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EMERGENCE OF QUANTUM THEORIES FROM CLASSICAL PROBABILITY:HISTORICAL ORIGINS, DEVELOPMENTS, AND OPEN PROBLEMS

LUIGI ACCARDI* AND YUN GANG LU

Dedicated to Professor Leonard Gross on the occasion of his 88th birthday

Abstract. After briefly mentioning some achievements of quantum probability during the past 20 years, we concentrate our attention on the emergence of natural extensions of usual quantum theory from the combination of classical probability with the theory of orthogonal polynomials and we discuss some implications, both for mathematics and physics of this fact.

1. Introduction

Since the ramifications of quantum probability (QP) now intersect practically all branches of mathematics, it is impossible to cover the main achievements obtained in all sub–fields of QP in the past 20 years in a single survey of acceptable dimensions. Quantum Markov chains have found applications from mathematical physics (where a sub–class of them is called *finitely correlated states*), to quantum information theory (where a sub–class of them is called *matrix product states*), to computational physics where the *density matrix renormalization group method* is entirely based on them. The fact that special cases of the same general mathematical structure take different names according to the sociological group using them is not a good sign on the ethical level of contemporary mathematics. Important advances have been made in the theory of quantum Markov fields, both in their general structure and in the appearance of the role of *boundary conditions* in special models. These results can open the way to a full understanding of the role of these conditions in the general theory as well as to the solution of the decades long problem of constructing a quantum version of Dobrushin theory with *good quantum properties*. The stochastic limit of quantum theory (SLQT) has radically changed the theory of open quantum systems showing that the Friedrichs–van Hove limit, concerning the reduced evolution of an open quantum system can be extended to a limit of the full Heisenberg evolution which, after the limit, becomes a quantum stochastic evolution i.e. satisfies a *quantum stochastic differential equation (QSDE)*. The subsequent discovery that QSDE are equivalent to *white noise Hamiltonian equations (WNHE)* created a bridge between QSDE and the operator

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distribution formalism used in quantum field theory and, most important, opened the way to a truly multi–dimensional stochastic analysis in which the driving noise is multi–dimensional white noise. The full development of this, i.e. beyond mere stochastic integration, is an important open problem for the future. Moreover, the highly non–trivial connection between the operator coefficients of the WNHE and those of the associated QSDE gives a microscopic interpretation of the latter coefficients in terms of the Hamiltonian coefficients which are commonly used in physics.

The notion of local equilibrium, also emerged from the SLQT, has led to a local extension of the KMS equilibrium condition which, in turn, has led to a mathematical extension of the Bose–Einstein condensation (BEC) phenomenon showing that, in a local equilibrium state BEC can take place also in excited states and not only in the ground state, as considered up to now. The same notion also motivated a finer analysis of the quantum Markov semi–groups emerging from SLQT that has given a mathematical foundation to the intuitive idea that, since the quantum Markov generators deduced in the weak coupling limit and in the low density limit describe completely different situations, this should be reflected in the structure of the corresponding invariant states. The subtle role of degeneracy of the Bohr frequencies, first emerged in the *dark states* introduced in quantum optics, plays a crucial role in this structure and is understood only in simple, but non–trivial, situations.

The notion of complementarity (in the finite–dimensional case re–baptized as *mutually unbiased bases* in quantum information) has found a natural extension within the theory of interacting Fock spaces. Sophisticated central limit theorems have shown that, even in the simplest case of quantum Bernoulli processes, but with non–standard embeddings into the infinite tensor product structure, one can obtain in the limit some previously introduced deformations of the Boson commutation relations or some known processes like Parthasaraty's *quantum Azema martingale*. The program of non-linear quantization, in its Lie algebra version, has led among other things to unexpected connections with string theory through the emergence, within this program, of a second quantized version of the Virasoro-Zamolodchikov hierarchy.

All these topics will not be discussed, or barely mentioned, in the present paper. In it we will concentrate on a single topic: *how quantum theory naturally emerges from classical probability* as a special case of a much wider class of quantum structures canonically associated to classical random variables or fields.

In the future it will be possible to explain to mathematicians, in a clear, short and most of all not artificial way, how quantum theories are fundamental objects of mathematics emerging from probability, algebra and analysis, independently of any physical consideration.

