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NONLINEAR PERTURBATIONS OF BLMP AND YTSF EQUATIONS

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Abstract. Some nonlinear evolutive equations of Mathematical Physics present infinitely many solutions described in many paper by different methods. For example Körteweg De Vries equation and Kadomtsev - Petviashvili equation are completely integrable despite the presence of quasilinear terms. In the present paper we perturb these kind of equations by positive nonlinear terms having polynomial growth. Assuming that the quasilinear term in the original equation has divergence form, we may apply test function method and establish a range of exponents for the perturbation so that a non-existence result of global weak solutions holds. Concerning initial data condition, an important difference with other equations studied by similar methods (wave, Tricomi and so on) appears. Indeed, we present a class of quasilinear equations for which the sign assumption on the initial data can be omitted and non-existence results still hold. Our basic examples are the perturbation of Boiti Leon Manna Pempinelli equation and Yu Toda Sasa Fukuyama equation. Finally we suggest open problems for other equations and other kind of perturbations.

1. Introduction.

In 1986 Boiti, Leon, Manna, Pempinelli in [1] proposed the following 2D-variant of Korteweg De Vries equation:

$$\partial_y(\partial_t + \partial_x^3)u + 3\partial_x u \partial_y \partial_x u + 3\partial_x^2 u \partial_y u = 0\,. \tag{1}$$

It describes the interaction of two different waves along the two axes and it appears in fluido dynamics and plasma physics. Let us observe that the linear part of the equation is a derivation of Airy operator $\partial_t + \partial_x^3$, moreover all the terms of (1) do not depend on the function u, but only on its derivatives. In the last decade, the 3D and 4D versions of this equation were given respectively in [6] and in [10] starting a new area of interest. In [8] we considered the N-dimensional version of BLMP equation and we add perturbations which depends on u. More precisely, let $N \geq 2$, for $(t, x) \in \mathbb{R} \times \mathbb{R}$ and $\xi \in \mathbb{R}^{N-1}$, setting $u = u(t, x, \xi) : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, we put

$$S(\nabla_\xi)u = \partial_{\xi_1} u + \cdots + \partial_{\xi_{N-1}} u\,,$$

and call N-dimensional version of (1) the following:

$$(\partial_t + \partial_x^3)S(\nabla_\xi)u(t, x, \xi) + 3(\partial_x u S(\nabla_\xi)\partial_x u + \partial_x^2 u S(\nabla_\xi)u) = 0\,.$$

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In [8] we studied non-existence of weak solutions of this equation perturbed by positive extra sources which growth polynomially in u and its derivatives.

Let $N \geq 2$, $\alpha, \gamma \geq 0$ and $p, q > 1$, then we considered

$$\begin{cases} (\partial_t + \partial_x^3)S(\nabla_\xi)u + 3(\partial_x u S(\nabla_\xi)\partial_x u + \partial_x^2 u S(\nabla_\xi)u) = \alpha|u|^p + \gamma|D_{x,\xi}u|^q\,, \\ u(0,x,\xi) = u_0(x,\xi)\,. \end{cases} \tag{2}$$

We observe that

$$(\partial_x u S(\nabla_\xi)\partial_x u + \partial_x^2 u S(\nabla_\xi)u) = \partial_x(\partial_x u S(\nabla_\xi)u)$$

In [8] we established that for non-zero $u_0 \in L^1$, if $2 < q = 2p < 2\frac{N+3}{N+2}$ and $\alpha = \gamma = 1$ then (2) has no global weak solution for any $u_0 \neq 0$. In the present paper we continue this analysis looking to the case $q \neq 2p$. This result will be a corollary of a non-existence result for weak solutions of a class of equations having the form

$$\begin{aligned} &\partial_t P_1(x,\xi,\partial_x,D_\xi)u + P_2(x,\xi,\partial_x,D_\xi)u + \nabla_{x,\xi}\cdot(G(u,\partial_x u,D_\xi u)) = \\ &= \alpha|u|^p + \alpha_1|\partial_x u|^{p_1} + \alpha_2|D_\xi u|^{p_2}\,, \end{aligned}$$

where $\alpha, \alpha_1, \alpha_2 > 0$, while P_1, P_2 are linear operators and $Gu = (G_1, \ldots, G_N)$ is a vector valued function possibly nonlinear. In the present paper, Theorem 2.4, we give an upper bound of p, p_1, p_2 so that a blow up occurs. For simplicity in Theorem 2.4 we treat only the case G dependent on the derivatives of u; instead in Theorem 4.3 we will deal with G dependent also on u.

A first application of these theorems is BLMP equation, indeed

$$G_1 = G_1(\partial_x u, D_\xi u) = 3\partial_x u S(\nabla_\xi)u, \quad G_i = 0 \quad i = 2,\ldots N\,.$$

This structure stands up also for 3D Yu Toda Sasa Fukuyama equation, introduced in [11] and well studied in many paper to describe shallow water in reacting mixtures:

$$-4\partial_t\partial_x u + \partial_z\partial_x^3 u + 3\partial_y^2 u + 4\partial_x u\partial_z\partial_x u + 2\partial_x^2 u\partial_z u = 0\,.$$

Here

$$\begin{aligned} &P_1 = -4\partial_x\,, \qquad P_2 = \partial_z\partial_x^3 + 3\partial_y^2\,, \\ &G = G(\partial_x u, D_{y,z}u) = (2\partial_x u\partial_z u, 0, (\partial_x u)^2)\,, \end{aligned}$$

indeed $\nabla\cdot G = \partial_x(2\partial_x u\partial_z u) + \partial_z(\partial_x u)^2 = 2\partial_x^2 u\partial_z u + 4\partial_x u\partial_x\partial_z u$. In this paper we consider its perturbed version:

$$\begin{aligned} &-4\partial_t\partial_x u + \partial_z\partial_x^3 u + 3\partial_y^2 u + 4\partial_x u\partial_z\partial_x u + 2\partial_x^2 u\partial_z u = \\ &= \alpha|u|^p + \alpha_1|\partial_x u|^{p_1} + \alpha_2|\nabla_{y,z}u|^{p_2}\,, \end{aligned}$$

with $\alpha, \alpha_1, \alpha_2 > 0$.

The 2D version of this equation can be deduced from the 3D case:

$$-4\partial_t\partial_x u - \partial_x^4 u + 3\partial_y^2 u - 6(\partial_x u)^2 - 6u\partial_x^2 u = \alpha|u|^p + \alpha_1|\partial_x u|^{p_1} + \alpha_2|\nabla_{y,z}u|^{p_2}\,,$$

see [3] for $\alpha = \alpha_1 = \alpha_2 = 0$. In this case

$$\begin{aligned} &P_1 = -4\partial_x\,, \qquad P_2 = -\partial_x^4 + 3\partial_y^2\,, \\ &G = G(u,\partial_x u, D_{y,z}u) = (-6u\partial_x u, 0)\,. \end{aligned}$$

Here the quasilinear term depends also on u.

Our approach works for many other equations of Mathematical Physics, among these we will see Kadomtsev Petviashvili equation, Calogero Bogayavlenskii Schiff equation, Jimbo Miwa equation. For perturbed Airy and Korteweg-de-Vries see [8]. Our approach works for other equations in KdV and KP hierarchy, but the chosen examples show some important differences in applying test function method to peculiar cases.

