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Small data solutions for the Euler-Poisson-Darboux equation with a power nonlinearity

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Abstract

We study the Cauchy problem for the Euler-Poisson-Darboux equation, with a power nonlinearity:

$$u_{tt} - u_{xx} + \frac{\mu}{t} u_t = t^\alpha |u|^p , \quad t > t_0, \ x \in \mathbb{R} ,$$

where $\mu > 0$, $p > 1$ and $\alpha > -2$. Here either $t_0 = 0$ (singular problem) or $t_0 > 0$ (regular problem). We show that this model may be interpreted as a semilinear wave equation with borderline dissipation: the existence of global small data solutions depends not only on the power p, but also on the parameter μ. Global small data weak solutions exist if

$$(p - 1) \min\left\{1, \ \mu, \ \frac{\mu}{2} + \frac{1}{p}\right\} > 2 + \alpha.$$

In the case of $\alpha = 0$, the above condition is equivalent to $p > p_{\text{crit}} = \max\{p_{\text{Str}}(1 + \mu), 3\}$, where $p_{\text{Str}}(k)$ is the critical exponent conjectured by W.A. Strauss for the semilinear wave equation without dissipation (i.e. $\mu = 0$) in space dimension k. Varying the parameter μ, there is a continuous transition from $p_{\text{crit}} = \infty$ (for $\mu = 0$) to $p_{\text{crit}} = 3$ (for $\mu \geq 4/3$). The optimality of p_{crit} follows by known nonexistence counterpart results for $1 < p \leq p_{\text{crit}}$ (and for any $p > 1$ if $\mu = 0$).

As a corollary of our result, we obtain analogous results for generalized semilinear Tricomi equations and other models related to the Euler-Poisson-Darboux equation.

Keywords: semilinear wave equations, semilinear Euler-Poisson-Darboux equation, semilinear Tricomi equations, global existence, dissipation, critical exponent, Fujita exponent, Strauss exponent
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1. Introduction

In this paper, we study the existence of global-in-time small data (weak) solutions to the Cauchy problem for the Euler-Poisson-Darboux (E. P. D.) equation with a power nonlinearity:

$$\begin{cases} u_{tt} - u_{xx} + \frac{\mu}{t} u_t = f(u), & t > t_0, \ x \in \mathbb{R} , \\ u(t_0, x) = u_0(x) , \quad u_t(t_0, x) = u_1(x) . \end{cases} \tag{1}$$

Here $\mu > 0$ and $f(u) = |u|^p$ or, more in general, f is locally Lipschitz-continuous and

$$f(0) = 0, \quad |f(u) - f(w)| \leq C |u - w|\big(|u|^{p-1} + |w|^{p-1}\big) , \tag{2}$$

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for some $p > 1$. The initial time t_0 may be zero (singular Cauchy problem) or may be positive (regular Cauchy problem).

The study of the solution to the linear Cauchy problem, i.e., $f = 0$ in (1), goes back to the first investigations of Euler [15], Poisson [49] and Darboux [12] for the singular problem ($t_0 = 0$), and goes back to [4, 13] for the regular problem ($t_0 > 0$). Some blow-up results for (1) in the singular case $t_0 = 0$ goes back to [32] (see also [35]), whereas the study of the solution of the singular Cauchy problem for the E. P. D. equation with inhomogeneous term $f = f(t, x)$ goes back to [64].

The term $\mu t^{-1} u_t$ in (1) may be interpreted as a dissipation acting on the wave model, in the sense that the wave energy

$$E(t) = \frac{1}{2}\, \|u_t(t,\cdot)\|_{L^2}^2 + \frac{1}{2}\, \|u_x(t,\cdot)\|_{L^2}^2 \tag{3}$$

for the regular linear problem, i.e. $f = 0$ and $t_0 > 0$ in (1), dissipates as $t \to \infty$; in particular, $E(t) \le C\, t^{-\min\{\mu,2\}}$ if $(u_0, u_1) \in H^1 \times L^2$ (see [62]). The same effect appears for the damped wave equation $u_{tt} - u_{xx} + \mu u_t = 0$, but in this latter case, $E(t) \le C\, t^{-1}$, for any $\mu > 0$. This decay profile is a consequence of the "diffusion phenomenon" (see, for instance, [22, 25, 38, 40]): the asymptotic profile of the solution is described by the solution to the heat equation $\mu u_t - u_{xx} = 0$. The crucial difference is that the asymptotic profile of the solution to the E. P. D. equation is described by the solution to the heat equation $\mu u_t - t u_{xx} = 0$ only for sufficiently large μ.

A consequence of the diffusion phenomenon is that the critical exponent for global-in-time small data solutions to the semilinear damped wave equation $u_{tt} - \Delta u + u_t = |u|^p$ for $t > 0$ and $x \in \mathbb{R}^n$, is $1 + 2/n$ (see [57]), the same of the semilinear heat equation $u_t - \Delta u = |u|^p$. By critical exponent p_{crit} we mean that global-in-time small data solutions exist for $p > p_{\mathrm{crit}}$ in a suitable space, and, in general, do not exist for $p \in (1, p_{\mathrm{crit}}]$, under suitable data sign assumptions. The study of these kind of problems has been originated by the pioneering paper of H. Fujita [16] about the semilinear heat equation. In general, nonlinear phenomena may break the boot-strap argument which allows to prolong local-in-time solutions. H. Fujita investigated how this occurrence is prevented for sufficiently small initial data if, and only if, the power nonlinearity is larger than a given threshold exponent.

The critical exponent remains $1 + 2/n$ also for the damped wave equation $u_{tt} - \Delta u + b(t)u_t = |u|^p$, for a large class of coefficients $b(t)$ verifying $tb(t) \to \infty$ as $t \to \infty$ (see [8]), in particular for $b(t) = \mu(1+t)^\beta$, with $\mu > 0$ and $\beta \in (-1, 1)$ (see [36, 41]). We stress that the critical exponent remains $1 + 2/n$ in the latter case, even if μ is very small.

In the case $\mu = 0$ in (1) (wave equation) the critical exponent is ∞, in the sense that no global-in-time solution to (1) exists, for any $p > 1$, under a sign assumption on the initial data. On the other hand, for small data in suitable functional spaces, global-in-time (weak) solutions exist for the wave equation $u_{tt} - \Delta u = |u|^p$ in space dimension $n \ge 2$, if $p > p_{\mathrm{Str}}(n)$, where $p_{\mathrm{Str}}(k)$ is the critical exponent conjectured by W.A. Strauss [53] (see also [54]), i.e., the solution to $(p - 1)\gamma(k, p) = 2$, where we put

$$\gamma(k,p) = \frac{k-1}{2} + \frac{1}{p}. \tag{4}$$

The conjecture was supported by the result obtained in the pioneering paper by F. John [29] in space dimension $n = 3$ and by the blow-up result obtained by R.T. Glassey [20] in space dimension $n = 2$. It was later proved in a series of papers, see [28, 50, 51, 63] for blow-up results, and [1, 18, 19, 21, 33, 37, 55, 66] for existence results.

In our paper, we show that global-in-time (weak) solutions to (1) exist in $L^\infty([t_0,\infty), L^p)$ for $p > p_{\mathrm{crit}} = \max\{p_{\mathrm{Str}}(1+\mu), 3\}$, for any $\mu > 0$, under the assumption of small data, for both the singular and the regular problem. This shows a continuous transition with respect to μ from a *shifted Strauss exponent* $p_{\mathrm{Str}}(1 + \mu)$ for $\mu \in (0, 4/3]$ to the *Fujita exponent* 3 for $\mu \ge 4/3$, typical of semilinear diffusive models.

In view of this effect, we may say that the dissipation $t^{-1}\mu u_t$ in (1) is borderline, and that the E. P. D. equation *bridges the gap* between pure semilinear wave models ($\mu = 0$) and semilinear dissipative wave models for which the diffusion phenomenon holds. The transition from one model to the other is described by how p_{crit} shrinks as the dissipation parameter μ increases from zero up to some threshold.

The critical exponent of the regular problem for the multidimensional version of the E. P. D. equation

$$\begin{cases} u_{tt} - \Delta u + \dfrac{\mu}{t}\, u_t = f(u), & t \ge t_0 > 0,\ x \in \mathbb{R}^n\,, \\ u(t_0,x) = u_0(x)\,, \quad u_t(t_0,x) = u_1(x)\,. \end{cases} \tag{5}$$

is $p_{\rm crit} = \max\{p_{\rm Str}(n+2), 1+2/n\}$ (see [9], see also [7, 44]) in the special case $\mu = 2$ (via the change of variable $w(t,x) = t\,u(t,x)$, the E. P. D. equation with $\mu = 2$ reduces to a wave equation, see Remark 2.5). On the other hand, the critical exponent for (5) is $1 + 2/n$ if μ is sufficiently large, in particular, if $\mu \geq n+2$ (see [5]). Up to our knowledge, there is no corresponding result for the singular problem. For some results for global-in-time solutions for some semilinear singular Cauchy problems for the multidimensional E. P. D. equation we address the reader to [59, 65].

The result in [9] leaded to the conjecture that the critical exponent for (5) is $p_{\rm crit} = \max\{p_{\rm Str}(n+\mu), 1+2/n\}$, for any $\mu > 0$. That is, $p_{\rm crit} = 1+2/n$ for $\mu \geq \bar{\mu}$ and $p_{\rm crit} = p_{\rm Str}(n+\mu)$ for $\mu \leq \bar{\mu}$, where

$$\bar{\mu}(n) = n - 1 + \frac{4}{n+2}\,. \tag{6}$$

M. Ikeda and M. Sobajima [26] obtained blow-up in finite time for (5) with $f = |u|^p$ if $\mu \leq \bar{\mu}$ and $1 < p \leq p_{\rm Str}(n+\mu)$ for suitable compactly supported data (see also [58]), strengthening the conjecture. Their result extended the blow-up result obtained for $1 < p \leq p_{\rm Str}(n+2\mu)$ by N.- A. Lai, H. Takamura, K. Wakasa in [34]. For lifespan estimates of the local-in-time solutions we address the reader to [27, 30, 31, 60].

In this paper, we prove the above conjecture for (5) in space dimension $n = 1$, and we show the existence of global-in-time small data solutions in $L^\infty([t_0,\infty), L^p)$ for $p > p_{\rm crit} = \max\{p_{\rm Str}(1+\mu), 3\}$ also for the more challenging singular problem with $t_0 = 0$. Moreover, we extend this result to the E. P. D. equation with the more general right-hand side $t^\alpha f(u)$.