The main implication of this for physics is, as shown below, that *the natural environment for truly non–linear quantum theories cannot be based on Heisenberg commutation relations, but on the new commutation relations that will be discussed below.*

2. Orthogonal Polynomials and the Mathematical Roots of Quantization

To understand a mathematical structure means to frame it within a wider context that also suggests which are the natural directions to extend it in a fruitful way for different applications and provides a systematic approach for these extensions. In the past 20 years this happened in connection with one of the most important and mysterious mathematical structures emerged in the past century: **quantum theory**.

An aura of mystery has accompanied quantum theory since its origins: Heisenberg himself claimed that *The mathematical formalism of QM is far from intuition* and the well known Nelson's aphorism *First Quantization is a mystery. Second Quantization is a functor*, formulated about 50 years after Heisenberg's statement, confirms the permanence in time of this attitude.

Now quantization is no longer a mystery, we know that, from the combination of **classical probability** with a 150 year old branch of classical analysis, **the theory of orthogonal polynomials** (and in fact even in more general situations, see the paper [37]), a whole variety of quantum theories naturally emerge. In the following section we briefly outline how this happens.

2.1. The canonical quantum decomposition and the CAP operators.

Every classical random variable X with values in a real vector space V, finite or infinite dimensional (so that quantum field theory is included) and with moments of all orders, admits a **canonical quantum decomposition** (simply quantum decomposition in the following) as a sum of three operators which are natural generalizations of the usual **creation, annihilation and preservation (CAP)** operators of quantum mechanics. In general X has many quantum decompositions which have the same classical moments but may have different quantum moments. The term *canonical* here refers to the fact that the CAP operators and their relations emerge deductively and are uniquely determined by intrinsic properties of X, while all other quantum decompositions involve *ad hoc* assumptions or artificial constructions.

An additional non–trivial consequence of the above construction is that any **symmetric** classical random variable X always admits a canonically conjugate Hermitean operator P_X called the **momentum operator associated to** X. This is not true for non–symmetric classical random variable if one requires, as it is natural to do, that P_X is Hermitean. The pair satisfies the relation $[X, P_X] = i\hbar$ which differs from Heisenberg's one only in the fact that here $\hbar$ is a gradation preserving operator (a function of the number operator in the case of a single real valued random variable). One proves that, in analogy with usual quantum mechanics, X and P_X are related by a natural generalization of the Fourier–Gauss transform. For example in the paper [39] it is shown that, if the probability distribution of X is the standard semi–circle law, then P_X is the **Hilbert transform** associated to this probability measure (i.e. the limit of the Cauchy transform when the imaginary part of z tends to zero).

Given P_X, one can define the X–translation group, the X–free energy and the associated X–free evolution. In other words the new approach to orthogonal polynomials associates to any classical random variable or random field, not only the CAP operators and their commutation relations, **but the whole standard structure of quantum mechanics**.

For example for any X, the X–harmonic oscillator evolution has a relatively simple explicit form [24]. This, combined with the fact that under non–restrictive conditions one can construct a unitary gradation preserving isomorphism mapping the L^2–space of X onto $L^2(\mathbb{R})$ and X into multiplication by X, gives a machinery to build many non–trivial examples of quantum systems which are **integrable** in the sense that their evolution is simple and explicit in the X–representation, but may be very complex in its $L^2(\mathbb{R})$ representation. The method is quite general and infinitely many examples can be constructed, but the problem here is not to make explicit the, usually long and complex computations needed to build the $L^2(\mathbb{R})$–representation (as some papers in physics have already begun to do), but to find at least one example which is interesting for at least one real application.

The quantum decomposition of a classical random variable X has an unexpected implication that in [4] was called *Gaussianization of classical random variables* (or of probability measures): it is known that a Gaussian classical random variable is characterized by the property that its moments are expressed in terms of its covariance and its expectation value. The term *Gaussianization* was justified by the fact that classical moments of X, of any order, *are expressed in terms of singletons and pair partitions i.e. classical mean values and quantum pair correlations* (in the classical Gaussian case quantum pair correlations are replaced by classical ones and this characterizes classical Gaussianity).

2.2. Commutation relations. One proves that the CAP operators canonically associated to a classical random field satisfy natural **commutation relations (CR)**. In this sense **statistics implies algebra**: this is the *yin* of quantum probability.

The *yang*, **algebra implies statistics**, is well epitomized by the **commutator theorem** [15], that has played a key role in the development of the theory and intuitively espresses the converse of the above statement, namely that **the CR characterize a classical random field** X up to moment equivalence (i.e. they characterize the class of probability distributions with the same moments as X).

One of the consequences of the commutator theorem is that **Heisenberg commutation relations characterize the class of Gaussian probability measures** [11].

