in finite time or, in other words, it is the function that will be estimated from below iteratively. By using these notations, we may rewrite (2.12) as

$$\beta F_1'(t) + (2\beta + 1)F_1(t) + \int_{\mathbb{R}^n} u(t, x) \Psi(t, x) dx = \int_0^t \int_{\mathbb{R}^n} |u_t(s, x)|^p \Psi(s, x) dx ds + \varepsilon I_\beta[u_0, u_1, u_2]. \quad (2.13)$$

Furthermore, the differentiation of (2.13) with respect to t provides

$$\beta F_1''(t) + (2\beta + 1)F_1'(t) + F_1(t) - \int_{\mathbb{R}^n} u(t, x) \Psi(t, x) dx = \int_{\mathbb{R}^n} |u_t(t, x)|^p \Psi(t, x) dx. \quad (2.14)$$

Adding up (2.13) with (2.14), we immediately obtain

$$\begin{aligned} & \beta F_1''(t) + (3\beta + 1)F_1'(t) + (2\beta + 2)F_1(t) \\ &= \int_0^t \int_{\mathbb{R}^n} |u_t(s, x)|^p \Psi(s, x) dx ds + \int_{\mathbb{R}^n} |u_t(t, x)|^p \Psi(t, x) dx + \varepsilon I_\beta[u_0, u_1, u_2]. \end{aligned} \quad (2.15)$$

Next, let us set

$$G(t) \doteq F_1'(t) + 2F_1(t) - (\beta + 1)^{-1} \int_0^t \int_{\mathbb{R}^n} |u_t(s, x)|^p \Psi(s, x) dx ds - \varepsilon(\beta + 1)^{-1} J_\beta[u_1, u_2],$$

where

$$J_\beta[u_1, u_2] \doteq \int_{\mathbb{R}^n} (\beta u_2(x) + (\beta + 1)u_1(x)) \Phi(x) dx.$$

The auxiliary functional G , together with H whose definition is going to be introduced in few lines, is important to derive a first lower bound estimate for F_1 and the iteration frame for F_1 . Employing (2.15) and the nonnegativity of u_0 , we arrive at

$$\beta G'(t) + (\beta + 1)G(t) = (\beta + 1)^{-1} \int_{\mathbb{R}^n} |u_t(t, x)|^p \Psi(t, x) dx + \varepsilon \int_{\mathbb{R}^n} u_0(x) \Phi(x) dx \geq 0,$$

which implies in turn

$$G(t) \geq e^{-(1+1/\beta)t} G(0) = \varepsilon(\beta + 1)^{-1} e^{-(1+1/\beta)t} \int_{\mathbb{R}^n} u_2(x) \Phi(x) dx \geq 0,$$

where we used the nonnegativity of u_2 .

Combining the definition of G into the inequality $G(t) \geq 0$, we get

$$F_1'(t) + 2F_1(t) \geq (\beta + 1)^{-1} \int_0^t \int_{\mathbb{R}^n} |u_t(s, x)|^p \Psi(s, x) dx ds + \varepsilon(\beta + 1)^{-1} J_\beta[u_1, u_2] \doteq H(t). \quad (2.16)$$

This leads to

$$\begin{aligned} F_1(t) & \geq e^{-2t} F_1(0) + \frac{\varepsilon}{2(\beta + 1)} J_\beta[u_1, u_2] (1 - e^{-2t}) \\ &= \frac{\varepsilon}{2} (1 + e^{-2t}) \int_{\mathbb{R}^n} u_1(x) \Phi(x) dx + \frac{\varepsilon \beta}{2(\beta + 1)} (1 - e^{-2t}) \int_{\mathbb{R}^n} u_2(x) \Phi(x) dx \\ & \geq \frac{\varepsilon}{2} \int_{\mathbb{R}^n} u_1(x) \Phi(x) dx + \frac{\varepsilon \beta}{2(\beta + 1)} (1 - e^{-1}) \int_{\mathbb{R}^n} u_2(x) \Phi(x) dx \doteq C_2 \varepsilon \end{aligned} \quad (2.17)$$

for any $t \geq 1/2$. Here we remark that we may guarantee that $C_2 > 0$ because we assumed that at least one among the nonnegative function u_1 or u_2 does not vanish identically. By Hölder's inequality and (2.10), we have

$$(1 + \beta)H'(t) \geq C_1^{1-p}(t + R)^{-(n-1)(p-1)/2}(F_1(t))^p.$$

Thus, integrating the above inequality over $[0, t]$ and using (2.16), we obtain the iteration frame

$$F_1(t) \geq C_3 \int_0^t e^{2(\tau-t)} \int_0^\tau (s + R)^{-(n-1)(p-1)/2} (F_1(s))^p ds d\tau, \quad (2.18)$$

where $C_3 \doteq C_1^{1-p}/(1 + \beta)$. We point out that in order to get (2.18) we used the conditions $H(0) > 0$ and $F_1(0) \geq 0$.