On the one hand, this generalization is of interest for the possibility to obtain, by a change of variable, results for semilinear generalized Tricomi equations [56] $w_{tt} - t^{2\ell}\, w_{xx} = f(w)$, setting $\mu = \ell/(\ell+1)$ and $\alpha = 2\mu$, and for other models related, like the semilinear modified E. P. D. equation. On the other hand, this generalization provides more insights about how the size of μ influences the critical exponent $p_{\rm crit}$ (see Remark 2.1).

2. Results

We consider both the singular problem

$$\begin{cases} u_{tt} - u_{xx} + \dfrac{\mu}{t}\, u_t = t^\alpha\, f(u), & t > 0,\ x \in \mathbb{R}\,, \\ u(0,x) = u_0(x)\,, \quad u_t(0,x) = 0\,, \end{cases} \tag{7}$$

and the regular problem

$$\begin{cases} u_{tt} - u_{xx} + \dfrac{\mu}{t}\, u_t = t^\alpha\, f(u), & t \geq t_0 > 0,\ x \in \mathbb{R}\,, \\ u(t_0,x) = 0\,, \quad u_t(t_0,x) = u_1(x)\,. \end{cases} \tag{8}$$

We stress that the assumption $u_t(0,x) = 0$ is natural for the singular problem, even in the linear case $f = 0$, whereas for the regular problem both initial data may be considered [4]. However, for this latter, we assume $u(t_0,x) = 0$ for brevity.

For both the singular problem (7) and the regular problem (8), we prove the existence of global-in-time small data weak solutions (in $L^\infty([t_0,\infty), L^p)$ or in $L^\infty_{\rm loc}([t_0,\infty), L^p)$) for $p > p_{\rm crit}$, with

$$p_{\rm crit} = \max\left\{1 + \frac{2+\alpha}{\min\{1,\mu\}}, \quad p_{\rm Str}(1+\mu,\alpha)\right\}, \tag{9}$$

for any $\alpha > -2$, where $p_{\rm Str}(k,\alpha)$ is the solution, for a given $k > 1$, to

$$(p-1)\gamma(k,p) = 2 + \alpha,$$

and $\gamma(k,p)$ is given by (4). Explicitly,

$$\gamma(1+\mu,p) = \frac{\mu}{2} + \frac{1}{p}, \qquad \text{so that} \quad (p-1)\Big(\frac{\mu}{2} + \frac{1}{p}\Big)\Big|_{p = p_{\rm Str}(1+\mu,\alpha)} = 2 + \alpha\,.$$

Remark 2.1. We may interpret $1 + (2+\alpha)/\min\{1,\mu\}$ as a *modified Fujita exponent*, and $p_{\rm Str}(1+\mu,\alpha)$ as a *modified, shifted Strauss exponent*. The modification in the exponent is related to the presence of the coefficient t^α in front of the nonlinearity $f(u)$. The condition $p > p_{\rm crit}$ is equivalent to the inequality

$$(p-1)\min\left\{1,\ \mu,\ \frac{\mu}{2}+\frac{1}{p}\right\} > 2+\alpha, \tag{10}$$

and is related to the $L^1 - L^p$ decay rate determined in Proposition 3.3 in §3 for the regular linear problem with starting time $s > 0$:

$$\begin{cases} v_{tt} - v_{xx} + \dfrac{\mu}{t}\, v_t = 0, & t \geq s > 0,\ x\in\mathbb{R}\,,\\ v(s,x) = 0\,, \quad v_t(s,x) = v_1(x)\,. \end{cases} \tag{11}$$

Indeed, summing the power of s, and p times the power of t in (42) (ignoring the logarithmic terms), we find the number

$$1-(p-1)\min\left\{1,\ \mu,\ \frac{\mu}{2}+\frac{1}{p}\right\}.$$

We mention that the role of the power of the parameter s in the decay estimate to determine $p_{\rm crit}$ does not appear in problems with constant coefficient. Due to the invariance for time translations, the decay rate for these problems with starting time s is simply obtained replacing t by $t-s$ in the problem with starting time 0.

Theorem 2.1. *Let $\mu > 0$, $p > \max\{1, 1/\mu\}$, and define $q \in [1,p)$ such that*

$$q = \max\{1,1/\mu\} \quad \textit{if} \quad \frac{2}{p} \geq \min\{\mu, 2-\mu\}, \quad \textit{or} \quad \frac{1}{q}-\frac{1}{p} = \frac{\mu}{2}, \quad \textit{if} \quad \frac{2}{p} < \min\{\mu,2-\mu\}. \tag{12}$$

If

$$\frac{p-1}{p} \leq \alpha+2 < (p-1)\min\left\{1,\ \mu,\ \frac{\mu}{2}+\frac{1}{p}\right\}, \tag{13}$$

then there exists $\varepsilon > 0$ such that for any initial data

$$u_0 \in L^q\cap L^p, \quad \textit{with}\ \|u_0\|_{L^q} + \|u_0\|_{L^p} \leq \varepsilon, \tag{14}$$

there is a unique global-in-time weak solution $u \in L^\infty([0,\infty), L^p)$, to (7)*. Moreover, the solution to* (7) *satisfies the decay estimate*

$$\|u(t,\cdot)\|_{L^p} \leq C\, g(1+t)\,(\|u_0\|_{L^q} + \|u_0\|_{L^p}), \quad \textit{with} \quad g(1+t) = (1+t)^{-\min\{1,\mu,\frac{\mu}{2}+\frac{1}{p}\}+\frac{1}{p}}\, {\rm d}_1(t)\,{\rm d}_2(t)\,, \tag{15}$$

where $C > 0$, is independent of t and of the initial data, and ${\rm d}_1(t)$ and ${\rm d}_2(t)$ are small loss terms determined as follows: either ${\rm d}_1 = 1$ if $\mu \neq 1$ or ${\rm d}_1 = 1+\log(1+t)$ if $\mu = 1$; either ${\rm d}_2(t) = 1$ if $2/p \neq \min\{\mu, 2-\mu\}$, or we may take ${\rm d}_2(t) = c_\delta\,(1+t)^\delta$ for any small $\delta > 0$ if $1 \leq \mu = 2-2/p$, or ${\rm d}_2(t) = 1 + (\log(1+t))^{1-\frac{\mu}{2}}$ if $2/p = \mu < 1$.

Remark 2.2. We notice that we may compute

$$-\min\left\{1,\mu,\frac{\mu}{2}+\frac{1}{p}\right\} + \frac{1}{p} = \begin{cases} -\min\{1,\mu\} + \frac{1}{p} & \text{if } 2/p \geq \min\{\mu,2-\mu\},\\ -\frac{\mu}{2} & \text{if } 2/p \leq \min\{\mu, 2-\mu\}, \end{cases}$$

in (15). The two cases above correspond to the behavior of the multiplier associated to the fundamental solution to the linear regular problem at "intermediate frequencies" (see the proof of Proposition 3.3). They may also be considered as the cases of:

- *effective dissipation* if $2/p \geq \min\{\mu, 2-\mu\}$; the decay rate is analogous to the $L^1 - L^p$ decay rate of a heat equation (for $\mu \geq 1$, this decay rate is $t^{-1+\frac{1}{p}}$);
- *non effective dissipation* if $2/p \leq \min\{\mu, 2-\mu\}$; the decay rate is independent of p.

Remark 2.3. As mentioned in Remark 2.1, the right-hand inequality in (13), i.e., (10), is equivalent to $p > p_{\mathrm{crit}}$. The left-hand inequality in (13) is equivalent to $\alpha \geq -1 - 1/p$. This condition is fundamental in the proof of Theorem 2.1 to avoid a non integrable singularity at $t = 0$. The interval in (13) is nonempty if, and only if, $p > 1/\mu$ when $\mu \in (0, 1)$ and this motivates the assumption $p > \max\{1, 1/\mu\}$. The fact that $p > 1/\mu$ also implies that $q < p$ in (12), for any $\mu > 0$.

The condition $\alpha \geq -1 - 1/p$ does not appear in the subsequent Theorem 2.2, since the Cauchy problem is regular and there is no singularity at $t = t_0 > 0$. We stress that when $\alpha < -1 - 1/p$, estimate (17) is not necessarily a decay estimate for $p > p_{\mathrm{crit}}$. Indeed, for $p \in (p_{\mathrm{crit}}, 1/\mu]$, $g(t) = t^{\frac{1}{p}-\mu}$ in (17) does not vanish as $t \to \infty$. However, even if the norm $\|u(t,\cdot)\|_{L^p}$ does not vanish as $t \to \infty$ in this case, the function $t^{-\alpha}$ decays sufficiently fast to imply the existence of a global-in-time solution in $L^\infty_{\mathrm{loc}}([t_0,\infty), L^p)$.

Theorem 2.2. *Let $\mu > 0$, $\alpha > -2$ and $p > p_{\mathrm{crit}}$, where p_{crit} is as in* (9)*, or $p > 1$ if $\alpha \leq -2$. Then there exists $\varepsilon > 0$ such that for any initial data*

$$u_1 \in L^1, \quad \textit{with } \|u_1\|_{L^1} \leq \varepsilon, \tag{16}$$

there is a unique global-in-time weak solution $u \in L^\infty_{\mathrm{loc}}([t_0,\infty), L^p)$, to (8)*. Moreover, the solution to* (8) *satisfies the estimate*

$$\|u(t,\cdot)\|_{L^p} \leq C\, g(t)\, \|u_1\|_{L^1}, \quad \textit{with} \qquad g(t) = t^{-\min\{1,\mu,\frac{\mu}{2}+\frac{1}{p}\}+\frac{1}{p}}\, \mathrm{d}_1(t)\, \mathrm{d}_2(t), \tag{17}$$

where $C = C(t_0) > 0$, is independent of t, and of the initial data, and $\mathrm{d}_1(t)$ and $\mathrm{d}_2(t)$ are logarithmic loss terms determined as follows: either $\mathrm{d}_1 = 1$ if $\mu \neq 1$ or $\mathrm{d}_1 = 1 + \log(1+t)$ if $\mu = 1$; either $\mathrm{d}_2(t) = 1$ if $2/p \neq \min\{\mu, 2-\mu\}$, or $\mathrm{d}_2(t) = 1 + (\log(1+t))^{1-\frac{\mu}{2}}$ if $2/p = \min\{\mu, 2-\mu\}$.

