

This article has been published in a revised form in

Discrete and Continuous Dynamical Systems, Series S

<https://www.aims sciences.org/article/doi/10.3934/dcdss.2020364>

This version is free to download for private research and study only. Not for redistribution, re-sale or use in derivative works.

EQUIPARTITION OF ENERGY FOR NONAUTONOMOUS DAMPED WAVE EQUATIONS

MARCELLO D'ABBICCO AND GIOVANNI GIRARDI

Dipartimento di Matematica
University of Bari Aldo Moro
Via E. Orabona 4, 70125 Bari, Italy

GISÉLE RUIZ GOLDSTEIN AND JEROME A. GOLDSTEIN*

Department of Mathematical Sciences
University of Memphis
373 Dunn Hall, Memphis, TN 38152-3240, USA

SILVIA ROMANELLI

Dipartimento di Matematica
University of Bari Aldo Moro
Via E. Orabona 4, 70125 Bari, Italy

Dedicated to Michel Pierre on his seventieth birthday

ABSTRACT. The kinetic and potential energies for the damped wave equation

$$u'' + 2Bu' + A^2u = 0 \quad (\text{DWE})$$

are defined by

$$K(t) = \|u'(t)\|^2, \quad P(t) = \|Au(t)\|^2,$$

where A, B are suitable commuting selfadjoint operators. Asymptotic equipartition of energy means

$$\lim_{t \rightarrow \infty} \frac{K(t)}{P(t)} = 1 \quad (\text{AEE})$$

for all (finite energy) non-zero solutions of (DWE). The main result of this paper is the proof of a result analogous to (AEE) for a nonautonomous version of (DWE).

1. Introduction. Denote by $\mathbb{R}_+ = [0, +\infty)$ and by $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ a complex Hilbert space. Let $A = A^* \geq 0$ be an injective selfadjoint operator on $\mathcal{H}$. The corresponding initial value problem for the wave equation is

$$u'' + A^2u(t) = 0, \quad u(0) = f, \quad u'(0) = g, \quad (1)$$

where $f \in \mathcal{D}(A)$ and $g \in \mathcal{H}$. Let $u \in C^2(\mathbb{R}, \mathcal{H})$ be a $\mathcal{D}(A)$ valued strong solution (which corresponds to $f \in \mathcal{D}(A^2)$ and $g \in \mathcal{D}(A)$). Let us introduce the vector variable

$$U = \begin{pmatrix} Au \\ u' \end{pmatrix} \quad (2)$$

2010 *Mathematics Subject Classification.* Primary: 34G10, 35L90; Secondary: 76D33.

Key words and phrases. Equipartition of energy, nonautonomous systems, asymptotics, damped wave equations.

* Corresponding author: Jerome A. Goldstein .

which satisfies the first order differential equation

$$U'(t) = \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix} U(t) =: MU(t)$$

where

$$M = \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix} = A \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = -M^*$$

on $\mathcal{H}^2 = \mathcal{H} \oplus \mathcal{H}$. The functional calculus for selfadjoint operators allows us to show (see [7])

$$e^{tM} = \begin{pmatrix} \cos(tA) & \sin(tA) \\ -\sin(tA) & \cos(tA) \end{pmatrix}.$$

This can be verified by differentiating both sides with respect to t . It is easily seen that $\{e^{tM} : t \in \mathbb{R}\}$ is a (C_0) unitary group on $\mathcal{H}^2$, and thus the total energy

$$E(t) = K(t) + P(t) := \|u'(t)\|^2 + \|Au(t)\|^2 \quad (3)$$

is conserved for all strong solutions $u \in C^2(\mathbb{R}, \mathcal{H})$ which correspond to $f \in \mathcal{D}(A^2)$ and $g \in \mathcal{D}(A)$ in (1). The same is true for all mild solutions which correspond to $\left\{ e^{tM} \begin{pmatrix} Af \\ g \end{pmatrix} : t \in \mathbb{R} \right\}$ for $f \in \mathcal{D}(A)$ and $g \in \mathcal{H}$. Here $K(t)$ [resp. $P(t)$] represents the kinetic [resp. potential] energy, and $E(t) = E(0)$ is the total energy which is conserved.

Equipartition of energy, i.e.,

$$\lim_{t \rightarrow \pm\infty} K(t) = \lim_{t \rightarrow \pm\infty} P(t) = \frac{E}{2}$$

(with $E = E(0)$) for all finite energy mild solutions holds iff

$$\langle e^{itA}h, k \rangle \rightarrow 0$$

as $t \rightarrow \pm\infty$ for all $h, k \in \mathcal{H}$. By the Riemann-Lebesgue Lemma, a sufficient condition for this condition to hold is that A is spectrally absolutely continuous (SAC). In order to define this, we write

$$A = \int_{[0, \infty)} \lambda dE(\lambda)$$

by the spectral theorem using a resolution of the identity. Then A is SAC means that the bounded monotone function

$$\lambda \rightarrow \|E(\lambda)f\|^2$$

is absolutely continuous on $[0, \infty]$ (with limits 0 at 0 and $\|f\|^2$ at ∞) for all $f \in \mathcal{H}$. This was proved in [5, 6].

The unique mild solution to (1) can be written by d'Alembert's formula as

$$u(t) = e^{itA}F + e^{-itA}G$$

where

$$\begin{aligned} F, G &\in \mathcal{D}(A), f \in \mathcal{D}(A), g \in \mathcal{H}, \\ AF &= \frac{1}{2}(Af - ig), AG = \frac{1}{2}(Af + ig), \\ f &= F + G, g = i(AF - AG). \end{aligned}$$

Then

$$\begin{aligned} K(t) &= \|u'(t)\|^2 = \|e^{itA}AF - e^{-itA}AG\|^2, \\ P(t) &= \|Au(t)\|^2 = \|e^{itA}AF + e^{-itA}AG\|^2. \end{aligned}$$

The law of cosines together with unitarity of e^{itA} implies

$$E(t) = K(t) + P(t) = 2(\|AF\|^2 + \|AG\|^2) = E(0),$$

and

$$K(t) - P(t) = -4 \operatorname{Re} \langle e^{2itA} AF, AG \rangle.$$

Thus energy is conserved, and energy is asymptotically equipartioned iff

$$K(t) - P(t) \rightarrow 0 \quad \text{as } t \rightarrow \pm\infty$$

for all $F, G \in \mathcal{D}(A)$. In the SAC case, the Riemann-Lebesgue Lemma gives the conclusion. The factorization

$$0 = \left(\frac{d^2}{dt^2} - A^2\right)u = \left(\frac{d}{dt} - A\right)\left(\frac{d}{dt} + A\right)u$$

leads to the d'Alembert formula

$$u(t) = e^{itA}AF + e^{-itA}AG$$

since

$$\mathcal{N} \left[\left(\frac{d}{dt} - A\right)\left(\frac{d}{dt} + A\right) \right] = \mathcal{N} \left(\frac{d}{dt} - A\right) + \mathcal{N} \left(\frac{d}{dt} + A\right).$$

This uses the fact that the space-time operators in parenthesis are injective and commute. Here $\mathcal{N}$ denotes the null space. Thus $\{e^{\pm itA}h : t \in \mathbb{R}\}$ enables us to find the general mild solution of

$$u' = \pm iAu, \quad u(0) = h,$$

as h varies. Care must be used in finding a nonautonomous version of the preceding.

