Asymptotic profiles for a wave equation with parameter dependent logarithmic damping

Ruy Coimbra Charão*
Department of Mathematics
Federal University of Santa Catarina
88040-270, Florianopolis, Brazil
and
Marcello D'Abbicco†
Department of Mathematics
University of Bari
70125 Bari, Italy
and
Ryo Ikehata‡
Department of Mathematics
Division of Educational Sciences
Graduate School of Humanities and Social Sciences
Hiroshima University
Higashi-Hiroshima 739-8524, Japan

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Abstract

We study a nonlocal wave equation with logarithmic damping which is rather weak in the low frequency zone as compared with frequently studied strong damping case. We consider the Cauchy problem for this model in $\mathbf{R}^n$ and we study the asymptotic profile and optimal estimates of the solutions and the total energy as $t \to \infty$ in L^2-sense. In that case some results on hypergeometric functions are useful.

Declaration

*Corresponding author: ruy.charao@ufsc.br; ruycharao@gmail.com
†marcello.dabbicco@uniba.it
‡ikehatar@hiroshima-u.ac.jp

1 Introduction

We consider a new type of wave equation with a logarithmic damping term

$$u_{tt} - \Delta u + L_\theta u_t = 0, \quad (t,x) \in (0,\infty) \times \mathbf{R}^n, \tag{1.1}$$

with initial data

$$u(0,x) = 0, \quad u_t(0,x) = u_1(x), \quad x \in \mathbf{R}^n, \tag{1.2}$$

where u_1 is chosen as

$$u_1 \in L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$$

and a parameter dependent operator

$$L_\theta : D(L_\theta) \subset L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$$

is defined for each $\theta > 0$ as follows:

$$D(L_\theta) := \left\{ f \in L^2(\mathbf{R}^n) \;\middle|\; \int_{\mathbf{R}^n} (\log(1+|\xi|^{2\theta}))^2 |\hat{f}(\xi)|^2 d\xi < +\infty \right\},$$

and for $f \in D(L_\theta)$

$$(L_\theta f)(x) := \mathcal{F}^{-1}_{\xi\to x}\left(\log(1+|\xi|^{2\theta})\hat{f}(\xi)\right)(x).$$

The case $\theta = 1$ has been introduced by Charão-Ikehata [5].

Symbolically writing, one can see that

$$L_\theta = \log(I + A^\theta),$$

where $A = -\Delta$ is the Laplacian operator. Here, we denote the Fourier transform $\mathcal{F}_{x\to\xi}(f)(\xi)$ of $f(x)$ by

$$\mathcal{F}_{x\to\xi}(f)(\xi) = \hat{f}(\xi) := \int_{\mathbf{R}^n} e^{-ix\cdot\xi} f(x)dx$$

as usual with $i := \sqrt{-1}$, and $\mathcal{F}^{-1}_{\xi\to x}$ expresses its inverse Fourier transform. Since the new operator L_θ is constructed by a nonnegative-valued multiplication one, it is nonnegative and self-adjoint in $L^2(\mathbf{R}^n)$. Then, by a similar argument to [21, Proposition 2.1] based on the Lumer-Phillips Theorem one can find that the problem (1.1)-(1.2) has a unique mild solution

$$u \in C([0,\infty); H^1(\mathbf{R}^n)) \cap C^1([0,\infty); L^2(\mathbf{R}^n))$$

satisfying the energy inequality

$$E_u(t) \le E_u(0), \tag{1.3}$$

where

$$E_u(t) := \frac{1}{2}\left(\|u_t(t,\cdot)\|^2_{L^2} + \|\nabla u(t,\cdot)\|^2_{L^2}\right).$$

A main topic of this paper is to find an asymptotic profile of solutions in the L^2 topology as $t \to \infty$ to problem (1.1)-(1.2), and to apply it to get the optimal rate of decay of solutions in terms of the L^2 and energy norms. It should be noticed that when one studies the asymptotic profile of solutions to problem (1.1)-(1.2) under the moment condition

$$\int_{\mathbf{R}^n} u_1(x)\,dx \neq 0,$$

it suffices to assume that the initial amplitude satisfies $u(0,x) = 0$, without loss of generality.

The asymptotic profile $\nu(t,x)$ as $t \to \infty$ of the solution $u(t,x)$ to the equation (1.1) with $\theta = 1$ is already known by [5], and has been represented as

$$\nu(t,x) = \left(\int_{\mathbf{R}^n} u_1(x)dx\right) \mathcal{F}^{-1}_{\xi\to x}\left((1+|\xi|^2)^{-\frac{t}{2}} \frac{\sin(|\xi|t)}{|\xi|}\right),$$

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and the dissipative structure of the solution $u(t,x)$ as $t \to \infty$ is basically dominated by the factor

$$(1+|\xi|^2)^{-\frac{t}{2}} = e^{-\frac{t}{2}\log(1+|\xi|^2)}.$$

In this case, it should be noticed that the behavior of the factor $e^{-\frac{t}{2}\log(1+|\xi|^2)}$ for small $|\xi|$ is similar to the Gauss kernel $e^{-\frac{t}{2}|\xi|^2}$ in the Fourier space because of the fact that

$$\lim_{|\xi|\to 0} \frac{\log(1+|\xi|^2)}{|\xi|^2} = 1.$$

So, the recent result due to Charão-Ikehata [5] is included, in a sense, in the framework of [17], which dealt with the equation (1.1) with L replaced by $A = -\Delta$ (strong damped waves).

A similar consideration remains valid if $\theta \in (1/2, 1)$, in the sense that, due to

$$\lim_{|\xi|\to 0} \frac{\log(1+|\xi|^{2\theta})}{|\xi|^{2\theta}} = 1, \tag{1.4}$$

the results obtained for the wave model with logarithmic damping $L_\theta u_t$ are analogous the results which may be obtained for the wave with fractional damping $A^\theta u_t$, namely,

$$u_{tt} + Au + A^\theta u_t = 0, \quad (t,x) \in (0,\infty) \times \mathbf{R}^n. \tag{1.5}$$

However, when $\theta > 1$, the asymptotic profile for the wave model with logarithmic damping $L_\theta u_t$ is, in general, different from the asymptotic profile for the wave equation with fractional damping $A^\theta u_t$. For this latter a regularity-loss structure appears (see [18]), due to the different behavior at high frequencies:

$$\lim_{|\xi|\to\infty} \frac{\log(1+|\xi|^{2\theta})}{|\xi|^2} = 0, \qquad \lim_{|\xi|\to\infty} \frac{|\xi|^{2\theta}}{|\xi|^2} = \infty \quad (\theta > 1).$$

This shows a crucial difference between (1.5) and (1.1) in the high frequency region, when $\theta > 1$. The regularity-loss structure that appears in the equation (1.5) with $\theta > 1$, do not appear in (1.1) (see Theorem 1.1).

Finally, we observe that the dissipative structure of the solution $u(t,x)$ to problem (1.1)–(1.2) as $t \to \infty$ is associated basically with the function

$$\psi(t,\xi) = e^{-\frac{t}{2}\log(1+|\xi|^{2\theta})} = (1+|\xi|^{2\theta})^{-\frac{t}{2}},$$

and to get the exact asymptotic behavior we handle with several results from the hypergeometric functions combined with the Gautschi inequality.

Now we mention some previous related works.
After two pioneering papers due to Ponce [25] and Shibata [26] studying the strongly damped wave equation

$$u_{tt} + Au + Au_t = 0 \tag{1.6}$$

appear from the viewpoint of $L^p - L^q$ estimates, it seems that one of the main topics on the equation (1.6) has been shifted to study the asymptotic profile and optimal rate of decay of various norms of solutions. The first trial from the point of view of the asymptotic profile of solutions (as $\to \infty$) has been done in abstract form by [21], and in concrete form by [17]. In this connection, before [21, 17], in [4] sharp energy decay estimates of the total energy are derived by a new type of energy method in the Fourier space combined with the Haraux-Komornik inequality. Although the sharpness of the results has been already discussed in the higher dimensional case such as $n \geq 3$ in [17], the low dimensional case ($n = 1, 2$) has been completed in the paper [19] at last by observing a strong singularity. Quite recently, in the papers [1, 2, 3] and [23] higher order asymptotic expansions of the solutions as $t \to \infty$ to the equation (1.6) are investigated by finding optimal rates of decay and/or blowup in infinite time.

On the other hand, a complete generalization to the structurally damped wave equation ($\alpha, \nu > 0$):

$$u_{tt} + \alpha A^\sigma u + \nu A^\delta u_t = 0 \tag{1.7}$$

with a parameter $\sigma \in (\delta, 2\delta)$ can be done in the papers [12] and [20] (only for $\sigma = 1$) from the viewpoint of capturing the leading terms of the solutions as time goes to infinity. In this connection, we have to cite a paper due to Narazaki-Reissig [24] which studies $L^1 - L^1$ estimates for solutions of the equation (1.7) with $\sigma = \alpha = 1$ and $\delta \in (0,1)$ (see also $L^1 - L^1$ estimates in [6] for waves with dissipative terms and in [14] for σ-evolution equations).

We stress that the case $0 < 2\delta \leq \sigma$ in (1.7) corresponds to a very different asymptotic profile of the solution, and in general, two different diffusive profiles for the solution may appear, see D'Abbicco-Ebert [7, 8, 9, 10].

Theorem 1.1 *Let $n \geq 1$, $\theta > 1/2$ and $u_1 \in L^2(\mathbf{R}^n) \cap L^{1,1}(\mathbf{R}^n)$. Then, the unique solution $u(t,x)$ to the problem* (1.1)-(1.2) *satisfies*

$$\left\| u(t,\cdot) - \left(\int_{\mathbf{R}^n} u_1(x)dx \right) \mathcal{F}^{-1}_{\xi\to x} \left((1+|\xi|^{2\theta})^{-\frac{t}{2}} \frac{\sin(|\xi|t)}{|\xi|} \right) \right\|_{L^2} \leq I_0 t^{-\beta(n,\theta)}, \quad (t \gg 1),$$

where $\beta = \beta(n,\theta)$ depends only on n and θ and is given by

$$\beta = \frac{n}{4\theta} \ \ for \ \ \theta \geq 1 \quad and \quad \beta = \frac{n+4\theta-4}{4\theta} \ \ for \ \ 1/2 < \theta < 1, \tag{1.8}$$

and

$$I_0 = \|u_1\|_{1,1} + \|u_1\|_{L^2}.$$

We stress that $\beta > 0$, exception given for the case $n = 1$ and $\theta \in (1/2, 3/4]$.

