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Semiclassical analysis for pseudo-relativistic Hartree equations

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Abstract

In this paper we study the semiclassical limit for the pseudo-relativistic Hartree equation

$$\sqrt{-\varepsilon^2\Delta + m^2}u + Vu = (I_\alpha * |u|^p)\,|u|^{p-2}u, \quad \text{in } \mathbb{R}^N,$$

where $m > 0$, $2 \leq p < \frac{2N}{N-1}$, $V : \mathbb{R}^N \to \mathbb{R}$ is an external scalar potential, $I_\alpha(x) = \frac{c_{N,\alpha}}{|x|^{N-\alpha}}$ is a convolution kernel, $c_{N,\alpha}$ is a positive constant and $(N-1)p - N < \alpha < N$. For $N = 3$, $\alpha = p = 2$, our equation becomes the pseudo-relativistic Hartree equation with Coulomb kernel.

Keywords: Pseudo-relativistic Hartree equations, semiclassical limit

1. Introduction

In this paper we study the semiclassical limit ($\varepsilon \to 0^+$) for the pseudo-relativistic Hartree equation

$$i\varepsilon\frac{\partial\psi}{\partial t} = \left(\sqrt{-\varepsilon^2\Delta + m^2} - m\right)\psi + V\psi - \left(\frac{1}{|x|} * |\psi|^2\right)\psi, \quad x \in \mathbb{R}^3 \qquad (1)$$

where $\psi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$ is the wave field, $m > 0$ is a physical constant, ε is the semiclassical parameter $0 < \varepsilon \ll 1$, a dimensionless scaled Planck constant (all other physical constant are rescaled to be 1), V is bounded external potential in $\mathbb{R}^3$. Here the pseudo-differential operator $\sqrt{-\varepsilon^2\Delta + m^2}$ is simply defined in Fourier variables by the symbol $\sqrt{\varepsilon^2|\xi|^2 + m^2}$ (see [23]).

Equation (1) has interesting applications in the quantum theory for large systems of self-interacting, relativistic bosons with mass $m > 0$. As recently shown by Elgart and Schlein [16], equation (1) emerges as the correct evolution equation for the mean-field dynamics of many-body quantum systems modelling

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pseudo-relativistic boson stars in astrophysics. The external potential, $V = V(x)$, accounts for gravitational fields from other stars. In what follows, we will assume that V is a smooth, bounded function (see [24, 19, 17, 18, 21, 28]). The pseudo-relativistic Hartree equation can be also derived coupling together a pseudo-relativistic Schrödinger equation with a Poisson equation (see for instance [1, 32]), i.e.

$$\begin{cases} i\varepsilon\frac{\partial\psi}{\partial t} = \left(\sqrt{-\varepsilon^2\Delta + m^2} - m\right)\psi + V\psi - U\psi, \\ -\Delta U = |\psi|^2 \end{cases}$$

See also [14, 20] for recent developments for models involving pseudo-relativistic Bose gases.

Solitary wave solutions $\psi(t,x) = e^{it\lambda/\varepsilon}u(x)$, $\lambda > 0$ to equation (1) lead to solve the non local single equation

$$\sqrt{-\varepsilon^2\Delta + m^2}u + Vu = \left(\frac{1}{|x|} * |u|^2\right)u, \quad \text{in } \mathbb{R}^3 \tag{2}$$

where for simplicity we write V instead of $V + (\lambda - m)$.

More generally, in this paper we will study the generalized pseudo-relativistic Hartree equation

$$\sqrt{-\varepsilon^2\Delta + m^2}u + Vu = (I_\alpha * |u|^p)\,|u|^{p-2}u, \quad \text{in } \mathbb{R}^N, \tag{3}$$

where $m > 0$, $2 \le p < \frac{2N}{N-1}$, $V\colon \mathbb{R}^N \to \mathbb{R}$ is an external scalar potential,

$$I_\alpha(x) = \frac{c_{N,\alpha}}{|x|^{N-\alpha}} \quad (x \ne 0), \quad \alpha \in (0, N)$$

is a convolution kernel and $c_{N,\alpha}$ is a positive constant; for our purposes we can choose $c_{N,\alpha} = 1$. For $N = 3$, $\alpha = p = 2$, equation (3) becomes the pseudo-relativistic Hartree equation (2) with Coulomb kernel.

We refer to [34, 9, 6, 30] for the semiclassical analysis of the non-relativistic Hartree equation. The study of the pseudo-relativistic Hartree equation (2) without external potential V starts in the pioneering paper [24] where Lieb and Yau, by minimization on the sphere $\{\phi \in L^2(\mathbb{R}^3) \mid \int_{\mathbb{R}^3} |\phi|^2 = M\}$, proved that a radially symmetric ground state exists in $H^{1/2}(\mathbb{R}^3)$ whenever $M < M_c$, the so-called Chandrasekhar mass. Later Lenzmann proved in [22] that this ground state is unique (up to translations and phase change) provided that the mass M is sufficiently small; some results about the non-degeneracy of the ground state solution are also given.

Successively, in [10] Coti-Zelati and Nolasco proved existence of a positive radially symmetric ground state solution for a pseudo-relativistic Hartree equation without external potential V, involving a more general radially symmetric convolution kernel. See the recent paper [11] dealing existence of ground states with given fixed "mass-charge".

In [27] Melgaard and Zongo established that (2) has a sequence of radially symmetric solutions of higher and higher energy, assuming that V is radially symmetric potential.

The requirement that V has radial symmetry was dropped in the recent paper [8], where a positive ground state solution for the pseudo-relativistic Hartree equation (3) is constructed under the assumption $(N-1)p - N < \alpha < N$.

To the best of our knowledge the study of the semiclassical limit for the pseudo-relativistic Hartree equation has been considered by Aki, Markowich and Sparber in [1]. Using Wigner trasformation techniques, they showed that its semiclassical limit yields the well known relativistic Vlasov-Poisson system.

In the present paper we are interested to study the pseudo-relativistic Hartree equation in the semiclassical limit regime ($0 < \varepsilon \ll 1$), using variational methods. Replacing $u(y)$ by $\varepsilon^{\frac{\alpha}{2(1-p)}} u(\varepsilon y)$, equation (3) becomes equivalent to following Hartree equation

$$\sqrt{-\Delta + m^2} u + V_\varepsilon(y) u = \left(I_\alpha * |u|^p\right) |u|^{p-2} u, \quad \text{in } \mathbb{R}^N. \tag{4}$$

where $V_\varepsilon(y) = V(\varepsilon y)$. In what follows we will assume that

(V) $V \colon \mathbb{R}^N \to \mathbb{R}$ is a continuous and bounded function such that $V_{\min} = \inf_{\mathbb{R}^N} V > -m$ and there exists a bounded open set $O \subset R^N$ with the property that

$$V_0 = \inf_O V < \min_{\partial O} V.$$

Let us define

$$\mathscr{M} = \left\{ y \in O \mid V(y) = V_0 \right\}.$$

We will establish the existence of a single-spike solution concentrating around a point close to $\mathscr{M}$. Precisely, our main result is the following.

Theorem 1.1. *Retain assumption (**V**) and assume that $2 \le p < 2N/(N-1)$ and $(N-1)p - N < \alpha < N$. Then, for every sufficiently small $\varepsilon > 0$, there exists a solution $u_\varepsilon \in H^{1/2}(\mathbb{R}^N)$ of equation (4) such that u_ε has a local maximum point y_ε satisfying*

$$\lim_{\varepsilon \to 0} \operatorname{dist}(\varepsilon y_\varepsilon, \mathscr{M}) = 0,$$

and for which

$$u_\varepsilon(y) \le C_1 \exp\left(-C_2 |y - y_\varepsilon|\right)$$

for suitable constants $C_1 > 0$ and $C_2 > 0$. Moreover, for any sequence $\{\varepsilon_n\}_n$ with $\varepsilon_n \to 0$, there exists a subsequence, still denoted by the same symbol, such that there exist a point $y_0 \in \mathscr{M}$ with $\varepsilon_n y_{\varepsilon_n} \to y_0$, and a positive least-energy solution $U \in H^{1/2}(\mathbb{R}^N)$ of the equation

$$\sqrt{-\Delta + m^2}\, U + V_0 U = \left(I_\alpha * U^p\right) U^{p-1}$$

for which we have

$$u_{\varepsilon_n}(y) = U\left(y - y_{\varepsilon_n}\right) + \mathscr{R}_n(y) \tag{5}$$

where $\lim_{n \to +\infty} \|\mathscr{R}_n\|_{H^{1/2}} = 0$.

To prove the main result, we replace the *nonlocal* problem (3) in $\mathbb{R}^N$ with a local Neumann problem in the half space $\mathbb{R}^{N+1}_+$ as in [10] (see [4]). We will find critical points of the Euler functional associated to the local Neumann problem by means of a variational approach introduced in [2, 3] (see also [7]) for nonlinear Schrödinger equations and extended in [9] to deal with non-relativistic Hartree equations.

In the present paper the presence of a pseudo-differential operator combined with a nonlocal term requires new ideas. As a first step, we need to perform a deep analysis of the local realization of the following limiting problem

$$\sqrt{-\Delta + m^2}u + au = (I_\alpha * |u|^p)\, |u|^{p-2}u \tag{6}$$

with $a > -m$. This equation does not have a unique (up to translation) positive, ground state solution, apart from the case $p = 2$, $N = 3$. Nevertheless we can prove that the set of positive, ground state solutions to the local realization of equation (6) satisfies some compactness properties. This is the crucial tool for finding single-peak solutions which are close to a set of prescribed functions. Even if we use a purely variational approach, we will take into account the shape and the location of the expected solutions as in the reduction methods.

Recently the existence of a spike-pattern solution for fractional nonlinear Schrödinger equation has been proved by Davila, del Pino and Wei in the semiclassical limit regime (see [15]). The authors perform a refined Lyapunov-Schmidt reduction, taking into advantage the fact that the limiting fractional problem has an unique, positive, radial, ground state solution, which is nondegenerate.

Notation

- We will use $|\cdot|_q$ for the norm in L^q, and $\|\cdot\|$ for the norm in $H^1(\mathbb{R}^{N+1}_+)$.
- Generic positive constants will be denoted by the (same) letter C.
- The symbol $\mathbb{R}^{N+1}_+$ denotes the half-space $\{(x, y) \mid x > 0,\ y \in \mathbb{R}^N\}$. We will identify the boundary $\partial\mathbb{R}^{N+1}_+$ with $\mathbb{R}^N$.
- The symbol $*$ will denote the convolution of two functions.
- For any subset A of $\mathbb{R}^N$ and any $\varrho > 0$, we set $A^\varrho = \{y \mid \mathrm{dist}(y, A) \le \varrho\}$.
- For any subset A of $\mathbb{R}^N$ and any $\varrho > 0$, we set $A_\varrho = \{y \mid \varrho y \in A\}$.

