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Semi-classical Analysis Around Local Maxima and Saddle Points for Degenerate Nonlinear Choquard Equations

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Abstract

We study existence of semi-classical states for the nonlinear Choquard equation:

$$-\varepsilon^2 \Delta v + V(x)v = \frac{1}{\varepsilon^\alpha}(I_\alpha * F(v))f(v) \quad \text{in } \mathbb{R}^N,$$

where $N \geq 3$, $\alpha \in (0, N)$, $I_\alpha(x) = A_\alpha/|x|^{N-\alpha}$ is the Riesz potential, $F \in C^1(\mathbb{R}, \mathbb{R})$, $F'(s) = f(s)$ and $\varepsilon > 0$ is a small parameter. We develop a new variational approach, in which our deformation flow is generated through a flow in an augmented space to get a suitable compactness property and to reflect the properties of the potential. Furthermore our flow keeps the size of the tails of the function small and it enables us to find a critical point without introducing a penalization term. We show the existence of a family of solutions concentrating to a local maximum or a saddle point of $V(x) \in C^N(\mathbb{R}^N, \mathbb{R})$ under general conditions on $F(s)$. Our results extend the results by Moroz and Van Schaftingen (Calc Var Partial Differ Equ 52:199–235, 2015) for local minima (see also Cingolani and Tanaka (Rev Mat Iberoam 35(6):1885–1924, 2019)) and Wei and Winter (J Math Phys 50:012905, 2009) for non-degenerate critical points of the potential.

Keywords Nonlinear Choquard equation · Semi-classical states · Non-local nonlinearities · Deformation argument

Mathematics Subject Classification 35Q55 · 35Q40 · 35J20 · 58E05

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1 Introduction

In the recent years a large amount of papers has been devoted to investigate concentration phenomena of solutions to nonlinear Schrödinger equations with local sources around potential wells, namely local minima of some external potential functions. Starting to the celebrated papers by Floer and Weinstein [31] and Rabinowitz [58], several variational approaches were implemented and some efforts were done to obtain optimal results. We mention for instance [7, 16, 18–21, 29, 36, 37]. A more difficult problem seems to detect concentration phenomena around local maxima or saddle points of the potential type function. Some results are known for nonlinear Schrödinger equations under nondegeneracy conditions of the local maxima which allow to perform Lyapunov Schmidt reduction arguments [2, 3, 31, 41, 51]. More recently, del Pino and Felmer in [30] introduced a new reduction and proved a concentration result for solutions of nonlinear Schrödinger equation around local maxima and saddle points of the potential, assuming Ambrosetti-Rabinowitz type conditions and monotonicity conditions on the nonlinearity, which are crucial to apply a Nehari manifold approach. We refer to [28] for a generalization of the result of [30]. The more general result is contained in [8, 9] where Byeon and the second author succeeded to show the existence of families of solutions to nonlinear Schrödinger equations with local nonlinearity of Berestycki-Lions type concentrating at critical points which are given by minimax method with suitable linking properties, e.g. local maxima, mountain pass critical points, non-degenerate critical points. See also [6, 10–12, 39].

The goal of the present paper is to develop a new theoretical approach to obtain existence of solutions which concentrate at local maxima or saddle points of potential functions, under quite optimal assumptions on the nonlinearity and without any nondegeneracy conditions for class of nonlinear Schrödinger equations having local or nonlocal source.

As prototype of nonlocal problem in the source, we focus our analysis on the following class of equations

$$\begin{cases} -\varepsilon^2 \Delta v + V(x)v = \dfrac{1}{\varepsilon^\alpha}(I_\alpha * F(v))f(v) \quad \text{in } \mathbb{R}^N, \\ v > 0 \quad \text{in } \mathbb{R}^N, \quad v \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a small positive parameter, $N \geq 3$, $\alpha \in (0, N)$,

$$I_\alpha(x) = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^\alpha |x|^{N-\alpha}} : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$$

is the Riesz potential, $F(s) \in C^1(\mathbb{R}, \mathbb{R})$ and $f(s) = F'(s)$. We recall that in 1954 the Eq. (1.1) with $N = 3$, $\alpha = 2$ and $F(s) = \frac{1}{2}|s|^2$ was introduced by Pekar [52] to describe the quantum theory of a polaron at rest. In 1976, (1.1) appeared in the work of Choquard on the modeling of an electron trapped in its own hole, in a certain approximation to the Hartree-Fock theory of plasma (see also [32]). More recently it has found a great attention due to models of self-gravitational collapse of a quantum mechanical wave function, proposed by Roger Penrose [53–55] and in that context it is known as as Schrödinger-Newton equation (see also [46, 60]).

In literature, (1.1) is usually referred as nonlinear Choquard equation or Schrödinger equation with Hartree type potential. From a mathematical point of view, the early existence and symmetry results are due to Lieb [42] and Lions [43]. Successively, Ma and Zhao [44] classified all positive solutions to (1.1) for power nonlinearity and showed that they must be radially symmetric and monotonically decreasing about some fixed point. Recently Moroz and Van Schaftingen [48] investigated existence, some qualitative properties and decay asymptotics of positive ground state solutions to (1.1) for $\varepsilon > 0$ fixed when F satisfies the Berestycki-Lions type conditions. Other results are contained in [4, 13, 17, 18, 24, 27, 40, 47, 50, 57].

In the present paper we are interested in the study the existence of concentrating family of solutions of (1.1) at local maxima or saddle point of $V(x)$ as $\varepsilon \to 0$.

Denoting $u(x) = v(\varepsilon x)$, the Eq. (1.1) is equivalent to

$$\begin{cases} -\Delta u + V(\varepsilon x)u = (I_\alpha * F(u))f(u) \quad \text{in } \mathbb{R}^N, \\ u > 0 \quad \text{in } \mathbb{R}^N. \quad u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.2}$$

Thus we try to find critical points of the corresponding functional:

$$I_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + V(\varepsilon x)u^2 - \frac{1}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u) : H^1(\mathbb{R}^N) \to \mathbb{R}$$

and we ask the existence of a concentrating family (u_ε) of solutions of (1.2) as $\varepsilon \to 0$.

Firstly the concentration at nondegenerate critical points of the potential $V(x)$ has been studied by Wei and Winter [62] using Lyapunov Schmidt reduction when $N = 3$, $\alpha = 2$ and $F(s) = s^2$. The case of local minima (possibly degenerate) of V when $N = 3$ and $F(s) = s^2$ has been considered in [22] by means of a penalization approach (see also [14, 59, 63]). More recently, Moroz and Van Schaftingen [49] proved existence of a single-peak solution of (1.1) concentrating at a local minima of $V(x)$ for $f(s) = |s|^{p-2}s$, $p \in [2, \frac{N+\alpha}{N-2})$ via a new non-local penalization method. [64] extended the result in [49] and showed the existence under (f4) below, $\lim_{t\to\infty} \frac{f(t)}{t^{\frac{\alpha+2}{N-2}}} = 0$ and

$$\lim_{t\to 0} \frac{f(t)}{t} = 0. \tag{1.3}$$

They also proved the existence of multi-peak solutions, whose each peak concentrates at different local minimum of $V(x)$ as $\varepsilon \to 0$. We note that conditions $p \geq 2$ or (1.3) is important in their arguments as it enables them to use linearized problems at infinity. See also [1, 45, 56] dealing with critical Choquard equations.

In [23] we developed a new variational approach which is applicable to a wide class of nonlinearities including $F(s) = |s|^p$, $p \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$. In particular, we can deal with the sublinear case $p \in (\frac{N+\alpha}{N}, 2)$, differently to [49]. We obtained the multiplicity of concentrating solutions via the cup-length of a critical set $\mathrm{Crit}_{V_0} = \{x \in \Omega;\ V(x) =$

$V_0\}$, where $\Omega \subset \mathbb{R}^N$ is a bounded set such that $V_0 \equiv \inf_{x\in\Omega} V(x) < \inf_{x\in\partial\Omega} V(x)$. See also [38] for the effect of the topology of the potential wells on the existence of multi-bumps solutions.

The main purpose of this paper is to study the existence of concentrating family of solutions of nonlinear Choquard equation (1.1) at a local maximum or saddle point of $V(x)$. To our knowledge, the only concentration result dealing nondegenerate local maxima is due to Wei and Winter [62], when $N = 3$, $\alpha = 2$ and $F(s) = s^2$.

The existence of concentrating families of solutions at local maxima and saddle points of $V(x)$ is a more involved open problem and deformation argument using the standard gradient flow associated to $I_\varepsilon(u)$ does not seem enough. We also note that non-degeneracy of solutions of the limit problem $-\Delta u + V(x_0)u = (I_\alpha * F(u)) f(u)$ is not known except the case $N = 3$, $\alpha = 2$, $F(u) = |u|^2$ and it seems difficult to apply Lyapunov Schmidt reduction methods in general.

To show the existence of concentrating family of solutions, in this paper we develop a new deformation argument, which is partially inspired by [8, 25, 33, 35].

Our deformation argument is developed for $V(x) \in C^1(\mathbb{R}^N, \mathbb{R})$ through a deformation in an augmented space $\mathbb{R}^N \times H^1(\mathbb{R}^N)$ and it has the following new features:

(i) Our deformation flow is developed through a deformation for an augmented functional:

$$J_\varepsilon(z, u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x + z)u(x)^2 - \frac{1}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)$$

for all $(z, u) \in \mathbb{R}^N \times H^1(\mathbb{R}^N)$. We use the following translation of $u \in H^1(\mathbb{R}^N)$ as a part of our new deformation argument:

$$t \mapsto u\left(x - \frac{h}{\varepsilon}t\right);\ (-\delta, \delta) \to \mathbb{R}, \tag{1.4}$$

where $h \in \mathbb{R}^N$. If $u_\varepsilon(x)$ "concentrates" at some point $p_0 \in \mathbb{R}^N$ in the original scale for (1.1), that is, $u_\varepsilon(x) \sim v(x - \frac{p_0}{\varepsilon})$ for some function $v(x)$, then as $\varepsilon \sim 0$

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} I_\varepsilon\left(u_\varepsilon\left(x - \frac{h}{\varepsilon}t\right)\right) &= \frac{1}{2}\frac{d}{dt}\Big|_{t=0}\int_{\mathbb{R}^N} V(\varepsilon x)u_\varepsilon\left(x - \frac{h}{\varepsilon}t\right)^2 dx \\
&\sim \frac{1}{2}\frac{d}{dt}\Big|_{t=0}\int_{\mathbb{R}^N} V(\varepsilon x)v\left(x - \frac{p_0 + ht}{\varepsilon}\right)^2 dx \\
&= \frac{1}{2}\frac{d}{dt}\Big|_{t=0}\int_{\mathbb{R}^N} V(\varepsilon x + p_0 + ht)v(x)^2\, dx \\
&= \frac{1}{2}(\nabla V(p_0), h)\int_{\mathbb{R}^N} v(x)^2\, dx.
\end{aligned}$$

Thus, if $\nabla V(p_0) \neq 0$, choosing $h = \nabla V(p_0)$, the traslation flow (1.4) gives a decreasing flow for $I_\varepsilon(u)$ in a small neighborhood of u_ε. Thus $\nabla V(p_0)$ gives a useful information for deformation argument. However we note that in $H^1(\mathbb{R}^N)$

the flow (1.4) is continuous but not of class C^1 in general and it cannot be obtained through the standard deformation theory, where the flow is obtained as a solution of ODE in a Banach space and it must be of class C^1.
Our augmented functional $J_\varepsilon(z,u)$ enjoys the following property:

$$J_\varepsilon(z,u) = I_\varepsilon\left(u\left(x-\frac{z}{\varepsilon}\right)\right) \quad \text{for all } z\in\mathbb{R}^N \text{ and } u\in\mathbb{R}$$

and the traslation flow (1.4) can be obtained as a composition of a C^1-flow in the augmented space

$$t\mapsto (ht, u(x));\ (-\delta,\delta)\to\mathbb{R}^N\times H^1(\mathbb{R}^N)$$

and a projection

$$\pi_\varepsilon : (z,u)\mapsto u\left(x-\frac{z}{\varepsilon}\right);\ \mathbb{R}^N\times H^1(\mathbb{R}^N)\to H^1(\mathbb{R}^N).$$

We also note that the standard deformation flow $\eta(t) : (-\delta,\delta)\to H^1(\mathbb{R}^N)$ for $I_\varepsilon(u)$ in $H^1(\mathbb{R}^N)$ also can be obtained as a composition of a flow $(-\delta,\delta)\to\mathbb{R}^N\times H^1(\mathbb{R}^N)$; $t\mapsto(0,\eta(t))$ and the projection π_ε.
In the following sections, first we construct a deformation flow $\widetilde{\eta}$ for the augmented functional $J_\varepsilon(z,u)$ in $\mathbb{R}^N\times H^1(\mathbb{R}^N)$ and we construct a deformation flow for $I_\varepsilon(u)$ as a composition $(\pi_\varepsilon\circ\widetilde{\eta})(t)$. We also note that our new construction of a deformation flow works under weaker version of Palais-Smale type condition (see Proposition 4.5, 4.7 and 6.1).