The combination of (2.17) and (2.18) shows

$$\begin{aligned} F_1(t) &\geq C_2^p C_3 \varepsilon^p \int_{1/2}^t e^{2(\tau-t)} \int_{1/2}^\tau (s + R)^{-(n-1)(p-1)/2} ds d\tau \\ &\geq C_2^p C_3 \varepsilon^p (t + R)^{-(n-1)(p-1)/2} \int_{t/2}^t e^{2(\tau-t)} (\tau - 1/2) d\tau \\ &\geq 4^{-1} C_2^p C_3 \varepsilon^p (t + R)^{-(n-1)(p-1)/2} (t - 1) (1 - e^{-t}). \end{aligned}$$

for $t \geq 1$. In particular, for $t \geq 1$ the factor containing the exponential function in the last line of the previous chain of inequalities can be estimate from below by a constant, namely,

$$F_1(t) \geq K_0 (t + R)^{-\alpha_0} (t - 1)^{\gamma_0} \quad \text{for any } t \geq 1, \quad (2.19)$$

where the multiplicative constant is $K_0 \doteq C_2^p C_3 (1 - e^{-1}) \varepsilon^p / 4$ and the exponents are defined by $\alpha_0 \doteq (n - 1)(p - 1)/2$ and $\gamma_0 \doteq 1$.

2.2. Iteration argument

The previous subsection is devoted to determine the iteration frame and a first lower bound for F_1 . Our next goal is to derive a sequence of lower bounds for F_1 by using (2.18). More precisely, we prove that

$$F_1(t) \geq K_j (t + R)^{-\alpha_j} (t - L_j)^{\gamma_j} \quad \text{for any } t \geq L_j, \quad (2.20)$$

where $\{K_j\}_{j \in \mathbb{N}}$, $\{\alpha_j\}_{j \in \mathbb{N}}$ and $\{\gamma_j\}_{j \in \mathbb{N}}$ are sequences of nonnegative real numbers that will be determined throughout this subsection and $\{L_j\}_{j \in \mathbb{N}}$ is the sequence of the partial products of the infinite product

$$\prod_{k=0}^{\infty} \ell_k \quad \text{with } \ell_k \doteq 1 + p^{-k} \text{ for any } k \in \mathbb{N},$$

that is,

$$L_j \doteq \prod_{k=0}^j \ell_k \quad \text{for any } j \in \mathbb{N}.$$

Clearly (2.19) implies (2.20) for $j = 0$. We are going to show (2.20) via an inductive argument. Also, it remains to verify only the inductive step. Let us assume that (2.20) holds for $j \geq 0$. Then, in order to prove the inductive step, we shall prove (2.20) for $j + 1$. After shrinking the domain of integration

in (2.18), if we plug (2.20) in (2.18), we find

$$\begin{aligned} F_1(t) &\geq C_3 K_j^p \int_{L_j}^t e^{2(\tau-t)} \int_{L_j}^{\tau} (s+R)^{-(n-1)(p-1)/2-\alpha_j p} (s-L_j)^{\gamma_j p} ds d\tau \\ &\geq C_3 K_j^p (t+R)^{-(n-1)(p-1)/2-\alpha_j p} \int_{L_j}^t e^{2(\tau-t)} \int_{L_j}^{\tau} (s-L_j)^{\gamma_j p} ds d\tau \\ &\geq \frac{C_3 K_j^p}{\gamma_j p + 1} (t+R)^{-(n-1)(p-1)/2-\alpha_j p} \int_{t/\ell_{j+1}}^t e^{2(\tau-t)} (\tau-L_j)^{\gamma_j p+1} d\tau \end{aligned}$$