2. The nonautonomous framework. Let $B = B^*$ on $\mathcal{H}$. Then $u(t) = e^{itB}h$ is the unique mild solution of

$$u' - iBu = 0, \quad u(0) = h$$

as $h \in \mathcal{H}$ varies. Moreover, by the Spectral Theorem,

$$B = U_0 M_b U_0^{-1}$$

where $U_0 : \mathcal{H} \rightarrow L^2 = L^2(\Omega, \Sigma, \mu)$ is unitary from $\mathcal{H}$ to some concrete complex L^2 space, $b : \Omega \rightarrow \mathbb{R}$ is Σ -measurable and

$$(M_b g)(w) = b(w)g(w)$$

for $g \in D(M_b) = \{g \in L^2(\Omega, \Sigma, \mu) : gb \in L^2\}$. If $\{B(t) : t \in \mathbb{R}\}$ is a family of commuting selfadjoint operators on $\mathcal{H}$, then

$$B(t) = U_0 M_{b(t)} U_0^{-1}$$

for U_0, L^2 as above and for $b(t) : \Omega \rightarrow \mathbb{R}$, Σ -measurable for all $t \in \mathbb{R}$. For simplicity, we will restrict to nonnegative time, $t \geq 0$.

In [4] the authors consider the nonautonomous wave equation

$$u'' + A^2(t)u(t) = 0, \quad u(0) = f, \quad u'(0) = g. \quad (4)$$

In particular, they assume the following about $A = A(t)$.

HYP 1. Let $\mathcal{A} = \{A(t) : t \in \mathbb{R}_+\}$ be a family of nonnegative selfadjoint operators acting on $\mathcal{H}$ such that the domains

$$\mathcal{D}_0 := \mathcal{D}(A(t)), \mathcal{D}_1 := \mathcal{D}(A^2(t))$$

are both independent of t . Moreover, the commutator $[A(t), A(s)] = A(t)A(s) - A(s)A(t) = 0$ in the sense that the bounded operators $e^{i\sigma A(s)}$ and $e^{i\tau A(t)}$ commute for all $t, s, \tau, \sigma \in \mathbb{R}_+$. Further, assume that

$$A(\cdot)f \in C^1(\mathbb{R}_+, \mathcal{H})$$

for all $f \in \mathcal{D}_0$, and there exists a bounded function $k_1 \in C([0, \infty), \mathbb{R})$ with $k_1(t) \geq \epsilon > 0$ such that

$$\|A(t)f\| \leq k_1(t)\|A(0)f\|, \quad (5)$$

for all $f \in \mathcal{D}_0$ and all $t \in \mathbb{R}_+$.

Note that (5) implies that $0 \in \rho(A(t))$ holds either for all $t \in \mathbb{R}_+$ or for no $t \in \mathbb{R}_+$.

By assuming HYP 1 one gains some useful properties which follow by the Spectral Theorem. Let us define

$$C(t) := \int_0^t A(s)ds, \quad (6)$$

and then $A = A(t)$ is unitarily equivalent to a multiplication operator, i.e., $A(t) = U_0 M_{a(t)} U_0^{-1}$. Hence, one has

$$C(t) = U_0 M_{\int_0^t a(s)ds} U_0^{-1}. \quad (7)$$

In particular, we may view $a(t) : \Omega \rightarrow (0, \infty)$ as $a(t, \omega)$ with $a(\cdot, \cdot) : \mathbb{R}_+ \times \Omega \rightarrow (0, \infty)$. By HYP 1 and an old theorem of J. L. Doob [3], without loss of generality we may assume $a(\cdot, \cdot)$ is jointly measurable on $\mathbb{R}_+ \times \Omega$ in the (Borel sets $\times \Sigma$) sense. Note that here each $a(t, \cdot)$ is defined on $\Omega \setminus N_t$ where $\mu(N_t) = 0$ for each $t \in \mathbb{R}_+$. There are uncountably many null sets N_t , but Doob's theorem says that this is not a problem; they can be chosen so that $a(t, \omega)$ is jointly measurable and certain integrals over $\mathbb{R}_+ \times \Omega$ will exist.

Another form of the Spectral Theorem says that, due to the commuting hypothesis in HYP 1, there is a function

$$F : \mathbb{R}_+ \times (0, \infty) \rightarrow (0, \infty) \quad (8)$$

such that for each $t \in \mathbb{R}_+$,

$$F(t, A(0)) = A(t); \quad (9)$$

moreover, $F(t, x)$ is a C^1 -function of $t \in \mathbb{R}_+$ for each fixed x .

Now using (6) and (7), we have

$$C(t) = U_0 M_{P(t)} U_0^{-1}$$

where

$$P(t) = \int_0^t a(s)ds.$$

Also, by (8) and (9), for $t \in \mathbb{R}_+$, we find

$$C(t) = G(t, A(0))$$

where

$$G(t, x) = \int_0^t F(s, x)ds.$$

Now, let $F_1, F_2 \in \mathcal{D}_1$ and $t \geq 0$. Let

$$w(t) := e^{iC(t)} F_1 + e^{-iC(t)} F_2, \quad (10)$$

$$\tilde{w}(t) := e^{iC(t)} F_1 - e^{-iC(t)} F_2. \quad (11)$$

Suppressing the argument t , we get

$$w' = iAe^{iC} F_1 - iAe^{-iC} F_2$$

and

$$w'' = e^{iC}(iA' - A^2)F_1 + e^{-iC}(-iA' - A^2)F_2.$$

A similar calculation gives

$$\tilde{w}'' = e^{iC}(iA' - A^2)F_1 - e^{-iC}(-iA' - A^2)F_2.$$

It follows that

$$\begin{pmatrix} w \\ \tilde{w} \end{pmatrix}''(t) = \begin{pmatrix} -A(t)^2 & iA'(t) \\ iA'(t) & -A(t)^2 \end{pmatrix} \begin{pmatrix} w \\ \tilde{w} \end{pmatrix}(t) =: Q(t) \begin{pmatrix} w \\ \tilde{w} \end{pmatrix}(t). \quad (12)$$

We can rewrite this as

$$W'' = QW$$

where $W = \begin{pmatrix} w \\ \tilde{w} \end{pmatrix}$ and

$$Q(t) = -A(t)^2 + iA'(t) \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

where $A(t)^2$ is identified with $A(t)^2 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$ for convenience.

Several conclusions follow. In the autonomous case $Q = -A^2$, and each component Au, u' of $U = \begin{pmatrix} Au \\ u' \end{pmatrix}$ satisfies the same wave equation $v'' + A^2v = 0$. But in the nonautonomous case, the two components w and $\tilde{w}$ of W satisfy different equations, namely

$$\begin{aligned} w'' + A^2w &= iA'\tilde{w} \\ \tilde{w}'' + A^2\tilde{w} &= iA'w. \end{aligned}$$

In other words,

$$z_{\pm}(t) = e^{\pm iB(t)} f_{\pm}$$

for $f_{\pm} \in \mathcal{D}_1$ satisfies

$$z_{\pm}'' + A^2z_{\pm} = \pm iA'z_{\pm}$$

and these are different second order single equations. Furthermore, $Q(t)$ is normal but not selfadjoint because the imaginary part, $A' \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, is nonzero.