Theorem 1.2 *Let $n \geq 1$, and let $u_1 \in L^2(\mathbf{R}^n) \cap L^{1,1}(\mathbf{R}^n)$. Assume that $\theta > 1/2$. Then, the unique solution $u(t,x)$ to problem* (1.1)-(1.2) *satisfies*

(i) $n \geq 3 \Rightarrow C_n|P_1|t^{-\frac{n-2}{4\theta}} \leq \|u(t,\cdot)\|_{L^2} \leq C_n^{-1} I_0 t^{-\frac{n-2}{4\theta}}$ $(t \gg 1)$,

(ii) $n = 2 \Rightarrow C_2|P_1|\sqrt{\log t} \leq \|u(t,\cdot)\|_{L^2} \leq C_2^{-1} I_0 \sqrt{\log t}$ $(t \gg 1)$,

(iii) $n = 1 \Rightarrow C_1|P_1|\sqrt{t} \leq \|u(t,\cdot)\|_{L^2} \leq C_1^{-1} I_0 \sqrt{t}$ $(t \gg 1)$,

where I_0 is defined in Theorem 1.1,

$$P_1 = \int_{\mathbf{R}^n} u_1(x)\,dx,$$

and C_n $(n \in \mathbf{N})$ are constants which depend on θ and are independent from any t and initial data.

We stress that $\beta(n,\theta) \to (n-2)/2$ as $\theta \to 1/2$ in Theorem 1.1, so that, in view of Theorem 1.2, it is not possible to describe the asymptotic behavior of the solution to problem (1.1)-(1.2) as we did in the case $\theta > 1/2$.

Remark 1.1 The reason why we are particularly interested about the estimates of the solution itself (not the time and/or spatial derivatives of the solution) is that the solution itself sometimes includes a kind of singularity near 0 frequency part, and this observation clearly appears by measuring the solution itself in terms of L^2-norm, and as for the time and/or spatial derivatives of the solution we may be able to treat sometimes by another well-known method. Anyway, we want to observe how a singularity appears in the solution throughout our series of papers. It is also interesting to note that although the estimate for the L^2-norm depends on θ, for low dimension $n = 1$ and 2 the blow-up explosion rate at infinity is the same as for the case $\theta = 1$ (see [5]).

Unlike the L^2-norm of the solution, the energy norm decays for all dimension n. We observe that to obtain the optimal behavior in time for the L^2-norm of $u_t(t,\cdot)$ we use the asymptotic profile given by (4.3) (see Proposition 4.5). In particular the following result holds.

Theorem 1.3 *Let $n \geq 1$, and let $u_1 \in \left(L^2(\mathbf{R}^n) \cap L^{1,1}(\mathbf{R}^n)\right)$. Assume that $\theta > 1/2$. Then, the unique solution $u(t,x)$ to problem* (1.1)-(1.2) *satisfies*

$$C_{n,\theta}|P_1|\,t^{-\frac{n}{4\theta}} \leq \|u_t(t,\cdot)\|_{L^2} + \|\nabla u(t,\cdot)\|_{L^2} \leq C_{n,\theta}^{-1} I_0 t^{-\frac{n}{4\theta}}, \quad t \gg 1,$$

where $C_{n,\theta}$ is a positive constant, and I_0 is a constant defined in Theorem 1.1.

Remark 1.2 The optimality of the estimates obtained in Theorems 1.2 and 1.3 hints to the possibility to compute the critical exponent of global-in-time solutions to

$$u_{tt} + Au + L_\theta u_t = |u|^p, \quad (t,x) \in (0,\infty) \times \mathbf{R}^n.$$

Following as in [13], it is easy to show that global-in-time energy solutions exist for initial small data in $L^1 \cap L^2$, if $p > p_c = 1 + (1+2\theta)/(n-1)$, in space dimension $n = 2$, for any $\theta > 1/2$, and if $p \in (p_c, 3]$, in space dimension $n = 3$, if $\theta \in (1/2, 3/2)$. It remains open to check whether some kind of nonexistence of global-in-time solutions result may hold for $p \le p_c$ or, otherwise, if the existence exponent may be improved by some means (in the case of σ-evolution equation with structural damping as in (1.7), see [11] for a different kind of exponent).

This paper is organized as follows. In section 2 we prepare several important lemmas, which will be used later. In section 3 we shall derive the asymptotic profile of the solution as $t \to \infty$, and Theorem 1.1 can be proved at a stroke. Section 4 is divided into two subsections, and in subsection 4.1, we study the upper and lower bound of the time estimates to the profile of the solution found in Section 3, and in subsection 4.2, we study the optimality of the decay rate of the total energy by deriving the leading term of the time derivative of the solution to problem (1.1). The result in subsection 4.2 seems new in the framework of this type of equations.

Notation. Throughout this paper, $\|\cdot\|_q$ stands for the usual $L^q(\mathbf{R}^n)$-norm. For simplicity of notation, in particular, we use $\|\cdot\|$ instead of $\|\cdot\|_2$. Furthermore, we denote $\|\cdot\|_{H^l}$ as the usual H^l-norm. Furthermore, we define a relation $f(t) \sim g(t)$ as $t \to \infty$ by: there exist constant $C_j > 0$ $(j = 1,2)$ such that

$$C_1 g(t) \le f(t) \le C_2 g(t) \quad (t \gg 1).$$

We also introduce the following weighted functional spaces

$$L^{1,\gamma}(\mathbf{R}^n) := \left\{ f \in L^1(\mathbf{R}^n) \;\middle|\; \|f\|_{1,\gamma} := \int_{\mathbf{R}^n} (1+|x|^\gamma)|f(x)|dx < +\infty \right\}.$$

Finally, we denote the surface area of the n-dimensional unit ball by $\omega_n := \displaystyle\int_{|\omega|=1} d\omega$.

2 Preliminaries

2.1 Basic integral estimates

Taking advantage of the theory of hypergeometric functions, we are interested in studying the asymptotic behavior as $t \to \infty$ of special integrals.

Lemma 2.1 *Let $0 \le x_1 < x_2 \le \infty$, $\mu \in \boldsymbol{R}$, and $t \in (0,\infty)$. Also, assume that $\mu > 0$ if $x_1 = 0$ and that $t > \mu$ if $x_2 = \infty$. We consider the integral*

$$I_{\mu;x_1,x_2}(t) = \int_{x_1}^{x_2} \frac{x^{\mu-1}}{(1+x)^t}\, dx.$$

Then the following asymptotic behavior holds, for any $x_2 \in (x_1, \infty]$:

$$\lim_{t\to\infty} t^\mu\, I_{\mu;0,x_2}(t) = \Gamma(\mu), \qquad \forall \mu > 0, \quad \text{if } x_1 = 0, \tag{2.1}$$

$$\lim_{t\to\infty} (x_1+1)^{t-1}\, t\, I_{\mu;x_1,x_2}(t) = x_1^{\mu-1}, \quad \text{if } x_1 > 0. \tag{2.2}$$

The limits in (2.1) *and in* (2.2) *are uniform with respect to x_2 on closed subsets of $(x_1, \infty]$.*

The relation of $I_{\mu;x_1,x_2}(t)$ with the hypergeometric functions is based on the fact that (see, e.g., [15, 3.149]):

$$I_{\mu;0,x_2}(t) = \frac{x_2^\mu}{\mu}\, {}_2\mathrm{F}_1(t,\mu;\mu+1;-x_2) \tag{2.3}$$

$$I_{\mu;x_1,\infty}(t) = \frac{x_1^{\mu-t}}{t-\mu}\, {}_2\mathrm{F}_1(t,t-\mu;t-\mu+1;-1/x_1). \tag{2.4}$$

The definition by series of the hypergeometric functions ${}_2\mathrm{F}_1(a,b;c;z)$ is:

$$ {}_2\mathrm{F}_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \, . $$

Proof. We first notice that, by difference, it is sufficient to prove (2.2) with $x_2 = \infty$, due to $(x_2+1)^{-t} = o((x_1+1)^{-t})$ as $t \to \infty$. This latter property implies that the limit in (2.2) is uniform with respect to x_2 on closed subsets of $(x_1, \infty]$.

Then we remark that (see, for instance, [27]):

$$ I_{\mu;0,\infty}(t) = \int_0^{\infty} (1+\rho)^{-t} \, \rho^{\mu-1} \, d\rho = B(\mu, t-\mu) = \frac{\Gamma(\mu)\,\Gamma(t-\mu)}{\Gamma(t)} \, , $$

where B is the Beta function and Γ the Gamma function. As a consequence of the Gautschi inequality, it holds

$$ \lim_{t\to\infty} \frac{t^s \, \Gamma(t-s)}{\Gamma(t)} = 1, \quad \forall s \in (0,1), $$

so that

$$ \lim_{t\to\infty} t^{\mu} \, I_{\mu;0,\infty}(t) = \Gamma(\mu) \, . $$

By difference, now (2.1) follows as a consequence of (2.2) with $x_2 = \infty$. In particular, the limit in (2.1) is uniform with respect to x_2 on closed subsets of $(0, \infty]$.

Using (2.4), thanks to the formula

$$ {}_2\mathrm{F}_1(a,b;c;z) = (1-z)^{c-a-b} \, {}_2\mathrm{F}_1(c-a, c-b; c; z), $$

we see that

$$ I_{\mu;x_1,\infty}(t) = \frac{x_1^{\mu-1} \, (x_1+1)^{1-t}}{t-\mu} \, {}_2\mathrm{F}_1(1-\mu, 1; t+1-\mu; -1/x_1), $$

and the proof follows by noticing that ${}_2\mathrm{F}_1(1-\mu, 1; t+1-\mu; -1/x_1) \to 1$, as $t \to \infty$.

