

SEGREGATED SOLUTIONS FOR NONLINEAR SCHRÖDINGER SYSTEMS WITH WEAK INTERSPECIES FORCES

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ABSTRACT. We find positive non-radial solutions for a system of Schrödinger equations in a weak fully attractive or repulsive regime in presence of an external radial trapping potential that exhibits a maximum or a minimum at infinity.

1. INTRODUCTION

We are interested in finding positive solutions to the system

$$-\Delta u_i + \lambda_i u_i + V_i(x)u_i = \mu_i u_i^3 + u_i \sum_{\substack{j=1 \\ j \neq i}}^d \beta_{ij} u_j^2 \text{ in } \mathbb{R}^n, \quad i = 1, \dots, d \quad (1.1)$$

where $\mu_i > 0$, $\lambda_i > 0$, $\beta_{ij} = \beta_{ji} \in \mathbb{R}$, $V_i \in C^0(\mathbb{R}^n)$, $d \in \mathbb{N}$, $n = 2, 3$. This system has been proposed as a mathematical model for multispecies Bose-Einstein condensation in m different states:

$$-i\partial_t \phi_i = \Delta \phi_i - V_i(x)\phi_i + \mu_i |\phi_i|^2 \phi_i + \sum_{\substack{j=1 \\ j \neq i}}^m \beta_{ij} |\phi_j|^2 \phi_i, \quad i = 1, \dots, d \quad (1.2)$$

where the complex valued functions ϕ_i 's are the wave functions of the i -th condensate, $|\phi_i|$ is the amplitude of the i -th density, μ_i describes the interaction between particles of the same component and β_{ij} , $i \neq j$, describes the interaction between particles of different components, which can be *attractive* if $\beta_{ij} > 0$ or *repulsive* if $\beta_{ij} < 0$. To obtain solitary wave solutions of the Gross-Pitaevskii system (1.2) we set $\phi_i(t, x) = e^{-i\lambda_i t} u_i(x)$ and we find real functions u_i 's which solve the system (1.1). We refer to [4, 5, 6, 10] for a detailed physical motivation.

There are different kind of solutions to (1.1). The trivial solution has all trivial components, i.e. $u_i \equiv 0$ for any i . A non-trivial solution has some trivial components, i.e. $u_i \equiv 0$ for some i and in this case the system (1.1) reduces to a system with a less number of components. The most interesting solutions are the so-called *fully non-trivial* solutions whose components are all non-trivial, i.e. $u_i \not\equiv 0$ for any i . Among the set

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of fully non-trivial solutions we can distinguish among *synchronized* solutions and *non-synchronized* solutions. We say that $\mathbf{u} = (u_1, \dots, u_m)$ is a *synchronized* solution to the system (1.1) (if, for example, $\lambda_i = \lambda$ and $V_i(x) = 0$ for any index i), if $\mathbf{u} = (\gamma_1 U, \dots, \gamma_k U)$ where $\gamma_i > 0$ and U is a positive solution to the single equation

$$-\Delta U + \lambda U = U^3 \text{ in } \mathbb{R}^n. \quad (1.3)$$

In this case the system (1.1) reduces to the algebraic system

$$\gamma_i = \mu_i \gamma_i^3 + \gamma_i \sum_{\substack{j=1 \\ j \neq i}}^m \beta_{ij} \gamma_j^2, \quad i = 1, \dots, m.$$

Now, let us recall some known results. First, let us focus on the system with two components

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^3 + \beta uv^2 \text{ in } \mathbb{R}^n, \\ -\Delta v + \lambda_2 v = \mu_2 v^3 + \beta u^2 v \text{ in } \mathbb{R}^n \end{cases} \quad (1.4)$$

It is immediate to check that if $\lambda = \lambda_1 = \lambda_2$, the system (1.4) has a synchronized solution

$$(u, v) = (\gamma_1 U, \gamma_2 U), \quad \gamma_1 = \sqrt{\frac{\beta - \mu_2}{\beta^2 - \mu_1 \mu_2}}, \quad \gamma_2 = \sqrt{\frac{\beta - \mu_1}{\beta^2 - \mu_1 \mu_2}} \quad (1.5)$$

where U solves (1.3) provided

$$-\sqrt{\mu_1 \mu_2} < \beta < \min\{\mu_1, \mu_2\} \text{ or } \beta > \max\{\mu_1, \mu_2\}. \quad (1.6)$$

In the attractive case, i.e. $\beta > 0$, all the positive solutions of (1.4) are radially symmetric (up to translation) and both components are decreasing in the radial variable (see for example [14]). On the other hand, in the repulsive case, i.e. $\beta < 0$, there exists a large variety of solutions. In fact radial solutions has been found by Wei & Weth [16], who proved that if $\beta \leq -1$ for any integer k there exists a radial solution (u_1, u_2) to the system (1.4) with $\lambda_i = 1$ and $\mu_i = 1$ such that the difference $u_1 - u_2$ has exactly $k - 1$ zeroes and converges as $\beta \rightarrow -\infty$ to a function W which is radial changing-sign solutions of the scalar equation (1.3). Bartsch, Dancer & Wang [2] extend this result to a larger range of parameters β, μ_1, μ_2 . The case of an arbitrary number of components $d \geq 3$ has been studied by Terracini & Verzini [13]. Problem (1.4) can also have non-radial solutions. When the coupling parameter $\beta < 0$ is small, Lin & Wei [7] found solutions with one component peaking at the origin and the other having a finite number of peaks on a k -polygon in the plane \mathbb{R}^2 . Wei & Weth [15] proved the existence of infinitely many non radial solutions which are invariant under the action of a finite subgroup of $\mathcal{O}(n)$ (for example they satisfy (2.7)).

The general case has been firstly considered by Lin & Wei [8] where they study the autonomous system

$$-\Delta u_i + \lambda_i u_i = \mu_i u_i^3 + \sum_{\substack{j=1 \\ j \neq i}}^d \beta_{ij} u_j^2 \text{ in } \mathbb{R}^n, \quad i = 1, \dots, d \quad (1.7)$$

They prove the existence of a ground state solution whose components are positive, radially symmetric and strictly decreasing when all the β_{ij} 's are positive and the matrix $(\beta_{ij})_{i,j=1,\dots,d}$ (here we set $\beta_{ii} = \mu_i$) is positively definite and also that the ground state solution does not exist anymore if all the β_{ij} 's are negative. A systematic analysis of system (1.7) under the assumption that it admits mixed couplings and the components are organized into different groups has been recently developed by Wei & Wu [17]. The first result concerning existence of non-radial solutions for systems with 3 components goes back to Lin & Wei [8] who proved the existence of non-radial solutions to (1.7) if all the coupling parameters are small and repulsion is much stronger than attraction. Recently, Peng, Wang & Wang [12] consider system (1.7) in the case the repulsive couplings are small and obtain solutions with some of the components synchronized between them while being segregated with the rest of the components.

All the previous results deal with the autonomous case. The non-autonomous case has been studied by Peng & Wang [11], who considered the system

$$\begin{cases} -\Delta u_1 + V_1(x)u_1 = \mu_1 u_1^3 + \beta u_1 u_2^2 & \text{in } \mathbb{R}^n, \\ -\Delta u_2 + V_2(x)u_2 = \mu_2 u_2^3 + \beta u_1^2 u_2 & \text{in } \mathbb{R}^n \end{cases}$$

and found infinitely many non-radial positive solutions when the potentials V_1 and V_2 are radially symmetric and satisfy

$$V_i(x) \sim 1 + \frac{\mathbf{v}_\infty^i}{|x|^{q_i}} \text{ as } |x| \rightarrow \infty, \quad i = 1, 2. \quad (1.8)$$

They build synchronized solutions (if $q_1 < q_2$, $\mathbf{v}_\infty^1 > 0$ or $q_1 > q_2$, $\mathbf{v}_\infty^2 > 0$ and the coupling parameter β satisfies (1.6)) and segregated solutions (if $q_1 = q_2$, $\mathbf{v}_\infty^1, \mathbf{v}_\infty^2 > 0$ and $\beta < \beta_0$ for some $\beta_0 > 0$). The synchronized solutions look like a sum of k copies of the synchronized solution (1.5)

$$(u_1, u_2) \sim \left(\gamma_1 \sum_{i=1}^k U(x - \rho \xi_i), \gamma_2 \sum_{i=1}^k U(x - \rho \xi_i) \right) \text{ as } k \rightarrow \infty$$

where the peaks satisfy

$$\xi_i = \left(\cos \frac{2(i-1)\pi}{k}, \sin \frac{2(i-1)\pi}{k}, 0 \right), \text{ where the radius } \rho \sim Rk \ln k \text{ for some } R > 0.$$

Here U is the unique positive radial solution to the single equation

$$-\Delta U + U = U^3 \text{ in } \mathbb{R}^n. \quad (1.9)$$

On the other hand, the profile of each component of the segregated solutions looks like a sum of k copies of the solution to (1.9)

$$(u_1, u_2) \sim \left(\sum_{i=1}^k U(x - \rho_1 \xi_i), \sum_{i=1}^k U(x - \rho_2 \eta_i) \right) \text{ as } k \rightarrow \infty$$

where the peaks ξ_i of the first component are as above, while the peaks η_i of the second component are nothing but the peaks of the first one rotated by an angle $\frac{\pi}{k}$ and the radii $\rho_i \sim R_i k \ln k$ for some $R_i > 0$. The proof of their result relies on the idea by Wei

& Yan [18] who found infinitely many solutions (positive if $\mathbf{v}_\infty > 0$ and sign-changing if $\mathbf{v}_\infty < 0$) to the single Schrödinger equation

$$-\Delta u + V(x)u = |u|^{p-1}u \text{ in } \mathbb{R}^n$$

when the potential V is radially symmetric and satisfy (1.10).

We will focus on the existence of segregated solutions and we will ask a couple of questions which naturally arise.

- (Q1) Peng & Wang's result holds when the system has only two components. Can we find segregated solutions when the system has at least three components?
 (Q2) Peng & Wang's result holds when all the \mathbf{v}_∞^i 's in (1.8) are positive. Do there exist any solutions when some \mathbf{v}_∞^i 's are negative?

We will give some partial positive answers in the particular case when all the parameters μ_i 's are equal to 1, all the β_{ij} 's are equal to a real number β and all the potentials V_i 's coincide with the radial potential $V \in C^1(\mathbb{R}^n)$ which satisfies for some $\mathbf{v}_\infty \in \mathbb{R}$, $\nu > 1$ and $\epsilon > 0$

$$V(|x|) = 1 + \frac{\mathbf{v}_\infty}{|x|^\nu} + \mathcal{O}\left(\frac{1}{|x|^{\nu+\epsilon}}\right) \quad C^1\text{-uniformly as } |x| \rightarrow +\infty, \quad (1.10)$$

so that the system (1.1) reduces to

$$-\Delta u_i + V(x)u_i = u_i^3 + \beta u_i \sum_{\substack{j=1 \\ j \neq i}}^d u_j^2 \text{ in } \mathbb{R}^n, \quad i = 1, \dots, d. \quad (1.11)$$

Our main result is the following one.

Theorem 1.1. *Let $d \geq 3$ and $\nu > \frac{2}{d-2}$. There exists $k_0 > 0$ such that for any integer $k \geq k_0$ there exists $\beta_k > 0$ such that for any $\beta \in (0, \beta_k)$ if $\mathbf{v}_\infty > 0$ or for any $\beta \in (-\beta_k, 0)$ if $\mathbf{v}_\infty < 0$ the system (1.11) has a positive solution (u_1, \dots, u_d) whose components satisfy*

$$u_i(x) \equiv u(\Theta_i x)$$

where

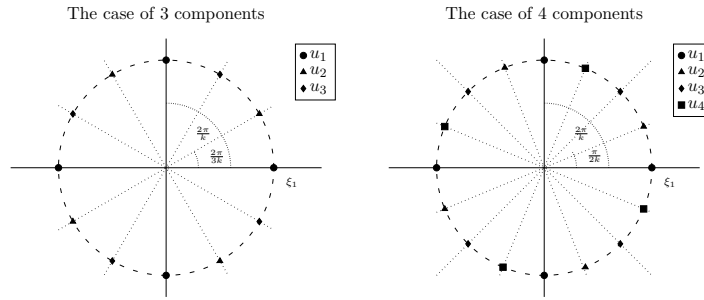
$$\Theta_i := \begin{pmatrix} \cos \frac{2(i-1)\pi}{dk} & \sin \frac{2(i-1)\pi}{dk} & 0 \\ -\sin \frac{2(i-1)\pi}{dk} & \cos \frac{2(i-1)\pi}{dk} & 0 \\ 0 & 0 & I_{(n-2) \times (n-2)} \end{pmatrix}$$

and

$$u_1(x) \sim \sum_{\ell=1}^k U(x - \rho \xi_\ell) \text{ as } k \rightarrow \infty$$

with

$$\xi_\ell = \left(\cos \frac{2(\ell-1)\pi}{k}, \sin \frac{2(\ell-1)\pi}{k}, 0 \right), \quad \rho \sim Rk \ln k \text{ as } k \rightarrow \infty \text{ for some } R > 0.$$



The proof of our result is given in Section 2. First, using the symmetry, we write the system (1.11) as a single non-local equation (2.4). Next, we build a solution whose main term is the sum of a large number of copies of solutions to problem (1.9) (see (2.8)) whose peaks are the vertices of a regular polygon with k edges at distance $\rho = \rho(k)$ from the origin. Due to the linear coupling term, the ansatz has to be improved by adding the solution of the linear problem (2.12). Then we perform a classical Ljapunov-Schmidt procedure to reduce the problem to that of finding a radius ρ which is the zero of the one-dimensional function (2.32). This *reduced 1D function* consists of four main terms (see Lemma 2.9). The first term $\mathbf{v}_\infty \frac{k}{\rho^{\nu+1}}$ arises from the potential effect and its sign depends on \mathbf{v}_∞ , the second term (which contains $e^{-\frac{2\pi\rho}{k}}$) is due to the interplay between peaks of the same component and is always negative, the third term (which contains $\beta e^{-\frac{4\pi\rho}{dk}}$) is due to the interaction among the peaks of different components and its sign depends on the coupling parameter β . If $d \geq 3$ the third term prevails the second one, while if $d = 2$ the second term dominates the third one. The asymptotic expansion of the last term Υ which is produced by the correction of the ansatz is really difficult to catch and, unfortunately, we believe its presence is not an innocence matter, since it could give a contribution to the second and the third terms. We are only able to provide the rough estimate (2.33). That is why we need to choose the coupling parameter β small enough so that Υ is an higher order term in the expansion of the reduced 1D function. It is clear that it would be extremely interesting to find the leading term of Υ . At this aim it is worthwhile to point out that such an expansion relies on the asymptotic decay of the solution of (2.12) and this is the key point we are not able to solve.

Now, let us go back to the reduced 1D function (2.32). We observe that Peng and Wang (using a different approach) prove that if the system has only two components the interaction among peaks of different components is negligible with respect to the interaction among peaks of the same component and since such an interaction is always negative (and does not depend on β), the existence of segregated solutions is ensured as soon as the potential V has a minimum at infinity, i.e. $\mathbf{v}_\infty > 0$. We conjecture that if the number of components is higher the interaction among peaks of different components should prevail. In such a case the existence of segregated solutions would depend on the behaviour of the potential V at ∞ and the sign of the coupling parameter β , namely they should exist if either in an attractive regime (i.e. $\beta > 0$) V has a minimum at ∞ or in a repulsive regime (i.e. $\beta < 0$) V has a maximum at ∞ (i.e. $\mathbf{v}_\infty < 0$). Actually, this is what happens when the coupling parameter β is small as in Theorem 1.1.