So the relevant second order equation here that is wellposed is a second order system in $\mathcal{H}$ or, equivalently, a second order equation in $\mathcal{H}^2$. For this system wellposedness was proved in [4]. Asymptotic equipartition of energy can be considered here even though the equation does not conserve energy.

3. The nonautonomous undamped system. We consider the system

$$W''(t) = Q(t)W(t)$$

in $\mathcal{H}^2$, where

$$Q(t) = -A(t)^2 + iA'(t) \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

As mentioned before, in the autonomous case, $A' \equiv 0$ and both w and $\tilde{w}$ satisfy

$$u'' + A^2 u = 0.$$

But when A depends in a nontrivial way on t , neither w nor $\tilde{w}$ satisfies a single second order equation. Still the pair

$$W = \begin{pmatrix} w \\ \tilde{w} \end{pmatrix}$$

does satisfy a second order equation generated by the evolution operator family $\{Z(t, s) : t \geq s \geq 0\}$ which can be described as follows. Now $W'' = QW$ implies

$$\tilde{W}' := \begin{pmatrix} W \\ W' \end{pmatrix}' = \begin{pmatrix} 0 & I \\ Q & 0 \end{pmatrix} \begin{pmatrix} W \\ W' \end{pmatrix} =: \tilde{Q} \begin{pmatrix} W \\ W' \end{pmatrix} = \tilde{Q}\tilde{W}.$$

Let

$$Z(t, s) = \int_s^t \tilde{Q}(\tau) d\tau.$$

Then

$$\tilde{W}(t) = Z(t, s)\tilde{W}(s)$$

for $t \geq s \geq 0$ and $Z(t, s)$ is *not* unitary because $\tilde{Q}$ is normal but not skewadjoint for each $t \in \mathbb{R}_+$.

Under the assumptions HYP 1 the authors in [4] proved the following wellposedness result.

Theorem 3.1. (*Wellposedness Theorem*) [4] *Let HYP1 hold. Then the problem*

$$\frac{d^2 W}{dt^2}(t) = Q(t)W(t), \quad W(0) = H_1, \quad W'(0) = H_2$$

with $H_1, H_2 \in \mathcal{D}_1^2$ has a unique strong solution in $C^2(\mathbb{R}_+, \mathcal{H}^2)$ of the form

$$W(t) = \begin{pmatrix} w(t) \\ \tilde{w}(t) \end{pmatrix}$$

where

$$w(t) = e^{iC(t)}F_1 + e^{-iC(t)}F_2, \quad \tilde{w}(t) = e^{iC(t)}F_1 - e^{-iC(t)}F_2,$$

$$H_1 = \begin{pmatrix} F_1 + F_2 \\ F_1 - F_2 \end{pmatrix}, \quad H_2 = iA(0) \begin{pmatrix} F_1 - F_2 \\ F_1 + F_2 \end{pmatrix}.$$

Here $C(t)$ is defined in (6), F_1, F_2 are as in Section 2, and each component of $\frac{1}{2}((iA(0))^{-1}H_2 + H_1)$ is F_1 , and each component of $-\frac{1}{2}(iA(0))^{-1}H_2 - H_1$ is F_1 .

The point is that solutions are built from linear combinations of $e^{iC(t)}F_1$ and $e^{-iC(t)}F_2$, which is the spirit of a nonautonomous d'Alembert's formula.

In [4] the authors considered the “partial” energies

$$E_1(t) := \|w'(t)\|^2, \quad E_2(t) := \|\tilde{w}'(t)\|^2, \quad (13)$$

$$E_3(t) := \|A(t)w(t)\|^2, \quad E_4(t) := \|A(t)\tilde{w}(t)\|^2. \quad (14)$$

They introduced the following additional assumptions for $A = A(t)$.

HYP 2. Assume HYP 1 holds and that the function k_1 of HYP 1 satisfies

$$0 < \epsilon_1 \leq k_1(t) \leq \frac{1}{\epsilon_1}$$

for some $\epsilon_1 > 0$ and all $t \in \mathbb{R}_+$. Assume that for every $f \in \mathcal{D}_0$,

$$A'(\cdot)f \in L^1(\mathbb{R}_+, \mathcal{H}).$$

It follows that

$$A(t)f = \int_0^t A'(s)f ds + A(0)f \rightarrow \int_0^\infty A'(s)f ds + A(0)f =: A(\infty)f$$

as $t \rightarrow \infty$, for all $f \in \mathcal{D}_0$, and we further assume

$$\int_0^\infty \|A(t)f - A(\infty)f\| dt < \infty$$

for all $f \in \mathcal{D}_0$ and that $A(\infty)$ (which is symmetric) is selfadjoint, spectrally absolutely continuous (SAC) and HYP 1 holds for $\{A(t) : t \in [0, \infty]\}$.

Finally they made the following assumption on $F = F(t, x)$ defined in (9).

HYP 3. Recall from HYP 1 and HYP 2 that

$$A(t) = F(t, A(0))$$

for $0 \leq t \leq \infty$ for a function $F = F(t, x)$

$$F : [0, \infty] \times (0, \infty) \rightarrow (0, \infty).$$

We assume $F \in C^1([0, \infty] \times (0, \infty))$.

Under these assumptions one can prove the following result (see [4]).

Theorem 3.2 (Asymptotic Equipartition of Energy). Assume HYP 1, HYP 2 and HYP 3. Then $A(t)$ is spectrally absolutely continuous for $0 \leq t \leq \infty$. For all finite energy solutions, as $t \rightarrow \infty$,

$$2E_j(t) \rightarrow \|A(\infty)F_1\|^2 + \|A(\infty)F_2\|^2 \quad \text{for } j = 1, 2, 3, 4,$$

for the partial energies of (13), where F_1, F_2 are as in Theorem 3.1.

Now that we have established the context for the nonautonomous undamped wave equation, we introduce damping.

In Section 4 we introduce damping in the autonomous case. The new results begin in Section 5.

4. The damped autonomous wave equation. The idea of passing from analyzing

$$u'' + A_1^2 u = 0 \quad (15)$$

to analyzing

$$u'' + 2Bu + A_2^2 u = 0 \quad (16)$$

is to let $v(t) = e^{-tB}u(t)$ when u is a (strong) solution of (15). Here A_1, A_2, B are commuting selfadjoint operators with A_1, A_2 injective. A straightforward calculation shows that

$$v'' + 2Bv' + (A_1^2 - B^2)v$$

holds, i.e., (16) holds with $A_2^2 = A_1^2 - B^2$ and A_2^2 is nonnegative (or positive) if $B^2 \leq A_1^2$ (or $B^2 < A_1^2$).