□

By the change of variable

$$ \int_{\eta}^{\eta_2} (1+r^{2\theta})^{-t} r^p dr = \frac{1}{2\theta} \int_{\eta^{2\theta}}^{\eta_2^{2\theta}} (1+r)^{-t} r^{\frac{p+1}{2\theta}-1} dr, $$

we obtain the following.

Corollary 2.1 *Let $\theta > 0$ and $p > -1$. Then*

$$ \lim_{t\to\infty} t^{\frac{p+1}{2\theta}} \, I_{p,\theta}(t) = \frac{1}{2\theta} \, \Gamma\left(\frac{p+1}{2\theta}\right), \quad \forall p > -1, \tag{2.5} $$

$$ \lim_{t\to\infty} (1+\eta^{2\theta})^{t-1} \, t \, J_{\eta;p,\theta}(t) = \frac{\eta^{p+1-2\theta}}{2\theta}, \quad \forall p \in \boldsymbol{R}, \, \eta > 0, \tag{2.6} $$

where

$$ I_{p,\theta}(t) = \int_0^{\eta_2} (1+r^{2\theta})^{-t} r^p dr \, , \quad \textit{for some } \eta_2 \in (0, \infty], $$

$$ J_{\eta;p,\theta}(t) = \int_{\eta}^{\eta_2} (1+r^{2\theta})^{-t} r^p dr \, , \quad \textit{for some } \eta_2 \in (\eta, \infty]. $$

2.2 Inequalities and asymptotics

Remark 2.1 Let $\theta > 0$. Then, it is important to note that the inequality

$$ |\xi|^{2\theta} - \log(1+|\xi|^{2\theta}) \geq 0 \tag{2.7} $$

holds for all $\xi \in \mathbf{R}^n$. Moreover, for each $\theta \geq 1/2$ there exists a number $\delta_0 = \delta_0(\theta)$, $0 < \delta_0 < 1$, such that the inequality

$$ 4|\xi|^2 - \log^2(1+|\xi|^{2\theta}) > 0 $$

holds for $\xi \in \mathbf{R}^n$ such that $0 < |\xi| \leq \delta_0$.

Remark 2.2 An explicit computation of the possible zeroes of the function

$$r \mapsto 4r^2 - \log^2(1+r^{2\theta}) = (2r - \log(1+r^{2\theta}))(2r + \log(1+r^{2\theta}))$$

is not necessary to prove our estimates, however we may include a few details for the sake of completeness. The zeros of the derivative of the function $h(r) = 2r - \log(1+r^{2\theta})$ are given by the equation $g(r) = 0$, where

$$g(r) = r^{2\theta} - \theta r^{2\theta-1} + 1.$$

Since $\min g = g(\theta - 1/2)$, the equation above admits two different real solutions if, and only if, $g(\theta-1/2) < 0$, that is, θ verifies the inequality $(\theta - 1/2)^{\theta-1/2} > \sqrt{2}$. In this latter case, noticing that $g(\theta) > 0$, we find that the local minimum of h is attained at some $\bar{r}$ in $(\theta - 1/2, \theta)$. Indeed, $\bar{r} \to \theta$ as θ becomes larger. Due to $h(\theta) < 2\theta(1 - \log\theta)$ we find that the minimum of h is negative at least for $\theta \geq e$.

Lemma 2.2 *Let $\theta \geq 1/2$. Then, the real functions $a(\xi)$ and $b(\xi)$ given by*

$$a(\xi) = \frac{\log(1+|\xi|^{2\theta})}{2} \quad \text{and} \quad b(\xi) = \frac{1}{2}\sqrt{4|\xi|^2 - \log^2(1+|\xi|^{2\theta})} \tag{2.8}$$

are well defined for $\xi \in \mathbf{R}^n$ such that $0 \leq |\xi| \leq \delta_0$.

To study an asymptotic profile of the solution to problem (1.1)–(1.2) we consider a decomposition of the Fourier transformed initial data.

Remark 2.3 Using the Fourier transform we can get a decomposition of the initial data $\hat{u}_1$ as follows

$$\hat{u}_1(\xi) = A_1(\xi) - iB_1(\xi) + P_1, \quad \xi \in \mathbf{R}^n,$$

where P_1, A_1, B_1 are defined by

$$P_1 = \int_{\mathbf{R}^n} u_1(x)dx, \quad A_1(\xi) = \int_{\mathbf{R}^n} u_1(x)\big(1 - \cos(\xi x)\big)dx, \quad B_1(\xi) = \int_{\mathbf{R}^n} u_1(x)\sin(\xi x)dx.$$

The next lemma according to the above decomposition appears in Ikehata [16].

Lemma 2.3 *Let $\kappa \in [0,1]$. For $u_1 \in L^{1,\kappa}(\mathbf{R}^n)$ and $\xi \in \mathbf{R}^n$ it holds that*

$$|A_1(\xi)| \leq K|\xi|^\kappa \|u_1\|_{L^{1,\kappa}} \quad \text{and} \quad |B_1(\xi)| \leq M|\xi|^\kappa \|u_1\|_{L^{1,\kappa}},$$

with positive constants K and M depending only on n.

3 Asymptotic profiles of solutions

The associated Cauchy problem to (1.1)-(1.2) in the Fourier space is given by

$$\begin{aligned} &\hat{u}_{tt}(t,\xi) + |\xi|^2\hat{u}(t,\xi) + \log(1+|\xi|^{2\theta})\hat{u}_t(t,\xi) = 0, \\ &\hat{u}(0,\xi) = 0, \quad \hat{u}_t(0,\xi) = \hat{u}_1(\xi). \end{aligned} \tag{3.1}$$

The characteristics roots λ_+ and λ_- of the characteristic polynomial

$$\lambda^2 + \log(1+|\xi|^{2\theta})\lambda + |\xi|^2 = 0, \quad \xi \in \mathbf{R}^n$$

associated to the equation (3.1) are given by

$$\lambda_\pm = \frac{-\log(1+|\xi|^{2\theta}) \pm \sqrt{\log^2(1+|\xi|^{2\theta}) - 4|\xi|^2}}{2}. \tag{3.2}$$

For $\theta \geq 1/2$ it should be mentioned that there is a number $\delta_0 > 0$ such that (see (2.1))

$$\log^2(1+|\xi|^{2\theta}) - 4|\xi|^2 < 0$$

for $\xi \in \mathbf{R}^n$ with $0 < |\xi| \leq \delta_0$. Therefore the characteristics roots are complex-valued and the real part is negative for $\theta \geq 12$ and $\xi \in \mathbf{R}^n$, $0 < |\xi| \leq \delta_0$.

Remark 3.1 We note that the function $f(r) = \log^2(1 + r^{2\theta}) - 4r^2,\ r \geq 0$ has derivative singular on $r = 0$ if $0 < \theta < 1/2$. Thus, in this case the characteristics roots given by (3.2) are real on the zone of low frequency and the structure of solutions will be different and we need to use a special method to deal with. Therefore, we do not study this case here and it remains open. The threshold case $\theta = 1/2$ also needs a different treatment. Indeed, for example, to obtain the important Proposition 3.1 we use the crucial fact that (see (3.10))

$$\lim_{r \to +0} \frac{\log^2(1 + r^{2\theta})}{r^2} = 0.$$

that holds only to $\theta > 1/2$ while for $\theta = 1/2$ this limit is 1 and for $\theta < 1/2$ the limit is $+\infty$ (see Remark 3.2).

From now on we will consider only the case $\theta > 1/2$. Then, for $\theta > 1/2$ we can write down $\lambda_\pm$ in the following form

$$\lambda_\pm = -a(\xi) \pm ib(\xi),$$

where $a(\xi)$ and $b(\xi)$ are defined by (2.8) in Lemma 2.2. In this case the solution of the equation (3.1) is given explicitly by

$$\hat{u}(t, \xi) = \frac{\hat{u}_1(\xi)}{b(\xi)} \sin(b(\xi)t) e^{-a(\xi)t}$$

for $\xi \in \mathbf{R}^n, |\xi| \leq \delta_0$ and $t \geq 0$.

Next, in order to find a better expression for $\hat{u}(t, \xi)$ we apply the mean value theorem to get

$$\sin\left(b(\xi)t\right) = \sin(|\xi|t) + t\left(b(\xi) - |\xi|\right)\cos(\mu(\xi)t), \tag{3.3}$$

with

$$\mu(\xi) := \eta_1 b(\xi) + (1 - \eta_1)|\xi|$$

for some $\eta_1 \in (0, 1)$, and

$$\frac{1}{\sqrt{1 - g(r)}} = 1 + \frac{\log^2(1 + r^{2\theta})}{8r^2} \frac{1}{\sqrt{(1 - \eta_2 g(r))^3}} \tag{3.4}$$

with some $\eta_2 \in (0, 1)$, where $r := |\xi|$, and

$$g(r) := \frac{\log^2(1 + r^{2\theta})}{4r^2}.$$

The identity (3.4) was obtained by applying the mean value theorem to the function

$$G(s) = \frac{1}{\sqrt{(1 - sg(r))^3}},\ 0 \leq s \leq 1.$$

Then by using Remark 2.3, (3.3) and (3.4) $\hat{u}(t, \xi)$ can be re-written as

$$\begin{aligned}\hat{u}(t, \xi) = {} & P_1 e^{-a(\xi)t} \frac{\sin(tr)}{r} + P_1 \frac{\log^2(1 + r^{2\theta})}{8r^3} \frac{1}{\sqrt{(1 - \eta_2 g(r))^3}} e^{-a(\xi)t} \sin(tr) \\ & + \left(\frac{A_1(\xi) - iB_1(\xi)}{b(\xi)} \right) e^{-a(\xi)t} \sin(b(\xi)t) + P_1 t e^{-a(\xi)t} \left(\frac{b(\xi) - r}{b(\xi)} \right) \cos(\mu(\xi)t).\end{aligned} \tag{3.5}$$

It should be remarked that (3.5) holds for small frequency parameters $\xi \in \mathbf{R}^n_\xi$ satisfying $0 < |\xi| \leq \delta_0$.

