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Integral representation formulae for the solution of a wave equation with time-dependent damping and mass in the scale-invariant case

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Abstract

This paper is devoted to derive integral representation formulae for the solution of an inhomogeneous linear wave equation with time-dependent damping and mass terms, that are scale-invariant with respect the so-called hyperbolic scaling. Yagdjian's integral transform approach is employed for this purpose. The main step in our argument consists in determining the kernel functions for the different integral terms, which are related to the source term and to initial data. We will start with the one dimensional case (in space). We point out that we may not apply in a straightforward way Duhamel's principle to deal with the source term since the coefficients of lower order terms make our model not invariant by time translation. On the contrary, we shall begin with the representation formula for the inhomogeneous equation with vanishing data by using a revised Duhamel's principle. Then, we will derive the representation of the solution in the homogeneous case with nontrivial data. After deriving the formula in the one dimensional case, the classical approach by spherical means is used in order to deal with the odd dimensional case. Finally, using the method of descent, the representation formula in the even dimensional case is proved.

Keywords Integral transform, Hypergeometric function, Spherical means, Method of descent, Wave equation, Time-dependent and scale-invariant lower order terms

AMS Classification (2010) Primary: 35A08, 35C15; Secondary: 33C05, 35L05, 35L15

1 Introduction

In the last years, several papers have been devoted to the study of the semilinear wave equations (and weakly coupled systems) with time-dependent damping and mass and power nonlinearity in the scale-invariant case, namely,

$$\begin{cases} u_{tt}-\Delta u+\frac{\mu}{1+t}u_t+\frac{\nu^2}{(1+t)^2}u=|u|^p, & x\in\mathbb{R}^n,\ t>0,\\ u(0,x)=\varepsilon u_0(x), & x\in\mathbb{R}^n,\\ u_t(0,x)=\varepsilon u_1(x), & x\in\mathbb{R}^n, \end{cases} \tag{1}$$

where μ,ν^2 are non negative constants, $p>1$ and ε is positive constant describing the smallness of Cauchy data (cf. [2, 25, 4, 3, 24, 10, 7, 23, 9] for the massless case and [1, 17] for the weakly coupled system). If we introduce the quantity

$$\delta \doteq (\mu-1)^2-4\nu^2, \tag{2}$$

then, it is known that the critical exponent for (1) depends on δ.

On the one hand, for $\delta\geqslant(n+1)^2$ the critical exponent is $p_{\mathrm{Fuj}}\left(n+\frac{\mu-1}{2}-\frac{\sqrt{\delta}}{2}\right)$, where $p_{\mathrm{Fuj}}(n)\doteq 1+\frac{2}{n}$ is the Fujita exponent, see [12, 13, 18]. On the other hand, for δ nonnegative and sufficiently small (depending on the spatial dimension n), it has been proved that for any $1<p\leqslant p_{\mathrm{Str}}(n+\mu)$ local in time solutions of (1) blow up in finite times under suitable integral sign assumptions on initial data (see [19, 20]). Here $p_{\mathrm{Str}}(n)$ denotes the Strauss exponent, that is the positive root of the quadratic equation $(n-1)p^2-(n+1)p-2=0$. The global in time existence of small data solutions for $p>p_{\mathrm{Str}}(n+\mu)$ has been proved only in the special case $\delta=1$ for radial symmetric solutions and for dimensions $n\geqslant 3$ (cf. [14] for the case n odd and [15] for the case n even). However, in the general case $\delta\neq 1$, the global in time existence of small data solutions for $p>p_{\mathrm{Str}}(n+\mu)$ is still open. Furthermore, in the case $\delta<0$ both the blow-up part and the global (in time) existence part are open, although a partial result is proved for the necessity part in [5].

In the proof of the global existence results for $\delta\geqslant(n+1)^2$ (when the critical exponent is the shift of Fujita exponent), L^2-L^2 estimates for the solution of the corresponding homogeneous equation and for its derivatives play a fundamental

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role. In particular, these estimates are derived by using the explicit representation formula of the fundamental solutions of the corresponding homogeneous problem, which contains in their expression some cylindrical functions due to the scale-invariance of the model. In other terms, it is used an approach based on Fourier integral operators.

In some sense, the fact that the above mentioned shift of Fujita exponent is critical for large values of δ can be proved by using tools which are suitable for the semilinear classical damped wave equation, such as $L^2 - L^2$ decay estimates with additional regularity for initial data for the global existence part or scaling arguments for the blow-up part (namely, the so-called test function method, cf. [11]). Unfortunately, this tools are not suitable when the behavior of the semilinear model in (1) is closer to the semilinear wave equation (δ nonnegative and "small") and we expect to find as critical exponent a shift of Strauss exponent. Therefore, it might be useful to derive results and tools which are widely employed to deal with the classical wave equation.

After this preface, we understand why it could be useful to derive an explicit integral representation formula for the solution of the linear wave equation with time-dependent damping and mass term in the scale-invariant case. More specifically, in this paper we will derive an explicit representation formula for the solution of the linear Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + \frac{\mu}{1+t} u_t + \frac{\nu^2}{(1+t)^2} u = f(t,x), & x \in \mathbb{R}^n, \ t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \\ u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{3}$$

where μ, ν^2 are non negative constants.

In the series of papers [26, 27, 33, 28, 34, 29, 30, 31, 32], many representations formulae for solutions of Cauchy problems for linear hyperbolic PDEs with variable coefficients have been derived. The general scheme is substantially the same: the representation formula is obtained considering the composition of two operators. The external operator is an integral transformation, whose kernel is determined by the time-dependent coefficients and/or by lower order terms, while the internal operator is a solution operator for a family of parameter dependent Cauchy problems (this is somehow a *revised Duhamel's priciple*). In particular, if the considered PDE is a wave equation with time-dependent speed of propagation, then, this solution operator maps a given function into the solution of the Cauchy problem for the classical free wave equation with the given function as first initial data and with vanishing second initial data.

Using *Yagdjian's Integral Transform approach*, we will provide an explicit representation formula for the solution of (3) in all spatial dimensions. More specifically, we begin by studying the one-dimensional case; then, we get the representation formula for odd dimensions via spherical means' method and, finally, by method of descent we find the representation formula for even dimensions.

Let us state the main results of this paper. We start with the case $n = 1$.

Theorem 1.1. *Let $n = 1$ and let μ, ν^2 be nonnegative constants. Let us assume $f \in \mathscr{C}^{0,1}_{t,x}([0,\infty) \times \mathbb{R})$ and $u_0 \in \mathscr{C}^2(\mathbb{R})$, $u_1 \in \mathscr{C}^1(\mathbb{R})$. Then, a representation formula for the solution of* (3) *is given by*

$$\begin{aligned} u(t,x) &= \frac{1}{2}(1+t)^{-\frac{\mu}{2}}\big(u_0(x+t) + u_0(x-t)\big) + \frac{1}{2\sqrt{\delta}} \int_{x-t}^{x+t} u_0(y) K_0(t,x;y;\mu,\nu^2)\, \mathrm{d}y \\ &\quad + \frac{1}{2\sqrt{\delta}} \int_{x-t}^{x+t} \big(u_1(y) + \mu\, u_0(y)\big) K_1(t,x;y;\mu,\nu^2)\, \mathrm{d}y + \frac{1}{2\sqrt{\delta}} \int_0^t \int_{x-t+b}^{x+t-b} f(b,y) E(t,x;b,y;\mu,\nu^2)\, \mathrm{d}y\, \mathrm{d}b, \end{aligned} \tag{4}$$

where the kernel functions are defined as follows

$$E(t,x;b,y;\mu,\nu^2) \doteq (1+t)^{-\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}} (1+b)^{\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}} \left((t+b+2)^2 - (y-x)^2\right)^{\frac{\sqrt{\delta}-1}{2}} \mathsf{F}\left(\tfrac{1-\sqrt{\delta}}{2}, \tfrac{1-\sqrt{\delta}}{2}; 1; \tfrac{(t-b)^2-(y-x)^2}{(t+b+2)^2-(y-x)^2}\right), \tag{5}$$

$$K_0(t,x;y;\mu,\nu^2) \doteq -\frac{\partial}{\partial b} E(t,x;b,y;\mu,\nu^2)\Big|_{b=0}, \tag{6}$$

$$K_1(t,x;y;\mu,\nu^2) \doteq E(t,x;0,y;\mu,\nu^2) \tag{7}$$

and $\mathsf{F}(\alpha,\beta;\gamma;z)$ denotes Gauss hypergeometric function.

Before stating the representation formula in the multidimensional case, let us introduce the following notations: if $f = f(t,x)$ is defined for $t \geqslant 0, x \in \mathbb{R}^n$, then, we denote by $w[f] = w[f](t,x;b)$ the solution to the parameter dependent Cauchy problem for the free wave equation

$$\begin{cases} w_{tt} - \Delta w = 0, & x \in \mathbb{R}^n, \ t > 0, \\ w(0,x) = f(b,x), & x \in \mathbb{R}^n, \\ w_t(0,x) = 0, & x \in \mathbb{R}^n, \end{cases} \tag{8}$$

with parameter $b \geqslant 0$. When the function f depends only on the spatial variable, the Cauchy problem depends no longer on the parameter b, namely, if $\varphi = \varphi(x)$, then, $w[\varphi] = w[\varphi](t,x)$ denotes the solution to the Cauchy problem for the free wave equation

$$
\begin{cases} w_{tt} - \Delta w = 0, & x \in \mathbb{R}^n, \ t > 0, \\ w(0,x) = \varphi(x), & x \in \mathbb{R}^n, \\ w_t(0,x) = 0, & x \in \mathbb{R}^n. \end{cases} \tag{9}
$$

Assuming that the function f (resp. φ) is sufficiently smooth with respect to the spatial variable, then, the representation formula for $w[f]$ (resp. $w[\varphi]$) is well-known and depends on the parity of n (see for example [6, Section 2.4]). More precisely, when $n \geqslant 3$ is an odd integer it holds

$$
\begin{aligned} w[f](t,x;b) &= \frac{1}{(n-2)!!} \left(\frac{\partial}{\partial t}\right) \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} \left(t^{n-2} \fint_{\partial B_t(x)} f(b,z)\, \mathrm{d}\sigma_z \right), \\ w[\varphi](t,x) &= \frac{1}{(n-2)!!} \left(\frac{\partial}{\partial t}\right) \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} \left(t^{n-2} \fint_{\partial B_t(x)} \varphi(z)\, \mathrm{d}\sigma_z \right), \end{aligned} \tag{10}
$$

while if $n \geqslant 2$ is an even integer, then,

$$
\begin{aligned} w[f](t,x;b) &= \frac{1}{n!!} \left(\frac{\partial}{\partial t}\right) \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} \left(t^{n} \fint_{B_t(x)} \frac{f(b,z)}{(t^2-|z-x|^2)^{1/2}}\, \mathrm{d}z \right), \\ w[\varphi](t,x) &= \frac{1}{n!!} \left(\frac{\partial}{\partial t}\right) \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} \left(t^{n} \fint_{B_t(x)} \frac{\varphi(z)}{(t^2-|z-x|^2)^{1/2}}\, \mathrm{d}z \right), \end{aligned} \tag{11}
$$

where $\fint_A$ denotes the integral average over A and $j!!$ is the double factorial, which is defined for any $j \in \mathbb{N}, j \geq 1$ by

$$
j!! \doteq \begin{cases} j(j-2)\cdots 1 & \text{if } j \text{ is odd}, \\ j(j-2)\cdots 2 & \text{if } j \text{ is even}. \end{cases}
$$

We may now state the representation formulae in the multidimensional case. We consider separately the case when n is an odd integer and the case when n is an even integer.

Theorem 1.2. *Let $n \geqslant 3$ be an odd integer and let μ, ν^2 be nonnegative constants. Let us assume $f \in \mathscr{C}^{\frac{n+1}{2}}([0,\infty)\times\mathbb{R}^n)$ and $u_0 \in \mathscr{C}^{\frac{n+1}{2}+1}(\mathbb{R}^n)$, $u_1 \in \mathscr{C}^{\frac{n+1}{2}}(\mathbb{R}^n)$. Then, a representation formula for the solution of* (3) *is given by*

$$
\begin{aligned} u(t,x) = (1+t)^{-\frac{\mu}{2}} w[u_0](t,x) &+ \frac{1}{2^{\sqrt{\delta}-1}} \int_0^t w[u_0](s,x) K_0(t,0;s;\mu,\nu^2)\, \mathrm{d}s + \frac{1}{2^{\sqrt{\delta}-1}} \int_0^t w[u_1+\mu\, u_0](s,x) K_1(t,0;s;\mu,\nu^2)\, \mathrm{d}s \\ &+ \frac{1}{2^{\sqrt{\delta}-1}} \int_0^t \int_0^{t-b} w[f](s,x;b) E(t,0;b,s;\mu,\nu^2)\, \mathrm{d}s\, \mathrm{d}b, \end{aligned} \tag{12}
$$

where $w[u_0], w[u_1+\mu\, u_0]$ and $w[f]$ are defined by (10).

Theorem 1.3. *Let $n \geqslant 2$ be an even integer and let μ, ν^2 be nonnegative constants. Let us assume $f \in \mathscr{C}^{\frac{n}{2}+1}([0,\infty)\times\mathbb{R}^n)$ and $u_0 \in \mathscr{C}^{\frac{n}{2}+2}(\mathbb{R}^n)$, $u_1 \in \mathscr{C}^{\frac{n}{2}+1}(\mathbb{R}^n)$. Then, a representation formula for the solution of* (3) *is given by* (12)*, but with $w[u_0], w[u_1+\mu\, u_0]$ and $w[f]$ defined in this case by* (11).

The paper is organized as follows: in Section 2 we prove Theorem 1.1 considering first the inhomogeneous problem with vanishing data and, then, we use this case to study the corresponding homogeneous problem with nontrivial data; in Section 3 we consider the odd dimensional case and we prove Theorem 1.2; in particular, we use the method of spherical means to associate this case to the one-dimensional one; in Section 4 we consider the even dimensional case and we use the method of descent so that we reduce the problem to the one considered in Section 3; finally, in Section 5 we point out some final remarks to our results and the relations of (4) and (12) with the representation formulae for other models with variable coefficients.

2 One dimensional case

In this section we will prove Theorem 1.1. Since the Cauchy problem (3) is linear, we may consider separately the case with vanishing initial data and the homogeneous case. In particular, we will show that

$$
u^{\mathrm{ih}} = u^{\mathrm{ih}}(t,x) = \frac{1}{2\sqrt{\delta}} \int_0^t \int_{x-t+b}^{x+t-b} f(b,y) E(t,x;b,y;\mu,\nu^2)\, \mathrm{d}y\, \mathrm{d}b, \tag{13}
$$

solves

$$\begin{cases} u_{tt} - u_{xx} + \frac{\mu}{1+t} u_t + \frac{\nu^2}{(1+t)^2} u = f(t,x), & x \in \mathbb{R},\ t > 0, \\ u(0,x) = 0, & x \in \mathbb{R}, \\ u_t(0,x) = 0, & x \in \mathbb{R}, \end{cases} \tag{14}$$

while

$$\begin{aligned} u^{\mathrm{h}} = u^{\mathrm{h}}(t,x) = \frac{1}{2}(1+t)^{-\frac{\mu}{2}} \big(u_0(x+t) + u_0(x-t)\big) + \frac{1}{2\sqrt{\delta}} \int_{x-t}^{x+t} u_0(y) K_0(t,x;y;\mu,\nu^2)\,\mathrm{d}y \\ + \frac{1}{2\sqrt{\delta}} \int_{x-t}^{x+t} \big(u_1(y) + \mu\, u_0(y)\big) K_1(t,x;y;\mu,\nu^2)\,\mathrm{d}y \end{aligned} \tag{15}$$

solves

$$\begin{cases} u_{tt} - u_{xx} + \frac{\mu}{1+t} u_t + \frac{\nu^2}{(1+t)^2} u = 0, & x \in \mathbb{R},\ t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ u_t(0,x) = u_1(x), & x \in \mathbb{R}. \end{cases} \tag{16}$$

The remaining part of the section is organized as follows: in Subsection 2.1 we prove some fundamental properties of the kernel function $E = E(t,x;b,y)$; then, in Subsection 2.2 we prove that v^{ih} solves (14) in the classical sense (punctually); finally, in Subsection 2.3 we use the representation formula for the inhomogeneous problem with vanishing data in the 1d case to derive a representation formula for the corresponding homogeneous case.

2.1 The kernel function and its properties

In this subsection we investigate some properties of the kernel function $E = E(t,x;b,y)$. Let us begin by proving that E is a solution of the corresponding homogeneous wave equation with scale-invariant damping and mass with respect to the variables (t,x).

For the sake of readability,we introduce the function

$$z = z(t,x;b,y) \doteq \frac{(t-b)^2 - (y-x)^2}{(t+b+2)^2 - (y-x)^2}. \tag{17}$$

Proposition 2.1. *Let $b \in [0,t]$ and $y \in [x-t+b, x+t-b]$. Then,*

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + \frac{\mu}{1+t}\frac{\partial}{\partial t} + \frac{\nu^2}{(1+t)^2} \right) E(t,x;b,y;\mu,\nu^2) = 0. \tag{18}$$

Proof. Let us remark that for $b \in [0,t]$ and $y \in [x-t+b, x+t-b]$ it holds $z = z(t,x;b,y) \in [0,1)$. In particular, we may compute the hypergeometric function in (5) without considering the analytic continuation. Let us begin by computing the derivatives of E involved in (18).