In many other cases an algebraic property is equivalent to a statistical one. For example the property that X has vanishing odd moments of all orders is equivalent to the vanishing of the preservation operator in its quantum decomposition [15].

It turns out that **there are two types of commutation relations (CR)** called type I and type II CR [22]. Type I CR are naturally associated to gradation preserving operators and they are common to all Interacting Fock spaces (IFS) (see section 2.8 below). Type II CR are a new type of commutation relations and they arise from the fact that the multiplication operators by components of X along arbitrary vectors in V mutually commute [22] (one of these relations, commutativity of creators, familiar in Boson quantum mechanics, was first discovered in [11]). In this sense **commutativity implies** (a very specific form of) **non–commutativity**.

Type II commutation relations, with the exception of commutativity of creators, have not been considered up to now.

The reason of this fact is explained by another important result of the theory, namely the **characterization of product measures** through the property that **the associated CAP operators** corresponding to different degrees of freedom **commute** (see [15] with improvements in [29], [31]: another example of *algebra implies statistics*). Another characterization of product measures in terms of a weaker algebraic property has recently been obtained in [36]). Recall that the probability distribution of the V–valued random variable X is called a product measure if there exists a linear basis of V such that the components of X along the vectors of this basis are moment independent. In this case the random variable X is called **linearly equivalent** to a family of independent random variables. If this is the case, one proves that the **type II CR are identically satisfied**.

Since commutativity of the CAP operators corresponding to different degrees of freedom is **always satisfied** in Boson quantum mechanics, this explains why type II commutation relations did not arise in physics.

This also explains why **the quantum mechanics of interacting systems cannot be described by Heisenberg commutation relations**.

In fact, product measures characterize independent random variables, which in physics describe non–interacting systems. But the most interesting systems, not only in physics but also in a multiplicity of fields where probability is applied, are the interacting, i.e. non–product ones. The above mentioned characterizations clearly indicate that **interactions are coded into the commutation relations** and not only in the interaction potential, as considered up to now.

Therefore, the study of those quantum theories canonically associated to probability measures that are not linearly equivalent to product measures corresponds to the study of interacting systems. Since interactions are reflected in non–linearities in the equations of motion it is natural to call the quantum theories associated to them **non–linear quantizations**.

In this direction an important **open problem** is the discovery of a class of probability measures that are not equivalent to product measures but that shares with the Gaussian class (whose elements are of product type) the property of allowing explicit calculation of the most interesting statistical quantities.

2.3. Quadratic and higher order quantizations. There are many reasons to conjecture that the natural candidate for this role is **quadratic quantization** which, in its infinite dimensional version includes the Virasoro algebra. Quadratic quantization, in the case of one degree of freedom, is at the moment the best understood among non–linear quantizations. This line of research originated in the paper [5], where the quadratic commutation relations were introduced and the existence of the Fock representation was proved. In the paper [8] the quadratic commutation relations were identified to be a representation of the current algebra over $\mathbb{R}$ of $sl(2,\mathbb{R})$ and the theory of its factorizable representations was framed in the general context of the theory of quantum Levy processes [57] which allowed to obtain, as an application of this theory, a class of representations more general than the usual Fock one. The explicit form of the quadratic scalar product, that puts in clear evidence the non–linear character of the theory, was deduced in the paper [41] by applying the Faá–Di Bruno formula to the explicit form of the quadratic exponential vectors that was found before in [6] (see also [19], [20], [21], [44], [55] for improvements and extensions).

For the developments that were going to come, probably the most important result in the paper [8] was the proof of the fact that the vacuum distributions of the generalized field operators are in one–to–one correspondence with the **three non–standard Meixner classes** (Gamma, negative binomial (or Pascal) and proper Meixner) just as in the usual Bosonic case they are in one–to–one correspondence with the two standard Meixner classes (Gauss and Poisson). This gives a strong intuitive support to the claim that quadratic quantization is **the first mathematical realization of the square of the free non–relativistic Boson field** (or equivalently white noise). In fact, since usual white noise is an operator valued distribution process with Gaussian vacuum distribution, it is natural to surmise that its square, if well defined, should be an operator valued distribution process with Gamma vacuum distribution. A corollary of the above mentioned result of [8] is that the vacuum distribution of the quadratic field operator (a particular generalized field operator) is precisely a Gamma distribution. Thus, in quadratic quantization, the Gamma plays the role of the Gaussian and the negative binomial that of the Poisson.