In Section 2 we will state the general result with G independent of u, proved in Section 3. In Section 4 we generalize to G depending on u and we describe in details the two chosen examples. The last two sections are devoted to other physical examples, conclusions and open problems.

Notations.

- $\nabla \cdot F$ is the divergence of a vector valued function $F = (F_1, \dots, F_K)$ in $\mathbb{R}^K$. The operator $S(\nabla)$ acts on real valued functions $f : \mathbb{R}^K \to \mathbb{R}$ as $S(\nabla)f = \nabla \cdot F$ with $F = (f, \dots, f)$.
- Let $A \subset \mathbb{R}^n$. With $C_c^\infty(A, \mathbb{R}_+)$ we denote the space of smooth positive functions with compact support in the domain A. We can identify these functions with their trivial C^∞ extension in $\bar{A}$.
- Let L be a linear differential operator on $\mathbb{R}^n$with domain $D(L)$ which structure depends on the regularity of the coefficients of L. We say that $g \in D(L^*)$ if there exists a unique $\Xi \in L^1_{loc}$ such that $\int Lfg dx = \int f \Xi dx$ for any $f \in D(L)$. Hence we define $\Xi = L^* g$ and it holds
$$\int_{\mathbb{R}^n} (Lf) g \, dx = \int_{\mathbb{R}^n} f L^* g \, dx \, .$$
- For $q > 1$ we denote by $q' = \frac{q}{q-1}$ the conjugate exponent. In particular for $a, b > 0$, we will use the Young inequality:
$$ab \le \epsilon a^q + C_\epsilon b^{q'} \, ,$$
being $\epsilon > 0$ chosen in the proof and $C_\epsilon > 0$ accordingly determined.
- We write $f \le Cg$ with a constant $C > 0$ that may changes in a chain of inequalities.

2. **Definitions and main result.** Let us consider the equation
$$\begin{aligned} &\partial_t P_1(x, \xi, \partial_x, D_\xi)u + P_2(x, \xi, \partial_x, D_\xi)u + N(\partial_x u, D_\xi u) \\ &= \alpha |u|^p + \alpha_1 |\partial_x u|^{p_1} + \alpha_2 |D_\xi u|^{p_2} \, , \end{aligned} \tag{3}$$
where P_i is a linear differential operator of order $k_i \ge 1$:
$$P_i(x, \xi, \partial_x, D_\xi) = \sum_{1 \le |(\beta, \gamma)| \le k_i} p^{(i)}_{\beta, \gamma}(x, \xi) \partial_x^\beta D_\xi^\gamma \, , \tag{4}$$
with multi-index $(\beta, \gamma) \in \mathbb{N} \times \mathbb{N}^{N-1}$. On the contrary, $N(\partial_x u, D_\xi u)$ is a nonlinear term. The variables $(t, x, \xi) \in \mathbb{R}^{N+1} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1}$ have dual variables, in the sense of Fourier transform, denoted by $(\tau, \tilde{x}, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1}$.

We suppose that

(H_1) $\alpha > 0$, $\alpha_1 > 0$ and $\alpha_2 > 0$.

(H_2) $\partial_t P_1(x, \xi, \partial_x, D_\xi)u + P_2(x, \xi, \partial_x, D_\xi)$ is m-th order quasi-homogeneous operator with *quasi-homogeneous dimension* Q for scaling powers $(\delta, \delta_1, \delta_2)$. This means that there exist $\delta > 0$ and $\delta_1, \delta_2 > 0$ such that
$$P_1(\lambda^{-\delta_1} x, \lambda^{-\delta_2} \xi, \lambda^{\delta_1} \tilde{x}, \lambda^{\delta_2} \eta) = \lambda^{m-\delta} P_1(x, \xi, \tilde{x}, \eta) \, ,$$
$$P_2(\lambda^{-\delta_1} x, \lambda^{-\delta_2} \xi, \lambda^{\delta_1} \tilde{x}, \lambda^{\delta_2} \eta) = \lambda^{m} P_2(x, \xi, \tilde{x}, \eta) \, ,$$

for any $\lambda > 0$, $x, \tilde{x} \in \mathbb{R}$ and $\xi, \eta \in \mathbb{R}^{N-1}$. Hence we put

$$Q = \delta + \delta_1 + \delta_2(N-1).$$

we can also say that $P_1(x, \xi, \partial_x, D_\xi)$ has quasi-homeogeneous dimension $Q_1 = \delta_1 + \delta_2(N-1)$ with quasi-homogeneous order $m - \delta$ and $P_2(x, \xi, \partial_x, D_\xi)$ has quasi-homeogeneous dimension Q_1 with order m.

(H_3) For $i = 1, 2$, it holds

$$\partial_x^\beta D_\xi^\gamma p_{\beta,\gamma}^{(i)}(x, \xi) = 0\,, \tag{5}$$

for any $(\beta, \gamma) \in \mathbb{N} \times \mathbb{N}^{N-1}$ such that $1 \leq |\beta| + |\gamma| \leq k_i$.

(H_4) There exists a vector valued function

$$G(\partial_x u, D_\xi u) = (G_1(\partial_x u, D_\xi u), \ldots, G_N(\partial_x u, D_\xi u))\,,$$

such that

$$N(\partial_x u, D_\xi u) = \nabla_{x,\xi} \cdot (G(\partial_x u, D_\xi u))\,.$$

(H_5) We assume that there exist $\beta_1, \beta_2 \in \mathbb{R}$, and $1 \leq q_1 \leq p_1$, $1 \leq q_2 \leq p_2$ such that

$$|G(\partial_x u, D_\xi u)| \leq \beta_1^2 |\partial_x u|^{q_1} + \beta_2^2 |D_\xi u|^{q_2}\,, \quad \beta_1^2, \beta_2^2 > 0\,.$$

The assumption (H_1) will lead to non-existence results even if we start from a quasilinear equation having infinite solution as in KdV hierarchy.

The assumption (H_2) will be here important to develop the test function method, according to [5] from which we inherit the notations. In particular, let $L = \partial_t P_1 + P_2$ or $L = \partial_t P_1$, then

$$L^* S^I_{\lambda^\delta} S^{II}_{\lambda^{\delta_1}} S^{III}_{\lambda^{\delta_2}} g = \lambda^m S^I_{\lambda^\delta} S^{II}_{\lambda^{\delta_1}} S^{III}_{\lambda^{\delta_2}} L^* g \quad \text{for } g \in D(L^*).$$

where $S^I_\lambda g(t, x, \xi) := g(\lambda t, x, \xi)$, $S^{II}_\lambda g(t, x, \xi) := g(t, \lambda x, \xi)$ and $S^{III}_\lambda g(t, x, \xi) := g(t, x, \lambda\xi)$. For L and g independent of t the same relation holds, simply neglecting the first scaling operator. In particular

$$P_1^* S^{II}_{\lambda^{\delta_1}} S^{III}_{\lambda^{\delta_2}} g = \lambda^{m-\delta} S^{II}_{\lambda^{\delta_1}} S^{III}_{\lambda^{\delta_2}} P_1^* g \quad \text{for } g \in D(P_1^*),$$
$$P_2^* S^{II}_{\lambda^{\delta_1}} S^{III}_{\lambda^{\delta_2}} g = \lambda^{m} S^{II}_{\lambda^{\delta_1}} S^{III}_{\lambda^{\delta_2}} P_1^* g \quad \text{for } g \in D(P_1^*).$$

The assumptions (H_3) implies a reduction of the supports after application of P_1^* and P_2^*. Similarly the assumption (H_4) leads to a holed support after integration by parts. Assumption (H_3) deals with linear operators and it has been mentioned in [5] and developed in [4]. The assumptions (H_4) (H_5) give one of the novelty of this paper, since they concern the quasilinear part of the equation. The test function method has been applied for elliptic quasilinear equations in [9], here we exploit the effect of a perturbation of a divergence form quasilinear term.