Remark 2.4. Let us determine p_{crit} according to the value of μ and $\alpha > -2$. We stress that

$$1 + \frac{2+\alpha}{\min\{1,\mu\}} > p_{\mathrm{Str}}(1+\mu,\alpha) \iff \min\{1,\mu\} < \frac{\mu}{2} + \frac{1}{p_{\mathrm{crit}}} \iff \frac{2}{p_{\mathrm{crit}}} > \min\{\mu, 2-\mu\}.$$

It holds $p_{\mathrm{crit}} = p_{\mathrm{Str}}(1+\mu,\alpha)$ if, and only if, $\alpha \geq -1$ and $-\alpha \leq \mu \leq \bar{\mu}$, where

$$\bar{\mu} = \frac{2(2+\alpha)}{3+\alpha}. \tag{18}$$

It holds $p_{\mathrm{crit}} = 3 + \alpha$ if, and only if, either $\mu \geq \bar{\mu}$, when $\alpha > -1$, or $\mu \geq 1$ when $\alpha \leq -1$. It holds $p_{\mathrm{crit}} = 1 + (2+\alpha)/\mu$ if, and only if, $0 < \mu \leq -\alpha$ if $\alpha \in (-1, 0)$, or $\mu \leq 1$ if $\alpha \leq -1$.

If $\alpha = 0$, then $p_{\mathrm{crit}} = \max\{p_{\mathrm{Str}}(1+\mu), 3\}$, and $p_{\mathrm{crit}} = p_{\mathrm{Str}}(1+\mu)$ if, and only if, $\mu \in (0, 4/3]$.

By the change of variable

$$w(t,x) = u(\Lambda(t), x), \quad \text{where } \Lambda(t) = \frac{t^{\ell+1}}{\ell+1}, \tag{19}$$

the singular Cauchy problem (7) for the E. P. D. equation is equivalent to the weakly hyperbolic semilinear Cauchy problem for the generalized Tricomi equation

$$\begin{cases} w_{tt} - t^{2\ell}\, w_{xx} + \dfrac{\mu_*}{t}\, w_t = t^{\alpha_*} f(w), & t > 0,\ x \in \mathbb{R}, \\ w(0,x) = w_0(x), \quad w_t(0,x) = 0, \end{cases} \tag{20}$$

with $\ell > -1$ and $\mu_* > -\ell$, where

$$\mu = \frac{\ell + \mu_*}{\ell+1}, \quad \alpha = \frac{\alpha_* - 2\ell}{\ell+1}. \tag{21}$$

Therefore, as a corollary of Theorem 2.1, we can prove the existence of global-in-time (weak) solutions to problem (20).

Corollary 2.1. *Let $\ell > -1$, $\mu_* > -\ell$ and $p > \max\{1, (\ell+1)/(\ell+\mu_*)\}$. Assume that*

$$\frac{p-1}{p}\,(\ell+1) \leq \alpha_* + 2 < (p-1)\min\left\{\ell+1,\ \ell+\mu_*,\ \frac{\ell+\mu_*}{2} + \frac{\ell+1}{p}\right\}, \tag{22}$$

Let $q \in [1, p)$ be such that

$$q = \max\left\{1, \frac{\ell+1}{\ell+\mu_*}\right\} \quad \text{if} \quad \frac{2}{p} \geq \frac{\ell+\min\{\mu_*, 2-\mu_*\}}{\ell+1}, \quad \text{or} \quad \frac{1}{q} - \frac{1}{p} = \frac{\mu}{2}, \quad \text{if} \quad \frac{2}{p} < \frac{\ell-\max\{\mu_*, 2-\mu_*\}}{\ell+1}. \tag{23}$$

Then there exists $\varepsilon > 0$ such that for any initial data

$$w_0 \in L^q \cap L^p, \quad \text{with } \|w_0\|_{L^q} + \|w_0\|_{L^p} \leq \varepsilon, \tag{24}$$

there exists a unique global-in-time weak solution $w \in L^\infty([0,\infty), L^p)$, to (20). *Moreover, for any $\delta > 0$, the solution to* (20) *satisfies the decay estimate*

$$\|w(t,\cdot)\|_{L^p} \leq C\, g_*(1+t)\,(\|w_0\|_{L^q} + \|w_0\|_{L^p}), \quad \text{with} \quad g_*(1+t) = (1+t)^{-\min\left\{\ell+1, \ell+\mu_*, \frac{\ell+\mu_*}{2}+\frac{\ell+1}{p}\right\}+\frac{\ell+1}{p}}\, \mathrm{d}_1(t)\, \mathrm{d}_2(t), \tag{25}$$

where $C > 0$, is independent of t, and of the initial data, and $\mathrm{d}_1(t)$ and $\mathrm{d}_2(t)$ are small loss terms determined as follows: either $\mathrm{d}_1 = 1$ if $\mu_* \neq 1$ or $\mathrm{d}_1 = 1 + \log(1+t)$ if $\mu_* = 1$; either $\mathrm{d}_2(t) = 1$ if $2/p \neq \frac{\ell-\max\{\mu_*, 2-\mu_*\}}{\ell+1}$, or we may take $\mathrm{d}_2(t) = c_\delta\,(1+t)^\delta$ for any small $\delta > 0$, if $\mu_* \geq 1$ and $\frac{2}{p} = \frac{\ell+2-\mu_*}{\ell+1}$, or $\mathrm{d}_2(t) = 1 + (\log(1+t))^{1-\frac{\mu}{2}}$ if $\mu_* < 1$ and $\frac{2}{p} = \frac{\ell+\mu_*}{\ell+1}$.

For any $\alpha_* > -2$, the right-hand side of (22) is equivalent to $p > p_{\mathrm{crit}}$, where

$$p_{\mathrm{crit}} = \max\left\{1 + \frac{2+\alpha_*}{\ell+\min\{1,\mu_*\}},\ p_{\mathrm{Str}}\left(\frac{2\ell+\mu_*+1}{\ell+1}, \frac{\alpha_*-2\ell}{\ell+1}\right)\right\}, \tag{26}$$

and the left-hand side of (22) is equivalent to $p \leq 1 + (2+\alpha_*)/(\ell-\alpha_*-1)$ if $\ell > \alpha_*+1$.

The nonexistence of global-in-time weak solutions to (20) for $\mu_* = 0$ and $p \in (1, 1+(2+\alpha_*)/\ell]$, under suitable sign condition on the data, is proved in Theorem 3.1 in [11]. In the special case $\mu_* = \alpha_* = 0$, Corollary 2.1 provides the global existence of solutions to (20) in $L^\infty([0,\infty), L^p)$, for $p > p_{\mathrm{crit}} = 1 + 2/\ell$, and small data w_0. The global-in-time existence of small data solutions to (20) for $p > 1 + 2/\ell$, in this special case $\mu_* = \alpha_* = 0$ has been recently proved in [23], see also [17].

Similarly, by the change of variable (19), the regular Cauchy problem (8) for the E. P. D. equation is equivalent to the strictly hyperbolic semilinear Cauchy problem for the generalized Tricomi equation

$$\begin{cases} w_{tt} - t^{2\ell}\, w_{xx} + \dfrac{\mu_*}{t}\, w_t = t^{\alpha_*}\, f(w), & t \geq t_1 > 0,\ x \in \mathbb{R}, \\ w(t_1, x) = 0, \quad w_t(t_1, x) = w_1(x), & \end{cases} \tag{27}$$

where μ and α are given by (21), $t_1 = \Lambda^{-1}(t_0) = ((\ell+1)t_0)^{\frac{1}{\ell+1}}$, and $w_1(x) = t_1^\ell\, u_1(x)$.

As a corollary of Theorem 2.2, we can prove the existence of global-in-time (weak) solutions to problem (27).

Corollary 2.2. *Let $\ell > -1$, $\mu_* > -\ell$, $\alpha_* > -2$, and $p > p_{\mathrm{crit}}$, where p_{crit} is as in* (26)*, or $p > 1$ if $\alpha_* \leq -2$. Then there exists $\varepsilon > 0$ such that for any initial data*

$$w_1 \in L^1, \quad \text{with } \|w_1\|_{L^1} \leq \varepsilon, \tag{28}$$

there exists a unique global-in-time weak solution $w \in L^\infty_{\mathrm{loc}}([t_1,\infty), L^p)$, to (27). *Moreover, the solution to* (27) *satisfies the estimate*

$$\|w(t,\cdot)\|_{L^p} \leq C\, g_*(t)\, \|w_1\|_{L^1}, \quad \text{with} \quad g_*(t) = t^{-\min\left\{\ell+1, \ell+\mu_*, \frac{\ell+\mu_*}{2}+\frac{\ell+1}{p}\right\}+\frac{\ell+1}{p}}\, \mathrm{d}_1(t)\, \mathrm{d}_2(t), \tag{29}$$

where $C > 0$, is independent of t, and of the initial data, and $\mathrm{d}_1(t)$ and $\mathrm{d}_2(t)$ are logarithmic loss terms determined as follows: either $\mathrm{d}_1 = 1$ if $\mu_* \neq 1$ or $\mathrm{d}_1 = 1 + \log(1+t)$ if $\mu_* = 1$; either $\mathrm{d}_2(t) = 1$ if $2/p \neq \frac{\ell-\max\{\mu_*, 2-\mu_*\}}{\ell+1}$, or $\mathrm{d}_2(t) = 1 + (\log(1+t))^{1-\frac{\mu}{2}}$ if $\frac{2}{p} = \frac{\ell-\max\{\mu_*, 2-\mu_*\}}{\ell+1}$.

Remark 2.5. By the change of variable $v(t,x) = t^\beta\, w(t,x)$, Cauchy problem (27) with $f(w) = |w|^p$ is equivalent to

$$\begin{cases} v_{tt} - t^{2\ell}\, v_{xx} + \dfrac{\mu_\circ}{t}\, v_t + \dfrac{m}{t^2}\, w = t^{\alpha_\circ}\, |v|^p, & t \geq t_1 > 0,\ x \in \mathbb{R}\,, \\ v(t_1,x) = 0\,, \quad v_t(t_1,x) = v_1(x)\,, \end{cases} \tag{30}$$

where $\mu_\circ = \mu_* - 2\beta$, $m = -\beta(\mu_\circ + \beta - 1)$ and $\alpha_\circ = \alpha_* - \beta(p-1)$, and we put $v_1(x) = t_1^\beta\, w_1(x)$. Therefore, Theorem 2.2 may be easily applied to obtain the existence of global-in-time small data weak solutions to (30). For the ease of reading, we postpone the details to §5.