for any $t \geq L_{j+1}$. We point out that in the last step we could restrict the domain of integration with respect to τ from $[L_j, t]$ to $[t/\ell_{j+1}, t]$ since $t \geq L_{j+1}$ and $\ell_{j+1} > 1$ imply the inequality $L_j \leq t/\ell_{j+1} < t$. Also,

$$F_1(t) \geq \frac{C_3 K_j^p}{2(\gamma_j p + 1) \ell_{j+1}^{\gamma_j p+1}} (t+R)^{-(n-1)(p-1)/2-\alpha_j p} (t-L_{j+1})^{\gamma_j p+1} \left(1 - e^{2t(1/\ell_{j+1}-1)}\right)$$

for any $t \geq L_{j+1}$. We observe that for $t \geq L_{j+1} \geq \ell_{j+1}$ it is possible to estimate

$$\begin{aligned} 1 - e^{2t(1/\ell_{j+1}-1)} &\geq 1 - e^{-2(\ell_{j+1}-1)} \geq 2(\ell_{j+1}-1)(2-\ell_{j+1}) \\ &\geq 2(p^{j+1}-1)p^{-2(j+1)} \geq 2(p-1)p^{-2(j+1)}. \end{aligned} \quad (2.21)$$

Thus, for any $t \geq L_{j+1}$ we have proved

$$F_1(t) \geq \frac{(p-1)p^{-2(j+1)} C_3 K_j^p}{(\gamma_j p + 1) \ell_{j+1}^{\gamma_j p+1}} (t+R)^{-(n-1)(p-1)/2-\alpha_j p} (t-L_{j+1})^{\gamma_j p+1},$$

which is exactly (2.20) for $j+1$, provided that

$$K_{j+1} \doteq \frac{(p-1)p^{-2(j+1)} C_3 K_j^p}{(\gamma_j p + 1) \ell_{j+1}^{\gamma_j p+1}}, \quad \alpha_{j+1} \doteq \frac{1}{2}(n-1)(p-1) + \alpha_j p, \quad \gamma_{j+1} \doteq \gamma_j p + 1.$$

By using recursively the previous relations for α_j and γ_j it is easy to get

$$\begin{aligned} \alpha_j &= p^j \left(\alpha_0 + \frac{n-1}{2} \right) - \frac{n-1}{2}, \\ \gamma_j &= p^j \left(\gamma_0 + \frac{1}{p-1} \right) - \frac{1}{p-1}. \end{aligned}$$

Besides, the inequality $\gamma_{j-1} p + 1 = \gamma_j \leq p^j \left(\gamma_0 + \frac{1}{p-1} \right)$ implies immediately

$$K_j \geq (p-1) C_3 \left(\gamma_0 + \frac{1}{p-1} \right)^{-1} K_{j-1}^p p^{-3j} \ell_j^{-\gamma_j}.$$

Due to the choice of ℓ_j , it holds

$$\lim_{j \rightarrow \infty} \ell_j^{\gamma_j} = \lim_{j \rightarrow \infty} \exp \left(\left(\gamma_0 + \frac{1}{p-1} \right) p^j \log(1+p^{-j}) \right) = e^{\gamma_0+1/(p-1)}.$$

Therefore, there exists a suitable constant $M = M(n, p) > 0$ such that $\ell_j^{-\gamma_j} \geq M$ for any $j \in \mathbb{N}$. So, combining this inequality with the previous estimate from below of K_j , we have

$$K_j \geq \underbrace{(p-1) M C_3 \left(\gamma_0 + \frac{1}{p-1} \right)^{-1}}_{\doteq D} K_{j-1}^p p^{-3j} \quad \text{for any } j \in \mathbb{N}.$$

If we apply the logarithmic function to both sides of the inequality $K_j \geq DK_{j-1}^p p^{-3j}$ and we use iteratively the resulting inequality, we obtain

$$\begin{aligned} \log K_j &\geq p^j \log K_0 - 3 \left(\sum_{k=0}^{j-1} (j-k)p^k \right) \log p + \left(\sum_{k=0}^{j-1} p^k \right) \log D \\ &\geq p^j \left(\log K_0 - \frac{3p \log p}{(p-1)^2} + \frac{\log D}{p-1} \right) + \frac{3j \log p}{p-1} + \frac{3p \log p}{(p-1)^2} - \frac{\log D}{p-1} \end{aligned}$$

for any $j \in \mathbb{N}$, where in the second step we use the identity

$$\sum_{k=0}^{j-1} (j-k)p^k = \frac{1}{p-1} \left(\frac{p^{j+1} - p}{p-1} - 1 \right).$$

