

Multiple solutions for some symmetric supercritical problems*

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Abstract

The aim of this paper is investigating the existence of one or more critical points of a family of functionals which generalizes the model problem

$$\bar{J}(u) = \frac{1}{p} \int_{\Omega} \bar{A}(x, u) |\nabla u|^p dx - \int_{\Omega} G(x, u) dx$$

in the Banach space $X = W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is an open bounded domain, $1 < p < N$ and the real terms $\bar{A}(x, t)$ and $G(x, t)$ are C^1 Carathéodory functions on $\Omega \times \mathbb{R}$.

We prove that, even if the coefficient $\bar{A}(x, t)$ makes the variational approach more difficult, if it satisfies “good” growth assumptions then at least one critical point exists also when the nonlinear term $G(x, t)$ has a suitable supercritical growth. Moreover, if the functional is even, it has infinitely many critical levels.

The proof, which exploits the interaction between two different norms on X , is based on a weak version of the Cerami–Palais–Smale condition and a suitable intersection lemma which allow us to use a Mountain Pass Theorem.

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1 Introduction

Here, we look for critical points of the nonlinear functional

$$\mathcal{J}(u) = \int_{\Omega} A(x, u, \nabla u) dx - \int_{\Omega} G(x, u) dx, \quad u \in \mathcal{D} \subset W_0^{1,p}(\Omega),$$

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which generalizes the model problem

$$\bar{J}(u) = \frac{1}{p} \int_{\Omega} \bar{A}(x, u) |\nabla u|^p dx - \int_{\Omega} G(x, u) dx, \quad u \in \mathcal{D} \subset W_0^{1,p}(\Omega), \quad (1.1)$$

where Ω is an open bounded domain in \mathbb{R}^N , $1 < p < N$, $A : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$, respectively $\bar{A} : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$, and $G : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ are given functions.

We note that, even in the simplest case $A(x, u, \nabla u) = \frac{1}{p} \bar{A}(x, t) |\nabla u|^p$ and $G(x, t) \equiv 0$, with $\bar{A}(x, t)$ smooth, bounded away from zero but $\frac{\partial \bar{A}}{\partial t}(x, t) \not\equiv 0$, the functional \bar{J} is defined in $W_0^{1,p}(\Omega)$ but is Gâteaux differentiable only along directions of $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

In the past, such a problem has been overcome by introducing suitable definitions of critical point for \mathcal{J} and related existence results have been stated (see, e.g., [?, ?, ?, ?]). Here, as in [?], suitable assumptions assure that the functional \mathcal{J} is C^1 in $X = W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ (see Proposition ??) and its Euler–Lagrange equation is

$$\begin{cases} -\operatorname{div}(a(x, u, \nabla u)) + A_t(x, u, \nabla u) = g(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

where

$$A_t(x, t, \xi) = \frac{\partial A}{\partial t}(x, t, \xi), \quad a(x, t, \xi) = \left(\frac{\partial A}{\partial \xi_1}(x, t, \xi), \dots, \frac{\partial A}{\partial \xi_N}(x, t, \xi) \right), \quad G(x, t) = \int_0^t g(x, s) ds. \quad (1.3)$$

We note that, from a physical point of view, problem (??) is interesting for its applications. For example, if $\Omega = \mathbb{R}^N$ and $A(x, u, \nabla u) = (1 + |u|^2) |\nabla u|^2$, model equations of (??) appear in Mathematical Physics and describe several physical phenomena in the theory of superfluid film and in dissipative quantum mechanics (for more details, see [?] and references therein).

In order to find solutions of (??), i.e. critical points of \mathcal{J} in X , we cannot apply directly existence and multiplicity results similar to the classical Ambrosetti–Rabinowitz theorems (see [?, ?]). Indeed, our functional \mathcal{J} does not satisfy the Palais–Smale condition in X as it has Palais–Smale sequences which converge in $W_0^{1,p}(\Omega)$ but are unbounded in $L^\infty(\Omega)$ (see, e.g., [?, Example 4.3]). Hence, we have to weaken the definition of Palais–Smale condition (see Definition ??) and use it for stating a generalized version of the Mountain Pass Theorems (see Theorems ?? and ??).

In [?] the existence of critical points of the functional \mathcal{J} , i.e. solutions of (??), has been already proved if $p > 1$, $A(x, t, \xi)$ satisfies suitable assumptions and $G(x, t)$ has a p -superlinear growth which has to be subcritical if $p < N$. Anyway, even if the dependence from t of the principal part $A(x, t, \xi)$ makes the variational approach more difficult, it can allow the nonlinear term $G(x, t)$ to be supercritical. In fact, the aim of this paper is to extend the main statements in [?] to a function $G(x, t)$ with critical or supercritical growth if $1 < p < N$: roughly speaking, we prove that the more $A(x, t, \xi)$ is unbounded and grows with respect to t , the more $G(x, t)$ can have a supercritical growth.

Since our main theorems need a list of hypotheses, we give their complete statements in Section ?? (see Theorems ?? and ??), anyway, here, in order to highlight how our approach improves previous results, we consider the particular setting

$$A(x, t, \xi) = \frac{1}{p} (A_1(x) + A_2(x) |t|^{ps}) |\xi|^p \quad \text{and} \quad g(x, t) = |t|^{\mu-2} t$$

with $1 < p < N$, $s \geq 0$, $\mu \geq 1$, so that problem (??) reduces to

$$\begin{cases} -\operatorname{div}((A_1(x) + A_2(x)|u|^{ps})|\nabla u|^{p-2}\nabla u) + sA_2(x)|u|^{ps-2}u|\nabla u|^p = |u|^{\mu-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.4)$$

If $s = 0$, then problem (??) has been widely studied (see, e.g., [?] and references therein). On the contrary, if $s > 0$ we obtain the following result.

Theorem 1.1. *Let $A_1, A_2 \in L^\infty(\Omega)$ be two given functions such that*

$$A_1(x) \geq \alpha_0, \quad A_2(x) \geq \alpha_0 \quad \text{for a.e. } x \in \Omega, \quad (1.5)$$

for a constant $\alpha_0 > 0$. Assume that

$$2 < 1 + p < p(s + 1) < \mu < p^*(s + 1), \quad (1.6)$$

where p^* is the critical exponent. Then, problem (??) has infinitely many weak bounded solutions.

To our knowledge, there are very few results dealing with quasilinear supercritical problems. Usually, they make use of a suitable change of variables which reduces the supercritical problem to a subcritical one (see, e.g., [?]). Unluckily, such an approach works only if $A(x, t, \xi)$ has a very particular form, and so, for example, it is not allowed also in the simplest case $A_2(x) = 1$ but $A_1(x)$ not constant. Different arguments can be found in [?] where, by using a sequence of truncated functionals, the authors prove that problem (??) with, e.g., $p = 2$, has at least one positive solution if (??) and the further condition $2(s + 1) < 2^*$ hold, which imply $N < 6$ (see [?, Theorem 2.1]). Differently from [?], here we use variational methods which exploit the interaction between two different norms and we do not require this additional restriction (see also [?] where, in the same setting of Theorem ??, the existence of at least one positive solution of problem (??) is proved).

This paper is organized as follows. In Section ?? we introduce the weak Cerami–Palais–Smale condition and prove some related abstract existence and multiplicity results which generalize the Mountain Pass Theorem (see [?, Theorem 2.2]) and its symmetric version (see [?, Theorem 9.12]). In Section ??, after introducing the hypotheses for $A(x, t, \xi)$ and $G(x, t)$, we give the variational formulation of our problem and prove that \mathcal{J} satisfies the weak Cerami–Palais–Smale condition. Finally, in Section ?? the main results are stated and proved.

2 Abstract setting

We denote $\mathbb{N} = \{1, 2, \dots\}$ and, throughout this section, we assume that:

- $(X, \|\cdot\|_X)$ is a Banach space with dual $(X', \|\cdot\|_{X'})$,
- $(W, \|\cdot\|_W)$ is a Banach space such that $X \hookrightarrow W$ continuously, i.e. $X \subset W$ and a constant $\sigma_0 > 0$ exists such that
$$\|u\|_W \leq \sigma_0 \|u\|_X \quad \text{for all } u \in X, \quad (2.1)$$
- $J : \mathcal{D} \subset W \rightarrow \mathbb{R}$ and $J \in C^1(X, \mathbb{R})$ with $X \subset \mathcal{D}$,
- $K_J = \{u \in X : dJ(u) = 0\}$ is the set of the critical points of J in X .