The equation in (30) is called modified E. P. D. equation when $\ell = 0$ and $\beta = \mu/2$ (see [4]). It is also called wave equation with scale-invariant mass and dissipation when $\ell = 0$ and $\beta < \mu/2$. For several studies on this model and its multidimensional version, we address the reader to [3, 10, 14, 42, 43, 45, 46, 47, 48] and the references therein.

3. Estimates for the linear problem

The E. P. D. equation is not invariant by time-translation, due to the time-dependent coefficient μt^{-1} in front of u_t. For this reason, we study the regular linear Cauchy problem (11), where the starting time is a parameter $s > 0$, in view of the application of Duhamel's principle to both the inhomogeneous singular and regular Cauchy problems.

The dependence on the parameter s of the estimates obtained for the solution to (11) plays a crucial role in the argument employed to prove the existence of global-in-time solutions: a precise evaluation of the dependence on the parameter s in the estimates is fundamental to find the critical exponent in the application to the semilinear problem (see Remark 2.1).

In order to prove our results, we will use the following multiplier theorem.

Proposition 3.1. *[see [39, Theorem 4.2] and the references therein] For any $\xi \in \mathbb{R}$, let*

$$m(\xi) = \psi(|\xi|)\, |\xi|^{-k} e^{\pm i|\xi|},$$

where $k > 0$ and $\psi \in C^\infty$ vanishes near the origin and is 1 for large values of $|\xi|$. Then $m \in M_q^p$ if, and only if, $1/q - 1/p \leq k$ when $1 < q \leq p < \infty$, and if, and only if, $1/q - 1/p < k$, when $q = 1$ or $p = \infty$.

We say that m is a multiplier in M_q^p, for some $1 \leq q \leq p \leq \infty$ if for any $f \in L^q$ it holds $T_m f = \mathfrak{F}^{-1}(m\hat{f}) \in L^p$; the quantity

$$\|m\|_{M_q^p} = \sup_{\|f\|_{L^q}=1} \|T_m f\|_{L^p}, \tag{31}$$

is a norm on M_q^p. In particular, $M_p^p \subset M_2^2 = L^\infty$ and $M_1^p = \mathfrak{F}(L^p)$ for $p > 1$ (see [24, Theorem 1.4]).

To write the Fourier transform with respect to the space variable, of the fundamental solution to (11), we will use the Bessel functions of first kind, whose definition by series is

$$J_\rho(z) = \sum_{m=0}^\infty \frac{(-1)^m}{m!\Gamma(m+\rho+1)}\, (z/2)^{2m+\rho}, \tag{32}$$

for $\rho \neq -1, -2, \ldots,$. We will also use the asymptotic expansion (see [61, §7.21]) of the Bessel functions $J_\rho(z)$ for large values of z,

$$\begin{aligned} J_\rho(z) &= (z\pi/2)^{-\frac12}\, \cos(z - \rho\pi/2 - \pi/4)\, R_{|\rho|,0}(z) - (z\pi/2)^{-\frac12}\, \sin(z - \rho\pi/2 - \pi/4)\, R_{|\rho|,1}(z), \quad \text{where} \\ R_{|\rho|,j}(z) &= \sum_{m=0}^\infty (-1)^m (|\rho|, 2m+j)(2z)^{-2m-j}. \end{aligned} \tag{33}$$

In particular,

$$|J_\rho(z)| \leq \begin{cases} C\, z^\rho & \text{for } z \in (0,1], \\ C\, z^{-\frac12} & \text{for } z \in [1,\infty). \end{cases} \tag{34}$$

3.1. Estimates for the linear singular problem

Before studying how the parameter s influences the estimates for problem (11), by straightforward calculations we obtain $L^q - L^p$ estimates for the singular linear Cauchy problem

$$\begin{cases} v_{tt} - v_{xx} + \dfrac{\mu}{t}\, v_t = 0, & t > 0,\ x \in \mathbb{R}, \\ v(0,x) = v_0(x), \quad v_t(0,x) = 0. \end{cases} \tag{35}$$

Proposition 3.2. *Let $\mu > 0$, $p \in (1,\infty)$ and $q \in [1,p]$. Assume that $1 - 1/p < \mu/2$ if $q = 1$, or that $1/q - 1/p \leq \mu/2$ otherwise. Then the solution to* (35) *verifies the following $L^q - L^p$ decay estimate:*

$$\|v(t,\cdot)\|_{L^p} \leq C\, t^{-\frac{1}{q}+\frac{1}{p}}\, \|v_0\|_{L^q}, \tag{36}$$

for some $C > 0$, independent of t.

Proof. Let $K(t)$ be the fundamental solution to (35). The Fourier transform of $K(t)$ with respect to the space variable x solves the Cauchy problem

$$\begin{cases} \hat{K}_{tt} + \xi^2 \hat{K} + \dfrac{\mu}{t}\, \hat{K}_t = 0, & t > 0, \\ \hat{K}(0) = 1, \quad \hat{K}_t(0) = 0. \end{cases} \tag{37}$$

The equation in (37) is equivalent to a Bessel's differential equation [61, §4.3] of order $\pm\nu$, where $\nu := (\mu-1)/2$:

$$\tau^2 y'' + \tau y' + (\tau^2 - \nu^2) y = 0, \qquad \tau > 0. \tag{38}$$

Indeed, if we define $\tau = t|\xi|$ and $w(t|\xi|) = \hat{K}(t)$, then Cauchy problem (37) may be written as

$$\begin{cases} w'' + w + \dfrac{\mu}{\tau}\, w' = 0, & \tau > 0, \\ w(0) = 1, \quad w'(0) = 0. \end{cases} \tag{39}$$

The equation in (39) becomes the Bessel's differential equation (38), if we put $y(\tau) = \tau^\nu\, w(\tau)$. Therefore, the solution to (39) is

$$w(\tau) = 2^\nu\, \Gamma(1+\nu)\, \tau^{-\nu}\, J_\nu(\tau),$$

since it verifies $w(0) = 1$ and $w'(0) = 0$; replacing $\tau = t|\xi|$, we get (see also [2])

$$\hat{K}(t) = 2^\nu\, \Gamma(1+\nu)\, (t|\xi|)^{-\nu}\, J_\nu(t|\xi|). \tag{40}$$

By homogeneity,

$$\|\hat{K}(t)\|_{M_q^p} = t^{-\frac{1}{q}+\frac{1}{p}}\, \|\hat{K}_0\|_{M_q^p},$$

where we put $K_0 = K(1)$, so that it is sufficient to prove that $\hat{K}_0 \in M_q^p$ if $1/q - 1/p \leq \mu/2$ when $q > 1$, or if $1 - 1/p < \mu/2$ when $q = 1$.

Indeed, these statements immediately follow by the explicit expression (see [52])

$$K_0 = c_{1,\mu}\, (1 - x^2)_+^{-1+\frac{\mu}{2}},$$

thanks to Young's theorem on convolution. However, we may also provide an alternative proof which only relies on the expression of $\hat{K}_0$, to emphasize the differences with the strategy employed to derive the analogous estimates for (11).

Let $\chi \in C_c^\infty$, even, be such that $\chi = 1$ in a neighborhood of the origin, say $\chi(\xi) = 1$ for $\xi \in [0,1/2]$ and $\chi(\xi) = 0$ for $\xi \geq 1$.

We first prove that $\hat{K}_0 \chi \in M_q^p$, for any $q \in [1,p]$. By (34) we find that $\hat{K}_0 \chi$ is bounded. Using the property of the Bessel functions $z J_\rho' = -\rho J_\rho + z\, J_{\rho-1}$, we obtain

$$\partial_\xi J_\rho(|\xi|) = J_\rho'(|\xi|)\, \mathrm{sign}\xi = \big(-|\xi|^{-1} \rho J_\rho(|\xi|) + J_{\rho-1}(|\xi|)\big)\, \mathrm{sign}\xi, \tag{41}$$

so that, recalling that χ is supported in $\{\xi : |\xi| \leq 1\}$ and it is smooth, we derive

$$\left|\partial_\xi(\hat{K}_0(\xi)\chi(\xi))\right| \leq C\,|\xi|^{-1}.$$

If $q = p$, by Mikhlin-Hörmander theorem (see [24, Theorem 2.5]), we obtain $\hat{K}_0\chi \in M_p^p$ for any $p \in (1, \infty)$. Due to $\chi \in C_c^\infty$, it also follows (see [24, Theorem 1.8]) that $\hat{K}_0\chi \in M_q^p$ for $1 \leq q < p < \infty$.

To prove that $(1-\chi)\hat{K}_0 \in M_q^p$ if $1/q - 1/p \leq \mu/2$ when $q > 1$, or if $1 - 1/p < \mu/2$ when $q = 1$, we rely on Proposition 3.1. Indeed, it is sufficient to use (33), and to notice that $|\xi|^{-\nu-k}\,e^{i|\xi|}\,(1-\chi) \in M_q^p$ for any $1 \leq q \leq p \leq \infty$ if $k = 1, 2, \ldots$, whereas $|\xi|^{-\nu}\,e^{i|\xi|}\,(1-\chi) \in M_q^p$ if, and only if, $1 - 1/p < \mu/2$ if $q = 1$, or $1/q - 1/p \leq \mu/2$, otherwise. This concludes the proof.

3.2. Estimates for the linear regular problem depending on the parameter s

For the sake of brevity, we only consider $L^1 - L^p$ estimates for the solution to (11), since these estimates will be used to prove Theorems 2.1 and 2.2. More general $L^q - L^p$ estimates may be obtained by minor modifications. For some $L^{p'} - L^p$ estimates, with $2 \leq p < \infty$ and $p' = p/(p-1)$, we address the reader to [62, Theorem 3.5].

Proposition 3.3. *Let $\mu \in \mathbb{R}$ and $p \in (1, \infty]$. Then the solution to* (11) *verifies the following $L^1 - L^p$ estimate:*

$$\|v(t,\cdot)\|_{L^p} \leq C\,(t/s)^{-\min\{1,\mu,\frac{\mu}{2}+\frac{1}{p}\}}\,t^{\frac{1}{p}}\,\mathrm{d}_1(t/s)\,\mathrm{d}_2(t/s)\,\|v_1\|_{L^1}\,, \tag{42}$$

for some $C > 0$, independent of s, t, where $\mathrm{d}_1(t/s)$ *and* $\mathrm{d}_2(t/s)$ *are logarithmic loss terms determined as in Theorem 2.2: either* $\mathrm{d}_1 = 1$ *if* $\mu \neq 1$, *or* $\mathrm{d}_1(t/s) = 1 + \log(1 + t/s)$ *if* $\mu = 1$*; either* $\mathrm{d}_2 = 1$ *if* $2/p \neq \min\{\mu, 2-\mu\}$ *or* $\mathrm{d}_2(t/s) = 1 + (\log(t/s))^{1-\frac{1}{p}}$ *if* $2/p = \min\{\mu, 2-\mu\}$.