It is also true for the more general variational system

$$-\Delta u_i + V(x)u_i = |u_i|^{p-1}u_i + \beta|u_i|^{\frac{p-3}{2}}u_i \sum_{\substack{j=1 \\ j \neq i}}^d |u_j|^{\frac{p-1}{2}}u_j \text{ in } \mathbb{R}^n, \quad i = 1, \dots, d, \quad (1.12)$$

when $p > 3$ if $n = 2$ or $p \in \left(3, \frac{n+2}{n-2}\right)$ if $n \geq 3$ (the system (1.12) with $p = 3$ reduces to (1.11)). In fact, in Section 3 we show the following result.

Theorem 1.2. *Assume (1.10) and*

- (i) $d > \frac{p+1}{2}$, $p > 5$ and either $\mathbf{v}_\infty > 0$ if $\beta > 0$ or $\mathbf{v}_\infty < 0$ if $\beta < 0$
- (ii) $d \leq \frac{p-1}{2}$ and $\mathbf{v}_\infty > 0$.

There exists $k_0 > 0$ such that for any integer $k \geq k_0$ the system (1.11) has a solution (u_1, \dots, u_d) whose components satisfy the properties listed in Theorem 1.1.

The situation in this case is easier, because the correction of the ansatz is not needed and the reduced 1D function reads as (3.5) in case (i) or (3.6) in case (ii). To conclude, we highlight the fact that if we were able to approximate the 1D function in Lemma 2.9 as (3.5) or (3.6), then we would have the existence of solutions of system (1.11) without any assumptions on the smallness of the interspecies force β .

The paper is organized as follows. Section 2 and Section 3 contain the proof of Theorem 1.1 and Theorem 1.2, respectively. Appendix A contains some auxiliary results. Appendix B contains the proof of Proposition 2.2.

Notation. In what follows we agree that $f \lesssim g$ means $|f| \leq c|g|(1 + o(1))$ for some positive constant c independent from k and $f \sim g$ means $f = g(1 + o(1))$.

2. PROOF OF THEOREM 1.1

2.1. Reducing the system to a non-local equation. First of all, we introduce some symmetries which allow to reduce the system (1.11) to a non-local equation.

Given $\theta \in [0, 2\pi]$, let $\Theta_\theta : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the rotation of an angle θ in the first two components, i.e.

$$\Theta_\theta := \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & I_{n-2 \times n-2} \end{pmatrix}. \quad (2.1)$$

Now, let $k \geq 2$ and set

$$\hat{\Theta}_i := \Theta_{\frac{2\pi}{dk}(i-1)}, \quad \text{for any } i = 1, \dots, d. \quad (2.2)$$

Note that $\hat{\Theta}_1 = I_{n \times n}$.

We look for a solution to (1.11) as $\mathbf{u} = (u_1, \dots, u_d) \in (H^1(\mathbb{R}^n))^d$ whose components satisfy

$$u_i(x) \equiv u(\hat{\Theta}_i x) \text{ for any } i = 1, \dots, d. \quad (2.3)$$

It is immediate to check that such a \mathbf{u} solves the system (1.11) if and only if u solves the non-local equation

$$-\Delta u + V(x)u = u^3 + \beta u \sum_{j=2}^d u^2(\hat{\Theta}_j x), \quad \text{in } \mathbb{R}^n. \quad (2.4)$$

2.2. The ansatz for the non-local equation. We look for a solution to (2.4) in the space

$$\mathcal{H} := \left\{ u \in W^{1,2}(\mathbb{R}^n) : u \text{ satisfies (2.6) and (2.7)} \right\}, \quad (2.5)$$

i.e.

$$u(x_1, \dots, x_i, \dots) = u(x_1, \dots, -x_i, \dots) \text{ for any } i = 2, \dots, n \quad (2.6)$$

and

$$u(x) = u\left(\Theta_{\frac{2\pi}{k}(h-1)}x\right) \text{ for any } h = 1, \dots, k \text{ (see (2.1)).} \quad (2.7)$$

We are going to find a solution of (2.4) whose main term looks like

$$W_\rho(x) := \sum_{h=1}^k U_h(x) \text{ and } U_h(x) := U(x - \rho\xi_h) \text{ solves (1.9),} \quad (2.8)$$

where the peaks satisfy

$$\xi_h := \Theta_{\frac{2\pi}{k}(h-1)}\xi_1, \quad h = 1, \dots, k, \text{ with } \xi_1 := (1, 0, 0) \in \mathbb{R}^2 \times \mathbb{R}^{n-2}, \quad (2.9)$$

and

$$\rho \in \mathcal{D}_k := [r_1 k \ln k, r_2 k \ln k] \text{ for some } r_2 > r_1 > 0. \quad (2.10)$$

It is useful to remind that U is the unique positive radial solution to (1.9) and decays exponentially together with its radial derivatives U' , i.e.

$$\lim_{r \rightarrow +\infty} r^{\frac{n-1}{2}} e^r U(r) = \mathbf{u} > 0 \text{ and } \lim_{r \rightarrow +\infty} \frac{U'(r)}{U(r)} = -1, \quad (2.11)$$

where \mathbf{u} is a positive constant depending on n .

It turns out that the error which comes from the interaction term

$$\beta W_\rho(x) \sum_{i=2}^d W_\rho^2(\hat{\Theta}_i x)$$

is too large and we need to refine the ansatz adding the function $\Psi_\rho = \beta Y_\rho$ where $Y_\rho \in \mathcal{H}$ solves

$$\mathcal{L}(Y_\rho) = W_\rho \sum_{i=2}^d W_\rho^2(\hat{\Theta}_i \cdot) - \gamma_\rho \partial_\rho W_\rho \text{ in } \mathbb{R}^n \quad (2.12)$$

where the linear operator \mathcal{L} is defined in (2.17) and

$$\gamma_\rho := \frac{\int_{\mathbb{R}^n} W_\rho(x) \sum_{i=2}^d W_\rho^2(\hat{\Theta}_i(x)) \partial_\rho W_\rho(x) dx}{\int_{\mathbb{R}^n} (\partial_\rho W_\rho(x))^2 dx}. \quad (2.13)$$

The existence of Ψ_ρ follows by Proposition (2.2).

Then we will build a solution to (2.4) as

$$u = W_\rho + \Psi_\rho + \Phi, \text{ with } W_\rho \text{ as in (2.8) and } \Psi_\rho \text{ as in (2.12)}. \quad (2.14)$$

Moreover the higher order term $\Phi \in \mathcal{H}$ satisfies the orthogonality condition

$$\int_{\mathbb{R}^n} \partial_\rho W_\rho \Phi = 0. \quad (2.15)$$

2.3. Rewriting the single equation via the finite dimensional reduction method.

It is useful to rewrite problem (2.4) in terms of Φ , i.e.

$$\mathcal{L}(\Phi) = \mathcal{N}(\Phi) + \mathcal{E} \text{ in } \mathbb{R}^n, \quad (2.16)$$

where the linear operator \mathcal{L} is defined by

$$\mathcal{L}(\Phi) = -\Delta\Phi + V(x)\Phi - 3W_\rho^2\Phi - \beta\Phi \sum_{i=2}^d W_\rho^2(\hat{\Theta}_i x) - 2\beta W_\rho \sum_{i=2}^d W_\rho(\hat{\Theta}_i x)\Phi(\hat{\Theta}_i x), \quad (2.17)$$

the error term \mathcal{E} is defined by

$$\begin{aligned} \mathcal{E} := & (1 - V(x))W_\rho \\ & + \Delta W_\rho - W_\rho + W_\rho^3 - \underbrace{\mathcal{L}(\Psi_\rho) + \beta W_\rho \sum_{i=2}^d W_\rho^2(\hat{\Theta}_i x) + \Psi_\rho^3 + 3W_\rho \Psi_\rho^2}_{= \gamma_\rho \partial_\rho W_\rho \text{ (see (2.13))}} \\ & + \beta \Psi_\rho \sum_{i=2}^d 2W_\rho(\hat{\Theta}_i x)\Psi_\rho(\hat{\Theta}_i x) + \beta \Psi_\rho \sum_{i=2}^d \Psi_\rho^2(\hat{\Theta}_i x) + \beta W_\rho \sum_{i=2}^d \Psi_\rho^2(\hat{\Theta}_i x) \end{aligned} \quad (2.18)$$

and the higher order term $\mathcal{N}(\Phi)$ is defined by

$$\begin{aligned} \mathcal{N}(\Phi) := & \Phi^3 + 3W_\rho\Phi^2 + 3\Psi_\rho\Phi^2 + 3\Psi_\rho^2\Phi + 6W_\rho\Psi_\rho\Phi \\ & + \beta\Phi \sum_{i=2}^d \left[2W_\rho(\hat{\Theta}_i x) \left(\Psi_\rho(\hat{\Theta}_i x) + \Phi(\hat{\Theta}_i x) \right) + \left(\Psi_\rho(\hat{\Theta}_i x) + \Phi(\hat{\Theta}_i x) \right)^2 \right] \\ & + \beta\Psi_\rho \sum_{i=2}^d 2\Phi(\hat{\Theta}_i x) \left(W_\rho(\hat{\Theta}_i x) + \Psi_\rho(\hat{\Theta}_i x) \right) + \beta\Psi_\rho \sum_{i=2}^d \Phi^2(\hat{\Theta}_i x) \\ & + \beta W_\rho \sum_{i=2}^d \Phi^2(\hat{\Theta}_i x) + 2\beta W_\rho \sum_{i=2}^d \Psi_\rho(\hat{\Theta}_i x)\Phi(\hat{\Theta}_i x). \end{aligned} \quad (2.19)$$

In order to solve (2.16) we use the classical Lyapunov-Schmidt procedure:

- (i) first, given $\rho \in \mathcal{D}_k$ (see (2.10)) we find a function $\Phi \in \mathcal{H}$ such that, for a certain $\mathfrak{c} \in \mathbb{R}$, it solves the *intermediate nonlinear problem*

$$\mathcal{L}(\Phi) = \mathcal{N}(\Phi) + \mathcal{E} + \mathfrak{c}\partial_\rho W_\rho \text{ in } \mathbb{R}^n \text{ and } \int_{\mathbb{R}^n} \partial_\rho W_\rho \Phi = 0. \quad (2.20)$$

- (ii) next, we find $\rho \in \mathcal{D}_k$ (see (2.10)) so that \mathfrak{c} in (2.20) is equal to zero.

2.4. The linear theory. We will find the solution Φ to (2.20) in a suitable Banach space where the linear operator \mathcal{L} defined in (2.17) is invertible.

We introduce the Banach space (see also [9, Section 3])

$$\mathcal{B}_* := \{h \in L^\infty(\mathbb{R}^n) : \|h\|_* < +\infty\}, \quad \|h\|_* := \sup_{x \in \mathbb{R}^n} \left(\sum_{i=1}^d \sum_{j=1}^k e^{-\alpha|x-\rho\eta_{ij}|} \right)^{-1} |h(x)|.$$

for some $\alpha \in (0, 1)$. The points $\eta_{i\ell} := \hat{\Theta}_i^{-1}\xi_\ell$ are the peaks of the bubble $U(\hat{\Theta}_i \cdot -\rho\xi_\ell)$, namely $U(\hat{\Theta}_i x - \rho\xi_\ell) = U(x - \rho\hat{\Theta}_i^{-1}\xi_\ell)$.

It is worthwhile to point out that all the peaks η_{ij} 's are different among them.

Lemma 2.1. *It holds true*

$$\min_{h=2,\dots,k} |\xi_1 - \xi_h| = |\xi_1 - \xi_2| = 2 \sin \frac{\pi}{k} \quad (2.21)$$

and

$$\min_{\substack{i=2,\dots,d \\ h,\ell=1,\dots,k}} |\xi_h - \eta_{i\ell}| = |\xi_1 - \eta_{21}| = 2 \sin \frac{\pi}{dk}. \quad (2.22)$$

Proof. (2.21) is immediate. Let us check (2.22). By (2.2) and (2.7)

$$\eta_{i\ell} = \Theta_i^{-1}\xi_\ell = \Theta_{\frac{2\pi}{k}(\ell-1) - \frac{2\pi}{dk}(i-1)}\xi_1 \quad \text{and} \quad \xi_h = \Theta_{\frac{2\pi}{k}(h-1)}\xi_1$$

and they coincide if and only if

$$\frac{2\pi}{dk}(i-1) = \frac{2\pi}{k}(\ell-h) \iff i = d(\ell-h) + 1 \iff i = 1 \text{ and } \ell = h.$$

Moreover, when $i \geq 2$ the distance $|\xi_h - \eta_{i\ell}|$ is minimal when the angle

$$\frac{2\pi}{dk}(i-1) - \frac{2\pi}{k}(\ell-h) = \frac{2\pi}{k} \left(\frac{1}{d}(i-1) - (\ell-h) \right)$$

is minimal, that is if $\ell = h$ and $i = 2$ and in this case

$$|\xi_h - \eta_{ih}| = |\xi_1 - \eta_{21}| = 2 \sin \frac{\pi}{dk}.$$

□

It is also useful to remark that

$$\|h\|_{L^\infty} \leq C\|h\|_* \quad \text{and} \quad \|h\|_{L^q} \leq C\|h\|_*, \quad \forall 1 \leq q < \infty. \quad (2.23)$$

Actually, the linear operator \mathcal{L} defined in (2.17) is invertible in \mathcal{B}_* as shown in the following proposition whose proof can be obtained arguing as in [3, Section 3] and is postponed in the Appendix B.

Proposition 2.2. *For any compact set $B \subset \mathfrak{B}$ (see Remark 2.3), there exists $k_0 > 0$ such that for any $\beta \in B$, for any $k \geq k_0$, $\rho \in \mathcal{D}_k$ and $h \in \mathcal{B}_*$ which satisfy (2.6) and (2.7), the linear problem*

$$\mathcal{L}(\Phi) = h + c\partial_\rho W_\rho \text{ in } \mathbb{R}^n \text{ and } \int_{\mathbb{R}^n} \partial_\rho W_\rho \Phi = 0$$

admits a unique solution $\Phi = \Phi(\rho, k) \in \mathcal{H} \cap \mathcal{B}_*$ and $\mathbf{c} = \mathbf{c}(\rho, k) \in \mathbb{R}$ such that

$$\|\Phi\|_* \lesssim \|h\|_* \quad \text{and} \quad |\mathbf{c}| \lesssim \|h\|_*. \quad (2.24)$$

Remark 2.3. Let Λ_κ be the sequence of eigenvalues of the problem

$$-\Delta\varphi + \varphi = \Lambda_\kappa U^2\varphi \text{ in } \mathbb{R}^n.$$

It is well known that $\Lambda_1 = 1$ is simple and the associated eigenfunction is U . The second eigenvalue Λ_2 has multiplicity n and the associated eigenfunctions are $\partial_{x_i}U$, $i = 1, \dots, n$. Set $\mathfrak{B} := \mathbb{R} \setminus \{\Lambda_\kappa, \kappa \in \mathbb{N}\}$.