Furthermore, fixing $\beta \in \mathbb{R}$, we define

- $K_J^\beta = \{u \in X : J(u) = \beta, dJ(u) = 0\}$ the set of the critical points of J in X at level β ,
- $J^\beta = \{u \in X : J(u) \leq \beta\}$ the sublevel of J with respect to β ,

and, taking $r > 0$, by pointing out the two different norms $\|\cdot\|_W$ and $\|\cdot\|_X$, we set

- $B_r^X = \{u \in X : \|u\|_X < r\}$, $B_r^W = \{u \in X : \|u\|_W < r\}$,
- $\bar{B}_r^X = \{u \in X : \|u\|_X \leq r\}$, $\bar{B}_r^W = \{u \in X : \|u\|_W \leq r\}$,
- $S_r^X = \{u \in X : \|u\|_X = r\}$, $S_r^W = \{u \in X : \|u\|_W = r\}$.

Anyway, in order to avoid any ambiguity and simplify, when possible, the notation, from now on by X we denote the space equipped with its given norm $\|\cdot\|_X$ while, if a different norm is involved, we write it explicitly.

For simplicity, taking $\beta \in \mathbb{R}$, we say that a sequence $(u_n)_n \subset X$ is a *Cerami–Palais–Smale sequence at level β* , briefly $(CPS)_\beta$ -sequence, if

$$\lim_{n \rightarrow +\infty} J(u_n) = \beta \quad \text{and} \quad \lim_{n \rightarrow +\infty} \|dJ(u_n)\|_{X'}(1 + \|u_n\|_X) = 0.$$

Moreover, β is a *Cerami–Palais–Smale level*, briefly (CPS) -level, if there exists a $(CPS)_\beta$ -sequence.

As $(CPS)_\beta$ -sequences may exist which are unbounded in $\|\cdot\|_X$ but converge with respect to $\|\cdot\|_W$, we have to weaken the classical Cerami–Palais–Smale condition in a suitable way according to the ideas already developed in previous papers (see, e.g., [?, ?, ?]).

Definition 2.1. The functional J satisfies the *weak Cerami–Palais–Smale condition at level β* ($\beta \in \mathbb{R}$), briefly $(wCPS)_\beta$ condition, if for every $(CPS)_\beta$ -sequence $(u_n)_n$, a point $u \in X$ exists, such that

- (i) $\lim_{n \rightarrow +\infty} \|u_n - u\|_W = 0$ (up to subsequences),
- (ii) $J(u) = \beta$, $dJ(u) = 0$.

If J satisfies the $(wCPS)_\beta$ condition at each level $\beta \in I$, I real interval, we say that J satisfies the $(wCPS)$ condition in I .

Since in [?] a Deformation Lemma has been proved if the functional J satisfies a weaker version of the $(wCPS)_\beta$ condition, namely any (CPS) -level is also a critical level, in particular we can state the following result.

Lemma 2.2 (Deformation Lemma). *Let $J \in C^1(X, \mathbb{R})$ and consider $\beta \in \mathbb{R}$ such that*

- J satisfies the $(wCPS)_\beta$ condition,
- $K_J^\beta = \emptyset$.

Then, fixing any $\bar{\varepsilon} > 0$, there exist a constant $\varepsilon > 0$ and a homeomorphism $\psi : X \rightarrow X$ such that $2\varepsilon < \bar{\varepsilon}$ and

$$(i) \quad \psi(J^{\beta+\varepsilon}) \subset J^{\beta-\varepsilon},$$

$$(ii) \quad \psi(u) = u \text{ for all } u \in X \text{ such that either } J(u) \leq \beta - \bar{\varepsilon} \text{ or } J(u) \geq \beta + \bar{\varepsilon}.$$

Moreover, if J is even on X , then ψ can be chosen odd.

Proof. It is enough to reason as in [?, Lemma 2.3] with $\beta_1 = \beta_2 = \beta$ and to note that the deformation $\psi : X \rightarrow X$ is a homeomorphism. \square

From Lemma ?? we obtain the following generalization of the Mountain Pass Theorem (compare it with [?, Theorem 1.7] and the classical statement in [?, Theorem 2.2]).

Theorem 2.3. *Let $J \in C^1(X, \mathbb{R})$ be such that $J(0) = 0$ and the (wCPS) condition holds in \mathbb{R}_+ . Moreover, assume that there exist a continuous map $\ell : X \rightarrow \mathbb{R}$, some constants $r_0, \varrho_0 > 0$, and $e \in X$ such that*

$$(i) \quad \ell(0) = 0 \quad \text{and} \quad \ell(u) \geq \|u\|_W \quad \text{for all } u \in X;$$

$$(ii) \quad u \in X, \quad \ell(u) = r_0 \quad \implies \quad J(u) \geq \varrho_0;$$

$$(iii) \quad \|e\|_W > r_0 \quad \text{and} \quad J(e) < \varrho_0.$$

Then, J has a Mountain Pass critical point $u_0 \in X$ such that $J(u_0) \geq \varrho_0$.

Furthermore, with the stronger assumption that J is symmetric, the following generalization of the symmetric Mountain Pass Theorem can be stated (see [?, Theorem 1.8] and compare with [?, Theorem 9.12] and [?, Theorem 2.4]).

Theorem 2.4. *Let $J \in C^1(X, \mathbb{R})$ be an even functional such that $J(0) = 0$ and the (wCPS) condition holds in \mathbb{R}_+ . Moreover, assume that $\varrho > 0$ exists so that:*

(\mathcal{H}_ϱ) *three closed subsets V_ϱ, Z_ϱ and \mathcal{M}_ϱ of X and a constant $R_\varrho > 0$ exist which satisfy the following conditions:*

(i) V_ϱ and Z_ϱ are subspaces of X such that

$$V_\varrho + Z_\varrho = X, \quad \text{codim } Z_\varrho < \dim V_\varrho < +\infty;$$

(ii) $\mathcal{M}_\varrho = \partial\mathcal{N}$, where $\mathcal{N} \subset X$ is a neighborhood of the origin which is symmetric and bounded with respect to $\|\cdot\|_W$;

$$(iii) \quad u \in \mathcal{M}_\varrho \cap Z_\varrho \quad \implies \quad J(u) \geq \varrho;$$

$$(iv) \quad u \in V_\varrho, \quad \|u\|_X \geq R_\varrho \quad \implies \quad J(u) \leq 0.$$

Then, if we put

$$\beta_\varrho = \inf_{\gamma \in \Gamma_\varrho} \sup_{u \in V_\varrho} J(\gamma(u)),$$

with

$$\Gamma_\varrho = \{\gamma : X \rightarrow X : \gamma \text{ odd homeomorphism, } \gamma(u) = u \text{ if } u \in V_\varrho \text{ with } \|u\|_X \geq R_\varrho\},$$

the functional J possesses at least a pair of symmetric critical points in X with corresponding critical level β_ϱ which belongs to $[\varrho, \varrho_1]$, where $\varrho_1 \geq \sup_{u \in V_\varrho} J(u) > \varrho$.

Remark 2.5. Since in Theorem ?? the vector space V_ϱ is finite dimensional, then condition $(\mathcal{H}_\varrho)(iv)$ implies that $\sup_{u \in V_\varrho} J(u) < +\infty$, furthermore it still holds if we replace $\|\cdot\|_X$ with $\|\cdot\|_W$.

If we can apply infinitely many times Theorem ??, then the following multiplicity abstract result can be stated.

Corollary 2.6. *Let $J \in C^1(X, \mathbb{R})$ be an even functional such that $J(0) = 0$, the (wCPS) condition holds in \mathbb{R}_+ and assumption (\mathcal{H}_ϱ) holds for all $\varrho > 0$.*

Then, the functional J possesses a sequence of critical points $(u_n)_n \subset X$ such that $J(u_n) \nearrow +\infty$ as $n \nearrow +\infty$.