We can write (16) as a system

$$V' = MV$$

for

$$V = \begin{pmatrix} A_2 v \\ v' \end{pmatrix},$$

where

$$M = \begin{pmatrix} 0 & A_2 \\ -A_2 & -2B \end{pmatrix}.$$

Consider the simple example of $M = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$. Then $M^* = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}$ and

$$MM^* = \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix} \neq M^*M = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}.$$

Thus $V' = M(t)V$ is a nonautonomous system with noncommuting operators $\{M(t)\}$ where $B(t) \neq 0$. So the Spectral Theorem cannot be applied. Instead we rely on a generalized d'Alembert formula. First we treat the autonomous case.

Solve

$$u'' + 2B + A^2u = 0 \tag{17}$$

with A, B commuting selfadjoint operators with $B \geq 0$. Then we obtain

$$u(t) = e^{itC_+} f_+ + e^{-itC_-} f_- \tag{18}$$

where $C_{\pm}$ satisfy

$$C^2 + 2BC + A^2C = 0,$$

which leads to

$$C_{\pm} = -B \pm (B^2 - A^2)^{\frac{1}{2}}.$$

Our convention for square roots is as follows. If $F = F^* \geq 0$, then $F^{\frac{1}{2}}$ is the unique nonnegative selfadjoint square root of F . Otherwise, F can be written uniquely as $F = F_+ - F_-$ where

$$F = \int_{\mathbb{R}} \lambda dE(\lambda), F_+ = \int_{[0, \infty)} \lambda dE(\lambda) \geq 0, F_- = \int_{(-\infty, 0)} \lambda dE(\lambda) \geq 0,$$

and we define

$$F^{\frac{1}{2}} := F_+^{\frac{1}{2}} + i(-F_-)^{\frac{1}{2}}.$$

Now (18) makes sense and gives solutions of (16) with $A_2 = A_1$. This is the general solution if 0 is not an eigenvalue of $B^2 - A^2$ (in the latter case $te^{itC_+}h$ is also a mild solution for some nonzero $h \in \mathcal{H}$).

Our strategy for the nonautonomous case is to study

$$v_{\pm}(t) = R_{\pm}(t)f_+ \pm R_{-}(t)f_-$$

for $t \geq s \geq 0$ where

$$R_{\pm}(t) = e^{i \int_s^t S_{\pm}(r) dr}$$

with

$$S_{\pm}(r) = -B(r) \pm (B(r)^2 - D(r)^2)^{\frac{1}{2}}, D^2(r) = A(r)^2 - B(r)^2.$$

Then, writing $v = v_+$ and $\tilde{v} = v_-$, $V = \begin{pmatrix} v \\ \tilde{v} \end{pmatrix}$ will satisfy an undamped nonautonomous wave equation in $\mathcal{H} \oplus \mathcal{H}$, that is, a second order wave equation in the

form of a 2×2 system. This can be treated using the methods developed in [4] and this leads to our asymptotic equipartition of energy results for the damped 2×2 nonautonomous wave equation system.

5. The damped nonautonomous wave equation. We consider the abstract nonautonomous wave equation with a damping term of the type

$$u'' + 2B(t)u' + A(t)^2u = 0. \quad (19)$$

where $A = A(t)$ satisfies the assumptions HYP 1 - HYP 3. We make the following assumptions on $B = B(t)$.

HYP 4. Let $\mathcal{B} = \{B(t) : t \in [0, \infty)\}$ be a family of nonnegative selfadjoint operators acting on $\mathcal{H}$ such that for all $t \in \mathbb{R}_+$, the domain $\mathcal{D}_2 := D(B(t))$ is independent of t and

$$\mathcal{D}_2 \supset D((A(t))^{\frac{1}{2}-\delta_1}) \quad \text{for all } t \in \mathbb{R}_+ \text{ and some } \delta_1 > 0.$$

For all $f \in \mathcal{D}_2$, $B(\cdot)f \in C^2(\mathbb{R}_+, \mathcal{H})$, and $B'(t)$ is symmetric for all $t \in \mathbb{R}_+$. Moreover, assume that for each $t \in [0, \infty)$ and $f \in \mathcal{D}_2$ one has

$$\|(B(t) + B'(t))f\| \geq k_1(t)\|(B(0) + B'(0))f\|,$$

where the function $k_1 = k_1(t)$ was introduced in the assumptions HYP 1, HYP 2. Finally, suppose that for all $C_1, C_2 \in \mathcal{A} \cup \mathcal{B}$, C_1 commutes with C_2 in the sense that

$$[(\lambda I - C_1)^{-1}, (\mu I - C_2)^{-1}] = 0, \quad \text{for all } \lambda, \mu \in \mathbb{C} \setminus \mathbb{R},$$

or equivalently,

$$[e^{isC_1}, e^{itC_2}] = 0, \quad \text{for all } s, t \in \mathbb{R}_+.$$

HYP 5. For all $f \in \mathcal{D}_2$, assume

$$B'(\cdot)f \in L^1(\mathbb{R}_+, \mathcal{H}), \quad B''(\cdot)f \in L^1(\mathbb{R}_+, \mathcal{H}).$$

and, as $t \rightarrow \infty$,

$$\begin{aligned} B(t)f &= \int_0^t B'(s)f ds + B(0)f \rightarrow \int_0^\infty B'(s)f ds + B(0)f =: B(\infty)f, \\ B'(t)f &= \int_0^t B''(s)f ds + B'(0)f \rightarrow \int_0^\infty B''(s)f ds + B'(0)f = 0. \end{aligned} \quad (20)$$

Moreover assume that $B(\infty)$ (which is symmetric) is selfadjoint, spectrally absolutely continuous (SAC) and HYP 4 holds for $\{B(t) : t \in [0, \infty)\}$.

Define the new operator $D = D(t) = D(t)^* \geq 0$ by

$$D(t)^2 := A(t)^2 - B(t)^2 - B'(t),$$

and suppose that there exists $\delta > 0$ such that for all $t \in [0, \infty)$

$$D(t)^2 := A(t)^2 - B(t)^2 - B'(t) \geq \delta^2 I.$$

In fact, we need only to assume this for t sufficiently large.

Remark 1. Let us observe that if we define

$$Q_\pm(t) := e^{\pm \int_s^t B(r) dr} \quad (21)$$

for $s \in \mathbb{R}_+$, then u is a solution of (DWE) if and only if $v := Q_+(t)u$ is a solution of

$$v_{tt} + D(t)^2 v = 0. \quad (22)$$

Note that $\hat{v}(t) := Q_-(t)u$ satisfies a different differential equation. Indeed it satisfies

$$\hat{v}_{tt} + \hat{D}(t)^2 \hat{v} = 0$$

where $\hat{D}(t)^2 = A(t)^2 - B(t)^2 + B'(t) \geq \delta^2 I$, for t large enough. In the autonomous case, when (DWE) becomes

$$u_{tt} + 2Bu_t + A^2u = 0, \quad (23)$$

for u a solution of (23),

$$v(t) := e^{tB}u(t)$$

satisfies

$$v_{tt} + D^2v = 0, \quad (24)$$

where $D^2 = A^2 - B^2$, and we assume $D^2 \geq \delta^2 I$ for some $\delta > 0$.