2. Preliminaries and variational setting

The realization of the operator $\sqrt{m^2 - \varepsilon^2\Delta}$ in Fourier variables seems not convenient for our purposes. Therefore, we prefer to make use of a *local* realization (see [10, 4]) by means of the *Dirichlet-to-Neumann* operator defined as follows.

For any $\varepsilon > 0$, given $u \in \mathcal{S}(\mathbb{R}^N)$, the Schwartz space of rapidly decaying smooth functions defined on $\mathbb{R}^N$, there exists one and only one function $v \in \mathcal{S}(\mathbb{R}^{N+1}_+)$ such that

$$\begin{cases} -\varepsilon^2 \Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ v(0,y) = u(y) & \text{for } y \in \mathbb{R}^N = \partial\mathbb{R}^{N+1}_+. \end{cases}$$

Setting

$$T_\varepsilon u(y) = -\varepsilon \frac{\partial v}{\partial x}(0,y),$$

we easily see that the problem

$$\begin{cases} -\varepsilon^2 \Delta w + m^2 w = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ w(0,y) = T_\varepsilon u(y) & \text{for } y \in \partial\mathbb{R}^{N+1}_+ = \{0\} \times \mathbb{R}^N \simeq \mathbb{R}^N \end{cases}$$

is solved by $w(x,y) = -\varepsilon \frac{\partial v}{\partial x}(x,y)$. From this we deduce that

$$T_\varepsilon(T_\varepsilon u)(y) = -\varepsilon \frac{\partial w}{\partial x}(0,y) = \varepsilon^2 \frac{\partial^2 v}{\partial x^2}(0,y) = \left(-\varepsilon^2 \Delta_y v + m^2 v\right)(0,y),$$

and hence $T_\varepsilon \circ T_\varepsilon = (-\varepsilon^2 \Delta_y + m^2)$, namely T_ε is a square root of the Schrödinger operator $-\varepsilon^2 \Delta_y + m^2$ on $\mathbb{R}^N = \partial\mathbb{R}^{N+1}_+$.

From the previous construction, we can replace the *nonlocal* problem (3) in $\mathbb{R}^N$ with the local Neumann problem in the half space $\mathbb{R}^{N+1}_+$

$$\begin{cases} -\varepsilon^2 \Delta v(x,y) + m^2 v(x,y) = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ -\varepsilon \frac{\partial v}{\partial x}(0,y) = -V(y)v(0,y) + \left(I_\alpha * |v(0,\cdot)|^p\right)|v(0,y)|^{p-2}v(0,y) & \text{for } y \in \mathbb{R}^N. \end{cases}$$

Setting $v_\varepsilon(x,y) = \varepsilon^{\frac{\alpha}{2(1-p)}} v(\varepsilon x, \varepsilon y)$ and $V_\varepsilon(y) = V(\varepsilon y)$, we are led to the *local* boundary-value problem

$$\begin{cases} -\Delta v_\varepsilon + m^2 v_\varepsilon = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ -\frac{\partial v_\varepsilon}{\partial x}(0,y) = -V_\varepsilon(y)v_\varepsilon(0,y) + \left(I_\alpha * |v_\varepsilon(0,\cdot)|^p\right)|v_\varepsilon(0,y)|^{p-2}v_\varepsilon(0,y) & \text{for } y \in \mathbb{R}^N. \end{cases}$$

We introduce the Sobolev space $H = H^1(\mathbb{R}^{N+1}_+)$, and recall that there is a continuous *trace operator* $\gamma \colon H \to H^{1/2}(\mathbb{R}^N)$. Moreover, this operator is surjective and the inequality

$$|\gamma(v)|_p^p \leq p|v|_{2(p-1)}^{p-1} \left|\frac{\partial v}{\partial x}\right|_2$$

holds for every $v \in H^1(\mathbb{R}^{N+1}_+)$: we refer to [33] for basic facts about the Sobolev space $H^{1/2}(\mathbb{R}^N)$ and the properties of the trace operator.

Reasoning as in [8, Page 5] and taking the Hardy-Littlewood-Sobolev inequality (see [25, Theorem 4.3]) into consideration, it follows easily that the functional $\mathscr{E}_\varepsilon \colon H \to \mathbb{R}$ defined by

$$\begin{aligned} \mathscr{E}_\varepsilon(v) = {} & \frac{1}{2}\int_{\mathbb{R}^{N+1}_+} |\nabla v|^2 \, dx \, dy + \frac{m^2}{2}\int_{\mathbb{R}^{N+1}_+} v^2 \, dx \, dy \\ & + \frac{1}{2}\int_{\mathbb{R}^N} V_\varepsilon(x)\gamma(v)^2 \, dy - \frac{1}{2p}\int_{\mathbb{R}^N} \left(I_\alpha * |\gamma(v)|^p\right)|\gamma(v)|^p \, dy \end{aligned}$$

is of class C^1, and its critical points are (weak) solutions to problem (4).

3. Compactness properties for the limiting problem

For $a > -m$, the equation

$$\sqrt{-\Delta + m^2}u + au = (I_\alpha * |u|^p)\,|u|^{p-2}u \tag{7}$$

plays the rôle of a limiting problem for (4). Its Euler functional $L_a \colon H \to \mathbb{R}$ is defined (via the local realization of Section 2) by

$$L_a(v) = \frac{1}{2}\int_{\mathbb{R}^{N+1}_+} \left(|\nabla v|^2 + m^2|v|^2\right)\,dx\,dy \\ + \frac{a}{2}\int_{\mathbb{R}^N} |\gamma(v)|^2\,dy - \frac{1}{2p}\int_{\mathbb{R}^N} \left(I_\alpha * |\gamma(v)|^p\right)|\gamma(v)|^p\,dy.$$

We define the ground-state level

$$m_a = \inf\left\{L_a(v) \mid L_a'(v) = 0, v \in H \setminus \{0\}\right\}$$

and the set S_a of elements $v \in H \setminus \{0\}$ such that $v > 0$, $L_a(v) = m_a$, and for every $x \geq 0$:

$$\max_{y \in \mathbb{R}^N} v(x,y) = v(x,0). \tag{8}$$

Proposition 3.1. *The set S_a is non-empty for any $a > -m$.*

PROOF. The proof is indeed standard, and we will be sketchy. First of all, we invoke [11, Lemma 2.1] to deduce that ground states of L_a correspond to ground states of the functional $\mathcal{L}_a \colon H^{1/2}(\mathbb{R}^N) \to \mathbb{R}$ defined as

$$\mathcal{L}_a(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left(\left|\sqrt{(m^2-\Delta)^{1/2} - m}u\right|^2 + (a+m)|u|^2\right) \\ - \frac{1}{2p}\int_{\mathbb{R}^N} \left(I_\alpha * |u|^p\right)|u|^p. \tag{9}$$

We claim that $\mathcal{L}_a$ possesses a ground state. We fix $a > -m$ and consider the minimization problem associated to (9)

$$\widetilde{m}_a = \inf_{u \in H^{1/2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\sqrt{(m^2-\Delta)^{1/2} - m}u|^2 + (a+m)\,|u|^2}{\left(\int_{\mathbb{R}^N} \left(I_\alpha * |u|^p\right)|u|^p\right)^{\frac{1}{p}}}. \tag{10}$$

Since $\sqrt{m^2 - \Delta} - m > 0$ in the sense of functional calculus and $a + m > 0$, it follows easily that $\widetilde{m}_a > 0$. As in [29, Proof of Proposition 2.2] we can show that $\widetilde{m}_a$ is attained. Since the quotient in (10) is homogeneous of degree zero, as in the local case we see that any minimizer of $\widetilde{m}_a$ is, up to a rescaling and a translation, a ground state for (9). Therefore the claim is proved, and in particular $S_a \neq \emptyset$. It is easy to check that ground states are non-negative, and, as in [10, Theorem 5.1], actually strictly positive.

Remark 3.2. *By [24, Formula (A.3)], the quotient to be minimized in (10) decreases under polarization. This implies, reasoning as in [29, Section 5] (see also [13]) that ground states are radially symmetric around a point of* $\mathbb{R}^N$.

For $U \in S_a$, we write $E_a = L_a(U)$. By an immediate extension of [31, Lemma 3.17], the map $a \mapsto E_a$ is strictly increasing and continuous. The following is the main result of this section.

Proposition 3.3. *The set* S_a *is compact in* H*, and for some* $C > 0$ *and any* $\sigma \in (-V_{\min}, m) \cap [0, +\infty)$ *we have*

$$v(x,y) \leq Ce^{-(m-\sigma)\sqrt{x^2+|y|^2}} e^{-\sigma x} \tag{11}$$

for every $v \in S_a$.

Proof. If $v \in S_a$, it follows easily from [10, Theorem 5.1] or [8, Theorem 7.1] that v decays exponentially fast at infinity and (11) holds. Moreover, since

$$m_a = L_a(v) = \left(\frac{1}{2} - \frac{1}{2p}\right)\left(|\nabla v|_2^2 + m^2|v|_2^2\right),$$

S_a is bounded in H. We claim that S_a is also bounded in $L^\infty(\mathbb{R}^{N+1}_+)$.