The proof of Theorem ?? is obtained reasoning as in [?, Theorem 1.8] by using Lemma ?? and the following result.

Lemma 2.7 (Intersection Lemma). *Let V, Z and \mathcal{M} be closed subsets of X which satisfy conditions (i) and (ii) in Theorem ?. Fixing $R > 0$ and defining*

$$\Gamma_R = \{\gamma : X \rightarrow X : \gamma \text{ odd homeomorphism, } \gamma(u) = u \text{ if } u \in V \text{ with } \|u\|_X \geq R\},$$

then

$$\gamma(V) \cap \mathcal{M} \cap Z \neq \emptyset \quad \text{for all } \gamma \in \Gamma_R.$$

Proof. Fixing any $\gamma \in \Gamma_R$, for simplicity we denote $Q = \gamma(V) \cap \mathcal{M} \cap Z$. It is enough to prove that

$$i_2(Q) \geq \dim V - \text{codim } Z \geq 1, \quad (2.2)$$

where $i_2(\cdot)$ is the Krasnoselskii genus (see, e.g., [?, Section II.5]).

In order to prove (??), firstly let us point out that hypotheses (i) and (ii) imply that Q is symmetric with respect to the origin but $0 \notin Q$. Moreover, Q is compact in X . In fact, we have $V = (V \cap \bar{B}_R^X) \cup (V \setminus B_R^X)$, with $V \cap \bar{B}_R^X$ compact (as $\dim V < +\infty$) and $\gamma(V \setminus B_R^X) = V \setminus B_R^X$ (by the definition of Γ_R). Hence, $Q = (\gamma(V \cap \bar{B}_R^X) \cap \mathcal{M} \cap Z) \cup ((V \setminus B_R^X) \cap \mathcal{M} \cap Z)$ is compact because $\gamma(V \cap \bar{B}_R^X) \cap \mathcal{M} \cap Z$ is compact (as closed subset of the compact set $\gamma(V \cap \bar{B}_R^X)$) and $(V \setminus B_R^X) \cap \mathcal{M} \cap Z$ is compact, too, as closed and bounded in the finite dimensional space V (since \mathcal{M} is bounded in $\|\cdot\|_W$ but in V the norms $\|\cdot\|_X$ and $\|\cdot\|_W$ are equivalent).

Then, by the continuity, monotonicity and subadditivity properties of the genus, an open neighborhood U of Q in X exists such that

$$i_2(Q) = i_2(\bar{U}) \geq i_2(\gamma(V) \cap \mathcal{M} \cap \bar{U}) \geq i_2(\gamma(V) \cap \mathcal{M}) - i_2(\gamma(V) \cap (\mathcal{M} \setminus U)). \quad (2.3)$$

Now, denoting by V^* the complement of Z , from hypothesis (i) it follows that $V^* \subset V$; furthermore, it has to be $\gamma(V) \cap (\mathcal{M} \setminus U) \subset V^* \setminus \{0\}$, hence

$$i_2(\gamma(V) \cap (\mathcal{M} \setminus U)) \leq \dim V^* = \text{codim } Z. \quad (2.4)$$

On the other hand, since γ is an odd homeomorphism on X , assumption (ii) implies that the set $V \cap \gamma^{-1}(\mathcal{M})$ is the boundary of a bounded symmetric neighborhood of the origin in V . Then, from [?, Proposition 5.2] we have

$$i_2(\gamma(V) \cap \mathcal{M}) = i_2(V \cap \gamma^{-1}(\mathcal{M})) = \dim V,$$

which, together with (??) and (??), implies (??). \square

3 Variational setting and first properties

From now on, let $\Omega \subset \mathbb{R}^N$ be an open bounded domain, $N \geq 2$, so we denote by:

- $L^q(\Omega)$ the Lebesgue space with norm $|u|_q = (\int_{\Omega} |u|^q dx)^{1/q}$ if $1 \leq q < +\infty$;
- $L^\infty(\Omega)$ the space of Lebesgue-measurable and essentially bounded functions $u : \Omega \rightarrow \mathbb{R}$ with norm

$$|u|_\infty = \operatorname{ess\,sup}_{\Omega} |u|;$$

- $W_0^{1,p}(\Omega)$ the classical Sobolev space with norm $\|u\|_W = |\nabla u|_p$ if $1 \leq p < +\infty$;
- $|C|$ the usual Lebesgue measure of a measurable set C in \mathbb{R}^N .

From now on, let $A : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ and $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be such that, considering the notation in (??), the following conditions hold:

- (H_0) $A(x, t, \xi)$ is a C^1 Carathéodory function, i.e.,
 $A(\cdot, t, \xi) : x \in \Omega \mapsto A(x, t, \xi) \in \mathbb{R}$ is measurable for all $(t, \xi) \in \mathbb{R} \times \mathbb{R}^N$,
 $A(x, \cdot, \cdot) : (t, \xi) \in \mathbb{R} \times \mathbb{R}^N \mapsto A(x, t, \xi) \in \mathbb{R}$ is C^1 for a.e. $x \in \Omega$;

- (H_1) a real number $p > 1$ and some positive continuous functions $\Phi_i, \phi_i : \mathbb{R} \rightarrow \mathbb{R}$, $i \in \{1, 2\}$, exist such that

$$\begin{aligned} |A_t(x, t, \xi)| &\leq \Phi_1(t) + \phi_1(t) |\xi|^p && \text{a.e. in } \Omega, \text{ for all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N, \\ |a(x, t, \xi)| &\leq \Phi_2(t) + \phi_2(t) |\xi|^{p-1} && \text{a.e. in } \Omega, \text{ for all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N; \end{aligned}$$

- (G_0) $g(x, t)$ is a Carathéodory function, i.e.,
 $g(\cdot, t) : x \in \Omega \mapsto g(x, t) \in \mathbb{R}$ is measurable for all $t \in \mathbb{R}$;
 $g(x, \cdot) : t \in \mathbb{R} \mapsto g(x, t) \in \mathbb{R}$ is continuous for a.e. $x \in \Omega$;

- (G_1) $a_1, a_2 > 0$ and $q \geq 1$ exist such that

$$|g(x, t)| \leq a_1 + a_2 |t|^{q-1} \quad \text{a.e. in } \Omega, \text{ for all } t \in \mathbb{R}.$$

Remark 3.1. From (G_1) it follows that there exist $a_3, a_4 > 0$ such that

$$|G(x, t)| \leq a_3 + a_4 |t|^q \quad \text{a.e. in } \Omega, \text{ for all } t \in \mathbb{R}. \quad (3.1)$$

We note that, unlike assumption (G_1) in [?], no upper bound on q is actually required.

In order to investigate the existence of weak solutions of the nonlinear problem (??), the notation introduced for the abstract setting at the beginning of Section ?? is referred to our problem with $W = W_0^{1,p}(\Omega)$ and the Banach space $(X, \|\cdot\|_X)$ defined as

$$X := W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \quad \|u\|_X = \|u\|_W + |u|_\infty \quad (3.2)$$

(here and in the following, $|\cdot|$ denotes the standard norm on any Euclidean space as the dimension of the considered vector is clear and no ambiguity arises).

Moreover, from the Sobolev Embedding Theorem, for any $r \in [1, p^*[$, $p^* = \frac{pN}{N-p}$ as $N > p$, a constant $\sigma_r > 0$ exists, such that

$$|u|_r \leq \sigma_r \|u\|_W \quad \text{for all } u \in W_0^{1,p}(\Omega)$$

and the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$ is compact.

From the definition of X , we have that $X \hookrightarrow W_0^{1,p}(\Omega)$ and $X \hookrightarrow L^\infty(\Omega)$ with continuous embeddings, and (??) holds with $\sigma_0 = 1$. If $p > N$ then $X = W_0^{1,p}(\Omega)$, as $W_0^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$; hence, classical Mountain Pass Theorems in [?] can be used.