Representation of $\partial_t^2 E(t,x;b,y;\mu,\nu^2)$ and $\partial_t E(t,x;b,y;\mu,\nu^2)$

Using the identities

$$\begin{aligned} \partial_t \big((t+b+2)^2 - (y-x)^2\big)^{\frac{\sqrt{\delta}-1}{2}} &= (\sqrt{\delta}-1)(t+b+2)\big((t+b+2)^2 - (y-x)^2\big)^{\frac{\sqrt{\delta}-1}{2}-1}, \\ \partial_t^2 \big((t+b+2)^2 - (y-x)^2\big)^{\frac{\sqrt{\delta}-1}{2}} &= (\sqrt{\delta}-1)\big((t+b+2)^2 - (y-x)^2\big)^{\frac{\sqrt{\delta}-1}{2}-1} \\ &\quad + (\sqrt{\delta}-1)(\sqrt{\delta}-1-2)(t+b+2)^2\big((t+b+2)^2 - (y-x)^2\big)^{\frac{\sqrt{\delta}-1}{2}-2}, \\ \partial_t \mathsf{F}\left(\tfrac{1-\sqrt{\delta}}{2}, \tfrac{1-\sqrt{\delta}}{2}; 1; z\right) &= \mathsf{F}_z\left(\tfrac{1-\sqrt{\delta}}{2}, \tfrac{1-\sqrt{\delta}}{2}; 1; z\right) \frac{\partial z}{\partial t}, \\ \partial_t^2 \mathsf{F}\left(\tfrac{1-\sqrt{\delta}}{2}, \tfrac{1-\sqrt{\delta}}{2}; 1; z\right) &= \mathsf{F}_{zz}\left(\tfrac{1-\sqrt{\delta}}{2}, \tfrac{1-\sqrt{\delta}}{2}; 1; z\right) \left(\frac{\partial z}{\partial t}\right)^2 + \mathsf{F}_z\left(\tfrac{1-\sqrt{\delta}}{2}, \tfrac{1-\sqrt{\delta}}{2}; 1; z\right) \frac{\partial^2 z}{\partial t^2}, \end{aligned}$$

we may calculate $\partial_t^2 E(t,x;b,y;\mu,\nu^2)$. Straightforward computations lead to

$$\begin{aligned}
\frac{\partial^2 E}{\partial t^2}(t,x;b,y;\mu,\nu^2) &= (1+b)^{\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}} \partial_t^2\Big((1+t)^{-\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}} \left((t+b+2)^2-(y-x)^2\right)^{\frac{\sqrt{\delta}-1}{2}} \mathsf{F}\Big(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\Big)\Big) \\
&= (1+b)^{\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}} (1+t)^{-\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}} \left((t+b+2)^2-(y-x)^2\right)^{\frac{\sqrt{\delta}-1}{2}} \\
&\quad \times \Big[\mathsf{F}_{zz}\Big(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\Big)\left(\frac{\partial z}{\partial t}\right)^2 + \mathsf{F}_z\Big(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\Big)\frac{\partial^2 z}{\partial t^2} \\
&\qquad + \big(-\tfrac{\mu}{2}+\tfrac{1-\sqrt{\delta}}{2}\big)\big(-\tfrac{\mu}{2}-1+\tfrac{1-\sqrt{\delta}}{2}\big)(1+t)^{-2}\mathsf{F}\Big(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\Big) \\
&\qquad + (\sqrt{\delta}-1)\left((t+b+2)^2-(y-x)^2\right)^{-1}\mathsf{F}\Big(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\Big) \\
&\qquad + (\sqrt{\delta}-1)(\sqrt{\delta}-1-2)\left((t+b+2)^2-(y-x)^2\right)^{-2}(t+b+2)^2\,\mathsf{F}\Big(\tfrac{\sqrt{\delta}-1}{2},\tfrac{\sqrt{\delta}-1}{2};1;z\Big) \\
&\qquad + 2(-\tfrac{\mu}{2}+\tfrac{1-\sqrt{\delta}}{2})(\sqrt{\delta}-1)(1+t)^{-1}\left((t+b+2)^2-(y-x)^2\right)^{-1}(t+b+2)\,\mathsf{F}\Big(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\Big) \\
&\qquad + 2(-\tfrac{\mu}{2}+\tfrac{1-\sqrt{\delta}}{2})(1+t)^{-1}\mathsf{F}_z\Big(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\Big)\frac{\partial z}{\partial t} \\
&\qquad + 2(\sqrt{\delta}-1)\left((t+b+2)^2-(y-x)^2\right)^{-1}(t+b+2)\,\mathsf{F}_z\Big(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\Big)\frac{\partial z}{\partial t}\Big]
\end{aligned}$$

and, similarly, to

$$\begin{aligned}
\frac{\partial E}{\partial t}(t,x;b,y;\mu,\nu^2) &= (1+b)^{\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}} \partial_t\Big((1+t)^{-\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}} \left((t+b+2)^2-(y-x)^2\right)^{\frac{\sqrt{\delta}-1}{2}} \mathsf{F}\Big(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\Big)\Big) \\
&= (1+b)^{\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}} (1+t)^{-\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}} \left((t+b+2)^2-(y-x)^2\right)^{\frac{\sqrt{\delta}-1}{2}} \\
&\quad \times \Big[\mathsf{F}_z\Big(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\Big)\frac{\partial z}{\partial t} + \big(-\tfrac{\mu}{2}+\tfrac{1-\sqrt{\delta}}{2}\big)(1+t)^{-1}\mathsf{F}\Big(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\Big) \\
&\qquad + (\sqrt{\delta}-1)\left((t+b+2)^2-(y-x)^2\right)^{-1}(t+b+2)\,\mathsf{F}\Big(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\Big)\Big]. \qquad (19)
\end{aligned}$$

Representation of $\partial_x^2 E(t,x;b,y;\mu,\nu^2)$

In order to calculate the partial derivative $\partial_x^2 E(t,x;b,y;\mu,\nu^2)$, we will employ the following relations

$$\begin{aligned}
\partial_x\big((t+b+2)^2-(y-x)^2\big)^{\frac{\sqrt{\delta}-1}{2}} &= (\sqrt{\delta}-1)(y-x)\big((t+b+2)^2-(y-x)^2\big)^{\frac{\sqrt{\delta}-1}{2}-1}, \\
\partial_x^2\big((t+b+2)^2-(y-x)^2\big)^{\frac{\sqrt{\delta}-1}{2}} &= (\sqrt{\delta}-1)(\sqrt{\delta}-1-2)(y-x)^2\big((t+b+2)^2-(y-x)^2\big)^{\frac{\sqrt{\delta}-1}{2}-2} \\
&\quad - (\sqrt{\delta}-1)\big((t+b+2)^2-(y-x)^2\big)^{\frac{\sqrt{\delta}-1}{2}-1}, \\
\partial_x\mathsf{F}\left(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\right) &= \mathsf{F}_z\left(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\right)\frac{\partial z}{\partial x}, \\
\partial_x^2\mathsf{F}\left(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\right) &= \mathsf{F}_{zz}\left(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\right)\left(\frac{\partial z}{\partial x}\right)^2 + \mathsf{F}_z\left(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\right)\frac{\partial^2 z}{\partial x^2}.
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{\partial^2 E}{\partial x^2}(t,x;b,y;\mu,\nu^2) &= (1+b)^{\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}} (1+t)^{-\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}} \partial_x^2\Big(\left((t+b+2)^2-(y-x)^2\right)^{\frac{\sqrt{\delta}-1}{2}} \mathsf{F}\Big(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\Big)\Big) \\
&= (1+b)^{\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}} (1+t)^{-\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}} \left((t+b+2)^2-(y-x)^2\right)^{\frac{\sqrt{\delta}-1}{2}} \\
&\quad \times \Big[\mathsf{F}_{zz}\Big(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\Big)\left(\frac{\partial z}{\partial x}\right)^2 + \mathsf{F}_z\Big(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\Big)\frac{\partial^2 z}{\partial x^2} \\
&\qquad + (\sqrt{\delta}-1)(\sqrt{\delta}-1-2)\left((t+b+2)^2-(y-x)^2\right)^{-2}(y-x)^2\,\mathsf{F}\Big(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\Big) \\
&\qquad - (\sqrt{\delta}-1)\left((t+b+2)^2-(y-x)^2\right)^{-1}\mathsf{F}\Big(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\Big) \\
&\qquad + 2(\sqrt{\delta}-1)\left((t+b+2)^2-(y-x)^2\right)^{-1}(y-x)\,\mathsf{F}_z\Big(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\Big)\frac{\partial z}{\partial x}\Big].
\end{aligned}$$

Combining now the expressions for the derivatives of E, we can now prove (18). Collecting the similar terms, we get

$$\begin{aligned}
&\frac{\partial^2 E}{\partial t^2}(t,x;b,y;\mu,\nu^2) - \frac{\partial^2 E}{\partial x^2}(t,x;b,y;\mu,\nu^2) + \frac{\mu}{1+t}\frac{\partial E}{\partial t}(t,x;b,y;\mu,\nu^2) + \frac{\nu^2}{(1+t)^2}E(t,x;b,y;\mu,\nu^2) \\
&= (1+b)^{\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}}(1+t)^{-\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}}\left((t+b+2)^2-(y-x)^2\right)^{\frac{\sqrt{\delta}-1}{2}} \\
&\quad \times \Bigg\{ \left[\left(\frac{\partial z}{\partial t}\right)^2 - \left(\frac{\partial z}{\partial x}\right)^2\right] \mathsf{F}_{zz}\left(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\right) \\
&\qquad + \Bigg[\frac{\partial^2 z}{\partial t^2} - \frac{\partial^2 z}{\partial x^2} + \big(2\big(-\tfrac{\mu}{2}+\tfrac{1-\sqrt{\delta}}{2}\big)+\mu\big)(1+t)^{-1}\frac{\partial z}{\partial t} \\
&\qquad\quad + 2(\sqrt{\delta}-1)\left((t+b+2)^2-(y-x)^2\right)^{-1}\left((t+b+2)\frac{\partial z}{\partial t} - (y-x)\frac{\partial z}{\partial x}\right)\Bigg]\mathsf{F}_z\left(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\right) \\
&\qquad + \Bigg[\underbrace{\Big(\big(-\tfrac{\mu}{2}+\tfrac{1-\sqrt{\delta}}{2}\big)\big(-\tfrac{\mu}{2}-1+\tfrac{1-\sqrt{\delta}}{2}\big)+\mu\big(-\tfrac{\mu}{2}+\tfrac{1-\sqrt{\delta}}{2}\big)+\nu^2\Big)}_{=0}(1+t)^{-2} \\
&\qquad\quad + \underbrace{\Big(2\big(-\tfrac{\mu}{2}+\tfrac{1-\sqrt{\delta}}{2}\big)(\sqrt{\delta}-1)+\mu(\sqrt{\delta}-1)\Big)}_{-(\sqrt{\delta}-1)^2}(1+t)^{-1}\left((t+b+2)^2-(y-x)^2\right)^{-1}(t+b+2) \\
&\qquad\quad + (\sqrt{\delta}-1)^2\left((t+b+2)^2-(y-x)^2\right)^{-1}\Bigg]\mathsf{F}\left(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\right)\Bigg\} \\
&= (1+b)^{\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}}(1+t)^{-\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}}\left((t+b+2)^2-(y-x)^2\right)^{\frac{\sqrt{\delta}-1}{2}} \\
&\quad \times \Bigg\{ \left[\left(\frac{\partial z}{\partial t}\right)^2 - \left(\frac{\partial z}{\partial x}\right)^2\right] \mathsf{F}_{zz}\left(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\right) \\
&\qquad + \Bigg[\frac{\partial^2 z}{\partial t^2} - \frac{\partial^2 z}{\partial x^2} + (1-\sqrt{\delta})(1+t)^{-1}\frac{\partial z}{\partial t} \\
&\qquad\quad + 2(\sqrt{\delta}-1)\left((t+b+2)^2-(y-x)^2\right)^{-1}\left((t+b+2)\frac{\partial z}{\partial t} - (y-x)\frac{\partial z}{\partial x}\right)\Bigg]\mathsf{F}_z\left(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\right) \\
&\qquad + \Bigg[(\sqrt{\delta}-1)^2\left((t+b+2)^2-(y-x)^2\right)^{-1} \\
&\qquad\quad - (\sqrt{\delta}-1)^2(1+t)^{-1}\left((t+b+2)^2-(y-x)^2\right)^{-1}(t+b+2)\Bigg]\mathsf{F}\left(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\right)\Bigg\}. \qquad (20)
\end{aligned}$$

Next, we will use that $\mathsf{F}\left(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\right)$ solves the differential equation

$$z(1-z)\mathsf{F}_{zz}\left(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\right) + \left[1-\left(2-\sqrt{\delta}\right)z\right]\mathsf{F}_z\left(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\right) - \left(\tfrac{1-\sqrt{\delta}}{2}\right)^2\mathsf{F}\left(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\right) = 0. \qquad (21)$$

For this purpose, we need first to rewrite the terms in the right hand side of the chain of equalities (20) that multiply $\mathsf{F}_{zz}\left(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\right)$, $\mathsf{F}_z\left(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\right)$ and $\mathsf{F}\left(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\right)$, respectively. Let us begin with the terms containing derivatives of the function $z = z(t,x;b,y)$. By elementary computations we get

$$\begin{aligned}
\frac{\partial z}{\partial t}(t,x;b,y) &= \frac{4(1+b)\left[(t-b)(t+b+2)+(y-x)^2\right]}{\left[(t+b+2)^2-(y-x)^2\right]^2}, \\
\frac{\partial^2 z}{\partial t^2}(t,x;b,y) &= \frac{8(1+b)\left[(1+t)\big((t+b+2)^2-(y-x)^2\big)-2\big((t-b)(t+b+2)+(y-x)^2\big)(t+b+2)\right]}{\left[(t+b+2)^2-(y-x)^2\right]^3}, \\
\frac{\partial z}{\partial x}(t,x;b,y) &= \frac{8(1+b)(1+t)(y-x)}{\left[(t+b+2)^2-(y-x)^2\right]^2}, \\
\frac{\partial^2 z}{\partial x^2}(t,x;b,y) &= -\frac{8(1+b)(1+t)\left[(t+b+2)^2+3(y-x)^2\right]}{\left[(t+b+2)^2-(y-x)^2\right]^3}.
\end{aligned}$$

Let us rewrite the factor that multiplies $\mathsf{F}_{zz}\left(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\right)$ in (20). Using the identity

$$\begin{aligned}
\big((A+B)^2-C^2\big)\big((A-B)^2-C^2\big) &= \big((A+C)^2-B^2\big)\big((A-C)^2-B^2\big) = \big((B+C)^2-A^2\big)\big((B+C)^2-A^2\big) \\
&= A^4+B^4+C^4-2A^2B^2-2A^2C^2-2B^2C^2 \qquad \text{for any } A,B,C\in\mathbb{R},
\end{aligned}$$

it follows

$$\left(\frac{\partial z}{\partial t}\right)^2-\left(\frac{\partial z}{\partial x}\right)^2=\frac{16(1+b)^2}{\big[(t+b+2)^2-(y-x)^2\big]^4}\Big\{\big[(t-b)(t+b+2)+(y-x)^2\big]^2-4(1+t)^2(y-x)^2\Big\}$$
$$=\frac{16(1+b)^2}{\big[(t+b+2)^2-(y-x)^2\big]^4}\Big\{\big[(1+t)^2-(1+b)^2+(y-x)^2\big]^2-4(1+t)^2(y-x)^2\Big\}$$
$$=\frac{16(1+b)^2}{\big[(t+b+2)^2-(y-x)^2\big]^4}\Big\{\big[((1+t)-(y-x))^2-(1+b)^2\big]\big[((1+t)+(y-x))^2-(1+b)^2\big]\Big\}$$
$$=\frac{16(1+b)^2}{\big[(t+b+2)^2-(y-x)^2\big]^4}\Big\{\big[((1+t)-(1+b))^2-(y-x)^2\big]\big[((1+t)+(1+b))^2-(y-x)^2\big]\Big\}$$
$$=\frac{16(1+b)^2\big[(t-b)^2-(y-x)^2\big]}{\big[(t+b+2)^2-(y-x)^2\big]^3}. \tag{22}$$

Since

$$z(1-z)=\frac{\big[(t-b)^2-(y-x)^2\big]\big[(t+b+2)^2-(t-b)^2\big]}{\big[(t+b+2)^2-(y-x)^2\big]^2}=\frac{4(1+t)(1+b)\big[(t-b)^2-(y-x)^2\big]}{\big[(t+b+2)^2-(y-x)^2\big]^2}, \tag{23}$$

combining (22) and (23), we find

$$\left(\frac{\partial z}{\partial t}\right)^2-\left(\frac{\partial z}{\partial x}\right)^2=\frac{4(1+b)\,z(1-z)}{(1+t)\big[(t+b+2)^2-(y-x)^2\big]}. \tag{24}$$

We rewrite now the factor multiplying $\mathsf{F}_z\left(\frac{1-\sqrt{\delta}}{2},\frac{1-\sqrt{\delta}}{2};1;z\right)$ in the right hand side of (20). We remark that

$$\frac{\partial^2 z}{\partial t^2}-\frac{\partial^2 z}{\partial x^2}=\frac{16(1+b)\big[(1+t)\big((t+b+2)^2+(y-x)^2\big)-\big((t-b)(t+b+2)+(y-x)^2\big)(t+b+2)\big]}{\big[(t+b+2)^2-(y-x)^2\big]^3}$$
$$=\frac{16(1+b)\big[(t+b+2)^2\big((1+t)-(t-b)\big)+(y-x)^2\big((1+t)-(t+b+2)\big)\big]}{\big[(t+b+2)^2-(y-x)^2\big]^3}=\frac{16(1+b)^2}{\big[(t+b+2)^2-(y-x)^2\big]^2}$$

and

$$(t+b+2)\,\frac{\partial z}{\partial t}-(y-x)\,\frac{\partial z}{\partial x}=(t+b+2)\frac{4(1+b)\big[(t-b)(t+b+2)+(y-x)^2\big]}{\big[(t+b+2)^2-(y-x)^2\big]^2}-(y-x)\frac{8(1+b)(1+t)(y-x)}{\big[(t+b+2)^2-(y-x)^2\big]^2}$$
$$=\frac{4(1+b)\big[(t-b)(t+b+2)^2+(y-x)^2\big((t+b+2)-2(1+t)\big]}{\big[(t+b+2)^2-(y-x)^2\big]^2}$$
$$=\frac{4(1+b)(t-b)}{\big[(t+b+2)^2-(y-x)^2\big]}.$$