(ii) Another new aspect of our deformation flow is that it keeps the size of the tail of functions small during deformation. That is, defining the size of a tail of a function u by

$$T_\varepsilon(u) = \int_{\mathbb{R}^N}\widetilde{\zeta}_{4/\sqrt{\varepsilon}}(x-\beta(u))(|\nabla u|^2+u^2)\,dx,$$

where $\widetilde{\zeta}_R(x)\in C^\infty(\mathbb{R}^N,\mathbb{R})$ satisfies $\widetilde{\zeta}_R(x)=1$ for $|x|\geq R$ and $\widetilde{\zeta}_R(x)=0$ for $|x|\leq R-1$ and $\beta(u)$ is the "center of mass" of u which will be defined in Sect. 3.3. We observe that for small κ_ε with $\kappa_\varepsilon\to 0$, the set $\{u : T_\varepsilon(u)\leq\kappa_\varepsilon\}$ is positively invariant under our deformation flow. See Proposition 6.1 and (6.3) in Sect. 6. This property ensures that if $u(x)$ concentrates around the center $\beta(u)$ of mass, deformed function $\eta(t,u)$ continues to concentrate around the center $\beta(\eta(t,u))$ of mass of the deformed functions $\eta(t,u)$. The standard deformation flow does not have this property. Such a property is usually obtained by using tail minimization methods for local problems, that is, we solve the elliptic boundary problem outside of a large ball centered at $\beta(u)$. We note that such a tail minimizing problem requires the unique solvability of the elliptic boundary problem and usually it is ensured for local problems, i.e., for nonlinear Schrödinger equations, under the condition $f\in C^1$. For non-local problems, e.g. nonlinear Choquard equations such an approach does not work because of non-local feature of the problem. In

Sects. 5 and 6 we develop a new deformation method in which the deformation flow is constructed through a deformation in an augmented space $\mathbb{R}^N \times H^1(\mathbb{R}^N)$. Our deformation method works for both of local and non-local problems. In a paper in preparation, we aim to apply this new approach to fractional problem (see [15] for concentration around local minima). See Remark 8.3 in Sect. 8 for an application to local problem (see also [26]).

Remark 1.1 In [8, 9], a related deformation argument is developed for nonlinear Schrödinger equation:

$$-\Delta u + V(\varepsilon x)u = g(u) \quad \text{in } \mathbb{R}^N \tag{1.5}$$

in a different way. Namely it is constructed as an iteration of 3 flows:

(1) The standard deformation flow $\eta_1(t,\cdot)$. Here $\eta_1(t,\cdot)$ is a solution of $\frac{d\eta_1}{dt} = -\varphi(\eta_1)\mathcal{V}(\eta_1)$, $\eta_1(0,u) = u$, where $\mathcal{V}(\cdot)$ is a pseudo-gradient vector associated to the functional corresponding to (1.2).
(2) The translation flow $\eta_2(t,u)(x) = u(x - \frac{h}{\varepsilon}t)$. Here $h = -\nabla V(\varepsilon\beta(u))$, where $\beta(u)$ is the center of mass of u.
(3) The tail minimizing operator $\tau_\varepsilon(u)$, which is defined by $\tau_\varepsilon(u) = v$, where v is a solution of the exterior problem:

$$\begin{cases} -\Delta v + V(\varepsilon x)v = g(v) & \text{in } |x - \beta(u)| > R, \\ v(x) = u(x) & \text{on } |x - \beta(u)| = R. \end{cases} \tag{1.6}$$

The procedure is rather complicated and in present paper we give an "easier" deformation argument through a construction flow in an augmented space $\mathbb{R}^N \times H^1(\mathbb{R}^N)$. We note that the exterior problem (1.6) is well-defined for local problem (1.5). But for non-local problem (1.2), the exterior problem is not well-defined because of non-locality of the problem.

To state our existence result for (1.2), we assume

(f1) $f(s) \in C(\mathbb{R},\mathbb{R})$;
(f2) there exists $C > 0$ such that for all $s \in \mathbb{R}$

$$|sf(s)| \le C\left(|s|^{\frac{N+\alpha}{N}} + |s|^{\frac{N+\alpha}{N-2}}\right);$$

(f3) $F(s) = \int_0^s f(t)\,dt$ satisfies

$$\lim_{s\to 0}\frac{F(s)}{|s|^{\frac{N+\alpha}{N}}} = 0, \quad \lim_{s\to\infty}\frac{F(s)}{|s|^{\frac{N+\alpha}{N-2}}} = 0;$$

(f4) $f(s)$ is odd and f is positive on $(0,\infty)$.

We remark that the conditions (f1)–(f4) are in the spirit of Berestycki and Lions [5, 34, 48] and in our previous work [23] for a continuous potential $V(x)$ we studied concentration at a local minimum under these conditions.

In the present paper we require much regularity on the potential $V(x)$. Precisely for $V(x)$ we assume

(V1) $V(x) \in C^N(\mathbb{R}^N, \mathbb{R})$, $\nabla V(x) \in L^{\frac{N}{2}}(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$;
(V2) $\inf_{x\in\mathbb{R}^N} V(x) \equiv \underline{V} > 0$, $\sup_{x\in\mathbb{R}^N} V(x) \equiv \overline{V} < \infty$;
(V3) there exists a bounded connected open set $\Omega \subset \mathbb{R}^N$ with a smooth boundary $\partial\Omega$ such that

$$\nabla V(x) \neq 0 \qquad \text{for all } x \in \partial\Omega.$$

We mainly study two situations where $V(x)$ has a local maximum in Ω or $V(x)$ has a mountain pass geometry in Ω. More precisely, we assume (LM) or (MP) below.

(LM) $V_0 \equiv \sup_{x\in\Omega} V(x) > \sup_{x\in\partial\Omega} V(x)$;
(MP) There exist $e_0, e_1 \in \Omega$ such that setting

$$V_0 \equiv \inf_{c\in\Lambda} \max_{\xi\in[0,1]} V(c(\xi)),$$
$$\Lambda = \{c(\xi) \in C([0,1], \Omega) : c(0) = e_0,\ c(1) = e_1\},$$

V_0 satisfies

(i) $V(e_0), V(e_1) < V_0$;
(ii) for $x \in \partial\Omega$ with $V(x) = V_0$,

$$-\nabla V(x) \notin \{\mu n(x) : \mu \geq 0\},$$

where $n(x) \in \mathbb{R}^N$ is the unit outer normal at $x \in \partial\Omega$.

We note that under the assumption (i), (ii) it is standard to see that V_0 is a critical value of $V(x)$.

Our main result is

Theorem 1.2 *Assume (f1)–(f4) and (V1)–(V3). Moreover suppose (LM) or (MP). Then* (1.1) *has at least one positive solution concentrating in*

$$\mathrm{Crit}_{V_0} \equiv \{x \in \Omega : V(x) = V_0,\ \nabla V(x) = 0\}.$$

That is, there exist $\varepsilon_0 > 0$ and a family $(u_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ of solutions of (1.2) *with the following property: for any sequence $(\varepsilon_j)_{j=1}^\infty \subset (0, \varepsilon_0]$ with $\varepsilon_j \to 0$ after extracting a subsequence—we denote it by ε_j for simplicity of notation—, there exist $(x_j)_{j=1}^\infty \subset \mathbb{R}^N$, $x_0 \in \mathrm{Crit}_{V_0}$ and a least energy solution $\omega_0 \in H^1(\mathbb{R}^N)$ of the limit problem $-\Delta u + V(x_0)u = (I_\alpha * F(u))f(u)$ in $\mathbb{R}^N$ such that*

$$\varepsilon_j x_j \to x_0,$$
$$u_{\varepsilon_j}(x - x_j) \to \omega_0(x) \quad \textit{strongly in } H^1(\mathbb{R}^N) \quad \textit{as } j \to \infty.$$

In (V1)–(V3), the assumption $V(x) \in C^N(\mathbb{R}^N, \mathbb{R})$ is used in order to show via Sard's Theorem that the set of critical values of $V(x)$ is of measure 0. For a potential $V(x)$ of class C^1, we can show the existence of a solution under the following assumption of isolatedness of critical points of $V(x)$

(V1') $V(x) \in C^1(\mathbb{R}^N, \mathbb{R})$, $\nabla V(x) \in L^{\frac{N}{2}}(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$;
(V1") critical points of $V(x)$ in Ω are isolated in Ω.

Namely we have

Theorem 1.3 *Assume (f1)–(f4) and (V1'), (V1"), (V2), (V3). Moreover suppose (LM) or (MP). Then the conclusion of Theorem 1.2 holds.*

Remark 1.4 If we assume (V1') without (V1") instead of (V1) in Theorem 1.2, a weaker version of the result holds. See Sect. 7.4.

This paper is organized as follows: In Sect. 2 we give some preliminary results. In Sect. 3 we study the limit problem. We introduce a Pohozaev type function $P_a(u)$ and a center $\beta(u)$ of mass, which are used in this paper repeatedly. In Sect. 4 we introduce a neighborhood of expected solutions and we show a concentration-compactness type results for functional $I_\varepsilon(u)$. We will develop a local deformation argument in this neighborhood in Sects. 5, 6, and 7. Here newly introduced ε-dependent distance $\mathrm{dist}_\varepsilon(\cdot,\cdot)$ in $H^1(\mathbb{R}^N)$ plays an important role. In Sect. 5 we introduce a functional $T_\varepsilon(u)$ to estimate the size of the tail of functions u and we construct a vector field, which decreases both of $T_\varepsilon(u)$ and $I_\varepsilon(u)$ and which enables us to generate a special deformation flow that keeps the tail of functions small. In Sect. 6 we give our new deformation result for $I_\varepsilon(u)$, which has new features stated above. Finally we give a proof of our main existence result in Sect. 7. In Sect. 8 we give a remark on concentration at a local minimum of $V(x)$.

2 Preliminaries

In what follows, we use notation: for $u \in H^1(\mathbb{R}^N)$

$$\|u\|_{H^1} = \left(\int_{\mathbb{R}^N} |\nabla u|^2 + u^2\right)^{1/2},$$

$$\|u\|_r = \left(\int_{\mathbb{R}^N} |u|^r\right)^{1/r} \quad \text{for } r \in [1,\infty), \qquad \|u\|_\infty = \operatorname*{ess\,sup}_{x\in\mathbb{R}^N} |u(x)|.$$

We also use notation for $p \in \mathbb{R}^N$, $u_0 \in H^1(\mathbb{R}^N)$, $r > 0$

$$B(p,r) = \{x \in \mathbb{R}^N : |x-p| < r\}, \qquad \overline{B}(p,r) = \{x \in \mathbb{R}^N : |x-p| \le r\},$$
$$B_{H^1}(u_0,r) = \{u \in H^1(\mathbb{R}^N) : \|u-u_0\|_{H^1} < r\}.$$

2.1 Estimates for Non-local Term

First we give some estimates for $\int_{\mathbb{R}^N}(I_\alpha * f)g$ and

$$\mathcal{D}(u) = \int_{\mathbb{R}^N}(I_\alpha * F(u))F(u).$$

For proofs, we refer to [23].

We denote various constants, which are independent of u, by $C, C', C'', \cdots$

Lemma 2.1 (c.f. Section 2.1 of [23]).

(i) *For $p, r > 1$ and $\alpha \in (0, N)$ with $\frac{1}{p} + \frac{1}{r} = \frac{N+\alpha}{N}$ there exists a constant $C = C(N, \alpha, p, r) > 0$ such that*

$$\left|\int_{\mathbb{R}^N}(I_\alpha * f)g\right| \le C\|f\|_p\|g\|_r$$

for all $f \in L^p(\mathbb{R}^N)$, $g \in L^r(\mathbb{R}^N)$.

(ii) *Assume $p, r > 1$ and $\alpha \in (0, N)$ with $\frac{1}{p} + \frac{1}{r} < \frac{N+\alpha}{N}$. Then for $L \ge 1$ there exists a constant $D_L = D_L(N, \alpha, p, r) > 0$ such that $D_L \to 0$ as $L \to \infty$ and*

$$\left|\int_{\mathbb{R}^N}(I_\alpha * f)g\right| \le D_L\|f\|_p\|g\|_r$$

for all $f \in L^p(\mathbb{R}^N)$, $g \in L^r(\mathbb{R}^N)$ with $\operatorname{dist}(\operatorname{supp} f, \operatorname{supp} g) \ge L$. □

In (ii), D_L is given by

$$D_L = \|I_\alpha^L\|_q,$$

where q satisfies $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$, in particular $q > \frac{N}{N-\alpha}$ and $I_\alpha^L(x)$ is defined by

$$I_\alpha^L(x) = \begin{cases} \frac{1}{|x|^{N-\alpha}} & \text{for } |x| \ge L, \\ 0 & \text{otherwise.} \end{cases}$$

Setting $\sigma(s) = s^2 + |s|^{\frac{2N}{N-2}}$ for $s \in \mathbb{R}$, under (f2) we have for $u, v \in H^1(\mathbb{R}^N)$

$$\|F(u)\|_{\frac{2N}{N+\alpha}} \le C\sigma(\|u\|_{H^1})^{\frac{N+\alpha}{2N}},$$

$$|\mathcal{D}(u)| \le C\|F(u)\|^2_{\frac{2N}{N+\alpha}} \le C'\sigma(\|u\|_{H^1})^{\frac{N+\alpha}{N}},$$

$$|\mathcal{D}'(u)v| \le C\|F(u)\|_{\frac{2N}{N+\alpha}}\|f(u)v\|_{\frac{2N}{N+\alpha}} \le C'\sigma(\|u\|_{H^1})^{\frac{N+\alpha}{2N}}(\|u\|_{H^1}^{\frac{\alpha}{N}} + \|u\|_{H^1}^{\frac{\alpha+2}{N-2}})\|v\|_{H^1}.$$

We also have

$$I_\varepsilon(u) \geq \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\underline{V}\|u\|_2^2 - C'\sigma(\|u\|_{H^1})^{\frac{N+\alpha}{N}},$$
$$I'_\varepsilon(u)u \geq \|\nabla u\|_2^2 + \underline{V}\|u\|_2^2 - C'\sigma(\|u\|_{H^1})^{\frac{N+\alpha}{N}}.$$

In particular, $I_\varepsilon(u)$ has mountain pass geometry uniformly in $\varepsilon \in (0, 1]$ and we have

Corollary 2.2 *There exist $\rho_0 > 0$ and $c_0 > 0$ such that for $\varepsilon \in (0, 1]$*

$$I_\varepsilon(u) \geq c_0\|u\|_{H^1}^2, \quad I'_\varepsilon(u)u \geq c_0\|u\|_{H^1}^2$$

for all $u \in H^1(\mathbb{R}^N)$ with $\|u\|_{H^1} \leq \rho_0$. □

For $R > 0$ we choose functions $\zeta_R(s), \widetilde{\zeta}_R(s) \in C^\infty(\mathbb{R}^N, \mathbb{R})$ such that

$$\zeta_R(x) = \begin{cases} 1 & \text{for } |x| \leq R, \\ 0 & \text{for } |x| \geq R+1, \end{cases} \qquad \widetilde{\zeta}_R(x) = \begin{cases} 0 & \text{for } |x| \leq R-1, \\ 1 & \text{for } |x| \geq R, \end{cases}$$
$$\zeta_R(x),\ \widetilde{\zeta}_R(x) \in [0, 1],\ |\nabla\zeta_R(x)|,\ |\nabla\widetilde{\zeta}_R(x)| \leq 2 \quad \text{for all } x \in \mathbb{R}^N. \tag{2.1}$$

We will use the following inequalities frequently: for $u \in H^1(\mathbb{R}^N)$, $R > 0$, $p \in \mathbb{R}^N$

$$\|\zeta_R(x-p)u\|_{H^1} \leq 3\|u\|_{H^1}, \quad \|\widetilde{\zeta}_R(x-p)u\|_{H^1} \leq 3\|u\|_{H^1}. \tag{2.2}$$

In fact,

$$\begin{aligned}\|\zeta_R(x-p)u\|_{H^1}^2 &= \|\nabla(\zeta_R(x-p)u)\|_2^2 + \|\zeta_R(x-p)u\|_2^2 \\ &\leq 2\|\zeta_R(x-p)\nabla u\|_2^2 + 2\|(\nabla\zeta_R(x-p))u)\|_2^2 + \|u\|_2^2 \\ &\leq 2\|\nabla u\|_2^2 + 9\|u\|_2^2 \leq 9\|u\|_{H^1}^2.\end{aligned}$$

We can show the second inequality in a similar way.