2.5. The weighted norm. We introduce the sector

$$\Sigma := \left\{ (r \cos \theta, r \sin \theta, x_3) : x_3 \in \mathbb{R}, r \geq 0, \theta \in \left[-\frac{\pi}{k}, \frac{\pi}{k} \right] \right\}$$

so that if u satisfies (2.7) then

$$\|u\|_* = \sup_{x \in \Sigma} \left(\sum_{j=1}^k \sum_{i=1}^d e^{-\alpha|x-\rho\eta_{ij}|} \right)^{-1} |u(x)|.$$

Moreover, it is useful to decompose Σ into d sectors (if d is odd $i \in I := \{-\frac{d-1}{2}, \dots, 0, \dots, \frac{d-1}{2}\}$) or in $d+1$ sectors (if d is even and $i \in I := \{-\frac{d}{2}, \dots, 0, \dots, \frac{d}{2}\}$)

$$\tilde{\Sigma}_i := \left\{ (r \cos \theta, r \sin \theta, x_3) \in \Sigma : x_3 \in \mathbb{R}, r \geq 0, \theta \in \left[-\frac{\pi}{dk} + \frac{2\pi}{dk}i, \frac{\pi}{dk} + \frac{2\pi}{dk}i \right] \right\}, \quad i \in I,$$

so that each sector $\tilde{\Sigma}_i$ contains only one point $\eta_{\ell\kappa}$ which will be denoted by $\tilde{\eta}_i$. In particular $\tilde{\eta}_0 = \xi_1$. Therefore, to estimate the weighted norm $\|u\|_*$ it will be enough to select the points $\tilde{\eta}_i$ which belong to the sector Σ , i.e.

$$\|u\|_* = \sup_{x \in \Sigma} \left(\sum_{j=1}^k \sum_{i=1}^d e^{-\alpha|x-\rho\eta_{ij}|} \right)^{-1} |u(x)| \leq \sup_{x \in \Sigma} \left(\sum_{i \in I} e^{-\alpha|x-\rho\tilde{\eta}_i|} \right)^{-1} |u(x)|.$$

Moreover, since

$$x \in \Sigma \Rightarrow |x - \rho\xi_h| \geq \frac{\rho}{2} |\xi_h - \xi_1| \text{ if } h \geq 2 \quad (2.25)$$

and

$$x \in \tilde{\Sigma}_i \Rightarrow |x - \rho\tilde{\eta}_\ell| \geq \frac{\rho}{2} |\tilde{\eta}_\ell - \tilde{\eta}_i| \text{ if } \ell \neq i, \quad (2.26)$$

in the sector Σ the main term of $W_\rho(x) = \sum_{j=1}^k U(x - \rho\xi_j)$ is the first bubble $U(x - \rho\xi_1)$ whose peak lies in Σ , while in each subsector $\tilde{\Sigma}_\ell$ the main term of $W_\rho(\Theta_i x) = \sum_{j=1}^k U(x - \rho\Theta_i^{-1}\xi_j)$ is the bubble $U(x - \rho\tilde{\eta}_\ell)$ whose peak $\eta_{ij} = \Theta_i^{-1}\xi_j = \tilde{\eta}_\ell$ is the unique which belongs to $\tilde{\Sigma}_\ell$. This can be made rigorous in the following lemma (whose proof can be found in [18, Lemma A.1]).

Lemma 2.4. For any $\sigma \in (0, 1)$ for any $j \geq 2$

$$U(x - \rho\xi_j) \lesssim e^{-\sigma\frac{\pi\rho}{k}} e^{-(1-\sigma)|x-\rho\xi_1|} \text{ for any } x \in \Sigma$$

and for any $\eta_{ij} \notin \tilde{\Sigma}_\ell$

$$U(x - \rho\eta_{ij}) \lesssim e^{-\sigma\frac{\pi\rho}{dk}} e^{-(1-\sigma)|x-\rho\tilde{\eta}_\ell|} \text{ for any } x \in \tilde{\Sigma}_\ell.$$

2.6. The size of the error. First, by Proposition (2.2), Lemma 2.5 and Lemma 2.6 the function $\Psi_\rho = \beta Y_\rho \text{in} \mathcal{H}$ which solves (2.12) satisfies

$$\|\Psi_\rho\|_* = \beta \|Y_\rho\|_* \lesssim \beta \|W_\rho \sum_{i=2}^d W_\rho^2(\hat{\Theta}_i \cdot)\|_* \lesssim e^{-(1-\alpha)\frac{2\pi\rho}{dk}}. \quad (2.27)$$

Lemma 2.5. *If $\alpha \in (0, 1)$ it holds true that*

$$\left\| W_\rho \sum_{i=2}^d W_\rho^2(\hat{\Theta}_i \cdot) \right\|_* \lesssim e^{-(1-\alpha)\frac{2\pi\rho}{dk}} \quad (2.28)$$

and

$$\left\| W_\rho(\cdot) \sum_{i=2}^d W_\rho(\hat{\Theta}_i \cdot) \right\|_* \lesssim e^{-(1-\alpha)\frac{2\pi\rho}{dk}}. \quad (2.29)$$

Proof. We only prove (2.28), because the proof of (2.29) is similar.

Let $x \in \Sigma$. There exists $\ell \in I$ such that $x \in \tilde{\Sigma}_\ell$ and so

$$\begin{aligned} & \left(\sum_{j \in I} e^{-\alpha|x-\rho\tilde{\eta}_j|} \right)^{-1} |W_\rho(x) \sum_{i=2}^d W_\rho^2(\hat{\Theta}_i x)| \\ &= \left(e^{-\alpha|x-\rho\tilde{\eta}_\ell|} + \sum_{\substack{j \in I \\ j \neq \ell}} e^{-\alpha|x-\rho\tilde{\eta}_j|} \right)^{-1} |W_\rho(x) \sum_{i=2}^d W_\rho^2(\hat{\Theta}_i x)| \\ &\leq e^{\alpha|x-\rho\tilde{\eta}_\ell|} |W_\rho(x) \sum_{i=2}^d W_\rho^2(\hat{\Theta}_i x)|. \end{aligned}$$

Now, by Lemma 2.4

$$\begin{aligned} W_\rho(x) \sum_{i=2}^d W_\rho^2(\hat{\Theta}_i x) &\lesssim U(x - \rho\tilde{\eta}_0) \sum_{\substack{i \in I \\ i \neq 0}} U^2(x - \rho\tilde{\eta}_i) \\ &\lesssim e^{-\alpha|x-\rho\tilde{\eta}_\ell|} \sum_{\substack{i \in I \\ i \neq 0}} e^{-|x-\rho\tilde{\eta}_0|-2|x-\rho\tilde{\eta}_i|+\alpha|x-\rho\tilde{\eta}_\ell|} \\ &\lesssim e^{-\alpha|x-\rho\tilde{\eta}_\ell|} e^{-(1-\alpha)\frac{2\pi\rho}{dk}}. \end{aligned}$$

Indeed if $\ell = 0$

$$\begin{aligned} |x - \rho\tilde{\eta}_0| + 2|x - \rho\tilde{\eta}_i| - \alpha|x - \rho\tilde{\eta}_\ell| &= (1 - \alpha)|x - \rho\tilde{\eta}_0| + 2|x - \rho\tilde{\eta}_i| \\ &\geq (1 - \alpha)\rho|\tilde{\eta}_i - \tilde{\eta}_0| - (1 - \alpha)|x - \rho\tilde{\eta}_i| + 2|x - \rho\tilde{\eta}_i| \\ &\geq (1 - \alpha)\rho \min_{\substack{i \in I \\ i \neq 0}} |\tilde{\eta}_i - \tilde{\eta}_0| \end{aligned}$$

and if $\ell \neq 0$ since $|x - \rho\tilde{\eta}_0| \geq |x - \rho\tilde{\eta}_\ell|$

$$\begin{aligned} |x - \rho\tilde{\eta}_0| + 2|x - \rho\tilde{\eta}_i| - \alpha|x - \rho\tilde{\eta}_\ell| &\geq (1 - \alpha)|x - \rho\tilde{\eta}_0| + 2|x - \rho\tilde{\eta}_i| \\ &\geq (1 - \alpha)\rho|\tilde{\eta}_i - \tilde{\eta}_0| - (1 - \alpha)|x - \rho\tilde{\eta}_i| + 2|x - \rho\tilde{\eta}_i| \\ &\geq (1 - \alpha)\rho \min_{\substack{i \in I \\ i \neq 0}} |\tilde{\eta}_i - \tilde{\eta}_0|. \end{aligned}$$

Finally, the claim follows. \square

Lemma 2.6. *It holds true that*

$$\int_{\mathbb{R}^n} W_\rho(x) \sum_{i=2}^d W_\rho^2(\hat{\Theta}_i x) \partial_\rho W_\rho(x) dx = \begin{cases} -\mathbf{c} \left(\frac{k}{\rho}\right)^2 e^{-4\rho\frac{\pi}{dk}} \ln \ln k + h.o.t. & \text{if } n = 3 \\ -\mathbf{c} \sqrt{\frac{k}{\rho}} e^{-4\rho\frac{\pi}{dk}} + h.o.t. & \text{if } n = 2 \end{cases}$$

for some positive constant \mathbf{C} .

Proof. We prove the claim when $n = 3$. The proof in the case $n = 2$ is similar. By Lemma A.1, Lemma A.2 and Lemma 2.4 and setting $\eta_{ij} := \hat{\Theta}_i^{-1}\xi_j$ and $U_{ij}(x) = U(x - \rho\eta_{ij})$

$$\begin{aligned} &\int_{\mathbb{R}^3} \left(\sum_{h=1}^k U_h \right) \sum_{i=2}^d \left(\sum_{j=1}^k U_j(\hat{\Theta}_i x) \right)^2 \partial_\rho W_\rho \\ &= k \int_{\Sigma} \left(U_1 + \sum_{h=2}^k U_h \right) \sum_{i=2}^d \left(\sum_{j=1}^k U_{ij} \right)^2 \left(\partial_\rho U_1 + \sum_{i=2}^k \partial_\rho U_i \right) \\ &= k \int_{\Sigma} U_1 \partial_\rho U_1 \sum_{i=2}^d \sum_{j=1}^k U_{ij}^2 + h.o.t. \\ &= k \sum_{\substack{\ell \in \mathcal{I} \\ \ell \neq 0}} \int_{\Sigma} U(x - \rho\xi_1) U'(x - \rho\xi_1) \frac{\langle x - \rho\xi_1, -\xi_1 \rangle}{|x - \rho\xi_1|} U^2(x - \rho\tilde{\eta}_\ell) dx + h.o.t. \\ &= k \sum_{\substack{\ell \in \mathcal{I} \\ \ell \neq 0}} \int_{\mathbb{R}^3} \underbrace{U(x + \rho(\tilde{\eta}_\ell - \xi_1)) U'(x + \rho(\tilde{\eta}_\ell - \xi_1)) \frac{\langle x + \rho(\tilde{\eta}_\ell - \xi_1), -\xi_1 \rangle}{|x + \rho(\tilde{\eta}_\ell - \xi_1)|}}_{= -\langle \frac{1}{2} \nabla_\zeta \Gamma_{22}(\zeta), \xi_1 \rangle, \zeta = \rho(\tilde{\eta}_\ell - \xi_1)} U^2(x) dx + h.o.t. \\ &= -k\mathbf{c} \sum_{\substack{\ell \in \mathcal{I} \\ \ell \neq 0}} \frac{1}{2} \left\langle \frac{\xi_1 - \tilde{\eta}_\ell}{|\xi_1 - \tilde{\eta}_\ell|}, \xi_1 \right\rangle \frac{e^{-2\rho|\xi_1 - \tilde{\eta}_\ell|} \ln(\rho|\xi_1 - \tilde{\eta}_\ell|)}{|\rho(\xi_1 - \tilde{\eta}_\ell)|^2} + h.o.t. \\ &= -\mathbf{c} \frac{d}{8\pi} k^2 \frac{e^{-4\rho\frac{\pi}{dk}}}{\rho^2} \ln \ln k + h.o.t. \end{aligned}$$

because of (2.22), the choice of $\rho \in \mathcal{D}_k$ (2.10) and the fact that

$$\min_{\substack{\ell \in \mathcal{I} \\ \ell \neq 0}} |\tilde{\eta}_\ell - \xi_1| = |\eta_{21} - \xi_1| \sim \frac{2\pi}{dk} \text{ and } \langle \xi_1 - \eta_{21}, \xi_1 \rangle = 1 - \cos \frac{2\pi}{dk} \sim \frac{1}{2} \left(\frac{2\pi}{dk} \right)^2. \quad (2.30)$$

□

Finally, we find the size of the error.

Lemma 2.7. *For any $\alpha \in (0, 1)$, it holds true that*

$$\|\mathcal{E}\|_* \lesssim \frac{1}{\rho^\nu} + e^{-(1-\alpha)\frac{4\pi\rho}{dk}}.$$

Proof. By (2.18) taking into account that the functions W_ρ , $\partial_\rho W_\rho$ and the weight in the norm are bounded we get

$$\begin{aligned} \|\mathcal{E}\|_* &\lesssim \underbrace{\|(1-V(x))W_\rho\|_*}_{:=E_1} + \underbrace{\|\Delta W_\rho - W_\rho + W_\rho^3\|_*}_{:=E_2} + \underbrace{|\gamma_\rho|}_{:=E_3} \\ &\quad + \underbrace{\|\Psi_\rho\|_*^3 + \|\Psi_\rho\|_*^2}_{\lesssim e^{-(1-\alpha)\frac{4\pi\rho}{dk}} \text{ because of (2.27)}} \\ &\lesssim \frac{1}{\rho^\nu} + e^{-(1-\alpha)\frac{4\pi\rho}{dk}}. \end{aligned}$$

- *Estimate of E_1 .* Let $x \in \Sigma$ then there is $\ell \in I$ such that $x \in \tilde{\Sigma}_\ell$. Then

$$\left(\sum_{j \in I} e^{-\alpha|x-\rho\tilde{\eta}_j|} \right)^{-1} |E_1(x)| \lesssim e^{\alpha|x-\rho\tilde{\eta}_\ell|} |V(x) - 1| e^{-|x-\rho\tilde{\eta}_0|}.$$

We note that if $|x| < \frac{\rho}{2}$ then $|x - \rho\tilde{\eta}_0| \geq \frac{\rho}{2}$. Now if $x \in \tilde{\Sigma}_0 \cap \{|x| < \frac{\rho}{2}\}$ then

$$-\alpha|x - \rho\tilde{\eta}_\ell| + |x - \rho\tilde{\eta}_0| = (1-\alpha)|x - \rho\tilde{\eta}_0| \geq (1-\alpha)\frac{\rho}{2}.$$

If $x \in \tilde{\Sigma}_\ell \cap \{|x| < \frac{\rho}{2}\}$ with $\ell \neq 0$ then $|x - \rho\tilde{\eta}_0| \geq |x - \rho\tilde{\eta}_\ell|$ and hence

$$\begin{aligned} -\alpha|x - \rho\tilde{\eta}_\ell| + |x - \rho\tilde{\eta}_0| &= (1-\alpha)|x - \rho\tilde{\eta}_0| + \alpha(|x - \rho\tilde{\eta}_0| - |x - \rho\tilde{\eta}_\ell|) \\ &\geq (1-\alpha)|x - \rho\tilde{\eta}_0| \geq (1-\alpha)\frac{\rho}{2}. \end{aligned}$$