Let $j_0 = j_0(n, p) \in \mathbb{N}$ be the smallest nonnegative integer such that

$$j_0 \geq \frac{\log D}{3 \log p} - \frac{p}{p-1}.$$

Then, for any $j \geq j_0$ it results

$$\log K_j \geq p^j \log \left(D^{1/(p-1)} p^{-3p/(p-1)^2} K_0 \right) = p^j \log(E\varepsilon^p)$$

for a suitable positive constant $E = E(n, p)$.

Let us denote

$$L \doteq \lim_{j \rightarrow \infty} L_j = \prod_{j=0}^{\infty} \ell_j \in \mathbb{R}.$$

Thanks to $\ell_j > 1$, it holds $L_j \uparrow L$ as $j \rightarrow \infty$. In particular, (2.20) holds for any $j \in \mathbb{N}$ and any $t \geq L$.

Combining the above results and using the explicit representation for α_j and γ_j , we get

$$\begin{aligned} F_1(t) &\geq \exp(p^j \log(E\varepsilon^p)) (t+R)^{-\alpha_j} (t-L)^{\gamma_j} \\ &\geq \exp \left(p^j \left(\log(E\varepsilon^p) - \left(\alpha_0 + \frac{n-1}{2} \right) \log(t+R) + \left(\gamma_0 + \frac{1}{p-1} \right) \log(t-L) \right) \right) \\ &\quad \times (t+R)^{(n-1)/2} (t-L)^{-1/(p-1)} \end{aligned}$$

for any $j \geq j_0$ and any $t \geq L$.

Then, since for $t \geq \max\{R, 2L\}$ we may estimate $R+t \leq 2t$ and $t-L \geq t/2$, we find

$$F_1(t) \geq \exp \left(p^j \log \left(E_1 \varepsilon^p t^{\gamma_0 + \frac{1}{p-1} - (\alpha_0 + \frac{n-1}{2})} \right) \right) (t+R)^n (t-L)^{-1/(p-1)} \quad (2.22)$$

for any $j \geq j_0$, where $E_1 \doteq 2^{-(\alpha_0 + (n-1)/2 + \gamma_0 + 1/(p-1))} E$. We rewrite the exponent for t in the last inequality as follows:

$$\gamma_0 + \frac{1}{p-1} - \left(\alpha_0 + \frac{n-1}{2} \right) = \frac{p}{2(p-1)} ((n+1) - (n-1)p) = \frac{p((n+1) - (n-1)p)}{2(p-1)}.$$

We notice that for $1 < p < p_{\text{Gla}}(n)$ (respectively, for $1 < p$ when $n=1$), this exponent for t is positive. Let us fix $\varepsilon_0 = \varepsilon_0(u_0, u_1, u_2, n, p, R, \beta) > 0$ such that

$$\varepsilon_0^{-\frac{2(p-1)}{(n+1) - (n-1)p}} \geq E_1^{\frac{2(p-1)}{p((n+1) - (n-1)p)}} \max\{R, 2L\}.$$

Also, for any $\varepsilon \in (0, \varepsilon_0]$ and any $t > E_2 \varepsilon^{-\frac{2(p-1)}{(n+1) - (n-1)p}}$, where $E_2 \doteq E_1^{-\frac{2(p-1)}{p((n+1) - (n-1)p)}}$, we have

$$t \geq \max\{R, 2L\} \quad \text{and} \quad \log \left(E_1 \varepsilon^p t^{\frac{p((n+1) - (n-1)p)}{2(p-1)}} \right) > 0.$$

Consequently, for any $\varepsilon \in (0, \varepsilon_0]$ and any $t > E_2 \varepsilon^{-\frac{2(p-1)}{(n+1) - (n-1)p}}$ letting $j \rightarrow \infty$ in (2.22) we find that the lower bound for F_1 blows up. So, for any $\varepsilon \in (0, \varepsilon_0]$ the functional F_1 has to blow up in finite time

too and, furthermore, the lifespan of the local solution u can be estimated from above in the following way:

$$T(\varepsilon) \leq C\varepsilon^{-\left(\frac{1}{p-1} - \frac{n-1}{2}\right)^{-1}}.$$

We completed the proof of Theorem 1.1 in the case $1 < p < p_{\text{Gla}}(n)$. In the next section we will investigate the blow – up dynamic in the case $p = p_{\text{Gla}}(n)$.