Indeed, by [10, Theorem 3.2] it follows that $\gamma(v) \in L^q(\mathbb{R}^N)$ for any $q \in [2, \infty]$, then also $g(\cdot) = -a\gamma(v) + (I_\alpha * |\gamma(v)|^p)\, |\gamma(v)|^{p-2}\gamma(v) \in L^q(\mathbb{R}^N)$ for $q \in [2, \infty]$. Following [5], we let $u(x,y) = \int_0^x v(t,y)\, dt$. It follows that $u \in H^1((0,R) \times \mathbb{R}^N)$ for all $R > 0$. Arguing as in [10, Proposition 3.9], we can deduce that u is a weak solution of the Dirichlet problem

$$\begin{cases} -\Delta u + m^2 u = g & \text{in } \mathbb{R}^{N+1}_+ \\ u = 0 & \text{for } y \in \mathbb{R}^N. \end{cases} \tag{12}$$

where $g(x,y) = g(y)$ for every $x > 0$ and $y \in \mathbb{R}^N$. We sketch the proof for the sake of completeness. Pick an arbitrary function $\eta \in C_0^\infty(\mathbb{R}^{N+1}_+)$ and write $\omega_t(x,y) = \eta(x+t,y)$ for any $t \geq 0$. Then

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}^N} \nabla v(x,y)\cdot\nabla\eta(x+t,y)\, dy\, dx\, dt \\
&\qquad = \int_0^{+\infty}\int_x^{+\infty}\int_{\mathbb{R}^N} \nabla v(x,y)\cdot\nabla\eta(s,y)\, dy\, ds\, dx \\
&\qquad = \int_0^{+\infty}\int_0^{s}\int_{\mathbb{R}^N} \nabla v(x,y)\cdot\nabla\eta(s,y)\, dy\, dx\, ds \\
&\qquad = \int_0^{+\infty}\int_{\mathbb{R}^N} \nabla\left(\int_0^s v(x,y)\, dx\right)\cdot\nabla\eta(s,y)\, dy\, ds
\end{aligned}$$

and this readily implies that

$$\int_{\mathbb{R}^{N+1}_+} \left(\nabla v \cdot \nabla w_t + m^2 v w_t\right)\, dx\, dy = \int_{\mathbb{R}^N} g w_t\, dy.$$

An integration with respect to t from 0 to $+\infty$ gives

$$\int_{\mathbb{R}^{N+1}_+} \left(\nabla u \cdot \nabla \eta + m^2 u\eta - g\eta\right)\, dx\, dy = 0,$$

and hence the validity of (12) is proved.

Moreover for any given $R > 0$ we can define $u_{\text{odd}} \in H^1((-R,R) \times \mathbb{R}^N)$ and $g_{\text{odd}} \in \bigcap_{q \geq 2} L^q((-R,R) \times \mathbb{R}^N)$ by

$$u_{\text{odd}} = \begin{cases} u(x,y) & \text{if } x \geq 0 \\ -u(-x,y) & \text{if } x < 0, \end{cases} \qquad g_{\text{odd}}(x,y) = \begin{cases} g(y) & \text{if } x \geq 0 \\ -g(y) & \text{if } x < 0. \end{cases}$$

It is easy to check as before that

$$-\Delta u_{\text{odd}} + m^2 u_{\text{odd}} = g_{\text{odd}} \quad \text{in } \mathbb{R}^{N+1}.$$

Since $g_{\text{odd}} \in L^q((-R,R) \times \mathbb{R}^N)$ for any $q \in [2, +\infty[$, $R > 0$, we can invoke standard regularity results to conclude that

$$u_{\text{odd}} \in W^{2,q}((-R,R) \times \mathbb{R}^N)$$

for every $q \geq 2$ and every $R > 0$, and hence $u_{\text{odd}} \in C^{1,\beta}(\mathbb{R}^{N+1})$, $u \in C^{1,\beta}(\mathbb{R}^{N+1}_+)$ and $v = \frac{\partial u}{\partial x} \in C^{0,\beta}(\mathbb{R}^{N+1}_+)$ by Sobolev's Embedding Theorem. Therefore $g \in C^{0,\beta/(p-1)}(\mathbb{R}^N)$, and Schauder estimates yield $u \in C^{2,\beta/(p-1)}(\mathbb{R}^{N+1}_+)$ and $v \in C^{1,\beta/(p-1)}(\mathbb{R}^{N+1}_+)$. Moreover, the $C^{1,\beta}$-norm of v can be estimated by the L^q-norm of g, which immediately implies that S_a is a bounded subset of $L^\infty(\mathbb{R}^{N+1}_+)$.

Next, we claim that $\lim_{|(x,y)| \to +\infty} v(x,y) = 0$ uniformly with respect to $v \in S_a$. We assume by contradiction that this is false: there exist a number $\delta > 0$, a sequence of points $(x_n, y_n) \in \mathbb{R}^{N+1}_+$ and a sequence of elements $v_n \in S_a$ such that $x_n + |y_n| \to +\infty$ but $v_n(x_n, y_n) \geq \delta$ for every n. Let us write $z_n = (x_n, y_n)$, and call $\tilde{v}_n(z) = v_n(z + z_n)$ for $z = (x,y) \in \mathbb{R}^{N+1}_+$. By the previous arguments, $\{\tilde{v}_n\}_n$ is a bounded sequence in $H \cap L^\infty(\mathbb{R}^{N+1}_+)$. Moreover, up to a subsequence, we can assume that $v_n \rightharpoonup v$, $\tilde{v}_n \rightharpoonup \tilde{v}$ in H and locally uniformly in $\mathbb{R}^{N+1}_+$. As in [9, pag. 989], both v and $\tilde{v}$ weakly solve (7). We now show that they are non-trivial weak solutions. The conclusion is obvious for $\tilde{v}$, since $\tilde{v}_n(0) = v_n(z_n) \geq \delta$, so that $\tilde{v}(0) \geq \delta$. We consider instead v, and remark that [10, Eq. (3.16)] implies

$$\sup_{y \in \mathbb{R}^N} |v_n(x,y)| \leq C |\gamma(v_n)|_2 \, e^{-mx}$$

for some universal constant $C > 0$. Hence $\delta \leq v_n(z_n) \leq |\gamma(v_n)|_2 \, e^{-mx_n}$, and the boundedness of $\gamma(v_n)$ in L^2 yields the boundedness of $\{x_n\}_n$ in $\mathbb{R}$. Without loss of generality, we can assume that $x_n \to \bar{x} \in [0, +\infty)$. Therefore, by (8),

$$v_n(\bar{x}, 0) \geq v_n(\bar{x}, y_n) \geq v_n(x_n, y_n) + o(1) \geq \frac{\delta}{2}$$

by locally uniform convergence, and we conclude that v is also nontrivial.

Now, for every $n \in \mathbb{N}$,

$$L_a(v_n) = \left(\frac{1}{2} - \frac{1}{2p}\right) \left(\int_{\mathbb{R}^{N+1}_+} \left(|\nabla v_n|^2 + m^2 v_n^2\right) \, dx \, dy + a \int_{\mathbb{R}^N} \gamma(v_n)^2 \, dy \right) = m_a,$$

and

$$L_a(v) \geq m_a, \quad L_a(\tilde{v}) \geq m_a.$$

If $R > 0$ satisfies $2R \leq x_n + |y_n|$, then

$$\begin{aligned}
m_a &= L_a(v_n) \\
&\geq \left(\frac{1}{2} - \frac{1}{2p}\right) \liminf_{n \to +\infty} \int_{B(0,R)} \left(|\nabla v_n|^2 + m^2 v_n^2\right) \, dx \, dy \\
&\quad + a \int_{B(0,R) \cap (\{0\} \times \mathbb{R}^N)} \gamma(v_n)^2 \, dy \\
&\quad + \left(\frac{1}{2} - \frac{1}{2p}\right) \liminf_{n \to +\infty} \int_{B(0,R)} \left(|\nabla \tilde{v}_n|^2 + m^2 \tilde{v}_n^2\right) \, dx \, dy \\
&\quad + a \int_{B(0,R) \cap (\{0\} \times \mathbb{R}^N)} \gamma(\tilde{v}_n)^2 \, dy \\
&\geq \left(\frac{1}{2} - \frac{1}{2p}\right) \left(\int_{B(0,R)} \left(|\nabla \tilde{v}|^2 + m^2 \tilde{v}^2\right) \, dx \, dy + a \int_{B(0,R) \cap (\{0\} \times \mathbb{R}^N)} \gamma(\tilde{v})^2 \, dy \right) \\
&= L_a(v) + L_a(\tilde{v}) + o(1) = 2m_a + o(1)
\end{aligned}$$

as $R \to +\infty$. This contradiction proves that

$$\lim_{|(x,y)| \to +\infty} v(x,y) = 0 \quad \text{uniformly with respect to } v \in S_a. \tag{13}$$

From [10, page 70] it follows immediately that

$$\lim_{|y| \to +\infty} I_\alpha * |\gamma(v)|^p(y) = 0, \quad \text{uniformly w.r.t. } v \in S_a.$$

Pick $R_a > 0$, independent of $v \in S_a$, such that $|y| \geq R_a$ implies

$$|I_\alpha * |\gamma(v)|^p(y)| \, |\gamma(v)(y)|^{p-2} \leq \frac{a}{2}.$$

As a consequence,

$$\begin{cases} -\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ -\frac{\partial v}{\partial x} \leq -\frac{a}{2} v & \text{in } \{0\} \times \{|y| \geq R_a\} \end{cases}$$

As in [10, Theorem 5.1] or [8, Theorem 7.1], and recalling the uniform decay at infinity of (13), it follows that v decays exponentially fast at infinity, with constants that are uniform with respect to $v \in S_a$.

We are ready to conclude: let $\{v_n\}_n$ be a sequence from S_a. Our previous arguments show that $\{v_n\}_n$ converges — up to a subsequence — weakly to some $v \in H$, and this limit v is also a solution to equation (7). Fix

$$r > \max\left\{1, \frac{N}{N(2-p)+p}\right\}$$

and split I_α as $I_\alpha^1 + I_\alpha^2$, where $I_\alpha^1 \in L^r(\mathbb{R}^N)$ and $I_\alpha^2 \in L^\infty(\mathbb{R}^N)$. This induces a decomposition of the non-local term $\mathscr{N}(v) = \mathscr{N}^1(v) + \mathscr{N}^2(v)$ as

$$\begin{aligned}
\mathscr{N}(v) &= \frac{1}{2p} \int_{\mathbb{R}^N} \left(I_\alpha * |\gamma(v)|^p\right) |\gamma(v)|^p \, dy \\
\mathscr{N}^1(v) &= \frac{1}{2p} \int_{\mathbb{R}^N} \left(I_\alpha^1 * |\gamma(v)|^p\right) |\gamma(v)|^p \, dy \\
\mathscr{N}^2(v) &= \frac{1}{2p} \int_{\mathbb{R}^N} \left(I_\alpha^2 * |\gamma(v)|^p\right) |\gamma(v)|^p \, dy.
\end{aligned}$$

We obtain immediately that

$$\begin{aligned}
0 &= \lim_{n \to +\infty} \left(\int_{\mathbb{R}_+^{N+1}} \left(|\nabla v_n|^2 + m^2 v_n^2\right) \, dx\, dy - \mathscr{N}(v_n) \right) \\
&= \int_{\mathbb{R}_+^{N+1}} \left(|\nabla v|^2 + m^2 v^2\right) \, dx\, dy - \mathscr{N}(v). \qquad (14)
\end{aligned}$$