Therefore,

$$\frac{\partial^2 z}{\partial t^2}-\frac{\partial^2 z}{\partial x^2}+(1+t)^{-1}\frac{\partial z}{\partial t}-2\big((t+b+2)^2-(y-x)^2\big)^{-1}\left((t+b+2)\,\frac{\partial z}{\partial t}-(y-x)\,\frac{\partial z}{\partial x}\right)$$
$$=\frac{16(1+b)^2}{\big[(t+b+2)^2-(y-x)^2\big]^2}+\frac{4(1+b)\big[(t-b)(t+b+2)+(y-x)^2\big]}{(1+t)\big[(t+b+2)^2-(y-x)^2\big]^2}-\frac{8(1+b)(t-b)}{\big[(t+b+2)^2-(y-x)^2\big]^2}$$
$$=\frac{4(1+b)}{(1+t)\big[(t+b+2)^2-(y-x)^2\big]^2}\big\{4(1+b)(1+t)+(t-b)(t+b+2)+(y-x)^2-2(t-b)(1+t)\big\}$$
$$=\frac{4(1+b)}{(1+t)\big[(t+b+2)^2-(y-x)^2\big]^2}\big\{4(1+b)(1+t)+(t+b+2)^2-2(1+b)(t+b+2)+(y-x)^2-2(t-b)(1+t)\big\}$$
$$=\frac{4(1+b)}{(1+t)\big[(t+b+2)^2-(y-x)^2\big]^2}\big\{2(1+b)\big[2(1+t)-(t+b+2)\big]+(t+b+2)^2+(y-x)^2-2(t-b)(1+t)\big\}$$
$$=\frac{4(1+b)}{(1+t)\big[(t+b+2)^2-(y-x)^2\big]^2}\big\{2(1+b)(t-b)+(t+b+2)^2+(y-x)^2-2(t-b)(1+t)\big\}$$
$$=\frac{4(1+b)}{(1+t)\big[(t+b+2)^2-(y-x)^2\big]^2}\big\{-2(t-b)^2+(t+b+2)^2+(y-x)^2\big\} \tag{25}$$

and

$$\begin{aligned}
&-(1+t)^{-1}\frac{\partial z}{\partial t}+2\big((t+b+2)^2-(y-x)^2\big)^{-1}\left((t+b+2)\,\frac{\partial z}{\partial t}-(y-x)\,\frac{\partial z}{\partial x}\right)\\
&\quad=-\frac{4(1+b)\big[(t-b)(t+b+2)+(y-x)^2\big]}{(1+t)\big[(t+b+2)^2-(y-x)^2\big]^2}+\frac{8(1+b)(t-b)}{\big[(t+b+2)^2-(y-x)^2\big]^2}\\
&\quad=\frac{4(1+b)}{(1+t)\big[(t+b+2)^2-(y-x)^2\big]^2}\big\{-(t-b)(t+b+2)-(y-x)^2+2(t-b)(1+t)\big\}\\
&\quad=\frac{4(1+b)}{(1+t)\big[(t+b+2)^2-(y-x)^2\big]^2}\big\{(t-b)^2-(y-x)^2\big\}=\frac{4(1+b)\,z}{(1+t)\big[(t+b+2)^2-(y-x)^2\big]}.
\end{aligned}\tag{26}$$

Moreover,

$$1-2z=\frac{(t+b+2)^2-2(t-b)^2+(y-x)^2}{(t+b+2)^2-(y-x)^2}.\tag{27}$$

Also, combining (25), (26) and (27), we have

$$\begin{aligned}
&\frac{\partial^2 z}{\partial t^2}-\frac{\partial^2 z}{\partial x^2}+(1-\sqrt{\delta})(1+t)^{-1}\frac{\partial z}{\partial t}+2(\sqrt{\delta}-1)\big((t+b+2)^2-(y-x)^2\big)^{-1}\left((t+b+2)\,\frac{\partial z}{\partial t}-(y-x)\,\frac{\partial z}{\partial x}\right)\\
&\quad=\frac{4(1+b)\,(1-2z)}{(1+t)\big[(t+b+2)^2-(y-x)^2\big]}+\frac{4\sqrt{\delta}(1+b)\,z}{(1+t)\big[(t+b+2)^2-(y-x)^2\big]}\\
&\quad=\frac{4(1+b)\,\big[1-2z+\sqrt{\delta}z\big]}{(1+t)\big[(t+b+2)^2-(y-x)^2\big]}=\frac{4(1+b)\,\big[1-\big(2-\sqrt{\delta}\big)z\big]}{(1+t)\big[(t+b+2)^2-(y-x)^2\big]},
\end{aligned}\tag{28}$$

which is the factor that multiplies the term $\mathsf{F}_z\left(\frac{1-\sqrt{\delta}}{2},\frac{1-\sqrt{\delta}}{2};1;z\right)$ in (20). Finally, we determine the factor multiplying $\mathsf{F}\left(\frac{1-\sqrt{\delta}}{2},\frac{1-\sqrt{\delta}}{2};1;z\right)$ in (20), namely

$$\begin{aligned}
(\sqrt{\delta}-1)^2\big((t+b+2)^2-(y-x)^2\big)^{-1}\left(1-\frac{t+b+2}{1+t}\right)&=-(\sqrt{\delta}-1)^2\frac{(1+b)}{(1+t)\big[(t+b+2)^2-(y-x)^2\big]}\\
&=-\frac{4(1+b)}{(1+t)\big[(t+b+2)^2-(y-x)^2\big]}\big(\tfrac{\sqrt{\delta}-1}{2}\big)^2.
\end{aligned}\tag{29}$$

Summarizing, if we plug (24), (28) and (29) in (20), we arrive at

$$\begin{aligned}
&\frac{\partial^2 E}{\partial t^2}(t,x;b,y;\mu,\nu^2)-\frac{\partial^2 E}{\partial x^2}(t,x;b,y;\mu,\nu^2)+\frac{\mu}{1+t}\frac{\partial E}{\partial t}(t,x;b,y;\mu,\nu^2)+\frac{\nu^2}{(1+t)^2}E(t,x;b,y;\mu,\nu^2)\\
&\quad=\frac{4(1+b)}{(1+t)\big[(t+b+2)^2-(y-x)^2\big]}\bigg\{z(1-z)\mathsf{F}_{zz}\left(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\right)+\big[1-\big(2-\sqrt{\delta}\big)z\big]\mathsf{F}_z\left(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\right)\\
&\qquad\qquad-\big(\tfrac{\sqrt{\delta}-1}{2}\big)^2\mathsf{F}\left(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\right)\bigg\}=0,
\end{aligned}$$

where in the last step we used (21). This completes the proof (18). □

So far, we proved that the kernel function E is a solution of the homogeneous wave equation with scale-invariant damping and mass with respect to (t,x). As consequence, we prove now that E is a solution of the adjoint equation of the homogeneous wave equation with scale-invariant damping and mass with respect to (b,y).

Corollary 2.2. *Let $b\in[0,t]$ and $y\in[x-t+b,x+t-b]$. Then,*

$$\left(\frac{\partial^2}{\partial b^2}-\frac{\partial^2}{\partial y^2}-\frac{\mu}{1+b}\frac{\partial}{\partial b}+\frac{\mu+\nu^2}{(1+t)^2}\right)E(t,x;b,y;\mu,\nu^2)=0.\tag{30}$$

Proof. Let us begin by remarking the identity

$$E(t,x;b,y;\mu,\nu^2)=(1+b)^{\mu}(1+t)^{-\mu}E(b,y;t,x;\mu,\nu^2).$$

Hence, Proposition 2.1 implies

$$\begin{aligned}
\frac{\partial^2 E}{\partial y^2}(t,x;b,y;\mu,\nu^2) &= (1+b)^{\mu}(1+t)^{-\mu}\frac{\partial^2 E}{\partial y^2}(b,y;t,x;\mu,\nu^2)\\
&= (1+b)^{\mu}(1+t)^{-\mu}\left[\frac{\partial^2}{\partial b^2}+\frac{\mu}{1+b}\frac{\partial}{\partial b}+\frac{\nu^2}{(1+b)^2}\right]E(b,y;t,x;\mu,\nu^2)\\
&= (1+b)^{\mu}(1+t)^{-\mu}\left[\frac{\partial^2}{\partial b^2}+\frac{\mu}{1+b}\frac{\partial}{\partial b}+\frac{\nu^2}{(1+b)^2}\right]\Big((1+b)^{-\mu}(1+t)^{\mu}E(t,x;b,y;\mu,\nu^2)\Big)\\
&= (1+b)^{\mu}\Bigg[(1+b)^{-\mu}\frac{\partial^2}{\partial b^2}-2\mu(1+b)^{-\mu-1}\frac{\partial}{\partial b}+\mu(\mu+1)(1+b)^{-\mu-2}\\
&\qquad +\mu(1+b)^{-\mu-1}\frac{\partial}{\partial b}-\mu^2(1+b)^{-\mu-2}+\nu^2(1+b)^{-\mu-2}\Bigg]E(t,x;b,y;\mu,\nu^2)\\
&= \left[\frac{\partial^2}{\partial b^2}-\frac{\mu}{1+b}\frac{\partial}{\partial b}+\frac{\mu+\nu^2}{(1+b)^2}\right]E(t,x;b,y;\mu,\nu^2),
\end{aligned}$$

which is exactly (30). □

Lemma 2.3. *Let $b\in[0,t]$. Then,*

$$\left[\frac{\partial E}{\partial t}(t,x;b,y;\mu,\nu^2)-\frac{\partial E}{\partial x}(t,x;b,y;\mu,\nu^2)\right]_{y=x+t-b}+2^{\sqrt{\delta}-2}\mu(1+t)^{-\frac{\mu}{2}-1}(1+b)^{\frac{\mu}{2}}=0,$$

$$\left[\frac{\partial E}{\partial t}(t,x;b,y;\mu,\nu^2)+\frac{\partial E}{\partial x}(t,x;b,y;\mu,\nu^2)\right]_{y=x-t+b}+2^{\sqrt{\delta}-2}\mu(1+t)^{-\frac{\mu}{2}-1}(1+b)^{\frac{\mu}{2}}=0.$$

Proof. Using the identity $\mathsf{F}_z(\alpha,\beta;\gamma;z)=\frac{\alpha\beta}{\gamma}\mathsf{F}(\alpha+1,\beta+1;\gamma+1;z)$, by (19) it follows

$$\begin{aligned}
\frac{\partial E}{\partial t}(t,x;b,y;\mu,\nu^2) &= (1+b)^{\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}}(1+t)^{-\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}}\left((t+b+2)^2-(y-x)^2\right)^{\frac{\sqrt{\delta}-1}{2}}\\
&\times\Bigg[(1-\sqrt{\delta})^2(1+b)\big((t-b)(t+b+2)+(y-x)^2\big)\big((t+b+2)^2-(y-x)^2\big)^{-2}\mathsf{F}\Big(\tfrac{3-\sqrt{\delta}}{2},\tfrac{3-\sqrt{\delta}}{2};2;z\Big)\\
&\quad+\big(-\tfrac{\mu}{2}+\tfrac{1-\sqrt{\delta}}{2}\big)(1+t)^{-1}\mathsf{F}\Big(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\Big)\\
&\quad+(\sqrt{\delta}-1)\left((t+b+2)^2-(y-x)^2\right)^{-1}(t+b+2)\,\mathsf{F}\Big(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\Big)\Bigg].
\end{aligned}$$

As $z(t,x;b,x\pm(t-b))=0$ and $\mathsf{F}(\alpha,\beta;\gamma;0)=1$, then,

$$\begin{aligned}
&\frac{\partial E}{\partial t}(t,x;b,y;\mu,\nu^2)\Big|_{y=x\pm(t-b)}\\
&= 2^{\sqrt{\delta}-1}(1+b)^{\frac{\mu}{2}-1}(1+t)^{-\frac{\mu}{2}-1}\Big[2^{-3}(1-\sqrt{\delta})^2(t-b)+\big(-\tfrac{\mu}{2}+\tfrac{1-\sqrt{\delta}}{2}\big)(1+b)+2^{-2}(\sqrt{\delta}-1)(t+b+2)\Big].
\end{aligned}\tag{31}$$

On the other hand,

$$\begin{aligned}
\frac{\partial E}{\partial x}(t,x;b,y;\mu,\nu^2) &= (1+b)^{\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}}(1+t)^{-\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}}\partial_x\Big(\left((t+b+2)^2-(y-x)^2\right)^{\frac{\sqrt{\delta}-1}{2}}\mathsf{F}\Big(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\Big)\Big)\\
&= (1+b)^{\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}}(1+t)^{-\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}}\left((t+b+2)^2-(y-x)^2\right)^{\frac{\sqrt{\delta}-1}{2}}\\
&\times\Big[\mathsf{F}_z\Big(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\Big)\frac{\partial z}{\partial x}+(\sqrt{\delta}-1)\left((t+b+2)^2-(y-x)^2\right)^{-1}(y-x)\,\mathsf{F}\Big(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\Big)\Big]\\
&= (1+b)^{\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}}(1+t)^{-\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}}\left((t+b+2)^2-(y-x)^2\right)^{\frac{\sqrt{\delta}-1}{2}}\\
&\times\Big[2(1-\sqrt{\delta})^2(1+b)(1+t)\left((t+b+2)^2-(y-x)^2\right)^{-2}(y-x)\,\mathsf{F}\Big(\tfrac{3-\sqrt{\delta}}{2},\tfrac{3-\sqrt{\delta}}{2};2;z\Big)\\
&\quad+(\sqrt{\delta}-1)\left((t+b+2)^2-(y-x)^2\right)^{-1}(y-x)\,\mathsf{F}\Big(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\Big)\Big]
\end{aligned}$$

implies

$$\frac{\partial E}{\partial x}(t,x;b,y;\mu,\nu^2)\Big|_{y=x\pm(t-b)}=\pm2^{\sqrt{\delta}-1}(1+b)^{\frac{\mu}{2}-1}(1+t)^{-\frac{\mu}{2}-1}\Big[2^{-3}(1-\sqrt{\delta})^2(t-b)+2^{-2}(\sqrt{\delta}-1)(t-b)\Big].\tag{32}$$

Consequently, combining (31) and (32), we have

$$
\begin{aligned}
&\left[\frac{\partial E}{\partial t}(t,x;b,y;\mu,\nu^2) \mp \frac{\partial E}{\partial x}(t,x;b,y;\mu,\nu^2)\right]_{y=x\pm(t-b)} \\
&\quad = 2^{\sqrt{\delta}-1}(1+b)^{\frac{\mu}{2}-1}(1+t)^{-\frac{\mu}{2}-1}\Big[2^{-3}(1-\sqrt{\delta})^2(t-b) + \big(-\tfrac{\mu}{2}+\tfrac{1-\sqrt{\delta}}{2}\big)(1+b) + 2^{-2}(\sqrt{\delta}-1)(t+b+2) \\
&\qquad\quad - 2^{-3}(1-\sqrt{\delta})^2(t-b) - 2^{-2}(\sqrt{\delta}-1)(t-b)\Big] \\
&\quad = -2^{\sqrt{\delta}-2}\mu(1+b)^{\frac{\mu}{2}}(1+t)^{-\frac{\mu}{2}-1},
\end{aligned}
$$

which are the desired estimates. The proof is complete. □

2.2 The inhomogeneous problem with vanishing data

In this subsection we prove that u^{ih} is a solution of (14). Let us determine first the time derivative of u^{ih}. Hence,

$$
\begin{aligned}
\partial_t u^{\text{ih}}(t,x) &= \frac{1}{2^{\sqrt{\delta}}}\int_0^t \partial_t\bigg(\int_{x-t+b}^{x+t-b} f(b,y)E(t,x;b,y;\mu,\nu^2)\,\mathrm{d}y\bigg)\mathrm{d}b \\
&= \frac{1}{2^{\sqrt{\delta}}}\int_0^t\int_{x-t+b}^{x+t-b} f(b,y)\frac{\partial E}{\partial t}(t,x;b,y;\mu,\nu^2)\,\mathrm{d}y\,\mathrm{d}b + \frac{1}{2^{\sqrt{\delta}}}\int_0^t f(b,x+t-b)E(t,x;b,x+t-b;\mu,\nu^2)\,\mathrm{d}b \\
&\quad + \frac{1}{2^{\sqrt{\delta}}}\int_0^t f(b,x-t+b)E(t,x;b,x-t+b;\mu,\nu^2)\,\mathrm{d}b.
\end{aligned}
$$

Therefore, it follows immediately that $u^{\text{ih}}(0,x) = \partial_t u^{\text{ih}}(0,x) = 0$. It remains to prove that u^{ih} solves the differential equation. Moreover,

$$
\begin{aligned}
E(t,x;b,x\pm(t-b);\mu,\nu^2) &= (1+t)^{-\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}}(1+b)^{\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}}\big((t+b+2)^2-(t-b)^2\big)^{\frac{\sqrt{\delta}-1}{2}}\mathsf{F}\Big(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;0\Big) \\
&= (1+t)^{-\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}}(1+b)^{\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}}\big(4(1+t)(1+b)\big)^{\frac{\sqrt{\delta}-1}{2}} \\
&= 2^{\sqrt{\delta}-1}(1+t)^{-\frac{\mu}{2}}(1+b)^{\frac{\mu}{2}} \qquad (33)
\end{aligned}
$$

implies

$$
\begin{aligned}
\partial_t u^{\text{ih}}(t,x) &= \frac{1}{2^{\sqrt{\delta}}}\int_0^t\int_{x-t+b}^{x+t-b} f(b,y)\frac{\partial E}{\partial t}(t,x;b,y;\mu,\nu^2)\,\mathrm{d}y\,\mathrm{d}b \\
&\quad + \frac{1}{2}\int_0^t \big[f(b,x+t-b)+f(b,x-t+b)\big](1+t)^{-\frac{\mu}{2}}(1+b)^{\frac{\mu}{2}}\,\mathrm{d}b. \qquad (34)
\end{aligned}
$$