As a consequence of assumptions HYP 1, HYP 2, HYP 4 and HYP 5 the operator $D(t)$ satisfies the assumptions HYP 1 and HYP 2. Then

$$D'(\cdot)f \in L^1(\mathbb{R}_+, \mathcal{H})$$

and

$$D(t)f = \int_0^t D'(s)f ds + D(0)f \rightarrow \int_0^\infty D'(s)f ds + D(0)f =: D(\infty)f. \quad (25)$$

Moreover by the Spectral Theorem we know that, due to the commuting hypothesis in HYP 4, there is a function

$$H : \mathbb{R}_+ \times (0, \infty) \rightarrow (0, \infty) \quad (26)$$

such that for each $t \in \mathbb{R}_+$,

$$H(t, D(0)) = D(t); \quad (27)$$

moreover $H(t, x)$ is a C^1 -function of $t \in \mathbb{R}_+$ for each fixed x .

We make the following additional assumptions on $D = D(t)$.

HYP 6. *We assume that*

$$\int_0^\infty \|D(t)f - D(\infty)f\| dt < \infty$$

for all $f \in \mathcal{D}_0$, where $D(\infty)$ is defined in (25); moreover we suppose that $D(\infty)$ (which is symmetric) is selfadjoint, spectrally absolutely continuous (SAC) and HYP 1 holds for $\{D(t) : t \in [0, \infty]\}$ and k_1 as in HYP 1. In addition, we suppose that the function $H = H(t, x)$ defined in (27) belongs to $C^1([0, \infty] \times (0, \infty))$.

Remark 2. Consider a spectral representation where all the commuting (essentially) selfadjoint operators $A(t)$, $B(t)$, $B'(t)$ are multiplication operators on $L^2 = L^2(\Lambda, \Sigma, \mu)$ by $a(t, \lambda)$, $b(t, \lambda)$, $\frac{d}{dt}b(t, \lambda)$ and in which $f \in \mathcal{D}(A(0)^2)$ is represented by $\hat{f} : \Lambda \rightarrow \mathbb{C}$. Then there is a unitary operator $U : \mathcal{H} \rightarrow L^2$ such that

$$U^{-1}c(t)U = C(t) \text{ for } C = A, B, B', \text{ and}$$

$c = a$, b , $\frac{d}{dt}b$ are real functions on Λ acting as multiplication operators, and $\hat{f} = Uf$. Then

$$\begin{aligned} & \|D(t)f - D(\infty)f\| \\ &= \|(A(t)^2 - B(t)^2 - B'(t))^{\frac{1}{2}}f - (A(\infty)^2 - B(\infty)^2)^{\frac{1}{2}}f\| \\ &= \|(a(t)^2 - b(t)^2 - \frac{d}{dt}b(t))^{\frac{1}{2}}\hat{f} - (a(\infty)^2 - b(\infty)^2)^{\frac{1}{2}}\hat{f}\|_{L^2} \leq ct^{-1-\epsilon} \end{aligned}$$

provided $\epsilon > 0$ and

$$|a(t) - a(\infty)|, |b(t) - b(\infty)|, \left| \frac{d}{dt}b(t) \right| \leq Kt^{-1-\epsilon} \text{ a.e.}$$

for some constants $\epsilon > 0$, $k > 0$, and $c = c(k, \epsilon)$ is independent of f .

Besides $B(t)f \rightarrow B(\infty)f$ we are assuming $B'(t) \rightarrow 0$ in the sense that

$$\langle B'(t)^2 f, f \rangle \rightarrow 0,$$

as $t \rightarrow \infty$ for each $f \in \mathcal{D}(A(0)^2) = \mathcal{D}_1$. The above proof can be slightly modified to establish the sufficiency of the following slightly more general condition: there exists $\omega \in L^1(1, \infty)$ such that

$$\|A(t)f - A(\infty)f\|, \|B(t)f - B(\infty)f\|, \|B'(t)f\| \leq \|f\|_{\mathcal{D}_0} \omega(t)$$

for all $t \geq 1$ and all $f \in \mathcal{D}_0$.

Note that (22) is the equation dealt with in [4], the first paper on asymptotic equipartition of energy for nonautonomous wave equations (related to but different from (DWE)).

The autonomous case is solved by the appropriate d'Alembert formula. We use this strategy to deal with the nonautonomous case.

To that end, let $F_1, F_2 \in \mathcal{D}_2$, $s, t \in \mathbb{R}_+$ with s fixed. Define

$$R_{\pm}(t) := e^{\pm i \int_s^t D(r) dr}, \quad (28)$$

$$v(t) := R_+(t)F_1 + R_-(t)F_2, \quad (29)$$

$$\tilde{v}(t) := R_+(t)F_1 - R_-(t)F_2. \quad (30)$$

Neither v and $\tilde{v}$ solve (22) in general. But in the autonomous case one has

$$v(t) = e^{itD}F_1 + e^{-itD}F_2,$$

and

$$\tilde{v}(t) = e^{itD}F_1 - e^{-itD}F_2$$

both satisfy (24).

We have

$$v'(t) = iD(t)R_+(t)F_1 - iD(t)R_-(t)F_2$$

and

$$v''(t) = R_+(t)(iD'(t) - D(t)^2)F_1 + R_-(t)(-iD'(t) - D(t)^2)F_2.$$

A similar calculation gives

$$\tilde{v}''(t) = R_+(t)(iD'(t) - D(t)^2)F_1 - R_-(t)(-iD'(t) - D(t)^2)F_2.$$

It follows that

$$\begin{pmatrix} v \\ \tilde{v} \end{pmatrix}''(t) = \begin{pmatrix} -D(t)^2 & iD'(t) \\ iD'(t) & -D(t)^2 \end{pmatrix} \begin{pmatrix} v \\ \tilde{v} \end{pmatrix}(t). \quad (31)$$

Note that since $D(\cdot)$ satisfies HYP 1, by Theorem 3.1 the initial value problem for (31) is wellposed.

For any $t \in \mathbb{R}_+$ define

$$\begin{aligned} k(t) &:= \|D(t)R_+(t)F_1\|^2 + \|D(t)R_-(t)F_2\|^2 \\ &= \|D(t)F_1\|^2 + \|D(t)F_2\|^2 \geq \delta^2(\|F_1\|^2 + \|F_2\|^2). \end{aligned}$$

We introduce the kinetic and potential energies, K_V , $K_{\hat{V}}$ and P_V , $P_{\hat{V}}$, for

$$V(t) = \begin{pmatrix} v \\ \tilde{v} \end{pmatrix}(t), \quad \hat{V}(t) = \begin{pmatrix} \tilde{v} \\ v \end{pmatrix}(t) \quad t \in [0, \infty],$$

defined by

$$\begin{aligned} K_V(t) &:= \|v'(t)\|^2 = k(t) - 2\operatorname{Re}\langle D(t)R_+(t)F_1, D(t)R_-(t)F_2 \rangle, \\ K_{\hat{V}}(t) &:= \|\tilde{v}'(t)\|^2 = k(t) + 2\operatorname{Re}\langle D(t)R_+(t)F_1, D(t)R_-(t)F_2 \rangle, \\ P_V(t) &:= \|D(t)v(t)\|^2 = k(t) + 2\operatorname{Re}\langle D(t)R_+(t)F_1, D(t)R_-(t)F_2 \rangle, \\ P_{\hat{V}}(t) &:= \|D(t)\tilde{v}(t)\|^2 = k(t) - 2\operatorname{Re}\langle D(t)R_+(t)F_1, D(t)R_-(t)F_2 \rangle. \end{aligned}$$

Notice that $K_V(t) = P_{\hat{V}}(t)$ and $K_{\hat{V}}(t) = P_V(t)$. Thus for problem (22) one has

$$K(t) = P(t) \tag{32}$$

for all $t \in [0, \infty]$, where $K(t) = K_V(t) + K_{\hat{V}}(t)$ is the total kinetic energy and $P(t) = P_V(t) + P_{\hat{V}}(t)$ is the total potential energy.