The basic idea of quadratic quantization is that you first make an heuristic guess on which the (re–normalized) commutation relations of the square of white noise should be and then you try to find a Hilbert space representation of the resulting Lie algebra.

The fact that this idea can be applied also to higher powers of white noise, was the starting point of a new line of research which led to an unexpected connection with the **Virasoro-Zamolodchikov hierarchy** [18], previously introduced in string theory. In fact, the Lie algebras appearing are the *second quantized* version of the Lie algebras in the above mentioned hierarchy.

The complexity and extension of this line of research and the specialistic nature of the results obtained, which mainly concern re–normalization theory, contractions and co–homology of infinite dimensional Lie algebras, require a separate survey. We limit ourselves to say that, a posteriori, the connection with the Virasoro algebra is very natural because this algebra, in its Boson representation, is a central extension of a sub–$*$–Lie algebra of the ∞–dimensional version of the Schrödinger algebra and the generators of the Schrödinger algebra consist of those of the Heisenberg algebra together with those of the quadratic algebra. Thus, in order to understand the structure of the Virasoro algebra, a deeper understanding of the structure of the quadratic algebra is a necessary preliminary step. The paper [38] is devoted to this problem and shows that this algebra can be identified to a *kind of non–commutative* $\mathrm{sl}(2,\mathbb{C})$. In the case of 1 degree of freedom the problem was solved in [23], (see also [55]) where the group manifold as well as the composition law of the quadratic Heisenberg group, denoted QHeis(1), was constructed and recently it was discovered [56] that QHeis(1) is isomorphic as a Lie group to the projective $SU(1,1)$ and in addition the holomorphic representation of this group was constructed.

2.4. The multi–dimensional Favard Lemma. The results discussed below have emerged, over a long period of time from the collaboration of many researchers, starting with the first paper [4] dealing with the 1–dimensional case, its finite–dimensional extension and the intrinsic formulation of Jacobi 3–diagonal relation (see [9] for the finite–dimensional case, [10] for its infinite–dimensional extension and [37] for the general formulation involving only discrete increasing filtrations of Hilbert spaces and Hermitean operators increasing by 1 the filtration). There is a concrete hope that this more general formulation of the Jacobi relations might open the way to the solution of the 20 year old problem of extending the quantum decomposition to measures without moments.

According to the Jacobi 3–diagonal relation, any probability measure μ on $\mathbb{R}$ with all finite moments uniquely defines two sequences, called the *Jacobi sequences* of μ,

$$\{(\omega_n)_{n\in\mathbb{N}},\ (\alpha_n)_{n\in\mathbb{N}}\}, \qquad \omega_n \in \mathbb{R}_+,\ \alpha_n \in \mathbb{R}, \qquad n = 0,1,2,\cdots \tag{2.1}$$

subjected to the only constraint that, for any $n,\ k \in \mathbb{N}$,

$$\omega_n = 0 \implies \omega_{n+k} = 0 \qquad ; \qquad \forall k \in \mathbb{N} \tag{2.2}$$

Favard Lemma is the converse of this statement i.e., given two sequences satisying (2.1) and (2.2), it gives an inductive way to uniquely reconstruct:

- **(i):** a state on the algebra $\mathcal{P}$ of polynomials in one indeterminate,
- **(ii):** the orthogonal decomposition of $\mathcal{P}$ canonically associated to this state.

Favard Lemma implies that, for each $n \in \mathbb{N}$, the pair $(\alpha_n\ ,\ \omega_n)$ condensates the **minimal information** gained from the knowledge of the n–th moment with respect to the knowledge of all the k–th moments with $k \leq n-1$. Here the word **minimal** is essential.

In the theory of multi–dimensional orthogonal polynomials there are several versions of Favard Lemma; however, in all of them **the minimality issue is not addressed**. In the paper [22] it is proved that in the multi–dimensional case the secondary sequence is replaced by a sequence (α_n) of Hermitean matrices (this was known before in a different language) and the principal one by a sequence $(\tilde{\Omega}_n)$ of **positive-definite matrix valued kernels** (in the infinite dimensional case matrices are replaced by operators).

The crucial new point, emerged from the analysis in [22] is that, while in case of polynomials in 1 indeterminate the two sequences (ω_n) and (α_n) are independent and arbitrary modulo the unique constraint (2.2), starting from dimension 2, only the real part of the kernels $(\tilde{\Omega}_n)$ is arbitrary modulo the multi–dimensional analogue of the constraint (2.2), while between the imaginary part of $(\tilde{\Omega}_n)$ and the matrices (α_n) **there are inductive linear constraints** which make the constructive part of the theory much more complex than in the 1–dimensional case.