We may calculate now the second order derivative with respect to t. Differentiating the last relation, we get

$$
\begin{aligned}
\partial_t^2 u^{\text{ih}}(t,x) &= \frac{1}{2^{\sqrt{\delta}}}\int_0^t \partial_t\bigg(\int_{x-t+b}^{x+t-b} f(b,y)\frac{\partial E}{\partial t}(t,x;b,y;\mu,\nu^2)\,\mathrm{d}y\bigg)\mathrm{d}b \\
&\quad + \frac{1}{2}\int_0^t \partial_t\Big(\big[f(b,x+t-b)+f(b,x-t+b)\big](1+t)^{-\frac{\mu}{2}}(1+b)^{\frac{\mu}{2}}\Big)\,\mathrm{d}b + f(t,x) \\
&= \frac{1}{2^{\sqrt{\delta}}}\int_0^t\int_{x-t+b}^{x+t-b} f(b,y)\frac{\partial^2 E}{\partial t^2}(t,x;b,y;\mu,\nu^2)\,\mathrm{d}y\,\mathrm{d}b \\
&\quad + \frac{1}{2^{\sqrt{\delta}}}\int_0^t \Big[f(b,y)\frac{\partial E}{\partial t}(t,x;b,y;\mu,\nu^2)\Big]_{y=x+t-b}\mathrm{d}b + \frac{1}{2^{\sqrt{\delta}}}\int_0^t \Big[f(b,y)\frac{\partial E}{\partial t}(t,x;b,y;\mu,\nu^2)\Big]_{y=x-t+b}\mathrm{d}b \\
&\quad - \frac{\mu}{4}\int_0^t \big[f(b,x+t-b)+f(b,x-t+b)\big](1+t)^{-\frac{\mu}{2}-1}(1+b)^{\frac{\mu}{2}}\,\mathrm{d}b \\
&\quad + \frac{1}{2}\int_0^t \Big[\frac{\partial f}{\partial x}(b,x+t-b) - \frac{\partial f}{\partial x}f(b,x-t+b)\Big](1+t)^{-\frac{\mu}{2}}(1+b)^{\frac{\mu}{2}}\,\mathrm{d}b + f(t,x). \qquad (35)
\end{aligned}
$$

The next step is to calculate the derivative of order two with respect to x of u^{ih}. Let us begin with the derivative of order one:

$$\begin{aligned}\partial_x u^{\mathrm{ih}}(t,x) &= \frac{1}{2^{\sqrt{\delta}}}\int_0^t\int_{x-t+b}^{x+t-b} f(b,y)\frac{\partial E}{\partial x}(t,x;b,y;\mu,\nu^2)\,\mathrm{d}y\,\mathrm{d}b + \frac{1}{2^{\sqrt{\delta}}}\int_0^t f(b,x+t-b)E(t,x;b,x+t-b;\mu,\nu^2)\,\mathrm{d}b \\ &\quad - \frac{1}{2^{\sqrt{\delta}}}\int_0^t f(b,x-t+b)E(t,x;b,x-t+b;\mu,\nu^2)\,\mathrm{d}b \\ &= \frac{1}{2^{\sqrt{\delta}}}\int_0^t\int_{x-t+b}^{x+t-b} f(b,y)\frac{\partial E}{\partial x}(t,x;b,y;\mu,\nu^2)\,\mathrm{d}y\,\mathrm{d}b \\ &\quad + \frac{1}{2}\int_0^t \big[f(b,x+t-b)-f(b,x-t+b)\big](1+t)^{-\frac{\mu}{2}}(1+b)^{\frac{\mu}{2}}\,\mathrm{d}b,\end{aligned}$$

where in the second equality we used again (33). A further derivation with respect to x of the last expression provides

$$\begin{aligned}\partial_x^2 u^{\mathrm{ih}}(t,x) &= \frac{1}{2^{\sqrt{\delta}}}\int_0^t\int_{x-t+b}^{x+t-b} f(b,y)\frac{\partial^2 E}{\partial x^2}(t,x;b,y;\mu,\nu^2)\,\mathrm{d}y\,\mathrm{d}b \\ &\quad + \frac{1}{2^{\sqrt{\delta}}}\int_0^t \left[f(b,y)\frac{\partial E}{\partial x}(t,x;b,y;\mu,\nu^2)\right]_{y=x+t-b}\mathrm{d}b - \frac{1}{2^{\sqrt{\delta}}}\int_0^t \left[f(b,y)\frac{\partial E}{\partial x}(t,x;b,y;\mu,\nu^2)\right]_{y=x-t+b}\mathrm{d}b \\ &\quad + \frac{1}{2}\int_0^t \left[\frac{\partial f}{\partial x}(b,x+t-b)-\frac{\partial f}{\partial x}f(b,x-t+b)\right](1+t)^{-\frac{\mu}{2}}(1+b)^{\frac{\mu}{2}}\,\mathrm{d}b. \qquad (36)\end{aligned}$$

Combining (13), (34), (35) and (36), we arrive at

$$\begin{aligned}&\partial_t^2 u^{\mathrm{ih}}(t,x) - \partial_x^2 u^{\mathrm{ih}}(t,x) + \tfrac{\mu}{1+t}\partial_t u^{\mathrm{ih}}(t,x) + \tfrac{\nu^2}{(1+t)^2}u^{\mathrm{ih}}(t,x) \\ &= \frac{1}{2^{\sqrt{\delta}}}\int_0^t\int_{x-t+b}^{x+t-b} f(b,y)\left[\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}+\frac{\mu}{1+t}\frac{\partial}{\partial t}+\frac{\nu^2}{(1+t)^2}\right]E(t,x;b,y;\mu,\nu^2)\,\mathrm{d}y\,\mathrm{d}b + f(t,x) \\ &\quad + \frac{1}{2^{\sqrt{\delta}}}\int_0^t \left[f(b,y)\left(\frac{\partial E}{\partial t}(t,x;b,y;\mu,\nu^2)-\frac{\partial E}{\partial x}(t,x;b,y;\mu,\nu^2)+2^{\sqrt{\delta}-2}\mu(1+t)^{-\frac{\mu}{2}-1}(1+b)^{\frac{\mu}{2}}\right)\right]_{y=x+t-b}\mathrm{d}b \\ &\quad + \frac{1}{2^{\sqrt{\delta}}}\int_0^t \left[f(b,y)\left(\frac{\partial E}{\partial t}(t,x;b,y;\mu,\nu^2)+\frac{\partial E}{\partial x}(t,x;b,y;\mu,\nu^2)+2^{\sqrt{\delta}-2}\mu(1+t)^{-\frac{\mu}{2}-1}(1+b)^{\frac{\mu}{2}}\right)\right]_{y=x-t+b}\mathrm{d}b.\end{aligned}$$

However, by Proposition 2.1 and Lemma 2.3 it follows that the all integrands on the right hand side of the last equality vanish. Consequently,

$$\partial_t^2 u^{\mathrm{ih}}(t,x) - \partial_x^2 u^{\mathrm{ih}}(t,x) + \tfrac{\mu}{1+t}\partial_t u^{\mathrm{ih}}(t,x) + \tfrac{\nu^2}{(1+t)^2}u^{\mathrm{ih}}(t,x) = f(t,x).$$

So, we proved that u^{ih} solves (14).

2.3 The homogeneous problem with nontrivial data

In this subsection we will prove that u^{h} defined in (15) is a solution of (16). For this purpose, we consider the function $w = w(t,x) \doteq u(t,x) - u_0(x) - tu_1(x)$. If u solves (15), then, w solves (14) with

$$f = f(t,x) = u_0''(x) + tu_1''(x) - \left(\frac{\nu^2}{(1+t)^2}u_0(x) + \frac{\mu}{1+t}u_1(x) + \frac{\nu^2\,t}{(1+t)^2}u_1(x)\right).$$

Therefore, according to the representation formula derived in Subsection 2.2, we obtain

$$\begin{aligned}w(t,x) &= \frac{1}{2^{\sqrt{\delta}}}\int_0^t\int_{x-t+b}^{x+t-b}\left[u_0''(y)+bu_1''(y)-\left(\frac{\nu^2}{(1+b)^2}u_0(y)+\frac{\mu}{1+b}u_1(y)+\frac{\nu^2\,b}{(1+b)^2}u_1(y)\right)\right]E(t,x;b,y;\mu,\nu^2)\,\mathrm{d}y\,\mathrm{d}b \\ &\doteq I_1+I_2+I_3+I_4+I_5.\end{aligned}$$

Now we will manipulate I_1, I_2 in order to get the cancellation of some terms in the expression of w and, then, u. Let us begin with $I_1 = \frac{1}{2\sqrt{\delta}} \int_0^t \int_{x-t+b}^{x+t-b} u_0''(y) E(t,x;b,y;\mu,\nu^2)\,\mathrm{d}y\,\mathrm{d}b$. Using twice integration by parts and Corollary 2.2, we find

$$\begin{aligned}
&\int_{x-t+b}^{x+t-b} u_0''(y) E(t,x;b,y;\mu,\nu^2)\,\mathrm{d}y \\
&= \Big[u_0'(y) E(t,x;b,y;\mu,\nu^2) - u_0(y)\frac{\partial E}{\partial y}(t,x;b,y;\mu,\nu^2)\Big]_{y=x-t+b}^{y=x+t-b} + \int_{x-t+b}^{x+t-b} u_0(y)\frac{\partial^2 E}{\partial y^2}(t,x;b,y;\mu,\nu^2)\,\mathrm{d}y \\
&= \Big[u_0'(y) E(t,x;b,y;\mu,\nu^2) - u_0(y)\frac{\partial E}{\partial y}(t,x;b,y;\mu,\nu^2)\Big]_{y=x-t+b}^{y=x+t-b} + \int_{x-t+b}^{x+t-b} u_0(y)\frac{\partial^2 E}{\partial b^2}(t,x;b,y;\mu,\nu^2)\,\mathrm{d}y \\
&\quad + \int_{x-t+b}^{x+t-b} u_0(y)\Big[-\frac{\mu}{1+b}\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2) + \frac{\mu+\nu^2}{(1+b)^2}E(t,x;b,y;\mu,\nu^2)\Big]\,\mathrm{d}y.
\end{aligned}$$

Hence,

$$\begin{aligned}
I_1 &= \frac{1}{2\sqrt{\delta}}\int_0^t \Big[u_0'(y) E(t,x;b,y;\mu,\nu^2) - u_0(y)\frac{\partial E}{\partial y}(t,x;b,y;\mu,\nu^2)\Big]_{y=x-t+b}^{y=x+t-b}\mathrm{d}b \\
&\quad + \frac{1}{2\sqrt{\delta}}\int_0^t\int_{x-t+b}^{x+t-b} u_0(y)\frac{\partial^2 E}{\partial b^2}(t,x;b,y;\mu,\nu^2)\,\mathrm{d}y\,\mathrm{d}b \\
&\quad - \frac{1}{2\sqrt{\delta}}\int_0^t\int_{x-t+b}^{x+t-b} u_0(y)\frac{\mu}{1+b}\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2)\,\mathrm{d}y\,\mathrm{d}b \\
&\quad + \frac{1}{2\sqrt{\delta}}\int_0^t\int_{x-t+b}^{x+t-b} u_0(y)\frac{\mu+\nu^2}{(1+b)^2}E(t,x;b,y;\mu,\nu^2)\,\mathrm{d}y\,\mathrm{d}b \doteq J_1+J_2+J_3+J_4.
\end{aligned}$$

Let us rewrite J_2, J_3 in a more suitable way. By using Fubini's theorem and integration by parts, we get

$$\begin{aligned}
J_2 &= \frac{1}{2\sqrt{\delta}}\int_0^t\int_{x-t+b}^{x+t-b} u_0(y)\frac{\partial^2 E}{\partial b^2}(t,x;b,y;\mu,\nu^2)\,\mathrm{d}y\,\mathrm{d}b = \frac{1}{2\sqrt{\delta}}\int_{x-t}^{x+t} u_0(y)\int_0^{t-|x-y|}\frac{\partial^2 E}{\partial b^2}(t,x;b,y;\mu,\nu^2)\,\mathrm{d}b\,\mathrm{d}y \\
&= \frac{1}{2\sqrt{\delta}}\int_{x-t}^{x+t} u_0(y)\Big[\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2)\Big]_{b=0}^{b=t-|x-y|}\mathrm{d}y, \\
J_3 &= -\frac{1}{2\sqrt{\delta}}\int_0^t\int_{x-t+b}^{x+t-b} u_0(y)\frac{\mu}{1+b}\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2)\,\mathrm{d}y\,\mathrm{d}b = -\frac{1}{2\sqrt{\delta}}\int_{x-t}^{x+t} u_0(y)\int_0^{t-|x-y|}\frac{\mu}{1+b}\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2)\,\mathrm{d}b\,\mathrm{d}y \\
&= -\frac{1}{2\sqrt{\delta}}\int_{x-t}^{x+t} u_0(y)\Big[\frac{\mu}{1+b}E(t,x;b,y;\mu,\nu^2)\Big]_{b=0}^{b=t-|x-y|}\mathrm{d}y + \frac{1}{2\sqrt{\delta}}\int_{x-t}^{x+t} u_0(y)\int_0^{t-|x-y|}\frac{\partial}{\partial b}\Big(\frac{\mu}{1+b}\Big)E(t,x;b,y;\mu,\nu^2)\,\mathrm{d}b\,\mathrm{d}y \\
&= -\frac{1}{2\sqrt{\delta}}\int_{x-t}^{x+t} u_0(y)\Big[\frac{\mu}{1+b}E(t,x;b,y;\mu,\nu^2)\Big]_{b=0}^{b=t-|x-y|}\mathrm{d}y - \frac{1}{2\sqrt{\delta}}\int_0^t\int_{x-t+b}^{x+t-b} u_0(y)\frac{\mu}{(1+b)^2}E(t,x;b,y;\mu,\nu^2)\,\mathrm{d}b\,\mathrm{d}y.
\end{aligned}$$

In particular, from the last relation we see that there is a cancellation between one term in J_3 and another one in J_4. Combining the expressions for J_2, J_3 that we have just proved, we obtain

$$I_1 = J_1+J_2+J_3+J_4 = J_1 + \frac{1}{2\sqrt{\delta}}\int_{x-t}^{x+t} u_0(y)\Big[\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2) - \frac{\mu}{1+b}E(t,x;b,y;\mu,\nu^2)\Big]_{b=0}^{b=t-|x-y|}\mathrm{d}y - I_3. \tag{37}$$

Now, we remark that

$$u_0'(x\pm(t-b)) = \mp\frac{\partial}{\partial b}\big(u_0(x\pm(t-b))\big).$$

Hence, we have

$$
\begin{aligned}
&\int_0^t \Big[u_0'(y)E(t,x;b,y;\mu,\nu^2)\Big]_{y=x-t+b}^{y=x+t-b} \mathrm{d}b \\
&= -\int_0^t \Big[\frac{\partial}{\partial b}\big(u_0(x+t-b)\big)E(t,x;b,x+t-b;\mu,\nu^2) + \frac{\partial}{\partial b}\big(u_0(x-t+b)\big)E(t,x;b,x-t+b;\mu,\nu^2)\Big]\mathrm{d}b \\
&= -\Big[u_0(x+t-b)E(t,x;b,x+t-b;\mu,\nu^2) + u_0(x-t+b)E(t,x;b,x-t+b;\mu,\nu^2)\Big]_{b=0}^{b=t} \\
&\quad + \int_0^t \Big[u_0(x+t-b)\frac{\partial}{\partial b}\Big(E(t,x;b,x+t-b;\mu,\nu^2)\Big) + u_0(x-t+b)\frac{\partial}{\partial b}\Big(E(t,x;b,x-t+b;\mu,\nu^2)\Big)\Big]\mathrm{d}b \\
&= -2\, u_0(x)E(t,x;t,x;\mu,\nu^2) + \Big[u_0(x+t)E(t,x;0,x+t;\mu,\nu^2) + u_0(x-t)E(t,x;0,x-t;\mu,\nu^2)\Big] \\
&\quad + \int_0^t \Big[u_0(y)\Big(\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2) - \frac{\partial E}{\partial y}(t,x;b,y;\mu,\nu^2)\Big)\Big]_{y=x+t-b} \mathrm{d}b \\
&\quad + \int_0^t \Big[u_0(y)\Big(\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2) + \frac{\partial E}{\partial y}(t,x;b,y;\mu,\nu^2)\Big)\Big]_{y=x-t+b} \mathrm{d}b.
\end{aligned}
$$

Since

$$
E(t,x;t,x;\mu,\nu^2) = (1+t)^{-\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}}(1+t)^{\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}}\left(2^2(1+t)^2\right)^{\frac{\sqrt{\delta}-1}{2}} \mathsf{F}\left(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;0\right) = 2^{\sqrt{\delta}-1}, \tag{38}
$$

$$
E(t,x;0,x\pm t;\mu,\nu^2) = (1+t)^{-\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}}\left((t+2)^2-t^2\right)^{\frac{\sqrt{\delta}-1}{2}} \mathsf{F}\left(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;0\right) = 2^{\sqrt{\delta}-1}(1+t)^{-\frac{\mu}{2}},
$$

plugging the right hand side of the last chain of equalities in J_1, we get

$$
\begin{aligned}
J_1 &= \frac{1}{2^{\sqrt{\delta}}}\int_0^t \Big[u_0'(y)E(t,x;b,y;\mu,\nu^2) - u_0(y)\frac{\partial E}{\partial y}(t,x;b,y;\mu,\nu^2)\Big]_{y=x-t+b}^{y=x+t-b} \mathrm{d}b \\
&= -u_0(x) + \frac{1}{2}(1+t)^{-\frac{\mu}{2}}\left[u_0(x+t)+u_0(x-t)\right] \\
&\quad + \frac{1}{2^{\sqrt{\delta}}}\int_0^t \Big[u_0(y)\Big(\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2) - 2\frac{\partial E}{\partial y}(t,x;b,y;\mu,\nu^2)\Big)\Big]_{y=x+t-b} \mathrm{d}b \\
&\quad + \frac{1}{2^{\sqrt{\delta}}}\int_0^t \Big[u_0(y)\Big(\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2) + 2\frac{\partial E}{\partial y}(t,x;b,y;\mu,\nu^2)\Big)\Big]_{y=x-t+b} \mathrm{d}b.
\end{aligned} \tag{39}
$$