3. Blow – up result in the critical case

3.1. Iteration frame

From the last section, we know that the first lower bound for functional F_1 is given by

$$F_1(t) \geq C_2\varepsilon$$

for any $t \geq 1/2$, with a positive constant C_2 .

In this section, we consider the case $p = p_{\text{Gla}}(n) = (n+1)/(n-1)$ when $n \geq 2$. In this special case, the iteration frame (2.18) takes the form

$$\begin{aligned} F_1(t) &\geq C_3 \int_0^t e^{2(\tau-t)} \int_0^\tau (s+R)^{-1} (F_1(s))^p ds d\tau \\ &\geq C_4 \int_1^t e^{2(\tau-t)} \int_1^\tau \frac{(F_1(s))^p}{s} ds d\tau \end{aligned} \quad (3.23)$$

for some suitable positive constant C_4 and for any $t \geq 1$.

3.2. Iteration argument

Analogously to what we did in Subsection 2.2 we derive now a sequence of lower bounds for F_1 by using the iteration frame (3.23). More specifically, we want to show that

$$F_1(t) \geq Q_j (\log(t/L_j))^{\sigma_j} \quad \text{for any } t \geq L_j, \quad (3.24)$$

where $\{Q_j\}_{j \in \mathbb{N}}$ and $\{\sigma_j\}_{j \in \mathbb{N}}$ are sequences of nonnegative real numbers to be determined and $\{L_j\}_{j \in \mathbb{N}}$ is defined as in Section 2. When $j = 0$, we have $Q_0 \doteq C_2\varepsilon$ and $\sigma_0 \doteq 0$ according to (2.17).

As in the subcritical case, we are going to prove (3.24) by using an inductive argument. We assume the validity of (3.24) for $j \geq 0$ and we have to prove it for $j+1$, prescribing the values of Q_{j+1} and of σ_{j+1} . Shrinking the domain of integration in (3.23) and plugging (3.24) in (3.23), we obtain

$$\begin{aligned} F_1(t) &\geq C_4 Q_j^p \int_{L_j}^t e^{2(\tau-t)} \int_{L_j}^\tau \frac{(\log(s/L_j))^{\sigma_j p}}{s} ds d\tau \\ &\geq \frac{C_4 Q_j^p}{\sigma_j p + 1} \int_{L_j}^t e^{2(\tau-t)} (\log(\tau/L_j))^{\sigma_j p + 1} d\tau \end{aligned}$$

for any $t \geq L_{j+1}$. Since for $t \geq L_{j+1}$ it holds $L_j \leq t/\ell_{j+1}$, a restriction of the domain of integration in the last inequality yields

$$\begin{aligned} F_1(t) &\geq \frac{C_4 Q_j^p}{\sigma_j p + 1} \int_{t/\ell_{j+1}}^t e^{2(\tau-t)} (\log(\tau/L_j))^{\sigma_j p + 1} d\tau \\ &\geq \frac{C_4 Q_j^p}{\sigma_j p + 1} (\log(t/L_{j+1}))^{\sigma_j p + 1} \int_{t/\ell_{j+1}}^t e^{2(\tau-t)} d\tau \\ &\geq \frac{C_4 Q_j^p}{2(\sigma_j p + 1)} (\log(t/L_{j+1}))^{\sigma_j p + 1} \left(1 - e^{2t(1/\ell_{j+1} - 1)}\right) \\ &\geq \frac{C_4 Q_j^p (p-1)p^{-2(j+1)}}{\sigma_j p + 1} (\log(t/L_{j+1}))^{\sigma_j p + 1}, \end{aligned}$$

where we used once again (2.21) in the last inequality. So, we proved (3.24) for $j + 1$, provided that

$$Q_{j+1} \doteq \frac{C_4 Q_j^p (p-1) p^{-2(j+1)}}{\sigma_j p + 1}, \quad \sigma_{j+1} \doteq \sigma_j p + 1.$$

Repeating the same procedure seen in Section 2, we get easily

$$\begin{aligned} \sigma_j &= \frac{p^j - 1}{p - 1}, \\ Q_j &\geq C_4 (p-1)^2 p^{-3j} Q_{j-1}^p \doteq \tilde{D} p^{-3j} Q_{j-1}^p. \end{aligned}$$