We now introduce a candidate to be a leading term as $t \to \infty$ of the solution in the following simple form:

$$P_1 e^{-a(\xi)t} \frac{\sin(|\xi|t)}{|\xi|}, \tag{3.6}$$

where $a(\xi) = \dfrac{\log(1 + |\xi|^{2\theta})}{2}$.

Our goal in this section is to get decay estimates in time to the remainder terms defined in (3.5). To proceed with that we define the next *three* functions which imply remainders with respect to the leading term (3.5).

- $K_1(t,\xi) = \Big(\dfrac{A_1(\xi) - iB_1(\xi)}{b(\xi)}\Big) e^{-a(\xi)t}\sin(b(\xi)t)$;
- $K_2(t,\xi) = P_1 e^{-a(\xi)t}\sin(rt)\dfrac{\log^2(1+r^{2\theta})}{8r^3}\dfrac{1}{\sqrt{(1-\eta_2 g(r))^3}}, \quad r = |\xi| > 0$;
- $K_3(t,\xi) = t\,P_1 e^{-a(\xi)t}\left(\dfrac{b(\xi)-|\xi|}{b(\xi)}\right)\cos(\mu(\xi)t)$,

where $a(\xi)$ and $b(\xi)$ are defined in Lemma 2.2. Note that using these $K_j(t,\xi)$ $(j = 1,2,3)$ the solution $\hat{u}(t,\xi)$ to problem (3.1) can be expressed as

$$\hat{u}(t,\xi) - P_1 e^{-a(\xi)t}\frac{\sin(tr)}{r} = \sum_{j=1}^{3} K_j(t,\xi). \tag{3.7}$$

Let us check, in fact, that $\{K_j(t,\xi)\}$ become error terms by using previous lemmas studied in Section 2. First we obtain decay rates for each one of these functions on the zone of low frequency $|\xi| \ll 1$.

We begin with the estimate for $K_1(t,\xi)$. For this function we prepare the following expression for $1/b(\xi)$ based on (3.4):

$$\frac{1}{b(\xi)} = \frac{1}{r} + \frac{\log^2(1+r^{2\theta})}{8r^3}\frac{1}{\sqrt{(1-\eta_2 g(r))^3}}, \quad r = |\xi| > 0. \tag{3.8}$$

Then,

$$\begin{aligned} K_1(t,\xi) := {} & \frac{A_1(\xi) - iB_1(\xi)}{|\xi|} e^{-a(\xi)t}\sin(b(\xi)t) \\ & + \big(A_1(\xi) - iB_1(\xi)\big)\frac{\log^2(1+r^{2\theta})}{8r^3}\frac{e^{-a(\xi)t}\sin(b(\xi)t)}{\sqrt{(1-\eta_2 g(r))^3}} =: K_{1,1}(t,\xi) + K_{1,2}(t,\xi). \end{aligned}$$

It is easy to check the following estimate based on Lemma 2.3 with $k = 1$ and Corollary 2.1 with $p = n-1$:

$$\begin{aligned} \int_{|\xi|\le\delta} |K_{1,1}(t,\xi)|^2 d\xi &\le (M+K)^2\|u_1\|_{1,1}^2 \int_{|\xi|\le\delta} e^{-t\log(1+|\xi|^{2\theta})} d\xi \\ &= \omega_n (M+K)^2\|u_1\|_{1,1}^2 \int_0^\delta (1+r^{2\theta})^{-t} r^{n-1} dr \\ &\le C\omega_n (M+K)^2 t^{-\frac{n}{2\theta}}\|u_1\|_{1,1}^2, \quad (t \gg 1). \end{aligned} \tag{3.9}$$

Remark 3.2 We note that

$$\lim_{r\to+0}\frac{\log^2(1+r^{2\theta})}{r^2} = 1$$

for $\theta = 1/2$ and

$$\lim_{r\to+0}\frac{\log^2(1+r^{2\theta})}{r^2} = +\infty$$

for $0 \le \theta < 1/2$. Due to theses limits the corresponding cases for θ are more difficult to treat.

Now, using the important fact

$$\lim_{r\to+0}\frac{\log^2(1+r^{2\theta})}{r^2} = 0 \tag{3.10}$$

for $\theta > 1/2$, we see that there is a constant δ, $0 < \delta \le 1$, such that for all $0 < r \le \delta$ it holds that

$$g(r) = \frac{\log^2(1+r^{2\theta})}{4r^2} \le 1/2. \tag{3.11}$$

Then, this implies

$$\frac{1}{\sqrt{(1-\eta_2 g(r))^3}} \le 2\sqrt{2}. \tag{3.12}$$

Thus, from (3.11), (3.12) and Corollary 2.1, together with Lemma 2.3 for $k = 1$, similarly to (3.9) one can also derive

$$\begin{aligned}\int_{|\xi|\le\delta}|K_{1,2}(t,\xi)|^2 d\xi &\le 8^{-1}(M+K)^2\|u_1\|_{1,1}^2\int_{|\xi|\le\delta}\left(\frac{\log^2(1+r^{2\theta})}{r^2}\right)^2 e^{-2ta(\xi)}d\xi\\ &\le \frac{1}{2}(M+K)^2\|u_1\|_{1,1}^2\int_{|\xi|\le\delta}e^{-2ta(\xi)}d\xi\\ &= \frac{1}{2}(M+K)^2\|u_1\|_{1,1}^2\omega_n\int_0^\delta(1+r^{2\theta})^{-t}r^{n-1}d\xi\\ &\le C\omega_n(M+K)^2\|u_1\|_{1,1}^2 t^{-\frac{n}{2\theta}},\quad (t\gg 1). \end{aligned}\tag{3.13}$$

By combining (3.9) and (3.13) we have the following estimate for $K_1(t,\xi)$,

$$\int_{|\xi|\le\delta}|K_1(t,\xi)|^2 d\xi \le C_{1,n}\|u_1\|_{1,1}^2 t^{-\frac{n}{2\theta}},\quad (t\gg 1).\tag{3.14}$$

Similarly to the computations to (3.14) and using (3.12) one can also obtain the following estimate for $K_2(t,\xi)$

$$\int_{|\xi|\le\delta}|K_2(t,\xi)|^2 d\xi \le C_{1,n}\,|P_1|^2\,t^{-\frac{n}{2\theta}},\quad (t\gg 1),\tag{3.15}$$

in the case of $\theta > 3/4$, due to

$$\lim_{r\to+0}\frac{\log^2(1+r^{2\theta})}{r^3} = 0.\tag{3.16}$$

However, if $\theta \in (1/2, 3/4]$, we have a different estimate. In this case, we estimate

$$\left(\frac{\log^2(1+r^{2\theta})}{r^3}\right)^2 \le r^{8\theta-6},\tag{3.17}$$

so that

$$\begin{aligned}\int_{|\xi|\le\delta}|K_2(t,\xi)|^2 d\xi &\le 8^{-1}\,|P_1|^2\int_{|\xi|\le\delta}r^{8\theta-6}\,e^{-2ta(\xi)}d\xi\\ &= 8^{-1}\,|P_1|^2\omega_n\int_0^\delta(1+r^{2\theta})^{-t}r^{8\theta-6+n-1}dr\\ &\le C\omega_n|P_1|^2\,t^{-\frac{n+8\theta-6}{2\theta}},\quad (t\gg 1),\end{aligned}\tag{3.18}$$

where we used that $8\theta-6+n > 0$ for any $\theta > 1/2$ if $n \ge 2$, and for any $\theta > 5/8$ if $n = 1$. On the other hand, if $n = 1$ and $\theta \in (1/2, 5/8]$, we estimate

$$\sin^2(tr)\left(\frac{\log^2(1+r^{2\theta})}{r^3}\right)^2 \le t\,r^{8\theta-5},\tag{3.19}$$

so that

$$\begin{aligned}\int_{|\xi|\le\delta}|K_2(t,\xi)|^2 d\xi &\le 4^{-1}\,t\,|P_1|^2\int_0^\delta(1+r^{2\theta})^{-t}r^{8\theta-5}dr\\ &\le C\omega_n|P_1|^2\,t^{-3+\frac{2}{\theta}},\quad (t\gg 1).\end{aligned}\tag{3.20}$$

Finally, we have to deal with the case of $K_3(t,\xi)$. This part is crucial in this paper.

We need suitable estimates on this term because the multiplication by t is included in its definition. To do that we observe that it is not difficult to see the following expression:

$$b(\xi) - r = r\left(-\frac{\frac{\log^2(1+r^{2\theta})}{4r^2}}{1+\sqrt{1-\frac{\log^2(1+r^{2\theta})}{4r^2}}}\right) = r\left(-\frac{g(r)}{1+\sqrt{1-g(r)}}\right),\quad (r = |\xi| \ne 0),$$

where again

$$g(r) = \frac{\log^2(1+r^{2\theta})}{4r^2}.$$

This implies

$$\frac{b(\xi)-r}{b(\xi)} = g(r)\left(\frac{-1}{1-g(r)+\sqrt{1-g(r)}}\right) =: g(r)h(r).$$

We make a next identity to gain $r^{4\theta-2}$ near $r=0$:

$$\left|\frac{b(\xi)-r}{b(\xi)}\right| = r^{4\theta-2}|h(r)|(\frac{g(r)}{r^{4\theta-2}}) \quad r \neq 0.$$

Notice that

$$\lim_{r\to+0}|h(r)| = \frac{1}{2},$$

because of the fact

$$\lim_{r\to+0} g(r) = 0$$

for $\theta > 1/2$. Furthermore, one can check that

$$\lim_{r\to+0}\frac{g(r)}{r^{4\theta-2}} = \frac{1}{4},$$

for $2\theta > 1$. Therefore, from these facts one can find $C>0$ and $\delta_1,\ 0<\delta_1\leq 1<\delta_0$ such that for all $r\in(0,\delta_1)$

$$\left|\frac{b(\xi)-r}{b(\xi)}\right| \leq Cr^{4\theta-2}. \tag{3.21}$$

By (3.21) and the definition of $K_3(t,\xi)$ one can estimate $K_3(t,\xi)$ as follows:

$$\int_{|\xi|\leq\delta_1}|K_3(t,\xi)|^2d\xi \leq |P_1|^2t^2\int_{|\xi|\leq\delta_1} r^{8\theta-4}e^{-2ta(\xi)}d\xi \leq C|P_1|^2t^{-\frac{n+4\theta-4}{2\theta}}, \tag{3.22}$$

for each $\theta > 1/2 > 3/8$, where one has just used Corollary 2.1 and the definition of $a(\xi)$ in Lemma 2.2.