Before plugging this expression in (37), let us rewrite the integral term in the extreme $b = t-|x-y|$ on the right hand side of (37) in a more convenient way, namely,

$$
\begin{aligned}
&\int_{x-t}^{x+t} u_0(y)\Big[\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2) - \frac{\mu}{1+b}E(t,x;b,y;\mu,\nu^2)\Big]_{b=t-|x-y|} \mathrm{d}y \\
&= \int_x^{x+t} u_0(y)\Big[\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2) - \frac{\mu}{1+b}E(t,x;b,y;\mu,\nu^2)\Big]_{b=t+x-y} \mathrm{d}y \\
&\quad + \int_{x-t}^{x} u_0(y)\Big[\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2) - \frac{\mu}{1+b}E(t,x;b,y;\mu,\nu^2)\Big]_{b=t-x+y} \mathrm{d}y \\
&= \int_0^t u_0(x+t-b)\Big[\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2) - \frac{\mu}{1+b}E(t,x;b,y;\mu,\nu^2)\Big]_{y=x+t-b} \mathrm{d}y \\
&\quad + \int_0^t u_0(x-t+b)\Big[\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2) - \frac{\mu}{1+b}E(t,x;b,y;\mu,\nu^2)\Big]_{y=x-t+b} \mathrm{d}y.
\end{aligned} \tag{40}
$$

Combining (37), (39) and (40), it follows

$$
\begin{aligned}
I_1+I_3 &= -u_0(x)+\frac{1}{2}(1+t)^{-\frac{\mu}{2}}\left[u_0(x+t)+u_0(x-t)\right]\\
&\quad+\frac{1}{2\sqrt{\delta}}\int_{x-t}^{x+t}u_0(y)\Big[-\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2)+\frac{\mu}{1+b}E(t,x;b,y;\mu,\nu^2)\Big]_{b=0}\mathrm{d}y\\
&\quad+\frac{1}{2\sqrt{\delta}}\int_0^t\Big[u_0(y)\Big(2\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2)-2\frac{\partial E}{\partial y}(t,x;b,y;\mu,\nu^2)-\frac{\mu}{1+b}E(t,x;b,y;\mu,\nu^2)\Big)\Big]_{y=x+t-b}\mathrm{d}b\\
&\quad+\frac{1}{2\sqrt{\delta}}\int_0^t\Big[u_0(y)\Big(2\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2)+2\frac{\partial E}{\partial y}(t,x;b,y;\mu,\nu^2)-\frac{\mu}{1+b}E(t,x;b,y;\mu,\nu^2)\Big)\Big]_{y=x-t+b}\mathrm{d}b.
\end{aligned}
$$

Next, we shall prove that the functions that multiply $u_0(y)$ in the last two integrals in the previous formula for I_1+I_3 are identically zero on the domain of integration. Using the identities

$$
\begin{aligned}
\frac{\partial z}{\partial b}(t,x;b,y) &= \frac{4(1+t)\big[(y-x)^2-(t-b)(t+b+2)\big]}{\big[(t+b+2)^2-(y-x)^2\big]^2},\\
\frac{\partial z}{\partial y}(t,x;b,y) &= -\frac{8(y-x)(1+t)(1+b)}{\big[(t+b+2)^2-(y-x)^2\big]^2},
\end{aligned}
$$

and the recursive relation for the derivative of a hypergeometric function, we get

$$
\begin{aligned}
\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2) &= (1+t)^{-\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}}(1+b)^{\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}}\big((t+b+2)^2-(y-x)^2\big)^{\frac{\sqrt{\delta}-1}{2}}\\
&\quad\times\Big[\mathsf{F}_z\left(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\right)\frac{\partial z}{\partial b}+\left(\tfrac{\mu}{2}+\tfrac{1-\sqrt{\delta}}{2}\right)(1+b)^{-1}\mathsf{F}\left(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\right)\\
&\qquad+(\sqrt{\delta}-1)\big((t+b+2)^2-(y-x)^2\big)^{-1}(t+b+2)\,\mathsf{F}\left(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\right)\Big]\\
&= (1+t)^{-\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}}(1+b)^{\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}}\big((t+b+2)^2-(y-x)^2\big)^{\frac{\sqrt{\delta}-1}{2}}\\
&\quad\times\Big[(1-\sqrt{\delta})^2(1+t)\big((y-x)^2-(t-b)(t+b+2)\big)\big((t+b+2)^2-(y-x)^2\big)^{-2}\mathsf{F}\left(\tfrac{3-\sqrt{\delta}}{2},\tfrac{3-\sqrt{\delta}}{2};2;z\right)\\
&\qquad+\left(\tfrac{\mu}{2}+\tfrac{1-\sqrt{\delta}}{2}\right)(1+b)^{-1}\mathsf{F}\left(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\right)\\
&\qquad+(\sqrt{\delta}-1)\big((t+b+2)^2-(y-x)^2\big)^{-1}(t+b+2)\,\mathsf{F}\left(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\right)\Big],\\
\frac{\partial E}{\partial y}(t,x;b,y;\mu,\nu^2) &= (1+t)^{-\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}}(1+b)^{\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}}\big((t+b+2)^2-(y-x)^2\big)^{\frac{\sqrt{\delta}-1}{2}}\\
&\quad\times\Big[\mathsf{F}_z\left(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\right)\frac{\partial z}{\partial y}-(\sqrt{\delta}-1)\big((t+b+2)^2-(y-x)^2\big)^{-1}(y-x)\,\mathsf{F}\left(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\right)\Big]\\
&= (1+t)^{-\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}}(1+b)^{\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}}\big((t+b+2)^2-(y-x)^2\big)^{\frac{\sqrt{\delta}-1}{2}}\\
&\quad\times\Big[-2(1-\sqrt{\delta})^2(y-x)(1+t)(1+b)\big((t+b+2)^2-(y-x)^2\big)^{-2}\mathsf{F}\left(\tfrac{3-\sqrt{\delta}}{2},\tfrac{3-\sqrt{\delta}}{2};2;z\right)\\
&\qquad-(\sqrt{\delta}-1)\big((t+b+2)^2-(y-x)^2\big)^{-1}(y-x)\,\mathsf{F}\left(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\right)\Big],\\
\frac{\mu}{1+b}E(t,x;b,y;\mu,\nu^2) &= \mu(1+t)^{-\frac{\mu}{2}+\frac{1-\sqrt{\delta}}{2}}(1+b)^{\frac{\mu}{2}-1+\frac{1-\sqrt{\delta}}{2}}\big((t+b+2)^2-(y-x)^2\big)^{\frac{\sqrt{\delta}-1}{2}}\mathsf{F}\left(\tfrac{1-\sqrt{\delta}}{2},\tfrac{1-\sqrt{\delta}}{2};1;z\right).
\end{aligned}
$$

Evaluating these functions in $y=x\pm(t-b)$, we get

$$
\begin{aligned}
\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2)\Big|_{y=x\pm(t-b)} &= 2^{\sqrt{\delta}-1}(1+t)^{-\frac{\mu}{2}-1}(1+b)^{\frac{\mu}{2}-1}\\
&\quad\times\Big[-2^{-3}(1-\sqrt{\delta})^2(t-b)+\left(\tfrac{\mu}{2}+\tfrac{1-\sqrt{\delta}}{2}\right)(1+t)+2^{-2}(\sqrt{\delta}-1)(t+b+2)\Big],\\
\frac{\partial E}{\partial y}(t,x;b,y;\mu,\nu^2)\Big|_{y=x\pm(t-b)} &= \mp 2^{\sqrt{\delta}-1}(1+t)^{-\frac{\mu}{2}-1}(1+b)^{\frac{\mu}{2}-1}\Big[2^{-3}(1-\sqrt{\delta})^2(t-b)+2^{-2}(\sqrt{\delta}-1)(t-b)\Big],\\
\frac{\mu}{1+b}E(t,x;b,y;\mu,\nu^2)\Big|_{y=x\pm(t-b)} &= 2^{\sqrt{\delta}-1}\mu(1+t)^{-\frac{\mu}{2}}(1+b)^{\frac{\mu}{2}-1}.
\end{aligned}
$$

Therefore,

$$\Big(2\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2)\mp 2\frac{\partial E}{\partial y}(t,x;b,y;\mu,\nu^2)-\frac{\mu}{1+b}E(t,x;b,y;\mu,\nu^2)\Big)_{y=x\pm(t-b)}$$
$$=2^{\sqrt{\delta}-1}(1+t)^{-\frac{\mu}{2}-1}(1+b)^{\frac{\mu}{2}-1}\Big[2\big(\tfrac{\mu}{2}+\tfrac{1-\sqrt{\delta}}{2}\big)(1+t)+2^{-1}(\sqrt{\delta}-1)(2t+2)-\mu(1+t)\Big]=0. \tag{41}$$

Summarizing, we proved

$$I_1+I_3=-u_0(x)+\frac{1}{2}(1+t)^{-\frac{\mu}{2}}\big[u_0(x+t)+u_0(x-t)\big]+\frac{1}{2\sqrt{\delta}}\int_{x-t}^{x+t}u_0(y)\Big[-\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2)\Big]_{b=0}\mathrm{d}y$$
$$+\frac{1}{2\sqrt{\delta}}\int_{x-t}^{x+t}\mu\,u_0(y)E(t,x;0,y;\mu,\nu^2)\,\mathrm{d}y. \tag{42}$$

We consider now the term $I_2=\frac{1}{2\sqrt{\delta}}\int_0^t\int_{x-t+b}^{x+t-b}bu_1''(y)E(t,x;b,y;\mu,\nu^2)\,\mathrm{d}y\,\mathrm{d}b$. We will proceed similarly as for I_1. Integration by parts leads to

$$\int_{x-t+b}^{x+t-b}b\,u_1''(y)E(t,x;b,y;\mu,\nu^2)\,\mathrm{d}y$$
$$=\Big[u_1'(y)\,bE(t,x;b,y;\mu,\nu^2)-u_1(y)\,b\frac{\partial E}{\partial y}(t,x;b,y;\mu,\nu^2)\Big]_{y=x-t+b}^{y=x+t-b}+\int_{x-t+b}^{x+t-b}u_1(y)\,b\frac{\partial^2E}{\partial y^2}(t,x;b,y;\mu,\nu^2)\,\mathrm{d}y$$
$$=\Big[u_1'(y)\,bE(t,x;b,y;\mu,\nu^2)-u_1(y)\,b\frac{\partial E}{\partial y}(t,x;b,y;\mu,\nu^2)\Big]_{y=x-t+b}^{y=x+t-b}+\int_{x-t+b}^{x+t-b}u_1(y)\,b\frac{\partial^2E}{\partial b^2}(t,x;b,y;\mu,\nu^2)\,\mathrm{d}y$$
$$+\int_{x-t+b}^{x+t-b}u_1(y)\Big[-\frac{\mu b}{1+b}\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2)+\frac{(\mu+\nu^2)\,b}{(1+b)^2}E(t,x;b,y;\mu,\nu^2)\Big]\,\mathrm{d}y.$$

Thus,

$$I_2=\frac{1}{2\sqrt{\delta}}\int_0^t\Big[u_1'(y)\,bE(t,x;b,y;\mu,\nu^2)-u_1(y)\,b\frac{\partial E}{\partial y}(t,x;b,y;\mu,\nu^2)\Big]_{y=x-t+b}^{y=x+t-b}\mathrm{d}b$$
$$+\frac{1}{2\sqrt{\delta}}\int_0^t\int_{x-t+b}^{x+t-b}u_1(y)\,b\frac{\partial^2E}{\partial b^2}(t,x;b,y;\mu,\nu^2)\,\mathrm{d}y\,\mathrm{d}b$$
$$-\frac{1}{2\sqrt{\delta}}\int_0^t\int_{x-t+b}^{x+t-b}u_1(y)\frac{\mu b}{1+b}\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2)\,\mathrm{d}y\,\mathrm{d}b$$
$$+\frac{1}{2\sqrt{\delta}}\int_0^t\int_{x-t+b}^{x+t-b}u_1(y)\frac{(\mu+\nu^2)\,b}{(1+b)^2}E(t,x;b,y;\mu,\nu^2)\,\mathrm{d}y\,\mathrm{d}b\doteq\widetilde{J}_1+\widetilde{J}_2+\widetilde{J}_3+\widetilde{J}_4.$$

Employing Fubini's theorem and integration by parts, we get

$$\widetilde{J}_2=\frac{1}{2\sqrt{\delta}}\int_0^t\int_{x-t+b}^{x+t-b}u_1(y)\,b\frac{\partial^2E}{\partial b^2}(t,x;b,y;\mu,\nu^2)\,\mathrm{d}y\,\mathrm{d}b=\frac{1}{2\sqrt{\delta}}\int_{x-t}^{x+t}u_1(y)\int_0^{t-|x-y|}b\frac{\partial^2E}{\partial b^2}(t,x;b,y;\mu,\nu^2)\,\mathrm{d}b\,\mathrm{d}y$$
$$=\frac{1}{2\sqrt{\delta}}\int_{x-t}^{x+t}u_1(y)\Big[b\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2)\Big]_{b=0}^{b=t-|x-y|}\mathrm{d}y-\frac{1}{2\sqrt{\delta}}\int_{x-t}^{x+t}u_1(y)\int_0^{t-|x-y|}\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2)\,\mathrm{d}b\,\mathrm{d}y$$
$$=\frac{1}{2\sqrt{\delta}}\int_{x-t}^{x+t}u_1(y)\Big[b\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2)-E(t,x;b,y;\mu,\nu^2)\Big]_{b=0}^{b=t-|x-y|}\mathrm{d}y$$
$$=\frac{1}{2\sqrt{\delta}}\int_{x-t}^{x+t}u_1(y)E(t,x;0,y;\mu,\nu^2)\,\mathrm{d}y+\frac{1}{2\sqrt{\delta}}\int_x^{x+t}u_1(y)\Big[b\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2)-E(t,x;b,y;\mu,\nu^2)\Big]_{b=t-y+x}\mathrm{d}y$$
$$+\frac{1}{2\sqrt{\delta}}\int_{x-t}^{x}u_1(y)\Big[b\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2)-E(t,x;b,y;\mu,\nu^2)\Big]_{b=t-x+y}\mathrm{d}y$$
$$=\frac{1}{2\sqrt{\delta}}\int_{x-t}^{x+t}u_1(y)E(t,x;0,y;\mu,\nu^2)\,\mathrm{d}y+\frac{1}{2\sqrt{\delta}}\int_0^t u_1(x+t-b)\Big[b\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2)-E(t,x;b,y;\mu,\nu^2)\Big]_{y=x+t-b}\mathrm{d}b$$
$$+\frac{1}{2\sqrt{\delta}}\int_0^t u_1(x-t+b)\Big[b\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2)-E(t,x;b,y;\mu,\nu^2)\Big]_{y=x-t+b}\mathrm{d}b$$

and

$$\begin{aligned}
\widetilde{J}_3 &= -\frac{1}{2^{\sqrt{\delta}}}\int_0^t\int_{x-t+b}^{x+t-b} u_1(y)\frac{\mu b}{1+b}\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2)\,\mathrm{d}y\,\mathrm{d}b = -\frac{1}{2^{\sqrt{\delta}}}\int_{x-t}^{x+t} u_1(y)\int_0^{t-|x-y|}\frac{\mu b}{1+b}\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2)\,\mathrm{d}b\,\mathrm{d}y\\
&= -\frac{1}{2^{\sqrt{\delta}}}\int_{x-t}^{x+t} u_1(y)\left[\frac{\mu b}{1+b}E(t,x;b,y;\mu,\nu^2)\right]_{b=0}^{b=t-|x-y|}\mathrm{d}y + \frac{1}{2^{\sqrt{\delta}}}\int_{x-t}^{x+t} u_1(y)\int_0^{t-|x-y|}\frac{\partial}{\partial b}\left(\frac{\mu b}{1+b}\right)E(t,x;b,y;\mu,\nu^2)\,\mathrm{d}b\,\mathrm{d}y\\
&= -\frac{1}{2^{\sqrt{\delta}}}\int_{x-t}^{x+t} u_1(y)\left[\frac{\mu b}{1+b}E(t,x;b,y;\mu,\nu^2)\right]_{b=t-|x-y|}\mathrm{d}y\\
&\quad + \frac{1}{2^{\sqrt{\delta}}}\int_{x-t}^{x+t} u_1(y)\int_0^{t-|x-y|}\left(-\frac{\mu b}{(1+b)^2}+\frac{\mu}{1+b}\right)E(t,x;b,y;\mu,\nu^2)\,\mathrm{d}b\,\mathrm{d}y\\
&= -\frac{1}{2^{\sqrt{\delta}}}\int_{x}^{x+t} u_1(y)\left[\frac{\mu b}{1+b}E(t,x;b,y;\mu,\nu^2)\right]_{b=t-y+x}\mathrm{d}y - \frac{1}{2^{\sqrt{\delta}}}\int_{x-t}^{x} u_1(y)\left[\frac{\mu b}{1+b}E(t,x;b,y;\mu,\nu^2)\right]_{b=t-x+y}\mathrm{d}y\\
&\quad - \frac{1}{2^{\sqrt{\delta}}}\int_0^t\int_{x-t+b}^{x+t-b} u_1(y)\frac{\mu b}{(1+b)^2}E(t,x;b,y;\mu,\nu^2)\,\mathrm{d}y\,\mathrm{d}b - I_4\\
&= -\frac{1}{2^{\sqrt{\delta}}}\int_0^t u_1(x+t-b)\left[\frac{\mu b}{1+b}E(t,x;b,y;\mu,\nu^2)\right]_{y=x+t-b}\mathrm{d}y - \frac{1}{2^{\sqrt{\delta}}}\int_0^t u_1(x-t+b)\left[\frac{\mu b}{1+b}E(t,x;b,y;\mu,\nu^2)\right]_{y=x-t+b}\mathrm{d}y\\
&\quad - \frac{1}{2^{\sqrt{\delta}}}\int_0^t\int_{x-t+b}^{x+t-b} u_1(y)\frac{\mu b}{(1+b)^2}E(t,x;b,y;\mu,\nu^2)\,\mathrm{d}y\,\mathrm{d}b - I_4.
\end{aligned}$$