Finally, we will see that for some not-Kovalevskian operators a standard positivity condition on initial data could be omitted while proving a nonexistence result.

Under the same assumptions on variables and operators, we associate to (3) the initial value problem:

$$\begin{cases} (\partial_t P_1 + P_2) u(t, x, \xi) + N(\partial_x u, D_\xi u) = \alpha^2 |u|^p + \alpha_1^2 |\partial_x u|^{p_1} + \alpha_2^2 |D_\xi u|^{p_2}\,, \\ u(0, x, \xi) = u_0(x, \xi)\,, \end{cases} \tag{6}$$

where $u_0 : \mathbb{R}^N \to \mathbb{R}$.

Definition 2.1. Let $\mathbf{p} = (p, p_1, p_2)$. Fix $T > 0$. We set

$$X_{\mathbf{p},G}(T) = \left\{ \quad u \in L^p_{loc}([0,T]\times\mathbb{R}^N) \quad \text{s.t.} \quad \begin{array}{r} \partial_x u \in L^{p_1}_{loc}([0,T]\times\mathbb{R}^N) \\ D_\xi u \in L^{p_2}_{loc}([0,T]\times\mathbb{R}^N) \\ G(\partial_x u, D_\xi u) \text{ is well defined} \\ \text{in distribution sense} \end{array} \right\}.$$

For $X_{\mathbf{p},G}(+\infty)$ we mean the same set with functions defined in $[0,+\infty)\times\mathbb{R}^N$.

We mention that, due to assumption (H_5), if $u \in X_{\mathbf{p},G}(T)$ then

$$G(\partial_x u, D_\xi u) \in L^1_{loc}([0,T]\times\mathbb{R}^N)\,.$$

Definition 2.2. Let be $T > 0$. Let $u_0 \in L^1_{loc}(\mathbb{R}^N)$. We say that $u : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}$ is a *local weak solution* of (6) if $u \in X_{\mathbf{p},G}(T)$ and for any $\eta \in \mathcal{C}^\infty_c([0,T),\mathbb{R}_+)$, for any $\Phi \in \mathcal{C}^\infty_c(\mathbb{R}^N,\mathbb{R}_+)$ one has

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^N} (\alpha|u|^p + \alpha_1|\partial_x u|^{p_1} + \alpha_2|D_\xi u|^{p_2})\,\eta(t)\Phi(x,\xi)\,dx\,dt\,d\xi \\
&+ \int_{\mathbb{R}^N} u_0(x,\xi)\eta(0)P_1^*\Phi(x,\xi)\,dx\,d\xi = \\
&\int_0^T \int_{\mathbb{R}^N} u(-\eta'(t)P_1^*\Phi(x,\xi) + \eta(t)P_2^*\Phi(x,\xi))\,dx\,dt\,d\xi \\
&- \int_0^T \int_{\mathbb{R}^N} \eta(t)G(\partial_x u, D_\xi u)\cdot\nabla_{x,\xi}\Phi(x,\xi)\,dx\,dt\,d\xi\,.
\end{aligned}$$

Definition 2.3. We denote by $T_{\scriptscriptstyle MAX} \in [0,+\infty]$ the lifespan of the solution of (6):

$$T_{\scriptscriptstyle MAX} := \sup\{T > 0 \text{ such that } u \in X_{\mathbf{p},G}(T) \text{ is a local weak solution of (6) }\}\,.$$

For $T_{\scriptscriptstyle MAX} = +\infty$ we have a *global weak solution.*

Theorem 2.4. *Assume* (H_1)-...-(H_5) *and*

$$p(Q - m) \le Q\,, \tag{7}$$

$$p_1(Q - \min\{\delta_1,\delta_2\}) \le Qq_1\,, \tag{8}$$

$$p_2(Q - \min\{\delta_1,\delta_2\}) \le Qq_2\,. \tag{9}$$

Then (6) *does not admit global weak solutions provided*

$$m \ge \delta \qquad \text{and} \qquad 0 \ne u_0 \in L^1(\mathbb{R}^N)\,.$$

The same result holds, if

$$P_1u_0 \in L^1(\mathbb{R}^N) \text{ with } \int_{\mathbb{R}^N} P_1u_0(x,\xi)\,dx\,d\xi > 0\,. \tag{10}$$

Remark 2.5. We observe that the condition on p and the ones on p_1, p_2 are uncoupled.

Remark 2.6. We shall see in the proof that if in (H_4) we have $G = (G_1, 0, \dots, 0)$, then δ_2 in (8) and (9) can be neglected. Similarly if $G = (0, G_2, \dots, G_N)$, then δ_1 can be erased in (8) and (9).

Remark 2.7. We shall see that for $m > \delta$ the assumption (H_3) with $i = 1$ will not be used in the estimate of the initial data. In the other cases it is necessary to assume (10). Assumption (10) have a particular relevance for $m < \delta$. Moreover for $m = \delta$ it can be used when (H_3) with $i = 1$ is not satisfied. Since for BLMP and YTSF equations (H_3) holds, and $m > \delta$, here we do not discuss the converse case whose proof can be obtained by a slightly modification of the proof of Theorem 2.4. For example in [8] we considered Airy operator $\partial_t + \partial_x^3$ for which $P_1 = 1$.

Remark 2.8. In Theorem 2.4 for $p_1 = q_1$ and $p_2 = q_2$ and $Q > m$, the non-existence conditions reduces to $p(Q - m) < Q$, that is a Fujita type exponent

$$p < 1 + m/(Q - m)\,.$$

3. **Proof of Theorem 2.4.** Let us consider $\phi \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R}_+)$ such that $\phi(\mathbb{R}) \subset [0, 1]$, $\phi = 1$ in $[-1/2, 1/2]$ and $\operatorname{supp}\phi \subset [-1, 1]$. Similarly we take $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^{N-1}, \mathbb{R}_+)$ such that $\psi = 1$ for $|\xi| \in [-1/2, 1/2]$ with $0 \le \psi(\xi) \le 1$ for any $\xi \in \mathbb{R}^{N-1}$ and $\operatorname{supp}\psi \subset \{|\xi| \le 1\}$. Finally $\eta \in \mathcal{C}_c^\infty([0,\infty), \mathbb{R}_+)$ is a decreasing function such that

$$\eta(t) = \begin{cases} 1 & t \le 1/2 \\ \eta(t) & 1/2 \le t \le 1 \\ 0 & t \ge 1 \end{cases},$$