Note that in the case when $\theta > 1/2$ we see

$$\frac{n+4\theta-4}{2\theta} > \frac{n-2}{2\theta}.$$

Now, by summarizing above discussion one can arrived at the following crucial lemma based on (3.7), (3.14), (3.15), (3.22).

Proposition 3.1 *Let $n\geq 1$ and $\theta > 1/2$. Then, there exists a small constant $\delta_1\in(0,1]$ such that*

$$\int_{|\xi|\leq\delta_1}\left|\hat{u}(t,\xi) - P_1e^{-a(\xi)t}\frac{\sin(tr)}{r}\right|^2d\xi \leq C\Big(|P_1|^2 + \|u_1\|_{1,1}^2\Big)t^{-2\beta(n,\theta)}, \quad (t\gg 1),$$

where the constant $\beta=\beta(n,\theta)$ depending only on n and θ is given by (1.8), that is,

$$\beta = \frac{n}{4\theta}\ \ for\ \ \theta\geq 1 \quad and \quad \beta = \frac{n+4\theta-4}{4\theta}\ \ for\ \ 1/2<\theta<1.$$

We stress that $\beta>0$, exception given for the case $n=1$ and $\theta\in(1/2,3/4]$. The constant $C=C_n>0$ in above estimate depends only on the dimension n. □

Remark 3.3 We observe that to the case $n=1$ and $\theta\in(1/2,3/4]$ the blowup order $\mathcal{O}(t^{\frac{3}{2\theta}-2})$ is slower than $\mathcal{O}(t)$, which is the optimal blowup order of the leading term coming from Proposition 4.2.

Next, let us prepare the so-called high frequency estimates to the L^2-norm of the solution $\hat{u}(\xi,t)$. In fact, the solution decays very fast, as usual, to the case $\theta \geq 1/2$ on the high frequency region $|\xi| \geq \delta_1$.

We note that on this region the characteristics roots can be real on part of the region $\{|\xi| \geq \delta_1\}$ depending on the size of θ, as for example $\theta \geq 3$. In fact, for $\theta = 3$, according to Remark 2.2 on the zone of high frequency $r_1 < |\xi| < r_2$ the characteristic roots are real, and the structure of the solution in the Fourier space is diffusion-like instead of wave-like type.

Thus, we apply another method to get the precise decay rate of the L^2-norm of solutions on the zone of high frequency, based on the following.

Lemma 3.1 *Assume that the roots $\lambda_\pm$ of*

$$\lambda^2 + a\lambda + b = 0$$

verify $\Re\lambda_- \leq \Re\lambda_+ < 0$. Then the solution to

$$y'' + ay' + b = 0, \quad y(0) = 0, \quad y'(0) = y_1,$$

verifies the decay estimate

$$|y(t)| \leq e^{\lambda_+ t}\, t\, |y_1|, \qquad |y'(t)| \leq e^{\lambda_+ t}\, (1 + t|\lambda_-|)\, |y_1|,$$

for any $t \geq 0$.

Proof. If $\lambda_+ = \lambda_-$, then the solution is $y = te^{\lambda_+ t}\, y_1$, and the proof is concluded. Otherwise, the solution is

$$y = \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}\, y_1 = e^{\lambda_+ t}\, \frac{1 - e^{(\lambda_- - \lambda_+)t}}{\lambda_+ - \lambda_-}\, y_1 = e^{\lambda_+ t}\, t\, y_1 \int_0^1 e^{\theta(\lambda_- - \lambda_+)t}\, d\theta,$$

where we used the Taylor expansion

$$e^x = 1 + \int_0^1 x\, e^{\theta x}\, d\theta.$$

As a consequence, using $\Re(\lambda_- - \lambda_+) \leq 0$, we derive $|y(t)| \leq e^{\lambda_+ t}\, t\, |y_1|$. Using

$$y' = \frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-}\, y_1 = e^{\lambda_+ t} y_1 + \lambda_- y$$

we conclude the proof.

□

Proposition 3.2 *Let $n \geq 1$, $\theta \geq 1/2$ and $\Omega_h = \{\xi \in \boldsymbol{R}^n : \; |\xi| \geq \delta_1\}$, where $\delta_1 > 0$ is defined in Proposition 3.1. Then, it holds that*

$$\int_{\Omega_h} |\hat{u}_t(t,\xi)|^2 d\xi \leq C\, \|u_1\|^2\, e^{-\gamma t}, \quad (t \to \infty),$$

and

$$\int_{\Omega_h} |\hat{u}(t,\xi)|^2 d\xi \leq C\, \|u_1\|^2\, e^{-\gamma t}, \quad (t \to \infty),$$

for some constant $\gamma > 0$ and $C > 0$.

Proof. By Lemma 3.1, we immediately obtain

$$\int_{\Omega_h} |\hat{u}(t,\xi)|^2 d\xi \leq t^2\, e^{-2\gamma t}\, \|u_1\|^2,$$

where

$$\gamma = \min_{\Omega_h} \Re(-\lambda_+) > 0.$$

The proof follows estimating $t^2\, e^{-\gamma t} \leq C$. To obtain the estimate for the time-derivative $\hat{u}_t$, we need a preliminary step. Let $B = B(\theta) > 1$ be such that $\log(1 + r^{2\theta}) \leq r$ for any $r \geq B$. As a consequence:

$$4r^2 - \log^2(1 + r^{2\theta}) \geq r\,(2r + \log(1 + r^{2\theta})) \geq 2r^2 \tag{3.23}$$

for any $r \geq B$. We divide Ω_h into two subzones. We define

$$\Omega_{h,1} = \{\xi \in \boldsymbol{R}^n :\ \delta_1 \leq |\xi| \leq B\}, \quad \Omega_{h,2} = \{\xi \in \boldsymbol{R}^n :\ |\xi| \geq B\}.$$

For any $\xi \in \Omega_{h,1}$, we apply Lemma 3.1. Using $|\lambda_-|^2 \leq C(1+|\xi|^2) \leq C(1+B^2)$, we obtain

$$\int_{\Omega_{h,1}} |\hat{u}_t(t,\xi)|^2 d\xi \leq e^{-2\gamma t} \left(\|u_1\|^2 + C(1+B^2)\, t^2 \|u_1\|^2 \right) \leq C_1\, e^{-\gamma t}\, \|u_1\|^2.$$

On the other hand, for any $\xi \in \Omega_{h,2}$, it holds

$$\hat{u}(t,\xi) = \hat{u}_1(\xi)\, e^{-a(\xi)t}\, \frac{\sin b(\xi)t}{b(\xi)},$$

where $a(\xi)$ and $b(\xi)$ are defined in (2.8). Hence, we may estimate

$$\begin{aligned} \int_{\Omega_{h,2}} |\hat{u}_t(t,\xi)|^2 d\xi &\leq \int_{\Omega_{h,2}} \left(1 + |\frac{a(\xi)}{b(\xi)}|^2 \right)^2 e^{-2a(\xi)t}\, |\hat{u}_1(\xi)|^2 d\xi \\ &\leq 4 \int_{\Omega_{h,2}} e^{-2a(\xi)t}\, |\hat{u}_1(\xi)|^2 d\xi \leq 4e^{-2\gamma t}\, \|u_1\|^2. \end{aligned}$$

Here, one has just used the fact that (3.23) implies

$$\left| \frac{a(\xi)}{b(\xi)} \right|^2 \leq \frac{\log^2(1+|\xi|^{2\theta})}{4|\xi|^2 - \log^2(1+|\xi|^{2\theta})} \leq \frac{\log^2(1+|\xi|^{2\theta})}{2|\xi|^2} \leq 1$$

for any $\xi \in \Omega_{h,2}$. □

In order to get Theorem 1.1 we need one more proposition in Ω_h.

Proposition 3.3 *Let $\theta > 1/2$. Then, there exists $\alpha \in (0,1)$ such that*

$$\int_{\Omega_h} |P_1 e^{-a(\xi)t} \frac{\sin(t|\xi|)}{|\xi|}|^2 d\xi \leq C|P_1|^2 e^{-\alpha t}, \quad (t \gg 1).$$

Proof. Indeed,

$$\begin{aligned} \int_{\Omega_h} |e^{-a(\xi)t} \frac{\sin(t|\xi|)}{|\xi|}|^2 d\xi &\leq \delta_1^{-2} \int_{\Omega_h} e^{-\log(1+|\xi|^{2\theta})t} d\xi = \delta_1^{-2} \int_{\Omega_h} (1+|\xi|^{2\theta})^{-t} d\xi \\ &= \delta_1^{-2} \omega_n \int_{\delta_1}^{\infty} (1+r^{2\theta})^{-t} r^{n-1} dr \leq Ct^{-1}\, (1+\delta_1^{2\theta})^{-t} \leq Ce^{-\alpha t}, \quad t \gg 1, \end{aligned}$$

for some constant $C = C(n,\theta) > 0$ and a suitable $\alpha \in (0,1)$, where we have applied Corollary 2.1 in the last inequality (see also Lemma 2.2 in [5]). This concludes the proof. □

Finally, Theorem 1.1 is a direct consequence of Propositions 3.1, 3.2 and 3.3. We shall draw its outline of proof in the case when $\theta > 1/2$.