By Theorem 3.2 we have that

$$\lim_{t \rightarrow \infty} \frac{K_V(t)}{P_V(t)} = 1 = \lim_{t \rightarrow \infty} \frac{K_{\hat{V}}(t)}{P_{\hat{V}}(t)} \tag{33}$$

for all finite energy solutions corresponding to $(F_1, F_2) \neq (0, 0)$ if and only if

$$\lim_{t \rightarrow \infty} \operatorname{Re}\langle R_+(t)F_1, R_-(t)F_2 \rangle = 0 \tag{34}$$

for such (F_1, F_2) . The condition (34) follows by the Riemann-Lebesgue lemma, by assuming HYP 6 (see [5]).

Recall the following elementary result (see [8]).

Lemma 5.1. *Let $t_0 \in \tilde{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, and let f, g, h, l_1, l_2 be real functions defined in a neighborhood of t_0 , with $g, l_2 > 0$ and $h, l_1 \geq 0$ such that*

$$\lim_{t \rightarrow t_0} \frac{f(t) + l_1(t) - l_2(t)}{g(t)} = 1.$$

Then

$$\lim_{t \rightarrow t_0} \frac{f(t) + h(t) + l_1(t)}{g(t) + h(t) + l_2(t)} = 1.$$

Proof. The proof follows by the following computation.

$$\begin{aligned} & \left| \frac{f(t) + h(t) + l_1(t)}{g(t) + h(t) + l_2(t)} - 1 \right| \\ &= \frac{g(t)}{g(t) + h(t) + l_2(t)} \left| \frac{f(t)}{g(t)} + \frac{h(t) + l_1(t) - (g(t) + h(t) + l_2(t))}{g(t)} \right| \\ &\leq \left| \frac{f(t) + l_1(t) - l_2(t)}{g(t)} - 1 \right| \rightarrow 0 \quad \text{as } t \rightarrow t_0. \end{aligned}$$

□

Observe that from (31) it follows that

$$V''(t) = \begin{pmatrix} -D(t)^2 & iD'(t) \\ iD'(t) & -D(t)^2 \end{pmatrix} V(t), \quad t \in \mathbb{R}_+. \quad (35)$$

Now define

$$q(t) := Q_-(t)v, \quad \tilde{q}(t) := Q_-(t)\tilde{v}.$$

Thus,

$$q'(t) = iD(t)\tilde{q} - B(t)q, \quad (36)$$

$$\tilde{q}'(t) = iD(t)q - B(t)\tilde{q}. \quad (37)$$

Consequently we find

$$q''(t) = (iD'(t) - 2iB(t)D(t))\tilde{q} - (D(t)^2 - B(t)^2 + B'(t))q$$

and

$$\tilde{q}''(t) = (iD'(t) - 2iB(t)D(t))q - (D(t)^2 - B(t)^2 + B'(t))\tilde{q}.$$

Thus we get

$$\begin{pmatrix} q \\ \tilde{q} \end{pmatrix}''(t) = \begin{pmatrix} -D(t)^2 + B(t)^2 - B'(t) & iD'(t) - 2iD(t)B(t) \\ iD'(t) - 2iD(t)B(t) & -D(t)^2 + B(t)^2 - B'(t) \end{pmatrix} \begin{pmatrix} q \\ \tilde{q} \end{pmatrix}(t).$$

Let us introduce the “partial” energies

$$E_1(t) = \|Q_+(t)q'\|^2, \quad E_2(t) = \|Q_+(t)\tilde{q}'\|^2, \quad E_3(t) = \|Q_+(t)A(t)q\|^2, \\ E_4(t) = \|Q_+(t)A(t)\tilde{q}\|^2,$$

with $t \in \mathbb{R}_+$.

Remark 3. The definition of $E_j(t)$, $1 \leq j \leq 4$ includes the “operator weight function” $Q_+(t)$. In case

$$B(t) = b > 0,$$

the weight $Q(t) = e^{(t-s)b}$ can be factored outside of the norm and thus disappears in the ratio

$$\frac{\tilde{K}(t)}{\tilde{P}(t)},$$

where $\tilde{K}(t) = E_1(t) + E_2(t)$ is the kinetic energy and $\tilde{P}(t) = E_3(t) + E_4(t)$ is the potential energy. But if we defined the various energies $E_j(t)$, $j \in \{1, 2, 3, 4\}$, without using the factor $Q(t)$, we would be unable to prove the desired asymptotic equipartition of energy results.

Using (36) and (37) we find

$$E_1(t) = \|v'(t)\|^2 + \|B(t)v\|^2 - 2\operatorname{Re}\langle v', B(t)v \rangle,$$

$$E_2(t) = \|\tilde{v}'(t)\|^2 + \|B(t)\tilde{v}\|^2 - 2\operatorname{Re}\langle \tilde{v}', B(t)\tilde{v} \rangle,$$

$$E_3(t) = \|D(t)v\|^2 + \|B(t)v\|^2 + \langle B'(t)v, v \rangle,$$

$$E_4(t) = \|D(t)\tilde{v}\|^2 + \|B(t)\tilde{v}\|^2 + \langle B'(t)\tilde{v}, \tilde{v} \rangle.$$

We are now able to prove our main result.

Theorem 5.2. *Assume that all the assumptions HYP 1-HYP 6 hold. Then for all finite energy solutions, one has*

$$\lim_{t \rightarrow \infty} \frac{E_1(t)}{E_3(t)} = 1, \quad \lim_{t \rightarrow \infty} \frac{E_2(t)}{E_4(t)} = 1.$$

Proof. We have that

$$\lim_{t \rightarrow \infty} \frac{E_1(t)}{E_3(t)} = \lim_{t \rightarrow \infty} \frac{K_V(t) + \|B(t)v\|^2 - 2\operatorname{Re}\langle v', B(t)v \rangle}{P_V(t) + \|B(t)v\|^2 + \langle B'(t)v, v \rangle}, \quad (38)$$

and

$$\lim_{t \rightarrow \infty} \frac{E_2(t)}{E_4(t)} = \lim_{t \rightarrow \infty} \frac{K_{\tilde{V}}(t) + \|B(t)\tilde{v}\|^2 - 2\operatorname{Re}\langle \tilde{v}', B(t)\tilde{v} \rangle}{P_{\tilde{V}}(t) + \|B(t)\tilde{v}\|^2 + \langle B'(t)\tilde{v}, \tilde{v} \rangle}. \quad (39)$$