Let $B, R > 1$. Taking $(\delta, \delta_1, \delta_2)$ given by (H_1), we put

$$\eta_B(t) = \eta\left(B^{-\delta}t\right), \quad \phi_R(x) = \phi\left(R^{-\delta_1}x\right), \quad \psi_R(\xi) = \psi\left(R^{-\delta_2}\xi\right).$$

We also use the following notation:

$$C_R \doteq \left\{(x, \xi) \in \mathbb{R}^N : |x| \le R^{\delta_1}, \quad |\xi| \le R^{\delta_2}\right\}.$$
$$HC_R \doteq \left\{(x, \xi) \in C_R : R^{\delta_1}/2 \le |x| \text{ andx } R^{\delta_2}/2 \le |\xi|\right\}.$$

Assume by contradiction that (6) admits global weak solution, by using these test functions in Definition 2.2 we get the relation

$$I_{B,R} + D_R = L_{B,R} - N_{B,R}\,, \tag{11}$$

with $L_{B,R} = L^1_{B,R} + L^2_{B,R}$ and $N_{B,R} = N^1_{B,R} + N^2_{B,R}$ given by

$$I_{B,R} = \int_0^{B^\delta}\int_{\mathbb{R}^N} (\alpha|u|^p + \alpha_1|\partial_x u|^{p_1} + \alpha_2|D_\xi u|^{p_2})\,\eta_B(t)\phi_R(x)\psi_R(\xi)\,dx\,dt\,d\xi\,,$$

$$D_R = \int_{\mathbb{R}^N} u_0(x, \xi)P_1^*(\phi_R(x)\psi_R(\xi))\,dx\,d\xi\,,$$

$$L^1_{B,R} = -B^{-\delta}\int_{B^\delta/2}^{B^\delta} (\eta')_B(t)\int_{\mathbb{R}^N} uP_1^*(\phi_R(x)\psi_R(\xi))\,dx\,dt\,d\xi\,,$$

$$L^2_{B,R} = \int_0^{B^\delta}\int_{\mathbb{R}^N} u\eta(t)P_2^*(\phi_R(x)\psi_R(\xi))\,dx\,dt\,d\xi\,,$$

$$N^1_{B,R} = R^{-\delta_1}\int_0^{B^\delta} \eta_B(t)\int_{\mathbb{R}^N} \psi_R(\xi)G_1(\partial_x u, D_\xi u)(\partial_x\phi)_R(x)\,dx\,dt\,d\xi\,,$$

$$N^2_{B,R} = R^{-\delta_2}\int_0^{B^\delta} \eta_B(t)\int_{\mathbb{R}^N} \phi_R(x)\tilde{G}(\partial_x u, D_\xi u)\cdot(\nabla_\xi\psi)_R(\xi)\,dx\,dt\,d\xi\,,$$

where $\tilde{G} = (G_2, \ldots G_N)$. Since $u \in X_{\mathbf{p},G}(+\infty)$, these terms are well-defined.

Due to assumptions (H_3) and (H_4) we can restrict the $\mathbb{R}^N$ integrals for D_R, $L^i_{B,R}$, $N^i_{B,R}$ over HC_R. We also put

$$I^\sharp_{B,R} = \int_0^{B^\delta}\int_{HC_R} (\alpha|u|^p + \alpha_1|\partial_x u|^{p_1} + \alpha_2|D_\xi u|^{p_2})\,\eta_B(t)\phi_R(x)\psi_R(\xi)\,dx\,dt\,d\xi\,.$$

3.1. **Estimate for D_R.** Since P_1 is quasi-homogeneous of dimension $\delta_1+\delta_2(N-1)$ and quasi-homogeneous order $m-\delta$, we have

$$D_R = R^{-m+\delta}\int_{HC_R} u_0(x,\xi) S^{II}_{R^{-\delta_2}} S^{III}_{R^{-\delta_3}} P_1^*(\phi(x)\psi(\xi))\, dx\, d\xi\,.$$

For $m>\delta$ one has

$$|D_R| \le R^{-m+\delta}\|P_1^*(\phi\psi)\|_\infty \|u_0\|_1 \to 0 \text{ for } R\to+\infty\,.$$

In the case $m=\delta$, we use assumption (H_3):

$$|D_R| \le \|P_1^*(\phi\psi)\|_\infty \|u_0\|_{L^1(HC_R)} \to 0 \text{ for } R\to+\infty$$

by Lebesque convergence theorem.

Finally, in the general case, we can only say that

$$D_R = \int_{C_R} P_1(u_0(x))\phi_R(x)\psi_R(\xi)\, dx\, d\xi\,.$$

Let $D := \int_{\mathbb{R}^N} P_1(u_0(x))\, dx\, d\xi > 0$, by using Lebesgue convergence theorem, this time we see that there exists $\bar{R}>0$ such that $D_R \ge D/2 > 0$ for any $R\ge\bar{R}$.

3.2. **Estimate for $L_{B,R}$.** By using Hölder inequality, due to (H_3) assumption, we have

$$\begin{aligned}|L^1_{B,R}| \le & CB^{-\delta}\left(\int_{B^\delta/2}^{B^\delta}\int_{HC_R}|u|^p\eta_B(t)\phi_R(x)\psi_R(\xi)\, dx\, d\xi\, dt\right)^{1/p}\times\\ &\times\left(\int_{B^\delta/2}^{B^\delta}\int_{HC_R}\frac{|\eta_B'(t)P_1^*(\phi_R(x)\psi_R(\xi))|^{p'}}{|\eta_B(t)\phi_R(x)\psi_R(\xi)|^{p'-1}}\, dx\, d\xi\, dt\right)^{1/p'},\end{aligned}$$

and

$$\begin{aligned}|L^2_{B,R}| \le & C\left(\int_0^{B^\delta}\int_{HC_R}|u|^p\eta_B(t)\phi_R(x)\psi_R(\xi)\, dx\, d\xi\, dt\right)^{1/p}\times\\ &\times\left(\int_0^{B^\delta}\int_{HC_R}\frac{|\eta_B(t)P_2^*(\phi_R\psi_R)|^{p'}}{|\eta_B(t)\phi_R(x)\psi_R(\xi)|^{p'-1}}\, dx\, d\xi\, dt\right)^{1/p'}.\end{aligned}$$

We can substitute $\eta\phi\psi$ with $(\eta\phi\psi)^\sigma$ with large $\sigma > mp'$ so that the functions in $L^i_{B,R}$, with $i=1,2$ are finite, see Lemma 2.1 in [4]. Due to $\alpha>0$ and (H_2) we get

$$\begin{aligned}&|L^1_{B,R}|\\ &\le CB^{-\delta}(I^\sharp_{B,R})^{1/p}\times\\ &\times\left(R^{(-m+\delta)p'}\int_{B^\delta/2}^{B^\delta}\frac{|\eta_B'(t)|^{p'}}{|\eta_B(t)|^{p'-1}}\int_{HC_R} S^{II}_{R^{-\delta_1}} S^{III}_{R^{-\delta_2}}\frac{|P_1^*(\phi(x)\psi(\xi))|^{p'}}{|\phi(x)\psi(\xi)|^{p'-1}}\, dx\, d\xi\, dt\right)^{1/p'}\\ &\le C(I^\sharp_{B,R})^{1/p}B^{-\delta}R^{(-m+\delta)}B^{\delta/p'}R^{(\delta_1+(N-1)\delta_2)/p'}\times\\ &\times\left(\int_{1/2}^{1}\int_{HC_1}\frac{|\eta'(t)P_1^*(\phi(x)\psi(\xi))|^{p'}}{|\eta(t)\phi(x)\psi(\xi)|^{p'-1}}\, dx\, d\xi\, dt\right)^{1/p'}.\end{aligned}$$