Now, we consider the functional $\mathcal{J} : X \rightarrow \mathbb{R}$ defined as

$$\mathcal{J}(u) = \int_{\Omega} A(x, u, \nabla u) dx - \int_{\Omega} G(x, u) dx, \quad u \in X. \quad (3.3)$$

Taking any $u, v \in X$, by direct computations it follows that its Gâteaux differential in u along the direction v is

$$\langle d\mathcal{J}(u), v \rangle = \int_{\Omega} (a(x, u, \nabla u) \cdot \nabla v + A_t(x, u, \nabla u)v) dx - \int_{\Omega} g(x, u)v dx. \quad (3.4)$$

The following proposition extends [?, Proposition 3.1] in which the regularity of \mathcal{J} is stated only if $G(x, t)$ has a subcritical growth.

Proposition 3.2. *Let us assume that conditions (H_0) – (H_1) , (G_0) – (G_1) hold and two positive continuous functions $\Phi_0, \phi_0 : \mathbb{R} \rightarrow \mathbb{R}$ exist such that*

$$|A(x, t, \xi)| \leq \Phi_0(t) + \phi_0(t) |\xi|^p \quad \text{a.e. in } \Omega, \text{ for all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N. \quad (3.5)$$

If $(u_n)_n \subset X$, $u \in X$ are such that

$$\|u_n - u\|_W \rightarrow 0, \quad u_n \rightarrow u \text{ a.e. in } \Omega \quad \text{if } n \rightarrow +\infty \quad (3.6)$$

$$\text{and } M > 0 \text{ exists so that } |u_n|_{\infty} \leq M \text{ for all } n \in \mathbb{N}, \quad (3.7)$$

then

$$\mathcal{J}(u_n) \rightarrow \mathcal{J}(u) \quad \text{and} \quad \|d\mathcal{J}(u_n) - d\mathcal{J}(u)\|_{X'} \rightarrow 0 \quad \text{if } n \rightarrow +\infty.$$

Hence, \mathcal{J} is a C^1 functional on X with Fréchet differential defined as in (??).

Proof. As in the first part of the proof of [?, Proposition 3.1], from assumptions (H_0) – (H_1) and (??) the functional

$$\mathcal{A} : u \in X \mapsto \mathcal{A}(u) = \int_{\Omega} A(x, u, \nabla u) dx \in \mathbb{R}$$

is such that $\mathcal{A}(u_n) \rightarrow \mathcal{A}(u)$ and $\|d\mathcal{A}(u_n) - d\mathcal{A}(u)\|_{X'} \rightarrow 0$, with

$$\langle d\mathcal{A}(u), v \rangle = \int_{\Omega} a(x, u, \nabla u) \cdot \nabla v dx + \int_{\Omega} A_t(x, u, \nabla u)v dx, \quad u, v \in X.$$

On the other hand, from (G_0) and (??) it follows that $G(x, u_n) \rightarrow G(x, u)$ and $g(x, u_n) \rightarrow g(x, u)$ a.e. in Ω , then (G_1) , (??), (??) and Lebesgue's Dominated Convergence Theorem imply that also the functional

$$\mathcal{G} : u \in X \mapsto \mathcal{G}(u) = \int_{\Omega} G(x, u) dx \in \mathbb{R}$$

is such that $\mathcal{G}(u_n) \rightarrow \mathcal{G}(u)$ and $\|d\mathcal{G}(u_n) - d\mathcal{G}(u)\|_{X'} \rightarrow 0$, with

$$\langle d\mathcal{G}(u), v \rangle = \int_{\Omega} g(x, u) v dx \quad \text{for all } u, v \in X.$$

Then, the conclusion follows. \square

In order to prove more properties of the functional \mathcal{J} in (??), we require that some constants $\alpha_i > 0$, $i \in \{1, 2, 3\}$, $\eta_j > 0$, $j \in \{1, 2\}$, and $s \geq 0$, $\mu > p$, $R_0 \geq 1$, exist such that the following hypotheses are satisfied:

$$(H_2) \quad A(x, t, \xi) \leq \eta_1 a(x, t, \xi) \cdot \xi \quad \text{a.e. in } \Omega \text{ if } |(t, \xi)| \geq R_0;$$

$$(H_3) \quad |A(x, t, \xi)| \leq \eta_2 \quad \text{a.e. in } \Omega \text{ if } |(t, \xi)| \leq R_0;$$

$$(H_4) \quad a(x, t, \xi) \cdot \xi \geq \alpha_1 (1 + |t|^{ps}) |\xi|^p \quad \text{a.e. in } \Omega, \text{ for all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N;$$

$$(H_5) \quad a(x, t, \xi) \cdot \xi + A_t(x, t, \xi)t \geq \alpha_2 a(x, t, \xi) \cdot \xi \quad \text{a.e. in } \Omega \text{ if } |(t, \xi)| \geq R_0;$$

$$(H_6) \quad \mu A(x, t, \xi) - a(x, t, \xi) \cdot \xi - A_t(x, t, \xi)t \geq \alpha_3 a(x, t, \xi) \cdot \xi \quad \text{a.e. in } \Omega \text{ if } |(t, \xi)| \geq R_0;$$

$$(H_7) \quad \text{for all } \xi, \xi^* \in \mathbb{R}^N, \xi \neq \xi^*, \text{ it is}$$

$$[a(x, t, \xi) - a(x, t, \xi^*)] \cdot [\xi - \xi^*] > 0 \quad \text{a.e. in } \Omega, \text{ for all } t \in \mathbb{R};$$

$$(G_2) \quad g(x, t) \text{ satisfies the Ambrosetti–Rabinowitz condition, i.e.}$$

$$0 < \mu G(x, t) \leq g(x, t)t \quad \text{for a.e. } x \in \Omega \text{ if } |t| \geq R_0.$$

Remark 3.3. If in (H_5) we take $t = 0$ and $|\xi| \geq R_0$, we deduce that $\alpha_2 \leq 1$. Moreover, from hypotheses (H_5) and (H_6) it follows that

$$\mu A(x, t, \xi) \geq (\alpha_2 + \alpha_3) a(x, t, \xi) \cdot \xi \quad \text{a.e. in } \Omega \text{ if } |(t, \xi)| \geq R_0; \quad (3.8)$$

hence, if also (H_4) holds, for a.e. $x \in \Omega$ we have that

$$A(x, t, \xi) \geq \alpha_1 \frac{\alpha_2 + \alpha_3}{\mu} (1 + |t|^{ps}) |\xi|^p \geq 0 \quad \text{if } |(t, \xi)| \geq R_0. \quad (3.9)$$

Thus, from (??) and (H_3) , for a.e. $x \in \Omega$ we obtain that

$$A(x, t, \xi) \geq \alpha_1 \frac{\alpha_2 + \alpha_3}{\mu} (1 + |t|^{ps}) |\xi|^p - \eta_3 \quad \text{for all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N \quad (3.10)$$

for a suitable $\eta_3 > 0$.

Remark 3.4. From (H_1) – (H_6) , since (??) is verified, then

$$|A(x, t, \xi)| \leq \eta_1 (\Phi_2(t) + \phi_2(t)) |\xi|^p + \eta_1 \Phi_2(t) + \eta_2 \quad (3.11)$$

a.e. in Ω , for all $(t, \xi) \in \mathbb{R} \times \mathbb{R}^N$. Whence, the growth condition (??) holds and Proposition ?? applies.

Remark 3.5. With respect to estimate (??), more precise growth conditions on $A(x, t, \xi)$ can be deduced. In fact, taken $|(t, \xi)| \geq R_0$, hypotheses (H_2) and (H_6) imply

$$\mu A(x, t, \xi) \geq \frac{1 + \alpha_3}{\eta_1} A(x, t, \xi) + A_t(x, t, \xi)t \quad \text{a.e. in } \Omega.$$

Hence, we have

$$\left(\mu - \frac{1 + \alpha_3}{\eta_1}\right) A(x, t, \xi) \geq A_t(x, t, \xi)t \quad \text{a.e. in } \Omega \text{ if } |(t, \xi)| \geq R_0, \quad (3.12)$$

where, without loss of generality, just taking η_1 large enough, we can always have

$$\mu > \frac{1 + \alpha_3}{\eta_1}.$$

Thus, by means of (??), (??) and (??), direct calculations allow one to prove the existence of a constant $\eta_4 > 0$ so that

$$A(x, t, \xi) \leq \eta_4 |t|^{\mu - \frac{1 + \alpha_3}{\eta_1}} |\xi|^p \quad \text{a.e. in } \Omega, \text{ if } |t| \geq 1 \text{ and } |\xi| \geq R_0. \quad (3.13)$$

Whence, (??) and (??) imply

$$a(x, t, \xi) \cdot \xi \leq \frac{\eta_4 \mu}{\alpha_2 + \alpha_3} |t|^{\mu - \frac{1 + \alpha_3}{\eta_1}} |\xi|^p \quad \text{a.e. in } \Omega, \text{ if } |t| \geq 1 \text{ and } |\xi| \geq R_0. \quad (3.14)$$

At last, (H_4) and (??) imply that

$$0 \leq ps \leq \mu - \frac{1 + \alpha_3}{\eta_1}. \quad (3.15)$$

We note that, if

$$0 \leq s < \frac{\mu}{p}, \quad (3.16)$$

then, without loss of generality, we can always choose η_1 in (H_2) large enough so that (??) holds.