Thus the precise multi–dimensional analogue of the principal Jacobi sequence (ω_n) is given by the sequence of real parts of the kernels $(\tilde{\Omega}_n)$.

The paper [44] exploits the canonical embedding of the n–th space of the orthogonal polynomial gradation into $(\mathbb{C}^d)^{\widehat{\otimes} n}$ (symmetric tensor product) [22] to write its scalar product in the form

$$\langle \, \cdot \, , \, \cdot \, \rangle_n = \langle \, \cdot \, , \Omega_n \, \cdot \, \rangle_{\widehat{\otimes} n} \tag{2.3}$$

where Ω_n is a positive Hermitean operator on $(\mathbb{C}^d)^{\widehat{\otimes} n}$ and $\langle \, \cdot \, , \, \cdot \, \rangle_{\widehat{\otimes} n}$ is the scalar product on $(\mathbb{C}^d)^{\widehat{\otimes} n}$ induced by the Euclidean scalar product $\langle \, \cdot \, , \, \cdot \, \rangle_{\mathbb{C}^d}$ on $\mathbb{C}^d$ $(\langle u^{\otimes n}, v^{\otimes n} \rangle_{\widehat{\otimes} n} := \prod_{j=1}^n \langle u, v \rangle_{\mathbb{C}^d})$. With these notations, the analogue of the constraint (2.2) found in [22] is refined as follows:

For any $M \in \{0, 1, \dots, dim\left((\mathbb{C}^d)^{\widehat{\otimes} n}\right)\}$

$$\mathrm{Rank}(\Omega_n) \le M \;\Rightarrow\; \mathrm{Rank}(\Omega_{n+k}) \le M \quad ; \quad \forall k \in \mathbb{N}$$

Moreover the support of the measure inducing the scalar product is contained in an algebraic variety uniquely determined by the kernels of the Ω_n [44]. Since the Ω_n are uniquely determined by the $\tilde{\Omega}_n$ (in fact, appropriately defining the composition of kernels, one can symbolically write $\Omega_n = \tilde{\Omega}_n!$) the above statements can be expressed in terms of the $\tilde{\Omega}_n$.

A more general problem is to classify the canonical forms of the $(\tilde{\Omega}_n)$ under the action of the group of invertible linear transformations induced by the linear group on the test function space (i.e. the space where the classical random variable takes its values).

Such a classification has been obtained for product measures [26] but the non–product case is open.

2.5. Classification of probability measures on $\mathbb{R}$ in terms of the Lie algebras generated by their CAP operators. Let X be a classical real valued random variable with all moments and let $\{a^+ \, , \; a^0 \, , \; a^-\}$ be its canonical CAP operators. We know that, in the case of usual Boson quantum mechanics, the complex Lie algebra generated by $\{a^+ \, , \; a^-\}$ is 3–dimensional.

A natural question is then: for which classical random variables X the associated $\{a^+ \, , \; a^-\}$ generates a **finite dimensional Lie algebra**?

A **necessary condition** for this to happen is that, denoting (ω_n) the principal Jacobi sequence of X, there exists $n, K \in \mathbb{N}$ such that

$$(\partial^n \omega)_m = 0 \qquad ; \qquad \forall m \ge K + 1 \tag{2.4}$$

The condition is **sufficient if and only if** in addition

$$n \in \{0, 1, 2\} \qquad ; \qquad K \in \mathbb{N} \text{ –arbitrary}$$

This result suggests that, among all the probability measures μ on $\mathbb{R}$, those which are *in some sense the simplest ones* are those whose principal Jacobi sequence (ω_n) satisfies at least the necessary condition (2.4). Since this condition depends on two parameters, it is natural to define **an index of information complexity** (or simply **information complexity**), denoted $C(\omega)$, of a principal Jacobi sequence (ω_m) as $+\infty$ if there exist no $n, K \in \mathbb{N}$ (ω_n) satisfying (2.4) and, if there are,

$$C(\mu) := (k, K), \iff \tag{2.5}$$

$$k = \min\{n \in \mathbb{N} \ : \ (\partial^{n+1}\omega)_m = 0 \ , \ \forall m \geq K+1 \ , \ (\partial^{n+1}\omega)_K \neq 0\}$$

One then says that a classical real valued random variable X belongs to the **information complexity class** (k, K) $(k, K \in \mathbb{N})$ if (k, K) is the information complexity index of its principal Jacobi sequence.