Lemma 2.3 (c.f. Corollary 2.6 of [23]). *For a fixed $M > 0$ there exists $C > 0$ such that for any $R, L \geq 1$ and $u \in H^1(\mathbb{R}^N)$ with $\|u\|_{H^1} \leq M$*

(i) $|(\mathcal{D}'(u) - \mathcal{D}'(\zeta_R u))\zeta_R u| \leq C(D_L + \sigma(\|u\|_{H^1(|x|\in[R,R+L])})^{\frac{N+\alpha}{2N}})$.
(ii) $|(\mathcal{D}'(u) - \mathcal{D}'(\widetilde{\zeta}_{R+L} u))\zeta_{R+L} u| \leq C(D_L + \sigma(\|u\|_{H^1(|x|\in[R,R+L])})^{\frac{N+\alpha}{2N}})$.

Here $D_L > 0$ is given in Lemma 2.1. In particular $D_L \to 0$ as $L \to \infty$.

Proof We set

$$\chi_1(x) = \begin{cases} 1 & \text{if } |x| \leq R, \\ 0 & \text{otherwise,} \end{cases} \quad \chi_2(x) = \begin{cases} 1 & \text{if } |x| \in [R, R+L], \\ 0 & \text{otherwise,} \end{cases}$$

$$\chi_3(x) = \begin{cases} 1 & \text{if } |x| \geq R + L, \\ 0 & \text{otherwise.} \end{cases}$$

We also set for $i = 1, 2, 3$

$$\begin{aligned} F_i &= \chi_i(x)F(u(x)), \quad \widetilde{F}_i = \chi_i(x)F(\zeta_R u(x)), \\ f_i &= \chi_i(x)f(u(x)), \quad \widetilde{f}_i = \chi_i(x)f(\zeta_R u(x)), \\ \widetilde{u}_i &= \chi_i(x)\zeta_R u(x). \end{aligned}$$

Since $F_1 = \widetilde{F}_1$, $f_1 = \widetilde{f}_1$, $\widetilde{F}_3 = \widetilde{f}_3 = \widetilde{u}_3 = 0$, $L > 1$, we have

$$\begin{aligned} &\frac{1}{2}(\mathcal{D}'(u) - \mathcal{D}'(\zeta_R u))\zeta_R u \\ &= \int_{\mathbb{R}^N} (I_\alpha * (F_1 + F_2 + F_3))(f_1\widetilde{u}_1 + f_2\widetilde{u}_2) - \int_{\mathbb{R}^N} (I_\alpha * (F_1 + \widetilde{F}_2))(f_1\widetilde{u}_1 + \widetilde{f}_2\widetilde{u}_2) \\ &= \int_{\mathbb{R}^N} (I_\alpha * F_1)(f_2 - \widetilde{f}_2)\widetilde{u}_2 + \int_{\mathbb{R}^N} (I_\alpha * F_2)(f_1\widetilde{u}_1 + f_2\widetilde{u}_2) \\ &\quad + \int_{\mathbb{R}^N} (I_\alpha * \widetilde{F}_2)(f_1\widetilde{u}_1 + \widetilde{f}_2\widetilde{u}_2) + \int_{\mathbb{R}^N} (I_\alpha * F_3)(f_1\widetilde{u}_1 + f_2\widetilde{u}_2). \end{aligned}$$

Since $\|F_2\|_{\frac{2N}{N+\alpha}}, \|\widetilde{F}_2\|_{\frac{2N}{N+\alpha}}, \|\widetilde{f}_2\widetilde{u}_2\|_{\frac{2N}{N+\alpha}}, \|f_2\widetilde{u}_2\|_{\frac{2N}{N+\alpha}} \leq C\sigma(\|u\|_{H^1(|x|\in[R,R+L])})^{\frac{N+\alpha}{2N}}$, $\|F_1\|_{\frac{2N}{N+\alpha}}, \|F_3\|_{\frac{2N}{N+\alpha}}, \|f_1\widetilde{u}_1\|_{\frac{2N}{N+\alpha}} \leq C\sigma(\|u\|_{H^1})^{\frac{N+\alpha}{2N}} \leq C\sigma(M)^{\frac{N+\alpha}{2N}}$ and

$$\left|\int_{\mathbb{R}^N} (I_\alpha * F_3)(f_1\widetilde{u}_1)\right| \leq D_L\|F_3\|_{L^{\frac{2N}{N+\alpha}}(|x|\geq R+L)}\|f_1\widetilde{u}_1\|_{L^{\frac{2N}{N+\alpha}}(|x|\leq R)},$$

We can see that (i) holds. We can show (ii) in a similar way. □

The above lemma gives a useful localization property of $\mathcal{D}(u)$.

Finally in this section we give the following lemma on the behavior of bounded Palais-Smale sequences, which will help us to get concentration-compactness type result in Sect. 4.

Lemma 2.4 *There exists $\rho_1 > 0$ with the following property: if $(\varepsilon_j)_{j=1}^\infty \subset (0, 1]$, a bounded sequence $(u_j)_{j=1}^\infty \subset H^1(\mathbb{R}^N)$ and $(y_j)_{j=1}^\infty \subset \mathbb{R}^N$ satisfy*

$$\begin{aligned} &I'_{\varepsilon_j}(u_j) \to 0 \quad \textit{strongly in } (H^1(\mathbb{R}^N))^*, \\ &u_j(x + y_j) \rightharpoonup u_0 \quad \textit{weakly in } H^1(\mathbb{R}^N) \end{aligned}$$

for some $u_0 \in H^1(\mathbb{R}^N)$ with $\|u_0\|_{H^1} \leq \rho_1$, then $u_0 = 0$.

Proof We set $v_j(x) = u_j(x + y_j)$. Let $L \in \mathbb{N}$. Since $(u_j)_{j=1}^\infty$ is bounded in $H^1(\mathbb{R}^N)$, we have for $C > 0$ independent of j and L

$$\sum_{i=1}^{L} \|v_j\|^2_{H^1(|x|\in[Li,L(i+1)])} \le \|v_j\|^2_{H^1} \le C.$$

Thus there exists $i_j \in \{1, 2, \cdots, L\}$ such that $\|v_j\|^2_{H^1(|x|\in[Li_j,L(i_j+1)])} \le \frac{C}{L}$. Extracting a subsequence if necessary, we may assume that for any $L \in \mathbb{N}$ there exists $k_L \in \{1, 2, \cdots, L\}$ such that

$$\|v_j\|^2_{H^1(|x|\in[Lk_L,L(k_L+1)])} \le \frac{C}{L} \quad \text{for all } j \in \mathbb{N}. \tag{2.3}$$

Let $\zeta_R(s)$ be a function satisfying (2.1) and set

$$v_j^{(L)}(x) = \zeta_{Lk_L}(x) v_j(x).$$

We have from (2.3)

$$\begin{aligned} &\left| \int_{\mathbb{R}^N} \nabla v_j \nabla v_j^{(L)} + V(\varepsilon_j x + y_j) v_j v_j^{(L)} - \int_{\mathbb{R}^N} |\nabla v_j^{(L)}|^2 + V(\varepsilon_j x + y_j)(v_j^{(L)})^2 \right| \\ &\quad \le \left| \int_{\mathbb{R}^N} \nabla(v_j - v_j^{(L)}) \nabla v_j^{(L)} + V(\varepsilon_j x + y_j)(v_j - v_j^{(L)}) v_j^{(L)} \right| \le a_L, \\ &|(\mathcal{D}'(v_j) - \mathcal{D}'(v_j^{(L)})) v_j^{(L)}| \le a_L, \end{aligned}$$

where a_L is independent of j and satisfies $a_L \to 0$ as $L \to \infty$. Here we apply Lemma 2.3 (i) with $R = Lk_L$ and L. Thus we have

$$\begin{aligned} I'_{\varepsilon_j}(v_j^{(L)}(x - y_j))(v_j^{(L)}(x - y_j)) &= \int_{\mathbb{R}^N} |\nabla v_j^{(L)}|^2 + V(\varepsilon_j x + y_j)(v_j^{(L)})^2 - \mathcal{D}'(v_j^{(L)}) v_j^{(L)} \\ &\le I'_{\varepsilon_j}(u_j)(v_j(x - y_j)) + 2a_L = o(1) + 2a_L. \end{aligned} \tag{2.4}$$

Since $v_j^{(L)} \to u_0^{(L)} \equiv \zeta_{Lk_L}(x) u_0(x)$ strongly in $L^p(\mathbb{R}^N)$ for $p \in (2, \frac{2N}{N-2})$ and $\|I'_{\varepsilon_j}(u_j)\|_{(H^1(\mathbb{R}^N))^*} \to 0$,

$$\limsup_{j\to\infty} \int_{\mathbb{R}^N} |\nabla v_j^{(L)}|^2 + V(\varepsilon_j x + y_j)(v_j^{(L)})^2 \le \mathcal{D}'(u_0^{(L)}) u_0^{(L)} + 2a_L.$$

Let $\rho_0 > 0$ be the number given in Corollary 2.2. Since $\|u_0^{(L)}\|_{H^1} \le C\|u_0\|_{H^1}$, choosing $\rho_1 > 0$ small, we have for L large

$$\limsup_{j\to\infty} \|v_j^{(L)}\|_{H^1} \le \rho_0, \quad \text{provided } \|u_0\|_{H^1} \le \rho_1.$$

By Corollary 2.2 and (2.4),

$$c_0 \limsup_{j\to\infty} \|v_j^{(L)}\|_{H^1}^2 \le \limsup_{j\to\infty} I'_{\varepsilon_j}(v_j^{(L)}(x-y_j))(v_j^{(L)}(x-y_j)) \le 2a_L.$$

Thus

$$\|\zeta_{Lk_L}(x)u_0\|_{H^1}^2 = \|u_0^{(L)}\|_{H^1}^2 \le \limsup_{j\to\infty} \|v_j^{(L)}\|_{H^1}^2 \le \frac{2}{c_0}a_L.$$

Since L is arbitrary, we have $u_0 = 0$. □

3 Limit Problems

3.1 Limit Problems

For $a > 0$ we define

$$L_a(u) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{a}{2}\|u\|_2^2 - \frac{1}{2}\mathcal{D}(u) : H^1(\mathbb{R}^N) \to \mathbb{R}.$$

Critical points of $L_a(u)$ is a solution of

$$-\Delta u + au = (I_\alpha * F(u))f(u) \quad \text{in } \mathbb{R}^N, \tag{3.1}$$

which appears as a limit equation for (1.2). That is, for a family $(u_\varepsilon(x))$ of solutions of (1.2) and $(x_\varepsilon) \subset \mathbb{R}^N$ with $x_\varepsilon \to x_0$, if there exists a limit $v_0(x) = \lim_{\varepsilon\to 0} u_\varepsilon(x + \frac{x_\varepsilon}{\varepsilon})$, then v_0 is a critical point of $L_{V(x_0)}(u)$, that is, a solution of (3.1) with $a = V(x_0)$. We denote by E_a the least energy level for (3.1):

$$E_a = \inf\{L_a(u) : u \ne 0,\ L'_a(u) = 0\}.$$

In [48], the existence of a least energy solution is proved under the conditions (f1)–(f3) and

(f4') there exists $s_0 \in \mathbb{R} \setminus \{0\}$ such that $F(s_0) > 0$.

They also proved that under (f1)–(f3), (f4') every ground state solution of (3.1) is radially symmetric with respect to some point in $\mathbb{R}^N$. It is also shown that any solution of (3.1) satisfies the Pohozaev identity:

$$P_a(u) = 0,$$

where

$$P_a(u) = \frac{N-2}{2}\|\nabla u\|_2^2 + \frac{N}{2}a\|u\|_2^2 - \frac{N+\alpha}{2}\mathcal{D}(u). \tag{3.2}$$

The least energy level E_a is characterized as

$$E_a = \inf\{L_a(u) : u \in H^1(\mathbb{R}^N) \setminus \{0\},\ P_a(u) = 0\}. \tag{3.3}$$

For $c > 0$ we set

$$\mathcal{S}_a^c = \{u \in H^1(\mathbb{R}^N) \setminus \{0\} : L_a'(u) = 0,\ L_a(u) \le c,\ |u(0)| = \max_{x \in \mathbb{R}^N} |u(x)|\}.$$

Arguing as in [48], we can show that

Lemma 3.1 *$\mathcal{S}_a^c$ is compact in $H^1(\mathbb{R}^N)$ provided $c < 2E_a$.*

3.2 Scaling Argument for $L_a(u)$

As in [23], to see the scaling property of the limit function $L_a(u)$, we consider for $u \in H^1(\mathbb{R}^N) \setminus \{0\}$

$$d(\lambda) = L_a(u(x/\lambda)) = \frac{1}{2}\|\nabla u\|_2^2 \lambda^{N-2} + \frac{a}{2}\|u\|_2^2 \lambda^N - \frac{1}{2}\mathcal{D}(u)\lambda^{N+\alpha} : (0, \infty) \to \mathbb{R}.$$

We have

(i) $d(\lambda) \to +0$ as $\lambda \to +0$;
(ii) $d(\lambda) \to -\infty$ as $\lambda \to \infty$;
(iii) $d(\lambda)$ has a unique critical point $\lambda_0(u)$, which is a maximum of $d(\lambda)$;
(iv) $d'(\lambda) = 0$ if and only if $P_a(u(x/\lambda)) = 0$.