We complete the proof by showing that $\lim_{n \to +\infty} \mathscr{N}(v_n) = \mathscr{N}(v)$. Now, by the Hardy-Littlewood-Sobolev inequality (see [25, Theorem 4.3])

$$\begin{aligned}
&\left|\mathscr{N}^1(v_n) - \mathscr{N}^1(v)\right| \\
\le& \int_{\mathbb{R}^N \times \mathbb{R}^N} I_\alpha^1(x-y) \left| |\gamma(v_n)(x)|^p |\gamma(v_n)(y)|^p - |\gamma(v)(x)|^p |\gamma(v)(y)|^p \right| dxdy \\
=& \int_{\mathbb{R}^N \times \mathbb{R}^N} I_\alpha^1(x-y) \Big| |\gamma(v_n)(x)|^p |\gamma(v_n)(y)|^p - |\gamma(v_n)(x)|^p |\gamma(v)(y)|^p \\
&+ |\gamma(v_n)(x)|^p |\gamma(v)(y)|^p - |\gamma(v)(x)|^p |\gamma(v)(y)|^p \Big| dx\, dy \\
\le& \int_{\mathbb{R}^N \times \mathbb{R}^N} I_\alpha^1(x-y) |\gamma(v_n)(x)|^p \left| |\gamma(v_n)(y)|^p - |\gamma(v)(y)|^p \right| \, dx\, dy \\
&+ \int_{\mathbb{R}^N \times \mathbb{R}^N} I_\alpha^1(x-y) |\gamma(v)(y)|^p \left| |\gamma(v_n)(x)|^p - |\gamma(v)(x)|^p \right| \, dx\, dy \\
=& \ 2 \int_{\mathbb{R}^N \times \mathbb{R}^N} I_\alpha^1(x-y) |\gamma(v_n)(x)|^p \left| |\gamma(v_n)(y)|^p - |\gamma(v)(y)|^p \right| dxdy \\
\le& \ 2C |I_\alpha^1|_r \, |\gamma(v_n)|^p_{\frac{2rp}{2r-1}} \left| |\gamma(v_n)|^p - |\gamma(v)|^p \right|_{\frac{2r}{2r-1}} = o(1),
\end{aligned}$$

since $|\gamma(v_n)|^p - |\gamma(v)|^p \to 0$ strongly in $L^{\frac{2r}{2r-1}}_{\mathrm{loc}}(\mathbb{R}^N)$ by the choice of r. On the other hand,

$$\begin{aligned}&\left|\mathscr{N}^2(v_n)-\mathscr{N}^2(v)\right|\\&\qquad\leq\|I_\alpha^2\|_\infty\int_{\mathbb{R}^N\times\mathbb{R}^N}\left||\gamma(v_n)(x)|^p|\gamma(v_n)(y)|^p-|\gamma(v)(x)|^p|\gamma(v)(y)|^p\right|dx\,dy\end{aligned}$$

and the conclusion follows as before. Since $\lim_{n\to+\infty}\mathscr{N}(v_n)=\mathscr{N}(v)$, equation (14) yields $\lim_{n\to+\infty}\|v_n\|^2=\|v\|^2$, and the proof is complete.

4. The penalization scheme

For

$$\delta=\frac{1}{10}\operatorname{dist}(\mathscr{M},\mathbb{R}^N\setminus O)\quad\text{and}\quad\beta\in(0,\delta)$$

we fix a cut-off $\varphi\in C_0^\infty(\mathbb{R}_+^{N+1})$ such that $0\leq\varphi\leq1$ everywhere, $\varphi(x,y)=1$ if $x+|y|\leq\beta$, and $\varphi(x,y)=0$ if $x+|y|\geq2\beta$. Setting $\varphi_\varepsilon(x,y)=\varphi(\varepsilon x,\varepsilon y)$, for any $U\in S_{V_0}$ and any point $y_0\in\mathscr{M}^\beta$ we define

$$U_\varepsilon^{y_0}(x,y)=\varphi_\varepsilon\left(x,y-\frac{y_0}{\varepsilon}\right)U\left(x,y-\frac{y_0}{\varepsilon}\right)$$

We also define, for all $\varepsilon>0$,

$$\chi_\varepsilon(y)=\begin{cases}0&\text{if }y\in O_\varepsilon\\\varepsilon^{-6/\mu}&\text{if }y\notin O_\varepsilon\end{cases}$$

and

$$Q_\varepsilon(v)=\left(\int_{\mathbb{R}^N}\chi_\varepsilon\gamma(v)^2\,dy-1\right)_+^{\frac{2p+1}{2}}$$

for $v\in H$. Finally, let

$$\Gamma_\varepsilon(v)=\mathscr{E}_\varepsilon(v)+Q_\varepsilon(v),\quad v\in H.$$

We want to find a solution, for $\varepsilon>0$ sufficiently small, near the set

$$X_\varepsilon=\left\{U_\varepsilon^{y_0}\mid y_0\in\mathscr{M}^\beta\text{ and }U\in S_{V_0}\right\}.$$

We define the (trivial) path $\psi_\varepsilon(s)=sU_\varepsilon^{y_0}$ for every $s\in[0,1]$.

Lemma 4.1. *There exists $T>0$ such that $\Gamma_\varepsilon(\psi_\varepsilon(T))<-2$ for all ε sufficiently small. Moreover,*

$$\lim_{\varepsilon\to0}\max_{s\in[0,T]}\Gamma_\varepsilon(\psi_\varepsilon(s))=E_{V_0}$$

where we recall that $E_{V_0}=L_{V_0}(U)$ for $U\in S_{V_0}$.

Proof. Indeed, by our definition of the penalization term Q_ε, by a simple change of variables and by the exponential decay of U at infinity,

$$\begin{aligned}
\Gamma_\varepsilon(\psi_\varepsilon(s)) &= \mathscr{E}_\varepsilon(\psi_\varepsilon(s)) \\
&= \frac{s^2}{2}\int_{\mathbb{R}^{N+1}_+} |\nabla\psi_\varepsilon(s)|^2 + \frac{m^2 s^2}{2}\int_{\mathbb{R}^{N+1}_+} \psi_\varepsilon(s)^2 + \frac{s^2}{2}\int_{\mathbb{R}^N} V_\varepsilon \gamma(\psi_\varepsilon(s))^2 \\
&\quad - \frac{s^{2p}}{2p}\int_{\mathbb{R}^N} \left(I_\alpha * |\gamma(\psi_\varepsilon(s))|^p\right)|\gamma(\psi_\varepsilon(s))|^p \\
&= \left(\frac{1}{2}\int_{\mathbb{R}^{N+1}_+} |\nabla U|^2 + \frac{m^2}{2}\int_{\mathbb{R}^{N+1}_+} U^2 + \frac{1}{2}\int_{\mathbb{R}^N} V_0\gamma(U)^2 + o(1)\right)s^2 \\
&\quad - \left(\int_{\mathbb{R}^N} \left(I_\alpha * |U|^p\right)|U|^p + o(1)\right)\frac{s^{2p}}{2p}
\end{aligned}$$

where $o(1) \to 0$ as $\varepsilon \to 0$ uniformly with respect to s. The conclusion follows easily.

We are ready to introduce our mini-max scheme. For $\varepsilon > 0$ sufficiently small, we define the set of paths

$$\Phi_\varepsilon = \left\{\psi \in C([0,T],H) \mid \psi(0) = 0, \psi(T) = \psi_\varepsilon(T) = TU_\varepsilon^{y_0}\right\},$$

where $T > 0$ is the number we found in Lemma 4.1. To this set we associate the min-max level

$$C_\varepsilon = \inf_{\psi\in\Phi_\varepsilon}\max_{s\in[0,T]} \Gamma_\varepsilon(\psi(s)).$$

By well-known arguments (see for instance [7, Proposition 3.2] for a proof in a local setting that extends smoothly to our case) it is possible to prove that

$$\lim_{\varepsilon\to 0} C_\varepsilon = E_{V_0}.$$

For $\alpha \in \mathbb{R}$ define the sublevel

$$\Gamma_\varepsilon^\alpha = \left\{v \in H \mid \Gamma_\varepsilon(v) \leq \alpha\right\}.$$

Proposition 4.2. *Let $d > 0$ be small enough, and let $\{\varepsilon_j\}_j$ be such that $\lim_{j\to+\infty}\varepsilon_j = 0$ and let $\{v_{\varepsilon_j}\} \subset X^d_{\varepsilon_j}$ be such that*

$$\lim_{j\to+\infty} \Gamma_{\varepsilon_j}(v_{\varepsilon_j}) \leq E_{V_0}, \quad \lim_{j\to+\infty} \Gamma'_{\varepsilon_j}(v_{\varepsilon_j}) = 0.$$

Then there exist — up to a subsequence — $\{\tilde{y}_j\}_j \subset \mathbb{R}^N$, a point $\bar{y} \in \mathscr{M}$ and $U \in S_{V_0}$ such that

$$\begin{aligned}
&\lim_{j\to+\infty} |\varepsilon_j\tilde{y}_j - \bar{y}| = 0 \\
&\lim_{j\to+\infty} \left\|v_{\varepsilon_j} - \varphi_{\varepsilon_j}(\cdot,\cdot - \tilde{y}_j)U(\cdot,\cdot - \tilde{y}_j)\right\| = 0.
\end{aligned}$$

Proof. In the proof we will drop the index j and write ε instead of ε_j for simplicity. By Proposition 3.3, there exist $Z \in S_{V_0}$, $\{y_\varepsilon\} \subset \mathscr{M}^\beta$ and $\bar{y} \in \mathscr{M}^\beta$ such that $y_\varepsilon \to \bar{y}$ as $\varepsilon \to 0$ and

$$\left\| v_\varepsilon - \varphi_\varepsilon\left(\cdot, \cdot - \frac{y_\varepsilon}{\varepsilon}\right) Z\left(\cdot, \cdot - \frac{y_\varepsilon}{\varepsilon}\right) \right\| \leq 2d \quad \text{for every } \varepsilon \ll 1. \tag{15}$$

We set

$$v_{1,\varepsilon} = \varphi_\varepsilon\left(\cdot, \cdot - \frac{y_\varepsilon}{\varepsilon}\right) Z\left(\cdot, \cdot - \frac{y_\varepsilon}{\varepsilon}\right), \quad v_{2,\varepsilon} = v_\varepsilon - v_{1,\varepsilon}.$$