This notion was introduced in [28] where it was shown that, while the principal Jacobi sequences with index (k, K) can be fully described, about the structure of the corresponding symmetric measures one knows much less.

The **information complexity class** $(0, 0)$ is parametrized by a single positive real number ω. The case $\omega = 0$ characterizes the **atomic measures at single points**.

If $\omega > 0$, then the symmetric measures in the class $(0, 0)$ are exactly all possible parametrizations of the **semi–circle laws**. The **information complexity class** $(0, 1)$ coincides with all possible parametrizations of **Arcsine laws**.

For these reasons the class corresponding to all measures with $(0, K)$ were called in [28] the **semi–circle–arcsine class**. It is interesting to notice that the class $(0, 2)$ first appeared in the study of central limits of quantum random walks in the sense of Konno (see [46], [47]).

We conjecture that the notions of independence, canonically associated to the measures in the class $(0, K)$ with $K \geq 2$ in the sense of [4], can be represented in the form described by Lenczewski in [53].

The **information complexity class** $(1, 0)$ includes the mean zero Gaussians (the unique symmetric measures in this class) and the Poisson, but nothing is known about the classes $(1, K)$ with $K > 1$.

The corresponding $*$–Lie algebra is the **Heisenberg algebra**.

The **information complexity class** $(2, 0)$ includes the three non–standard Meixner classes and **the corresponding $*$–Lie algebra is** $sl(2, \mathbb{R})$. As we have seen in section 2.3, this class corresponds to quadratic quantization.

This gives a new, **purely algebraic characterization of this class**, completely different from the one obtained by Meixner, which is based on a special structure postulated for the generating functions of the orthogonal polynomials.

Starting from $n = 3$, the $*$–Lie algebra corresponding to the class $(n, 0)$ is infinite dimensional. This shows in particular that the necessary condition (2.4) is not sufficient.

A refinement of the complexity index can be obtained by considering, in each class (n, K), those measures such that the initial *segment* $(\omega_1, \dots, \omega_K)$ also satisfies a difference equation (possibly of order different from n).

This allows to introduce a complexity hierarchy also in the class of finitely supported measures and can be considered as a natural extension of the point of view advocated in [54] with reference to the Meixner measures.

The above results naturally suggest the program of an **algebraic classification of probability measures** on $\mathbb{R}$ in terms of their information complexity index from one side and of the Lie algebras canonically associated to them on the other side.

Very little is known, and only in special cases, for the analogue problem in dimensions ≥ 2.

2.6. Solutions of problems of classical probability with QP techniques. As strange as it might seem, the problem of expressing the Jacobi sequences and the associated orthogonal polynomials of powers of a given real valued classical random variable X in terms of the same objects associated to X, was solved only in the case of the square and, in this case only for the Jacobi sequences (this was true even for the Gaussian measure!).

In the papers [27] and [30] this problem was solved for arbitrary real valued classical random variable and for arbitrary powers. This gives rise to a large family of **new identities among orthogonal polynomials** and provides an illustration of how methods and techniques of quantum probability can help in solving open problems in classical probability and in classical analysis.

2.7. The emergence of general non–Gaussian analysis. Another fall out of the above mentioned research line is related to the notion of *normal order*, that extends to arbitrary random variables (with all moments) the notion of *Wick order* used in quantum mechanics. Using this notion some very fine inequalities of classical analysis have been extended to classes of probability measures more general than the Gaussian one and in some cases the new technique has allowed to deduce the best constants in them [58], [59], [42]), [50], [51], [52]. Given the importance of these inequalities in classical analysis, it is natural to consider any further step in this direction an important open problem of the theory.

Gaussian analysis, essentially originated from Euclidean field theory, has received a great deal of attention (see the recent exposition [60]). Various extensions of different results of Gaussian analysis are now appearing in several works in classical harmonic analysis. The algebraic approach to the theory of orthogonal polynomials provides a unified context that allows more systematic investigations in this direction.

2.8. Interacting Fock spaces (IFS). The notion of IFS has generated a great quantity of papers. In the following we only mention those which are more intimately related to orthogonal polynomials.