Let us consider $\widetilde{J}_1$. Since

$$\begin{aligned}
&\int_0^t\left[u_1'(y)\,bE(t,x;b,y;\mu,\nu^2)\right]_{y=x-t+b}^{y=x+t-b}\mathrm{d}b\\
&\quad= -\int_0^t\left[\frac{\partial}{\partial b}\big(u_1(x+t-b)\big)\,bE(t,x;b,x+t-b;\mu,\nu^2) + \frac{\partial}{\partial b}\big(u_1(x-t+b)\big)\,bE(t,x;b,x-t+b;\mu,\nu^2)\right]\mathrm{d}b\\
&\quad= -\Big[u_1(x+t-b)\,bE(t,x;b,x+t-b;\mu,\nu^2) + u_1(x-t+b)\,bE(t,x;b,x-t+b;\mu,\nu^2)\Big]_{b=0}^{b=t}\\
&\qquad+\int_0^t\left[u_1(x+t-b)\frac{\partial}{\partial b}\Big(bE(t,x;b,x+t-b;\mu,\nu^2)\Big) + u_1(x-t+b)\frac{\partial}{\partial b}\Big(bE(t,x;b,x-t+b;\mu,\nu^2)\Big)\right]\mathrm{d}b\\
&\quad= -2^{\sqrt{\delta}}\,tu_1(x) + \int_0^t\left[u_1(y)\left(E(t,x;b,y;\mu,\nu^2) + b\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2) - b\frac{\partial E}{\partial y}(t,x;b,y;\mu,\nu^2)\right)\right]_{y=x+t-b}\mathrm{d}b\\
&\qquad+\int_0^t\left[u_1(y)\left(E(t,x;b,y;\mu,\nu^2) + b\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2) + b\frac{\partial E}{\partial y}(t,x;b,y;\mu,\nu^2)\right)\right]_{y=x-t+b}\mathrm{d}b,
\end{aligned}$$

where in the last step we used (38), then,

$$\begin{aligned}
\widetilde{J}_1 &= \frac{1}{2^{\sqrt{\delta}}}\int_0^t\left[u_1'(y)\,bE(t,x;b,y;\mu,\nu^2) - u_1(y)\,b\frac{\partial E}{\partial y}(t,x;b,y;\mu,\nu^2)\right]_{y=x-t+b}^{y=x+t-b}\mathrm{d}b\\
&= -tu_1(x) + \frac{1}{2^{\sqrt{\delta}}}\int_0^t\left[u_1(y)\left(E(t,x;b,y;\mu,\nu^2) + b\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2) - 2b\frac{\partial E}{\partial y}(t,x;b,y;\mu,\nu^2)\right)\right]_{y=x+t-b}\mathrm{d}b\\
&\quad+\frac{1}{2^{\sqrt{\delta}}}\int_0^t\left[u_1(y)\left(E(t,x;b,y;\mu,\nu^2) + b\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2) + 2b\frac{\partial E}{\partial y}(t,x;b,y;\mu,\nu^2)\right)\right]_{y=x-t+b}\mathrm{d}b.
\end{aligned}$$

Summarizing,

$$\begin{aligned}
I_2 &= \widetilde{J}_1 + \widetilde{J}_2 + \widetilde{J}_3 + \widetilde{J}_4\\
&= \frac{1}{2^{\sqrt{\delta}}}\int_{x-t}^{x+t} u_1(y)E(t,x;0,y;\mu,\nu^2)\,\mathrm{d}y - tu_1(x) - I_4 - I_5\\
&\quad+\frac{1}{2^{\sqrt{\delta}}}\int_0^t\left[u_1(y)\left(2b\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2) - 2b\frac{\partial E}{\partial y}(t,x;b,y;\mu,\nu^2) - \frac{\mu b}{1+b}E(t,x;b,y;\mu,\nu^2)\right)\right]_{y=x+t-b}\mathrm{d}b\\
&\quad+\frac{1}{2^{\sqrt{\delta}}}\int_0^t\left[u_1(y)\left(2b\frac{\partial E}{\partial b}(t,x;b,y;\mu,\nu^2) + 2b\frac{\partial E}{\partial y}(t,x;b,y;\mu,\nu^2) - \frac{\mu b}{1+b}E(t,x;b,y;\mu,\nu^2)\right)\right]_{y=x-t+b}\mathrm{d}b.
\end{aligned}$$

Using the identity (41), from the last equality we get

$$I_2 + I_4 + I_5 = \frac{1}{2\sqrt{\delta}} \int_{x-t}^{x+t} u_1(y) E(t, x; 0, y; \mu, \nu^2)\,\mathrm{d}y - t u_1(x). \tag{43}$$

Finally, if we combine (42) and (43), we arrive at

$$\begin{aligned} w(t, x) &= I_1 + I_2 + I_3 + I_4 + I_5 \\ &= -u_0(x) - t u_1(x) + \frac{1}{2}(1+t)^{-\frac{\mu}{2}} \big[u_0(x+t) + u_0(x-t)\big] + \frac{1}{2\sqrt{\delta}} \int_{x-t}^{x+t} u_0(y) \Big[-\frac{\partial E}{\partial b}(t, x; b, y; \mu, \nu^2) \Big]_{b=0} \mathrm{d}y \\ &\quad + \frac{1}{2\sqrt{\delta}} \int_{x-t}^{x+t} \big(u_1(y) + \mu\, u_0(y)\big) E(t, x; 0, y; \mu, \nu^2)\,\mathrm{d}y. \end{aligned}$$

So, we proved that u^{h} defined in (15) solves (16).

2.4 Final remarks on the 1d case

Combining the results from Subsections 2.2 and 2.3, we see that $u = u^{\mathrm{h}} + u^{\mathrm{ih}}$ is a solution of the Cauchy problem (3) for $n = 1$ as stated in Theorem 1.1.

Let us remark that for $\mu = \nu^2 = 0$ we have $\delta = 1$ and $\mathsf{F}(0, 0; 1; z) = 1$, so that $E(t, x; b, y; 0, 0) = 1$. This means that (4) coincides with d'Alembert's representation formula for $\mu = \nu^2 = 0$. As in d'Alembert's representation formula for the classical wave equation, in the one dimensional case we have no loss of regularity for the solution in comparison with initial data. However, differently from d'Alembert's representation formula, we have that the first data appears, in general, also in an integral term.

3 Odd dimensional case

In this section we will prove Theorem 1.2 with the method of spherical means (see [8] for further details).

3.1 Spherical means

Let $u = u(t, x)$ solve (3). We define

$$I_r[u](t, x) \doteq \frac{1}{\omega_{n-1}} \int_{|\omega|=1} u(t, x + r\omega)\,\mathrm{d}\sigma_\omega = \frac{1}{\omega_{n-1} r^{n-1}} \int_{\partial B_r(x)} u(t, z)\,\mathrm{d}\sigma_z = ⨍_{\partial B_r(x)} u(t, z)\,\mathrm{d}\sigma_z,$$

where ω_{n-1} is the $(n-1)$-dimensional measure of the unit sphere of $\mathbb{R}^n$ and, analogously,

$$I_r[u_j](x) \doteq ⨍_{\partial B_r(x)} u_j(z)\,\mathrm{d}\sigma_z \qquad \text{for} \quad j = 0, 1.$$

Moreover, we introduce the operator Ω_r as follows:

$$\begin{aligned} \Omega_r[u](t, x) &\doteq \Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^{k-1} \Big(r^{2k-1} I_r[u](t, x)\Big), \\ \Omega_r[u_j](x) &\doteq \Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^{k-1} \Big(r^{2k-1} I_r[u_j](x)\Big) \qquad \text{for} \quad j = 0, 1, \end{aligned} \tag{44}$$

where k satisfies the relation $n = 2k + 1$. We remark that the equality

$$I_r[u_{tt}](t, x) = ⨍_{\partial B_r(x)} u_{tt}(t, z)\,\mathrm{d}\sigma_z = \Big(\frac{\partial}{\partial t}\Big)^2 ⨍_{\partial B_r(x)} u(t, z)\,\mathrm{d}\sigma_z = \Big(\frac{\partial}{\partial t}\Big)^2 \Big(I_r[u](t, x)\Big)$$

implies

$$\begin{aligned} \Omega_r[u_{tt}](t, x) &= \Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^{k-1} \Big(r^{2k-1} I_r[u_{tt}](t, x)\Big) = \Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^{k-1} \bigg(r^{2k-1} \Big(\frac{\partial}{\partial t}\Big)^2 \Big(I_r[u](t, x)\Big)\bigg) \\ &= \Big(\frac{\partial}{\partial t}\Big)^2 \Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^{k-1} \Big(r^{2k-1} \Big(I_r[u](t, x)\Big)\Big) = \Big(\frac{\partial}{\partial t}\Big)^2 \Omega_r[u](t, x). \end{aligned}$$

Similarly, one can prove

$$\Omega_r\left[\frac{\mu}{1+t}u_t\right](t,x) = \frac{\mu}{1+t}\left(\frac{\partial}{\partial t}\right)\Omega_r[u](t,x),$$
$$\Omega_r\left[\frac{\nu^2}{(1+t)^2}u\right](t,x) = \frac{\nu^2}{(1+t)^2}\,\Omega_r[u](t,x).$$

Due to the linearity of the operator Ω_r, we get that $\Omega_r[u]$ solves

$$\left(\frac{\partial}{\partial t}\right)^2\Omega_r[u](t,x) + \frac{\mu}{1+t}\left(\frac{\partial}{\partial t}\right)\Omega_r[u](t,x) + \frac{\nu^2}{(1+t)^2}\,\Omega_r[u](t,x) = \Omega_r[\Delta u](t,x) + \Omega_r[f](t,x).$$

Next, we shall express in a more convenient way the action of Ω_r on the Laplacian of u. This relation is well-known in the literature, but we will prove it in few steps for the ease of the reader. Let us calculate the derivative of order 2 of $I_r[u]$. By Green's formula we get

$$\begin{aligned}\frac{\partial}{\partial r}I_r[u](t,x) &= \frac{1}{\omega_{n-1}}\int_{|\omega|=1}\nabla u(t,x+r\omega)\cdot\frac{\partial}{\partial r}(x+r\omega)\,\mathrm{d}\sigma_\omega = \frac{1}{\omega_{n-1}}\int_{|\omega|=1}\nabla u(t,x+r\omega)\cdot\omega\,\mathrm{d}\sigma_\omega\\ &= \frac{r}{\omega_{n-1}}\int_{|\omega|\leqslant 1}\Delta u(t,x+r\omega)\,\mathrm{d}\omega = \frac{1}{\omega_{n-1}r^{n-1}}\int_{B_r(x)}\Delta u(t,z)\,\mathrm{d}z\\ &= \frac{1}{\omega_{n-1}r^{n-1}}\int_0^r\int_{\partial B_\varrho(x)}\Delta u(t,\omega)\,\mathrm{d}\sigma_\omega\,\mathrm{d}\varrho.\end{aligned}$$

A further differentiation with respect to r provides

$$\begin{aligned}\left(\frac{\partial}{\partial r}\right)^2 I_r[u](t,x) &= \frac{1}{\omega_{n-1}r^{n-1}}\int_{\partial B_r(x)}\Delta u(t,\omega)\,\mathrm{d}\sigma_\omega - \frac{n-1}{\omega_{n-1}r^n}\int_0^r\int_{\partial B_\varrho(x)}\Delta u(t,\omega)\,\mathrm{d}\sigma_\omega\,\mathrm{d}\varrho\\ &= \frac{1}{\omega_{n-1}r^{n-1}}\int_{\partial B_r(x)}\Delta u(t,\omega)\,\mathrm{d}\sigma_\omega - \frac{n-1}{r}\frac{\partial}{\partial r}I_r[u](t,x),\end{aligned}$$

that is,

$$\left(\frac{\partial}{\partial r}\right)^2 I_r[u](t,x) + \frac{n-1}{r}\frac{\partial}{\partial r}I_r[u](t,x) = \fint_{B_r(x)}\Delta u(t,\omega)\,\mathrm{d}\sigma_\omega = I_r[\Delta u](t,x).$$

The previous relation implies

$$\begin{aligned}\left(\frac{\partial}{\partial r}\right)^2\Omega_r[u](t,x) &= \left(\frac{\partial}{\partial r}\right)^2\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1}I_r[u](t,x)\right) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k}\left(r^{2k}\frac{\partial}{\partial r}I_r[u](t,x)\right)\\ &= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left[r^{2k-1}\left(\frac{\partial}{\partial r}\right)^2 I_r[u](t,x) + 2k\,r^{2k-2}\frac{\partial}{\partial r}I_r[u](t,x)\right]\\ &= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left[r^{2k-1}\left(\left(\frac{\partial}{\partial r}\right)^2 I_r[u](t,x) + \frac{n-1}{r}\frac{\partial}{\partial r}I_r[u](t,x)\right)\right] = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left[r^{2k-1}\left(I_r[\Delta u](t,x)\right)\right]\\ &= \Omega_r[\Delta u](t,x),\end{aligned}$$

where in the second equality we used the identity

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^2\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)^{k-1}\left(r^{2k-1}\phi(r)\right) = \left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)^{k}\left(r^{2k}\frac{\mathrm{d}\phi}{\mathrm{d}r}(r)\right)$$

whose validity can be proved by using an inductive argument (cf. [6, Lemma 2, Section 2.4.1]). If we introduce the function $v = v(r,t;x) = \Omega_r[u](t,x)$, then, v solves the following initial boundary value problem depending on the parameter $x\in\mathbb{R}^n$:

$$\begin{cases}\partial_t^2 v(r,t;x) - \partial_r^2 v(r,t;x) + \frac{\mu}{1+t}\partial_t v(r,t;x) + \frac{\nu^2}{(1+t)^2}v(r,t;x) = \Omega_r[f](t,x), & t>0, r>0,\\ v(r,0;x) = \Omega_r[u_0](x), & r>0,\\ \partial_t v(r,0;x) = \Omega_r[u_1](x), & r>0,\\ v(0,t;x) = 0, & t\geqslant 0.\end{cases} \tag{45}$$

In order to get the boundary condition in (45), we employed the following formula

$$\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)^{k-1}\left(r^{2k-1}\phi(r)\right)=\sum_{j=0}^{k-1}\beta_j^{(k)}r^{j+1}\frac{\mathrm{d}^j\phi}{\mathrm{d}r^j}(r), \tag{46}$$

where the constants $\{\beta_j^{(k)}\}_{j=0,\cdots,k-1}$ are independent of ϕ and, in particular, $\beta_0^{(k)}=(2k-1)!!$ (see also, for example, [6, Lemma 2, Section 2.4.1]).

Since $I_r[u](t,x)$ can be extended to an even function for $r<0$, $\Omega_r[u](t,x)$ has a natural extension as odd function with respect to r for $r<0$, due to (44). We denote the odd extensions of v, $\Omega_r[f]$ and $\Omega_r[u_j]$ for $j=0,1$ by

$$\widetilde{v}(r,t;x)\doteq\begin{cases}v(r,t;x) & \text{if } r\geqslant 0,\\ -v(-r,t;x) & \text{if } r\leqslant 0,\end{cases}\qquad \widetilde{\Omega}_r[f](t,x)\doteq\begin{cases}\Omega_r[f](t,x) & \text{if } r\geqslant 0,\\ -\Omega_{-r}[f](t,x) & \text{if } r\leqslant 0,\end{cases}$$

$$\widetilde{\Omega}_r[u_j](x)\doteq\begin{cases}\Omega_r[u_j](x) & \text{if } r\geqslant 0,\\ -\Omega_{-r}[u_j](x) & \text{if } r\leqslant 0,\end{cases}$$

respectively. Therefore, $\widetilde{v}$ solves the Cauchy problem depending on the parameter $x\in\mathbb{R}^n$

$$\begin{cases}\partial_t^2\widetilde{v}(r,t;x)-\partial_r^2\widetilde{v}(r,t;x)+\frac{\mu}{1+t}\partial_t\widetilde{v}(r,t;x)+\frac{\nu^2}{(1+t)^2}\widetilde{v}(r,t;x)=\widetilde{\Omega}_r[f](t,x), & t>0, r\in\mathbb{R},\\ \widetilde{v}(r,0;x)=\widetilde{\Omega}_r[u_0](x), & r\in\mathbb{R},\\ \partial_t\widetilde{v}(r,0;x)=\widetilde{\Omega}_r[u_1](x), & r\in\mathbb{R}.\end{cases} \tag{47}$$

Hence, (47) is a Cauchy problem for an inhomogeneous linear wave equation with scale-invariant damping and mass in the one dimensional case. Thanks to Theorem 1.1, we have an explicit representation formula for $\widetilde{v}$, namely,

$$\begin{aligned}\widetilde{v}(r,t;x)&=\frac{1}{2}(1+t)^{-\frac{\mu}{2}}\Big(\widetilde{\Omega}_{r+t}[u_0](x)+\widetilde{\Omega}_{r-t}[u_0](x)\Big)+\frac{1}{2\sqrt{\delta}}\int_{r-t}^{r+t}\widetilde{\Omega}_s[u_0](x)K_0(t,r;s;\mu,\nu^2)\,\mathrm{d}s\\ &\quad+\frac{1}{2\sqrt{\delta}}\int_{r-t}^{r+t}\Big(\widetilde{\Omega}_s[u_1](x)+\mu\,\widetilde{\Omega}_s[u_0](x)\Big)K_1(t,r;s;\mu,\nu^2)\,\mathrm{d}s+\frac{1}{2\sqrt{\delta}}\int_0^t\int_{r-t+b}^{r+t-b}\widetilde{\Omega}_s[f](b,x)E(t,r;b,s;\mu,\nu^2)\,\mathrm{d}s\,\mathrm{d}b.\end{aligned} \tag{48}$$

In the next subsection, we will apply a limit argument to (48) in order to derive a representation formula for (3) in the odd dimensional case.