Then, since $V \in L^\infty(\mathbb{R}^n)$ we get that in $\{|x| < \frac{\rho}{2}\}$

$$\left(\sum_{j \in I} e^{-\alpha|x-\rho\tilde{\eta}_j|} \right)^{-1} |E_1(x)| \lesssim e^{-(1-\alpha)\frac{\rho}{2}}.$$

If $|x| \geq \frac{\rho}{2}$ by (1.10) $|V(x) - 1| \lesssim \frac{1}{\rho^\nu}$ and so for $x \in \tilde{\Sigma}_0 \cap \{|x| \geq \frac{\rho}{2}\}$

$$-\alpha|x - \rho\tilde{\eta}_\ell| + |x - \rho\tilde{\eta}_0| = (1-\alpha)|x - \rho\tilde{\eta}_0| \geq 0.$$

For $x \in \tilde{\Sigma}_\ell \cap \{|x| \geq \frac{\rho}{2}\}$ with $\ell \neq 0$ since $|x - \rho\tilde{\eta}_0| \geq |x - \rho\tilde{\eta}_\ell|$ we get

$$-\alpha|x - \rho\tilde{\eta}_\ell| + |x - \rho\tilde{\eta}_0| = (1-\alpha)|x - \rho\tilde{\eta}_0| + \alpha(|x - \rho\tilde{\eta}_0| - |x - \rho\tilde{\eta}_\ell|) \geq 0.$$

Hence in $\{|x| \geq \frac{\rho}{2}\}$ we get

$$\left(\sum_{j \in I} e^{-\alpha|x-\rho\tilde{\eta}_j|} \right)^{-1} |E_1(x)| \lesssim \frac{1}{\rho^\nu}.$$

This implies

$$\|E_1\|_* \lesssim \frac{1}{\rho^\nu}.$$

- *Estimate of E_2 .* Let $x \in \Sigma$ then there is $\ell \in I$ such that $x \in \tilde{\Sigma}_\ell$. Now since

$$\Delta W_\rho - W_\rho + W_\rho^3 = \left(\sum_{h=1}^k U_h \right)^3 - \sum_{h=1}^k U_h^3$$

we get that

$$\left(\sum_{i \in I} e^{-\alpha|x-\rho\tilde{\eta}_i|} \right)^{-1} |E_2(x)| \lesssim e^{\alpha|x-\rho\tilde{\eta}_\ell|} \left| \left(\sum_{h=1}^k U_h \right)^3 - \sum_{h=1}^k U_h^3 \right|.$$

Now by Lemma 2.4

$$\begin{aligned} & \left(\left(\sum_{h=1}^k U_h \right)^3 - \sum_{h=1}^k U_h^3 \right) \\ &= \left(U_1^3 + 3U_1^2 \sum_{h \geq 2}^k U_h + 3U_1 \left(\sum_{h \geq 2}^k U_h \right)^2 + \left(\sum_{h \geq 2}^k U_h \right)^3 - U_1^3 - \sum_{h \geq 2}^k U_h^3 \right) \\ &\lesssim U_1^2 \sum_{h \geq 2}^k U_h \\ &\lesssim e^{-\alpha|x-\rho\tilde{\eta}_\ell|} \sum_{h \geq 2}^k e^{\alpha|x-\rho\tilde{\eta}_\ell| - 2|x-\rho\xi_1| - |x-\rho\xi_h|} \\ &\lesssim e^{-\alpha|x-\rho\tilde{\eta}_\ell|} e^{-\frac{2\pi\rho}{k}} \end{aligned}$$

because (remind that $\xi_1 = \tilde{\eta}_0$) if $x \in \tilde{\Sigma}_0$

$$\begin{aligned} & -\alpha|x-\rho\tilde{\eta}_\ell| + 2|x-\rho\xi_1| + |x-\rho\xi_h| \\ &= (2-\alpha)|x-\rho\xi_1| + |x-\rho\xi_h| \\ &\geq \rho|\xi_1 - \xi_h| + (1-\alpha)|x-\rho\xi_1| \geq \rho|\xi_1 - \xi_h| \end{aligned}$$

while if $x \in \tilde{\Sigma}_\ell$ with $\ell \neq 0$ then $|x-\rho\tilde{\eta}_0| \geq |x-\rho\tilde{\eta}_\ell|$ and

$$\begin{aligned} & -\alpha|x-\rho\tilde{\eta}_\ell| + 2|x-\rho\xi_1| + |x-\rho\xi_h| \\ &\geq \rho|\xi_1 - \xi_h| + |x-\rho\xi_1| - \alpha|x-\rho\tilde{\eta}_\ell| \\ &= \rho|\xi_1 - \xi_h| + (1-\alpha)|x-\rho\xi_1| + \alpha(|x-\rho\tilde{\eta}_0| - |x-\rho\tilde{\eta}_\ell|) \\ &\geq \rho|\xi_1 - \xi_h|. \end{aligned}$$

Then by (2.21)

$$\sum_{h=2}^k e^{\alpha|x-\rho\bar{\eta}_\ell|-2|x-\rho\xi_1|-|x-\rho\xi_h|} \lesssim \sum_{h=2}^k e^{-\rho|\xi_1-\xi_h|} \lesssim e^{-\rho\frac{2\pi}{k}}.$$

Therefore, the estimate

$$\|E_3\|_* \lesssim e^{-\frac{2\pi\rho}{k}}$$

follows.

- *Estimate of E_3 .* By Lemma 2.6 and the fact that $\int_{\mathbb{R}^n} (\partial_\rho W_\rho)^2 \sim ck$ for some positive constant c , we get

$$|\gamma_\rho| \lesssim \frac{k}{\rho^2} e^{-4\rho\frac{\pi}{dk}} \ln \ln k \lesssim e^{-(1-\alpha)\frac{4\pi\rho}{dk}}.$$

□

2.7. Solving the intermediate non-linear problem (2.20). The next step relies on a classical contraction mapping argument.

Proposition 2.8. *For any compact set $B \subset \mathfrak{B}$ (see Remark 2.3), there exists $k_0 > 0$ such that for any $\beta \in B$, for any $k \geq k_0$ and for any $\rho \in \mathcal{D}_k$, there is a unique $(\Phi, \mathfrak{c}) \in \mathcal{H} \times \mathbb{R}$ which solves (2.20). Moreover*

$$\|\Phi\|_* \lesssim \left(\frac{1}{\rho^\nu} + e^{-(1-\alpha)\frac{4\pi\rho}{dk}} \right). \quad (2.31)$$

Proof. For a given $R > 0$, let us consider the ball

$$\mathcal{B}_k := \left\{ \Phi \in L^\infty(\mathbb{R}^n) : \|\Phi\|_* \leq R \left(\frac{1}{\rho^\nu} + e^{-(1-\alpha)\frac{4\pi\rho}{dk}} \right) \right\}$$

which is a non-empty closed subset of \mathcal{B}_* . Let us also introduce the map $\mathcal{T} : \mathcal{B}_k \cap \mathcal{H} \rightarrow \mathcal{B}_k \cap \mathcal{H}$ as

$$\mathcal{T}(\Phi) := -\mathcal{L}^{-1}(\mathcal{N}(\Phi) + \mathcal{E}).$$

Now solving (2.20) is equivalent to find a fixed point to \mathcal{T} .

It is quite standard to prove that \mathcal{T} is a contraction mapping for some R provided k is large enough. Indeed, by Proposition 2.2

$$\|\mathcal{T}(\Phi)\|_* \lesssim (\|\mathcal{N}(\Phi)\|_* + \|\mathcal{E}\|_*) \quad \text{and} \quad \|\mathcal{T}(\Phi_1) - \mathcal{T}(\Phi_2)\|_* \lesssim \|\mathcal{N}(\Phi_1) - \mathcal{N}(\Phi_2)\|_*.$$

Moreover, by (2.19) and since $W_\rho \lesssim 1$

$$\|\mathcal{N}(\Phi)\|_* \lesssim \|\Psi_\rho\|_* \|\Phi\|_* + \|\Phi\|_*^2 + \|\Phi\|_*^3$$

and

$$\|\mathcal{N}(\Phi_1) - \mathcal{N}(\Phi_2)\|_* \lesssim (\|\Phi_1\|_* + \|\Phi_2\|_* + \|\Psi_\rho\|_*) \|\Phi_1 - \Phi_2\|_*.$$

Finally, by Lemma 2.7 and (2.27) the claim follows.

□

2.8. **The reduced problem.** Finally, the problem reduces to find ρ such that

$$\mathfrak{C}_k(\rho) := \int_{\mathbb{R}^3} (\mathcal{E} + \mathcal{N}(\Phi) - \mathcal{L}(\Phi)) \partial_\rho W_\rho = 0, \quad (2.32)$$

where $\Phi = \Phi(\rho, k)$ and $\mathfrak{c} = \mathfrak{c}(\rho, k)$ are the solutions of (2.20) found in Proposition 2.8.

Lemma 2.9. *For any compact set $B \subset \mathfrak{B}$ (see Remark 2.3), there exists $k_0 > 0$ such that for any $\beta \in B$, for any $k \geq k_0$ and $\rho \in \mathcal{D}_k$*

$$\mathfrak{C}_k(\rho) = \begin{cases} \mathfrak{v}_\infty \mathbf{A} \frac{k}{\rho^{\nu+1}} (1 + o(1)) - \mathbf{B} \frac{k}{\rho} e^{-2\rho \frac{\pi}{k}} (1 + o(1)) - \beta \mathbf{C} \left(\frac{k}{\rho} \right)^2 e^{-4\rho \frac{\pi}{dk}} \ln \ln k (1 + o(1)) \\ \quad + \beta^2 \Upsilon(k, \rho) \text{ if } n = 3 \\ \mathfrak{v}_\infty \mathbf{A} \frac{k}{\rho^{\nu+1}} (1 + o(1)) - \mathbf{B} \sqrt{\frac{k}{\rho}} e^{-2\rho \frac{\pi}{k}} (1 + o(1)) - \beta \mathbf{C} \sqrt{\frac{k}{\rho}} e^{-4\rho \frac{\pi}{dk}} (1 + o(1)) \\ \quad + \beta^2 \Upsilon(k, \rho) \text{ if } n = 2 \end{cases}$$

where \mathbf{A} , \mathbf{B} and \mathbf{C} are positive constants only depending on n and

$$|\Upsilon(k, \rho)| \lesssim k e^{-(1-\alpha) \frac{4\pi\rho}{dk}}. \quad (2.33)$$

Proof. We know that

$$\partial_\rho W_\rho = \sum_{h=1}^k \partial_\rho U_h = \sum_{h=1}^k \langle \nabla U_h, (-\xi_h) \rangle.$$

First, let us estimate the leading term

$$\begin{aligned} \int_{\mathbb{R}^3} \mathcal{E} \partial_\rho W_\rho &:= \underbrace{\int_{\mathbb{R}^3} (1 - V(x)) W_\rho \partial_\rho W_\rho}_{=I_1} + \underbrace{\int_{\mathbb{R}^3} (\Delta W_\rho - W_\rho + W_\rho^3) \partial_\rho W_\rho}_{=I_2} \\ &\quad + \underbrace{\gamma_\rho \int_{\mathbb{R}^3} (\partial_\rho W_\rho)^2}_{=I_3} \\ &\quad + \underbrace{\int_{\mathbb{R}^3} 3W_\rho \Psi_\rho^2 \partial_\rho W_\rho + \int_{\mathbb{R}^3} \beta W_\rho \sum_{i=2}^d \Psi_\rho^2(\hat{\Theta}_i x) \partial_\rho W_\rho}_{=: \Upsilon(k, \rho) = I_4} \\ &\quad + \underbrace{\int_{\mathbb{R}^3} 2\beta \Psi_\rho \sum_{i=2}^d W_\rho(\hat{\Theta}_i x) \Psi_\rho(\hat{\Theta}_i x) \partial_\rho W_\rho}_{=I_5} \\ &\quad + \underbrace{\int_{\mathbb{R}^3} \left(\Psi_\rho^3 + \beta \Psi_\rho \sum_{i=2}^d \Psi_\rho^2(\hat{\Theta}_i x) \right) \partial_\rho W_\rho}_{=I_6} \end{aligned}$$

- *Estimate of I_1 .*

By Lemma 2.4

$$\begin{aligned}
 I_1 &= k \int_{\Sigma} (1 - V(|x|)) \left(U_1 + \sum_{h=2}^k U_h \right) \left(\partial_{\rho} U_1 + \sum_{i=2}^k \partial_{\rho} U_i \right) \\
 &= k \int_{\Sigma} (1 - V(|x|)) U_1 \partial_{\rho} U_1 + h.o.t. \\
 &= k \int_{\Sigma} (1 - V(|x|)) U(x - \rho \xi_1) U'(x - \rho \xi_1) \frac{\langle x - \rho \xi_1, -\xi_1 \rangle}{|x - \rho \xi_1|} dx + h.o.t. \\
 &= k \int_{\mathbb{R}^3} (1 - V(|y + \rho \xi_1|)) U(y) U'(y) \left(-\frac{y_1}{|y|} \right) dy + h.o.t. \\
 &= -k \int_{\mathbb{R}^3} (1 - V(|y + \rho \xi_1|)) \frac{1}{2} \partial_{y_1} U^2(y) dy + h.o.t. \\
 &= -\frac{k}{2} \int_{\mathbb{R}^3} \partial_{y_1} V(|y + \rho \xi_1|) U^2(y) dy + h.o.t. \\
 &= \frac{k}{2} \frac{\nu}{\rho^{\nu+1}} \mathbf{v}_{\infty} \int_{\mathbb{R}^3} U^2(y) dy + h.o.t.
 \end{aligned}$$

because by (1.10)

$$\partial_{x_1} V(|x|) = -\mathbf{v}_{\infty} \nu \frac{1}{|x|^{\nu+1}} \frac{x_1}{|x|} + \mathcal{O}\left(\frac{1}{|x|^{\nu+1+\epsilon}}\right).$$

- *Estimate of I_2 .*

By Lemma A.1, Lemma A.2 and Lemma 2.4

$$\begin{aligned}
 I_2 &= k \int_{\Sigma} \left(\left(U_1 + \sum_{h=2}^k U_h \right)^3 - U_1^3 - \sum_{h=2}^k U_h^3 \right) \left(\partial_{\rho} U_1 + \sum_{i=2}^k \partial_{\rho} U_i \right) \\
 &= 3k \sum_{h=2}^k \int_{\Sigma} U_1^2 U_h \partial_{\rho} U_1 + h.o.t. \\
 &= 3k \sum_{h=2}^k \int_{\Sigma} U^2(x - \rho \xi_1) U'(x - \rho \xi_1) \frac{\langle x - \rho \xi_1, -\xi_1 \rangle}{|x - \rho \xi_1|} U(x - \rho \xi_h) dx + h.o.t. \\
 &= k \sum_{h=2}^k \int_{\mathbb{R}^3} \underbrace{3U^2(x + \rho(\xi_h - \xi_1)) U'(x + \rho(\xi_h - \xi_1)) \frac{\langle x + \rho(\xi_h - \xi_1), -\xi_1 \rangle}{|x + \rho(\xi_h - \xi_1)|}}_{= -\langle \nabla_{\zeta} \Gamma_{31}(\zeta), \xi_1 \rangle, \zeta = \rho(\xi_h - \xi_1)} U(x) dx + h.o.t. \\
 &= -ck \sum_{h=2}^k \left\langle \frac{\xi_1 - \xi_h}{|\xi_1 - \xi_h|}, \xi_1 \right\rangle \frac{e^{-\rho|\xi_1 - \xi_h|}}{(\rho|\xi_1 - \xi_h|)^2} + h.o.t. \\
 &= -\frac{1}{2} ck \frac{e^{-2\rho\frac{\pi}{k}}}{\rho} + h.o.t.
 \end{aligned}$$

because of (2.21) and the fact that

$$|\xi_2 - \xi_1| \sim \frac{2\pi}{k}d \text{ and } \langle \xi_1 - \xi_2, \xi_1 \rangle = 1 - \cos \frac{2\pi}{k} \sim \frac{1}{2} \left(\frac{2\pi}{k} \right)^2. \quad (2.34)$$

- *Estimate of I_3 .* The term I_3 is estimated in Lemma 2.6.
- *Estimate of I_4 .* We know that $\Psi_\rho = \beta Y_\rho$ where Y_ρ solves (2.12)

$$I_4 = \beta^2 \left(\int_{\mathbb{R}^3} 3W_\rho Y_\rho^2 \partial_\rho W_\rho + \int_{\mathbb{R}^3} \beta W_\rho \sum_{i=2}^d Y_\rho^2(\hat{\Theta}_i x) \partial_\rho W_\rho \right) \lesssim \beta^2 k e^{-(1-\alpha)\frac{4\pi\rho}{dk}}$$

because $\|Y_\rho\|_* \lesssim e^{-(1-\alpha)\frac{4\pi\rho}{dk}}$ (see (2.27)).