The proof follows from (33). In fact

$$\langle B'(t)v, v \rangle \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

due to condition (20) in HYP 5. Moreover,

$$\begin{aligned} \langle v', B(t)v \rangle &= \langle iD(t)R_+(t)F_1 - iD(t)R_-(t)F_2, B(t)R_+(t)F_1 + B(t)R_-(t)F_2 \rangle \\ &= \langle iD(t)F_1, B(t)F_1 \rangle - \langle iD(t)F_2, B(t)F_2 \rangle \\ &\quad + \langle iD(t)R_+(t)F_1, B(t)R_-(t)F_2 \rangle - \langle iD(t)R_-(t)F_2, B(t)R_+(t)F_1 \rangle. \end{aligned} \quad (40)$$

The real parts of the first two terms in (40) are equal to 0. Indeed, since $J(t) := D(t)B(t)$ is selfadjoint for all $t \in [0, \infty]$, we have

$$\langle D(t)h, B(t)h \rangle = \langle J(t)h, h \rangle \in \mathbb{R} \quad \text{for each } h \in \mathcal{D}_1.$$

Moreover we have

$$\langle iD(t)R_+(t)F_1, B(t)R_-(t)F_2 \rangle = \langle e^{2i \int_s^t D(\tau) d\tau} h, k \rangle,$$

for some $h, k \in \mathcal{H}$. By [5] the expression

$$\langle e^{2i \int_s^t D(\tau) d\tau} h, k \rangle$$

tends to 0 as $t \rightarrow \infty$ for all $h, k \in \mathcal{H}$ iff

$$\langle e^{itD(\infty)} h, k \rangle \rightarrow 0$$

as $t \rightarrow \infty$ for such $h, k \in \mathcal{H}$ (since $\int_s^t D(\tau) d\tau = t(\frac{1}{t}) \int_s^t D(\tau) d\tau$ and $(\frac{1}{t} \int_s^t D(\tau) d\tau) f$ tends to $D(\infty)f$ for all $f \in \mathcal{D}_0$), iff

$$\langle e^{itD(\infty)} h, h \rangle \rightarrow 0$$

as $t \rightarrow \infty$ for all $h \in \mathcal{H}$.

By HYP 6 we have that $D(\infty)$ is spectrally absolutely continuous. Thus by the Spectral Theorem,

$$D(\infty) = \int_0^\infty \lambda dE(\lambda),$$

and

$$\langle e^{itD(\infty)} h, h \rangle = \int_0^\infty e^{it\lambda} d(\|E_\lambda h\|^2) = \int_0^\infty e^{it\lambda} g_h(\lambda) d\lambda \rightarrow 0$$

as $t \rightarrow \infty$, where $g_h \in L^1(\mathbb{R}, d\lambda)$ is the associated density, which exists because $D(\infty)$ is spectrally absolutely continuous. Then the Riemann-Lebesgue lemma yields the conclusion. The same reasoning can be applied to prove that the last term in (40) tends to 0. This completes the proof of Theorem 5.2. $\square$

6. **Some examples.** Let $\mathcal{H} = L^2(\mathbb{R}^n)$, $\mathcal{D}_0 = H^\alpha(\mathbb{R}^n)$, $\mathcal{D}_2 = H^{2\theta}(\mathbb{R}^n)$, where $0 < \alpha$ and $0 \leq \theta < \alpha/2$. Consider the selfadjoint operators (see [7])

$$A_\alpha(t) = (I - a(t)^2 \Delta)^{\frac{\alpha}{2}}, \quad B_\theta(t) = b(t)(-\Delta)^\theta, \quad (41)$$

defined respectively on $\mathcal{D}(A_\alpha(t)) = \mathcal{D}_0$ and $\mathcal{D}(B_\theta(t)) = \mathcal{D}_2$. The corresponding operator $D(t)$ is

$$D(t) = ((I - a(t)^2 \Delta)^\alpha - b(t)^2 (-\Delta)^{2\theta} - b'(t)(-\Delta)^\theta)^{\frac{1}{2}},$$

defined on $\mathcal{D}_0$.

Concerning $a = a(t)$ and $b = b(t)$ we make four assumptions:

- $a(t)$ is positive, nonincreasing on $\mathbb{R}_+$ and

$$0 < a(\infty) := \lim_{t \rightarrow \infty} a(t).$$

- $a \in C^1(\mathbb{R}_+)$, and $(a - a(\infty))', a' \in L^1(\mathbb{R}_+)$.
- b is positive and nonincreasing and $b(t) < a(t)^2$ on $\mathbb{R}_+$.
Moreover, $c(t) := \int_t^\infty |b'(s)| ds \in L^1(\mathbb{R}_+)$.
- $b \in C^2(\mathbb{R}_+)$, $b', b'' \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ (and hence $b'(t) \rightarrow 0$ as $t \rightarrow \infty$).

Under these hypotheses, the operators $A_\alpha(t)$ and $B_\theta(t)$ satisfy the HYP 1 through HYP 6. Here are the proofs.

(HYP 1). The selfadjoint operator $A_\alpha(t)$ is positive, bounded below by $(I - a(\infty)^2 \Delta)^{\frac{\alpha}{2}}$ and bounded above by $L_\alpha := (I - a(0)^2 \Delta)^{\frac{\alpha}{2}}$. Thus there is a constant $C_1 > 0$ such that

$$\frac{1}{C_1} \|L_\alpha f\|_{L^2}^2 \leq \|A_\alpha(t)f\|_{L^2}^2 \leq C_1 \|L_\alpha f\|_{L^2}^2, \quad (42)$$

for any $t \in \mathbb{R}_+$ and all $f \in \mathcal{D}_0$. Also, HYP 1 is satisfied for $k_1 = 1$ since $a(t)$ is nonincreasing.

(HYP 2). For all $f \in \mathcal{D}_0$ we have

$$\begin{aligned} A_\alpha(\tau)f - A_\alpha(\infty)f &= - \int_\tau^\infty A'_\alpha(t)f dt, \\ A'_\alpha(t)f &= \frac{\alpha}{2} A_{\alpha-2}(t)(-2a(t)a'(t)\Delta)f; \end{aligned} \quad (43)$$

then since HYP 1 holds for the operator $A_{\alpha-2}$, we get

$$\|A'_\alpha(t)f\|_{L^2} \leq \sqrt{C_1} \alpha a(0)(-a'(t)) \|L_{\alpha-2} \Delta f\|_{L^2}.$$

Due to our assumption $a' \in L^1(\mathbb{R}_+)$, for every $f \in \mathcal{D}_0$ we find $A'_\alpha(\cdot)f \in L^1(\mathbb{R}_+, L^2(\mathbb{R}^n))$ and

$$\|A_\alpha(t)f - A_\alpha(\infty)f\|_{L^2} \leq \sqrt{C_1} \alpha a(0)(a(t) - a(\infty)) \|L_{\alpha-2} \Delta f\|_{L^2};$$

since we are assuming that $a - a(\infty)$ belongs to $L^1(\mathbb{R}_+)$ we conclude

$$\int_0^\infty \|A(t)f - A(\infty)f\|_{L^2} dt < \infty$$

for any $f \in \mathcal{D}_0$. Hence HYP 2 follows.