In particular, we have

Proposition 3.2 *For a least energy solution $\omega_0(x)$ of (3.1), that is, $L_a'(\omega_0) = 0$, $L_a(\omega_0) = E_a$, we have*

$$L_a\left(\omega_0\left(\frac{x}{s}\right)\right) < E_a \quad \text{for } s \in (0, \infty) \setminus \{1\},$$
$$P_a\left(\omega_0\left(\frac{x}{s}\right)\right) \begin{cases} > 0 & \text{for } s \in (0, 1), \\ < 0 & \text{for } s \in (1, \infty). \end{cases}$$

3.3 Center of Mass

Here we introduce a center of mass $\beta(u)$ in a neighborhood of a shifted compact set. We will use the following

Proposition 3.3 *Let $\widehat{D} \subset H^1(\mathbb{R}^N) \setminus \{0\}$ be a compact set. We set for $\rho > 0$*

$$\widetilde{D} = \{\omega(x - p) : \omega \in \widehat{D},\ p \in \mathbb{R}^N\},$$
$$\widetilde{D}_\rho = \{u \in H^1(\mathbb{R}^N) : \operatorname{dist}_{H^1}(u, \widetilde{D}) < \rho\}.$$

Then there exist $\rho_2 > 0$, $R_0 > 0$ and C^1-function $\beta : \widetilde{D}_{\rho_2} \to \mathbb{R}^N$ such that

(i) *For* $u(x) = \omega(x-p) + \varphi(x) \in \widetilde{D}_{\rho_2}$ *with* $\omega \in \widehat{D}$, $p \in \mathbb{R}^N$, $\|\varphi\|_{H^1} < \rho_2$,

$$|\beta(u) - p| \le R_0.$$

(ii) $\beta(u)$ *is shift-invariant, that is,*

$$\beta(u(x-q)) = \beta(u) + q$$

for all $u \in \widetilde{D}_{\rho_2}$ *and* $q \in \mathbb{R}^N$.

(iii) *If* $u, v \in \widetilde{D}_{\rho_2}$ *satisfy*

$$u(x) = v(x) \quad \text{in } B(\beta(u), 4R_0), \tag{3.4}$$

then $\beta(u) = \beta(v)$.

(iv) *There exists* $C > 0$ *independent of* u *such that*

$$\|\beta'(u)\|_{(H^1(\mathbb{R}^N))^*} \le C \quad \text{for all } u \in \widetilde{D}_{\rho_2}.$$

A similar center of mass is given in [8, 9], which is locally Lipschitz continuous. Here we modify and improve the argument in [8, 9] and give a center of mass $\beta(u)$, which is of class C^1.

Proof We set $r_* = \inf_{\omega \in \widehat{D}} \|\omega\|_{H^1} > 0$. Since $\widehat{D}$ is compact, there exists $R_* > 0$ such that

$$\|\omega\|_{H^1(|x| \le R_*)} \ge \frac{2}{3} r_*, \quad \|\omega\|_{H^1(|x| \ge R_*)} \le \frac{1}{6} r_* \quad \text{for all } \omega \in \widehat{D}.$$

For $u = \omega(x-p) + \varphi(x)$ with

$$p \in \mathbb{R}^N, \ \omega \in \widehat{D} \text{ and } \|\varphi\|_{H^1} < \frac{1}{6} r_* \tag{3.5}$$

we have

$$\|u(x)\|_{H^1(|x-p| \le R_*)} \ge \frac{1}{2} r_*, \qquad \|u(x)\|_{H^1(|x-p| \ge R_*)} \le \frac{1}{3} r_*. \tag{3.6}$$

We set for $q \in \mathbb{R}^N$ and $u \in \widetilde{D}_{r_*/6}$

$$\Phi(q, u) = \int_{\mathbb{R}^N} \zeta_{R_*}(x-q)(|\nabla u|^2 + u^2)\, dx,$$

where $\zeta_{R_*}(x-q)$ is introduced in (2.1). By (3.6), we have for $u(x) = \omega(x-p)+\varphi(x) \in \widetilde{D}_{r_*/6}$

$$\begin{aligned}
&\Phi(p, u) \geq \left(\frac{1}{2}r_*\right)^2, \\
&\Phi(q, u) \leq \left(\frac{1}{3}r_*\right)^2 \quad \text{for } |q-p| \geq 2R_* + 1, \\
&\Phi(q, u(x-q')) = \Phi(q-q', u(x)) \quad \text{for all } q, q' \in \mathbb{R}^N.
\end{aligned}$$

In fact, $\operatorname{supp} \zeta_{R_*}(x-q) \subset \{x : |x-p| \geq R_*\}$ for $|q-p| \geq 2R_* + 1$. We choose and fix a function $\psi(s) \in C^\infty([0,\infty), \mathbb{R})$ such that

$$\psi(s) = \begin{cases} 1 & s \in [(\frac{1}{2}r_*)^2, \infty), \\ 0 & s \in [0, (\frac{1}{3}r_*)^2], \end{cases} \qquad \psi(s) \in [0,1] \quad \text{for all } s \in \mathbb{R}.$$

Then we have for $u = \omega(x-p) + \varphi(x) \in \widetilde{D}_{r_*/6}$ with (3.5)

$$\psi(\Phi(p,u)) = 1 \quad \text{and} \quad \psi(\Phi(q,u)) = 0 \quad \text{for } |q-p| \geq 2R_* + 1. \tag{3.7}$$

We set

$$\beta(u) = \frac{\int_{\mathbb{R}^N} q\psi(\Phi(q,u))\,dq}{\int_{\mathbb{R}^N} \psi(\Phi(q,u))\,dq} : \widetilde{D}_{r_*/6} \to \mathbb{R}^N.$$

Then we have

$$\begin{aligned}
&|\beta(u) - p| \leq 2R_* + 1, \\
&\beta(u(x-q')) = \beta(u(x)) + q'.
\end{aligned} \tag{3.8}$$

Thus, setting $R_0 = 2R_* + 1$, $\rho_2 = r_*/6$, we have (i)–(ii).

Next we prove (iii). We suppose that $u(x) = \omega(x-p) + \varphi(x)$, $v(x) = \omega'(x-p') + \varphi'(x) \in \widetilde{D}_{r_*/6}$ satisfy (3.4). By (3.7) and (3.8),

$$\operatorname{supp} \psi(\Phi(\cdot,u)) \subset \overline{B}(p, R_0) \subset \overline{B}(\beta(u), 2R_0). \tag{3.9}$$

Similarly $\operatorname{supp} \psi(\Phi(\cdot,v)) \subset \overline{B}(p', R_0) \subset \overline{B}(\beta(v), 2R_0)$.

By (3.4), we have $v(x) = u(x)$ on $\overline{B}(p, R_0)$, from which we have $\psi(\Phi(p,v)) = \psi(\Phi(p,u)) = 1$. Thus $p \in \operatorname{supp} \psi(\Phi(\cdot,v))$ and we have $|p-p'| \leq R_0$. And thus $\operatorname{supp} \psi(\Phi(\cdot,v)) \subset \overline{B}(p', R_0) \subset \overline{B}(p, 2R_0)$. Since $v = u$ on $\overline{B}(p, 3R_0) \subset \overline{B}(\beta(u), 4R_0)$, we have $\psi(\Phi(\cdot,v)) = \psi(\Phi(\cdot,u))$ on $\mathbb{R}^N$. Thus we have $\beta(v) = \beta(u)$.

Finally we prove (iv). We set $A = \int_{\mathbb{R}^N} \psi(\Phi(q,u))\,dq$. For $h \in H^1(\mathbb{R}^N)$ we compute that

$$\begin{aligned}\beta'(u)h &= \frac{1}{A}\int_{\mathbb{R}^N} q\psi'(\Phi(q,u))\partial_u\Phi(q,u)h\,dq \\ &\quad - \frac{1}{A^2}\int_{\mathbb{R}^N} q\psi(\Phi(q,u))\,dq \int_{\mathbb{R}^N}\psi'(\Phi(q,u))\partial_u\Phi(q,u)h\,dq \\ &= \frac{1}{A}\int_{\mathbb{R}^N}(q-\beta(u))\psi'(\Phi(q,u))\partial_u\Phi(q,u)h\,dq.\end{aligned}$$

By (3.9),

$$\begin{aligned}|\beta'(u)h| &\le \frac{2R_0}{A}\int_{\mathbb{R}^N}|\psi'(\Phi(q,u))\partial_u\Phi(q,u)h|\,dq \\ &\le \frac{2R_0}{A}|B(\beta(u),2R_0)|\,\|\psi'\|_\infty \max_{q\in\overline{B}(\beta(u),2R_0)}|\partial_u\Phi(q,u)h|.\end{aligned}$$

Noting $|\partial_u\Phi(q,u)h| = 2|\int_{\mathbb{R}^N}\zeta_{R_*}(x-q)(\nabla u\nabla h + uh)| \le 2\|u\|_{H^1}\|h\|_{H^1}$, we have (iv). □

In the following sections, we develop a deformation argument for $I_\varepsilon(u)$ in $\widetilde{D}_{\rho_2}$ for a suitable choice of $\widehat{D}$.

4 A Neighborhood of Expected Solutions

In this section we set up a neighborhood of expected solutions, in which we will develop a deformation argument in Sect. 6.

4.1 A Neighborhood Ω of Concentrating Points

In this section, we show that we may assume the following (V4) in addition to (V1)–(V3) and (LM) (or (MP)).

(V4) For any $p \in \Omega$, $2E_{V(p)} > E_{V_0}$.

In fact, since E_a is a continuous function of $a \in (0,\infty)$, there exists $\alpha > 0$ such that

$$2E_{V_0-\alpha} > E_{V_0}.$$

On the other hand, since $V(x)$ is of class C^N, the set of critical values of $V(x)$ is of measure 0 in $\mathbb{R}$ by Sard Theorem. Therefore we may assume $V_0-\alpha$ is a regular value of $V(x)$. We set

$$\Omega_\alpha = \{x \in \Omega :\ V(x) > V_0 - \alpha\}.$$

Then, $V(x)$ satisfies (V1)–(V4).

We observe that if $V(x)$ satisfies (LM) ((MP) respectively) in Ω, then $V(x)$ satisfies (LM) ((MP) respectively) in Ω_α. We show just for (MP).

We may assume $V(e_0)$, $V(e_1) < V_0 - \alpha$. We set

$$\begin{aligned} M_i &= \{x \in \overline{\Omega} : V(x) = V_0 - \alpha,\ x \text{ and } e_i \text{ are path connected in} \\ &\qquad \{x \in \Omega : V(x) \le V_0 - \alpha\}\} \quad \text{for } i = 0, 1, \\ \widetilde{\Lambda} &= \{c(\xi) \in ([0,1], \overline{\Omega_\alpha}) : c(0) \in M_0,\ c(1) \in M_1,\ V(c(\xi)) > V_0 - \alpha \text{ for } \xi \in (0,1)\}. \end{aligned}$$

Then we can easily see that $V_0 = \inf_{c \in \widetilde{\Lambda}} \max_{\xi \in [0,1]} V(c(\xi))$. Clearly there exists paths $(c_k)_{k=1}^\infty \subset \widetilde{\Lambda}$ with

$$c_k(0) \in M_0, \quad c_k(1) \in M_1, \quad \max_{\xi \in [0,1]} V(c_k(\xi)) \to V_0 \quad \text{as } k \to \infty.$$

Since M_0, M_1 are compact, we may assume after extracting a subsequence

$$c_k(0) \to \widetilde{e}_0 \in M_0, \quad c_k(1) \to \widetilde{e}_1 \in M_1 \quad \text{as } k \to \infty.$$

Choose $\widetilde{\widetilde{e}}_0, \widetilde{\widetilde{e}}_1 \in \Omega_\alpha$ so that $\widetilde{\widetilde{e}}_0$ is close to $\widetilde{e}_0$ and $\widetilde{\widetilde{e}}_1$ is close to $\widetilde{e}_1$. Replacing Ω, e_0, e_1, Λ with Ω_α, $\widetilde{\widetilde{e}}_0$, $\widetilde{\widetilde{e}}_1$ and $\widetilde{\widetilde{\Lambda}} = \{c(\xi) \in C([0,1], \Omega_\alpha) : c(0) = \widetilde{\widetilde{e}}_0,\ c(1) = \widetilde{\widetilde{e}}_1\}$. we can see that (MP) holds.