- *Estimate of I_5 .* We observe that

$$|\Psi_\rho(\hat{\Theta}_i x)| \leq \left(\sum_{\ell=1}^d \sum_{j=1}^k e^{-\alpha|\hat{\Theta}_i x - \rho\eta_{\ell j}|} \right) \|\Psi_\rho\|_* = \left(\sum_{\ell=1}^d \sum_{j=1}^k e^{-\alpha|x - \rho\eta_{\ell j}|} \right) \|\Psi_\rho\|_*.$$

Therefore, since $|\partial_\rho W_\rho| \lesssim W_\rho$ by (2.29)

$$\begin{aligned} I_5 &\lesssim \int_{\mathbb{R}^3} 2|\Psi_\rho| W_\rho \sum_{i=2}^d W_\rho(\hat{\Theta}_i x) |\Psi_\rho(\hat{\Theta}_i x)| \\ &\lesssim k \|\Psi_\rho\|_*^2 \left\| \underbrace{W_\rho \sum_{i=2}^d W_\rho(\hat{\Theta}_i x) \left(\sum_{\ell=1}^d \sum_{j=1}^k e^{-\alpha|x - \rho\eta_{\ell j}|} \right)}_{= \|W_\rho \sum_{i=2}^d W_\rho(\hat{\Theta}_i x)\|_{L^\infty(\mathbb{R}^n)}} \right\|_* \\ &\lesssim k e^{-(1-\alpha)\frac{4\pi\rho}{dk}} e^{-(1-\alpha)\frac{2\pi\rho}{kd}} \lesssim k e^{-(1-\alpha)\frac{6\pi\rho}{dk}}. \end{aligned}$$

- *Estimate of I_6 .* Using again that $|\partial_\rho W_\rho| \lesssim 1$, by (2.27)

$$I_6 \lesssim k \|\Psi_\rho\|_*^3 \lesssim k e^{-(1-\alpha)\frac{6\pi\rho}{dk}}.$$

Next, by (2.19)

$$\int_{\mathbb{R}^3} \mathcal{N}(\Phi) \partial_\rho W_\rho \lesssim k (\|\Phi\|_*^3 + \|\Phi\|_*^2 + \|\Psi_\rho\|_* \|\Phi\|_*) \lesssim k e^{-(1-\alpha)\frac{6\pi\rho}{dk}}.$$

Finally, it only remains to estimate (see (2.17))

$$\begin{aligned} \int_{\mathbb{R}^3} \mathcal{L}(\Phi) \partial_\rho W_\rho &= \underbrace{\int_{\mathbb{R}^3} (V(x) - 1) \Phi \partial_\rho W_\rho}_{:=L_1} + \underbrace{\int_{\mathbb{R}^3} (-\Delta \Phi + \Phi - 3W_\rho^2 \Phi) \partial_\rho W_\rho}_{:=L_2} \\ &\quad - \underbrace{\int_{\mathbb{R}^3} \beta \Phi \sum_{i=2}^d W_\rho^2(\hat{\Theta}_i x) \partial_\rho W_\rho - 2 \int_{\mathbb{R}^3} \beta W_\rho \sum_{i=2}^d W_\rho(\hat{\Theta}_i x) \Phi(\hat{\Theta}_i x) \partial_\rho W_\rho}_{:=L_3}. \end{aligned}$$

- *Estimate of L_1 .* Since $|\partial_\rho W_\rho| \lesssim W_\rho$ and since $\nu > 1$

$$\int_{\mathbb{R}^3} (V(x) - 1) \Phi \partial_\rho W_\rho \lesssim k \|(V(x) - 1) W_\rho\|_* \|\Phi\|_* \lesssim k \frac{1}{\rho^\nu} \|\Phi\|_*.$$

- *Estimate of L_2 .* Remind that

$$\partial_\rho W_\rho = \sum_{h=1}^k \partial_\rho U_h = \sum_{h=1}^k \langle \nabla U_h, (-\xi_h) \rangle$$

and also

$$-\Delta \partial_i U_h + \partial_i U_h = \partial_i U_h^3 \quad \Rightarrow \quad -\Delta \partial_\rho W_\rho + \partial_\rho W_\rho = \sum_{h=1}^k 3U_h^2 \langle \nabla U_h, (-\xi_h) \rangle$$

and so

$$\begin{aligned} L_2 &= \int_{\mathbb{R}^3} \left(\sum_{h=1}^k 3U_h^2 \langle \nabla U_h, (-\xi_h) \rangle - 3W_\rho^2 \partial_\rho W_\rho \right) \Phi \\ &\lesssim k \|\Phi\|_* \left\| \sum_{h=1}^k 3U_h^2 \langle \nabla U_h, (-\xi_h) \rangle - 3W_\rho^2 \partial_\rho W_\rho \right\|_* \\ &\lesssim k \|\Phi\|_* e^{-\frac{2\pi\rho}{k}} \lesssim k e^{-(1-\alpha)\frac{6\pi\rho}{dk}}. \end{aligned}$$

Indeed, if $x \in \Sigma$ then there is $\ell \in I$ such that $x \in \tilde{\Sigma}_\ell$ (taking into account that $|\langle \nabla U_h, (-\xi_h) \rangle| \lesssim U_h$) arguing exactly as in the estimate of the term E_2 in Lemma 2.7 we get

$$\begin{aligned} &e^{\alpha|x-\rho\tilde{\eta}_\ell|} \left(\sum_{h=1}^k 3U_h^2 \langle \nabla U_h, (-\xi_h) \rangle - 3W_\rho^2 \partial_\rho W_\rho \right) \\ &= e^{\alpha|x-\rho\tilde{\eta}_\ell|} \left[3U_1^2 \langle \nabla U_1, (-\xi_1) \rangle + \sum_{h=2}^k 3U_h^2(x) \langle \nabla U_h(x), (-\xi_h) \rangle \right. \\ &\quad \left. - 3 \left(U_1 + \sum_{h=2}^k U_h \right)^2 \left(\langle \nabla U_1, (-\xi_1) \rangle + \sum_{h=2}^k \langle \nabla U_h, (-\xi_h) \rangle \right) \right] \\ &\lesssim e^{\alpha|x-\rho\tilde{\eta}_\ell|} \left(U_1^2 \sum_{h=2}^k U_h + U_1 \left(\sum_{h=2}^k U_h \right)^2 \right) \lesssim e^{-\frac{2\pi\rho}{k}}. \end{aligned}$$

- *Estimate of L_3 .* Since $\partial_\rho W_\rho \lesssim W_\rho$ by Lemma 2.5

$$L_3 \lesssim k \|\Phi\|_* \left(\|W_\rho \sum_{i=2}^d W_\rho^2(\hat{\Theta}_i x)\|_* + \|W_\rho \sum_{i=2}^d W_\rho(\hat{\Theta}_i x)\|_* \right) \lesssim k e^{-(1-\alpha)\frac{6\pi\rho}{dk}}.$$

We combine all the previous estimates with the size of Φ (2.31) and the size of Ψ_ρ in (2.27) and we get the claim. \square

2.9. The proof of Theorem 1.1: completed. We have to find $\rho = \rho(k)$ such that $\mathfrak{C}_k(\rho) = 0$ (see (2.32)). We only consider the case $n = 3$. When $n = 2$ we argue in a similar way. Now, by Lemma 2.9

$$\begin{aligned} \mathfrak{C}_k(\rho) &= \mathbf{v}_\infty \mathbf{A} \frac{k}{\rho^{\nu+1}} (1 + o(1)) - \mathbf{B} \frac{k}{\rho} e^{-2\rho \frac{\pi}{k}} (1 + o(1)) - \beta \mathbf{C} \left(\frac{k}{\rho} \right)^2 e^{-4\rho \frac{\pi}{dk}} \ln \ln k (1 + o(1)) \\ &\quad + \beta^2 \Upsilon(k, \rho). \end{aligned}$$

If $d = 3$ and $\nu > 2$ or $d \geq 4$ and $\nu > 1$ we choose

$$|\beta| \lesssim \beta_k := \frac{1}{k^b} \text{ with } 1 < b < \nu \frac{d-2}{2} \quad (2.35)$$

such that

$$\beta^2 \Upsilon(k, \rho), \frac{k}{\rho} e^{-2\rho \frac{\pi}{k}} = o \left(\beta \left(\frac{k}{\rho} \right)^2 e^{-4\rho \frac{\pi}{dk}} \ln \ln k \right) \quad (2.36)$$

and so $\mathfrak{C}_k(\rho)$ reduces to

$$\mathfrak{C}_k(\rho) = \mathbf{v}_\infty \mathbf{A} \frac{k}{\rho^{\nu+1}} (1 + o(1)) - \beta \mathbf{C} \left(\frac{k}{\rho} \right)^2 e^{-4\rho \frac{\pi}{dk}} \ln \ln k (1 + o(1)).$$

Therefore, if $\rho = rk \ln k$ we compute

$$\mathfrak{C}_k(rk \ln k) = \mathbf{v}_\infty \mathbf{A} \frac{1}{r^{\nu+1} (\ln k)^{\nu+1}} \frac{1}{k^\nu} (1 + o(1)) - \beta \mathbf{C} \frac{\ln \ln k}{r^2 (\ln k)^2} \frac{1}{k^{\frac{4\pi}{d} r}} (1 + o(1))$$

and

$$\mathfrak{C}_k(rk \ln k) < 0 \text{ if } r < \frac{d\nu}{4\pi} \text{ and } \mathfrak{C}_k(rk \ln k) > 0 \text{ if } r > \frac{d\nu}{4\pi}.$$

Therefore, if \mathbf{v}_∞ and β have the same sign, for any $\epsilon > 0$ there exists $r(k) \in \left(\frac{d\nu}{4\pi} - \epsilon, \frac{d\nu}{4\pi} + \epsilon \right)$, such that $\mathfrak{C}_k(r(k)k \ln k) = 0$ and the claim is proved.

It only remains to prove that (2.36) holds for any $\rho = rk \ln k$ and $r \in \left(\frac{d\nu}{4\pi} - \epsilon, \frac{d\nu}{4\pi} + \epsilon \right)$ for some $\epsilon > 0$ small enough. We have

$$\frac{\beta^2 \Upsilon(k, \rho)}{\beta \left(\frac{k}{\rho} \right)^2 e^{-4\rho \frac{\pi}{dk}} \ln \ln k} \lesssim \beta k e^{\alpha \frac{4\pi\rho}{dk}} \lesssim k^{1-b+\alpha r \frac{4\pi}{d}} = o(1) \text{ if } b > 1$$

and

$$\frac{\frac{k}{\rho} e^{-2\rho \frac{\pi}{k}}}{\beta \left(\frac{k}{\rho} \right)^2 e^{-4\rho \frac{\pi}{dk}} \ln \ln k} \lesssim \frac{1}{\beta} e^{\frac{4\pi\rho}{dk} - \frac{2\pi\rho}{k}} \lesssim k^{b+r \left(\frac{4\pi}{d} - 2\pi \right)} = o(1) \text{ if } b + \nu \left(1 - \frac{d}{2} \right) < 0.$$

Finally, the positivity of the solutions can be proved arguing as in [11] since β is small. That completes the proof.

3. PROOF OF THEOREM 1.2

We consider the more general system

$$-\Delta u_i + V(x)u_i = |u_i|^{p-1}u_i + \beta|u_i|^{q-1}u_i \sum_{\substack{j=1 \\ j \neq i}}^d |u_j|^{r-1}u_j \text{ in } \mathbb{R}^n, \quad i = 1, \dots, d \quad (3.1)$$

with $p \in \left(1, \frac{n+2}{n-2}\right)$, if $n \geq 3$ or $p > 1$ if $n = 1, 2$ and $q, r > 1$. Arguing as above we look for a solution to (3.1) as $\mathbf{u} = (u_1, \dots, u_d) \in (H^1(\mathbb{R}^n))^d$ whose components satisfy (2.3) and u solves the non-local equation

$$-\Delta u + V(x)u = |u|^{p-1}u + \beta|u|^{q-1}u \sum_{i=2}^d (|u|^{r-1}u)(\hat{\Theta}_i x), \quad \text{in } \mathbb{R}^n. \quad (3.2)$$

We build a solution to (3.2) as $u = W_\rho + \Phi$, where W_ρ is defined in (2.14). In this case U (see also (2.11)) is the unique positive radial solution to

$$-\Delta U + U = U^p \text{ in } \mathbb{R}^n.$$

The higher order term $\Phi \in \mathcal{H}$ satisfies the orthogonality condition (2.15). We follow the same strategy developed in the previous sections. Let us point out that we are not refining the ansatz as in the previous case. This will create some constraints on the choice of the exponents p, q, r and the number of equations d . We will focus on these constraints.

First, we write the non-local equation (3.2) in terms of Φ , i.e.

$$\mathcal{L}(\Phi) = \mathcal{N}(\Phi) + \mathcal{E} \text{ in } \mathbb{R}^n,$$

where the linear operator \mathcal{L} is defined by

$$\mathcal{L}(\Phi) = -\Delta \Phi + \Phi - pW_\rho^{p-1}\Phi,$$

the error term \mathcal{E} is defined by

$$\mathcal{E} := (1 - V(x))W_\rho + \Delta W_\rho - W_\rho + W_\rho^p + \beta W_\rho^q \sum_{i=2}^d W_\rho^r(\hat{\Theta}_i x) \quad (3.3)$$

and the higher order term $\mathcal{N}(\Phi)$ is defined by

$$\begin{aligned} \mathcal{N}(\Phi) := & |W_\rho + \Phi|^p - W_\rho^p - pW_\rho^{p-1}\Phi + (V(x) - 1)\Phi \\ & + \beta \left(|W_\rho + \Phi|^q \sum_{i=2}^d |W_\rho + \Phi|^r(\hat{\Theta}_i x) - W_\rho^q \sum_{i=2}^d W_\rho^r(\hat{\Theta}_i x) \right). \end{aligned}$$

First of all, we invert the linear operator \mathcal{L} in the Banach space introduced in [9, Section 3]

$$\mathcal{B}_* := \{h \in L^\infty(\mathbb{R}^n) : \|h\|_* < +\infty\}, \quad \|h\|_* := \sup_{x \in \mathbb{R}^n} \left(\sum_{j=1}^k e^{-\alpha|x-\rho\xi_j|} \right)^{-1} |h(x)|.$$

Here the ξ_j 's are defined in (2.9) and $\alpha \in (0, 1)$. We point out that in this case it is not necessary to refine the choice of the weight in the norm adding the points η_{ij} 's.