Remark 3.6. In the model case $A(x, t, \xi) = \frac{1}{p} \bar{A}(x, t) |\xi|^p$ conditions (H_2) and (H_7) are trivially verified, so the set of assumptions reduce to the following one:

$(H_0)'$ $\bar{A}(x, t)$ is a C^1 Carathéodory function in $\Omega \times \mathbb{R}$;

$(H_1)'$ two positive continuous functions $\Phi_i : \mathbb{R} \rightarrow \mathbb{R}$, $i \in \{1, 2\}$, exist such that

$$|\bar{A}_t(x, t)| \leq \Phi_1(t), \quad |\bar{A}(x, t)| \leq \Phi_2(t) \quad \text{a.e. in } \Omega, \text{ for all } t \in \mathbb{R};$$

$(H_4)'$ $\bar{A}(x, t) \geq \alpha_1(1 + |t|^{ps})$ a.e. in Ω , for all $t \in \mathbb{R}$;

$(H_5)'$ $\bar{A}(x, t) + \frac{1}{p} \bar{A}_t(x, t)t \geq \alpha_2 \bar{A}(x, t)$ a.e. in Ω if $|t| \geq R_0$;

$(H_6)'$ $\left(\frac{\mu}{p} - 1\right) \bar{A}(x, t) - \frac{1}{p} \bar{A}_t(x, t)t \geq \alpha_3 \bar{A}(x, t)$ a.e. in Ω if $|t| \geq R_0$.

In particular, if we consider $\bar{A}(x, t) = A_1(x) + A_2(x)|t|^{ps}$ as in (??), the previous hypotheses hold if $A_1, A_2 \in L^\infty(\Omega)$ are such that (??) is satisfied and

$$2 < 1 + p < p(s + 1) < \mu. \quad (3.17)$$

Remark 3.7. Conditions (G_0) and (G_2) imply that a function $\eta \in L^\infty(\Omega)$, $\eta(x) > 0$ a.e. in Ω , and a constant $a_5 \geq 0$ exist such that

$$G(x, t) \geq \eta(x) |t|^\mu - a_5 \quad \text{a.e. in } \Omega, \text{ for all } t \in \mathbb{R}. \quad (3.18)$$

Hence, if also (G_1) holds, from (??), (??) and (??) it follows

$$ps < \mu \leq q.$$

If the assumptions in this section hold with $s = 0$ in (H_4) and $q < p^*$ in (G_1) , from [?, Proposition 4.6] it follows that the functional \mathcal{J} in (??) satisfies the *(wCPS)* condition in \mathbb{R} . Here, in order to extend such a result to the case $s > 0$, and then considering $G(x, t)$ with a critical or supercritical growth, we need the following application of the Rellich Embedding Theorem.

Lemma 3.8. *Taking $1 < p < N$ and $s \geq 0$, let $(u_n)_n \subset X$ be a sequence such that*

$$\left(\int_{\Omega} (1 + |u_n|^{ps}) |\nabla u_n|^p dx \right)_n \quad \text{is bounded.} \quad (3.19)$$

Then, $u \in W_0^{1,p}(\Omega)$ exists such that $|u|^s u \in W_0^{1,p}(\Omega)$, too, and, up to subsequences, if $n \rightarrow +\infty$ we have

$$u_n \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega), \quad (3.20)$$

$$|u_n|^s u_n \rightharpoonup |u|^s u \text{ weakly in } W_0^{1,p}(\Omega), \quad (3.21)$$

$$u_n \rightarrow u \text{ a.e. in } \Omega, \quad (3.22)$$

$$u_n \rightarrow u \text{ strongly in } L^r(\Omega) \text{ for each } r \in [1, p^*(s + 1)]. \quad (3.23)$$

Proof. Firstly, we note that

$$|\nabla(|u|^s u)|^p = (s + 1)^p |u|^{ps} |\nabla u|^p \quad \text{a.e. in } \Omega \text{ for all } u \in X, \quad (3.24)$$

then from (??) the sequences $(u_n)_n$ and $(|u_n|^s u_n)_n$ are bounded in $W_0^{1,p}(\Omega)$; hence, $u, v \in W_0^{1,p}(\Omega)$ exist such that, up to subsequences, we have (??), (??), (??) with $r < p^*$, and also $|u_n|^s u_n \rightharpoonup v$ weakly in $W_0^{1,p}(\Omega)$ and $|u_n|^s u_n \rightarrow v$ a.e. in Ω . Thus, $v = |u|^s u$ and (??) holds.

At last, if $s > 0$, (??) holds also if $p^* \leq r < p^*(s + 1)$ from interpolation as $u \in L^{p^*(s+1)}(\Omega)$ and $(u_n)_n$ is bounded in $L^{p^*(s+1)}(\Omega)$. \square

Now, we recall a particular version of [?, Theorem II.5.1] which we will use for proving the boundedness of the weak limit of a *(CPS)*-sequence (see [?, Lemma 4.5]).

Lemma 3.9. *Let p, r be so that $1 < p \leq r < p^*$, $p < N$ and take $v \in W_0^{1,p}(\Omega)$. Assume that $\bar{a} > 0$ and $k_0 \in \mathbb{N}$ exist such that the inequality*

$$\int_{\Omega_k^+} |\nabla v|^p dx \leq \bar{a} \left(|\Omega_k^+| + \int_{\Omega_k^+} v^r dx \right)$$

holds for all $k \geq k_0$, with $\Omega_k^+ = \{x \in \Omega : v(x) > k\}$. Then, $\text{ess sup}_\Omega v$ is bounded from above by a positive constant which can be chosen so that it depends only on $|\Omega|$, N , p , r , \bar{a} , k_0 , $|v|_{p^*}$.

Now, we are ready to prove that \mathcal{J} satisfies the weak Cerami–Palais–Smale condition in X . If $1 < p < N$, this new result extends [?, Proposition 4.6] where the exponent q in (G_1) is subcritical, i.e., $q < p^*$. On the contrary, here we assume the weaker condition

$$q < p^*(s + 1). \quad (3.25)$$

Hence, without loss of generality, we can always assume q large enough such that

$$p(s + 1) < q < p^*(s + 1). \quad (3.26)$$

Proposition 3.10. *Assume that hypotheses (H_0) – (H_7) , (G_0) – (G_2) and (??) hold with $1 < p < N$. Then, the functional \mathcal{J} satisfies the (wCPS) condition in \mathbb{R} .*

Proof. Let $\beta \in \mathbb{R}$ be fixed and consider a $(CPS)_\beta$ -sequence $(u_n)_n \subset X$, i.e.,

$$\mathcal{J}(u_n) \rightarrow \beta \quad \text{and} \quad \|d\mathcal{J}(u_n)\|_{X'}(1 + \|u_n\|_X) \rightarrow 0. \quad (3.27)$$

We divide our proof in the following steps:

1. $(u_n)_n$ is bounded in $W_0^{1,p}(\Omega)$, or more precisely (??) holds; thus from Lemma ?? a function $u \in W_0^{1,p}(\Omega)$ exists such that $|u|^s u \in W_0^{1,p}(\Omega)$ and (??)–(??) hold, up to subsequences;
2. $u \in L^\infty(\Omega)$;
3. if $k \geq \max\{|u|_\infty, R_0\} + 1$ ($R_0 \geq 1$ as in the set of hypotheses) then

$$\mathcal{J}(T_k u_n) \rightarrow \beta \quad \text{and} \quad \|d\mathcal{J}(T_k u_n)\|_{X'} \rightarrow 0,$$

where $T_k : \mathbb{R} \rightarrow \mathbb{R}$ is the truncation function defined as

$$T_k t = \begin{cases} t & \text{if } |t| \leq k \\ k \frac{t}{|t|} & \text{if } |t| > k \end{cases};$$

4. $\|T_k u_n - u\|_W \rightarrow 0$ if $n \rightarrow +\infty$, then $\|u_n - u\|_W \rightarrow 0$ if $n \rightarrow +\infty$, too;
5. $\mathcal{J}(u) = \beta$ and $d\mathcal{J}(u) = 0$.