(HYP 3). To show

$$A_\alpha(t) = F_\alpha(t, A_\alpha(0)),$$

it is enough to show this for $\alpha = 2$, since $I \leq A_\alpha(t) = A_2(t)^{\frac{\alpha}{2}}$, for $\alpha > 0$. Since $A_2(t) = I - a(t)^2\Delta$, set $\tilde{F}_2(t, z) = 1 + a(t)^2z$ for $t, z \geq 0$. Then,

$$\tilde{F}_2(t, z) - 1 = (\tilde{F}_2(0, z) - 1) \frac{a(t)^2}{a(0)^2};$$

so

$$\tilde{F}_2(t, z) = \tilde{F}_2(0, z) \frac{a(t)^2}{a(0)^2} - \frac{a(t)^2}{a(0)^2} + 1,$$

and hence

$$A_2(t) = \tilde{F}_2(t, -\Delta) = \tilde{F}_2(0, -\Delta) \frac{a(t)^2}{a(0)^2} - \frac{a(t)^2}{a(0)^2} + 1 =: F_2(t, A_2(0)).$$

Consequently

$$A_\alpha(t) = A_2(t)^{\frac{\alpha}{2}} = (F_2(t, A_\alpha(0)^{\frac{2}{\alpha}})^{\frac{\alpha}{2}} =: F_\alpha(t, A_\alpha(0));$$

whence

$$F_\alpha(t, z) = F_2(t, z^{\frac{2}{\alpha}})^{\frac{\alpha}{2}}$$

for $t \geq 0, z \geq 1$. The regularity of F_2 (and hence of F_α) follows from the assumed regularity of A_α , since the regularity of $F_\alpha(t, z)$ follows from that of $F_2(t, z)$ for $(t, z) \in [0, \infty) \times [1, \infty)$.

(HYP 4). Since $\theta < \alpha/2$ the canonical embedding of $H^\alpha(\mathbb{R}^n)$ into $H^{2\theta}(\mathbb{R}^n)$ is continuous. Thus, HYP 4 follows easily from our hypotheses.

(HYP 5). This follows from the verification of HYP 4 together with the assumption that $b', b'' \in L^1(\mathbb{R}_+)$.

(HYP 6). Our assumptions imply that $D(t)$ is a positive selfadjoint operator with domain $\mathcal{D}_0$ for all $t \in [0, \infty]$. By the closed graph theorem and the uniform continuity of all the coefficients, there is a positive constant M such that

$$\frac{1}{M} \|D(0)f\|_{L^2} \leq \|D(t)f\|_{L^2} \leq M \|D(0)f\|_{L^2}$$

holds for all $f \in \mathcal{D}_0$ and for all $t \in [0, \infty]$. By the Heinz inequality (see [9]), for each $s \in (0, 1)$, $\mathcal{D}(D(t)^s) = H^{2\alpha s}(\mathbb{R}^n)$ and

$$\frac{1}{M^s} \|D(0)^s f\|_{L^2} \leq \|D(t)^s f\|_{L^2} \leq M^s \|D(0)^s f\|_{L^2},$$

for all $f \in \mathcal{D}(D(0)^s)$ and all $t \in [0, \infty]$.

The rest of the proof of HYP 6 is like the verification of HYP 1. We can safely omit further details.

The equations discussed above

$$\frac{\partial^2 u}{\partial t^2} + B_\theta(t) \frac{\partial u}{\partial t} + A_\alpha(t)u = 0$$

are all pseudo differential equations. To get a partial differential equation we must choose α to be an integer satisfying $\alpha \geq 1$ unless $\theta = 0$, the generalized telegraph equation case. This follows from our assumptions $\theta < \alpha/2$. The lowest order PDE for which $\theta > 0$ corresponds to $(\alpha, \theta) = (6, 1)$, namely

$$\frac{\partial^2 u}{\partial t^2} - b(t) \frac{\partial}{\partial t}(\Delta u) + (1 - a(t)^2 \Delta)^3 u = 0.$$

The details involved in establishing wellposedness of the initial value problem for these equations involve checking HYP 1 - HYP 6 for these cases. This is similar to,

but easier than the verification of HYP 1 - HYP 6 in the A_α case. We omit these calculations.

The interest in studying this kind of example comes from the literature. In fact in the last years several authors investigated asymptotic estimates of the solution to some Cauchy problems associated to models in the form

$$u_{tt} - A^2(t)u + 2B(t)u_t = 0,$$

where the differential operators $A(t)$ and $B(t)$ are eventually constant with respect to t and involve fractional powers of the Laplace operator. Then they applied these estimates to get global existence results for the Cauchy problem associated to the corresponding nonlinear problem, with nonlinearity $\pm|u|^{p-1}u$ or $|u_t|^{p-1}u_t$ (see for example [1] and [2]).

Acknowledgement. Part of this paper was done during the visits as Visiting Professors of University of Bari Aldo Moro of G. R. Goldstein and J. A. Goldstein. The results obtained are part of the research plan of M. D’Abbicco, G. Girardi and S. Romanelli as members of G.N.A.M.P.A. (Istituto Nazionale di Alta Matematica).

REFERENCES

- [1] M. D’Abbicco, M. R. Ebert and S. Lucente, [Self-similar asymptotic profile of the solution to a nonlinear evolution equation with critical dissipation](#), *Math. Methods Appl. Sci.*, **40** (2017), 6480–6494.
- [2] M. D’Abbicco, G. Girardi and M. Reissig, [A scale of critical exponents for semilinear waves with time-dependent damping and mass terms](#), *Nonlinear Anal.*, **179** (2019), 15–40.
- [3] J. L. Doob, *Stochastic Processes*, John Wiley and Sons, Inc., New York, Chapman and Hall, Ltd., 1953.
- [4] G. R. Goldstein, J. A. Goldstein and F. Travessini, [Equipartition of energy for nonautonomous wave equations](#), *Discrete Contin. Dyn. Syst. Ser. S*, **10** (2017), 75–85.
- [5] J. A. Goldstein, [An asymptotic property of solutions of wave equations](#), *Proc. Amer. Math. Soc.*, **23** (1969), 359–363.
- [6] J. A. Goldstein, [An asymptotic property of solutions of wave equations. II](#), *J. Math. Anal. Appl.*, **32** (1970), 392–399.
- [7] J. A. Goldstein *Semigroups of Linear Operators and Applications*, 2nd edition, Dover Publications, Inc., Mineola, New York, 2017.
- [8] J. A. Goldstein and G. Reyes, [Equipartition of operator-weighted energies in damped wave equations](#), *Asymptot. Anal.*, **81** (2013), 171–187.
- [9] T. Kato *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, 1995.

Received for publication December 2019.

E-mail address: marcello.dabbicco@uniba.it

E-mail address: giovanni.girardi@uniba.it

E-mail address: ggoldste@memphis.edu

E-mail address: jgoldste@memphis.edu

E-mail address: silvia.romanelli@uniba.it