Proposition 3.1. *There exist $k_0 > 0$ such that for any $k \geq k_0$, $\rho \in \mathcal{D}_k$ and $h \in \mathcal{B}_*$ which satisfy (2.6) and (2.7), the linear problem*

$$\mathcal{L}(\Phi) = h + \mathfrak{c} \partial_\rho W_\rho \text{ in } \mathbb{R}^n \text{ and } \int_{\mathbb{R}^n} \partial_\rho W_\rho \Phi = 0$$

admits a unique solution $\Phi = \Phi_{\rho,k} \in \mathcal{H} \cap \mathcal{B}_$ and $\mathfrak{c} = \mathfrak{c}(\rho, k) \in \mathbb{R}$ such that*

$$\|\Phi\|_* \lesssim \|h\|_* \quad \text{and} \quad |\mathfrak{c}| \lesssim \|h\|_*.$$

Next, we estimate the error (3.3).

Lemma 3.2. *It holds true that*

$$\|\mathcal{E}\|_* \lesssim \frac{1}{\rho^\nu} + e^{-\min\{1, (p-1-\alpha)\} \frac{2\pi\rho}{k}} + e^{-(\min\{q,r\}-\alpha) \frac{2\pi\rho}{dk}}.$$

Proof. We know that

$$\|\mathcal{E}\|_* \lesssim \underbrace{\|(1-V(x))W_\rho\|_*}_{:=E_1} + \underbrace{\|\Delta W_\rho - W_\rho + W_\rho^p\|_*}_{:=E_2} + \underbrace{\|W_\rho^q \sum_{i=2}^d W_\rho^r(\hat{\Theta}_i x)\|_*}_{:=E_3}.$$

E_1 and E_2 can be estimated as in Lemma 2.7. Let us estimate the term E_3 . If $x \in \Sigma$ we have

$$\begin{aligned} e^{\alpha|x-\rho\xi_1|} W_\rho^q(x) \sum_{\ell=2}^d W_\rho^r(\hat{\Theta}_\ell x) &\lesssim e^{\alpha|x-\rho\xi_1|} U^q(x - \rho\xi_1) \sum_{\substack{\ell \in I \\ \ell \neq 0}} U^r(x - \rho\tilde{\eta}_\ell) \\ &\lesssim \sum_{\substack{\ell \in I \\ \ell \neq 0}} e^{-(q-\alpha)|x-\rho\xi_1| - r|x-\rho\tilde{\eta}_\ell|} \\ &\lesssim e^{-(q-\alpha) \frac{2\pi\rho}{dk}}, \end{aligned}$$

because

$$\begin{aligned} (q-\alpha)|x-\rho\xi_1| + r|x-\rho\tilde{\eta}_\ell| &\geq (q-\alpha)\rho|\tilde{\eta}_0 - \tilde{\eta}_\ell| + (r-q+\alpha)|x-\rho\tilde{\eta}_\ell| \\ &\geq (q-\alpha)\rho \sin \frac{2\pi}{dk} \text{ if } q \leq r \end{aligned}$$

and

$$\begin{aligned} (q-\alpha)|x-\rho\xi_1| + r|x-\rho\tilde{\eta}_\ell| &\geq r\rho|\tilde{\eta}_0 - \tilde{\eta}_\ell| + (q-\alpha-r)|x-\rho\tilde{\eta}_\ell| \\ &\geq r\rho \sin \frac{2\pi}{dk} \text{ if } q > r. \end{aligned}$$

Then the estimate

$$\|E_3\|_* \lesssim \begin{cases} e^{-(q-\alpha) \frac{2\pi\rho}{dk}} & \text{if } q \leq r \\ e^{-r \frac{2\pi\rho}{dk}} & \text{if } q > r \end{cases}$$

follows. □

Now, we use a standard contraction mapping argument to solve the intermediate non-linear problem (2.20).

Proposition 3.3. *There exists k_0 such that for any $k \geq k_0$ and for any $\rho \in \mathcal{D}_k$, there is a unique $(\Phi, \mathbf{c}) \in \mathcal{H} \times \mathbb{R}$ which solves (2.20). Moreover*

$$\|\Phi\|_* \lesssim \frac{1}{\rho^\nu} + e^{-\frac{2\pi\rho}{k}\sigma}, \quad \sigma := \min \left\{ 1, (p-1-\alpha), \frac{q-\alpha}{d}, \frac{r-\alpha}{d} \right\}. \quad (3.4)$$

Finally, we have to find ρ such that

$$\mathfrak{C}_k(\rho) := \int_{\mathbb{R}^n} (\mathcal{E} + \mathcal{N}(\Phi) - \mathcal{L}(\Phi)) = 0.$$

and in the next lemma we estimate $\mathfrak{C}_k(\rho)$.

Lemma 3.4. *It holds true that*

(1) if $q+1 \neq r$

$$\begin{aligned} \mathfrak{C}_k(\rho) &= \mathbf{v}_\infty \mathbf{A}_1 \frac{k}{\rho^{\nu+1}} (1 + o(1)) - \mathbf{A}_2 \left(\frac{k}{\rho} \right)^{\frac{n-1}{2}} e^{-\frac{2\pi\rho}{k}} (1 + o(1)) \\ &\quad - \beta \mathbf{A}_3 \left(\frac{k}{\rho} \right)^{\min\{q+1, r\} \frac{n-1}{2}} e^{-\min\{q+1, r\} \frac{2\pi\rho}{dk}} (1 + o(1)) + \Xi_k(\rho), \end{aligned}$$

(2) if $q+1 = r > \frac{n+1}{n-1}$

$$\begin{aligned} \mathfrak{C}_k(\rho) &= \mathbf{v}_\infty \mathbf{A}_1 \frac{k}{\rho^{\nu+1}} (1 + o(1)) - \mathbf{A}_2 \left(\frac{k}{\rho} \right)^{\frac{n-1}{2}} e^{-\frac{2\pi\rho}{k}} (1 + o(1)) \\ &\quad - \beta \mathbf{A}_3 \left(\frac{k}{\rho} \right)^{r \frac{n-1}{2}} e^{-r \frac{2\pi\rho}{dk}} (1 + o(1)) + \Xi_k(\rho), \end{aligned}$$

(3) if $q+1 = r = \frac{n+1}{n-1}$

$$\begin{aligned} \mathfrak{C}_k(\rho) &= \mathbf{v}_\infty \mathbf{A}_1 \frac{k}{\rho^{\nu+1}} (1 + o(1)) - \mathbf{A}_2 \left(\frac{k}{\rho} \right)^{\frac{n-1}{2}} e^{-\frac{2\pi\rho}{k}} (1 + o(1)) \\ &\quad - \beta \mathbf{A}_3 \left(\frac{k}{\rho} \right)^{r \frac{n-1}{2}} e^{-r \frac{2\pi\rho}{dk}} \ln \ln k (1 + o(1)) + \Xi_k(\rho), \end{aligned}$$

(4) if $q+1 = r < \frac{n+1}{n-1}$

$$\begin{aligned} \mathfrak{C}_k(\rho) &= \mathbf{v}_\infty \mathbf{A}_1 \frac{k}{\rho^{\nu+1}} (1 + o(1)) - \mathbf{A}_2 \left(\frac{k}{\rho} \right)^{\frac{n-1}{2}} e^{-\frac{2\pi\rho}{k}} (1 + o(1)) \\ &\quad - \beta \mathbf{A}_3 \left(\frac{k}{\rho} \right)^{r(n-1) + \frac{n-1}{2}} e^{-r \frac{2\pi\rho}{dk}} (1 + o(1)) + \Xi_k(\rho), \end{aligned}$$

where the \mathbf{A}_i 's are positive constants

$$\Xi_k(\rho) = k \left(\|\Phi\|_*^2 + \left(\frac{1}{\rho^\nu} + e^{-\frac{2\pi\rho}{k}\tau} \right) \|\Phi\|_* \right), \quad \tau := \min \left\{ 1, p-1-\alpha, \frac{q-1-\alpha}{d}, \frac{r-1-\alpha}{d} \right\}$$

and Φ satisfies (3.4).

Proof. We argue as in Lemma 2.9. The leading term is $\int_{\mathbb{R}^3} \mathcal{E} \partial_\rho W_\rho$:

$$\begin{aligned} \int_{\mathbb{R}^3} \mathcal{E} \partial_\rho W_\rho &:= \underbrace{\int_{\mathbb{R}^3} (1 - V(x)) W_\rho \partial_\rho W_\rho}_{=I_1} + \underbrace{\int_{\mathbb{R}^3} (\Delta W_\rho - W_\rho + W_\rho^p) \partial_\rho W_\rho}_{=I_2} \\ &\quad + \underbrace{\int_{\mathbb{R}^n} W_\rho^q \sum_{i=2}^d W_\rho^r (\hat{\Theta}_i x) \partial_\rho W_\rho}_{=I_3} \end{aligned}$$

The first term I_1 is given in Lemma 2.9. We estimate I_2 (see the estimate of I_2 in Lemma 2.9)

$$\begin{aligned} I_2 &= k \int_{\Sigma} \left(\left(U_1 + \sum_{h=2}^k U_h \right)^p - U_1^p - \sum_{h=2}^k U_h^p \right) \left(\partial_\rho U_1 + \sum_{i=2}^k \partial_\rho U_i \right) \\ &= k \sum_{h=2}^k \int_{\Sigma} p U_1^{p-1} U_h \partial_\rho U_1 + h.o.t. \\ &= k \sum_{h=2}^k \int_{\Sigma} p U^{p-1}(x - \rho \xi_1) U'(x - \rho \xi_1) \frac{\langle x - \rho \xi_1, -\xi_1 \rangle}{|x - \rho \xi_1|} U(x - \rho \xi_h) dx + h.o.t. \\ &= k \underbrace{\sum_{h=2}^k \int_{\mathbb{R}^3} p U^{p-1}(x + \rho(\xi_h - \xi_1)) U'(x + \rho(\tilde{\eta}_\ell - \xi_1)) \frac{\langle x + \rho(\xi_h - \xi_1), -\xi_1 \rangle}{|x + \rho(\xi_h - \xi_1)|} U(x) dx}_{= \langle \nabla_\zeta \Gamma_{p,1}(\zeta), \xi_1 \rangle, \zeta = \rho(\xi_h - \xi_1)} + h.o.t. \\ &= -ck \sum_{h=2}^k \left\langle \frac{\xi_1 - \xi_h}{|\xi_1 - \xi_h|}, \xi_1 \right\rangle \frac{e^{-\rho|\xi_1 - \xi_h|}}{(\rho|\xi_h - \xi_1|)^{\frac{n-1}{2}}} + h.o.t. \\ &= -A_2 \left(\frac{k}{\rho} \right)^{\frac{n-1}{2}} e^{-2\rho \frac{\pi}{k}} + h.o.t. \text{ because of (2.34)} \end{aligned}$$

and also I_3 (see the estimate of I_3 in Lemma 2.9)

$$\begin{aligned}
I_3 &= \beta \int_{\mathbb{R}^3} \left(\sum_{h=1}^k U_h \right)^q \sum_{i=2}^d \left(\sum_{j=1}^k U_j(\hat{\Theta}_i x) \right)^r \partial_\rho W_\rho \\
&= \beta k \int_{\Sigma} \left(U_1 + \sum_{h=2}^k U_h \right)^q \sum_{i=2}^d \left(\sum_{j=1}^k U_{ij} \right)^r \left(\partial_\rho U_1 + \sum_{i=2}^k \partial_\rho U_i \right) \\
&= \beta k \int_{\Sigma} U_1^q \partial_\rho U_1 \sum_{i=2}^d \sum_{j=1}^k U_{ij}^r + h.o.t. \\
&= \beta k \sum_{\substack{\ell \in \mathcal{I} \\ \ell \neq 0}} \int_{\Sigma} U^q(x - \rho \xi_1) U'(x - \rho \xi_1) \frac{\langle x - \rho \xi_1, -\xi_1 \rangle}{|x - \rho \xi_1|} U^r(x - \rho \tilde{\eta}_\ell) dx + h.o.t. \\
&= \beta k \sum_{\substack{\ell \in \mathcal{I} \\ \ell \neq 0}} \int_{\mathbb{R}^n} \underbrace{U^q(x + \rho(\tilde{\eta}_\ell - \xi_1)) U'(x + \rho(\tilde{\eta}_\ell - \xi_1)) \frac{\langle x + \rho(\tilde{\eta}_\ell - \xi_1), -\xi_1 \rangle}{|x + \rho(\tilde{\eta}_\ell - \xi_1)|} U^r(x) dx}_{= -\frac{1}{q+1} \langle \nabla_\zeta \Gamma_{q+1,r}(\zeta), \xi_1 \rangle, \zeta = \rho(\tilde{\eta}_\ell - \xi_1)} + h.o.t. \\
&= \begin{cases} -\beta \mathbf{A}_3 \left(\frac{k}{\rho} \right)^{\min\{q+1,r\} \frac{n-1}{2}} e^{-\min\{q+1,r\} \frac{2\pi\rho}{dk}} + h.o.t. & \text{if } r \neq q+1 \\ -\beta \mathbf{A}_3 \left(\frac{k}{\rho} \right)^{r(n-1) + \frac{n+1}{2}} e^{-r \frac{2\pi\rho}{dk}} + h.o.t. & \text{if } r = q+1 < \frac{n+1}{n-1} \\ -\beta \mathbf{A}_3 \left(\frac{k}{\rho} \right)^{\frac{r(n-1)}{2}} e^{-r \frac{2\pi\rho}{dk}} \ln \ln k + h.o.t. & \text{if } r = q+1 = \frac{n+1}{n-1} \\ -\beta \mathbf{A}_3 \left(\frac{k}{\rho} \right)^{\frac{r(n-1)}{2}} e^{-r \frac{2\pi\rho}{dk}} + h.o.t. & \text{if } r = q+1 > \frac{n+1}{n-1} \end{cases}
\end{aligned}$$

because of (2.30).