Outline of proof of Theorem 1.1. Let $n \geq 1,\ \theta > 1/2$ and set

$$\nu_\theta(t,\xi) := P_1 e^{-a(\xi)t} \frac{\sin(t|\xi|)}{|\xi|}.$$

Then, one can estimate as follows.

$$\begin{aligned} &\int_{\mathbf{R}^n} |\hat{u}(t,\xi) - \nu_\theta(t,\xi)|^2 d\xi \\ &= \left(\int_{|\xi|\leq\delta_1} + \int_{\Omega_h} \right) |\hat{u}(t,\xi) - \nu_\theta(t,\xi)|^2 d\xi := I_1(t) + I_2(t). \end{aligned} \tag{3.24}$$

To begin with, by using Proposition 3.1 one has

$$I_1(t) \le C\Big(|P_1|^2 + \|u_1\|_{1,1}^2\Big) t^{-2\beta(n,\theta)} \quad (t \gg 1), \tag{3.25}$$

where $\beta(n,\theta)$ is defined in (1.8).

Secondarily, from Propositions 3.2 and 3.3, for $n \ge 1$ and $\theta \ge 1/2$ we have

$$I_2(t) \le C \int_{\Omega_h} |\hat{u}(t,\xi)|^2 d\xi + C \int_{\Omega_h} |\nu_\theta(t,\xi)|^2 d\xi \le C\, \|u_1\|^2\, e^{-\gamma t} + C\, |P_1|^2 e^{-\alpha t}, \quad (t \to \infty). \tag{3.26}$$

The statement of Theorem 1.1 can be proved by combining (3.24), (3.25), and (3.26). □

4 Optimal asymptotic behavior

In this section we study the optimality of various estimates of the integrals closely related with the leading terms obtained in previous sections.
We first prepare the following proposition in the large dimensional case.

4.1 Optimal behavior of the L^2-norm

Proposition 4.1 *Let $n > 2$ and $\theta \ge 1/2$. Then there exists $t_0 > 0$ such that for $t \ge t_0$ it holds that*

$$C^{-1} t^{-\frac{n-2}{2\theta}} \ge \int_{\mathbf{R}^n} e^{-t \log(1+|\xi|^{2\theta})} \frac{|\sin(|\xi| t)|^2}{|\xi|^2} d\xi \ge C t^{-\frac{n-2}{2\theta}},$$

with C a positive constant depending only on n and θ.

Proof. First, we may note that

$$\begin{aligned} M(t) :&= \int_{\mathbf{R}^n} e^{-t\ \log(1+|\xi|^{2\theta})} \frac{|\sin(|\xi| t)|^2}{|\xi|^2} d\xi \\ &= \omega_n \int_0^\infty e^{-t\ \log(1+r^{2\theta})} r^{n-3} |\sin(rt)|^2 dr \\ &\ge \omega_n \int_0^\infty e^{-t\ r^{2\theta}} r^{n-3} |\sin(rt)|^2 dr. \end{aligned}$$

Now we apply the change of variable $s = t^{1/2\theta} r$, for a fixed $t > 0$, to arrive at

$$M(t) \ge \omega_n t^{-\frac{n-2}{2\theta}} \int_0^\infty e^{-s^{2\theta}} s^{n-3} \sin^2\big(t^{\frac{2\theta-1}{2\theta}} s\big) ds.$$

For $\theta = 1/2$ the result directly follows from this last estimate. For $\theta > 1/2$ we use the fundamental identity

$$2\sin^2 x = (1 - \cos 2x)\,,$$

to obtain

$$\begin{aligned} M(t) &\ge \frac{1}{2} \omega_n t^{-\frac{n-2}{2\theta}} \int_0^\infty e^{-s^{2\theta}} s^{n-3} \Big(1 - \cos(2t^{\frac{2\theta-1}{2\theta}} s)\Big) ds \\ &= \frac{1}{2} \omega_n t^{-\frac{n-2}{2\theta}} \big(A_{n,\theta} - F_{n,\theta}(t)\big), \end{aligned}$$

where

$$A_{n,\theta} = \int_0^\infty e^{-s^{2\theta}} s^{n-3} ds, \quad F_{n,\theta}(t) = \int_0^\infty e^{-s^{2\theta}} s^{n-3} \cos\big(2t^{\frac{2\theta-1}{2\theta}} s\big) ds.$$

Due to the fact $e^{-s^{2\theta}} s^{n-3} \in L^1(\mathbf{R})$ $(n > 2)$, we can apply the Riemann-Lebesgue theorem to get

$$F_{n,\theta}(t) \to 0, \quad t \to \infty.$$

Then we conclude the existence of $t_0 > 0$ such that $F_{n,\theta}(t) \leq \dfrac{A_{n,\theta}}{2}$ for all $t \geq t_0$. Thus, the half part of proposition is proved with $C = \dfrac{\omega_n A_{n,\theta}}{4}$.

Now we prove the estimate from above of the proposition. Indeed,

$$\begin{aligned} M(t) &\leq \int_{\mathbf{R}^n} e^{-t\ \log(1+|\xi|^{2\theta})}|\xi|^{-2}d\xi \\ &= \omega_n \int_0^\infty e^{-t\ \log(1+r^{2\theta})} r^{n-3} dr \\ &= \omega_n \int_0^\infty (1+r^{2\theta})^{-t} r^{n-3} dr \\ &\leq C_{n,\theta}\, t^{-\frac{n-2}{2\theta}}, \quad t \gg 1, \end{aligned}$$

where one has just used Corollary 2.1. This estimate completes the proof of the proposition. □

Proposition 4.2 *Let $n = 1$ and $\theta > 0$. Then it is true that*

$$\int_{\mathbf{R}} (1+|\xi|^{2\theta})^{-t} \frac{\sin^2(t|\xi|)}{|\xi|^2} d\xi \sim t, \quad (t \gg 1).$$

The proof of this proposition is the same to the case $\theta = 1$ which appears in a Lemma by Charão-Ikehata [5]. In fact, the expression $(1 + r^{2\theta})^{-t}$, $\theta \neq 0$ does not change the proof of the case $\theta = 1$. That is, the result of the lemma is independent of θ.

Next we also deal with the two dimensional case.

The following proposition has a version for the case $\theta = 1$ in Charão-Ikehata [5] and its proof is also independent of θ.

Proposition 4.3 *Let $n = 2$ and $\theta > 0$. Then it is true that*

$$\int_{\mathbf{R}^2} (1+|\xi|^{2\theta})^{-t} \frac{\sin^2(t|\xi|)}{|\xi|^2} d\xi \sim \log t, \quad (t \gg 1).$$

Finally, let us now prove Theorem 1.2 at a stroke.

Proof of Theorem 1.2 completed. It follows from the Plancherel theorem and triangle inequality, with some constant $C_n > 0$ one can get

$$C_n\|u(t,\cdot)\| \geq |P_1| \|(1+|\xi|^{2\theta})^{-\frac{t}{2}} \frac{\sin(t|\xi|)}{|\xi|}\| - \|\hat{u}(t,\cdot) - P_1(1+|\xi|^{2\theta})^{-\frac{t}{2}} \frac{\sin(t|\xi|)}{|\xi|}\|.$$

and

$$C_n\|u(t,\cdot)\| \leq |P_1| \|(1+|\xi|^{2\theta})^{-\frac{t}{2}} \frac{\sin(t|\xi|)}{|\xi|}\| + \|\hat{u}(t,\cdot) - P_1(1+|\xi|^{2\theta})^{-\frac{t}{2}} \frac{\sin(t|\xi|)}{|\xi|}\|.$$

These inequalities together with Theorem 1.1, Propositions 4.1, 4.2 and 4.3 imply the desired estimates. This part is, nowadays, well-known (see [17, 19]). We stress that $6 - 8\theta < 2$, for any $\theta > 1/2$, and this gives $t^{-\frac{n-(6-8\theta)_+}{2\theta}} = o(t^{-\frac{n-2}{2\theta}})$. In the case $n = 1$ and $\theta \in (1/2, 5/8)$, we also notice that $1/\theta - 3/2 < 1/2$, for any $\theta > 1/2$. □

4.2 Optimal behavior of energy norm

In this subsection, we get the optimal decay estimates of the total energy itself. Again the total energy $E_u(t)$ is defined by

$$2E_u(t) := \|u_t(t,\cdot)\|^2 + \|\nabla u(t,\cdot)\|^2.$$

We will study the topic based on the expression defined by (3.5).

First of all, the estimate of the part $\|\nabla u(t,\cdot)\| = \||\xi|\hat{u}(t,\cdot)\|$ is simple. It is sufficient to multiply the expression (3.5) by $|\xi|$ and to make estimates similarly to get decay rates of the L^2-norm of $u(t,\xi)$. The result will be

$$C^{-1}t^{-\frac{n}{2\theta}}|P_1|^2 \le \|\nabla u(t,\cdot)\|^2 \le CI_1^2\, t^{-\frac{n}{2\theta}} \quad (t \gg 1) \tag{4.1}$$

with C a positive constant depending on n and θ.

The delicate part is to get the precise estimate of the L^2-norm of $\hat{u}_t(t,\xi)$. In order to do that, we take the time derivative of $\hat{u}(t,\xi)$ given by (3.5) to obtain the following expression to the time derivative of the solution with $r = |\xi|$,

$$\begin{aligned}
\hat{u}_t(t,\xi) = &-P_1 a(\xi) e^{-a(\xi)t}\frac{\sin(tr)}{r} + P_1 e^{-a(\xi)t}\cos(tr)\\
&- P_1 \frac{\log^2(1+r^{2\theta})}{8r^3}\frac{a(\xi)}{\sqrt{(1-\eta_2 g(r))^3}} e^{-a(\xi)t}\sin(tr)\\
&+ P_1 \frac{\log^2(1+r^{2\theta})}{8r^3}\frac{1}{\sqrt{(1-\eta_2 g(r))^3}} e^{-a(\xi)t} r\cos(tr)\\
&- \left(\frac{A_1(\xi)-iB_1(\xi)}{b(\xi)}\right) a(\xi) e^{-a(\xi)t}\sin(b(\xi)t) + \Big(A_1(\xi)-iB_1(\xi)\Big) e^{-a(\xi)t}\cos(b(\xi)t)\\
&- tP_1 a(\xi) e^{-a(\xi)t}\left(\frac{b(\xi)-r}{b(\xi)}\right)\cos(\mu(\xi)t) - tP_1 e^{-a(\xi)t}\left(\frac{b(\xi)-r}{b(\xi)}\right)\mu(\xi)\sin(\mu(\xi)t)\\
&P_1 e^{-a(\xi)t}\left(\frac{b(\xi)-r}{b(\xi)}\right)\cos(\mu(\xi)t).
\end{aligned} \tag{4.2}$$

It should be remarked that (4.2) holds for small frequency parameters $|\xi| \ll 1$ and all $\theta > 1/2$. A candidate to be a leading term as $t \to \infty$ of the velocity $u_t(t,\xi)$ is given by a simple form:

$$P_1 e^{-a(\xi)t}\cos(|\xi|t) \tag{4.3}$$

where $a(\xi) = \dfrac{\log(1+|\xi|^{2\theta})}{2}$ and $P_1 := \displaystyle\int_{\mathbf{R}^n} u_1(x)dx$.