The orthogonal polynomial gradations and the quantum structures canonically associated to it are included into a more general structure, that of **Interacting Fock space (IFS)**, which was introduced long before its connections with orthogonal polynomials were discovered (see [2] for the original emergence of this notion and [3] for the first systematic exposition). In fact, the main result of the first paper on the new approach to orthogonal polynomials [4] can be expressed by saying that orthogonal polynomials in one indeterminate and 1–mode IFS are isomorphic categories with morphisms given by the gradation preserving isometries which intertwine the position operators. It became soon clear that there is no hope to obtain a similar identification in dimensions ≥ 2; in fact, many IFS that naturally arise in the stochastic limit of quantum theory [1] are not of orthogonal polynomial type and the same is true for many IFS arising from quantum central limit theorems [33].

However, it took some time to understand that the obstruction to such identification is given by the Type II commutation relations which, contrarily to the Type I, present in both structures, are very specific to the orthogonal polynomial framework. The role of the multi–dimensional Favard Lemma, discussed in section 2.4, is precisely that of **characterizing, within the class of IFS, those which correspond to some family of orthogonal polynomials**.

However, many resulst on orthogonal polynomials remain true in the IFS framework and this fact helps in understanding from a general point of view many properties of orthogonal polynomials. For example, also in IFS the scalar product can be uniquely reconstructed by a sequence $(\tilde{\Omega}_n)$ of positive-definite operator valued kernels, which arise from Type I commutation relations. The only difference is that these kernels are only subject the multi–dimensional extension of the constraint (2.2) and not to the additional constraints that come from Type II commutation relations. In particular, in the case of IFS, commutativity of creators, that in the latter case follows from the Type II commutation relations, is usually not satisfied. Those IFS that enjoy this property are called **symmetric IFS**. In this sense IFS are simpler objects with respect to the sub–class of IFS coming from orthogonal polynomials.

One can say that an IFS is the most general $\mathbb{N}$–graded Hilbert (or pre–Hilbert or semi–Hilbert) space $(\mathcal{H}, \langle\,\cdot\,,\,\cdot\,\rangle) = \bigoplus_{n\in\mathbb{N}}(\mathcal{H}_n, \langle\,\cdot\,,\,\cdot\,\rangle_n)$ which admits a creation map from the complexification of a real vector space V into the adjointable linear operators of $\mathcal{H}$ increasing by 1 the gradation. The adjoints of creators are called annihilators and enjoy properties that naturally extend the properties of usual creators and annihilators.

Definition 2.1. Let V be a vector space. An **interacting Fock space on** V is a pair:

$$\left\{(H_n, \langle\,\cdot\,,\,\cdot\,\rangle_n)_{n\in\mathbb{N}}, a^+\right\} \tag{2.6}$$

such that:

– $\left(H_n, \langle\,\cdot\,,\,\cdot\,\rangle_n\right)_{n\in\mathbb{N}}$ is a sequence of semi–Hilbert spaces with

$$H_0 =: \mathbb{C}\cdot\Phi_0\ , \quad \|\Phi_0\| = 1$$

Φ_0 is called the **vacuum or Fock vector**;

– denoting $\langle\,\cdot\,,\,\cdot\,\rangle$ the unique semi–Hilbert space scalar product on the vector space direct sum of the family $\left(H_n\right)_{n\in\mathbb{N}}$ which makes this direct sum

$$(H\ ,\ \langle\,\cdot\,,\,\cdot\,\rangle) := \bigoplus_{n\in\mathbb{N}}(H_n, \langle\,\cdot\,,\,\cdot\,\rangle_n) \tag{2.7}$$

an orthogonal sum and whose restriction on each H_n^0 coincides with $\langle\,\cdot\,,\,\cdot\,\rangle_n$, the linear operator

$$a^+\ :\ V \to \mathcal{L}_a\left((H\ ,\ \langle\,\cdot\,,\,\cdot\,\rangle)\right)$$

satisfies the following conditions:

$$\begin{aligned} H_{n+1} &= \text{lin. span }\left\{a^+(V)H_n\right\} \\ &= \text{lin. span }\left\{a^+(v)H_n\ :\ v\in V\right\}\ ;\ \forall n\in\mathbb{N} \end{aligned} \tag{2.8}$$