Moreover, the higher order terms $\int_{\mathbb{R}^n} \mathcal{N}(\Phi) \partial_\rho W_\rho$ and $\int_{\mathbb{R}^n} \mathcal{L}(\Phi) \partial_\rho W_\rho$ can be estimated as

$$\int_{\mathbb{R}^n} \mathcal{N}(\Phi) \partial_\rho W_\rho \lesssim k \left(\|\Phi\|_*^2 + \left\| W_\rho^{q-1} \sum_{\ell=2}^d |W_\rho(\hat{\Theta}_\ell x)|^r \right\|_* \|\Phi\|_* + \left\| W_\rho^q \sum_{\ell=2}^d |W_\rho(\hat{\Theta}_\ell x)|^{r-1} \right\|_* \|\Phi\|_* \right)$$

where (arguing as in Lemma 2.5)

$$\left\| W_\rho^{q-1} \sum_{\ell=2}^d |W_\rho(\hat{\Theta}_\ell x)|^r \right\|_* \lesssim e^{-(\min\{q-1,r\} - \alpha) \frac{2\pi\rho}{dk}}$$

and

$$\left\| W_\rho^q \sum_{\ell=2}^d |W_\rho(\hat{\Theta}_\ell x)|^{r-1} \right\|_* \lesssim e^{-(\min\{q,r-1\} - \alpha) \frac{2\pi\rho}{dk}},$$

and

$$\int_{\mathbb{R}^n} \mathcal{L}(\Phi) \partial_\rho W_\rho \lesssim k \left(\frac{1}{\rho^\nu} \|\Phi\|_* + e^{-\min\{1, (p-1-\alpha)\} \frac{2\pi\rho}{k}} \|\Phi\|_* \right),$$

since we have (see the estimate of L_2 in Lemma 2.9)

$$\begin{aligned} & \sum_{h=1}^k U_h^{p-1} \langle \nabla U_h, (-\xi_h) \rangle - W_\rho^{p-1} \partial_\rho W_\rho \\ &= U_1^{p-1} \langle \nabla U_1, (-\xi_1) \rangle + \sum_{h=2}^k U_h^{p-1} \langle \nabla U_h, (-\xi_h) \rangle \\ & - \left(U_1 + \sum_{h=2}^k U_h \right)^{p-1} \left(\langle \nabla U_1, (-\xi_1) \rangle + \sum_{h=2}^k \langle \nabla U_h, (-\xi_h) \rangle \right) \\ & \lesssim U_1^{p-1} \sum_{h=2}^k U_h + U_1 \sum_{h=2}^k U_h^{p-1}. \end{aligned}$$

□

At this point it is clear that a solution to the non-local equation (3.2) does exist if we find ρ so that $\mathfrak{C}_k(\rho) = 0$. At this aim, it is useful that the last term $\Xi_k(\rho)$ is an higher order term in the expansion of \mathfrak{C}_k and this is achieved if we choose the exponents p , q and r and the number of equations d in a proper way. We will focus on the particular case $q = \frac{p-1}{2}$ and $r = \frac{p+1}{2}$.

Proof of Theorem 1.2: completed. If $d > \frac{p+1}{2}$ and $p > 5$ ($n = 2$ because the exponent 5 is critical in 3D), by (2) of Lemma 3.4

$$\mathfrak{C}_k(\rho) = \mathbf{v}_\infty \mathbf{A}_1 \frac{k}{\rho^{\nu+1}} (1 + o(1)) - \beta \mathbf{A}_3 \left(\frac{k}{\rho} \right)^{\frac{p+1}{4}} e^{-\frac{(p+1)\pi\rho}{dk}} (1 + o(1)). \quad (3.5)$$

Indeed, in this case $\frac{p+1}{2d} < 1$ and $\Xi_k(\rho)$ can be estimated using (3.4) with

$$\sigma = \frac{p-1}{2d}, \quad \tau = \frac{p-3}{2d}, \quad 2\sigma > \frac{p+1}{2d} \quad \text{and} \quad \sigma + \tau > \frac{p+1}{2d}.$$

By (3.5), there exists $\rho(k) \in \mathcal{D}_k$ (see (2.10)) such that $\mathfrak{C}_k(\rho(k)) = 0$ if either \mathbf{v}_∞ and β have the same sign.

On the other hand if $d \leq \frac{p-1}{2}$ and $p > 3$ the coupling term is an higher order term in the expansion (2), (3) or (4) of Lemma 3.4 and

$$\mathfrak{C}_k(\rho) = \mathbf{v}_\infty \mathbf{A}_1 \frac{k}{\rho^{\nu+1}} (1 + o(1)) - \beta \mathbf{A}_2 \left(\frac{k}{\rho} \right)^{\frac{n-1}{2}} e^{-\frac{2\pi\rho}{k}} (1 + o(1)). \quad (3.6)$$

Indeed in this case $1 < \frac{p+1}{2d}$ and $\Xi_k(\rho)$ can be estimated using (3.4) with

$$\sigma = 1, \quad \tau > 0, \quad 2\sigma > 1 \quad \text{and} \quad \sigma + \tau > 1.$$

By (3.6), there $\rho(k) \in \mathcal{D}_k$ (see (2.10)) such that $\mathfrak{C}_k(\rho(k)) = 0$ if $\mathbf{v}_\infty > 0$ whatever β is.

□

APPENDIX A. AUXILIARY RESULTS

We recall the result [1, Lemma 3.7].

Lemma A.1. *Let $W_1, W_2 : \mathbb{R}^n \rightarrow \mathbb{R}$ be two positive continuous radial functions such that*

$$W_i(x) \sim |x|^{-a_i} e^{-b_i|x|} \text{ as } |x| \rightarrow \infty$$

where $a_i \in \mathbb{R}$, $b_i > 0$. Then for some constant $\mathfrak{c} > 0$ we have

(i) If $b_1 < b_2$ then

$$\int_{\mathbb{R}^3} W_1(x + \zeta) W_2(x) dx \sim \mathfrak{c} e^{-b_1|\zeta|} |\zeta|^{a_1} \text{ as } |\zeta| \rightarrow \infty$$

Clearly, if $b_1 > b_2$ a similar expression holds, by replacing a_1 and b_1 with a_2 and b_2 .

(ii) If $b_1 = b_2 =: b$ then, suppose that $a_1 \leq a_2$

$$\int_{\mathbb{R}^3} W_1(x + \zeta) W_2(x) dx \sim \begin{cases} \mathfrak{c} e^{-b|\zeta|} |\zeta|^{-a_1 - a_2 + \frac{n+1}{2}} & \text{if } a_2 < \frac{n+1}{2} \\ \mathfrak{c} e^{-b|\zeta|} |\zeta|^{-a_1} \ln |\zeta| & \text{if } a_2 = \frac{n+1}{2} \\ \mathfrak{c} e^{-b|\zeta|} |\zeta|^{-a_1} & \text{if } a_2 > \frac{n+1}{2} \end{cases} \text{ as } |\zeta| \rightarrow \infty.$$

Let $s, t \geq 1$. Set

$$\Gamma_{s,t}(\zeta) := \int_{\mathbb{R}^3} U^s(x + \zeta) U^t(x) dx, \quad \zeta \in \mathbb{R}^n.$$

By Lemma (A.1) and (2.11) there exists $\mathfrak{c} > 0$ such that

(i) if $s < t$ then

$$\Gamma_{s,t}(\zeta) \sim \mathfrak{c} e^{-s|\zeta|} |\zeta|^{-s\frac{n-1}{2}} \text{ as } |\zeta| \rightarrow \infty$$

(ii) if $s = t$ then

$$\Gamma_{s,t}(\zeta) \sim \begin{cases} \mathfrak{c} e^{-s|\zeta|} |\zeta|^{-s(n-1) - \frac{n+1}{2}} & \text{if } s < \frac{n+1}{n-1} \\ \mathfrak{c} e^{-s|\zeta|} |\zeta|^{-s\frac{n-1}{2}} \ln |\zeta| & \text{if } s = \frac{n+1}{n-1} \\ \mathfrak{c} e^{-s|\zeta|} |\zeta|^{-s\frac{n-1}{2}} & \text{if } s > \frac{n+1}{n-1} \end{cases} \text{ as } |\zeta| \rightarrow \infty.$$

We are going to prove that all the previous estimates hold true in the C^1 -sense.

Lemma A.2. *It holds true that*

(i) if $s < t$ then

$$\nabla_{\zeta} \Gamma_{s,t}(\zeta) \sim -\mathfrak{c} s \frac{\zeta}{|\zeta|} e^{-s|\zeta|} |\zeta|^{-s\frac{n-1}{2}} \text{ as } |\zeta| \rightarrow \infty$$

(ii) if $s = t$ then, suppose that $a_1 \geq a_2$

$$\nabla_{\zeta} \Gamma_{s,t}(\zeta) \sim \begin{cases} -\mathbf{c}s \frac{\zeta}{|\zeta|} e^{-s|\zeta|} |\zeta|^{-s(n-1) - \frac{n+1}{2}} & \text{if } s < \frac{n+1}{n-1} \\ -\mathbf{c}s \frac{\zeta}{|\zeta|} e^{-s|\zeta|} |\zeta|^{-s\frac{n-1}{2}} \ln |\zeta| & \text{if } s = \frac{n+1}{n-1} \\ -\mathbf{c}s \frac{\zeta}{|\zeta|} e^{-s|\zeta|} |\zeta|^{-s\frac{n-1}{2}} & \text{if } s > \frac{n+1}{n-1} \end{cases} \quad \text{as } |\zeta| \rightarrow \infty.$$

Proof. We only prove the case $s < t$, being the proof of other cases similar. We point out that

$$\int_{\mathbb{R}^n} U^s(x + \zeta) \partial_{x_i} U^t(x) dx = - \int_{\mathbb{R}^n} \partial_{x_i} U^s(x + \zeta) U^t(x) dx = - \int_{\mathbb{R}^n} \partial_{\zeta_i} U^s(x + \zeta) U^t(x) dx$$

and we are going to prove that

$$\int_{\mathbb{R}^n} \partial_{\zeta_i} U^s(x + \zeta) U^t(x) dx \sim -\mathbf{c}s \frac{\zeta}{|\zeta|} e^{-s|\zeta|} |\zeta|^{-s\frac{n-1}{2}} \quad \text{as } |\zeta| \rightarrow \infty.$$

Set $f(\zeta) := g(\zeta)h(\zeta)$ with

$$g(\zeta) := \int_{\mathbb{R}^n} U^s(x + \zeta) U^t(x) dx \quad \text{and} \quad h(\zeta) := e^{s|\zeta|} |\zeta|^{s\frac{n-1}{2}}.$$

We know that

$$\lim_{|\zeta| \rightarrow \infty} f(\zeta) = \mathbf{c}, \quad (\text{A.1})$$

We are going to prove that

$$\lim_{|\zeta| \rightarrow \infty} \partial_{\zeta_i} f(\zeta) = 0. \quad (\text{A.2})$$

Since

$$\partial_{\zeta_i} f = g \partial_{\zeta_i} h + h \partial_{\zeta_i} g \quad (\text{A.3})$$

and

$$\partial_{\zeta_i} h(\zeta) = s \frac{\zeta_i}{|\zeta|} h(\zeta) (1 + o(1)), \quad (\text{A.4})$$

by (A.1), (A.2), (A.3) and (A.4) the claim follows. To prove (A.2), by Lemma A.3 we need to show that there exists $c > 0$ such that

$$|\partial_{\zeta_i \zeta_i}^2 f(\zeta)| \leq c \quad \text{if } |\zeta| \text{ is large enough.}$$

We have

$$\partial_{\zeta_i \zeta_i}^2 f = g \partial_{\zeta_i \zeta_i}^2 h + 2 \partial_{\zeta_i} g \partial_{\zeta_i} h + h \partial_{\zeta_i \zeta_i}^2 g.$$

It is easy to check that

$$|\partial_{\zeta_i} h|, |\partial_{\zeta_i \zeta_i}^2 h| = \mathcal{O}(h).$$

By (A.1) $g = \mathcal{O}\left(\frac{1}{h}\right)$. We only need to estimate $\partial_{\zeta_i} g$ and $\partial_{\zeta_i \zeta_i}^2 g$. We have

$$\begin{aligned} \partial_{\zeta_i} g(\zeta) &= \int_{\mathbb{R}^n} \partial_{\zeta_i} U^s(x + \zeta) U^t(x) dx \\ &= \int_{\mathbb{R}^s} s U^{s-1}(x + \zeta) U'(x + \zeta) \frac{x_1 + \zeta_i}{|x + \zeta|} U^t(x) dx \\ &\lesssim \int_{\mathbb{R}^n} U^{s-1}(x + \zeta) |U'(x + \zeta)| U^t(x) dx \text{ since } |U'| = \mathcal{O}(U) \\ &\lesssim \int_{\mathbb{R}^n} U^s(x + \zeta) U^t(x) dx \lesssim \frac{1}{h(\zeta)} \text{ because of (A.1)} \end{aligned}$$

and

$$\begin{aligned} \partial_{\zeta_i \zeta_i}^2 g(\zeta) &= \int_{\mathbb{R}^n} s \partial_{\zeta_i} \left(U^{s-1}(x + \zeta) U'(x + \zeta) \frac{x_1 + \zeta_i}{|x + \zeta|} \right) U^t(x) dx \\ &= \int_{\mathbb{R}^n} s U^{s-2} \left(U'(x + \zeta) \frac{x_1 + \zeta_i}{|x + \zeta|} \right)^2 U^t(x) dx \\ &\quad + \int_{\mathbb{R}^n} s U^{s-1}(x + \zeta) U''(x + \zeta) \left(\frac{x_1 + \zeta_i}{|x + \zeta|} \right)^2 U^t(x) dx \\ &\quad + \int_{\mathbb{R}^n} s U^{s-1}(x + \zeta) U'(x + \zeta) \left(\frac{1}{|x + \zeta|} - \frac{(x_1 + \zeta_i)^2}{|x + \zeta|^3} \right) U^t(x) dx \\ &\lesssim \int_{\mathbb{R}^n} U^s(x + \zeta) U^t(x) dx \text{ since } |U'|, |U''| = \mathcal{O}(U) \\ &\quad + \underbrace{\int_{\mathbb{R}^n} \frac{1}{|x + \zeta|} U^s(x + \zeta) U^t(x) dx}_{=\mathcal{O}\left(e^{-s|\zeta|} |\zeta|^{-s \frac{n-1}{2} - 1}\right)} \text{ because of Lemma A.1 with } b_1 = s < b_2 = t \text{ and } a_1 = -s \frac{n-1}{2} - 1 \\ &\lesssim \frac{1}{h(\zeta)} \text{ because of (A.1)}. \end{aligned}$$

That concludes the proof. □

Lemma A.3. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^2 -function such that*

$$\lim_{|x| \rightarrow \infty} f(x) = l \in \mathbb{R}$$

and there exists $c > 0$ such that

$$\sup_{x \in \mathbb{R}^n} |\partial_{x_1 x_1}^2 f(x)| \leq c$$

then

$$\lim_{|x| \rightarrow \infty} \partial_{x_1} f(x) = 0.$$

Proof. Let $\epsilon > 0$ be fixed. If $\delta = \frac{\epsilon^2}{4c}$ there exists $R > 0$ such that if $|x| \geq R$ then $|f(x) - l| \leq \delta$. Now take $e_1 := (1, 0, \dots, 0) \in \mathbb{R}^n$, $t \in (-1, 1)$ so that $|x + te_1| \geq R$ if $|x| \geq R + 1$ and apply mean value theorem

$$f(x + te_1) = f(x) + \partial_{x_1} f(x)t + \frac{1}{2} \partial_{x_1 x_1}^2 f(x + \epsilon t e_1) t^2$$

for some $\epsilon \in (0, 1)$. Then if $|x| \geq R + 1$

$$|\partial_{x_1} f(x)| \leq \frac{1}{|t|} 2\delta + \frac{1}{2} c |t| \text{ for any } t \in (-1, 1), t \neq 0 \Rightarrow |\partial_{x_1} f(x)| \leq 2\sqrt{\delta c} = \epsilon$$

and the claim follows. \square

APPENDIX B. PROOF OF PROPOSITION 2.2

Let us consider the Hilbert space $W^{1,2}(\mathbb{R}^n)$ equipped with the scalar product

$$\langle u, v \rangle = \int_{\mathbb{R}^n} (\nabla u \nabla v + uv) dx.$$

It is well known that the embedding $\mathcal{S} : W^{1,2}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$ is continuous. We define the adjoint operator $\mathcal{S}^* : L^2(\mathbb{R}^n) \rightarrow W^{1,2}(\mathbb{R}^n)$ by duality, i.e.

$$\mathcal{S}^* f = u \text{ if and only if } u \text{ is a weak solution to } -\Delta u + u = f \text{ in } \mathbb{R}^n.$$

Let us introduce the space

$$H_\rho^\perp := \left\{ \Phi \in \mathcal{H} : \langle \Phi, \mathcal{S}^*(\partial_\rho W_\rho) \rangle = \int_{\mathbb{R}^n} \Phi \partial_\rho W_\rho = 0 \right\}$$

where \mathcal{H} is defined in (2.5).