Remark 3.1. It is sufficient to prove Proposition 3.3 for $\mu \geq 1$. Indeed, let $\mu \in (-\infty, 1)$ in (11). If we define

$$v^\sharp(t,x) = t^{\mu-1}\,v(t,x)\,, \qquad \text{and} \quad \mu^\sharp = 2-\mu\,, \tag{43}$$

then Cauchy problem (11) becomes

$$\begin{cases} v_{tt}^\sharp - v_{xx}^\sharp + \dfrac{\mu^\sharp}{t}\,v_t^\sharp = 0, & t > s\,,\ x \in \mathbb{R}^n\,, \\ v^\sharp(s,x) = 0, \quad v_t^\sharp(s,x) = s^{1-\mu^\sharp}\,v_1(x). \end{cases} \tag{44}$$

Applying Proposition 3.3 to (44) with $\mu^\sharp > 1$, we obtain the statement of Proposition 3.3 for $\mu < 1$.

Proof. Let $K = K(t,s)$ be the fundamental solution to (11). The Fourier transform of K with respect to the space variable solves the problem

$$\begin{cases} \hat{K}_{tt} + \xi^2\hat{K} + \dfrac{\mu}{t}\,\hat{K}_t = 0, & t > s, \\ \hat{K}(s,s) = 0\,, \quad \hat{K}_t(s,s) = 1\,. \end{cases} \tag{45}$$

If we set

$$\tau = t|\xi|, \quad \sigma = s|\xi|, \quad w(t|\xi|) = \hat{K}(t,s),$$

we find the equivalent problem

$$\begin{cases} w'' + w + \dfrac{\mu}{\tau}\,w' = 0, & \tau \geq \sigma\,, \\ w(\sigma) = 0\,, \quad w'(\sigma) = |\xi|^{-1}\,. \end{cases} \tag{46}$$

If we put $\nu := (\mu-1)/2$ and $y(\tau) = \tau^\nu\,w(\tau)$, then from (46) we obtain the following Cauchy problem for the Bessel's differential equation (38) of order $\pm\nu$:

$$\begin{cases} \tau^2 y'' + \tau y' + (\tau^2 - \nu^2)y = 0\,, & \tau \geq \sigma\,, \\ y(\sigma) = 0, \quad y'(\sigma) = s\,\sigma^{\nu-1}. \end{cases} \tag{47}$$

We assume that $\nu > 0$ is not an integer, that is, $\mu > 1$ is not an odd integer. Then a system of linearly independent solutions to (47) is given by the pair of Bessel functions (of first kind) $J_{\pm\nu}(\tau)$. Hence, we put

$$y = C_+(\sigma)\, J_\nu(\tau) + C_-(\sigma)\, J_{-\nu}(\tau).$$

We postpone the case where ν is an integer to the end of the proof. In that case, we use a different system of linearly independent solutions to (47). However, only minor changes appear, unless $\nu = 0$, that is, $\mu = 1$.

Recalling that the Wronskian satisfies [61, §3.12]

$$W[J_\nu, J_{-\nu}](\sigma) = J_\nu(\sigma) J'_{-\nu}(\sigma) - J'_\nu(\sigma) J_{-\nu}(\sigma) = \frac{-2\sin(\nu\pi)}{\pi\sigma},$$

we obtain the solution

$$y = \frac{\pi}{2\sin(\nu\pi)}\, (J_{-\nu}(\sigma) J_\nu(\tau) - J_\nu(\sigma) J_{-\nu}(\tau))\, s\, \sigma^\nu,$$

so that, replacing $\sigma = s|\xi|$ and $\tau = t|\xi|$, we find

$$\hat{K}(t,s) = \frac{\pi}{2\sin(\nu\pi)}\, (J_{-\nu}(s|\xi|) J_\nu(t|\xi|) - J_\nu(s|\xi|) J_{-\nu}(t|\xi|))\, s^{\nu+1}\, t^{-\nu}.$$

We now want to estimate the multiplier norm (31) of $\hat{K}(t,s)$, depending on both s, t.

We define $a = s/t \in (0, 1]$. By a dilation argument, for any $t > 0$ it holds

$$\|\hat{K}(t,s)\|_{M_1^p} = s\, t^{-1+\frac{1}{p}}\, \|\hat{K}_a\|_{M_1^p}, \tag{48}$$

where

$$\hat{K}_a(\xi) = \frac{\pi}{2\sin(\nu\pi)}\, a^\nu\, (J_{-\nu}(a|\xi|) J_\nu(|\xi|) - J_\nu(a|\xi|) J_{-\nu}(|\xi|)). \tag{49}$$

Incidentally, we notice that, using Euler's reflection formula, for any given ξ,

$$\hat{K}_a(\xi) \sim \frac{\pi}{2\sin(\nu\pi)}\, \frac{2^\nu\, |\xi|^{-\nu}}{\Gamma(1-\nu)}\, J_\nu(|\xi|) = \frac{1}{2}\, \hat{K}_0(\xi), \quad \text{as } a \to 0,$$

with $\hat{K}_0$ as in the proof of Proposition 3.2. First, we consider the easier case $p \geq 2$.

For $|\xi| \leq 1$, $\hat{K}_a$ is uniformly bounded with respect to a; indeed, thanks to (34),

$$|\hat{K}_a(\xi)| \leq C\, a^\nu (a^{-\nu} + a^\nu) \leq 2C.$$

Let $|\xi| \in [1, a^{-1}]$. In this case, using (34), noticing that $a|\xi| \leq 1 \leq |\xi|$, we obtain

$$|\hat{K}_a(\xi)| \leq C\, |\xi|^{-\nu-\frac{1}{2}} = C\, |\xi|^{-\frac{\mu}{2}}.$$

On the other hand, for $|\xi| \in [a^{-1}, \infty)$, we use (34) to estimate

$$|\hat{K}_a(\xi)| \leq C\, a^{\nu-\frac{1}{2}} |\xi|^{-1} = C\, a^{\frac{\mu-2}{2}}\, |\xi|^{-1}.$$

In all the above estimates, $C > 0$ is independent of a. By the Hausdorff-Young inequality, we have $\|\hat{K}_a\|_{M_1^p} \leq C\, \|\hat{K}_a\|_{L^{p'}}$, where $p' = p/(p-1)$. Hence, we obtain

$$\|\hat{K}_a\|_{M_1^p} \leq C_1 + C_2 \Big(\int_1^{a^{-1}} |\xi|^{-\frac{\mu}{2}p'}\, d\xi \Big)^{\frac{1}{p'}} + C_3\, a^{\frac{\mu-2}{2}} \Big(\int_{a^{-1}}^\infty |\xi|^{-p'}\, d\xi \Big)^{\frac{1}{p'}} \leq C_1 + \tilde{C}_3 a^{\frac{\mu}{2}-\frac{1}{p'}} + \begin{cases} \tilde{C}_2\, a^{\frac{\mu}{2}-\frac{1}{p'}} & \text{if } p' < 2/\mu, \\ \tilde{C}_2\, (-\log a)^{\frac{1}{p'}} & \text{if } p' = 2/\mu, \\ \tilde{C}_2 & \text{if } p' > 2/\mu. \end{cases}$$

The first and the second term are dominated by the latter one in the sum above, so that we conclude

$$\|\hat{K}_a\|_{M_1^p} \leq \begin{cases} C\, a^{\frac{\mu}{2}-\frac{1}{p'}} & \text{if } 1 - 1/p > \mu/2, \\ C\, (\log(e + 1/a))^{\frac{1}{p'}} & \text{if } 1 - 1/p = \mu/2, \\ C & \text{if } 1 - 1/p < \mu/2, \end{cases} \tag{50}$$

with $C > 0$, independent of a. Now let $p \in (1,2)$. In order to prove (50) it is sufficient to prove that $\|\hat{K}_a\|_{M_1^p} \le C$, since $1 - 1/p < 1/2 \le \mu/2$.

In this case, we cannot use the Hausdorff-Young inequality, so we follow the proof of Proposition 3.2. However, in order to take into account of the influence from the parameter a, we fix three localizing functions $\chi_0, \chi_1, \chi_2 \in C^\infty$, with the following properties:

- $\chi_0(\xi) = 1$ for $|\xi| \le 1/2$, and χ_0 is supported in the "low frequencies zone" $\{\xi : |\xi| \le 1\}$;
- $\chi_2(\xi) = 1$ for $a|\xi| \ge 2$, and χ_2 is supported in the "high frequencies zone" $\{\xi : a|\xi| \ge 1\}$, say $\chi_2 = 1 - \chi_0(a|\xi|/2)$;
- it holds $1 = \chi_0^2 + \chi_1^2 + \chi_2^2$; in particular, χ_1 is supported in the "intermediate frequencies zone" $\{\xi : 1/2 \le |\xi| \le 2a^{-1}\}$.

Then (50) follows, if we prove that $\|\hat{K}_a\chi_j^2\|_{M_1^p} \le C$, for $j = 0, 1, 2$.

Thanks to Young inequality,

$$\|\hat{K}_a\chi_0^2\|_{M_1^p} \le C\|\mathfrak{F}^{-1}(\hat{K}_a\chi_0^2)\|_{L^p}.$$

The function $\hat{K}_a\chi_0^2$ is continuous and compactly supported. Using (41) and

$$\partial_\xi J_\rho(a|\xi|) = a\, J_\rho'(a|\xi|)\,\mathrm{sign}\xi = \big(-|\xi|^{-1}\rho J_\rho(a|\xi|) + aJ_{\rho-1}(a|\xi|)\big)\,\mathrm{sign}\xi,$$

we derive

$$\left|\partial_\xi(\hat{K}_a(\xi)\chi_0^2(\xi))\right| \le C\,|\xi|^{-1},$$

with C independent of a. Proceeding as in the proof of Proposition 3.2, by Mikhlin-Hörmander theorem, it follows that $\|\hat{K}_a\chi_0^2\|_{M_1^p} \le C$, with $C > 0$, independent of a, for any $p > 1$.