Since the last integral is finite we can conclude

$$|L^1_{B,R}| \le CB^{-\delta/p}R^{-m+Q/p'+\delta/p}(I^\sharp_{B,R})^{1/p}\,. \tag{12}$$

Similarly

$$|L^2_{B,R}| \le C B^{\delta/p'} R^{-m+Q/p'-\delta/p'} (I^\sharp_{B,R})^{1/p}\,. \tag{13}$$

Let $R = B$. After Young inequality, we may conclude that for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$|L_R| \le \epsilon I^\sharp_{R,R} + C_\epsilon R^{-mp'+Q}\,.$$

This means

$$|L_R| \le \epsilon I^\sharp_{R,R} + C_\epsilon R^{-m\frac{p}{p-1}+Q}\,. \tag{14}$$

3.3. **Estimate for $N_{B,R}$.** Due to assumption (H_5), we get

$$|N^1_{B,R}| \le R^{-\delta_1} \int_0^{B^\delta} \int_{HC_R} \left(\beta_1^2 |\partial_x u|^{q_1} + \beta_2^2 |D_\xi u|^{q_2}\right) \eta_B(t) |(\partial_x \phi)_R(x)| \psi_R(\xi) dx d\xi dt\,,$$

$$|N^2_{B,R}| \le R^{-\delta_2} \int_0^{B^\delta} \int_{HC_R} \left(\beta_1^2 |\partial_x u|^{q_1} + \beta_2^2 |D_\xi u|^{q_2}\right) \eta_B(t) \phi_R(x) |(\nabla_\xi \psi)_R(\xi)| dx d\xi dt\,.$$

For $\tilde{G} = 0$ the term $N^2_{B,R}$ disappears; similarly for $G_1 = 0$ we do not have $N^2_{B,R}$. For this reason, in such cases, the assumptions (9), respectively (8) can be neglected as explained in Remark 2.6.

Let $f, g \ge 0$ and $q_i < p_i$ for $i = 1, 2$; we can apply Hölder inequality in the form

$$\int f^q g \le \left(\int f^p g\right)^{q/p} \left(\int g\right)^{1-q/p} \qquad \text{if } q \le p\,.$$

After changing ϕ with ϕ^2 and ψ with ψ^2, we obtain

$$\begin{aligned}
|N^1_{B,R}| &\le C\, R^{-\delta_1} (I^\sharp_{B,R})^{q_1/p_1} \left(\int_0^{B^\delta} \int_{HC_R} \eta_B(t) |(\partial_x \phi)_R| \psi_R(\xi)\, dx\, d\xi\, dt\right)^{1-\frac{q_1}{p_1}} \\
&\quad + C\, R^{-\delta_1} (I^\sharp_{B,R})^{q_2/p_2} \left(\int_0^{B^\delta} \int_{HC_R} \eta_B(t) |(\partial_x \phi)_R| \psi_R(\xi)\, dx\, d\xi\, dt\right)^{1-\frac{q_2}{p_2}} \\
&\le C (I^\sharp_{B,R})^{q_1/p_1} B^{\delta\left(1-\frac{q_1}{p_1}\right)} R^{-\delta_1 + (Q-\delta)\left(1-\frac{q_1}{p_1}\right)} \\
&\quad + C\, (I^\sharp_{B,R})^{q_2/p_2} B^{\delta\left(1-\frac{q_2}{p_2}\right)} R^{-\delta_1 + (Q-\delta)\left(1-\frac{q_2}{p_2}\right)}\,.
\end{aligned}$$

Similarly

$$\begin{aligned}
|N^2_{B,R}| &\le C (I^\sharp_{B,R})^{q_1/p_1} B^{\delta\left(1-\frac{q_1}{p_1}\right)} R^{-\delta_2 + (Q-\delta)\left(1-\frac{q_1}{p_1}\right)} \\
&\quad + C\, (I^\sharp_{B,R})^{q_2/p_2} B^{\delta\left(1-\frac{q_2}{p_2}\right)} R^{-\delta_2 + (Q-\delta)\left(1-\frac{q_2}{p_2}\right)}\,.
\end{aligned}$$

We can conclude that

$$\begin{aligned}
|N^1_{B,R}| &\le \epsilon_{1,1} I^\sharp_{B,R} + C_{\epsilon_{1,1}} B^\delta R^{-\delta_1 \frac{p_1}{p_1-q_1} + (Q-\delta)} \qquad (15) \\
&\quad + \epsilon_{1,2} c_{B,R} + C_{\epsilon_{1,2}} B^\delta R^{-\delta_1 \frac{p_2}{p_2-q_2} + (Q-\delta)}\,, \\
|N^2_{B,R}| &\le \epsilon_{2,1} I^\sharp_{B,R} + C_{\epsilon_{2,1}} B^\delta R^{-\delta_2 \frac{p_1}{p_1-q_1} + (Q-\delta)} \qquad (16) \\
&\quad + \epsilon_{2,2} I^\sharp_{B,R} + C_{\epsilon_{2,2}} B^\delta R^{-\delta_2 \frac{p_2}{p_2-q_2} + (Q-\delta)}\,.
\end{aligned}$$

Taking $B = R$ and $\epsilon + \epsilon_{1,1} + \epsilon_{1,2} + \epsilon_{2,1} + \epsilon_{2,2} < 1$ in (14), (15), (16), we arrive at

$$I_{R,R} \le -D_R + C\, R^{-m\frac{p}{p-1}+Q} + \tag{17}$$

$$+ C\, R^{\delta+(Q-\delta)} \left(R^{-\delta_1 \frac{p_1}{p_1-q_1}} + R^{-\delta_2 \frac{p_1}{p_1-q_1}} \right) + \tag{18}$$

$$+ C\, R^{\delta+(Q-\delta)} \left(R^{-\delta_1 \frac{p_2}{p_2-q_2}} + R^{-\delta_2 \frac{p_2}{p_2-q_2}} \right) . \tag{19}$$

The case $p_1 = q_1$ or $p_2 = q_2$ is simpler, indeed $N_{R,R} \leq C\, I^{\sharp}_{R,R}(R^{-\delta_1} + R^{-\delta_2})$ so that for large $R > 1$ we can still absorb $N_{R,R}$ in the left side, this means that the previous inequality still holds.

3.4. **Non-existence, subcritical case.** Let us recall that if $u_0 \in L^1(\mathbb{R}^N)$ and $m \geq \delta$ the term $|D_R| \to 0$ for $R \to \infty$. When $m < \delta$ and $\int P_1(u_0)dxd\xi > 0$ we may simply neglect $D_R > 0$ for large R. Hence taking $R \to +\infty$ in (17), (18), (19), we find $u \equiv 0$ provided

$$\begin{aligned}
&p(Q-m) < Q\,,\\
&Q - \frac{p_1}{p_1-q_1} \min\{\delta_1,\delta_2\} < 0\,,\\
&Q - \frac{p_2}{p_2-q_2} \min\{\delta_1,\delta_2\} < 0\,.
\end{aligned}$$

Taking $u_0 \neq 0$ we get an absurd, hence the non-existence of global weak solutions for (6).