3.2 Representation formula via a limit argument

From (46) it follows that

$$u(t,x)=\lim_{r\to 0}I_r[u](t,x)=\lim_{r\to 0}\frac{1}{\beta_0^{(k)}r}\Omega_r[u](t,x)=\frac{1}{(n-2)!!}\lim_{r\to 0}\frac{\widetilde{v}(r,t;x)}{r}.$$

Our strategy consists in using (48) in order to calculate the previous limit. We will consider separately the four addends that appear in (48). Fixed $t>0$, since we will calculate the limit as $r\to 0$ we may assume without loss of generality that $r<t$, thus,

$$\frac{1}{r}\Big(\widetilde{\Omega}_{r+t}[u_0](x)+\widetilde{\Omega}_{r-t}[u_0](x)\Big)=\frac{1}{r}\Big(\Omega_{r+t}[u_0](x)-\Omega_{t-r}[u_0](x)\Big)\xrightarrow[r\to 0]{}2\frac{\partial}{\partial t}\Omega_t[u_0](x).$$

For the integral containing the kernel function K_0, we have

$$\begin{aligned}\frac{1}{r}\int_{r-t}^{r+t}\widetilde{\Omega}_s[u_0](x)K_0(t,r;s;\mu,\nu^2)\,\mathrm{d}s&=\frac{1}{r}\int_{-t}^{t}\widetilde{\Omega}_{s+r}[u_0](x)K_0(t,r;s+r;\mu,\nu^2)\,\mathrm{d}s\\ &=\frac{1}{r}\int_0^t\Big[\widetilde{\Omega}_{s+r}[u_0](x)K_0(t,r;s+r;\mu,\nu^2)+\widetilde{\Omega}_{r-s}[u_0](x)K_0(t,r;-s+r;\mu,\nu^2)\Big]\,\mathrm{d}s\\ &=\int_0^t\frac{1}{r}\Big[\widetilde{\Omega}_{s+r}[u_0](x)+\widetilde{\Omega}_{r-s}[u_0](x)\Big]K_0(t,r;s+r;\mu,\nu^2)\,\mathrm{d}s,\end{aligned}$$

where in the last step we used that $K_0(t,r;s+r;\mu,\nu^2)$ is even with respect to s; this follows immediately from the fact that $E(t,r;b,s+r;\mu,\nu^2)$ is even with respect to s and from the definition (6). Letting $r\to 0$ in the last expression we have

$$\frac{1}{r}\int_{r-t}^{r+t}\widetilde{\Omega}_s[u_0](x)K_0(t,r;s;\mu,\nu^2)\,\mathrm{d}s$$
$$=\int_0^t\frac{1}{r}\Big[\Omega_{s+r}[u_0](x)-\Omega_{s-r}[u_0](x)\Big]K_0(t,r;s+r;\mu,\nu^2)\,\mathrm{d}s\xrightarrow[r\to 0]{}2\int_0^t\frac{\partial}{\partial s}\,\Omega_s[u_0](x)K_0(t,0;s;\mu,\nu^2)\,\mathrm{d}s.$$

In an analogous way, $K_1(t,r;s+r;\mu,\nu^2)$ being an even function with respect to s, we get

$$\frac{1}{r}\int_{r-t}^{r+t}\widetilde{\Omega}_s[u_1+\mu u_0](x)K_1(t,r;s;\mu,\nu^2)\,\mathrm{d}s\xrightarrow[r\to 0]{}2\int_0^t\frac{\partial}{\partial s}\,\Omega_s[u_1+\mu u_0](x)K_1(t,0;s;\mu,\nu^2)\,\mathrm{d}s.$$

Finally, we consider the integral term involving the source term. It results

$$\begin{aligned}\frac{1}{r}\int_0^t\int_{r-t+b}^{r+t-b}\widetilde{\Omega}_s[f](b,x)E(t,r;b,s;\mu,\nu^2)\,\mathrm{d}s\,\mathrm{d}b&=\frac{1}{r}\int_0^t\int_{-t+b}^{t-b}\widetilde{\Omega}_{s+r}[f](b,x)E(t,r;b,s+r;\mu,\nu^2)\,\mathrm{d}s\,\mathrm{d}b\\&=\frac{1}{r}\int_0^t\int_0^{t-b}\Big[\widetilde{\Omega}_{s+r}[f](b,x)E(t,r;b,s+r;\mu,\nu^2)+\widetilde{\Omega}_{r-s}[f](b,x)E(t,r;b,-s+r;\mu,\nu^2)\Big]\,\mathrm{d}s\,\mathrm{d}b\\&=\frac{1}{r}\int_0^t\int_0^{t-b}\Big[\widetilde{\Omega}_{s+r}[f](b,x)+\widetilde{\Omega}_{r-s}[f](b,x)\Big]E(t,r;b,s+r;\mu,\nu^2)\,\mathrm{d}s\,\mathrm{d}b,\end{aligned}$$

where in the last step we used the property $E(t,r;b,s+r;\mu,\nu^2)=E(t,r;b,-s+r;\mu,\nu^2)$. Consequently, letting $r\to 0$, we have

$$\frac{1}{r}\int_0^t\int_{r-t+b}^{r+t-b}\widetilde{\Omega}_s[f](b,x)E(t,r;b,s;\mu,\nu^2)\,\mathrm{d}s\,\mathrm{d}b$$
$$=\int_0^t\int_0^{t-b}\frac{1}{r}\Big[\Omega_{s+r}[f](b,x)-\Omega_{s-r}[f](b,x)\Big]E(t,r;b,s+r;\mu,\nu^2)\,\mathrm{d}s\,\mathrm{d}b\xrightarrow[r\to 0]{}2\int_0^t\int_0^{t-b}\frac{\partial}{\partial s}\,\Omega_s[f](b,x)E(t,0;b,s;\mu,\nu^2)\,\mathrm{d}s\,\mathrm{d}b.$$

Summarizing, we proved

$$\begin{aligned}(n-2)!!\,u(t,x)=\lim_{r\to 0}\frac{\widetilde{v}(r,t;x)}{r}&=(1+t)^{-\frac{\mu}{2}}\frac{\partial}{\partial t}\,\Omega_t[u_0](x)+\frac{1}{2^{\sqrt{\delta}-1}}\int_0^t\frac{\partial}{\partial s}\,\Omega_s[u_0](x)K_0(t,0;s;\mu,\nu^2)\,\mathrm{d}s\\&\quad+\frac{1}{2^{\sqrt{\delta}-1}}\int_0^t\frac{\partial}{\partial s}\,\Omega_s[u_1+\mu u_0](x)K_1(t,0;s;\mu,\nu^2)\,\mathrm{d}s\\&\quad+\frac{1}{2^{\sqrt{\delta}-1}}\int_0^t\int_0^{t-b}\frac{\partial}{\partial s}\,\Omega_s[f](b,x)E(t,0;b,s;\mu,\nu^2)\,\mathrm{d}s\,\mathrm{d}b.\end{aligned}\tag{49}$$

According to what we recall in the introduction, more precisely the representation given in (10), we have

$$w[\varphi](t,x)=\frac{1}{(n-2)!!}\frac{\partial}{\partial t}\,\Omega_t[\varphi](x).$$

So, (49) implies easily (12).

4 Even dimensional case: method of descent

In this section we prove Theorem 1.3, by using the so-called *method of descent*. Let us consider $u=u(t,x)$ solution of (3) when $n\geqslant 2$ is an even integer. Then, we can consider formally u as a function defined on $[0,\infty)\times\mathbb{R}^{n+1}$, by setting

$$\bar{u}(t,x,x_{n+1})\doteq u(t,x)\qquad\text{for any }\ t\geqslant 0,(x,x_{n+1})\in\mathbb{R}^{n+1}.$$

Then, $\bar{u}$ solves

$$\begin{cases}\bar{u}_{tt}-\sum_{j=1}^{n+1}\bar{u}_{x_jx_j}+\frac{\mu}{1+t}\bar{u}_t+\frac{\nu^2}{(1+t)^2}\bar{u}=\bar{f}(t,x,x_{n+1}), & (x,x_{n+1})\in\mathbb{R}^{n+1},\ t>0,\\ \bar{u}(0,x,x_{n+1})=\bar{u}_0(x,x_{n+1}), & (x,x_{n+1})\in\mathbb{R}^{n+1},\\ \bar{u}_t(0,x,x_{n+1})=\bar{u}_1(x,x_{n+1}), & (x,x_{n+1})\in\mathbb{R}^{n+1},\end{cases}\tag{50}$$

where

$$\begin{aligned}&\bar{u}_0(x,x_{n+1})\doteq u_0(x),\quad \bar{u}_1(x,x_{n+1})\doteq u_1(x) &&\text{for any }\ (x,x_{n+1})\in\mathbb{R}^{n+1},\\ &\bar{f}(t,x,x_{n+1})\doteq f(t,x) &&\text{for any }\ t\geqslant 0,(x,x_{n+1})\in\mathbb{R}^{n+1}.\end{aligned}$$

Due to the fact that $n+1$ is an odd integer, we can use Theorem 1.2 to get a representation formula for $\bar{u}$. Let us underline that $\bar{u}$ depends only formally on x_{n+1}, so we can consider without loss of generality the restriction of $\bar{u}$ on the hyperplane $\{x_{n+1}=0\}$. For the sake of readability we will denote by $\bar{B}_r(z)$ the ball around z with radius r in $\mathbb{R}^{n+1}$ and we will keep the usual notation for balls in $\mathbb{R}^n$. According to (12) and (10), we have

$$\begin{aligned} u(t,x) &= \bar{u}(t,x,0) \\ &= \frac{1}{(n-1)!!}(1+t)^{-\frac{\mu}{2}}\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n}{2}-1}\left(t^{n-1}\fint_{\partial\bar{B}_t(x,0)}\bar{u}_0(z,z_{n+1})\,\mathrm{d}\sigma_{(z,z_{n+1})}\right) \\ &\quad+\frac{2^{1-\sqrt{\delta}}}{(n-1)!!}\int_0^t\left(\frac{\partial}{\partial s}\right)\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{\frac{n}{2}-1}\left(s^{n-1}\fint_{\partial\bar{B}_s(x,0)}\bar{u}_0(z,z_{n+1})\,\mathrm{d}\sigma_{(z,z_{n+1})}\right)K_0(t,0;s;\mu,\nu^2)\,\mathrm{d}s \\ &\quad+\frac{2^{1-\sqrt{\delta}}}{(n-1)!!}\int_0^t\left(\frac{\partial}{\partial s}\right)\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{\frac{n}{2}-1}\left(s^{n-1}\fint_{\partial\bar{B}_s(x,0)}\big(\bar{u}_1(z,z_{n+1})+\mu\bar{u}_0(z,z_{n+1})\big)\,\mathrm{d}\sigma_{(z,z_{n+1})}\right)K_1(t,0;s;\mu,\nu^2)\,\mathrm{d}s \\ &\quad+\frac{2^{1-\sqrt{\delta}}}{(n-1)!!}\int_0^t\int_0^{t-b}\left(\frac{\partial}{\partial s}\right)\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{\frac{n}{2}-1}\left(s^{n-1}\fint_{\partial\bar{B}_s(x,0)}\bar{f}(b,z,z_{n+1})\,\mathrm{d}\sigma_{(z,z_{n+1})}\right)E(t,0;b,s;\mu,\nu^2)\,\mathrm{d}s\,\mathrm{d}b. \end{aligned} \tag{51}$$

The next step is to rewrite the surface integrals in $\mathbb{R}^{n+1}$ as domain integrals in $\mathbb{R}^n$. We remark that

$$\partial\bar{B}_r(x,0)=\left\{(y,y_{n+1})\in\mathbb{R}^{n+1}:y_{n+1}=\pm(r^2-|y-x|^2)^{1/2}\right\}.$$

Therefore, $\partial\bar{B}_r(x,0)\cap\{(y,y_{n+1}):y_{n+1}\geqslant 0\}$ is the graph of the function

$$\gamma:y\in B_r(x)\to\gamma(y)\doteq(r^2-|y-x|^2)^{1/2}$$

and, similarly, $\partial\bar{B}_r(x,0)\cap\{(y,y_{n+1}):y_{n+1}\leqslant 0\}$ is the graph of the function $-\gamma$. Since,

$$\nabla\gamma(y)=-\frac{(y-x)}{(r^2-|y-x|^2)^{1/2}},$$

if φ is a function defined on $\mathbb{R}^n$ and $\bar{\varphi}$ denotes its trivial extension as a function of $n+1$ variables (we have in mind the cases in which φ is equal to u_0,u_1 or $f(t,\cdot)$), then,

$$\begin{aligned} r^{n-1}\fint_{\partial\bar{B}_r(x,0)}\bar{\varphi}(z,z_{n+1})\,\mathrm{d}\sigma_{(z,z_{n+1})} &= \frac{1}{\omega_n r}\int_{\partial\bar{B}_r(x,0)}\bar{\varphi}(z,z_{n+1})\,\mathrm{d}\sigma_{(z,z_{n+1})}=\frac{2}{\omega_n r}\int_{B_r(x)}\bar{\varphi}(z,\gamma(z))\sqrt{1+|\nabla\gamma(z)|^2}\,\mathrm{d}z \\ &= \frac{2}{\omega_n}\int_{B_r(x)}\frac{\varphi(z)}{(r^2-|y-x|^2)^{1/2}}\,\mathrm{d}z=\frac{2\omega_{n-1}}{\omega_n n}r^n\fint_{B_r(x)}\frac{\varphi(z)}{(r^2-|y-x|^2)^{1/2}}\,\mathrm{d}z, \end{aligned}$$

where the factor 2 in the second step is due to the fact that $\partial\bar{B}_r(x,0)$ consists of two hemispheres. It is well-known that the measure of the $(n-1)$-dimensional unit sphere of $\mathbb{R}^n$ is

$$\omega_{n-1}=\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})},$$

where Γ is the Euler integral function of the second kind. Consequently, using the recursive relation $\Gamma(z+1)=z\Gamma(z)$ iteratively and the values $\Gamma(1)=1$, $\Gamma(\frac{1}{2})=\sqrt{\pi}$, we get

$$\frac{2\omega_{n-1}}{\omega_n n}=\frac{2}{\sqrt{\pi}n}\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}=\frac{2}{\sqrt{\pi}n}\frac{\frac{n-1}{2}\cdot\frac{n-3}{2}\cdots\frac{1}{2}\cdot\Gamma(\frac{1}{2})}{\frac{n-2}{2}\cdot\frac{n-4}{2}\cdots\frac{2}{2}\cdot\Gamma(1)}=\frac{2}{\sqrt{\pi}n}\frac{2^{-\frac{n}{2}}(n-1)!!\sqrt{\pi}}{2^{-\frac{n}{2}+1}(n-2)!!}=\frac{(n-1)!!}{n!!}.$$

Therefore,

$$r^{n-1}\fint_{\partial\bar{B}_r(x,0)}\bar{\varphi}(z,z_{n+1})\,\mathrm{d}\sigma_{(z,z_{n+1})}=\frac{(n-1)!!}{n!!}r^n\fint_{B_r(x)}\frac{\varphi(z)}{(r^2-|y-x|^2)^{1/2}}\,\mathrm{d}z.$$

Hence, applying the previous relation to (51), we get finally

$$
\begin{aligned}
u(t,x) = \bar{u}(t,x,0) &= \frac{1}{n!!}(1+t)^{-\frac{\mu}{2}}\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n}{2}-1}\left(t^n \fint_{B_t(x)} \frac{u_0(z)}{(t^2-|y-x|^2)^{1/2}}\,\mathrm{d}z\right)\\
&+ \frac{2^{1-\sqrt{\delta}}}{n!!}\int_0^t \left(\frac{\partial}{\partial s}\right)\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{\frac{n}{2}-1}\left(s^n \fint_{B_s(x)} \frac{u_0(z)}{(s^2-|y-x|^2)^{1/2}}\,\mathrm{d}z\right) K_0(t,0;s;\mu,\nu^2)\,\mathrm{d}s\\
&+ \frac{2^{1-\sqrt{\delta}}}{n!!}\int_0^t \left(\frac{\partial}{\partial s}\right)\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{\frac{n}{2}-1}\left(s^n \fint_{B_s(x)} \frac{u_1(z)+\mu\, u_0(z)}{(s^2-|y-x|^2)^{1/2}}\,\mathrm{d}z\right) K_1(t,0;s;\mu,\nu^2)\,\mathrm{d}s\\
&+ \frac{2^{1-\sqrt{\delta}}}{n!!}\int_0^t\int_0^{t-b} \left(\frac{\partial}{\partial s}\right)\left(\frac{1}{s}\frac{\partial}{\partial s}\right)^{\frac{n}{2}-1}\left(s^n \fint_{B_s(x)} \frac{f(b,z)}{(s^2-|y-x|^2)^{1/2}}\,\mathrm{d}z\right) E(t,0;b,s;\mu,\nu^2)\,\mathrm{d}s\,\mathrm{d}b.
\end{aligned}
$$

So, combining (12) and (11), we proved Theorem 1.3.