To deal with the intermediate frequencies, we use different multiplier estimates for $J_{\pm\nu}(a|\xi|)$ and $J_{\mp\nu}(|\xi|)$, noticing that

$$\|\hat{K}_a\chi_1^2\|_{M_1^p} \le \frac{\pi}{2\sin(\nu\pi)}\, a^\nu\, (\|J_{-\nu}(a|\xi|)\chi_1\|_{M_p^p}\,\|J_\nu(|\xi|)\chi_1\|_{M_1^p} + \|J_\nu(a|\xi|)\chi_1\|_{M_p^p}\,\|J_{-\nu}(|\xi|)\chi_1\|_{M_1^p}).$$

Proceeding as before, we estimate

$$|J_{\pm\nu}(a|\xi|)\chi_1(\xi)| \le C(a|\xi|)^{\pm\nu}, \quad |\partial_\xi(J_{\pm\nu}(a|\xi|)\chi_1(\xi))| \le C(a|\xi|)^{\pm\nu}\,|\xi|^{-1},$$

with $C > 0$, independent of a. Since we are at intermediate frequencies, we may estimate $(a|\xi|)^\nu \le 2^\nu$ and $(a|\xi|)^{-\nu} \le 2^\nu\, a^{-\nu}$. Therefore, by Mikhlin-Hörmander multiplier theorem, we obtain

$$\|J_\nu(a|\xi|)\chi_1\|_{M_p^p} \le C \qquad \|J_{-\nu}(a|\xi|)\chi_1\|_{M_p^p} \le C\,a^{-\nu}.$$

On the other hand,

$$\|J_{\pm\nu}(|\xi|)\chi_1\|_{M_1^p} \le C,$$

with $C > 0$, independent of a. Indeed, using (33), the previous estimate follows from the fact that

$$\||\xi|^{-\frac{1}{2}-k}\,e^{i|\xi|}\,\chi_1\|_{M_1^p} \le C, \quad k = 0, 1, \ldots,$$

due to $p \in (1,2)$ (see Proposition 3.1). Summarizing,

$$\|\hat{K}_a\chi_1^2\|_{M_1^p} \le C\,, \tag{51}$$

with $C > 0$, independent of a.

At high frequencies, we use (33) for both $J_{\pm\nu}(a|\xi|)$ and $J_{\mp\nu}(|\xi|)$. By the cosine and sine addition formulas, a straightforward computation leads to

$$\hat{K}_a(\xi) = a^{\nu-\frac{1}{2}}\,|\xi|^{-1}\,R(a, |\xi|),$$

with

$$R(a,|\xi|) = \sin((1-a)|\xi|)(R_{|\nu|,0}(a|\xi|)R_{|\nu|,0}(|\xi|) + R_{|\nu|,1}(a|\xi|)R_{|\nu|,1}(|\xi|)) + \cos((1-a)|\xi|)(R_{|\nu|,0}(a|\xi|)R_{|\nu|,1}(|\xi|) - R_{|\nu|,1}(a|\xi|)R_{|\nu|,0}(|\xi|))$$

so that

$$\hat{K}_a(\xi) = \frac{1}{2}\, a^{\nu-\frac{1}{2}}\, |\xi|^{-1} \sin((1-a)|\xi|) + \ldots$$

By Proposition 3.1, we may estimate

$$a^{\nu-\frac{1}{2}-j}\||\xi|^{-1-k-j}\sin((1-a)|\xi|)\chi_2^2\|_{M_1^p} \le a^{\nu+k}\,\|(a|\xi|)^{-\frac{1}{2}-k-j}\chi_2\|_{M_p^p}\,\||\xi|^{-\frac{1}{2}}\sin((1-a)|\xi|)\chi_2\|_{M_1^p} \le C\,a^{\nu+k} \le C,$$

for $k+j=0,2,4,\ldots$, due to $p\in(1,2)$, and similarly for the cosine terms, for $k+j=1,3,5,\ldots$

Summarizing, we concluded the proof of (50). Recalling (48), and replacing $a=s/t$, we proved so far that

$$\|K(t,s)\|_{M_1^p} \le C\,s\,t^{-1+\frac{1}{p}} \times \begin{cases} (t/s)^{1-\frac{1}{p}-\frac{\mu}{2}} & \text{if } 1-1/p > \mu/2,\\ (\log(e+t/s))^{1-\frac{1}{p}} & \text{if } 1-1/p = \mu/2,\\ 1 & \text{if } 1-1/p < \mu/2, \end{cases}$$

and this concludes the proof of (42) for $\mu>1$, not an odd integer.

If $\mu\in 2\mathbb{N}+1$, that is, ν is a nonnegative integer, then we write the fundamental solution to (47) as

$$y = C_+(\sigma)\,J_\nu(\tau) + C_-(\sigma)\,\mathbf{Y}_\nu(\tau).$$

where

$$\mathbf{Y}_\nu = \lim_{k\to\nu}\frac{J_k-(-1)^\nu J_{-k}}{k-\nu} = (\partial_k J_k - (-1)^\nu\partial_k J_{-k})_{k=\nu}\,,$$

is a Bessel function of second kind. The Wronskian satisfies [61, §3.63] $W[J_\nu,\mathbf{Y}_\nu](\sigma)=2/\sigma$. Imposing the initial conditions, we derive

$$y = \frac{1}{2}\,(J_\nu(\sigma)\mathbf{Y}_\nu(\tau) - \mathbf{Y}_\nu(\sigma)J_\nu(\tau))\,s\,\sigma^\nu.$$

After replacing $\sigma=s|\xi|$ and $\tau=t|\xi|$, we find

$$\hat{K}(t,s) = \frac{1}{2}\,(J_\nu(s|\xi|)\mathbf{Y}_\nu(t|\xi|) - \mathbf{Y}_\nu(s|\xi|)J_\nu(t|\xi|))\,s^{\nu+1}\,t^{-\nu}.$$

Once again, we study $\hat{K}_a$ where

$$\hat{K}_a = \frac{a^\nu}{2}\,(J_\nu(a|\xi|)\mathbf{Y}_\nu(|\xi|) - \mathbf{Y}_\nu(a|\xi|)J_\nu(|\xi|)).$$

The estimates at high frequencies are analogous to the case of non-integer ν, due to the asymptotic expansion (see [61, §7.21]):

$$\mathbf{Y}_\nu(z) = (z/(2\pi))^{-\frac{1}{2}}\,\sin(z-\nu\pi/2-\pi/4)\,R_{\nu,0}(z) - (z/(2\pi))^{-\frac{1}{2}}\,\cos(z-\nu\pi/2-\pi/4)\,R_{\nu,1}(z).$$

Moreover, as $z\to 0$,

$$\mathbf{Y}_\nu(z) \sim -(\nu-1)!\,(z/2)^{-\nu}, \quad \nu\in\mathbb{N}\setminus\{0\}, \quad \text{but} \quad \mathbf{Y}_0(z)\sim 2\log(z/2),$$

and similarly for their derivative, using $\mathbf{Y}_\nu' = \nu z^{-1}\mathbf{Y}_\nu - \mathbf{Y}_{\nu+1}$.

At low and intermediate frequencies we may still proceed as we did for the case of non-integer ν if $\nu\in\mathbb{N}\setminus\{0\}$. For that reason, we consider in the following only the case $\nu=0$, that is, $\mu=1$. In this case, we shall take into account of the logarithmic term in

$$\hat{K}_a = \frac{1}{2}\,(J_0(a|\xi|)\mathbf{Y}_0(|\xi|) - \mathbf{Y}_0(a|\xi|)J_0(|\xi|)).$$

At low frequencies, cancelations occur, in the sense that

$$\hat{K}_a \sim -\log(a|\xi|/2) + \log(|\xi|/2) = -\log a, \quad \text{as } \xi \to 0. \tag{52}$$

At intermediate frequencies, use that $-\log(a|\xi|) \leq \log 2 - \log a$.

First, let $p \in [2, \infty]$. Then, we estimate

$$\begin{aligned} \|\hat{K}_a\|_{L^{p'}} &\leq C_1 \, \log(e + 1/a) + C_2 \, \log(e + 1/a) \Big(\int_1^{\frac{1}{a}} |\xi|^{-\frac{p'}{2}} \, d\xi \Big)^{\frac{1}{p'}} + C_3 \, a^{-\frac{1}{2}} \Big(\int_{\frac{1}{a}}^{\infty} |\xi|^{-p'} \, d\xi \Big)^{\frac{1}{p'}} \\ &\leq C_1 \, \log(e + 1/a) + \tilde{C}_3 a^{\frac{1}{p} - \frac{1}{2}} + \tilde{C}_2 \, \log(e + 1/a) \times \begin{cases} a^{\frac{1}{p} - \frac{1}{2}} & \text{if } p > 2, \\ (-\log a)^{\frac{1}{2}} & \text{if } p = 2. \end{cases} \end{aligned}$$

The first and the second term are dominated by the latter one in the sum above, so that we conclude

$$\|\hat{K}_a\|_{M_1^p} \leq \begin{cases} C \, a^{\frac{1}{p} - \frac{1}{2}} \, \log(e + 1/a) & \text{if } p > 2, \\ C \, (\log(e + 1/a))^{\frac{3}{2}} & \text{if } p = 2. \end{cases}$$

Now let $p \in (1, 2)$. Taking χ_j as in the case of non-integer ν, we claim that

$$\|\hat{K}_a \chi_j^2\|_{M_1^p} \leq C \, \log(e + 1/a), \quad j = 0, 1, \qquad \|\hat{K}_a \chi_2^2\|_{M_1^p} \leq C \, . \tag{53}$$

At low frequencies, using (52), we may estimate

$$|\partial_\xi^k \hat{K}_a(\xi)| \leq C \, \log(e + 1/a) \, |\xi|^{-k}, \quad k = 0, 1,$$

so that, following as in the proof of Proposition 3.2, we prove (53) for $j = 0$. At intermediate frequencies, we obtain

$$\begin{aligned} &\|J_0(a|\xi|)\chi_1\|_{M_p^p} \leq C, && \|\mathbf{Y}_0(|\xi|)\chi_1\|_{M_1^p} \leq C, \\ &\|\mathbf{Y}_0(a|\xi|)\chi_1\|_{M_p^p} \leq C \, \log(e + 1/a), && \|J_0(|\xi|)\chi_1\|_{M_p^p}\| \leq C, \end{aligned}$$

so that we prove (53) for $j = 1$. At high frequencies, we obtain (53), proceeding as we did for non-integer values of ν. This concludes the proof of (42) for $\mu = 1$.