3.5. **Non-existence, critical case.** Let us assume that at least one of the conditions (7), (8), (9) is satisfied with equality. From (17), (18), (19), we can only deduce that $I_{R,R}$ is bounded with respect to R. Indeed also $-D_R$ is bounded provided $m \geq \delta$ or (10). In turn this implies $u \in L^p([0,+\infty) \times \mathbb{R}^N)$, $\partial_x u \in L^{p_1}([0,+\infty) \times \mathbb{R}^N)$ and $D_\xi u \in L^{p_2}([0,+\infty) \times \mathbb{R}^N)$. By Lebesgue convergence theorem, we deduce that $I^{\sharp}_{R,R} \to 0$. Coming back to (12) and (13), by Lebesgue convergence theorem, we deduce that $L_{R,R} \to 0$. Here the assumption (H_3) is again crucial. Similarly $N_{R,R} \to 0$. In turn (11) gives $I_{R,R} \to 0$, hence $u \equiv 0$ and this is absurd for not vanishing initial data.

4. Examples.

4.1. **BLMP.** First of all we come back to the nonlinear perturbation of BLMP equation:

$$\begin{cases}
(\partial_t + \partial_x^3)S(\nabla_\xi)u + 3(\partial_x u S(\nabla_\xi)\partial_x u + \partial_x^2 u S(\nabla_\xi)u) = \\
= \alpha|u|^p + \alpha_1|\partial_x u|^{p_1} + \alpha_2|D_\xi u|^{p_2}\,,\\
u(0,x,\xi) = u_0(x,\xi)\,.
\end{cases} \tag{20}$$

We assume $\alpha, \alpha_1, \alpha_2 > 0$ such that (H_1) is satisfied. The operator $(\partial_t + \partial_x^3)S(\nabla_\xi)$ is quasi-homogeneous of order $m = 4$ with $\delta = 3$, $\delta_1 = \delta_2 = 1$ hence $Q = 3 + N$ and (H_3) is satisfied. We soonly observe that $m > \delta$. As seen in Section 1, the quasilinear part satisfies (H_4) with $G_1(\partial_x, D_\xi) = 3\partial_x u S(\nabla_\xi)u$ and $G_i = 0$ for $i = 2, \ldots, N$. The term $N^2_{R,R}$ can be neglected and non-existence conditions are given by

$$(N-1)p \leq N+3\,, \tag{21}$$

$$(N+2)p_1 \leq (N+3)q_1\,, \quad q_1 \leq p_1\,, \tag{22}$$

$$(N+2)p_2 \leq (N+3)q_2\,, \quad q_2 \leq p_2\,, \tag{23}$$

choosing $q_1, q_2 > 1$ such that (H_5) is satisfied, that is

$$|\partial_x u S(\nabla_\xi)u| \leq |\partial_x u|^{q_1} + |D_\xi u|^{q_2}\,.$$

By Young inequality, we fix $q_2 = q_1'$ hence we need $1 \le q_1 \le p_1$ and $1 \le q_2 = q_1' \le p_2$ that imply

$$1 \le p_2' \le p_1 \,,$$

and

$$\frac{N+2}{N+3} \le \frac{1}{p_1} + \frac{1}{p_2} \,.$$

Conversely if this last inequality holds, we can take $r \in \left[\frac{N+2}{N+3} - \frac{1}{p_2}, \frac{1}{p_1}\right]$ and for $q_1 = \frac{N+2}{N+3}\frac{1}{r}$ and $q_2 = q_1'$ we get (22) and (23). This shows an interaction between p_1 and p_2 for a quasilinear term independent of u; we can conclude that the quasilinear term strongly influences the non-existence critical exponents.

In [8] we have only considered the case $p_1 = p_2 = 2p$ and use $q_1 = q_2 = 2$ finding the same result: condition (21) and the stronger one

$$(N+2)2p \le 2(N+3) \,,$$

that is $p \le \frac{N+3}{N+2}$. In this paper we have a more general choice of p, that leads to the Fujita type exponent according to Remark 2.8. Since $1 \le p_2' \le p_1$ means $\frac{1}{p_1} + \frac{1}{p_2} \le 1$ we can summarize as it follows.

Corollary 4.1. *Let*

$$\begin{aligned} &(N-1)p \le N+3 \,, \\ &\frac{N+2}{N+3} \le \frac{1}{p_1} + \frac{1}{p_2} \le 1 \,. \end{aligned}$$

For any non-zero $u_0 \in L^1(\mathbb{R}^N)$, the Cauchy Problem (20) has no global weak solutions.

4.2. **3D-YTSF.** Let us consider the Cauchy problem

$$\begin{cases} -4\partial_t\partial_x u + \partial_z\partial_x^3 u + 3\partial_y^2 u + 4\partial_x u\partial_z\partial_x u + 2\partial_x^2 u\partial_z u = \\ = \alpha|u|^p + \alpha_1|\partial_x u|^{p_1} + \alpha_2|\nabla_{y,z} u|^{p_2} \,, \\ u(0,x,y,z) = u_0(x,y,z). \end{cases} \tag{24}$$

We assume $\alpha, \alpha_1, \alpha_2 > 0$ such that (H_1) is satisfied.

The structure of the linear part $\partial_t P_1 + P_2$ is satisfied with $P_1 = -4\partial_x$ and $P_2 = \partial_z\partial_x^3 + 3\partial_y^2$ so that $m = 4$, $\delta = 10/3$, $\delta_1 = 2/3$ and $\delta_2 = 2$ gives (H_2) with quasi-homogeneous dimension $Q = 8$. According to condition (7) and Remark 2.8, the first critical range is $1 \le p \le 2$. Since only non-zero order term are included in P_1 and P_2, the assumption (H_3) is satisfied. For (H_4) we see that

$$4\partial_x u\partial_z\partial_x u + 2\partial_x^2 u\partial_z u = \partial_x(2\partial_x u\partial_z)u + \partial_z(\partial_x u)^2 \,.$$

Hence

$$G = G(\partial_x u, D_{y,z}u) = (2\partial_x u\partial_z u, 0, (\partial_x u)^2)$$

satisfies (H_4). Moreover, it holds

$$|G(\partial_x u, D_{y,z}u)| \le C\left(|\partial_x u|^2 + |D_{y,z}u|^2\right),$$

that is we can take $q_1 = q_2 = 2$ and gain (H_5). Conditions (8) and (9) and the requirement $q_i \le p_i$ for $i = 1, 2$ correspond to $2 \le p_1 \le 24/11$, $2 \le p_2 \le 24/11$. In this range of p, p_1, p_2 Theorem 2.4 assures a non-existence result of weak solution to (24) for any initial data non-zero $u_0 \in L^1(\mathbb{R}^3)$. Summarizing, we have

Corollary 4.2. *Let*

$$1 < p \leq 2\,,$$
$$2 \leq p_1 \leq 24/11,$$
$$2 \leq p_2 \leq 24/11\,.$$

Given non-zero $u_0 \in L^1(\mathbb{R}^2)$ the Cauchy Problem (24) has no global weak solutions.