Our goal in this section is to get decay estimates in time to the remainder terms that appear in (4.2), and are defined by the next 8 functions which imply remainders with respect to the leading term (4.3).

- $F_1(t,\xi) = -P_1 \dfrac{\log^2(1+r^{2\theta})}{8r^3}\dfrac{a(\xi)}{\sqrt{(1-\eta_2 g(r))^3}} e^{-a(\xi)t}\sin(tr)$,
- $F_2(t,\xi) = P_1 \dfrac{\log^2(1+r^{2\theta})}{8r^3}\dfrac{1}{\sqrt{(1-\eta_2 g(r))^3}} e^{-a(\xi)t} r\cos(tr)$,
- $F_3(t,\xi) = -\left(\dfrac{A_1(\xi)-iB_1(\xi)}{b(\xi)}\right) a(\xi) e^{-a(\xi)t}\sin(b(\xi)t)$,
- $F_4(t,\xi) = \Big(A_1(\xi)-iB_1(\xi)\Big) e^{-a(\xi)t}\cos(b(\xi)t)$,
- $F_5(t,\xi) = tP_1 a(\xi) e^{-a(\xi)t}\left(\dfrac{b(\xi)-r}{b(\xi)}\right)\cos(\mu(\xi)t)$,
- $F_6(t,\xi) = tP_1 e^{-a(\xi)t}\left(\dfrac{b(\xi)-r}{b(\xi)}\right)\mu(\xi)\sin(\mu(\xi)t)$,
- $F_7(t,\xi) = P_1 e^{-a(\xi)t}\left(\dfrac{b(\xi)-r}{b(\xi)}\right)\cos(\mu(\xi)t)$,
- $F_8(t,\xi) = P_1 a(\xi) e^{-a(\xi)t}\dfrac{\sin(t|\xi|)}{|\xi|}$,

where $r := |\xi|$. Then, the remainder term for $\hat{u}_t(t,\xi)$ is given by

$$\hat{u}_t(t,\xi) - P_1 e^{-a(\xi)t}\cos(tr) = \sum_{j=1}^{8} F_j(t,\xi). \tag{4.4}$$

We need the following lemmas in order to show that the remainder terms decay faster than the leading term (asymptotic profile).

Lemma 4.1 *Let $n \geq 1$ and $\theta \geq 1/2$. Then there exists $t_0 > 0$ such that for $t \geq t_0$ it holds that*

$$Ct^{-\frac{n+4\theta-2}{2\theta}} \geq \int_{\mathbf{R}^n} a(\xi)^2 e^{-t\log(1+|\xi|^{2\theta})} \frac{|\sin(|\xi|t)|^2}{|\xi|^2} d\xi \geq C^{-1} t^{-\frac{n+4\theta-2}{2\theta}},$$

with C a positive constant depending only on n and θ.

Proof: First place we note that

$$\log(1+r^{2\theta}) \leq r^{2\theta},$$

for all $r \geq 0$ and $\theta > 1/2$.

Then, using the definition of $a(\xi)$ we can easily obtain the estimate from above.

$$\begin{aligned}\int_{\mathbf{R}^n} a(\xi)^2 e^{-t\log(1+|\xi|^{2\theta})} \frac{|\sin(|\xi|t)|^2}{|\xi|^2} d\xi &= \omega_n \int_0^\infty \frac{1}{4}\log^2(1+r^{2\theta}) e^{-t\log(1+r^{2\theta})} \frac{|\sin(rt)|^2}{r^2} r^{n-1} dr \\ &\leq \omega_n \int_0^\infty \frac{r^{4\theta}}{4}(1+r^{2\theta})^{-t} \frac{|\sin(rt)|^2}{r^2} r^{n-1} dr \\ &\leq \frac{\omega_n}{4} \int_0^\infty r^{n+4\theta-3}(1+r^{2\theta})^{-t} dr \\ &\leq C_{n,\theta}\, t^{-\frac{n+4\theta-2}{2\theta}}, \quad t \gg 1,\end{aligned}$$

with a positive constant $C_{n,\theta}$, where we have just used Corollary 2.1.

Next we want to get the lower bound. We first observe that

$$\lim_{r\to 0} \frac{\log(1+r^{2\theta})}{r^{2\theta}} = 1.$$

Thus, there exists $\delta_0 = \delta_0(\theta) > 0$ such that

$$1/2 \leq \frac{\log(1+r^{2\theta})}{r^{2\theta}} \leq 2 \tag{4.5}$$

for all $0 < r \leq \delta_0$. By using (4.5) we may obtain for $t \geq \frac{1}{2\delta_0^{2\theta}}$

$$\begin{aligned}I_{\sin}(t) &:= \int_{\mathbf{R}^n} a(\xi)^2 e^{-t\log(1+|\xi|^{2\theta})} \frac{|\sin(|\xi|t)|^2}{|\xi|^2} d\xi \\ &= \omega_n \int_0^\infty \frac{1}{4}\log^2(1+r^{2\theta}) e^{-t\log(1+r^{2\theta})} \frac{|\sin(rt)|^2}{r^2} r^{n-1} dr \\ &\geq \frac{\omega_n}{4} \int_0^{\delta_0} \frac{r^{4\theta+n-3}}{4} e^{-2tr^{2\theta}} \sin^2(rt) dr \\ &= \frac{\omega_n}{16\theta} 2^{-\frac{1}{2\theta}-\frac{4\theta+n-3}{2\theta}} t^{-\frac{4\theta+n-2}{2\theta}} \int_0^{\sqrt{2t}\delta_0^\theta} s^{\frac{3\theta+n-2}{\theta}} e^{-s^2} \sin^2(2^{-\frac{1}{2\theta}} t^{\frac{2\theta-1}{2\theta}} s^{\frac{1}{\theta}}) ds.\end{aligned}$$

We use the identity $\sin^2 x = \dfrac{1}{2}(1-\cos(2x))$ to get

$$\begin{aligned}I_{\sin}(t) &\geq \frac{\omega_n}{16\theta} 2^{-\frac{1}{2\theta}-\frac{4\theta+n-3}{2\theta}} t^{-\frac{4\theta+n-2}{2\theta}} \int_0^{\sqrt{2t}\delta_0^\theta} s^{\frac{3\theta+n-2}{\theta}} e^{-s^2} \frac{1}{2} ds \\ &- \frac{\omega_n}{16\theta} 2^{-\frac{1}{2\theta}-\frac{4\theta+n-3}{2\theta}} t^{-\frac{4\theta+n-2}{2\theta}} \int_0^{\sqrt{2t}\delta_0^\theta} s^{\frac{3\theta+n-2}{\theta}} e^{-s^2} \frac{1}{2}\cos(2^{1-\frac{1}{2\theta}} t^{\frac{2\theta-1}{2\theta}} s^{\frac{1}{\theta}}) ds \\ &\geq \frac{\omega_n}{32\theta} 2^{-\frac{1}{2\theta}-\frac{4\theta+n-3}{2\theta}} t^{-\frac{4\theta+n-2}{2\theta}} \int_0^1 s^{\frac{3\theta+n-2}{\theta}} e^{-s^2} ds \\ &- \frac{\omega_n}{32\theta} 2^{-\frac{1}{2\theta}-\frac{4\theta+n-3}{2\theta}} t^{-\frac{4\theta+n-2}{2\theta}} \int_0^\infty s^{\frac{3\theta+n-2}{\theta}} e^{-s^2} \cos(2^{1-\frac{1}{2\theta}} t^{\frac{2\theta-1}{2\theta}} s^{\frac{1}{\theta}}) ds.\end{aligned}$$

We note that $s^{\frac{3\theta+n-2}{\theta}}e^{-s^2}\in L^1(0,\infty)$ because of $\theta>1/2$.

Then, we define the following positive constant depending on θ and n

$$A_{n,\theta}=\int_0^1 s^{\frac{3\theta+n-2}{\theta}}e^{-s^2}ds.$$

Moreover, by the Riemann-Lebesgue Lemma one has

$$\lim_{t\to\infty}\int_0^\infty s^{\frac{3\theta+n-2}{\theta}}e^{-s^2}\cos(2^{1-\frac{1}{2\theta}}t^{\frac{2\theta-1}{2\theta}}s^{\frac{1}{\theta}})ds=0$$

because of $\theta>1/2$. So, we can choose $t_0\geq\frac{1}{2\delta_0^{2\theta}}$ such that

$$\int_0^\infty s^{\frac{3\theta+n-2}{\theta}}e^{-s^2}\cos(2^{1-\frac{1}{2\theta}}t^{\frac{2\theta-1}{2\theta}}s^{\frac{1}{\theta}})ds\leq\frac{A_{n,\theta}}{2}$$

for all $t\geq t_0$.