4.2 A Neighborhood of Expected Solutions

In what follows, we assume (V1)–(V4) hold for Ω and V_0 is a critical value of $V(x)$ in Ω. We write

$$b = E_{V_0}$$

and set

$$\mathcal{K}_b = \{(\xi, \omega) \in \Omega \times H^1(\mathbb{R}^N) : \nabla V(\xi) = 0,\ L'_{V(\xi)}(\omega) = 0,\ L_{V(\xi)}(\omega) = b\}.$$

We note that

$$\mathcal{K}_b = \{(\xi, \omega) \in \Omega \times H^1(\mathbb{R}^N) : DL(\xi, \omega) = 0,\ L(\xi, \omega) = b\},$$

where $D = (\partial_z, \partial_u)$ and

$$L(z, u) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}V(z)\|u\|_2^2 - \frac{1}{2}\mathcal{D}(u) : \mathbb{R}^N \times H^1(\mathbb{R}^N) \to \mathbb{R}.$$

We remark that $L(z, u)$ appears as a limit functional for $I_\varepsilon(u)$. In fact, for $z \in \mathbb{R}^N$ and $u(x) \in H^1(\mathbb{R}^N)$, we have

$$I_\varepsilon\left(u\left(x - \frac{z}{\varepsilon}\right)\right) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x + z)u(x)^2 - \frac{1}{2}\mathcal{D}(u) \to L(z, u) \quad \text{as } \varepsilon \to 0.$$

In what follows, we denote the projections to the first and second components by

$$P_1(z,u)=z, \quad P_2(z,u)=u.$$

Remark 4.1 (i) We have

$$\{(\xi,\omega)\in\Omega\times H^1(\mathbb{R}^N):\ V(\xi)=V_0,\ \nabla V(\xi)=0,$$
$$\omega \text{ is a least energy solution of } L'_{V_0}(\omega)=0\}\subset\mathcal{K}_b.$$

(ii) Since $E_a>b=E_{V_0}$ for $a>V_0$, $(\xi,\omega)\in\mathcal{K}_b$ implies $V(\xi)\le V_0$. Thus we have

$$P_1\mathcal{K}_b\cap\partial\Omega=\emptyset, \qquad P_1\mathcal{K}_b\subset\{\xi\in\Omega:\ V(\xi)\le V_0\}$$

and $P_1\mathcal{K}_b$ is compact in Ω by the assumption (V3).

(iii) If $(\xi,\omega)\in\mathcal{K}_b$ satisfies $V(\xi)=V_0$, we have $L_{V(\xi)}(\omega)=b$, that is, ω is a least energy solution of $L_{V_0}(\cdot)$. On contrary, if $V(\xi)<V_0$, we have $L_{V(\xi)}(\omega)=b>E_{V(\xi)}$ and ω is not a least energy solution of $L_{V(\xi)}(\cdot)$.

We set $Q=[0,1]^N$ and

$$\widehat{\mathcal{K}}_b=\{(\xi,\omega)\in\mathcal{K}_b:\ \|\omega\|_{L^2(Q)}=\max_{n\in\mathbb{Z}^N}\|\omega\|_{L^2(n+Q)}\}.$$

For $\varepsilon>0$ we set

$$\widehat{K}_b^{(\varepsilon)}=\left\{\omega\left(x-\frac{\xi}{\varepsilon}\right):\ (\xi,\omega)\in\widehat{\mathcal{K}}_b\right\}.$$

and we try to find a critical point of $I_\varepsilon(u)$ in a neighborhood of $\widehat{K}_b^{(\varepsilon)}$. We introduce $\mathcal{K}_b$ and $\widehat{\mathcal{K}}_b$ to obtain necessary compactness properties, in particular, to show Proposition 4.5 below.

For our minimax argument, we also introduce

$$\widehat{S}_b=\left\{\omega\left(\frac{x}{s}\right):\ \omega\in P_2\widehat{\mathcal{K}}_b,\ s\in\left[\frac{1}{2},\frac{3}{2}\right]\right\},$$
$$\widehat{\mathcal{Z}}_b=\{(\xi,w):\ \xi\in\overline{\Omega},\ w\in\widehat{S}_b\},$$
$$\widehat{Z}_b^{(\varepsilon)}=\left\{\omega\left(x-\frac{\xi}{\varepsilon}\right):\ (\xi,\omega)\in\widehat{\mathcal{Z}}_b\right\}. \tag{4.1}$$

It holds

$$\widehat{\mathcal{K}}_b\subset\widehat{\mathcal{Z}}_b, \quad \widehat{K}_b^{(\varepsilon)}\subset\widehat{Z}_b^{(\varepsilon)}.$$

By (V1)–(V4) and Lemma 3.1, we see

Lemma 4.2 *$\widehat{\mathcal{K}}_b$ and $\widehat{\mathcal{Z}}_b$ are compact in $\mathbb{R}^N \times H^1(\mathbb{R}^N)$. $\widehat{K}_b^{(\varepsilon)}$ and $\widehat{Z}_b^{(\varepsilon)}$ are also compact in $H^1(\mathbb{R}^N)$.*

Here and in what follows we indicate compact sets by $\widehat{\cdot}$.

To describe neighborhoods, we introduce an ε-dependent distance $\mathrm{dist}_\varepsilon(\cdot,\cdot)$ on $H^1(\mathbb{R}^N)$ by

$$\mathrm{dist}_\varepsilon(u(x), v(x)) = \inf_{h\in\mathbb{R}^N} \left(|h|^2 + \left\| u(x) - v\left(x - \frac{h}{\varepsilon}\right)\right\|_{H^1}^2 \right)^{1/2}.$$

The ε-dependent distance $\mathrm{dist}_\varepsilon(\cdot,\cdot)$ is a natural distance to consider concentration of a sequence $(u_{\varepsilon_j})_{j=1}^\infty \subset H^1(\mathbb{R}^N)$, $\varepsilon_j \to 0$ to a limit profile $(\xi,\omega) \in \widehat{\mathcal{K}}_b$ as $u_{\varepsilon_j}(x) \sim \omega(x - \frac{\xi}{\varepsilon_j})$. In fact, introducing $H_\varepsilon : H^1(\mathbb{R}^N) \to \mathbb{R}^N$ by

$$H_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} \nabla V(\varepsilon x) u(x)^2,$$

we have

Lemma 4.3 (i) *For $(\xi,\omega) \in \mathbb{R}^N \times H^1(\mathbb{R}^N)$, if $(u_j)_{j=1}^\infty \subset H^1(\mathbb{R}^N)$, $\varepsilon_j \to 0$ satisfies*

$$\mathrm{dist}_{\varepsilon_j}\left(u_j, \omega\left(x - \frac{\xi}{\varepsilon_j}\right)\right) \to 0, \tag{4.2}$$

then for $\varphi \in H^1(\mathbb{R}^N)$

$$I_{\varepsilon_j}(u_j) \to L(\xi,\omega),$$
$$I'_{\varepsilon_j}(u_j)\varphi\left(x - \frac{\xi}{\varepsilon_j}\right) \to \partial_u L(\xi,\omega)\varphi, \tag{4.3}$$
$$H_{\varepsilon_j}(u_j) \to \frac{1}{2}\nabla V(\xi)\|\omega\|_2^2 = \partial_z L(\xi,\omega). \tag{4.4}$$

(ii) *For (ξ,ω), $(\xi',\omega') \in \mathbb{R}^N \times H^1(\mathbb{R}^N)$ with $\omega, \omega' \neq 0$ and $\varepsilon_j \to 0$,*

$$\mathrm{dist}_{\varepsilon_j}\left(\omega\left(x - \frac{\xi}{\varepsilon_j}\right), \omega'\left(x - \frac{\xi'}{\varepsilon_j}\right)\right) \to 0 \tag{4.5}$$

holds if and only if

$$\xi' = \xi \quad \text{and} \quad \omega'(x) = \omega(x - h_0) \text{ for some } h_0 \in \mathbb{R}^N. \tag{4.6}$$

Proof (i) $\mathrm{dist}_{\varepsilon_j}(u_j(x), \omega(x - \frac{\xi}{\varepsilon_j})) \to 0$ holds if and only if there exists $(h_j)_{j=1}^{\infty} \subset \mathbb{R}^N$ and $(\varphi_j)_{j=1}^{\infty} \subset H^1(\mathbb{R}^N)$ such that

$$h_j \to 0, \quad \|\varphi_j\|_{H^1} \to 0$$

and

$$u_j(x) = \omega\left(x - \frac{\xi + h_j}{\varepsilon_j}\right) + \varphi_j\left(x - \frac{\xi + h_j}{\varepsilon_j}\right).$$

Thus

$$\begin{aligned} I_{\varepsilon_j}(u_j) &= \frac{1}{2}\|\nabla\omega + \nabla\varphi_j\|_{H^1}^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon_j x + \xi + h_j)|\omega(x) + \varphi_j(x)|^2 \\ &\quad - \frac{1}{2}\mathcal{D}(\omega + \varphi_j) \\ &\to L(\xi, \omega). \end{aligned}$$

(4.3) and (4.4) hold in a similar way. (ii) can be shown easily. □

We also note that $\mathrm{dist}_\varepsilon(\cdot, \cdot)$ is weaker than H^1-distance, namely there exist sequences $(u_j)_{j=1}^{\infty}, (v_j)_{j=1}^{\infty} \subset H^1(\mathbb{R}^N)$ such that for $\varepsilon_j \to 0$

$$\mathrm{dist}_{\varepsilon_j}(u_j, v_j) \to 0, \qquad \liminf_{j\to\infty} \|u_j - v_j\|_{H^1} > 0. \tag{4.7}$$

In fact, for $\omega \neq 0$, setting $u_j(x) = \omega(x - \frac{p_1}{\sqrt{\varepsilon_j}})$, $v_j(x) = \omega(x)$, where $p_1 = (1, 0, \cdots, 0)$, we have (4.7).

Lemma 4.3 (i) shows that for $(\xi, \omega) \in \widehat{\mathcal{K}}_b$, $(u_j)_{j=1}^{\infty}$ satisfying (4.2) is an ε-dependent Palais-Smale type sequence with the limit profile (ξ, ω). Conversely, in Proposition 4.5 below, we study the convergence of ε-dependent Palais-Smale type sequences with respect to the distance $\mathrm{dist}_\varepsilon(\cdot, \cdot)$.

We set for $\rho > 0$

$$\begin{aligned} N_\rho^{(\varepsilon)} &= \{u \in H^1(\mathbb{R}^N) : \mathrm{dist}_\varepsilon(u, \widehat{K}_b^{(\varepsilon)}) < \rho\} \\ &= \left\{\omega\left(x - \frac{\xi + h}{\varepsilon}\right) + \varphi\left(x - \frac{\xi + h}{\varepsilon}\right) : (\xi, \omega) \in \widehat{\mathcal{K}}_b,\ |h|^2 + \|\varphi\|_{H^1}^2 < \rho^2\right\}, \\ A_\rho^{(\varepsilon)} &= \{u \in H^1(\mathbb{R}^N) : \mathrm{dist}_\varepsilon(u, \widehat{Z}_b^{(\varepsilon)}) < \rho\} \\ &= \left\{\omega\left(x - \frac{\xi + h}{\varepsilon}\right) + \varphi\left(x - \frac{\xi + h}{\varepsilon}\right) : \omega \in \widehat{S}_b,\ \xi \in \overline{\Omega},\ |h|^2 + \|\varphi\|_{H^1}^2 < \rho^2\right\}. \end{aligned}$$

These sets are uniformly bounded with respect to $\varepsilon \in (0, 1]$ and we have

$$N_\rho^{(\varepsilon)} \subset A_\rho^{(\varepsilon)}.$$

In what follows, for suitable $0 < \rho < \rho'$ we develop a deformation argument in $A_{\rho'}^{(\varepsilon)}$ to find a critical point in $N_\rho^{(\varepsilon)}$.

Remark 4.4 The reason we introduce $A_\rho^{(\varepsilon)}$ is to construct neighborhoods which are suitable for our deformation arguments. Our neighborhood $A_\rho^{(\varepsilon)}$ includes a suitable initial path in $H^1(\mathbb{R}^N)$ which is related to a minimax argument in $\Omega \subset \mathbb{R}^N$. See Sect. 7.1 below. Our another neighborhood $N_\rho^{(\varepsilon)}$ is precisely an ε-neighborhood of expected solutions with the profile in $\widehat{\mathcal{K}}_b$.

4.3 Concentration-Compactness Type Results

In this section we give an ε-dependent concentration-compactness type results, which will be useful to develop deformation theory in Sect. 6.

Proposition 4.5 *There exists $\rho_3 > 0$ such that if $(\varepsilon_j)_{j=1}^\infty \subset (0,1]$ and $(u_j)_{j=1}^\infty \subset H^1(\mathbb{R}^N)$ satisfy $\varepsilon_j \to 0$, $u_j \in A_{\rho_3}^{(\varepsilon_j)}$ and*

$$I_{\varepsilon_j}(u_j) \to b, \tag{4.8}$$

$$I'_{\varepsilon_j}(u_j) \to 0 \quad \text{strongly in } (H^1(\mathbb{R}^N))^*, \tag{4.9}$$

$$H_{\varepsilon_j}(u_j) \to 0 \quad \text{in } \mathbb{R}^N \tag{4.10}$$

as $j \to \infty$, then

$$\mathrm{dist}_{\varepsilon_j}(u_j, \widehat{K}_b^{(\varepsilon_j)}) \to 0 \quad \text{as } j \to \infty.$$

In particular, for any $\rho > 0$ there exists $j_\rho \in \mathbb{N}$ such that

$$u_j \in \overline{N_\rho^{(\varepsilon_j)}} \quad \text{for } j \ge j_\rho.$$

Remark 4.6 To show the existence of a family concentrating at a local minimum of $V(x)$, in [23] we obtained a similar result for $(u_j)_{j=1}^\infty \subset N_{\rho_3}^{(\varepsilon_j)}$ but without the assumption (4.10). To study concentration at local maxima and saddle points, we need (4.10). In fact, if $(\xi, \omega) \in \Omega \times H^1(\mathbb{R}^N)$ satisfies

$$L(\xi,\omega) = b, \quad \partial_u L(\xi,\omega) = 0, \quad \|\omega\|_{L^2(Q)} = \max_{n\in\mathbb{Z}^N} \|\omega\|_{L^2(n+Q)},$$

then $u_j(x) = \omega(x - \frac{\xi}{\varepsilon_j})$ with $\varepsilon_j \to 0$ satisfies (4.8) and (4.9). However we don't have $\nabla V(\xi) = 0$ and the limit set

$$\{(\xi,\omega) : L(\xi,\omega) = b,\ \partial_u L(\xi,\omega) = 0,\ \|\omega\|_{L^2(Q)} = \max_{n\in\mathbb{Z}^N} \|\omega\|_{L^2(n+Q)}\} \tag{4.11}$$

is not compact in $\Omega \times H^1(\mathbb{R}^N)$ in general.