Due to the presence in the quasilinear term G of the strong perturbation $(\partial_x u)^2$, in this case there is no interaction between p_1 and p_2.

4.3. **2D-YTSF.** In order to apply our result to 2D-YTSF equations, we need to generalize Theorem 2.4 for nonlinear term N depending also on u. Consider

$$\begin{cases} (\partial_t P_1 + P_2)u(t,x,\xi) + N(u, D_{x,\xi}u) = \alpha|u|^p + \alpha_1|\partial_x u|^{p_1} + \alpha_2|D_\xi u|^{p_2}\,, \\ u(0,x,\xi) = u_0(x,\xi). \end{cases} \tag{25}$$

with $x \in \mathbb{R}$, $\xi \in \mathbb{R}^{N-1}$ and $P_1(x,\xi,\partial_x,D_\xi)$, $P_2(x,\xi,\partial_x,D_\xi)$ as in Section 2. The definition of weak solution and the space $X_{\mathbf{p},G}(T)$ requires just the small change in G variables, hence we do not rewrite it.

Theorem 4.3. *Suppose that*

(A_1) $\alpha > 0$, $\alpha_1 > 0$ and $\alpha_2 > 0$.
(A_2) The operator $\partial_t P_1(x,\xi,\partial_x,D_\xi)u + P_2(x,\xi,\partial_x,D_\xi)$ is m-th order quasi-homogeneous with quasi-homogeneous dimension $Q = Q = \delta + \delta_1 + \delta_2(N-1)$ for scaling powers $(\delta,\delta_1,\delta_2)$. Assume in addiction $m \geq \delta$ or (10).
(A_3) Suppose P_1, P_2 in the form (4) and $\partial_x^\beta D_\xi^\gamma p^{(i)}_{\beta,\gamma}(x,\xi) = 0$, for $i = 1,2$ and for any $(\beta,\gamma) \in \mathbb{N} \times \mathbb{N}^{N-1}$ such that $1 \leq |\beta| + |\gamma| \leq k_i$.
(A_4) There exists a vector valued function

$$G(u,\partial_x u, D_\xi u) = (G_1(u,\partial_x u, D_\xi u), \ldots, G_N(u,\partial_x u, D_\xi u))$$

such that

$$N(u, D_{x,\xi}u) = \nabla_{x,\xi} \cdot (G(u,\partial_x u, D_\xi u))\,.$$

(A_5) There exist $\beta,\beta_1,\beta_2 \geq 0$, and $q, q_1.q_2 > 1$ such that $1 \leq q \leq p$, $1 \leq q_1 \leq p_1$, $1 \leq q_2 \leq p_2$ and

$$|G(u,\partial_x u, D_\xi u)| \leq \beta|u|^q + \beta_1|\partial_x u|^{q_1} + \beta_2|D_\xi u|^{q_2}\,.$$

If

$$p(Q-m) \leq Q \tag{26}$$
$$p(Q - \min\{\delta_1,\delta_2\}) \leq Qq\,, \tag{27}$$
$$p_1(Q - \min\{\delta_1,\delta_2\}) \leq Qq_1\,, \tag{28}$$
$$p_2(Q - \min\{\delta_1,\delta_2\}) \leq Qq_2\,. \tag{29}$$

Then for any non-zero $u_0 \in L^1(\mathbb{R}^N)$ the Cauchy Problem (25) does not admit global weak solution.

In the proof of this theorem we can argue as in the proof of Theorem 2.4, the only difference is the presence in $N^i_{B,R}$ of a term dependent of $|u|^q$. Assuming $q \leq p$, one can stil combine Hölder and Young inequality reaching a vanishing term for $B = R \to \infty$ provided (27).

We are now ready to prove a non-existence result for nonlinear perturbation of 2D-YTSF Cauchy problem:

$$\begin{cases} \partial_t(-4\partial_x u) + 3\partial_y^2 u + \partial_x^4 u - 6\partial_x(u\partial_x u) = \alpha|u|^p + \alpha_1|\partial_x u|^{p_1} + \alpha_2|\partial_y u|^{p_2} \\ u(0,x,y) = u_0(x,y). \end{cases} \tag{30}$$

Here $P_1 = -4\partial_x$ while $P_2 = 3\partial_y y + \partial_x^4$ and $G_1 = -6u\partial_x u$. Let us verify the first set of assumptions:

(A_1) $\alpha, \alpha_1, \alpha_2 > 0$.

(A_2) $\partial_t P_1 + P_2$ is quasi-homogeneous of order $m = 4$ with $\delta = 3$, $\delta_1 = 1$ and $\delta_2 = 2$. Hence we put $Q = 6$ and observe that $m > \delta$.

(A_3) Being P_1 and P_2 with constant coefficients and no zero order term, this assumption is trivially satisfied.

(A_4) Since $N = -6\partial_x(u\partial_x u) = \nabla_{x,\xi} \cdot (-6u\partial_x u, 0)$ the quasilinear term is in divergence form.

(A_5) We can estimate

$$|G_1(u, \partial_x u)| \le \beta|u|^q + \beta_1|\partial_x u|^{q_1} \le \beta|u|^q + \beta_1|\partial_x u|^{q_1} + \beta_2|\partial_y u|^{q_2}$$

with $q \le p$ and $1 < q_1 = q' \le p_1$ and for any $q_2 \le p_2$.

Taking $\beta_2 = \alpha_2$ and $q_2 = p_2$ in (A_5) and the condition (29) is trivially satisfied. Hence we need

$$\begin{aligned} &2p \le 6 \Rightarrow p \le 3\,, \\ &5p \le 6q \le 6p\,, \\ &5p_1 \le 6q_1 \le 6p_1\,, \\ &\frac{1}{q} + \frac{1}{q_1} = 1\,. \end{aligned}$$

Summarizing, we have

Corollary 4.4. *If*

$$\begin{aligned} &1 < p \le 3, \\ &\frac{5}{6} \le \frac{1}{p} + \frac{1}{p_1} \le 1 \end{aligned}$$

then the Cauchy Problem (30) has no global weak solutions for any $p_2 > 1$ and $u_0 \in L^1(\mathbb{R}^2)$.

Here the interaction term $u\partial_x u$ in the nonlinear part gives an interaction between p and p_1.

5. **Other equations having same structure.**

There are many other equations having the structure considered in Section 2 and Section 4.3 with particular relevance in Mathematical Physics.