It is immediate to check that linear problem

$$\mathcal{L}(\Phi) = h \text{ in } \mathbb{R}^n, \int_{\mathbb{R}^n} \Phi \partial_\rho W_\rho = \int_{\mathbb{R}^n} h \partial_\rho W_\rho = 0 \quad (\text{B.1})$$

can be rewritten as

$$\Phi + \mathcal{K}(\Phi) = h^* := \mathcal{S}^* h, \quad h^*, \Phi \in H_\rho^\perp,$$

where

$$\mathcal{K}(\Phi) = \mathcal{S}^* \left[(V(x) - 1) \Phi - 3W_\rho^2 \Phi - \beta \Phi \sum_{i=2}^d W_\rho^2(\hat{\Theta}_i x) - 2\beta W_\rho \sum_{i=2}^d W_\rho(\hat{\Theta}_i x) \Phi(\hat{\Theta}_i x) \right]$$

is a compact operator since W_ρ decays exponentially and V satisfies (1.10). Therefore, by Fredholm alternative solving problem (B.1) is equivalent to prove that it has a unique

solution when $h = 0$. Now, it is important to point out that if $h \in \mathcal{B}_*$ then the solution $\Phi \in W^{1,2}(\mathbb{R}^n)$ to (B.1) also belongs to \mathcal{B}_* , i.e. it belongs to $L^\infty(\mathbb{R}^n)$. It is enough to apply the standard regularity theory. Indeed, by (2.23) we deduce that $h \in L^q(\mathbb{R}^n)$ for any $q > 1$. Moreover, since $W_\rho \in L^\infty(\mathbb{R}^n)$ and $\Phi \in L^2(\mathbb{R}^n)$, by (B.1) we immediately deduce that $\Phi \in W^{2,2}(\mathbb{R}^n)$, which is embedded in $L^\infty(\mathbb{R}^n)$ if $n = 2, 3$.

Finally, to prove the Proposition 2.2 it is enough to prove the a priori estimate (2.24).

At this aim it is useful to decompose \mathcal{L} as

$$\mathcal{L}(\Phi) = \underbrace{-\Delta\Phi + V(x)\Phi - 3W_\rho^2\Phi - \beta\Phi \sum_{i=2}^d W_\rho^2(\hat{\Theta}_i x)}_{\mathcal{L}_0(\Phi)} - \underbrace{2\beta W_\rho \sum_{i=2}^d W_\rho(\hat{\Theta}_i x)\Phi(\hat{\Theta}_i x)}_{\mathcal{L}_1(\Phi)}.$$

and to point out that the non-local linear part is small, since by (2.29)

$$\|\mathcal{L}_1(\Phi)\|_* \leq ce^{-(1-\alpha)\frac{2\pi\rho}{dk}} \|\Phi\|_*.$$

So our problem reduces to prove that if $\Phi \in H_\rho^\perp$ solves

$$\mathcal{L}_0(\Phi) = h + \mathfrak{c}\partial_\rho W_\rho \text{ in } \mathbb{R}^n \quad (\text{B.2})$$

for some $\mathfrak{c} \in \mathbb{R}$ then the priori estimate (2.24) holds and this is done using the same arguments of Lemma 4.3 of [3]. For sake of completeness, we give the proof below.

First, we prove that

$$|\mathfrak{c}| \lesssim \left[\left(\frac{1}{\rho^\nu} + e^{-(1-\alpha)\frac{2\pi}{kd}} \right) \|\Phi\|_* + \|h\|_* \right]. \quad (\text{B.3})$$

Indeed, by (B.2)

$$\int_{\mathbb{R}^n} \mathcal{L}_0(\Phi)\partial_\rho W_\rho - \int_{\mathbb{R}^n} h\partial_\rho W_\rho = \mathfrak{c} \int_{\mathbb{R}^n} (\partial_\rho W_\rho)^2,$$

where arguing as in Lemma 2.9

$$\left| \int_{\mathbb{R}^n} \mathcal{L}_0(\Phi)\partial_\rho W_\rho \right| \lesssim \left(\frac{k}{\rho^\nu} + ke^{-(1-\alpha)\frac{2\pi}{dk}} \right) \|\Phi\|_*,$$

moreover

$$\left| \int_{\mathbb{R}^n} h\partial_\rho W_\rho \right| \lesssim k\|h\|_*, \text{ because } |\partial_\rho W_\rho| \lesssim W_\rho$$

and

$$\int_{\mathbb{R}^n} (\partial_\rho W_\rho)^2 \sim ck \text{ for some positive constant } c.$$

Next, we show that there exist constants τ and c (all independent of k) such that for any $x \in \mathbb{R}^n \setminus \bigcup_{\substack{i=1,\dots,d \\ j=1,\dots,k}} B(\rho\eta_{ij}, \tau)$

$$|\Phi(x)| \leq C \left(\|\mathcal{L}_0(\Phi)\|_* + \sup_{\substack{i=1,\dots,d \\ j=1,\dots,k}} \|\Phi\|_{L^\infty(\partial B(\rho\eta_{ij}, \tau))} \right) \sum_{i=1}^d \sum_{j=1}^k e^{-\alpha|x-\rho\eta_{ij}|}, \quad (\text{B.4})$$

which immediately implies

$$\|\Phi\|_* \leq C \left(\|\mathcal{L}_0(\Phi)\|_* + \sup_{\substack{i=1,\dots,d \\ j=1,\dots,k}} \|\Phi\|_{L^\infty(\partial B(\rho\eta_{ij}, \tau))} \right). \quad (\text{B.5})$$

To prove the above pointwise estimate we first show the independence of τ on k for any $x \in \mathbb{R}^n \setminus \bigcup_{\substack{i=1,\dots,d \\ j=1,\dots,k}} B(\rho\eta_{ij}, \tau)$. Indeed reasoning as in [3] and using Lemma 3.4 of [3] we get that

$$\begin{aligned} W_\rho(\hat{\Theta}_i x) &= \sum_{j=1}^k U(x - \rho\eta_{ij}) \\ &\leq \sum_{|x - \rho\eta_{ij}| < \rho \sin \frac{\pi}{dk}} U(x - \rho\eta_{ij}) + \sum_{\ell=1}^{+\infty} \sum_{\ell \rho \sin \frac{\pi}{dk} \leq |x - \rho\eta_{ij}| < (\ell+1)\rho \sin \frac{\pi}{dk}} U(x - \rho\eta_{ij}) \\ &\lesssim U(\tau) + c \sum_{\ell=1}^{+\infty} \ell^{n-1} e^{-\ell \rho \frac{\pi}{dk}} \\ &\leq CU(\tau). \end{aligned}$$

Thus we can take τ sufficiently large (but independent of k) such that for any $x \in \mathbb{R}^n \setminus \bigcup_{\substack{i=1,\dots,d \\ j=1,\dots,k}} B(\rho\eta_{ij}, \tau)$

$$3W_\rho^2(x) \leq \frac{1}{2} \frac{V_0 - \alpha^2}{4}; \quad \beta \sum_{i=2}^d W_\rho^2(\hat{\Theta}_i x) \leq \frac{1}{2} \frac{V_0 - \alpha^2}{4}.$$

Now we let $\Pi_\pm(x) = \sum_{i=1}^d \sum_{j=1}^k e^{\pm\alpha|x - \rho\eta_{ij}|}$. For $x \in \mathbb{R}^n \setminus \bigcup_{\substack{i=1,\dots,d \\ j=1,\dots,k}} B(\rho\eta_{ij}, \tau)$ we get

$$\begin{aligned} \mathcal{L}_0(\Pi_\pm(x)) &= \sum_{i=2}^k \sum_{j=1}^k e^{\pm\alpha|x - \rho\eta_{ij}|} \left(-\alpha^2 \mp \frac{\alpha(n-1)}{|x - \rho\eta_{ij}|} + V(x) - 3W_\rho^2(x) - \beta \sum_{i=2}^d W_\rho^2(\hat{\Theta}_i x) \right) \\ &\geq \sum_{i=2}^k \sum_{j=1}^k e^{\pm\alpha|x - \rho\eta_{ij}|} \left(-\alpha^2 \mp \frac{\alpha(n-1)}{|x - \rho\eta_{ij}|} + V_0 - \frac{V_0 - \alpha^2}{4} \right) \end{aligned}$$

Hence

$$\mathcal{L}_0(\Pi_\pm(x)) \geq c_0 \Pi_\pm(x)$$

for some positive constant c_0 independent of k .

Then we can use $\Pi_\pm(x)$ as barriers and we can apply the maximum principle to the linear operator \mathcal{L}_0 obtaining

$$|\Phi(x)| \leq C \left(\|\mathcal{L}_0(\Phi)\|_* + \sup_{\substack{i=1,\dots,d \\ j=1,\dots,k}} \|\Phi\|_{L^\infty(\partial B(\rho\eta_{ij}, \tau))} \right) \sum_{i=1}^d \sum_{j=1}^k e^{-\alpha|x - \rho\eta_{ij}|} + \delta \sum_{i=1}^d \sum_{j=1}^k e^{\alpha|x - \rho\eta_{ij}|}$$

for any $\delta > 0$ and C independent of k and δ . Letting $\delta \rightarrow 0$ we get the estimate (B.4).

Finally, we prove (2.24) arguing by contradiction. Assuming that there is a sequence (Φ_n, h_n) satisfying (B.2) such that

$$\|\Phi_n\|_* = 1 \quad \text{and} \quad \|h_n\|_* = o(1) \quad \text{as } k_n \rightarrow +\infty.$$

In the sequel we will omit the dependence on n . By (B.3) and the fact that $\|\partial_\rho W_\rho\|_* \lesssim 1$, we also have $\|\mathcal{L}_0(\Phi)\|_* = o(1)$. Hence (B.5) implies the existence of a subsequence of η_{ij} such that

$$\|\Phi\|_{L^\infty(\partial B(\rho\eta_{ij}, \tau))} \geq C > 0 \tag{B.6}$$

for some fixed constant C which is independent of k .

Since $\|\Phi\|_{L^\infty(\mathbb{R}^n)} \leq 1$ by elliptic regularity estimates we get that $\|\Phi\|_{C^1(\mathbb{R}^n)} \leq C$. By applying the Ascoli-Arzelà's Theorem we get the existence of a subsequence of η_{ij} such that $\Phi(x + \rho\eta_{ij})$ converges (on compact sets) to Φ_∞ which is a bounded solution of

$$-\Delta\Phi_\infty + \Phi_\infty - 3U^2\Phi_\infty = 0 \quad \text{or} \quad -\Delta\Phi_\infty + \Phi_\infty - \beta U^2\Phi_\infty = 0.$$

In the first case $\Phi_\infty \equiv 0$ because $\Phi \in H_\rho^\perp$, while in the second case $\Phi_\infty \equiv 0$ because $\beta \neq \Lambda_\kappa$. Finally, a contradiction arises because of (B.6).

REFERENCES

- [1] A. Ambrosetti, E. Colorado, D. Ruiz, *Multi-bump solitons to linearly coupled systems of nonlinear Schrödinger equations*, Calc. Var. Partial Diff. Equ., **30**, 85–112, (2007).
- [2] T. Bartsch, N. Dancer, Z.-Q. Wang, *A Liouville theorem, a-priori bounds, and bifurcating branches of positive solutions for a nonlinear elliptic system*, Calc. Var. Partial Diff. Equ., **37**, 345–361 (2010).
- [3] M. del Pino, J. Wei, W. Yao, *Intermediate reduction method and infinitely many positive solutions of nonlinear Schrödinger equations with non-symmetric potentials*, Calc. Var. Partial Diff. Equ., **(53) (1-2)**, 473–523 (2015).
- [4] B.D. Esry, C.H. Greene, J.P. Burke Jr., J.L. Bohn, *Hartree–Fock theory for double condensates*, Phys. Rev. Lett., **78**, 3594–3597, (1997).
- [5] B.D. Esry, C.H. Greene, *Spontaneous spatial symmetry breaking in two-component Bose–Einstein condensates*, Phys. Rev. A, **59**, 1457–1460, (1999).
- [6] D.S. Hall, M.R. Matthews, J.R. Ensher, C.E. Wieman, E.A. Cornell, *Dynamics of component separation in a binary mixture of Bose–Einstein condensates*, Phys. Rev. Lett., **81**, 1539–1542, (1998).
- [7] T. C. Lin, J. Wei, *Solitary and self-similar solutions of two-component systems of nonlinear Schrödinger equations*, Physica D: Nonlinear Phenomena, **220**, 99–115 (2006).
- [8] Lin, Tai-Chia; Wei, Juncheng *Ground state of N coupled nonlinear Schrödinger equations in \mathbb{R}^n , $n \geq 3$* . Comm. Math. Phys. 255 (2005), no. 3, 629–653.
- [9] Musso, Monica; Pacard, Frank; Wei, Juncheng *Finite-energy sign-changing solutions with dihedral symmetry for the stationary nonlinear Schrödinger equation*. J. Eur. Math. Soc. (JEMS) 14 (2012), no. 6, 1923–1953.
- [10] A.S. Parkins, D.F. Walls, *The Physics of trapped dilute-gas Bose–Einstein condensates*, Phys. Rep., **303**, 1–80, (1998).
- [11] S. Peng, Z.-Q. Wang, *Segregated and Synchronized Vector Solutions for Nonlinear Schrödinger systems*, Archive for Rat. Mech. and Anal., **208**, 305–339, (2013).
- [12] Peng, Shuangjie; Wang, Qingfang; Wang, Zhi-Qiang *On coupled nonlinear Schroedinger systems with mixed couplings*. Trans. Amer. Math. Soc. 371 (2019), no. 11, 7559–7583.
- [13] S. Terracini, G. Verzini, *Multipulse phase in k-mixtures of Bose-Einstein condensates*, Arch. Ration. Mech. Anal., **194**, 717–741, (2009).
- [14] W. C. Troy, *Symmetry properties in systems of semilinear elliptic equations*. J. Differential Equations 42 (1981), 400–413.
- [15] J. Wei, T. Weth, *Nonradial symmetric bound states for s system of two coupled Schrödinger equations*, Rend. Lincei Mat. Appl., **18**, 279–293 (2007).

- [16] J. Wei, T. Weth, *Radial solutions and phase separation in a system of two coupled Schrödinger equations*, Arch. Rational Mech. Anal., **190**, 83–106 (2008)
- [17] Wei, Juncheng; Wu, Yuanze Ground states of nonlinear Schroedinger systems with mixed couplings. J. Math. Pures Appl. (9) 141 (2020), 50–88.
- [18] J. Wey, S. Yan, *Infinitely many positive solutions for the nonlinear Schrödinger equations in \mathbb{R}^3* , Calc. Var. and PDEs, **37**, 423–439 (2010).

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