Hence, applying again the monotonicity of the logarithmic function, in this case to the inequality $Q_j \geq \tilde{D} p^{-3j} Q_{j-1}^p$, we derive

$$\log Q_j \geq p^j \left(\log Q_0 - \frac{3p \log p}{(p-1)^2} + \frac{\log \tilde{D}}{p-1} \right) + \frac{3j \log p}{p-1} + \frac{3p \log p}{(p-1)^2} - \frac{\log \tilde{D}}{p-1}$$

for any $j \in \mathbb{N}$. Let $j_1 = j_1(n, p) \in \mathbb{N}$ be the smallest nonnegative integer such that

$$j_1 \geq \frac{\log \tilde{D}}{3 \log p} - \frac{p}{p-1}.$$

Then, for any $j \geq j_1$ it results

$$\log Q_j \geq p^j \log \left(\tilde{D}^{1/(p-1)} p^{-3p/(p-1)^2} Q_0 \right) = p^j \log(\tilde{E} \varepsilon)$$

for a suitable positive constant $\tilde{E} = \tilde{E}(n, p)$. Let us recall that L denotes the monotonic limit of the sequence $\{L_j\}_{j \in \mathbb{N}}$. Therefore, we have that (3.24) holds for any $j \in \mathbb{N}$ and any $t \geq L$.

Thus, applying the explicit representation for σ_j , we arrive at

$$\begin{aligned} F_1(t) &\geq \exp \left(p^j \log(\tilde{E} \varepsilon) \right) (\log(t/L))^{\sigma_j} \\ &= \exp \left(p^j \log \left(\tilde{E} \varepsilon (\log(t/L))^{1/(p-1)} \right) \right) (\log(t/L))^{-1/(p-1)}, \end{aligned} \tag{3.25}$$

for any $j \geq j_1$ and any $t \geq L$. In this case, we fix $\varepsilon_0 = \varepsilon_0(u_0, u_1, u_2, n, p, R, \beta) > 0$ in such a way that

$$\exp \left(\tilde{E}^{-p+1} \varepsilon_0^{-(p-1)} \right) \geq 1.$$

Consequently, for any $\varepsilon \in (0, \varepsilon_0]$ and any $t > L \exp(\tilde{E}^{-p+1} \varepsilon^{-(p-1)})$, we get

$$t \geq L \quad \text{and} \quad \log \left(\tilde{E} \varepsilon (\log(t/L))^{1/(p-1)} \right) > 0.$$

Hence, for any $\varepsilon \in (0, \varepsilon_0]$ and any $t > L \exp(\tilde{E}^{-p+1} \varepsilon^{-(p-1)})$ by letting $j \rightarrow \infty$ in (3.25) we see that the lower bound for F_1 blows up. Thus, for any $\varepsilon \in (0, \varepsilon_0]$ the functional F_1 has to blow up in finite time as well and, besides, the lifespan of the local solution u can be estimated from above in the following way

$$T(\varepsilon) \leq \exp \left(C \varepsilon^{-(p-1)} \right),$$

for a suitable constant C which is independent of ε . This completes the proof of Theorem 1.1 in the case $p = p_{\text{Gla}}(n)$.

4. Final remarks

In Theorem 1.1, we proved a blow – up result for $1 < p \leq p_{\text{Gla}}(n)$ under suitable sign and support assumptions for the Cauchy data. Furthermore, as byproduct of the iteration arguments we obtained upper bound estimates for the lifespan as well. In particular, we find the same range for p in the blow – up result as for the semilinear Cauchy problem (1.3), which is known to be sharp in the case of this last wave model.