We claim that

$$\Gamma_\varepsilon(v_\varepsilon) \geq \Gamma_\varepsilon(v_{1,\varepsilon}) + \Gamma_\varepsilon(v_{2,\varepsilon}) + O(\varepsilon). \tag{16}$$

Suppose that there exist $R > 0$ and points

$$\tilde{y}_\varepsilon \in B\left(\frac{y_\varepsilon}{\varepsilon}, \frac{2\beta}{\varepsilon}\right) \setminus B\left(\frac{y_\varepsilon}{\varepsilon}, \frac{\beta}{\varepsilon}\right)$$

such that

$$\liminf_{\varepsilon \to 0} \int_{B(\tilde{y}_\varepsilon, R)} \gamma(v_\varepsilon)^2 \, dy > 0.$$

Set $\tilde{v}_\varepsilon(x,y) = v_\varepsilon(x, y + \tilde{y}_\varepsilon)$ so that

$$\liminf_{\varepsilon \to 0} \int_{B(0,R)} \gamma(\tilde{v}_\varepsilon)^2 \, dy > 0. \tag{17}$$

Up to subsequences, we can assume that

$$\lim_{\varepsilon \to 0} \varepsilon \tilde{y}_\varepsilon = y_0 \in \overline{B(\bar{y}, 2\beta) \setminus B(\bar{y}, \beta)}.$$

The sequence $\{v_\varepsilon\}$ is bounded in H and hence in every $L^q(\mathbb{R}^N)$ with $q < 2N/(N-1)$. As a consequence, $\tilde{v}_\varepsilon \to \mathcal{W}$ weakly in H and strongly in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$ for every $q < 2N/(N-1)$. By (17), $\mathcal{W} \neq 0$. Moreover,

$$\sqrt{-\Delta + m^2}\mathcal{W} + V(y_0)\mathcal{W} = \left(I_\alpha * |\mathcal{W}|^p\right) |\mathcal{W}|^{p-2}\mathcal{W}.$$

Choosing $R \gg 1$,

$$\liminf_{\varepsilon \to 0} \int_{(0,+\infty) \times B(\tilde{y}_\varepsilon, R)} \left(|\nabla v_\varepsilon|^2 + m^2 v_\varepsilon^2\right) dx\, dy \geq \frac{1}{2} \int_{\mathbb{R}^{N+1}_+} \left(|\nabla \mathcal{W}|^2 + m^2 \mathcal{W}^2\right) dx\, dy.$$

Since $E_a > E_b$ whenever $a > b$, we have

$$L_{V(y_0)}(\mathcal{W}) \geq E_{V(y_0)} \geq E_{V_0}.$$

Hence, for some absolute constant $c_0 > 0$,

$$\liminf_{\varepsilon \to 0} \int_{(0,+\infty) \times B(\tilde{y}_\varepsilon, R)} \left(|\nabla v_\varepsilon|^2 + m^2 v_\varepsilon^2\right) dx\, dy \geq c_0 \cdot L_{V(y_0)}(\mathcal{W}) \geq c_0 \cdot E_{V_0} > 0,$$

and this is a contradiction to the exponential decay at infinity of Z and the fact that $y_0 \neq \bar{y}$.

Since such a sequence $\{\tilde{y}_\varepsilon\}$ cannot exist, a Lemma of P.-L. Lions (see [26, Lemma I.1]) implies that

$$\limsup_{\varepsilon\to 0} \int_{B(\frac{y_\varepsilon}{\varepsilon},\frac{2\beta}{\varepsilon})\setminus B(\frac{y_\varepsilon}{\varepsilon},\frac{\beta}{\varepsilon})} |\gamma(v_\varepsilon)|^{\frac{N+1}{N-1}}\,dy = 0.$$

This, the boundedness of $\{\gamma(v_\varepsilon)\}$ in L^2 and the Hardy-Littlewood-Sobolev inequality imply

$$\lim_{\varepsilon\to 0}\Bigg(\int_{\mathbb{R}^N}(I_\alpha * |\gamma(v_\varepsilon)|^p)|\gamma(v_\varepsilon)|^p\,dy \\ - \int_{\mathbb{R}^N}(I_\alpha * |\gamma(v_{1,\varepsilon})|^p)|\gamma(v_{1,\varepsilon})|^p\,dy - \int_{\mathbb{R}^N}(I_\alpha * |\gamma(v_{2,\varepsilon})|^p)|\gamma(v_{2,\varepsilon})|^p\,dy\Bigg) = 0.$$

If we write

$$\begin{aligned}
&\Gamma_\varepsilon(v_\varepsilon) = \Gamma_\varepsilon(v_{1,\varepsilon}) + \Gamma_\varepsilon(v_{2,\varepsilon}) \\
&+ \int_{(0,+\infty)\times\left(B(\frac{y_\varepsilon}{\varepsilon},\frac{2\beta}{\varepsilon})\setminus B(\frac{y_\varepsilon}{\varepsilon},\frac{\beta}{\varepsilon})\right)} \varphi_\varepsilon\left(x, y-\frac{y_\varepsilon}{\varepsilon}\right)\left(1-\varphi_\varepsilon\left(x,y-\frac{y_\varepsilon}{\varepsilon}\right)\right)|\nabla v_\varepsilon|^2\,dx\,dy \\
&+ \int_{B(\frac{y_\varepsilon}{\varepsilon},\frac{2\beta}{\varepsilon})\setminus B(\frac{y_\varepsilon}{\varepsilon},\frac{\beta}{\varepsilon})} V_\varepsilon\gamma\left(\varphi_\varepsilon\left(x,y-\frac{y_\varepsilon}{\varepsilon}\right)\right)\left(1-\gamma\left(\varphi_\varepsilon\left(x,y-\frac{y_\varepsilon}{\varepsilon}\right)\right)\right)\gamma(v_\varepsilon)^2\,dy \\
&- \frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha * |\gamma(v_\varepsilon)|^p)|\gamma(v_\varepsilon)|^p\,dy \\
&+ \frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha * |\gamma(v_{1,\varepsilon})|^p)|\gamma(v_{1,\varepsilon})|^p\,dy \\
&+ \frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha * |\gamma(v_{2,\varepsilon})|^p)|\gamma(v_{2,\varepsilon})|^p\,dy + o(1)
\end{aligned}$$

as $\varepsilon \to 0$, we deduce that (16) holds true. We now estimate $\Gamma_\varepsilon(v_{2,\varepsilon})$. There results

$$\begin{aligned}
\Gamma_\varepsilon(v_{2,\varepsilon}) &\geq \mathscr{E}_\varepsilon(v_{2,\varepsilon}) \\
&= \frac{1}{2}\int_{\mathbb{R}^{N+1}_+}|\nabla v_{2,\varepsilon}|^2\,dx\,dy + \frac{1}{2}\int_{\mathbb{R}^N} V_\varepsilon\gamma(v_{2,\varepsilon})^2\,dy \\
&\quad - \frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha * |\gamma(v_{2,\varepsilon})|^p)|\gamma(v_{2,\varepsilon})|^p\,dy. \qquad (18)
\end{aligned}$$

For some constant $C > 0$ and using again the boundedness of $\{\gamma(v_{2,\varepsilon})\}$ in L^2,

$$\int_{\mathbb{R}^N}(I_\alpha * |\gamma(v_{2,\varepsilon})|^p)|\gamma(v_{2,\varepsilon})|^p\,dy \leq C\|v_{2,\varepsilon}\|.$$

Now (15) implies that $\|v_{2,\varepsilon}\| \leq 4d$ for small values of ε. Taking $d > 0$ sufficiently small uniformly with respect to ε, we have

$$\frac{1}{2}\|v_{2,\varepsilon}\|^2 - \frac{1}{2p}\int_{\mathbb{R}^N}(I_\alpha * |\gamma(v_{2,\varepsilon})|^p)|\gamma(v_{2,\varepsilon})|^p\,dy \geq \frac{1}{8}\|v_{2,\varepsilon}\|^2.$$

Since the functional $\mathscr{E}_\varepsilon$ is uniformly bounded in X_ε^d for small $\varepsilon > 0$, the penalization term Q_ε is uniformly bounded in X_ε^d for small $\varepsilon > 0$ as well. As a consequence, for an absolute constant $C > 0$,

$$\int_{\mathbb{R}^N \setminus O_\varepsilon} \gamma(v_{2,\varepsilon})^2 \, dy \leq C \varepsilon^{\frac{6}{\mu}}, \tag{19}$$

and (18–19) imply $\Gamma(v_{2,\varepsilon}) \geq o(1)$ as $\varepsilon \to 0$.

Let us introduce

$$v_{1,\varepsilon}^1(x,y) = \begin{cases} v_{1,\varepsilon}(x,y) & \text{if } y \in O_\varepsilon \\ 0 & \text{otherwise.} \end{cases}$$

For $\mathfrak{W}_\varepsilon(x,y) = v_{1,\varepsilon}^1(x, y + y_\varepsilon/\varepsilon)$, we can proceed as before and conclude that $\mathfrak{W}_\varepsilon$ converges weakly in $L^q(\mathbb{R}_+^{N+1})$, $q < 2N/(N-1)$, to a solution $\mathfrak{W}$ of

$$\sqrt{-\Delta + m^2}\mathfrak{W} + V(\bar{y})\mathfrak{W} = (I_\alpha * |\mathfrak{W}|^p)|\mathfrak{W}|^{p-2}\mathfrak{W}.$$

We claim that $\mathfrak{W}_\varepsilon$ converges to $\mathfrak{W}$ strongly in H. As before, assume the existence of a radius $R > 0$ and of a sequence $\{z_\varepsilon\} \subset \mathbb{R}^N$ such that $z_\varepsilon \in B(y_\varepsilon/\varepsilon, 2\beta/\varepsilon)$,

$$\liminf_{\varepsilon \to 0} \left| z_\varepsilon - \varepsilon^{-1} y_\varepsilon \right| = 0 \quad \text{and} \quad \liminf_{\varepsilon \to 0} \int_{B(z_\varepsilon, R)} |\gamma(v_{1,\varepsilon}^1)|^2 \, dy > 0.$$