Recalling that the case $\mu < 1$ may be treated by the change of variable in Remark 3.1, this concludes the proof of Proposition 3.3.

Remark 3.2. We notice that we used the assumption $\mu > 1$, that is, $\nu > 0$, in (51). For negative, non-integer, ν, we should replace (51) by

$$\|\hat{K}_a \chi_1^2\|_{M_1^p} \leq C \, a^{2\nu} = C \, a^{\mu - 1} \, . \tag{54}$$

This modification, eventually, leads to prove Proposition 3.3 for $\mu < 1$, without the use of Remark 3.1.

In view of the estimates obtained in Proposition 3.3, the following straightforward consequence of Proposition 3.2 is of interest to study the semilinear problem (7).

Corollary 3.1. *Let $\mu > 0$ and $p > \max\{1, 1/\mu\}$. Assume that $v_0 \in L^q \cap L^p$, where q is defined as in* (12)*. Then the solution to* (35) *verifies the $L^q - L^p$ estimate*

$$\|v(t, \cdot)\|_{L^p} \leq C \, (\|v_0\|_{L^q} + \|v_0\|_{L^p}) \times \begin{cases} (1 + t)^{-\min\{1, \mu\} + \frac{1}{p}} & \text{if } 2/p > \min\{\mu, 2 - \mu\}, \\ (1 + t)^{-\frac{\mu}{2}} & \text{if } 2/p < \min\{\mu, 2 - \mu\}, \\ (1 + t)^{-\frac{\mu}{2}} & \text{if } \mu \in (0, 1) \text{ and } 2/p = \mu, \end{cases} \tag{55}$$

where $C > 0$ is independent of t and v_0. If $\mu \geq 1$ and $1 - 1/p = \mu/2$, for any small $\varepsilon \in (0, 1 - 1/p)$ there exists $C_\varepsilon > 0$ such that:

$$\|v(t, \cdot)\|_{L^p} \leq C_\varepsilon \, (1 + t)^{\varepsilon - \frac{\mu}{2}} \, (\|v_0\|_{L^1} + \|v_0\|_{L^p}). \tag{56}$$

Proof. If $t \in [0,1]$, then (55) and (56) follow by the (nonsingular) $L^p - L^p$ estimate in (36).

Estimate (55) for $t \geq 1$ follows by (36) with q as in (12). Indeed:

- $q = 1$ if $\mu \geq 1$ and $1 - 1/p < \mu/2$, and the decay rate for the $L^1 - L^p$ estimate in (36) is $t^{-1+\frac{1}{p}}$, as in (55);
- $q = 1/\mu$ if $\mu \in (0,1)$ and $1/p \leq \mu/2$, so that $1/q - 1/p = \mu/2$ and the decay rate for the $L^{\frac{1}{\mu}} - L^p$ estimate in (36) is $t^{-\frac{1}{\mu}+\frac{1}{p}}$, as in (55);
- q is obtained by $1/q - 1/p = \mu/2$ if $2/p < \min\{\mu, 2-\mu\}$, so that (55) follows immediately by (36), since $q > 1$.

On the other hand, estimate (56) for $t \geq 1$ follows by taking $q \in (1,p]$ such that $1 - 1/q = \varepsilon$ in (42), so that $t^{-\frac{1}{q}+\frac{1}{p}} = t^{\varepsilon-\frac{\mu}{2}}$, as in (56).

4. Proofs of Theorems 2.1 and 2.2, and of Corollaries 2.1 and 2.2

To prove Theorems 2.1 and 2.2, we use a contraction argument, exploiting the sharpness of the $L^1 - L^p$ decay estimates derived in Proposition 3.3, in particular the dependence on s in (42), to construct a suitable solution space, in which we may prove the global-in-time existence of small data solutions for $p > p_{\text{crit}}$.

Proof (Proof of Theorem 2.1). For a general $T > 0$, we define

$$X(T) = \{u \in L^\infty([0,T], L^p) : \|u\|_{X(T)} < \infty\},$$

equipped with the norm

$$\|u\|_{X(T)} = \sup_{t\in[0,T]} (g(1+t))^{-1} \|u(t,\cdot)\|_{L^p}, \tag{57}$$

where $g(1+t)$ is as in (15), for a sufficiently small $\delta > 0$ which we will fix later. Thanks to Corollary 3.1, there exists $C > 0$, independent of T, such that the solution to the linear singular problem (35) with $v_0 = u_0$ verifies the estimate

$$\|v\|_{X(T)} \leq C\left(\|u_0\|_{L^q} + \|u_0\|_{L^p}\right). \tag{58}$$

We want to prove that there exists a constant $C > 0$, independent of $T > 0$, such that the operator

$$F : X(T) \to X(T), \quad Fu(t,x) = \int_0^t K(t,s) * f(u(s,x))\, ds,$$

where $K = K(t,s)$ is the fundamental solution to (11), verifies the contractive estimate

$$\|Fu - Fw\|_{X(T)} \leq C\, \|u - w\|_{X(T)} (\|u\|_{X(T)}^{p-1} + \|w\|_{X(T)}^{p-1}). \tag{59}$$

Properties (58) and (59), imply that there exists $\varepsilon > 0$ such that if u_0 verifies (14), then there is a unique global-in-time solution to (7), verifying

$$\|u\|_{X(T)} \leq C\left(\|u_0\|_{L^1} + \|u_0\|_{L^p}\right),$$

for any $T > 0$, with $C > 0$, independent of T.

Indeed, let $R > 0$ be such that $CR^{p-1} < 1/2$. Then F is a contraction on $X_R(T) = \{u \in X(T) : \|u\|_{X(T)} \leq R\}$. The solution to (7) is a fixed point for $v(t,x) + Fu(t,x)$, so if $\|v\|_{X(T)} \leq R/2$, then $u \in X_R(T)$ and the uniqueness and existence of the solution in $X_R(T)$ follows by the Banach fixed point theorem on contractions. The condition $\|v\|_{X(T)} \leq R/2$ is obtained taking initial data as in (14), with $C\varepsilon \leq R/2$. Since C, R and ε do not depend on T, the solution is global-in-time.

We now prove the contractive estimate (59) for $u, w \in X(T)$. Using (2) and Hölder inequality, due to the fact that $u, w \in X(T)$, we may estimate

$$\|(f(u)-f(w))(s,\cdot)\|_{L^1} \leq C\, \|(u-w)(s,\cdot)\|_{L^p} \left(\|u(s,\cdot)\|_{L^p}^{p-1} + \|w(s,\cdot)\|_{L^p}^{p-1}\right) \leq C\, (g(1+s))^p\, \|u-w\|_{X(T)} (\|u\|_{X(T)}^{p-1} + \|w\|_{X(T)}^{p-1}). \tag{60}$$

Then, using (42) and (60) we obtain

$$\|(Fu-Fw)(t,\cdot)\|_{L^p} \leq C\, t^{-\min\{1,\mu,\frac{\mu}{2}+\frac{1}{p}\}+\frac{1}{p}}\, \mathrm{d}_1(t)\, \mathrm{d}_2(t)\, I(t)\, \|u-w\|_{X(T)} (\|u\|_{X(T)}^{p-1} + \|w\|_{X(T)}^{p-1}), \tag{61}$$

where

$$I(t) = \int_0^t s^{\min\left\{1,\mu,\frac{\mu}{2}+\frac{1}{p}\right\}+\alpha} (g(1+s))^p\, ds. \tag{62}$$

In order to prove (59) for $t \leq 1$ we use the left-hand side of (13) and $p > \max\{1, 1/\mu\}$ (see Remark 2.3) to estimate

$$\alpha \geq -1 - \frac{1}{p} > -1 - \mu.$$

Using $g(1+s) \leq 1$, and using again $\alpha \geq -1 - 1/p$, we find

$$I(t) \leq C\, t^{\min\left\{1,\mu,\frac{\mu}{2}+\frac{1}{p}\right\}+\alpha+1} \leq C\, t^{\min\left\{1,\mu,\frac{\mu}{2}+\frac{1}{p}\right\}-\frac{1}{p}}\,.$$

This concludes the proof of (59) for $t \leq 1$.

In order to prove (59) for $t \geq 1$, it is sufficient to show that $I(t)$ is uniformly bounded, with respect to t, i.e., that $I(\infty)$ is a convergent integral. As before, the convergence of the integral as $s \to 0$, is a consequence of $\alpha \geq -1 - 1/p$ and $p > \max\{1, 1/\mu\}$. Recalling the definition of g in (15), we find that the integral is convergent at infinity if, and only if,

$$\min\left\{1,\mu,\frac{\mu}{2}+\frac{1}{p}\right\} + \alpha - p\left(\min\left\{1,\mu,\frac{\mu}{2}+\frac{1}{p}\right\} - \frac{1}{p}\right) < -1, \tag{63}$$

provided that we take a sufficiently small δ in (15), if $p \in [2,\infty)$ and $\mu = 2 - 2/p$.

Condition (63) is equivalent to (10) and $p > p_{\mathrm{crit}}$ (see Remark 2.1). Therefore, we proved (59), and this concludes the proof.

The proof of Theorem 2.2 is simpler than the proof of Theorem 2.1. On the one hand, for both v and $Fu - Fw$ we may rely on the same estimates provided by Proposition 3.3. On the other hand, since the problem is not singular, due to $t_0 > 0$, we do not need to discuss the short time estimates to avoid possible singular behaviors.

Proof (Proof of Theorem 2.2). We follow the proof of Theorem 2.1 with the following modifications. The space

$$X(T) = \{u \in L^\infty([t_0,T], L^p) :\ \|u\|_{X(T)} < \infty\},$$

equipped with norm

$$\|u\|_{X(T)} = \sup_{t\in[t_0,T]} (g(t))^{-1}\, \|u(t,\cdot)\|_{L^p},$$

is defined for a general $T > t_0$, with $g(t)$ given by (17). Thanks to Proposition 3.3, there exists $C = C(t_0) > 0$, independent of T, such that the solution to the linear regular problem (11) with $s = t_0$ and $v_1 = u_1$ verifies the estimate

$$\|v\|_{X(T)} \leq C\, \|u_1\|_{L^1}\,. \tag{64}$$

We want to prove that the operator F verifies the contractive estimate (59). As in the proof of Theorem 2.1, properties (64) and (59) imply that there exists $\varepsilon > 0$ such that if u_1 verifies (16), then there is a unique global-in-time solution to (8), verifying $\|u\|_{X(T)} \leq C\, \|u_1\|_{L^1}$, for any $T > t_0$, with $C = C(t_0) > 0$, independent of T.