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References

- [1] R. Agemi, Blow-up of solutions to nonlinear wave equations in two space dimensions, *Manuscripta Math.* **73** (2) (1991) 153–162.
- [2] R. Agemi, Y. Kurokawa and H. Takamura, Critical curve for p - q systems of nonlinear wave equations in three space dimensions, *J. Differential Equations* **167** (1) (2000) 87–133.
- [3] M. O. Alves, A. H. Caixeta, M. A. J. Silva and J. H. Rodrigues, Moore-Gibson-Thompson equation with memory in a history framework: a semigroup approach, *Z. Angew. Math. Phys.* **69** (4) (2018) 19 pp.
- [4] A. H. Caixeta, I. Lasiecka and V. N. Domingos Cavalcanti, On long time behavior of Moore-Gibson-Thompson equation with molecular relaxation, *Evol. Equ. Control Theory* **5** (4) (2016) 661–676.
- [5] W. Chen, A. Palmieri, Nonexistence of global solutions for the semilinear Moore-Gibson-Thompson equation in the conservative case, preprint, arXiv:1909.08838, 2019.
- [6] F. Dell’Oro, I. Lasiecka and V. Pata, On the MGT equation with memory of type II, preprint (2019).
- [7] F. Dell’Oro, I. Lasiecka and V. Pata, The Moore-Gibson-Thompson equation with memory in the critical case, *J. Differential Equations* **261** (7) (2016) 4188–4222.
- [8] F. Dell’Oro and V. Pata, On the Moore-Gibson-Thompson equation and its relation to linear viscoelasticity, *Appl. Math. Optim.* **76** (3) (2017) 641–655.
- [9] G. C. Gorain, Stabilization for the vibrations modeled by the ‘standard linear model’ of viscoelasticity, *Proc. Indian Acad. Sci. Math. Sci.* **120** (4) (2010) 495–506.
- [10] K. Hidano and K. Tsutaya, Global existence and asymptotic behavior of solutions for nonlinear wave equations, *Indiana Univ. Math. J.* **44** (4) (1995) 1273–1305.
- [11] K. Hidano, C. Wang and K. Yokoyama, The Glassey conjecture with radially symmetric data, *J. Math. Pures Appl. (9)* **98** (5) (2012) 518–541.
- [12] M. Ikeda, Z. Tu and K. Wakasa, Small data blow-up of semi-linear wave equation with scattering dissipation and time-dependent mass, preprint, arXiv:1904.09574, 2019.
- [13] F. John, Blow-up for quasilinear wave equations in three space dimensions, *Comm. Pure Appl. Math.* **34** (1) (1981) 29–51.
- [14] P. M. Jordan, Second-sound phenomena in inviscid, thermally relaxing gases, *Discrete Contin. Dyn. Syst. Ser. B* **19** (7) (2014) 2189–2205.
- [15] B. Kaltenbacher and I. Lasiecka, Exponential decay for low and higher energies in the third order linear Moore-Gibson-Thompson equation with variable viscosity, *Palest. J. Math.* **1** (1) (2012) 1–10.
- [16] B. Kaltenbacher, I. Lasiecka and R. Marchand, Wellposedness and exponential decay rates for the Moore-Gibson-Thompson equation arising in high intensity ultrasound, *Control Cybernet.* **40** (4) (2011) 971–988.
- [17] N. A. Lai and H. Takamura, Blow-up for semilinear damped wave equations with subcritical exponent in the scattering case, *Nonlinear Anal.* **168** (2018) 222–237.
- [18] N. A. Lai and H. Takamura, Nonexistence of global solutions of nonlinear wave equations with weak time-dependent damping related to Glassey’s conjecture, *Differential Integral Equations* **32** (1/2) (2019) 37–48.
- [19] N. A. Lai and H. Takamura, Nonexistence of global solutions of wave equations with weak time-dependent damping and combined nonlinearity, *Nonlinear Anal. Real World Appl.* **45** (2019) 83–96.
- [20] N. A. Lai, H. Takamura and K. Wakasa, Blow-up for semilinear wave equations with the scale invariant damping and super-Fujita exponent, *J. Differential Equations* **263** (9) (2017) 5377–5394.
- [21] I. Lasiecka, Global solvability of Moore-Gibson-Thompson equation with memory arising in nonlinear acoustics, *J. Evol. Equ.* **17** (1) (2017) 411–441.
- [22] I. Lasiecka and X. Wang, Moore-Gibson-Thompson equation with memory, part I: exponential decay of energy, *Z. Angew. Math. Phys.* **67** (2) (2016) 23 pp.