For simplicity, here and in the following we will use the notation $(\varepsilon_n)_n$ for any infinitesimal sequence depending only on $(u_n)_n$ while d_i will denote any strictly positive constant independent of n .

Step 1. From (??), (??), (??), together with (H_1) , (H_3) , (H_6) , (??), (G_1) , (G_2) , by reasoning as in the proof of Step 1 in [?, Proposition 4.6] and using hypothesis (H_4) we have that

$$\begin{aligned} \mu\beta + \varepsilon_n &= \mu\mathcal{J}(u_n) - \langle d\mathcal{J}(u_n), u_n \rangle \geq \alpha_3 \int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla u_n dx - d_1 \\ &\geq \alpha_1 \alpha_3 \int_\Omega (1 + |u_n|^{ps}) |\nabla u_n|^p dx - d_1 \end{aligned}$$

which implies (??).

Step 2. Arguing by contradiction, let us assume that

$$\operatorname{ess\,sup}_{\Omega} u = +\infty; \quad (3.28)$$

thus, taking any $k \in \mathbb{N}$, $k > R_0$ ($R_0 \geq 1$ as in the hypotheses), we have that

$$|\Omega_k^+| > 0 \quad \text{with} \quad \Omega_k^+ = \{x \in \Omega : u(x) > k\}. \quad (3.29)$$

Now, for any $\tilde{k} > 0$ consider the new function $R_k^+ : t \in \mathbb{R} \rightarrow R_k^+ t \in \mathbb{R}$ such that

$$R_k^+ t = \begin{cases} 0 & \text{if } t \leq \tilde{k} \\ t - \tilde{k} & \text{if } t > \tilde{k} \end{cases}.$$

Taking $\tilde{k} = k^{s+1}$, from (??) it follows that

$$R_{k^{s+1}}^+ (|u_n|^s u_n) \rightharpoonup R_{k^{s+1}}^+ (|u|^s u) \quad \text{weakly in } W_0^{1,p}(\Omega);$$

then, the weak lower semicontinuity of $\|\cdot\|_W$ implies

$$\int_{\Omega} |\nabla R_{k^{s+1}}^+ (|u|^s u)|^p dx \leq \liminf_{n \rightarrow +\infty} \int_{\Omega} |\nabla R_{k^{s+1}}^+ (|u_n|^s u_n)|^p dx,$$

i.e.,

$$\int_{\Omega_k^+} |\nabla (u^{s+1})|^p dx \leq \liminf_{n \rightarrow +\infty} \int_{\Omega_{n,k}^+} |\nabla (u_n^{s+1})|^p dx \quad (3.30)$$

as $|t|^{st} > k^{s+1} \iff t > k$, with $\Omega_{n,k}^+ = \{x \in \Omega : u_n(x) > k\}$.

On the other hand, from $\|R_k^+ u_n\|_X \leq \|u_n\|_X$, (??) and (??) it follows that $n_k \in \mathbb{N}$ exists so that

$$|\langle d\mathcal{J}(u_n), R_k^+ u_n \rangle| < |\Omega_k^+| \quad \text{for all } n \geq n_k. \quad (3.31)$$

From (??), (H_5) with $\alpha_2 \leq 1$ (see Remark ??), (H_4) , (??), we have that

$$\begin{aligned} \langle d\mathcal{J}(u_n), R_k^+ u_n \rangle &= \int_{\Omega_{n,k}^+} \left(1 - \frac{k}{u_n}\right) (a(x, u_n, \nabla u_n) \cdot \nabla u_n + A_t(x, u_n, \nabla u_n) u_n) dx \\ &\quad + \int_{\Omega_{n,k}^+} \frac{k}{u_n} a(x, u_n, \nabla u_n) \cdot \nabla u_n dx - \int_{\Omega} g(x, u_n) R_k^+ u_n dx \\ &\geq \alpha_2 \int_{\Omega_{n,k}^+} a(x, u_n, \nabla u_n) \cdot \nabla u_n dx - \int_{\Omega} g(x, u_n) R_k^+ u_n dx \\ &\geq \alpha_1 \alpha_2 \int_{\Omega_{n,k}^+} u_n^{ps} |\nabla u_n|^p dx - \int_{\Omega} g(x, u_n) R_k^+ u_n dx \\ &= \frac{\alpha_1 \alpha_2}{(s+1)^p} \int_{\Omega_{n,k}^+} |\nabla (u_n^{s+1})|^p dx - \int_{\Omega} g(x, u_n) R_k^+ u_n dx. \end{aligned}$$

Thus, from (??) it follows that

$$\int_{\Omega_{n,k}^+} |\nabla(u_n^{s+1})|^p dx \leq \frac{(s+1)^p}{\alpha_1 \alpha_2} \left(|\Omega_k^+| + \int_{\Omega} g(x, u_n) R_k^+ u_n dx \right).$$

Now, from (G_1) , (??) and (??) it results

$$\int_{\Omega} g(x, u_n) R_k^+ u_n dx \rightarrow \int_{\Omega} g(x, u) R_k^+ u dx;$$

hence, by passing to the lower limit, (??) implies

$$\int_{\Omega_k^+} |\nabla(u^{s+1})|^p dx \leq \frac{(s+1)^p}{\alpha_1 \alpha_2} \left(|\Omega_k^+| + \int_{\Omega} g(x, u) R_k^+ u dx \right).$$

Therefore, as in Ω_k^+ it is $u > 1$, from (G_1) and direct computations it follows that

$$\int_{\Omega_k^+} |\nabla(u^{s+1})|^p dx \leq d_2 \left(|\Omega_k^+| + \int_{\Omega_k^+} u^q dx \right). \quad (3.32)$$

At last, if we set $v = |u|^s u$, as $v \in W_0^{1,p}(\Omega)$ and $\Omega_k^+ = \{x \in \Omega : v(x) > k^{s+1}\}$ (in particular, $v = u^{s+1}$ in Ω_k^+), from (??) we obtain

$$\int_{\Omega_k^+} |\nabla v|^p dx \leq d_2 \left(|\Omega_k^+| + \int_{\Omega_k^+} v^{\frac{q}{s+1}} dx \right).$$

Then, from (??) Lemma ?? applies and $\text{ess sup } v < +\infty$ in contradiction to (??). Similar arguments apply if $\text{ess sup } (-u) = +\infty$. Hence, $u \in L^\infty(\Omega)$.

Step 3. The proof can be obtained reasoning as in the proof of Step 3 in [?, Proposition 4.6] but using (??) and (??) instead of [?, (4.15)].

Steps 4, 5. The proofs are as in the corresponding steps of the proof of [?, Proposition 4.6]. \square

At last, in order to prove a multiplicity result, we introduce a suitable decomposition of X .

If $p = 2$, we deal with the Hilbert space $H_0^1(\Omega)$ so the classical choice is to consider the sequence of the eigenvalues of $-\Delta$ on Ω , with homogeneous Dirichlet data, and their (bounded) eigenfunctions, so that, for each $n \geq 1$, the Banach space X can be decomposed into the closed subspace spanned by the first n of such eigenfunctions and the corresponding complement (for the model problem in this case, see [?]).