We note that if b is corresponding to local minimum, $L(\xi,\omega)=b$, $\partial_u L(\xi,\omega)=0$ imply $V(\xi)=V_0\,(=\inf_{x\in\Omega}V(x))$, $L(\xi,\omega)=E_{V_0}$ and the set defined in (4.11) is compact.

Proof of Proposition 4.5 For $\rho'>0$ suppose that $(\varepsilon_j)_{j=1}^\infty$, $(u_j)_{j=1}^\infty$ satisfy $\varepsilon_j\to 0$, $u_j\in A_{\rho'}^{(\varepsilon_j)}$ and (4.8)–(4.10). Since $u_j\in A_{\rho'}^{(\varepsilon_j)}$, there exist $(\xi_j,\omega_j)\in\widehat{\mathcal{Z}}_b$, $\varphi_j\in H^1(\mathbb{R}^N)$ and $h_j\in\mathbb{R}^N$ such that

$$u_j(x)=\omega_j\left(x-\frac{\xi_j+h_j}{\varepsilon_j}\right)+\varphi_j\left(x-\frac{\xi_j+h_j}{\varepsilon_j}\right), \tag{4.12}$$

$$\|\varphi_j\|_{H^1}<\rho',\qquad |h_j|<\rho'. \tag{4.13}$$

Extracting a subsequence if necessary, we may assume for some $(\xi_0,\omega_0)\in\widehat{\mathcal{Z}}_b$, $\varphi_0\in H^1(\mathbb{R}^N)$ and $h_0\in\mathbb{R}^N$ such that

$$\begin{aligned}&\xi_j\to\xi_0,\qquad h_j\to h_0,\\&\omega_j\to\omega_0\quad\text{strongly in } H^1(\mathbb{R}^N),\\&\varphi_j\rightharpoonup\varphi_0\quad\text{weakly in } H^1(\mathbb{R}^N).\end{aligned}$$

We set

$$\begin{aligned}&\widetilde{\xi}_j\equiv\xi_j+h_j\to\widetilde{\xi}_0\equiv\xi_0+h_0,\\&\widetilde{\omega}_j(x)\equiv\omega_j(x)+\varphi_j(x)\rightharpoonup\widetilde{\omega}_0(x)=\omega_0+\varphi_0\quad\text{weakly in } H^1(\mathbb{R}^N).\end{aligned}\tag{4.14}$$

Suppose $\rho'\in(0,\rho_1)$, where $\rho_1>0$ is given by Lemma 2.4. Then we have
Step 1: $\widetilde{\omega}_j(x)\to\widetilde{\omega}_0(x)$ *strongly in* $H^1(\mathbb{R}^N)$.
It suffices to show that

$$\sup_{n\in\mathbb{Z}^N}\|\widetilde{\omega}_j-\widetilde{\omega}_0\|_{L^2(n+Q)}\to 0\quad\text{as } j\to\infty. \tag{4.15}$$

Since $(\widetilde{\omega}_j)_{j=1}^\infty$ is bounded in $H^1(\mathbb{R}^N)$, (4.15) implies for $p\in(2,\frac{2N}{N-2})$

$$\begin{aligned}&\widetilde{\omega}_j\to\widetilde{\omega}_0\quad\text{strongly in } L^p(\mathbb{R}^N),\\&\mathcal{D}'(\widetilde{\omega}_j)\widetilde{\omega}_j\to\mathcal{D}'(\widetilde{\omega}_0)\widetilde{\omega}_0\quad\text{as } j\to\infty.\end{aligned}$$

It follows from

$$I'_{\varepsilon_j}\left(\widetilde{\omega}_j\left(x-\frac{\widetilde{\xi}_j}{\varepsilon_j}\right)\right)\widetilde{\omega}_j\left(x-\frac{\widetilde{\xi}_j}{\varepsilon_j}\right)\to 0,\qquad I'_{\varepsilon_j}\left(\widetilde{\omega}_j\left(x-\frac{\widetilde{\xi}_j}{\varepsilon_j}\right)\right)\widetilde{\omega}_0\left(x-\frac{\widetilde{\xi}_j}{\varepsilon_j}\right)\to 0$$

that

$$\|\nabla\widetilde{\omega}_j\|_2^2+\int_{\mathbb{R}^N}V(\varepsilon_j x+\widetilde{\xi}_j)\widetilde{\omega}_j^2=\mathcal{D}'(\widetilde{\omega}_j)\widetilde{\omega}_j+o(1)=\mathcal{D}'(\widetilde{\omega}_0)\widetilde{\omega}_0+o(1),$$
$$\|\nabla\widetilde{\omega}_0\|_2^2+\int_{\mathbb{R}^N}V(\varepsilon_j x+\widetilde{\xi}_j)\widetilde{\omega}_0^2=\mathcal{D}'(\widetilde{\omega}_0)\widetilde{\omega}_0+o(1).$$

And thus $\widetilde{\omega}_j\to\widetilde{\omega}_0$ strongly in $H^1(\mathbb{R}^N)$.

If (4.15) does not hold, there exists $(n_j)_{j=1}^\infty\subset\mathbb{Z}^N$ such that

$$\|\widetilde{\omega}_j-\widetilde{\omega}_0\|_{L^2(n_j+Q)}\not\to 0. \tag{4.16}$$

By (4.14), we have $|n_j|\to\infty$. Thus letting $\widetilde{\omega}_j(x+n_j)\rightharpoonup\widetilde{\widetilde{\omega}}_0(x)$ weakly in $H^1(\mathbb{R}^N)$, we have from (4.13), (4.16) that $\widetilde{\widetilde{\omega}}_0\neq 0$ and

$$\|\widetilde{\widetilde{\omega}}_0\|_{H^1}\le\rho'. \tag{4.17}$$

On the other hand, since $\widetilde{\omega}_j(x+n_j)=u_j(x+\frac{\widetilde{\xi}_j}{\varepsilon_j}+n_j)$ and $I'_{\varepsilon_j}(u_j)\to 0$ strongly in $(H^1(\mathbb{R}^N))^*$, Lemma 2.4 and (4.17) imply $\widetilde{\widetilde{\omega}}_0=0$, which is in contradiction.

Step 2: $\nabla V(\widetilde{\xi}_0)=0$.

We have

$$H_{\varepsilon_j}(u_j)=\frac{1}{2}\int_{\mathbb{R}^N}\nabla V(\varepsilon_j x)u_j(x)^2=\frac{1}{2}\int_{\mathbb{R}^N}\nabla V(\varepsilon_j x)\widetilde{\omega}_j\left(x-\frac{\widetilde{\xi}_j}{\varepsilon_j}\right)^2$$
$$=\frac{1}{2}\int_{\mathbb{R}^N}\nabla V(\varepsilon_j x+\widetilde{\xi}_j)\widetilde{\omega}_j(x)^2\to\frac{1}{2}\nabla V(\widetilde{\xi}_0)\|\widetilde{\omega}_0\|_2^2\quad\text{as } j\to\infty$$

and thus (4.10) implies $\nabla V(\widetilde{\xi}_0)=0$.

Step 3: $DL(\widetilde{\xi}_0,\widetilde{\omega}_0)=0$ and $L(\widetilde{\xi}_0,\widetilde{\omega}_0)=b$.

For any $\varphi\in C_0^\infty(\mathbb{R}^N)$, we have

$$I'_{\varepsilon_j}(u_j)\varphi\left(x-\frac{\widetilde{\xi}_j}{\varepsilon_j}\right)=\int_{\mathbb{R}^N}\nabla\widetilde{\omega}_j\nabla\varphi+V(\varepsilon_j x+\widetilde{\xi}_j)\widetilde{\omega}_j\varphi-\mathcal{D}'(\widetilde{\omega}_j)\varphi$$
$$\to\int_{\mathbb{R}^N}\nabla\widetilde{\omega}_0\nabla\varphi+V(\widetilde{\xi}_0)\widetilde{\omega}_0\varphi-\mathcal{D}'(\widetilde{\omega}_0)\varphi.$$

Thus (4.9) implies $\partial_u L(\widetilde{\xi}_0,\widetilde{\omega}_0)=0$. It is easily seen that (4.8) implies $L(\widetilde{\xi}_0,\widetilde{\omega}_0)=b$.

Step 4: For $\rho'>0$ small, $\mathrm{dist}_{\varepsilon_j}(u_j,\widehat{K}_b^{(\varepsilon_j)})\to 0$

It is clear that $\widetilde{\xi}_j=\xi_j+h_j$ is in a ρ'-neighborhood of Ω and thus so is $\widetilde{\xi}_0$. Since $\nabla V(x)\neq 0$ on $\partial\Omega$, we have $(\widetilde{\xi}_0,\widetilde{\omega}_0)\in\mathcal{K}_b$ if $\rho'>0$ is sufficiently small. Thus there exists $h_0 \in \mathbb{R}^N$ such that $\widehat{\omega}_0(x) = \widetilde{\omega}_0(x - h_0)$ satisfies $(\widetilde{\xi}_0, \widehat{\omega}_0) \in \widehat{\mathcal{K}}_b$. We have

$$\begin{aligned}
\mathrm{dist}_{\varepsilon_j}(u_j, \widehat{K}_b^{(\varepsilon_j)}) &\le \mathrm{dist}_{\varepsilon_j}\left(u_j(x), \widehat{\omega}_0\left(x - \frac{\widetilde{\xi}_0}{\varepsilon_j}\right)\right) \\
&= \mathrm{dist}_{\varepsilon_j}\left(\widetilde{\omega}_j\left(x - \frac{\widetilde{\xi}_j}{\varepsilon_j}\right), \widehat{\omega}_0\left(x - \frac{\widetilde{\xi}_0}{\varepsilon_j}\right)\right) \\
&\le \mathrm{dist}_{\varepsilon_j}\left(\widetilde{\omega}_j\left(x - \frac{\widetilde{\xi}_j}{\varepsilon_j}\right), \widetilde{\omega}_0\left(x - \frac{\widetilde{\xi}_j}{\varepsilon_j}\right)\right) \\
&\quad + \mathrm{dist}_{\varepsilon_j}\left(\widetilde{\omega}_0\left(x - \frac{\widetilde{\xi}_j}{\varepsilon_j}\right), \widehat{\omega}_0\left(x - \frac{\widetilde{\xi}_0}{\varepsilon_j}\right)\right) \\
&\le \|\widetilde{\omega}_j - \widetilde{\omega}_0\|_{H^1} + \mathrm{dist}_{\varepsilon_j}\left(\widehat{\omega}_0\left(x - \frac{\widetilde{\xi}_j - \varepsilon_j h_0}{\varepsilon_j}\right), \widehat{\omega}_0\left(x - \frac{\widetilde{\xi}_0}{\varepsilon_j}\right)\right) \\
&\le \|\widetilde{\omega}_j - \widetilde{\omega}_0\|_{H^1} + |\widetilde{\xi}_j - \varepsilon_j h_0 - \widetilde{\xi}_0| \to 0 \quad \text{as } j \to \infty.
\end{aligned}$$

Thus choosing $\rho_3 > 0$ small, the proof is completed. □

Next we show that $I_\varepsilon(u)$ satisfies the Palais-Smale type condition in $A_{\rho_1}^{(\varepsilon)}$ for $\varepsilon \in (0, 1]$ fixed.

Proposition 4.7 *Let $\rho_1 > 0$ be the number given in Lemma 2.4. For $\varepsilon \in (0, 1]$ fixed, $I_\varepsilon(u)$ satisfies the Palais-Smale type condition in $A_{\rho_1}^{(\varepsilon)}$. That is, if $(u_j)_{j=1}^\infty \subset A_{\rho_1}^{(\varepsilon)}$ satisfies*

$$(H_\varepsilon(u_j), I_\varepsilon'(u_j)) \to 0 \quad \textit{strongly in } (\mathbb{R}^N \times H^1(\mathbb{R}^N))^*, \tag{4.18}$$

then $(u_j)_{j=1}^\infty$ has a strongly convergent subsequence in $H^1(\mathbb{R}^N)$. Moreover, after extracting a subsequence if necessary, assume $u_j \to u_0$ strongly as $j \to \infty$. Then u_0 satisfies $I_\varepsilon'(u_0) = 0$ and

$$H_\varepsilon(u_0) = 0. \tag{4.19}$$

Proof Since $(u_j)_{j=1}^\infty \subset A_{\rho_1}^{(\varepsilon)}$, there exist $(\xi_j, \omega_j) \in \widehat{\mathcal{Z}}_b$, $h_j \in \mathbb{R}^N$ and $\varphi_j \in H^1(\mathbb{R}^N)$ such that

$$\begin{aligned}
&u_j(x) = \omega_j\left(x - \frac{\xi_j + h_j}{\varepsilon_j}\right) + \varphi_j\left(x - \frac{\xi_j + h_j}{\varepsilon_j}\right), \\
&|h_j| \le \rho_1, \quad \|\varphi_j\|_{H^1} \le \rho_1.
\end{aligned}$$

Extracting a subsequence if necessary, we may assume for some $(\xi_0, \omega_0) \in \widehat{\mathcal{K}}_b$, $\varphi_0 \in H^1(\mathbb{R}^N)$ and $h_0 \in \mathbb{R}^N$

$$\begin{aligned}
&\xi_j \to \xi_0, \qquad h_j \to h_0, \\
&\omega_j \to \omega_0 \quad \text{strongly in } H^1(\mathbb{R}^N), \\
&\varphi_j \rightharpoonup \varphi_0 \quad \text{weakly in } H^1(\mathbb{R}^N).
\end{aligned}$$

Using Lemma 2.4 and arguing as in Step 1 of the proof of Proposition 4.5, we have the strong convergence of (u_j). (4.19) follows from $H_\varepsilon(u_j) \to 0$. □

Remark 4.8 In Proposition 4.7, the condition (4.18) can be relaxed to

$$I'_\varepsilon(u_j) \to 0$$

To see this fact, first we remark that $I'_\varepsilon(u_0) = 0$ implies $H_\varepsilon(u_0) = 0$. Indeed, from the regularity argument (c.f. [47, 50]), it follows from $I'_\varepsilon(u_0) = 0$ that $u_0 \in H^2(\mathbb{R}^N)$. On the other hand, we have for $j \in \{1, 2, \cdots, N\}$

$$\begin{aligned}
\int_{\mathbb{R}^N} \nabla u_0 \nabla u_{0x_j} &= \frac{1}{2}\int_{\mathbb{R}^N} \partial_{x_j}(|\nabla u_0|^2) = 0,\\
\int_{\mathbb{R}^N} V(\varepsilon x) u_0 u_{0x_j} &= \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)\partial_{x_j}(u_0^2) = -\frac{\varepsilon}{2}\int_{\mathbb{R}^N} \frac{\partial V}{\partial x_j}(\varepsilon x) u_0^2,\\
\mathcal{D}'(u)(u_{0x_j}) = \int_{\mathbb{R}^N} (I_\alpha * F(u_0))F'(u_0)u_{0x_j} &= \int_{\mathbb{R}^N} (I_\alpha * F(u_0))(F(u_0))_{x_j} = 0.
\end{aligned}$$

Thus $I'_\varepsilon(u_0) = 0$ implies $\int_{\mathbb{R}^N} \frac{\partial V}{\partial x_j}(\varepsilon x)u_0^2 = 0$ for $j = 1, 2, \cdots, N$. That is,

$$H_\varepsilon(u_0) = \frac{1}{2}\int_{\mathbb{R}^N} \nabla V(\varepsilon x) u_0^2 = 0.$$

If $I'_\varepsilon(u_j) \to 0$ strongly in $(H^1(\mathbb{R}^N))^*$, from the proof of Proposition 4.7 there exists a strongly convergent subsequence $(u_{j_k})_{k=1}^\infty$. Let $u_{j_k} \to u_0$ in $H^1(\mathbb{R}^N)$. Then we have $I'_\varepsilon(u_0) = 0$, $H_\varepsilon(u_{j_k}) \to H_\varepsilon(u_0)$. Since $I'_\varepsilon(u_0) = 0$ implies $H_\varepsilon(u_0) = 0$, we have $H_\varepsilon(u_{j_k}) \to 0$. Thus we have (4.18).