5 Final remarks

In this section, we list some straightforward consequences of Theorems 1.2 and 1.3 and some relations/connections of the representation formulae in (4) and (12) with representation formulae for other hyperbolic equations with time-dependent coefficients.

Loss of regularity First, we remark that in the multidimensional case $n \geq 2$ we have a loss of regularity for the solution of (3) in comparison with the regularity of initial data, differently from the one-dimensional case. Indeed, according to Theorem 1.2 in the odd dimensional case we have a loss of regularity of order $\frac{n-1}{2}$, while in the even dimensional case the loss of regularity has order $\frac{n}{2}$, according to Theorem 1.3.

Domain of dependence From (4) and from (12) (combined with (10) and (11)) we see that the domain of dependence in the point $(t_0,x_0) \in [0,\infty) \times \mathbb{R}^n$ for the solution of (3) is

$$
\Omega(t_0,x_0) = \Big\{(t,x) \in [0,\infty) \times \mathbb{R}^n : t \in [0,t_0],\ |x-x_0| \leqslant (t_0-t)\Big\}.
$$

In other words, $u(t_0,x_0)$ depends on the value of f in $\Omega(t_0,x_0)$ and the values of u_0,u_1 in $\Omega(t_0,x_0) \cap \{t=0\}$. So, in the case of scale-invariant models from the representation formulae that we proved in this work we found in a different way a property that is known to be true in a more general frame for hyperbolic models (see for example [22, Theorem 2.2 in Chapter 1]).

Finite speed of propagation of perturbations Of course, we may change our prospective and analyze how the initial data and the source term influence the behavior of the solution. Let us assume that u_0,u_1 are compactly supported in $B_R(0)$ and that $\operatorname{supp} f \subset K_R \doteq \{(t,x) \in [0,\infty) \times \mathbb{R}^n : |x| \leq R+t\}$. Then, the solution itself has support contained in the forward conical domain K_R. This follows immediately by (4) and (12). Indeed, in order to get not identically vanishing integrands in (4) and (12) or an actual influence of the traveling wave for $n=1$ or from the wave $(1+t)^{-\frac{\mu}{2}}w[u_0](t,x)$ for the multidimensional case, it must hold $(t,x) \in K_R$ necessarily. So, we have shown the validity of the property of finite speed of propagation of perturbations with constant speed 1 (also in this case the result is already known in the literature, e.g. [22, Corollary 2.3 in Chapter 1]).

Huygens' principle In general, we have seen the existence of a forward wave front in the case of compactly supported initial data and of source term supported in the conical domain correspondingly. However, in the case of a homogeneous problem ($f \equiv 0$) a backward wave front is not present generally, even in the odd dimensional case. If we denote

$$
u^{\mathrm{Huy}}(t,x) \doteq \begin{cases} 2^{-1}(1+t)^{-\frac{\mu}{2}}\big(u_0(x+t)+u_0(x-t)\big) & \text{if } n=1,\\ (1+t)^{-\frac{\mu}{2}}w[u_0](t,x) & \text{if } n \geq 2,\end{cases}
$$

$$
u^{\mathrm{nHuy}}(t,x) \doteq \begin{cases} 2^{-\sqrt{\delta}}\displaystyle\int_{x-t}^{x+t} u_0(y)K_0(t,x;y;\mu,\nu^2)\,\mathrm{d}y + 2^{-\sqrt{\delta}}\int_{x-t}^{x+t}\big(u_1(y)+\mu\, u_0(y)\big)K_1(t,x;y;\mu,\nu^2)\,\mathrm{d}y & \text{if } n=1,\\ 2^{1-\sqrt{\delta}}\displaystyle\int_0^t w[u_0](s,x)K_0(t,0;s;\mu,\nu^2)\,\mathrm{d}s + 2^{1-\sqrt{\delta}}\int_0^t w[u_1+\mu\, u_0](s,x)K_1(t,0;s;\mu,\nu^2)\,\mathrm{d}s & \text{if } n \geq 2,\end{cases}
$$

then, in the term u^{Huy} we have the existence of a backward wave front set in the odd dimensional case, that is,

$$
\operatorname{supp} u^{\mathrm{Huy}} \subset \{(t,x) \in [0,\infty) \times \mathbb{R}^n : t-R \leq |x| \leq t+R\},
$$

while in general for u^{nHuy} this is not true. We said in general, as in some special cases the kernel functions K_0 and K_1 may have simplified expressions. For example, when μ, ν^2 satisfy the condition $\delta = 1$, then, the expression of the kernel E is simpler than the general case, namely,

$$E(t, x; b, y; \mu, \nu^2) = (1+t)^{-\frac{\mu}{2}} (1+b)^{\frac{\mu}{2}}.$$

Therefore, for $\delta = 1$ and $n \geq 3$, n odd we get

$$u^{\mathrm{nHuy}}(t, x) = \frac{1}{(n-2)!!} (1+t)^{-\frac{\mu}{2}} \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-3}{2}} \left(t^{n-2} \fint_{\partial B_t(x)} \left(u_1(z) + \tfrac{\mu}{2} u_0(z) \right) \mathrm{d}\sigma_z \right),$$

where we applied simply the fundamental theorem of calculus due to the facts that $w[\varphi](s, x)$ is the s-derivative of a certain function involving spherical means in (10) and $K_0(t, 0; s; \mu, \nu^2) = -\frac{\mu}{2}(1+t)^{-\frac{\mu}{2}}$, $K_1(t, 0; s; \mu, \nu^2) = (1+t)^{-\frac{\mu}{2}}$ do not really depend on s. Also, when $\delta = 1$ and $n \geq 3$ is odd, the term u^{nHuy} provides a backward wave front as well and, hence, Huygens' principle holds. Curiously, in the one dimensional case even in the very special case $\delta = 1$ not only Huygens' principle but also the so-called *incomplete Huygens' principle* fails. The incomplete Huygens' principle, that was introduced in [30], means the presence of a backward wave front for the homogeneous equation when the second data u_1 is identically 0. This is due to the presence of the integral terms in (4) which do not cancel each others for $\delta = 1$ even though $u_1 = 0$ and $f = 0$.

Connections with other hyperbolic models We point out now that the range for the parameters of Gauss' hypergeometric functions in (5) is somehow related to the range of the corresponding parameters for the representation formula of the solution to the Cauchy problem for the Klein-Gordon equation in the anti-de Sitter space-time with complex mass, namely,

$$\begin{cases} w_{tt} - \mathrm{e}^{2t} \Delta w + M^2 w = g(t, x), & x \in \mathbb{R}^n, t > 0, \\ w(0, x) = w_0(x), & x \in \mathbb{R}^n, \\ w_t(0, x) = w_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{52}$$

where $M \in \mathbb{C}$. In fact, considering the change of variables

$$1 + t \doteq \mathrm{e}^{\tau}, \qquad \tau = \log(1+t)$$

and the transformation

$$u(t, x) \doteq \mathrm{e}^{-\frac{\mu-1}{2}\tau} v(\tau, x),$$

we have that u solves (3) if and only if v solves

$$\begin{cases} v_{\tau\tau} - \mathrm{e}^{2\tau} \Delta v - \frac{\delta}{4} v = \mathrm{e}^{\frac{\mu+3}{2}\tau} f(\mathrm{e}^{\tau} - 1, x), & x \in \mathbb{R}^n, \ \tau > 0, \\ v(0, x) = u_0(x), & x \in \mathbb{R}^n, \\ v_\tau(0, x) = \frac{\mu-1}{2} u_0(x) + u_1(x), & x \in \mathbb{R}^n. \end{cases}$$

In particular, the case $\delta = 0$ corresponds to the massless case $M = 0$ in (52). So, it is not surprising to find $(\frac{1}{2}, \frac{1}{2}; 1)$ as parameters in (5), having in mind the corresponding representation formula for the solution of the wave equation in the anti-de Sitter space-time (cf. [33, equations (1.2) and (1.6)]).

On the one hand, for $\delta > 0$ the Cauchy problem (3) can be transformed in a Cauchy problem as in (52) with an imaginary mass. Therefore, we find that $(\frac{1-\sqrt{\delta}}{2}, \frac{1-\sqrt{\delta}}{2}; 1)$ are real parameter as in the corresponding representation for the solution of (52) (cf. [28, page 682]). According to [16], the case $\delta > 0$ corresponds to the *dominant damping case*. Thus, we have that the dominant damping case for the scale-invariant wave equation is related to the Klein-Gordon equation in the anti-de Sitter space-time with imaginary mass. On the other hand, the case $\delta < 0$ (classified as *Klein-Gordon type case* in [16] for the scale-invariant model) is related in the same way to the Klein-Gordon equation in the anti-de Sitter space-time but now with positive mass. Hence, it is not surprising that in both cases we find an analogous situation for the parameters of the hypergeometric function: indeed, there exists a complex number with nontrivial imaginary part that appears in the hypergeometric function in the first two parameters. More precisely, these complex numbers are $\frac{1-i\sqrt{-\delta}}{2}$ for (3) and $\frac{1}{2} + iM$ for (52), cf. [34, equation (0.20)].

However, Klein-Gordon equation in the anti-de Sitter space-time (or de Sitter space-time if we consider the backwards Cauchy problem) is not the only equation which is related to (3). Besides the previous case, we may consider a different change of variables and transformation of the dependent variable in the case $\delta \in (0, 1]$, namely,

$$1 + t \doteq (1+\tau)^{\ell+1}, \quad x \doteq (\ell+1) y \quad \text{and} \quad v(\tau, y) \doteq (1+t)^{\frac{\mu-1+\sqrt{\delta}}{2}} u(t, x),$$

where $\ell \doteq \frac{1-\sqrt{\delta}}{\sqrt{\delta}}$. Then, u solves (3) if and only if v solves

$$\begin{cases} v_{\tau\tau} - (1+\tau)^{2\ell}\Delta_y v = \frac{1}{\delta}(1+\tau)^{\frac{\mu-1+\sqrt{\delta}}{2}(\ell+1)+2\ell} f((1+\tau)^{\ell+1}-1, (\ell+1)y), & y \in \mathbb{R}^n,\ \tau > 0, \\ v(0,y) = u_0(y), & y \in \mathbb{R}^n, \\ v_\tau(0,y) = \frac{\mu-1+\sqrt{\delta}}{2}(\ell+1)u_0((\ell+1)y) + (\ell+1)u_1((\ell+1)y), & y \in \mathbb{R}^n. \end{cases}$$

Employing the representation formula given in [19] for the solution of the Cauchy problem

$$\begin{cases} w_{tt} - (1+t)^{2\ell}\Delta w = g(t,x), & x \in \mathbb{R}^n, t > 0, \\ w(0,x) = w_0(x), & x \in \mathbb{R}^n, \\ w_t(0,x) = w_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{53}$$

it is possible to find the representation formula for (3) in the one-dimensional case. In turn, the representation formula for (53) in the case $n = 1$ is obtained in [19, Section 4] by following the works [26, 31] on the generalized Tricomi equation (Gellerstedt equation). For a summary overview on *Yagdjian's Integral Transform approach* applied to several hyperbolic equations with variable coefficients, one can see also [29].

Future applications of the representation formulae In the forthcoming paper [21], the representation formulae which are derived in this work will be applied to study the blow-up dynamic of the semilinear wave equation with damping and mass terms in the scale-invariant case and with nonlinearity of derivative type $|\partial_t u|^p$.

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References

[1] W. Chen, A. Palmieri, Weakly coupled system of semilinear wave equations with distinct scale-invariant terms in the linear part, preprint, arXiv:1809:10103v1, 2018.

[2] M. D'Abbicco, The threshold of effective damping for semilinear wave equations, *Math. Methods Appl. Sci.* **38**(6) (2015), 1032–1045.

[3] M. D'Abbicco, S. Lucente, NLWE with a special scale invariant damping in odd space dimension, *Discrete Contin. Dyn. Syst.* Dynamical systems, differential equations and applications. 10th AIMS Conference. Suppl. (2015), 312–319, doi: 10.3934/proc.2015.0312.

[4] M. D'Abbicco, S. Lucente, M. Reissig, A shift in the Strauss exponent for semilinear wave equations with a not effective damping, *J. Differential Equations* **259**(10) (2015), 5040–5073.

[5] M. D'Abbicco, A. Palmieri, $L^p - L^q$ estimates on the conjugate line for semilinear critical dissipative Klein-Gordon equations, preprint.

[6] L. C. Evans, *Partial Differential Equations*, AMS, Graduate Studies in Mathematics Vol. 19, First Edition, 1998.

[7] M. Ikeda, M. Sobajima, Life-span of solutions to semilinear wave equation with time-dependent critical damping for specially localized initial data, *Math. Ann.* (2018), https://doi.org/10.1007/s00208-018-1664-1.

[8] F. John, *Plane waves and spherical means applied to partial differential equations*, Interscience Publishers, New York-London, 1955.

[9] M. Kato, M. Sakuraba, Global existence and blow-up for semilinear damped wave equations in three space dimensions, preprint, arXiv:1807.04327v1, 2018.

[10] N. A. Lai, H. Takamura, K. Wakasa, Blow-up for semilinear wave equations with the scale invariant damping and super-Fujita exponent, *J. Differential Equations* **263**(9) (2017), 5377–5394.

[11] E. Mitidieri , S. I. Pohozaev, A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities, *Proc. Steklov Inst. Math.* **234** (2001), 1-362.

[12] W. Nunes do Nascimento, A. Palmieri, M. Reissig, Semi-linear wave models with power non-linearity and scale-invariant time-dependent mass and dissipation, *Math. Nachr.* **290**(11/12) (2017), 1779–1805.

[13] A. Palmieri, Global existence of solutions for semi-linear wave equation with scale-invariant damping and mass in exponentially weighted spaces, *J. Math. Anal. Appl.* **461**(2) (2018), 1215–1240.

[14] A. Palmieri, Global existence results for a semilinear wave equation with scale-invariant damping and mass in odd space dimension, in M. D'Abbicco et al. (eds.), *New Tools for Nonlinear PDEs and Application*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-10937-0_12

[15] A. Palmieri, A global existence result for a semilinear wave equation with scale-invariant damping and mass in even space dimension, preprint, arXiv:1804.03978v1, 2018.

[16] A. Palmieri, Global in time existence and blow-up results for a semilinear wave equation with scale-invariant damping and mass, PhD thesis, TU Bergakademie Freiberg, 2018, 279 pp.

[17] A. Palmieri, A note on a conjecture for the critical curve of a weakly coupled system of semilinear wave equations with scale-invariant lower order terms, preprint, arXiv:1812.06588v1, 2018.

[18] A. Palmieri, M. Reissig, Semi-linear wave models with power non-linearity and scale-invariant time-dependent mass and dissipation, II, *Math. Nachr.*, **291**(11/12) (2018), 1859–1892.

[19] A. Palmieri, M. Reissig, A competition between Fujita and Strauss type exponents for blow-up of semi-linear wave equations with scale-invariant damping and mass, *J. Differential Equations* **266** (2019), 1176-1220.

[20] A. Palmieri, Z. Tu, Lifespan of semilinear wave equation with scale invariant dissipation and mass and sub-Strauss power nonlinearity, *J. Math. Anal. Appl.* (2018), https://doi: 10.1016/j.jmaa.2018.10.015.

[21] A. Palmieri, Z. Tu, A blow-up result for a semilinear wave equation with scale-invariant damping and mass and nonlinearity of derivative type, in preparation.

[22] C. D. Sogge, *Lectures on non-linear wave equations. Second edition*, International Press, Boston, MA, 2008.

[23] Z. Tu, J. Lin, A note on the blowup of scale invariant damping wave equation with sub-Strauss exponent, preprint, arXiv:1709.00866v2, 2017.

[24] K. Wakasa, The lifespan of solutions to semilinear damped wave equations in one space dimension, *Commun. Pure Appl. Anal.* **15**(4) (2016), 1265-1283.

[25] Y. Wakasugi, Critical exponent for the semilinear wave equation with scale invariant damping, in: M. Ruzhansky , V. Turunen (Eds.) Fourier Analysis. Trends in Mathematics. Birkhäuser, Cham (2014), https://doi.org/10.1007/978-3-319-02550-6_19.

[26] K. Yagdjian, A note on the fundamental solution for the Tricomi-type equation in the hyperbolic domain, *J. Differential Equations* **206** (2004), 227-252.

[27] K. Yagdjian, The self-similar solutions of the Tricomi-type equations, *Z. Angew. Math. Phys.* **58** (2007), 612-645.

[28] K. Yagdjian, The semilinear Klein-Gordon equation in de Sitter spacetime, *Discrete Contin. Dyn. Syst. Ser. S* **2**(3) (2009), 679–696.

[29] K. Yagdjian, Fundamental solutions for hyperbolic operators with variable coefficients, *Rend. Istit. Mat. Univ. Trieste* **42** (2010), suppl., 221-243.

[30] K. Yagdjian, Huygens' principle for the Klein-Gordon equation in the de Sitter spacetime, *J. Math. Phys.* **54**(9) (2013), 091503, 18 pp.

[31] K. Yagdjian, Integral transform approach to generalized Tricomi equations, *J. Differential Equations* **259**(11) (2015),5927-5981.

[32] K. Yagdjian, Integral transform approach to solving Klein–Gordon equation with variable coefficients, *Math. Nachr.* **288**(17/18) (2015), 2129–2152

[33] K. Yagdjian, A. Galstian, Fundamental solutions of the wave equation in Robertson-Walker spaces, *J. Math. Anal. Appl.* **346**(2) (2008), 501-520.

[34] K. Yagdjian, A. Galstian, Fundamental Solutions for the Klein-Gordon Equation in de Sitter Spacetime, *Comm. Math. Phys.* **285** (2009), 293-344.