More in general, if $p > 1$ and $p \neq 2$, $W_0^{1,p}(\Omega)$ is just a reflexive Banach space and a ‘‘canonical’’ decomposition is not known. Anyway, as in [?, Section 5], a sequence of positive numbers $(\lambda_j)_j$ exists such that

- $0 < \lambda_1 < \lambda_2 \leq \dots \leq \lambda_j \leq \dots$ and $\lambda_j \nearrow +\infty$ as $j \rightarrow +\infty$;
- for each $j \in \mathbb{N}$ a function $\varphi_j \in W_0^{1,p}(\Omega)$ exists such that $|\varphi_j|_p = 1$, $\|\varphi_j\|_W = \lambda_j$ and $\varphi_i \neq \varphi_j$ if $i \neq j$;

- $\lambda_1 > 0$ is the first eigenvalue of $-\Delta_p$ in $W_0^{1,p}(\Omega)$ such that

$$\lambda_1 \int_{\Omega} |w|^p dx \leq \int_{\Omega} |\nabla w|^p dx \quad \text{for all } w \in W_0^{1,p}(\Omega) \quad (3.33)$$

and $\varphi_1 \in W_0^{1,p}(\Omega)$ is the unique corresponding eigenfunction such that $\varphi_1 > 0$, $|\varphi_1|_p = 1$ and $\|\varphi_1\|_W = \lambda_1$ (see, e.g., [?]);

- $\varphi_j \in L^\infty(\Omega)$ for each $j \in \mathbb{N}$;
- the sequence $(\varphi_j)_j$ generates the whole space $W_0^{1,p}(\Omega)$.

Moreover, fixing any $n \in \mathbb{N}$ and defining

$$V_n = \text{span}\{\varphi_1, \dots, \varphi_n\} = \{v \in W_0^{1,p}(\Omega) : \exists \beta_1, \dots, \beta_n \in \mathbb{R} \text{ s.t. } v = \sum_{i=1}^n \beta_i \varphi_i\},$$

a closed subspace W_n exists such that

$$W_0^{1,p}(\Omega) = V_n + W_n, \quad V_n \cap W_n = \{0\},$$

and

$$\lambda_{n+1} \int_{\Omega} |w|^p dx \leq \int_{\Omega} |\nabla w|^p dx \quad \text{for all } w \in W_n. \quad (3.34)$$

Then, V_n is a closed subspace of X , too, and we have that

$$X = V_n + W_n^X \quad \text{and} \quad V_n \cap W_n^X = \{0\}, \quad \text{with } W_n^X = W_n \cap L^\infty(\Omega), \quad (3.35)$$

whence,

$$\text{codim}W_n^X = \dim V_n = n. \quad (3.36)$$

4 Existence and multiplicity results

Finally, we can state our main theorems.

Theorem 4.1. *Assume that (H_0) – (H_7) , (G_0) – (G_2) and (??) hold. If, furthermore, $\alpha_4 > 0$ exists such that*

$$(H_8) \quad A(x, t, \xi) \geq \alpha_4(1 + |t|^{ps})|\xi|^p \quad \text{a.e. in } \Omega, \text{ for all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N;$$

$$(G_3) \quad \limsup_{t \rightarrow 0} \frac{g(x, t)}{|t|^{p-2}t} < p\alpha_4\lambda_1 \quad \text{uniformly with respect to a.e. } x \in \Omega, \text{ where } \lambda_1 \text{ is the first eigenvalue of } -\Delta_p \text{ in } W_0^{1,p}(\Omega),$$

then the functional \mathcal{J} defined in (??) possesses at least one nontrivial critical point, i.e., problem (??) admits at least a weak bounded nontrivial solution.

Remark 4.2. We note that the estimate in hypothesis (H_8) follows from (H_4) – (H_6) if $|(t, \xi)| \geq R_0$ (see inequality (??)). Here, we need such an estimate also for $|(t, \xi)| \leq R_0$.

Theorem 4.3. *Assume that (H_0) – (H_7) , (G_0) – (G_2) and $(??)$ hold. Moreover, if $A(x, \cdot, \cdot)$ is even and $g(x, \cdot)$ is odd for a.e. $x \in \Omega$, then functional \mathcal{J} in $(??)$ possesses a sequence of critical points $(u_n)_n \subset X$ such that $\mathcal{J}(u_n) \nearrow +\infty$, i.e., problem $(??)$ admits infinitely many weak bounded solutions.*

We note that if $s = 0$ in (H_4) and (H_8) or if $p \geq N$, then Theorems $??$ and $??$ have already been proved in $[?]$. So, here we consider $s > 0$ and $1 < p < N$.

Firstly, we define

$$\ell_{W,s}(u) = \max\{\|u\|_W, \| |u|^s u \|_W\} \quad \text{for all } u \in X. \quad (4.1)$$

Remark 4.4. From $(??)$ it follows that the map $u \mapsto \| |u|^s u \|_W$ is well-defined and continuous in $(X, \|\cdot\|_X)$; thus, also $\ell_{W,s} : X \rightarrow \mathbb{R}$ is continuous with respect to $\|\cdot\|_X$.

From now on, assume that (H_0) – (H_7) , (G_0) – (G_2) and $(??)$ hold.

In order to prove that \mathcal{J} satisfies some suitable geometric conditions, we need the following lemmas.

Proposition 4.5. *Fixing any $\varrho \in \mathbb{R}$ there exist $n \in \mathbb{N}$, $n = n(\varrho)$, and $r_n > 0$ such that*

$$u \in W_n^X, \quad \ell_{W,s}(u) = r_n \quad \implies \quad \mathcal{J}(u) \geq \varrho, \quad (4.2)$$

where the subspace W_n^X is as in $(??)$.

Proof. Firstly, taking any $u \in X$ we note that $(??)$ and $(??)$ imply

$$\mathcal{J}(u) \geq \alpha_1 \frac{\alpha_2 + \alpha_3}{\mu} \int_{\Omega} (1 + |u|^{ps}) |\nabla u|^p dx - a_4 \int_{\Omega} |u|^q dx - (\eta_3 + a_3) |\Omega|, \quad (4.3)$$

while from $(??)$ and $(??)$ it follows that

$$\int_{\Omega} (1 + |u|^{ps}) |\nabla u|^p dx \geq \frac{1}{(s+1)^p} [\ell_{W,s}(u)]^p. \quad (4.4)$$

On the other hand, from $(??)$, a constant $r > 0$ exists such that $\frac{r}{p} + \frac{q-r}{p^*(s+1)} = 1$, so, classical interpolation arguments apply and we have

$$|u|_q^q \leq |u|_{p^*(s+1)}^{q-r} |u|_p^r, \quad (4.5)$$

where, by the Sobolev Embedding Theorem and $(??)$, one has

$$|u|_{p^*(s+1)}^{q-r} = \| |u|^s u \|_{p^*}^{\frac{q-r}{s+1}} \leq c_* \| |u|^s u \|_W^{\frac{q-r}{s+1}} \leq c_* [\ell_{W,s}(u)]^{\frac{q-r}{s+1}}, \quad (4.6)$$

for a suitable constant $c^* > 0$.

Now, fixing any $n \in \mathbb{N}$, from $(??)$, $(??)$, $(??)$ and $(??)$ it follows that

$$|u|_q^q \leq c_* \lambda_{n+1}^{-\frac{r}{p}} \|u\|_W^r [\ell_{W,s}(u)]^{\frac{q-r}{s+1}} \leq c_* \lambda_{n+1}^{-\frac{r}{p}} [\ell_{W,s}(u)]^{r + \frac{q-r}{s+1}} \quad \text{for all } u \in W_n^X, \quad (4.7)$$

where from $(??)$ we have

$$r + \frac{q-r}{s+1} = \frac{q+rs}{s+1} > p.$$

Hence, (??), (??) and (??) imply that

$$\begin{aligned} \mathcal{J}(u) &\geq b_1 [\ell_{W,s}(u)]^p - b_2 \lambda_{n+1}^{-\frac{r}{p}} [\ell_{W,s}(u)]^{\frac{q+rs}{s+1}} - b_3 \\ &= b_1 [\ell_{W,s}(u)]^p \left(1 - \frac{b_2}{b_1} \lambda_{n+1}^{-\frac{r}{p}} [\ell_{W,s}(u)]^{\frac{q+rs}{s+1}-p} \right) - b_3 \quad \text{for all } u \in W_n^X, \end{aligned}$$

for suitable positive constants b_1, b_2, b_3 independent of n .