4.4 A Choice of Neighborhoods and Gradient Estimates

We choose $\rho_{**} > 0$ small so that in a neighborhood $A^{(\varepsilon)}_{\rho_{**}}$ of $\widehat{K}^{(\varepsilon)}_b$, we can develop a deformation argument for a proof of our main result.

We set

$$\begin{aligned}
\widetilde{S}_b &= \{w(x-p) : w \in \widehat{S}_b,\ p \in \mathbb{R}^N\},\\
\widetilde{S}_{b,\rho} &= \{u \in H^1(\mathbb{R}^N) : \operatorname{dist}_{H^1}(u, \widetilde{S}_b) < \rho\} \quad \text{for } \rho > 0.
\end{aligned}$$

Here $\widehat{S}_b$ is defined in (4.1). Applying the argument in Sect. 3.3 with $\widehat{D} = \widehat{S}_b$, $\widetilde{D} = \widetilde{S}_b$ and $\widetilde{D}_\rho = \widetilde{S}_{b,\rho}$, we can define the center of mass:

$$\beta : \widetilde{S}_{b,\rho_2} \to \mathbb{R}^N \quad \text{for small } \rho_2 > 0.$$

We choose and fix $\rho_*, \rho_{**} > 0$ such that

$$0 < 16\rho_* < \rho_{**} < \min\left\{\frac{1}{6}\rho_0, \rho_1, \rho_2, \rho_3\right\}, \tag{4.20}$$

where ρ_2 is given above and ρ_0 (ρ_1, ρ_3 respectively) is given in Corollary 2.2 (Lemma 2.4, Proposition 4.5 respectively). We will use relation $16\rho_* < \rho_{**}$ later in the proof of Lemma 6.9. We note that the center of mass $\beta(u)$ is defined on $A^{(\varepsilon)}_{\rho_{**}}$ and

$$\mathrm{dist}_{\mathbb{R}^N}(\varepsilon\beta(u), \Omega) < \varepsilon R_0 + \rho_{**} \quad \text{for } u \in A^{(\varepsilon)}_{\rho_{**}}. \tag{4.21}$$

In fact, by the definition of $A^{(\varepsilon)}_{\rho_{**}}$, we have for some $\xi \in \overline{\Omega}$, $\omega \in \widehat{S}_b$, $h \in \mathbb{R}^N$

$$|h|^2 + \left\| u(x) - \omega\left(x - \frac{\xi + h}{\varepsilon}\right)\right\|^2_{H^1} < \rho_{**}{}^2.$$

Thus by Proposition 3.3 (i) we have (4.21).

By Propositions 4.5 and 4.7, we have the following estimates.

Proposition 4.9 *For* $0 < \rho_* < \rho_{**}$ *with* (4.20). *Then we have*

(i) *There exist* $\varepsilon_0 > 0$, $\nu_0 > 0$ *and* $\delta_0 > 0$ *with the following properties: For* $\varepsilon \in (0, \varepsilon_0]$

$$\|(H_\varepsilon(u), I'_\varepsilon(u))\|_{(\mathbb{R}^N\times H^1(\mathbb{R}^N))^*} \equiv \left(|H_\varepsilon(u)|^2 + \|I'_\varepsilon(u)\|^2_{(H^1(\mathbb{R}^N))^*}\right)^{1/2} \geq \nu_0$$

for all $u \in A^{(\varepsilon)}_{\rho_{**}} \setminus \overline{N^{(\varepsilon)}_{\rho_*}}$ *with* $I_\varepsilon(u) \in [b - \delta_0, b + \delta_0]$.

(ii) *Suppose that for some* $\varepsilon \in (0, \varepsilon_0]$

$$(H_\varepsilon(u), I'_\varepsilon(u)) \neq 0 \quad \text{for all } u \in \overline{N^{(\varepsilon)}_{\rho_*}} \text{ with } I_\varepsilon(u) \in [b - \delta_0, b + \delta_0]. \tag{4.22}$$

Then there exists $\nu_\varepsilon > 0$ *such that*

$$\|(H_\varepsilon(u), I'_\varepsilon(u))\|_{(\mathbb{R}^N\times H^1(\mathbb{R}^N))^*} \geq \nu_\varepsilon, \tag{4.23}$$

for $u \in A^{(\varepsilon)}_{\rho_{**}}$ *with* $I_\varepsilon(u) \in [b - \delta_0, b + \delta_0]$.

In what follows we assume without loss of generality $\nu_\varepsilon \leq \nu_0$.

Proof (i), (ii) follow from Propositions 4.5 and 4.7 easily. □

We fix ε_0, $\nu_0 > 0$ and $\delta_0 > 0$ given in Proposition 4.9.

Remark 4.10 (4.22) can be replaced with $I'_\varepsilon(u) \neq 0$. We note that $I'_\varepsilon(u) = 0$ implies $H_\varepsilon(u) = 0$ (see Remark 4.8). (4.23) can be replaced by

$$\|I'_\varepsilon(u)\|_{(H^1(\mathbb{R}^N))^*} \geq \nu_\varepsilon.$$

In the following Sect. 5, we develop a special deformation argument for $I_\varepsilon(u)$.

5 A Functional Corresponding to the Tail of a Function

5.1 Functional $T_\varepsilon(u)$

To find a critical point of $I_\varepsilon(u)$ in a neighborhood $N_\rho^{(\varepsilon)}$ of expected solutions, it is important to control the size of u outside of a ball $B(\beta(u), \frac{4}{\sqrt{\varepsilon}})$.

We set for $u \in \widetilde{S}_{b,\rho_2}$ and $\varepsilon > 0$

$$T_\varepsilon(u) = \int_{\mathbb{R}^N} \widetilde{\zeta}_{4/\sqrt{\varepsilon}}(x - \beta(u))(|\nabla u|^2 + |u|^2) \in C^1(\widetilde{S}_{b,\rho_2}, \mathbb{R}). \tag{5.1}$$

We note that $T_\varepsilon(u)$ is translation invariant, that is,

$$T_\varepsilon(u(\cdot - h)) = T_\varepsilon(u) \quad \text{for all } h \in \mathbb{R}^N$$

and

$$\|u\|^2_{H^1(|x-\beta(u)| \geq \frac{4}{\sqrt{\varepsilon}})} \leq T_\varepsilon(u).$$

We use $T_\varepsilon(u)$ to estimate the size of u outside of a ball $B(\beta(u), \frac{4}{\sqrt{\varepsilon}})$.

In this section, we extend our idea in [23] to generate a special deformation flow for $I_\varepsilon(u)$, which keeps $T_\varepsilon(u)$ small along the flow.

5.2 A Special Vector Field in $A_{\rho_{**}}^{(\varepsilon)}$

To construct a deformation flow which keeps the size of tail $T_\varepsilon(u)$ small, we find a special vector field in this section.

We note $A_{\rho_{**}}^{(\varepsilon)}$ is bounded and so there exists $C > 0$ such that

$$\|u\|^2_{H^1} \leq C \quad \text{for all } u \in A_{\rho_{**}}^{(\varepsilon)}.$$

First we decompose $u \in A_{\rho_{**}}^{(\varepsilon)}$ into a center part $u^{(1)}$ and a tail part $u^{(2)}$. We denote the integer part of $a > 0$ by $[a]$.

Since

$$\sum_{k=0}^{[\varepsilon^{-1/4}]-1} \|u\|^2_{H^1(|x-\beta(u)| \in [\frac{2}{\sqrt{\varepsilon}} + \frac{k}{\varepsilon^{1/4}}, \frac{2}{\sqrt{\varepsilon}} + \frac{k+1}{\varepsilon^{1/4}}])} \leq \|u\|^2_{H^1} \leq C,$$

there exists $k \in \{1, 2, \cdots, [\varepsilon^{-1/4}] - 1\}$ such that

$$\|u\|^2_{H^1(|x-\beta(u)| \in [\frac{2}{\sqrt{\varepsilon}} + \frac{k}{\varepsilon^{1/4}}, \frac{2}{\sqrt{\varepsilon}} + \frac{k+1}{\varepsilon^{1/4}}])} \leq \frac{C}{[\varepsilon^{-1/4}]} \to 0 \quad \text{as } \varepsilon \to 0. \tag{5.2}$$

In what follows we denote by c_ε various constants which do not depend on u and satisfy $c_\varepsilon \to 0$ as $\varepsilon \to 0$. We set

$$u^{(1)}(x) = \zeta_{\frac{2}{\sqrt{\varepsilon}}+\frac{k}{\varepsilon^{1/4}}}(x-\beta(u))u(x), \qquad u^{(2)}(x) = \widetilde{\zeta}_{\frac{2}{\sqrt{\varepsilon}}+\frac{k+1}{\varepsilon^{1/4}}}(x-\beta(u))u(x),$$

where $\zeta_R(x)$, $\widetilde{\zeta}_R(x)$ are defined in (2.1). We also set

$$M_1(u) = \zeta_{1/\sqrt{\varepsilon}}(x-\beta(u))u, \qquad M_2(u) = (1-\zeta_{1/\sqrt{\varepsilon}}(x-\beta(u)))u. \tag{5.3}$$

These function also give decomposition of u into a center part and a tail part. Clearly we have $u = M_1(u)+M_2(u)$. By (2.2), we also have $\|M_1(u)\|_{H^1}, \|M_2(u)\|_{H^1} \le 3\|u\|_{H^1}$.

We note that $u^{(1)}$, $u^{(2)}$, $M_1(u)$, $M_2(u)$ depend on ε. But for simplicity of notation, we omit ε from the notation.

We use $-u^{(2)}$ to construct a deformation flow and we use $M_1(u)$ and $M_2(u)$ to estimate effects of $-u^{(2)}$.

$u^{(2)}$ has the following properties.

Lemma 5.1 *There exists $c_\varepsilon > 0$ independent of $u \in A^{(\varepsilon)}_{\rho_{**}}$ such that*

$$c_\varepsilon \to 0 \quad as\ \varepsilon \to 0$$

and for $\varepsilon > 0$ small $u \in A^{(\varepsilon)}_{\rho_{**}}$ satisifes the following properties (i)–(v).

(i)

$$\|u^{(2)}\|_{H^1},\ \|M_2(u)\|_{H^1} < \rho_0, \tag{5.4}$$
$$\|u - u^{(1)} - u^{(2)}\|_{H^1} \le c_\varepsilon, \tag{5.5}$$
$$|(u-u^{(2)}, u^{(2)})_{H^1}| \le c_\varepsilon, \tag{5.6}$$
$$|(I'_\varepsilon(u) - I'_\varepsilon(u^{(2)}))u^{(2)}| \le c_\varepsilon. \tag{5.7}$$

(ii) *For the center of mass $\beta(u)$ defined in Sect. 3.3,*

$$\beta'(u)u^{(2)} = 0. \tag{5.8}$$

(iii) *For $M_1(u)$, $M_2(u)$ defined in (5.3),*

$$\partial_u M_1(u)u^{(2)} = 0, \tag{5.9}$$
$$\partial_u(\|M_2(u)\|^2_{H^1})u^{(2)} \ge -c_\varepsilon. \tag{5.10}$$

(iv) *For $T_\varepsilon(u)$ defined in (5.1),*

$$T_\varepsilon(u) \le \|u^{(2)}\|^2_{H^1}, \tag{5.11}$$
$$T'_\varepsilon(u)u^{(2)} = 2T_\varepsilon(u). \tag{5.12}$$

(v) *For* $c_0 > 0$ *given in Corollary 2.2, we have*

$$I'_\varepsilon(u)u^{(2)} \geq c_0 T_\varepsilon(u) - c_\varepsilon. \tag{5.13}$$

From Lemma 5.1, we can observe a vector field $u \mapsto -u^{(2)}$ has good properties for deformation. By (ii), (iii), $-u^{(2)}$ does not effect the center part $M_1(u)$ and the center $\beta(u)$ of mass of u. By (5.12) and (5.13), $-u^{(2)}$ gives a direction which decreases both of $I_\varepsilon(u)$ and $T_\varepsilon(u)$ provided $T_\varepsilon(u) \geq \frac{c_\varepsilon}{c_0}$. Thus it is convenient to construct a deformation flow for $I_\varepsilon(u)$ which keeps the size $T_\varepsilon(u)$ of tail small.