To prove the contractive estimate (59) for $u, w \in X(T)$, we proceed as in the proof of Theorem 2.1, but due to $t \geq t_0 > 0$ we may avoid to discuss the behavior at short times. Moreover, we may remove the restriction $\alpha \geq -1 - 1/p$, which was used to avoid a nonintegrable singularity at $t = 0$. To prove (59) it is sufficient to show that

$$\int_{t_0}^\infty s^{\min\left\{1,\mu,\frac{\mu}{2}+\frac{1}{p}\right\}+\alpha} (g(s))^p\, ds \leq C(t_0),$$

and, recalling the definition of g in (17), this estimate is verified if, and only if, $p > p_{\mathrm{crit}}$ when $\alpha > -2$, whereas it holds for any $p > 1$ if $\alpha \leq -2$. This concludes the proof of Theorem 2.2.

Proof (Proof of Corollaries 2.1 and 2.2). The proof is a straightforward application of Theorems 2.1 and 2.2, with μ and α as in (21). The decay rate $g_*(1+t)$ in (25) is obtained by (15), using

$$\|w(t,\cdot)\|_{L^p} = \|u(\Lambda(t),\cdot)\|_{L^p} \leq C\, g(1+\Lambda(t))\,(\|u_0\|_{L^q} + \|u_0\|_{L^p}),$$

and replacing $u_0 = w_0$,

$$\min\{\mu, 2-\mu\} = \frac{\ell + \min\{\mu_*, 2-\mu_*\}}{\ell+1},$$

and

$$\Lambda(t)^{-\frac{\mu}{2}} = c_1\, t^{-\frac{\ell+\mu_*}{2}}, \quad \Lambda(t)^{-\min\{1,\mu\}+\frac{1}{p}} = c_2\, t^{-\ell-\min\{1,\mu_*\}+\frac{\ell+1}{p}}.$$

Similarly, the decay rate $g_*(t)$ in (29) is obtained by (17).

5. Concluding remarks and open problems

In this section we collect some open problems and we add some concluding remarks.

In a forthcoming paper, we will study the semilinear multidimensional E. P. D. equation. Indeed, the technique employed in Proposition 3.3 to study the linear regular problem (11) is not directly applicable to the multidimensional Cauchy problem (5), in general. A complete global existence result in space dimension $n \geq 2$, for small values of μ is still an open problem.

Also, a complete knowledge of blow-up results for the semilinear E. P. D. equation considered in this paper is lacking so far.

Open problem 1. Theorem 1.1 in [6] implies that there is no global-in-time weak solution to both the singular problem (7) and the regular problem (8), if $1 < p \leq 3+\alpha$, under suitable data sign assumption. If $\mu \in (0,1)$, thanks to the change of variable in Remark 3.1, the same theorem implies the nonexistence of global-in-time weak solutions to the regular problem (8), if $1 < p \leq 1 + (2+\alpha)/\mu$. We expect that this nonexistence result remains valid for the singular problem, as well. Moreover, we expect that it is possible to prove the nonexistence of global-in-time solutions to both the singular and the regular problem (8) for $1 < p \leq p_{\mathrm{Str}}(1+\mu,\alpha)$, possibly extending the result in [26] which holds for the regular problem (8) when $\alpha = 0$.

In (8), we assumed the initial condition $u(t_0,x) = 0$, for brevity. If we replace this condition by $u(t_0,x) = u_0(x)$, for some nontrivial u_0, then we may replace (16) in Theorem 2.2 by

$$u_0 \in L^1 \cap L^p, \quad u_1 \in L^1, \quad \text{with } \|u_0\|_{L^1} + \|u_0\|_{L^p} + \|u_1\|_{L^1} \leq \varepsilon. \tag{65}$$

Indeed, following as in the proof of Proposition 3.3, the solution to

$$\begin{cases} \tau^2 y'' + \tau y' + (\tau^2 - \nu^2) y = 0\,, & \tau \geq \sigma\,, \\ y(\sigma) = \sigma^\nu, \quad y'(\sigma) = 0, \end{cases} \tag{66}$$

when $\nu > 0$ is not an integer, is

$$y = -\frac{\pi}{2\sin(\nu\pi)}\,\big(J'_{-\nu}(\sigma) J_\nu(\tau) - J'_\nu(\sigma) J_{-\nu}(\tau)\big)\,\sigma^{\nu+1}\,,$$

so that, replacing $w(\tau) = \tau^{-\nu} y(\tau)$, $\sigma = s|\xi|$ and $\tau = t|\xi|$, we find

$$w = -\frac{\pi}{2\sin(\nu\pi)}\,\big(J'_{-\nu}(s|\xi|) J_\nu(t|\xi|) - J'_\nu(s|\xi|) J_{-\nu}(t|\xi|)\big)\, s^{\nu+1}\, t^{-\nu}\, |\xi|.$$

In particular, the contribution from $|\xi|$ in the expression above, together with the asymptotic behavior (33), motivates the assumption $u_0 \in L^p$ to obtain the $L^p - L^p$ high frequencies estimate. For the sake of brevity, we omit the details of the proof.

As a final remark, we provide some details about global existence of small data solutions for (30). The equation in (30) appears in a general formulation which includes the E. P. D. equation, the Tricomi generalized equation, the wave equation with scale-invariant damping and mass. We stress that we cannot consider the singular Cauchy problem corresponding to $t_1 = 0$ for this equation with our approach, since the coefficients of both u_t and u in the equation in (30) are singular at $t = 0$.

Taking into account of the expression $m = -\beta(\mu_\circ + \beta - 1)$, we shall assume $m \leq (\mu_\circ - 1)^2/4$ in (30), so that we may fix $\beta = (1 - \mu_\circ \pm \delta)/2$, where

$$\delta = \sqrt{(\mu_\circ - 1)^2 - 4m}\,. \tag{67}$$

On the other hand, $\mu_* = \mu_\circ + 2\beta = 1 \pm \delta$ and $\alpha_* = \alpha_\circ + \beta(p-1)$. We now consider the condition $p > p_{\mathrm{crit}}$, with p_{crit} as in (26), which is equivalent to the right-hand side of (22). Replacing the expressions for μ_* and α_*, we find

$$(p-1)\min\left\{\ell + \frac{\mu_\circ + 1 - \delta}{2},\ \frac{\ell + \mu_\circ}{2} + \frac{\ell + 1}{p}\right\} > \alpha_\circ + 2\,. \tag{68}$$

Therefore, as a consequence of Corollary 2.2, we may prove the following result for (30).

Corollary 5.1. *Let $\ell > -1$, $\mu_\circ \in \mathbb{R}$, $m \leq (\mu_\circ - 1)^2/4$, $\alpha_\circ \in \mathbb{R}$, and assume that p satisfies* (68)*, where δ is as in* (67)*. Then there exists $\varepsilon > 0$ such that for any initial data*

$$v_1 \in L^1, \quad \text{with } \|v_1\|_{L^1} \leq \varepsilon, \tag{69}$$

there exists a unique global-in-time weak solution $v \in L^\infty_{\mathrm{loc}}([t_1, \infty), L^p)$, to (30)*. Moreover, the solution to* (30) *satisfies the estimate*

$$\|v(t,\cdot)\|_{L^p} \leq C\, g_\circ(t)\, \|v_1\|_{L^1}, \quad \text{with} \quad g_\circ(t) = t^{-\frac{\mu_\circ}{2} - \min\left\{\ell + \frac{1-\delta}{2}, \frac{\ell}{2} + \frac{\ell+1}{p}\right\} + \frac{\ell+1}{p}}\, \mathrm{d}_1(t)\, \mathrm{d}_2(t), \tag{70}$$

where $C > 0$, is independent of t, and of the initial data, and $\mathrm{d}_1(t)$ and $\mathrm{d}_2(t)$ are logarithmic loss terms determined as follows: either $\mathrm{d}_1 = 1$ if $\delta \neq 0$ or $\mathrm{d}_1 = 1 + \log(1+t)$ if $\delta = 0$; either $\mathrm{d}_2(t) = 1$ if $2/p \neq \frac{\ell - 1 - \delta}{\ell + 1}$, or $\mathrm{d}_2(t) = 1 + (\log(1+t))^{1-\frac{\mu}{2}}$ if $\frac{2}{p} = \frac{\ell - 1 - \delta}{\ell + 1}$.

Proof. The proof follows by applying Corollary 2.2 with $\mu_* = 1 + \delta$ and $\alpha_* = \alpha_\circ + \beta(p-1)$, where $\beta = (1 - \mu_\circ + \delta)/2$. We stress that the condition $\mu_* > -\ell$ in Corollary 2.2 is satisfied, due to $\mu_* \geq 1$ and $\ell > -1$.

Replacing $v(t,x) = t^\beta\, w(t,x)$, we may compute

$$\|v(t,\cdot)\|_{L^p} = t^\beta\, \|w(t,\cdot)\|_{L^p} \leq C\, t^\beta\, g_*(t)\, \|v_1\|_{L^1} = t^{\frac{1-\mu_\circ+\delta}{2} - \min\left\{\ell+1, \frac{\ell+1+\delta}{2} + \frac{\ell+1}{p}\right\} + \frac{\ell+1}{p}}\, \mathrm{d}_1(t)\, \mathrm{d}_2(t)\, \|v_1\|_{L^1}\,,$$

and this concludes the proof.

Let $\ell = 0$, $\mu_\circ > 0$, and $\alpha_\circ > -2$. Assuming $\mu_\circ + 1 - \delta > 0$, that is, $-\mu_\circ < m \leq (\mu_\circ - 1)^2/4$, we find that (68) is equivalent to $p > p_{\mathrm{crit}}$, where

$$p_{\mathrm{crit}} = \max\left\{1 + \frac{2(2 + \alpha_\circ)}{\mu_\circ + 1 - \delta},\ p_{\mathrm{Str}}(1 + \mu_\circ, \alpha_\circ)\right\}.$$

We stress that $p_{\mathrm{Str}}(1 + \mu_\circ, \alpha_\circ)$ in the expression above is the same modified shifted Strauss exponent appearing in (9). That is, the role played by the mass term m in the quantity δ in (67) only influences the contribution to the critical exponent coming from the Fujita-type exponent $1 + 2(2 + \alpha_\circ)/(\mu_\circ + 1 - \delta)$.

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