Finally, we choose $r_n > 0$ so that

$$1 - \frac{b_2}{b_1} \lambda_{n+1}^{-\frac{r}{p}} r_n^{\frac{q+rs}{s+1}-p} = \frac{1}{2},$$

i.e.,

$$r_n = \left(\frac{b_1}{2b_2} \lambda_{n+1}^{\frac{r}{p}} \right)^{\frac{s+1}{q-p(s+1)+rs}}. \quad (4.8)$$

Thus, as $\lambda_n \nearrow +\infty$, (??) and (??) imply that $r_n \nearrow +\infty$, then from the estimate

$$\mathcal{J}(u) \geq \frac{b_1}{2} r_n^p - b_3 \quad \text{for all } u \in W_n^X \text{ with } \ell_{W,s}(u) = r_n,$$

the thesis follows. \square

At last, as in [?, Proposition 6.6], the following statement holds.

Proposition 4.6. *For any finite dimensional subspace V of X , there exists $R > 0$ such that*

$$\mathcal{J}(u) \leq 0 \quad \text{for all } u \in V \text{ with } \|u\|_X \geq R.$$

Hence, \mathcal{J} is bounded from above in V .

Proof of Theorem ??. From (G_3) , we can take $\bar{\lambda} \in \mathbb{R}$ so that

$$\limsup_{t \rightarrow 0} \frac{g(x,t)}{|t|^{p-2}t} < \bar{\lambda} < p\alpha_4\lambda_1. \quad (4.9)$$

Then, from (G_1) , (??) and standard computations, a suitable constant $b_1 > 0$ exists such that

$$G(x,t) \leq \frac{\bar{\lambda}}{p}|t|^p + b_1|t|^q \quad \text{a.e. in } \Omega, \text{ for all } t \in \mathbb{R};$$

hence, from (H_8) , (??) and (??), we obtain

$$\mathcal{J}(u) \geq \left(\alpha_4 - \frac{\bar{\lambda}}{p\lambda_1} \right) \|u\|_W^p + \frac{\alpha_4}{(s+1)^p} \| |u|^s u \|_W^p - b_1 |u|_q^q \quad \text{for all } u \in X,$$

with $\alpha_4 - \frac{\bar{\lambda}}{p\lambda_1} > 0$. Now, since (??) holds, by Sobolev Embedding Theorem and (??), we have

$$\int_{\Omega} |u|^q dx = \int_{\Omega} \| |u|^s u \|_{s+1}^{\frac{q}{s+1}} dx \leq b_2 \| |u|^s u \|_W^{\frac{q}{s+1}} \leq b_2 [\ell_{W,s}(u)]^{\frac{q}{s+1}}$$

for some $b_2 > 0$. Thus, from the previous estimates it follows that there exist $b_3, b_4 > 0$ such that

$$\mathcal{J}(u) \geq b_3 [\ell_{W,s}(u)]^p - b_4 [\ell_{W,s}(u)]^{\frac{q}{s+1}} \quad \text{for all } u \in X.$$

Whence, from (??) some strictly positive constants $r_0, \varrho_0 > 0$ can be chosen so that $\mathcal{J}(u) \geq \varrho_0$ if $\ell_{W,s}(u) = r_0$.

On the other hand, taking any $v^* \in X \setminus \{0\}$, by Proposition ?? with $V = \text{span}\{v^*\}$ and the equivalence of $\|\cdot\|_X$ and $\|\cdot\|_W$ in V , an element $e \in V$ exists such that $\|e\|_W > r_0$ and $\mathcal{J}(e) \leq 0$. Whence, as without loss of generality we can assume $\int_{\Omega} A(x, 0, 0) dx = 0$, it is $\mathcal{J}(0) = 0$, so Proposition ?? and Theorem ?? imply that \mathcal{J} has at least a nontrivial critical point. \square

Proof of Theorem ??. For simplicity, if $r > 0$ we set

$$\mathcal{M}_r = \{u \in X : \ell_{W,s}(u) = r\}.$$

We note that \mathcal{M}_r is the boundary of a neighborhood of the origin which is symmetric and bounded with respect to $\|\cdot\|_W$.

Then, fixing any $\varrho > 0$, from Proposition ?? an integer $n \in \mathbb{N}$ and a constant $r_n > 0$ exist such that (??) holds, i.e.

$$u \in \mathcal{M}_{r_n} \cap W_n^X \implies \mathcal{J}(u) \geq \varrho.$$

Now, taking any $m > n$, from (??) the m -dimensional space V_m is such that $\text{codim } W_n^X < \dim V_m$; thus, Proposition ?? and the previous remarks imply that assumption (\mathcal{H}_{ϱ}) in Theorem ?? holds. At last, without loss of generality we can assume $\int_{\Omega} A(x, 0, 0) dx = 0$, then $\mathcal{J}(0) = 0$ and for the arbitrariness of $\varrho > 0$ and Proposition ?? we have that Corollary ?? applies. \square

Proof of Theorem ??. The proof follows from Theorem ?? and Remark ?? with $g(x, t) = |t|^{\mu-2}t$ and so $q = \mu$. \square

References

- [1] A. Ambrosetti and P.H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.* **14** (1973), 349-381.
- [2] D. Arcoya and L. Boccardo, Critical points for multiple integrals of the calculus of variations, *Arch. Rational Mech. Anal.* **134** (1996), 249-274.
- [3] D. Arcoya and L. Boccardo, Some remarks on critical point theory for nondifferentiable functionals, *NoDEA Nonlinear Differential Equations Appl.* **6** (1999), 79-100.
- [4] D. Arcoya, L. Boccardo and L. Orsina, Critical points for functionals with quasilinear singular Euler-Lagrange equations, *Calc. Var. Partial Differential Equations* **47** (2013), 159-180.
- [5] P. Bartolo, V. Benci and D. Fortunato, Abstract critical point theorems and applications to some nonlinear problems with “strong” resonance at infinity, *Nonlinear Anal.* **7** (1983), 981-1012.

- [6] A.M. Candela and G. Palmieri, Multiple solutions of some nonlinear variational problems, *Adv. Nonlinear Stud.* **6** (2006), 269-286.
- [7] A.M. Candela and G. Palmieri, Infinitely many solutions of some nonlinear variational equations, *Calc. Var. Partial Differential Equations* **34** (2009), 495-530.
- [8] A.M. Candela and G. Palmieri, Some abstract critical point theorems and applications. In: *Dynamical Systems, Differential Equations and Applications* (X. Hou, X. Lu, A. Miranville, J. Su & J. Zhu Eds), *Discrete Contin. Dynam. Syst. Suppl.* **2009** (2009), 133-142.
- [9] A.M. Candela and G. Palmieri, Multiplicity results for some nonlinear elliptic problems with asymptotically p -linear terms, *Calc. Var. Partial Differential Equations* **56:72** (2017).
- [10] A.M. Candela and A. Salvatore, Positive solutions for some generalized p -Laplacian type problems, *Discrete Contin. Dyn. Syst. Ser. S* (to appear) .
- [11] A. Canino, Multiplicity of solutions for quasilinear elliptic equations, *Topol. Methods Nonlinear Anal.* **6** (1995), 357-370.
- [12] O.A. Ladyzhenskaya and N.N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, 1968.
- [13] P. Lindqvist, On the equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$, *Proc. Amer. Math. Soc.* **109** (1990), 157-164.
- [14] J.Q. Liu, Y.Q. Wang and Z.Q. Wang, Soliton solutions for quasilinear Schrödinger equations, II, *J. Differential Equations* **187** (2003), 473-493.
- [15] B. Pellacci and M. Squassina, Unbounded critical points for a class of lower semicontinuous functionals, *J. Differential Equations* **201** (2004), 25-62.
- [16] K. Perera, R.P. Agarwal and D. O'Regan, *Morse Theoretic Aspects of p -Laplacian Type Operators*, Math. Surveys Monogr. **161**, Amer. Math. Soc., Providence RI, 2010.
- [17] P.H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Regional Conf. Ser. in Math. **65**, Amer. Math. Soc., Providence, 1986.
- [18] M. Struwe, *Variational Methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, 4rd Edition, *Ergeb. Math. Grenzgeb. (4)* **34**, Springer-Verlag, Berlin, 2008.