Following the naming of the equation given in [3], we consider the perturbation of Kadomtsev - Petviashvili equation

$$\partial_t(\partial_x u) + 3\partial_y^2 u + \partial_x(u\partial_x u) = \alpha|u|^p + \alpha_1|\partial_x u|^{p_1} + \alpha_2|\partial_y u|^{p_2}\,, \tag{31}$$

with $\alpha, \alpha_1, \alpha_2 > 0$

We are dealing with dimension $N = 2$ and it can be rewritten as $\partial_t P_1 + P_2 + \nabla\cdot G = F$ where

$$P_1 = \partial_x\,,$$

$$P_2 = \partial_y^2\,,$$
$$G = (u\partial_x u, 0)\,.$$

Concerning the exponent for the quasi-homogeneous relation in (H_2) we have

$$m = 2, \quad \delta = \delta_2 = \delta_1 = 1, \quad Q = 3\,.$$

Choosing $q_1 = q'$, Theorem 4.3 gives a non-existence result for global weak solutions to (31) for non-zero initial data $u_0 \in L^1(\mathbb{R}^N)$ provided

$$1 < p \leq 3$$
$$\frac{2}{3} \leq \frac{1}{p} + \frac{1}{p_1} \leq 1$$

The interaction between u and $\partial_x u$ in G becomes an interaction between p and p_1. Since G is independent on $\partial_y u$ the conditions on p_2 are satisfied for a suitable choice of $q_2 \in \left[\frac{2}{3}p_2, p_2\right]$. The range $[2/3, 1]$ is larger than $[5/6, 1]$ given in Corollary 4.4, this shows the role of the term ∂_x^4 in the blow up dinamyc of 2D-YTSF equation, indeed such term is absent in KP equation.

Our next result concern the 3D Jimbo Miwa equation, introduced in the seminal paper [7]:

$$\partial_t(-2\partial_{x_2}u) - 3\partial_{x_1}\partial_{x_3}u + 3\partial_{x_1}^3\partial_{x_2}u + \partial_{x_1}(3\partial_{x_2}u\partial_{x_1}u) = F(u, \nabla u)\,.$$

In order to use Theorem 2.4, we rename this variables as $x_3 = x$, $x_2 = y$ and $x_1 = z$, obtaining

$$\partial_t(-2\partial_y u) - 3\partial_z\partial_x u + 3\partial_z^3\partial_y u + \partial_z(3\partial_y u\partial_z u) \\ = F(u, \partial_x u, \partial_y u, \partial_z u)\,. \tag{32}$$

It can be rewritten as $\partial_t P_1 + P_2 + \nabla\cdot G = F$ where

$$P_1 = -2\partial_y$$
$$P_2 = -3\partial_x\partial_z + 3\partial_z^3\partial_y$$
$$G = (3\partial_y u\partial_z u, 0, 0)\,.$$

We put $\xi = (y, z)$ and

$$F(u, \partial_x u, \partial_y u, \partial_z u) = \alpha|u|^p + \alpha_1|\partial_x u|^{p_1} + \alpha_2(|\partial_y u|^{p_2} + |\partial_z u|^{p_2})\,.$$

So that

$$m = 4, \quad \delta_2 = 1, \quad \delta_1 = \delta = 3, \quad Q = 3 + 3 + 1\cdot(3-1) = 8\,,$$

The estimate for G is trivial: $|G| \leq C|D_\xi u|^2$ hence $q_2 = 2$ while q_1 is arbitrarily chosen, so we take $q_1 = p_1$ and the condition for p_1 is trivially fulfilled. As a conclusion Theorem 2.4 gives a non-existence result for non-global weak solutions to (32) for non-zero initial data $u_0 \in L^1(\mathbb{R}^N)$ provided

$$1 \leq p \leq 2\,,$$
$$2 \leq p_2 \leq 16/7\,.$$

The next application of our result is to Calogero Bogoyavlenskii Schiff equation. Following [2], this equation is written in 2D variables in the form

$$\partial_t(\partial_x u) + \partial_x^3\partial_y u - 4\partial_x u\partial_{xy}^2 u - 2\partial_x^2 u\partial_y u = 0\,.$$

We write in divergence form the quasilinear terms and add the nonlinear perturbation obtaining

$$\partial_t(\partial_x u) + \partial_x^3\partial_y u - \partial_x(2\partial_x u\partial_y u) - \partial_y((\partial_x u)^2) = \alpha|u|^p + \alpha_1|\partial_x u|^{p_1} + \alpha_2|\partial_y u|^{p_2}\,.$$

Hence we have

$$\begin{aligned} P_1 &= \partial_x\,,\\ P_2 &= \partial_x^3\partial_y\,,\\ G(u) &= (-2\partial_x u\partial_y u, -(\partial_x u)^2)\,. \end{aligned}$$

We see that

$$m = 4\,,\quad \delta_1 = \delta_2 = 1\,,\quad \delta = 3 < m\,,\quad Q = 5$$

and

$$|G(u)| \le |\partial_x u|^2 + |\partial_y u|^2\,,$$

indeed the term $(\partial_x u)^2$ is dominant. A direct application of Theorem 2.4 gives non-existence result for

$$1 < p \le 5\,,\quad 2 \le p_1 \le 5/2\,,\quad 2 \le p_2 \le 5/2\,.$$

6. **Conclusion and open problems.**

Remark 6.1. One can observe that all the considered equations belongs to KdV or KP hierarchy. Since we use test function method, we cannot treat all the equations in the class together, indeed in such a case pseudo-differential operators would come into play and support information would be not preserved.

Now we list some other directions of this research. Anyway, in order to complete this analysis it is necessary to understand a local existence theory for (6) at least when P_1, P_2, N give the known equations of Mathematical Physics like the examples in Section 4 and Section 5.

- A larger discussion on the initial data conditions can be done assuming for example that (H_3) does not hold for $i = 1$ but u_0 is in a weighted space which weight depend on P_1 coefficients.
- One can perturb quasi homogeneous operators by means of low order terms. The possibility to weaken the (H_2) assumption can be studied starting from Theorem 4.2 of [5] or modified test function method introduced in [4]. Here we avoid such generalizations since we have in mind certain equations relevant in Mathematical Physics.
- One can split in a different way the variables and treat a more general nonlinear term sum of $\alpha_j|\partial_{\xi_j} u|^{q_j}$. For example in 3D-YTSF equation one can avoid to put $\xi = (y, z) \in \mathbb{R}^2$ and rebuild an analogous theorem for $\partial_t(P_1) + P_2 + P_3$ where $P_2 = 3\partial_y^2$ and $P_3 = \partial_z\partial_x^3$. In this work we do not treat such generalization since the case of $\xi \in \mathbb{R}^{N-1}$ includes any idea. Similarly for Jimbo Miwa equation.
- Concerning assumption (H_1) it is possible to take $\alpha = 0$ and control $L_{B,R}$ by means of Sobolev embedding, this will change the critical exponents. Some interplay between p, p_1, p_2 can appear in non-existence conditions. The case $\alpha_1 = 0$ is admissible if $\beta_1 = 0$ in the assumption (H_5). Indeed the perturbation $|\partial_x u|^{p_1}$ interacts with the quasilinear dependence of G with respect to $\partial_x u$. Similarly the case $\alpha_2 = 0$ is admissible if $\beta_2 = 0$.

- One can consider the most general case in which a source term $\alpha_3^2|u_t|^{p_3}$ appears.
- One can treat second order in time Cauchy Problems as for Boussinesq equations. In such case other conditions on initial data will appear.
- We believe it is possible to extend this method when $N = N(x, \xi, \partial_x u, D_\xi u)$, that is N has variable coefficients. In this case the growth order of N with respect to x, ξ may change the non-existence exponents. We neglect this extension since we started from some Mathematical Physics equations which quasilinear terms do not involve variable coefficients.

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