Combining these results with the last estimate for $I_{sin}(t)$ we arrive at the desired estimate:

$$I_{\sin}(t)\geq\frac{\omega_n}{32\theta}2^{-\frac{1}{2\theta}-\frac{4\theta+n-3}{2\theta}}t^{-\frac{4\theta+n-2}{2\theta}}(A_{n,\theta}-\frac{1}{2}A_{n,\theta}),\quad t\geq t_0,\ n\geq1.$$

□

Lemma 4.2 *Let $n\geq1$ and $\theta\geq1/2$. Then there exists $t_0>0$ such that for $t\geq t_0$ it holds that*

$$C^{-1}t^{-\frac{n}{2\theta}}\geq\int_{\mathbf{R}^n}e^{-t\log(1+|\xi|^{2\theta})}|\cos(|\xi|t)|^2d\xi\geq Ct^{-\frac{n}{2\theta}},$$

with a constant $C>0$ depending only on n and θ.

Proof: To get the upper estimate we have

$$\begin{aligned}I_{\cos}(t):=\int_{\mathbf{R}^n}e^{-t\log(1+|\xi|^{2\theta})}|\cos(|\xi|t)|^2d\xi&=\int_{\mathbf{R}^n}(1+|\xi|^{2\theta})^{-t}|\cos(|\xi|t)|^2d\xi\\&\leq\omega_n\int_0^\infty(1+|r|^{2\theta})^{-t}r^{n-1}dr\ \leq C\,t^{-\frac{n}{2\theta}},\quad t\gg1\end{aligned}$$

with a constant $C>0$ depending only on n and θ according to Corollary 2.1.

To get the estimate from below we use (4.5). Then one has

$$\begin{aligned}I_{\cos}(t)&=\omega_n\int_0^\infty e^{-t(1+|r|^{2\theta})}|\cos(rt)|^2r^{n-1}dr\geq\omega_n\int_0^1e^{-2tr^{2\theta}}\cos^2(rt)r^{n-1}dr\\&=\frac{\omega_n}{\theta}2^{-\frac{n}{2\theta}}t^{-\frac{n}{2\theta}}\int_0^{\sqrt{2t}}e^{-s^2}s^{\frac{n-\theta}{\theta}}\cos^2(2^{-\frac{1}{2\theta}}t^{\frac{2\theta-1}{2\theta}}s^{\frac{1}{\theta}})ds\\&\geq\frac{\omega_n}{\theta}2^{-\frac{n}{2\theta}}t^{-\frac{n}{2\theta}}\int_0^1e^{-s^2}s^{\frac{n-\theta}{\theta}}\cos^2(2^{-\frac{1}{2\theta}}t^{\frac{2\theta-1}{2\theta}}s^{\frac{1}{\theta}})ds,\quad t\gg1.\end{aligned}$$

Next, with the same argument as in Lemma 4.1 via the Riemann-Lebesgue Lemma used to prove the lower bound for $I_{\sin}(t)$, and the fact that $e^{-s^2}s^{\frac{n-\theta}{\theta}}\in L^1(0,1)$ one can obtain the estimate from below of this lemma.
□

Next we give various estimates to the functions $F_j(t,\xi)$, $j=1,\cdots,7$, since the estimate for the term $F_8(t,\xi)$ is already given by Lemma 4.1.

To estimate $F_1(t,\xi)$ one notes that due to the limit in (3.16), if $\theta>3/4$, then there exists $\delta_0\in(0,1]$ such that

$$\frac{\log^2(1+r^{2\theta})}{8r^3}\leq1/2\tag{4.6}$$

for $0 < r = |\xi| \le \delta_0$.

While, to the function $g(r)$ defined in Section 3 such that

$$g(r) = \frac{\log^2(1 + r^{2\theta})}{4r^2},$$

it happens that

$$\lim_{r\to 0} g(r) = 0.$$

Therefore, there exists a number $\delta_1 \in (0, 1]$ such that

$$(1 - \eta_2 g(r))^3 \ge (1 - g(r))^3 \ge (1/2)^3$$

for $0 < r \le \delta_1$.

Now, we define $\delta = min\{\delta_0\ ,\ \delta_1\}$. Then, using the last estimate and (4.6) in

$$\int_{|\xi|\le\delta} |F_1(t,\xi)|^2 d\xi \le |P_1|^2 \int_{|\xi|\le\delta} \Big|\frac{\log^2(1 + r^{2\theta})}{8r^3}\frac{a(\xi)}{\sqrt{(1-\eta_2 g(r))^3}} e^{-a(\xi)t}\sin(tr)\Big|^2 d\xi,$$

one can obtain that

$$\int_{|\xi|\le\delta} |F_1(t,\xi)|^2 d\xi \le C|P_1|^2 \int_{|\xi|\le 1} |\xi|^2 e^{-2a(\xi)t} d\xi,$$

with a constant $C > 0$ because of the fact that $a(\xi) = \frac{1}{2}\log(1 + |\xi|^{2\theta}) \le \frac{1}{2}|\xi|$ for $|\xi| \le 1$.

Using Corollary 2.1 it implies that

$$\int_{|\xi|\le\delta} |F_1(t,\xi)|^2 d\xi \le C|P_1|^2 t^{-\frac{n+2}{2\theta}}, \quad t \gg 1. \tag{4.7}$$

If $\theta \in (1/2, 3/4]$, then we use (3.17), so that $|P_1|^2 t^{-\frac{n+2}{2\theta}}$ is replaced by $|P_1|^2 t^{-\frac{n+2-(6-8\theta)_+}{2\theta}}$ in the estimate for F_1 (there is no need to distinguish $n \ge 2$ and $n = 1$ in this case). Again, we stress that $6 - 8\theta < 2$, for any $\theta > 1/2$. We estimate F_2 as we did for F_1.

The estimates for the other functions on low frequency zone are similarly done by using the method to estimate functions $K_j(t,\xi),\ j = 1, 2, 3$. In particular, when one estimates $F_6(t,\xi)$, it is necessary to use the inequality $|\mu(\xi)|^2 = |\eta_1 b(\xi) + (1-\eta_1)|\xi||^2 \le 10|\xi|^2$ for $|\xi| \le 1$ and estimate similar to (3.22). The result is that there exists a number $\delta > 0$ such that

$$\int_{|\xi|\le\delta} |F_j(t,\xi)|^2 d\xi \le C|P_1|^2 t^{-\frac{n+2}{2\theta}}, \quad t \gg 1, \tag{4.8}$$

for $j = 4, 5, 6, 7$.

Finally, the estimate for F_3 can be obtained by following the similar estimate to that of K_1 in (3.14). The result is

$$\int_{|\xi|\le\delta} |F_3(t,\xi)|^2 d\xi \le C||u_1||_{1,1}^2 t^{-\frac{n+2}{2\theta}}, \quad t \gg 1. \tag{4.9}$$

Combining these estimates above one can get the following result on the difference between $\hat{u}_t(t,\xi)$ and the asymptotic profile given by (4.3).

Proposition 4.4 *Let $n \ge 1$ and $\theta > 1/2$. Then, there exists a small constant $\delta \in (0, 1]$ such that*

$$\int_{|\xi|\le\delta} \big|\hat{u}_t(t,\xi) - P_1 e^{-a(\xi)t}\cos(tr)\big|^2 d\xi \le C\Big[\ \big(|P_1|^2\, t^{(6-8\theta)_+} + \|u_1\|_{1,1}^2\big) t^{-\frac{n+2}{2\theta}} + |P_1|^2 t^{-\frac{n+4\theta-2}{2\theta}}\ \Big] \quad (t \gg 1),$$

with some generous constant $C = C_{n,\theta} > 0$ depending only on θ and n.

Based on Proposition 4.4, one can get the crucial result on the behavior for the time derivative of the solution.

Proposition 4.5 *Let $n \geq 1$, $\theta > 1/2$. Then, in the case of $1/2 < \theta \leq 1$ it holds that*

$$\int_{\mathbf{R}^n} \left|\hat{u}_t(t,\xi) - P_1 e^{-a(\xi)t}\cos(tr)\right|^2 d\xi \leq C\Big[\ \|u_1\|_{1,1}^2 + \|u_1\|^2\Big] t^{-\frac{n+4\theta-2}{2\theta}} \qquad (t \gg 1),$$

and in the case of $\theta \geq 1$ it is true that

$$\int_{\mathbf{R}^n} \left|\hat{u}_t(t,\xi) - P_1 e^{-a(\xi)t}\cos(tr)\right|^2 d\xi \leq C\Big[\ \|u_1\|_{1,1}^2 + \|u_1\|^2\Big] t^{-\frac{n+2}{2\theta}} \qquad (t \gg 1),$$

with some generous constant $C = C_{n,\theta} > 0$ depending only on θ and n.

Proof: According to the Proposition 4.4, in order to prove the statement it suffices to get the estimates on the high frequency region $\Omega_h = \{\xi \in \mathbf{R}^n \,:\, |\xi| \geq \delta\}$. In fact,

$$\begin{aligned}
&\int_{\Omega_h} \left|\hat{u}_t(t,\xi) - P_1 e^{-a(\xi)t}\cos(tr)\right|^2 d\xi \\
&\quad \leq 2\int_{\Omega_h} |\hat{u}_t(t,\xi)|^2 d\xi + 2|P_1|^2 \int_{|\xi|\geq\delta} e^{-2a(\xi)t} d\xi \\
&\quad \leq \|u_1\|^2 e^{-\gamma t} + 2|P_1|^2 \int_\delta^\infty (1+r^{2\theta})^{-t} \omega_n r^{n-1} d\xi \\
&\quad \leq \|u_1\|^2 e^{-\gamma t} + 2|P_1|^2 C \frac{(1+\delta^{2\theta})^{-t}}{t}
\end{aligned} \tag{4.10}$$

which holds for $t \gg 1$, where we have just used Proposition 3.2 and Corollary 2.1. □

Proof of Theorem 1.3 : The proof for lower bound of decay can be done by (4.1) and the following estimate

$$||u_t(t,\cdot)|| \geq ||P_1 e^{-a(\xi)t}\cos(tr)|| - ||u_t(t,\cdot) - P_1 e^{-a(\xi)t}\cos(tr)||$$

combined with the estimates from below of Lemma 4.2 and Proposition 4.5.

The estimate from above can be obtained using by Proposition 3.2, the estimates for the functions $F_j(t,\xi)$ $(j = 1, \cdots, 8)$ and the estimate from above of Lemma 4.2. □

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