- [23] I. Lasiecka and X. Wang, Moore-Gibson-Thompson equation with memory, part II: General decay of energy, *J. Differential Equations* **259** (12) (2015) 7610–7635.
- [24] R. Marchand, T. McDevitt and R. Triggiani, An abstract semigroup approach to the third-order Moore-Gibson-Thompson partial differential equation arising in high-intensity ultrasound: structural decomposition, spectral analysis, exponential stability, *Math. Methods Appl. Sci.* **35** (15) (2012) 1896–1929.
- [25] K. Masuda, Blow-up solutions for quasi-linear wave equations in two space dimensions, *Lecture Notes Num. Appl. Anal.* **6** (1983) 87–91.
- [26] F. K. Moore and W. E. Gibson, *Propagation of weak disturbances in a gas subject to relaxation effect*, *J. Aero/Space Sci.* **27** (1960) 117–127.
- [27] A. Palmieri, A note on a conjecture for the critical curve of a weakly coupled system of semilinear wave equations with scale-invariant lower order terms, preprint, arXiv:1812.06588, 2018.
- [28] A. Palmieri and H. Takamura, Blow-up for a weakly coupled system of semilinear damped wave equations in the scattering case with power nonlinearities, *Nonlinear Anal.* **187** (2019) 467–492.
- [29] A. Palmieri and H. Takamura, Nonexistence of global solutions for a weakly coupled system of semilinear damped wave equations of derivative type in the scattering case, preprint, arXiv:1812.10653, 2018.
- [30] A. Palmieri and H. Takamura, Nonexistence of global solutions for a weakly coupled system of semilinear damped wave equations in the scattering case with mixed nonlinear terms, preprint, arXiv:1901.04038, 2019.
- [31] A. Palmieri and Z. Tu, Lifespan of semilinear wave equation with scale invariant dissipation and mass and sub-Strauss power nonlinearity, *J. Math. Anal. Appl.* **470** (1) (2019) 447–469.
- [32] A. Palmieri and Z. Tu, A blow-up result for a semilinear wave equation with scale-invariant damping and mass and nonlinearity of derivative type, preprint, arXiv:1905.11025, 2019.
- [33] M. Pellicer and B. Said-Houari, Wellposedness and decay rates for the Cauchy problem of the Moore-Gibson-Thompson equation arising in high intensity ultrasound, *Appl. Math. Optim.* (2017).
- [34] M. Pellicer and J. Solá-Morales, Optimal scalar products in the Moore-Gibson-Thompson equation, *Evol. Equ. Control Theory* **8** (1) (2019) 203–220.
- [35] R. Racke and B. Said-Houari, Global well-posedness of the Cauchy problem for the Jordan-Moore-Gibson-Thompson equation, *Konstanzer Schriften in Mathematik* 382 (2019), <http://nbn-resolving.de/urn:nbn:de:bsz:352-2-8ztzhsc03jj82>
- [36] M. A. Rammaha, Finite-time blow-up for nonlinear wave equations in high dimensions, *Comm. Partial Differential Equations* **12** (6) (1987) 677–700.
- [37] J. Schaeffer, Finite-time blow-up for $u_{tt} - \Delta u = H(u_r, u_t)$, *Commun. Partial Differential Equations* **11** (5) (1986) 513–543.
- [38] T. S. Sideris, Global behavior of solutions to nonlinear wave equations in three dimensions, *Comm. Partial Differential. Equations* **8** (12) (1983) 1219–1323.
- [39] H. Takamura and K. Wakasa, The sharp upper bound of the lifespan of solutions to critical semilinear wave equations in high dimensions, *J. Differential Equations* **251** (4/5) (2011) 1157–1171.
- [40] H. Takamura and K. Wakasa, Almost global solutions of semilinear wave equations with the critical exponent in high dimensions, *Nonlinear Anal.* **109** (2014) 187–229.
- [41] P. A. Thompson, *Compressible-fluid dynamics*, McGraw-Hill, New York, (1972).
- [42] N. Tzvetkov, Existence of global solutions to nonlinear massless Dirac system and wave equation with small data, *Tsukuba J. Math.* **22** (1) (1998) 198–211.
- [43] K. Wakasa and B. Yordanov, Blow-up of solutions to critical semilinear wave equations with variable coefficients, *J. Differential Equations* **266** (9) (2019) 5360–5376.
- [44] B. T. Yordanov and Q. S. Zhang, Finite time blow up for critical wave equations in high dimensions, *J. Funct. Anal.* **231** (2) (2006) 361–374.
- [45] Y. Zhou, Blow up of solutions to the Cauchy problem for nonlinear wave equations, *Chinese Ann. Math. Ser. B* **22** (3) (2001) 275–280.

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