For each $v \in V$, any choice of adjoint of $a^+(v)$ is denoted by $a^-(v)$ (or simply a_v). The operators $a^+(v)$ ($v \in V$) are called **creators** and their adjoints $a(v)$ **annihilators**. The spaces $\left(H_n\right)_{n\in\mathbb{N}}$ are called the n–**particle spaces**, (or sectors). If $n = 0$ one speaks of the vacuum space. If

$$\{(H_{1,n}, \langle \cdot , \cdot \rangle_{1,n})_{n\in\mathbb{N}}, a_1^*\}$$

is another IFS on a vector space V_1, an **isomorphism** from

$\{(H_n, \langle \cdot , \cdot \rangle_n)_{n\in\mathbb{N}}, a^+\}$ to $\{(H_{1,n}, \langle \cdot , \cdot \rangle_{1,n})_{n\in\mathbb{N}}, a_1^*\}$ is a real linear one–to–one map $U_1 : V \to V_1$ and a linear unitary isomorphism

$$U : \bigoplus_{n\in\mathbb{N}}(H_n, \langle \cdot , \cdot \rangle_n) \to \bigoplus_{n\in\mathbb{N}}(H_{1,n}, \langle \cdot , \cdot \rangle_{1,n})$$

such that U is gradation preserving and

$$Ua_v^+U^* = a_{1,U_1v}^* \qquad , \qquad \forall v \in V$$

In the Hilbert space framework the linear span in (2.8) is replaced by the closed linear span. Notice that the Fock prescription

$$a(v)\Phi_0 = 0 \qquad ; \qquad \forall v \in V \tag{2.9}$$

follows from Definition 2.1.

Every IFS (resp. symmetric IFS) admits a **tensor representation** in terms of the tensor algebra (resp. symmetric tensor algebra) over the test function space associated to it and this makes the notion of IFS more similar to that of usual Boson Fock spaces. More details on this representation and on other properties of IFS can be found in the Appendix of the paper [22].

In [4] it was proved that any symmetric probability measure μ on $\mathbb{R}$ with all moments can be obtained from a central limit theorem associated to a notion of independence defined in terms of the IFS canonically associated to μ. In other words: given μ, one can define a notion of stochastic independence with the property that μ plays the role of the *gaussian* for this notions. For this reason it was called a **universal central limit theorem**. A constructive proof of this result, based on interacting Fock spaces and valid also in the non symmetric case, was later proved in [13] and the corresponding notion of independence, called **projective independence** was introduced in [40].

The notion of IFS is strictly related with the notion of quantum decomposition. Recently it has been proved that there exist exactly three classes of IFS: Type I, in which all field operators, considered as real valued classical random variables with respect to their vacuum distribution, have the same quantum moments as the associated classical random variables. The usual Boson Fock space, as well as the so–called 1–*mode type IFS* belong to this classs. Type II, in which this property holds only for some of the field operators. Examples show that this class is non–empty, but at the moment we have only some mathematical constructions and more natural examples should be constructed. Type III, in which no field operator enjoys this property [32]. The *monotone IFS* belong to this class.

Another interesting property of IFS is their relation with the mathematical approach to the notion of complementarity in quantum physics (see [34] for a short description of its historical development). In fact, within the IFS context, one can extend this notion to that of n–complementarity. This gives a weak form of stochastic independence and allows to construct several examples in which n–complementarity, but **not tensor independence**, is realized. In [35] IFS were used to construct non–standard representations of the various notions of stochastic independence as well as of several new ones.

The recent paper [45] contains an exhaustive analysis of the notion of IFS, in particular the standard ones, as well as a generalization of this notion motivated by an application to **sub–product systems**, realized in the same paper.

The notion of interacting Fock space **(IFS)** arose from the solution, in the early 1990's, of the stochastic limit problem for quantum electrodynamics without dipole approximation [2]. The structure of the quantum noise obtained in the limit, radically new with respect to all previously known quantum noises and more complex due to the fact that its state space is a Hilbert module rather than a usual Hilbert space, motivated our attempt to axiomatize the main properties of this mathematical object in order to understand it more clearly and the result of this axiomatization was the notion of IFS.

3. Conclusions

In conclusion, with the emergence of QP, probability theory, traditionally related mainly to classical analysis and classical physics, has expanded its range creating deep and non-trivial connections with almost all branches of mathematics, physics and information theory. This growth has been reflected in the QP conferences and in their proceedings volumes (32 in the World Scientific QP-PQ series and 4 appeared with different publishers) as well as in the journal Infininite Dimensional Analysis, Quantum Probabiblity and Related Topics that, since more that 20 years is a reference point for everybody who is interested in the connections between probability and quantum theory. The intrinsic multi-disciplinary nature of the field is also reflected in the fact that nowadays several disciplines, from theoretical and simulation physics to mathematical physics, engineering, classical and quantum information begin to make extensive use of methods, ideas, techniques and results coming from quantum probability.

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