Without loss of generality, $\varepsilon z_\varepsilon \to z \in O$ as $\varepsilon \to 0$. Then $\widetilde{\mathfrak{W}}_\varepsilon(x,y) = \mathfrak{W}_\varepsilon(x, y + z_\varepsilon)$ converges weakly in $L^q(\mathbb{R}_+^{N+1})$, $q < 2N/(N-1)$, to some $\widetilde{\mathfrak{W}} \in H$ that solves

$$\sqrt{-\Delta + m^2}\widetilde{\mathfrak{W}} + V(z)\widetilde{\mathfrak{W}} = (I_\alpha * |\widetilde{\mathfrak{W}}|^p)|\widetilde{\mathfrak{W}}|^{p-2}\widetilde{\mathfrak{W}}.$$

and we obtain a contradiction as before. Again,

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} (I_\alpha * |\gamma(\mathfrak{W}_\varepsilon)|^p)|\gamma(\mathfrak{W}_\varepsilon)|^p \, dy = \int_{\mathbb{R}^N} (I_\alpha * |\gamma(\mathfrak{W})|^p)|\gamma(\mathfrak{W})|^p \, dy. \tag{20}$$

Hence

$$\begin{aligned}
\limsup_{\varepsilon \to 0} \Gamma_\varepsilon(v_{1,\varepsilon}^1) &\geq \liminf_{\varepsilon \to 0} \Gamma_\varepsilon(v_{1,\varepsilon}^1) \\
&\geq \liminf_{\varepsilon \to 0} \frac{1}{2} \int_{(0,+\infty) \times B(0,R)} |\nabla \mathfrak{W}_\varepsilon|^2 \, dx \, dy \\
&\quad + \frac{1}{2} \int_{B(0,R)} V(\varepsilon y + y_\varepsilon) |\gamma(\mathfrak{W}_\varepsilon)|^2 \, dy \\
&\quad - \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha * |\gamma(\mathfrak{W}_\varepsilon)|^p)|\gamma(\mathfrak{W}_\varepsilon)|^p \, dy \\
&\geq \frac{1}{2} \int_{(0,+\infty) \times B(0,R)} |\nabla \mathfrak{W}_\varepsilon|^2 \, dx \, dy \\
&\quad + \frac{1}{2} V(\bar{y}) \int_{B(0,R)} |\gamma(\mathfrak{W}_\varepsilon)|^2 \, dy \\
&\quad - \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha * |\gamma(\mathfrak{W})|^p)|\gamma(\mathfrak{W})|^p \, dy.
\end{aligned}$$

Since $R > 0$ is arbitrary,

$$\begin{aligned}\limsup_{\varepsilon\to 0}\Gamma_\varepsilon(v_{1,\varepsilon}^1) &\geq \frac{1}{2}\int_{\mathbb{R}^{N+1}_+}|\nabla \mathfrak{W}|^2\,dx\,dy + \frac{1}{2}V(\bar y)\int_{\mathbb{R}^N}|\gamma(\mathfrak{W})|^2\,dy \\ &\quad - \int_{\mathbb{R}^N}(I_\alpha * |\gamma(\mathfrak{W})|^p)|\gamma(\mathfrak{W})|^p\,dy \\ &= L_{V(\bar y)}(\mathfrak{W}) \\ &\geq E_{V_0}. \end{aligned} \tag{21}$$

Recalling (16), we find

$$\limsup_{\varepsilon\to 0}\left(\Gamma_\varepsilon(v_{2,\varepsilon}) + \Gamma_\varepsilon(v_{1,\varepsilon}^1)\right) = \limsup_{\varepsilon\to 0}\left(\Gamma_\varepsilon(v_{2,\varepsilon}) + \Gamma_\varepsilon(v_{1,\varepsilon})\right) \leq \limsup_{\varepsilon\to 0}\Gamma_\varepsilon(v_\varepsilon) \leq E_{V_0}.$$

Now $\Gamma_\varepsilon(u_{2,\varepsilon}) \geq o(1)$ yields

$$\lim_{\varepsilon\to 0}\Gamma_\varepsilon(v_{1,\varepsilon}^1) = E_{V_0}.$$

What we have just proved entails that $L_{V(\bar y)}(\mathfrak{W}) = E_{V_0}$, and then $\bar y \in \mathscr{M}$. As a consequence, $\mathfrak{W}$ is, up to a translation in the y-variable, an element of S_{V_0}, namely $\mathfrak{W}(x,y) = U(x, y-\mathfrak{z})$ for some $U \in S_{V_0}$ and some $\mathfrak{z} \in \mathbb{R}^N$.

Recalling that $V \geq V(\bar y)$ on the subset O and using the identity $L_{V(\bar y)}(\mathfrak{W}) = E_{V_0}$ we get

$$\begin{aligned}&\int_{\mathbb{R}^{N+1}_+}|\nabla \mathfrak{W}|^2\,dx\,dy + V_0\int_{\mathbb{R}^N}|\gamma(\mathfrak{W})|^2\,dy - 2p\int_{\mathbb{R}^N}(I_\alpha * |\gamma(\mathfrak{W})|^p)|\gamma(\mathfrak{W})|^p\,dy \\ &\quad\geq \limsup_{\varepsilon\to 0}\int_{\mathbb{R}^{N+1}_+}|\nabla \mathfrak{W}_\varepsilon|^2\,dx\,dy + \int_{\mathbb{R}^N}V(\varepsilon y + y_\varepsilon)|\gamma(\mathfrak{W}_\varepsilon)|^2\,dy \\ &\qquad - 2p\int_{\mathbb{R}^N}(I_\alpha * |\gamma(\mathfrak{W}_\varepsilon)|^p)|\gamma(\mathfrak{W}_\varepsilon)|^p\,dy \\ &\quad\geq \limsup_{\varepsilon\to 0}\int_{\mathbb{R}^{N+1}_+}|\nabla \mathfrak{W}_\varepsilon|^2\,dx\,dy + \int_{\mathbb{R}^N}V(\bar y)|\gamma(\mathfrak{W}_\varepsilon)|^2\,dy \\ &\qquad - 2p\int_{\mathbb{R}^N}(I_\alpha * |\gamma(\mathfrak{W}_\varepsilon)|^p)|\gamma(\mathfrak{W}_\varepsilon)|^p\,dy \\ &\quad\geq \int_{\mathbb{R}^{N+1}_+}|\nabla \mathfrak{W}|^2\,dx\,dy + V_0\int_{\mathbb{R}^N}|\gamma(\mathfrak{W})|^2\,dy - 2p\int_{\mathbb{R}^N}(I_\alpha * |\gamma(\mathfrak{W})|^p)|\gamma(\mathfrak{W})|^p\,dy, \end{aligned} \tag{22}$$

and therefore

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}V(\varepsilon y + y_\varepsilon)|\gamma(\mathfrak{W}_\varepsilon)|^2\,dy = \int_{\mathbb{R}^N}V(\bar y)|\gamma(\mathfrak{W})|^2\,dy.$$

Using again the fact that $V \geq V(\bar y)$ on the subset O we conclude that $\gamma(\mathfrak{W}_\varepsilon) \to$

$\gamma(\mathfrak{W})$ strongly in $L^2(\mathbb{R}^N)$. Finally, from (20), (21) and (22) we see that

$$\int_{\mathbb{R}^{N+1}_+} |\nabla \mathfrak{W}|^2 \,dx\,dy + \int_{\mathbb{R}^N} V(\bar{y})|\gamma(\mathfrak{W})|^2\,dy \\ \geq \limsup_{\varepsilon\to 0} \int_{\mathbb{R}^{N+1}_+} |\nabla \mathfrak{W}_\varepsilon|^2\,dx\,dy + \int_{\mathbb{R}^N} V(\bar{y})|\gamma(\mathfrak{W}_\varepsilon)|^2\,dy.$$

The strong convergence of $\mathfrak{W}_\varepsilon$ to $\mathfrak{W}$ in H is now proved. Thus

$$v^1_{1,\varepsilon}(x,y) = U\left(x, y - \frac{y_\varepsilon}{\varepsilon} - \mathfrak{z}\right) + o(1),$$

and straightforward algebraic manipulations show that

$$v_{1,\varepsilon}(x,y) = \varphi_\varepsilon\left(x, y - \frac{y_\varepsilon}{\varepsilon} - \mathfrak{z}\right) U\left(x, y - \frac{y_\varepsilon}{\varepsilon} - \mathfrak{z}\right) + o(1)$$

strongly in H. But $E_{V_0} \geq \lim_{\varepsilon\to 0}\Gamma_\varepsilon(v_\varepsilon)$ and $\lim_{\varepsilon\to 0}\Gamma_\varepsilon(v_{1,\varepsilon}) = E_{V_0}$, so that $\lim_{\varepsilon\to 0}\Gamma_\varepsilon(v_{2,\varepsilon}) = 0$ by (16). Using (18) and (19) we discover that $v_{2,\varepsilon} \to 0$ strongly in H. This completes the proof.

5. Critical points of the penalized functional

We are now ready to show that the penalized functional Γ_ε possesses a critical point for every $\varepsilon > 0$ sufficiently small.

Lemma 5.1. *For $d > 0$ sufficiently small, there exist positive constants ε_0 and ω such that $|\Gamma'_\varepsilon(v)| \geq \omega$ for every $v \in \Gamma_\varepsilon^{D_\varepsilon} \cap \left(X^d_\varepsilon \setminus X^{d/2}_\varepsilon\right)$ and $\varepsilon \in (0, \varepsilon_0)$.*

Proof. If not, for $d > 0$ so small that Proposition 4.2 applies, there exist sequences $\{\varepsilon_j\}_j$ with $\lim_j \varepsilon_j = 0$ and $\{v_{\varepsilon_j}\}_j$ with $v_{\varepsilon_j} \in X^d_{\varepsilon_j} \setminus X^{d/2}_{\varepsilon_j}$ satisfying

$$\lim_{j\to+\infty} \Gamma_{\varepsilon_j}(v_{\varepsilon_j}) \leq E_{V_0} \quad \text{and} \quad \lim_{j\to+\infty} \Gamma'_{\varepsilon_j}(v_{\varepsilon_j}) = 0.$$

Hence Proposition 4.2 applies and provides points $y_{\varepsilon_j} \in \mathbb{R}^N$, $\bar{y} \in \mathscr{M}$ and a ground state $U \in S_{V_0}$ such that

$$\lim_{j\to+\infty} |\varepsilon_j y_j - \bar{y}| = 0 \\ \lim_{j\to+\infty} \left\| v_{\varepsilon_j} - \varphi_{\varepsilon_j}(\cdot, \cdot - y_j) U(\cdot, \cdot - y_j)\right\| = 0. \tag{23}$$

The definition of X_{ε_j} implies $\lim_{j\to+\infty} \operatorname{dist}(v_{\varepsilon_j}, X_{\varepsilon_j}) = 0$, and this contradicts the assumption $v_{\varepsilon_j} \notin X^{d/2}_{\varepsilon_j}$.

Let now $d > 0$ be chosen so that Lemma 5.1 applies.

Proposition 5.2. *For $\varepsilon > 0$ sufficiently small, the functional Γ_ε has a critical point $v_\varepsilon \in X^d_\varepsilon \cap \Gamma^D_\varepsilon$.*

Proof. Pick $R_0 > 0$ so large that $O \subset \left(\{0\} \times \mathbb{R}^N\right) \cap B(0, R_0)$ and $\psi_\varepsilon(s) \in H_0^1(B(0, R/\varepsilon))$ for any $s \in [0, T]$, $R > R_0$ and $\varepsilon > 0$ sufficiently small. We write $D_\varepsilon = \max_{0 \leq s \leq T} \Gamma_\varepsilon(\psi_\varepsilon(s))$. By Lemma 4.1, there exists $\mathfrak{a} \in (0, E_{V_0})$ such that, for sufficiently small $\varepsilon > 0$,

$$\Gamma_\varepsilon(\psi_\varepsilon(s)) \geq D_\varepsilon - \mathfrak{a} \quad \text{implies} \quad \psi_\varepsilon(s) \in X_\varepsilon^{d/2} \cap H_0^1(B(0, R/\varepsilon)).$$

We claim that, for sufficiently small $\varepsilon > 0$ and $R > R_0$, there is a sequence $\{v_n^R\}_n \subset X_\varepsilon^{d/2} \cap \Gamma_\varepsilon^{D_\varepsilon} \cap H_0^1(B(0, R/\varepsilon))$ such that $\Gamma_\varepsilon'(v_n^R) \to 0$ is $H_0^1(B(0, R/\varepsilon))$ as $n \to +\infty$.

Arguing by contradiction, we assume that for sufficiently small $\varepsilon > 0$ there exists a number $a_R(\varepsilon) > 0$ such that

$$|\Gamma_\varepsilon'(v)| \geq a_R(\varepsilon)$$

on $X_\varepsilon^{d/2} \cap \Gamma_\varepsilon^{D_\varepsilon} \cap H_0^1(B(0, R/\varepsilon))$. With a slight abuse of notation, we will identify any $v \in H_0^1(B(0, R/\varepsilon))$ with its extension to H as the null function outside $B(0, R/\varepsilon)$. Applying Lemma 5.1, we find a number $\omega > 0$, independent of $\varepsilon > 0$, such that $|\Gamma_\varepsilon'(v)| \geq \omega$ for $v \in \Gamma_\varepsilon^{D_\varepsilon} \cap (X_\varepsilon^d \setminus X_\varepsilon^{d/2})$. By a classical deformation argument that starts from ψ_ε, there exist some $\mu \in (0, \mathfrak{a})$ and a path $\psi \in C([0, T], H)$ satisfying

$$\psi(s) = \psi_\varepsilon(s) \text{ for } \psi_\varepsilon(s) \in \Gamma_\varepsilon^{D_\varepsilon - \mathfrak{a}}, \quad \psi(s) \in X_\varepsilon^d \text{ for } \psi_\varepsilon(s) \notin \Gamma_\varepsilon^{D_\varepsilon - \mathfrak{a}}$$

and

$$\Gamma_\varepsilon(\psi(s)) < D_\varepsilon - \mu \quad \text{for every } s \in [0, T]. \tag{24}$$

Let $\zeta \in C_0^\infty(\mathbb{R}_+^{N+1})$ be a cut-off function such that $\zeta(x, y) = 1$ for $0 < x < \delta$ and $y \in O^\delta$, $\zeta(x, y) = 0$ for $x \geq 2\delta$ and $y \notin O^{2\delta}$, $\zeta(\cdot, \cdot) \in [0, 1]$, and $|\nabla \zeta| \leq 2/\delta$. For $\psi(s) \in X_\varepsilon^d$ we denote $\psi_1(s) = \zeta_\varepsilon \psi(s)$ and $\psi_2(s) = (1 - \zeta_\varepsilon)\psi(s)$, where $\zeta_\varepsilon(x, y) = \zeta(\varepsilon x, \varepsilon y)$. We remark that we understand the dependency on ε in the notation of ψ_1 and ψ_2. Observe that

$$\begin{aligned}
\Gamma_\varepsilon(\psi(s)) &= \Gamma_\varepsilon(\psi_1(s)) + \Gamma_\varepsilon(\psi_2(s)) + Q_\varepsilon(\psi(s)) - Q_\varepsilon(\psi_1(s)) - Q_\varepsilon(\psi_2(s)) \\
&\quad - \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha * |\gamma(\psi(s))|^p) |\gamma(\psi(s))|^p \\
&\quad + \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha * |\gamma(\psi_1(s))|^p) |\gamma(\psi_1(s))|^p \\
&\quad + \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha * |\gamma(\psi_2(s))|^p) |\gamma(\psi_2(s))|^p.
\end{aligned}$$

The elementary inequality $(h + k - 1)_+ \geq (h - 1)_+ + (k - 1)_+$ valid for $h \geq 0$ and $k \geq 0$ immediately implies that

$$Q_\varepsilon(\psi(s)) \geq Q_\varepsilon(\psi_1(s)) + Q_\varepsilon(\psi_2(s))$$

and, similarly to (19), we find that

$$\int_{\mathbb{R}^N \setminus O_\varepsilon} |\gamma(\psi(s))|^2 \, dy \leq C \varepsilon^{6/\mu}. \tag{25}$$

On the other hand, writing $\kappa = (I_\alpha * |\gamma(\psi(s))|^p)|\gamma(\psi(s))|^p - (I_\alpha * |\gamma(\psi_1(s))|^p)|\gamma(\psi_1(s))|^p - (I_\alpha * |\gamma(\psi_2(s))|^p)|\gamma(\psi_2(s))|^p$ for simplicity,

$$\int_{\mathbb{R}^N} \kappa = 2\int_{O_\varepsilon^{2\delta}\times(\mathbb{R}^N\setminus O_\varepsilon^{2\delta})} (I_\alpha * |\gamma(\psi(s))|^p)|\gamma(\psi(s))|^p - 2\int_{(O_\varepsilon^{2\delta}\setminus O_\varepsilon^{\delta})\times(\mathbb{R}^N\setminus O^{\delta})} (I_\alpha * |\gamma(\psi(s))|^p)|\gamma(\psi(s))|^p$$

and from (25) via interpolation we deduce that

$$\lim_{\varepsilon\to 0}\int_{O_\varepsilon^{2\delta}\times(\mathbb{R}^N\setminus O_\varepsilon^{2\delta})} (I_\alpha * |\gamma(\psi(s))|^p)|\gamma(\psi(s))|^p = 0 \tag{26}$$

$$\lim_{\varepsilon\to 0}\int_{(O_\varepsilon^{2\delta}\setminus O_\varepsilon^{\delta})\times(\mathbb{R}^N\setminus O_\varepsilon^{\delta})} (I_\alpha * |\gamma(\psi(s))|^p)|\gamma(\psi(s))|^p = 0. \tag{27}$$

Equations (26) and (27) yield

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} (I_\alpha * |\gamma(\psi(s))|^p)|\gamma(\psi(s))|^p - \int_{\mathbb{R}^N} (I_\alpha * |\gamma(\psi_1(s))|^p)|\gamma(\psi_1(s))|^p - \int_{\mathbb{R}^N} (I_\alpha * |\gamma(\psi_2(s))|^p)|\gamma(\psi_2(s))|^p = 0,$$

and hence, as $\varepsilon \to 0$,

$$\Gamma_\varepsilon(\psi(s)) \geq \Gamma_\varepsilon(\psi_1(s)) + \Gamma_\varepsilon(\psi_2(s)) + o(1).$$

By similar arguments,

$$\Gamma_\varepsilon(\psi_2(s)) \geq -\frac{1}{2p}\int_{(\mathbb{R}^N\setminus O_\varepsilon)\times(\mathbb{R}^N\setminus O_\varepsilon)} I_\alpha(x-y)|\gamma(\psi_2(s)(x)|^p|\gamma(\psi_2(s)(y)|^p\,dx\,dy \geq o(1),$$

and we finally conclude that

$$\Gamma_\varepsilon(\psi(s)) \geq \Gamma_\varepsilon(\psi_1(s)) + o(1)$$

as $\varepsilon \to 0$. If we define

$$\psi_1^1(s)(x,y) = \begin{cases} \psi_1(s)(x,y) & \text{if } x>0 \text{ and } y \in O^{2\delta} \\ 0 & \text{if } x>0 \text{ and } y \notin O^{2\delta}, \end{cases}$$

we immediately see that $\Gamma_\varepsilon(\psi_1(s)) \geq \Gamma_\varepsilon(\psi_1^1(s))$, and $\psi_1^1 \in \Phi_\varepsilon$ because $0 < \mathfrak{a} < E_{V_0}$. Now [12, Proposition 3.4] implies that, as $\varepsilon \to 0$,

$$\max_{0\leq s\leq T} \Gamma_\varepsilon(\psi(s)) \geq E_{V_0} + o(1),$$

and this contradicts (24).

For a fixed ε sufficiently small and for $R \gg 1$, we consider a sequence $\{v_n^R\}_n \subset X_\varepsilon^{d/2} \cap \Gamma_\varepsilon^{D_\varepsilon} \cap H_0^1(B(0,R/\varepsilon))$ such that $\Gamma'_\varepsilon(v_n^R) \to 0$ is $H_0^1(B(0,R/\varepsilon))$ as $n \to +\infty$. The boundedness of $\{v_n^R\}_n$ in $H_0^1(B(0,R/\varepsilon))$ and the Sobolev embedding theorem imply that $v_n^R \to v^R$ strongly in $L^q(B(0,R/\varepsilon))$ for any $q < 2N/(N-1)$. Since $\{v_n^R\}_n$ is a Palais-Smale sequence, a standard argument shows that $v_n^R \to v^R$ strongly in $H_0^1(B(0,R/\varepsilon))$. Hence the limit v^R is a weak solution to the problem

$$-\Delta v^R + m^2 v^R = 0 \quad \text{in } B\left(0, \tfrac{R}{\varepsilon}\right)$$

with

$$\begin{aligned} -\frac{\partial v^R}{\partial x}(0,y) \;=\; & -V_\varepsilon(y) v^R(0,y) + \left(I_\alpha * |v^R(0,\cdot)|^p\right) |v^R(0,y)|^{p-2} v^R(0,y) + \\ & + (2p+1) \left(\int_{\mathbb{R}^N} \chi_\varepsilon \gamma(v^R)^2 \, dy - 1 \right)_+^{\frac{2p-1}{2}} \chi_\varepsilon v^R(0,y) \end{aligned}$$

for $y \in \mathbb{R}^N$ with $|y| = R/\varepsilon$.

Since $v^R \in X_\varepsilon^d \cap \Gamma_\varepsilon^{D_\varepsilon} \cap H_0^1(B(0,R/\varepsilon))$, we deduce that both $\{\|v^R\|\}_R$ and $\{\Gamma_\varepsilon(v^R)\}_R$ are uniformly bounded for $\varepsilon > 0$ sufficiently small. Hence also $\{Q_\varepsilon(v^R)\}_R$ is uniformly bounded for $\varepsilon > 0$ sufficiently small. Now a Moser iteration scheme like [10, Theorem 3.2] yields that $\{v^R\}_R$ is bounded in L^∞ uniformly for $\varepsilon > 0$ sufficiently small. Taking into account that $\{Q_\varepsilon(v^R)\}_R$ is uniformly bounded in L^∞ and

$$\left(I_\alpha * |v^R(0,\cdot)|^p\right) |v^R(0,y)|^{p-1} \le \frac{1}{2}(V_\varepsilon + m)|v^R(0,y)|$$

when $|y| \ge 2R/\varepsilon$, we can perform a comparison argument as in [10, Theorem 5.1] and derive

$$|v^R(x,y)| \le C e^{-m(\sqrt{x^2+|y|^2} - 2R_0)}.$$

We assume, without loss of generality, that $\{v^R\}_R$ weakly converges to some v_ε in H as $R \to +\infty$ that solves

$$-\Delta v_\varepsilon + m^2 v_\varepsilon = 0 \quad \text{in } \mathbb{R}_+^{N+1} \tag{28}$$

with

$$\begin{aligned} -\frac{\partial v_\varepsilon}{\partial x}(0,y) \;=\; & -V_\varepsilon(y) v_\varepsilon(0,y) + \left(I_\alpha * |v_\varepsilon(0,\cdot)|^p\right) |v_\varepsilon(0,y)|^{p-2} v_\varepsilon(0,y) + \\ & + (2p+1) \left(\int_{\mathbb{R}^N} \chi_\varepsilon \gamma(v_\varepsilon)^2 \, dy - 1 \right)_+^{\frac{2p-1}{2}} \chi_\varepsilon v_\varepsilon(0,y) \end{aligned} \tag{29}$$

for $y \in \mathbb{R}^N$.

6. Proof of the Theorem 1.1

We can now collect all the results of the previous section to prove our main existence theorem. To begin with, Proposition 5.2 gives us a number $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, the penalized functional Γ_ε possesses a critical point $v_\varepsilon \in X_\varepsilon^d \cap \Gamma_\varepsilon^{D_\varepsilon}$. As in the proof of Proposition 3.3, we have $v_\varepsilon \in \bigcap_{q>2} L^q(\mathbb{R}_+^{N+1})$, and $\{v_\varepsilon\}$ is bounded $L^\infty([0,+\infty) \times \mathbb{R}^N)$. By the results of Proposition 4.2,

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}_+^{N+1} \setminus ([0,+\infty) \times (\mathscr{M}^{2\beta})_\varepsilon)} \left(|\nabla v_\varepsilon|^2 + V_\varepsilon |v_\varepsilon|^2 \right) \, dx \, dy = 0.$$

It now follows that

$$\lim_{\varepsilon \to 0} \sup_{(x,y) \in \mathbb{R}_+^{N+1} \setminus ([0,+\infty) \times (\mathscr{M}^{2\beta})_\varepsilon)} |v_\varepsilon(x,y)| = 0,$$

and as in the last step of the previous section we deduce an exponential decay of the trace u_ε away from $\mathbb{R}^N \setminus (\mathscr{M}^{2\beta})_\varepsilon$:

$$|u_\varepsilon(y)| \leq C_1 \exp\left(-C_2 \operatorname{dist}\left(y, (\mathscr{M}^{2\beta})_\varepsilon\right)\right).$$

Taking ε smaller, this estimate implies that $Q_\varepsilon(v_\varepsilon) = 0$, and (28)-(29) are the local Neumann problem in the half space $\mathbb{R}^N$ corresponding to the nonlocal problem (4). The conclusion now follows by reversing the local realization of the operator $\sqrt{-\Delta + m^2}$. Recalling (23) and all the scalings, we immediately deduce (5). This completes the proof.

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References

[1] G.L. Aki, P.A. Markowich, C. Sparber, *Classical limit for semirelativistic Hartree systems*, J. Math. Phys. **49** (2008), no. 10, 102110, 10 pp.

[2] J. Byeon, L. Jeanjean, *Standing waves for nonlinear Schrödinger equations with a general nonlinearity*, Archives for Rational Mechanics and Analysis, **185** (2007), no. 2, 185-200.

[3] J. Byeon, L. Jeanjean, *Erratum: Standing waves for nonlinear Schrödinger equations with a general nonlinearity*, Archives for Rational Mechanics and Analysis, **190** (2008), no. 3, 549-551.

[4] L. Caffarelli, L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations **32** (2007), 1245–1260.

[5] X. Cabré, S. Solà-Morales, *Layer solutions in a half-space for boundary reactions*, Comm. Pure Appl. Math. **58** (2005) no. 12, 1678–1732.

[6] S. Cingolani, M. Clapp, S. Secchi, *Intertwining semiclassical solutions to a Schrödinger-Newton system*, Discrete Continuous Dynmical Systems Series S 6 (2013), 898–908.

[7] S. Cingolani, L. Jeanjean, S. Secchi, *Multi-peak solutions for magnetic NLS equations without non-degeneracy conditions.* ESAIM: Control Optim. Calc. Variations **15** (2009), 653–675.

[8] S. Cingolani, S. Secchi, *Ground states for the pseudo-relativistic Hartree equation with external potential*, Proc. of the Royal Soc. Edinburgh, in press.

[9] S. Cingolani, S. Secchi, M. Squassina, *Semi-classical limit for Schrödinger equations with magnetic field and Hartree-type nonlinearities*, Proceedings of the Royal Society of Edinburgh **140A** (2010), 973–1009.

[10] V. Cozi Zelati, M. Nolasco, *Existence of ground states for nonlinear, pseudorelativistic Schrödinger equations*, Red. Lincei Mat. Appl. **22** (2011), 51–72.

[11] V. Cozi Zelati, M. Nolasco, *Ground states for pseudo-relativistic Hartree equations of critical type*, Rev. Mat. Iberoam. **22** (2013), 1421–1436.

[12] V. Coti Zelati, P.H. Rabinowitz, *Homoclinic orbits for second order Hamiltonian systems possessing superquadratic potentials.* J. Am. Math. Soc. **4** (1991), 693–727.

[13] P. D'Avenia, G. Siciliano, M. Squassina, *On fractional Choquard equations*, Mathematical Models and Methods in Applied Sciences, to appear.

[14] A. Dall'Acqua, T.O. Sorensen, E. Stockmeyer, *Hartree-Fock theory for pseudo-relativistic atoms*, Ann. Henri Poincaré **9** (2008), 711–742.

[15] J. Davila, M. del Pino, J. Wei, *Concentrating standing waves for the fractional nonlinear Schrödinger equation*, J. Differential Equations **256**(2014), 858–892.

[16] A. Elgart, B. Schlein, *Mean field dynamics of Boson stars*, Comm. Pure Appl. Math. **60** (2007), 500-545.

[17] J. Fröhlich, B.L.G. Jonsson, E. Lenzmann, *Dynamical collapse of white dwarfs in Hartree and Hartree-Fock theory*, Comm. Math Phys. **274** (2007), 737-750.

[18] J. Fröhlich, B.L.G. Jonsson, E. Lenzmann, *Effective dynamics for boson stars*, Nonlinearity **20**, No. 5 (2007), 1031–1075.

[19] J. Fröhlich, J. Jonsson, E. Lenzmann, *Boson stars as solitary waves*, Comm. Math. Phys. **274** (2007), 1–30.

[20] J. Fröhlich, E. Lenzmann, *Mean-field limit of quantum Bose gases and nonlinear Hartree equation*, in Séminaire: Équations aux Dérivées Partielles 2003–2004, Exp. No. XIX, 26 pp., École Polytech., Palaiseau, 2004.

[21] E. Lenzmann, *Well-posedness for Semi-relativistic Hartree equations with Critical type*, Math. Phys. Anal. Geom. **10** (2007), 43–64.

[22] E. Lenzmann, *Uniqueness of ground states for pseudo-relativistic Hartree equations*, Analysis and PDE **2** (2009), 1–27.

[23] E.H. Lieb, M. Loss, *Analysis*, Graduates Studies in Mathematics, no.14, American Mathematical Society, 1997.

[24] E.H. Lieb, H.-T. Yau, *The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics*, Comm. Math. Phys. **112** (1987), 147–174.

[25] E. H. Lieb, M. Loss. Analysis. Graduate Studies in Mathematics **14**. American Mathematical Society, 2001.

[26] P.L. Lions, *The concentration–compactness principle in the calculus of variations. The locally compact case. Part II*, Annales Inst. H. Poincaré Anal. Nonlin. **1** (1984), 223–283.

[27] M. Melgaard, F. Zongo, *Multiple solutions of the quasirelativistic Choquard equation*, J. Math. Phys. **53** (2012), no. 3, 033709, 12 pp.

[28] P. Markowich, N. Mauser, *The classical limit of a self- consistent quantum-Vlasov equation in 3-D*, Math. Models Meth. Appl. Sci. **3**, 109 (1993).

[29] V. Moroz, J. Van Schaftingen, *Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics*, J. Funct. Anal. **265**, No. 2, (2013), 153–184.

[30] V. Moroz, J. Van Schaftingen, *Semi-classical states for the Choquard equations*, Calc. Var. Partial Differential Equations, to appear.

[31] P. H. Rabinowitz, *On a class of nonlinear Schrödinger equations*, Z angew Math Phys **43** (1992), 270–291.

[32] W.A. Salem, T. Chen and V. Vougalter, *Existence and Nonlinear Stability of Stationary States for the Semi-Relativistic Schrödinger-Poisson System*, Ann. H. I. Poincaré **15** , (2014), 1171–1196.

[33] L. Tartar. An introduction to Sobolev spaces. Springer, 2007.

[34] J. Wei, M. Winter, *Strongly interacting bumps for the Schrödinger–Newton equation*, J. Math. Phys. **50** (2009), 012905.