Proof (i) $u \in A^{(\varepsilon)}_{\rho_{**}} \subset \widetilde{S}_{b,\rho_{**}}$ can be written as

$$u(x) = \omega(x-p) + \varphi(x),$$

where $\omega \in \widehat{S}_b$ and $\|\varphi\|_{H^1} < \rho_{**}$. Since $|\beta(u) - p| \leq R_0$ and $\widehat{S}_b$ is compact in $H^1(\mathbb{R}^N)$, we have $\|u\|_{H^1(|x-\beta(u)|\geq 1/\sqrt{\varepsilon})} \leq 2\rho_{**}$ for ε small. Thus by (2.2)

$$\|u^{(2)}\|_{H^1},\ \|M_2(u)\|_{H^1} \leq 3\|u\|_{H^1(|x-\beta(u)|\geq 1/\sqrt{\varepsilon})} \leq 6\rho_{**} < \rho_0.$$

By (5.2), we have uniformly in $u \in A^{(\varepsilon)}_{\rho_{**}}$,

$$\begin{aligned}\|u - u^{(1)} - u^{(2)}\|_{H^1} &= \|(1 - \zeta_{\frac{2}{\sqrt{\varepsilon}}+\frac{k}{\varepsilon^{1/4}}}(x-\beta(u)) - \widetilde{\zeta}_{\frac{2}{\sqrt{\varepsilon}}+\frac{k+1}{\varepsilon^{1/4}}}(x-\beta(u)))u\|_{H^1} \\ &\leq 3\|u\|_{H^1(|x-\beta(u)|\in[\frac{2}{\sqrt{\varepsilon}}+\frac{k}{\varepsilon^{1/4}}, \frac{2}{\sqrt{\varepsilon}}+\frac{k+1}{\varepsilon^{1/4}}])} \to 0 \quad \text{as } \varepsilon \to 0.\end{aligned}$$

We also have

$$|(u - u^{(2)}, u^{(2)})_{H^1}| \leq C\|u\|^2_{H^1(|x-\beta(u)|\in[\frac{2}{\sqrt{\varepsilon}}+\frac{k}{\varepsilon^{1/4}}, \frac{2}{\sqrt{\varepsilon}}+\frac{k+1}{\varepsilon^{1/4}}])} \to 0 \quad \text{as } \varepsilon \to 0.$$

Thus we have (5.5) and (5.6). In a similar way, using Lemma 2.3 (ii) with $R = \frac{2}{\sqrt{\varepsilon}} + \frac{k}{\varepsilon^{1/4}}$ and $L = \frac{1}{\varepsilon^{1/4}}$, we have (5.7).

(ii) Since $\operatorname{supp} u^{(2)} \subset \mathbb{R}^N \setminus B(\beta(u), \frac{2}{\sqrt{\varepsilon}})$ does not intersect $B(\beta(u), 3R_0)$ for $\varepsilon \in (0, \frac{1}{9R_0^2})$, we have by (iii) of Proposition 3.3

$$\beta(u + t\ell u^{(2)}) = \beta(u) \quad \text{for small } t.$$

Thus we have (5.8).
By $\operatorname{supp} u^{(2)} \subset \mathbb{R}^N \setminus B(\beta(u), \frac{2}{\sqrt{\varepsilon}})$ we note that

$$\zeta_{1/\sqrt{\varepsilon}}(x-\beta(u))u^{(2)}(x) = 0. \tag{5.14}$$

(iii) We have from (5.8), (5.14)

$$\partial_u M_1(u)u^{(2)} = -\zeta'_{1/\sqrt{\varepsilon}}(x-\beta(u))(\beta'(u)u^{(2)})u + \zeta_{1/\sqrt{\varepsilon}}(x-\beta(u))u^{(2)} = 0.$$

Thus we have (5.9).

For $M_2(u)$, we compute by (5.6)

$$\begin{aligned}
&\frac{1}{2}\partial_u(\|M_2(u)\|_{H^1}^2)u^{(2)} = (M_2(u), \partial_u M_2(u)u^{(2)})_{H^1} \\
=&((1-\zeta_{1/\sqrt{\varepsilon}}(x-\beta(u))u(x), \zeta'_{1/\sqrt{\varepsilon}}(x-\beta(u))(\beta'(u)u^{(2)})u \\
&+(1-\zeta_{1/\sqrt{\varepsilon}})(x-\beta(u))u^{(2)})_{H^1} \\
=&(u, u^{(2)})_{H^1} = \|u^{(2)}\|_{H^1}^2 + (u-u^{(2)}, u^{(2)})_{H^1} \geq (u-u^{(2)}, u^{(2)})_{H^1} \\
\geq& -c_\varepsilon.
\end{aligned}$$

Thus we have (5.10).

(iv) Since $u(x) = u^{(2)}(x)$ in supp $\widetilde{\zeta}_{4/\sqrt{\varepsilon}}(x-\beta(u)) = \mathbb{R}^N \setminus B(\beta(u), \frac{4}{\sqrt{\varepsilon}}-1)$, we have (5.11) and

$$\begin{aligned}
T'_\varepsilon(u)u^{(2)} =& -\int_{\mathbb{R}^N} \widetilde{\zeta}'_{4/\sqrt{\varepsilon}}(x-\beta(u))(\beta'(u)u^{(2)})(|\nabla u|^2+u^2) \\
&+2\int_{\mathbb{R}^N} \widetilde{\zeta}_{4/\sqrt{\varepsilon}}(x-\beta(u))(\nabla u\nabla u^{(2)}+uu^{(2)}) \\
=&2T_\varepsilon(u).
\end{aligned}$$

Thus we have (5.12).

(v) By (5.4), (5.7), (5.11) and Corollary 2.2,

$$I'_\varepsilon(u)u^{(2)} \geq I'_\varepsilon(u^{(2)})u^{(2)} - c_\varepsilon \geq c_0\|u^{(2)}\|_{H^1}^2 - c_\varepsilon \geq c_0 T_\varepsilon(u) - c_\varepsilon.$$

Thus we get (v).

□

Choice of κ_ε. By the compactness of $\widehat{S}_b$, we have

$$\sup_{\omega\in\widehat{S}_b} T_\varepsilon(\omega) \to 0 \quad \text{as } \varepsilon \to 0.$$

For $c_\varepsilon > 0$ given in Lemma 5.1, we set

$$\kappa_\varepsilon \equiv \max\left\{2\sup_{\omega\in\widehat{S}_b} T_\varepsilon(\omega), \frac{2c_\varepsilon}{c_0}\right\} \to 0 \quad \text{as } \varepsilon \to 0. \tag{5.15}$$

With this choice of κ_ε, we have the following corollary. In what follows, we use the following notation for $c \in \mathbb{R}$

$$\begin{aligned}
[I_\varepsilon \leq c] &= \{u \in H^1(\mathbb{R}^N) : I_\varepsilon(u) \leq c\}, \\
[T_\varepsilon \geq c] &= \{u \in H^1(\mathbb{R}^N) : T_\varepsilon(u) \geq c\}.
\end{aligned}$$

Corollary 5.2 *For $u \in A^{(\varepsilon)}_{\rho_{**}} \cap [T_\varepsilon \geq \kappa_\varepsilon]$, we have*

$$I'_\varepsilon(u)u^{(2)} \geq c_\varepsilon,$$

in particular, $I'_\varepsilon(u) \neq 0$ in $A^{(\varepsilon)}_{\rho_{**}} \cap [T_\varepsilon \geq \kappa_\varepsilon]$.

Proof By (v) of Lemma 5.1, we have for $u \in A^{(\varepsilon)}_{\rho_{**}} \cap [T_\varepsilon \geq \kappa_\varepsilon]$.

$$I'_\varepsilon(u)u^{(2)} \geq c_0 T_\varepsilon(u) - c_\varepsilon \geq c_0\kappa_\varepsilon - c_\varepsilon \geq c_0 \cdot \frac{2c_\varepsilon}{c_0} - c_\varepsilon = c_\varepsilon.$$

□

As a corollary to Proposition 4.9 (ii) and Corollary 5.2, we have

Corollary 5.3 *Suppose that for $\varepsilon > 0$*

$$(H_\varepsilon(u), I'_\varepsilon(u)) \neq (0,0) \quad \text{for } u \in \overline{N^{(\varepsilon)}_{\rho_*}} \cap [T_\varepsilon \leq \kappa_\varepsilon] \text{ with } I_\varepsilon(u) \in [b-\delta_0, b+\delta_0]. \tag{5.16}$$

Then there exists $\nu_\varepsilon > 0$ such that

$$\|(H_\varepsilon(u), I'_\varepsilon(u))\|_{(\mathbb{R}^N \times H^1(\mathbb{R}^N))^*} \geq \nu_\varepsilon \tag{5.17}$$

for $u \in A^{(\varepsilon)}_{\rho_{**}}$ with $I_\varepsilon(u) \in [b-\delta_0, b+\delta_0]$.

In fact, Corollary 5.2 and (5.16) imply (4.22). Thus Proposition 4.9 (ii) implies (5.17).

For later use, we state the following lemma, which states that the property $u \in [T_\varepsilon \leq \kappa_\varepsilon]$ ensures that u concentrates around the center of mass $\beta(u)$.

Proposition 5.4 *Assume $u \in A^{(\varepsilon)}_{\rho_{**}} \cap [T_\varepsilon \leq \kappa_\varepsilon]$. Then we have*

$$I_\varepsilon(u) \geq L(\varepsilon\beta(u), u) - c_\varepsilon - \frac{1}{2}\overline{V}\kappa_\varepsilon.$$

Here $c_\varepsilon > 0$ is independent of u and satisfies $c_\varepsilon \to 0$ as $\varepsilon \to 0$.

Proof For $u \in A^{(\varepsilon)}_{\rho_{**}} \cap [T_\varepsilon \le \kappa_\varepsilon]$ we compute

$$\begin{aligned}
I_\varepsilon(u) &= L(\varepsilon\beta(u), u) + \frac{1}{2}\int_{\mathbb{R}^N} (V(\varepsilon x) - V(\varepsilon\beta(u)))u^2 \\
&= L(\varepsilon\beta(u), u) + \frac{1}{2}\left(\int_{|x-\beta(u)|\le \frac{4}{\sqrt{\varepsilon}}} + \int_{|x-\beta(u)|\ge \frac{4}{\sqrt{\varepsilon}}}\right)(V(\varepsilon x) - V(\varepsilon\beta(u)))u^2 \\
&\ge L(\varepsilon\beta(u), u) - \frac{1}{2}\|V(y) - V(\varepsilon\beta(u))\|_{L^\infty(|y-\varepsilon\beta(u)|\le 4\sqrt{\varepsilon})}\|u\|_2^2 \\
&\quad - \frac{1}{2}\overline{V}\|u\|^2_{H^1(|x-\beta(u)|\ge \frac{4}{\sqrt{\varepsilon}})} \\
&\ge L(\varepsilon\beta(u), u) - \frac{1}{2}\|V(y) - V(\varepsilon\beta(u))\|_{L^\infty(|y-\varepsilon\beta(u)|\le 4\sqrt{\varepsilon})}\|u\|_2^2 - \frac{1}{2}\overline{V}T_\varepsilon(u).
\end{aligned}$$

Let $\Omega_{\varepsilon R_0+\rho_{**}}$ be a $(\varepsilon R_0 + \rho_{**})$-neighborhood of Ω, that is, $\Omega_{\varepsilon R_0+\rho_{**}} = \{x \in \mathbb{R}^N : \mathrm{dist}_{\mathbb{R}^N}(x, \Omega) \le \varepsilon R_0 + \rho_{**}\}$. We recall (4.21) and we note that $A^{(\varepsilon)}_{\rho_{**}}$ is uniformly bounded for all $\varepsilon \in (0, 1]$ and let $C = \sup_{\varepsilon\in(0,1], u\in A^{(\varepsilon)}_{\rho_{**}}} \|u\|_2^2 < \infty$. Setting

$$c_\varepsilon = \frac{1}{2}C \sup\{|V(y) - V(y')| : y, y' \in \Omega_{\varepsilon R_0+\rho_{**}},\ |y - y'| \le 4\sqrt{\varepsilon}\} \to 0 \quad \text{as } \varepsilon \to 0,$$

and noting $u \in [T_\varepsilon \le \kappa_\varepsilon]$, we have the conclusion of Proposition 5.4. □

6 Deformation Argument

6.1 Deformation Result

In this section we develop a special deformation argument for $I_\varepsilon(u)$, which keeps $T_\varepsilon(u)$ small. Our aim is to show the following deformation result.

Proposition 6.1 *Let $\varepsilon_0, \nu_0, \delta_0 > 0$ be numbers given in Proposition 4.9 and let $\kappa_\varepsilon > 0$ be a number given in (5.15), which satisfies $\kappa_\varepsilon \to 0$ as $\varepsilon \to 0$. Moreover suppose for some $\varepsilon \in (0, \varepsilon_0]$*

$$(H_\varepsilon(u), I'_\varepsilon(u)) \ne 0 \quad \text{for } u \in \overline{N^{(\varepsilon)}_{\rho_*}} \text{ with } I_\varepsilon(u) \in [b - \delta_0, b + \delta_0]. \tag{6.1}$$

Then for any $\delta_1 \in (0, \delta_0)$ there exist $\delta \in (0, \delta_1)$ and a continuous map $\eta(t, u) : [0, 1] \times A^{(\varepsilon)}_{\rho_{**}} \to A^{(\varepsilon)}_{\rho_{**}}$ such that

(i) *$\eta(0, u) = u$ for all $u \in A^{(\varepsilon)}_{\rho_{**}}$.*
(ii) *$\eta(t, u) = u$ for all $t \in [0, 1]$ if $I_\varepsilon(u) \notin [b - \delta_1, b + \delta_1]$ or $u \notin A^{(\varepsilon)}_{\frac{3\rho_{**}+\rho_*}{4}}$.*
(iii) *$t \mapsto I_\varepsilon(\eta(t, u))$ is a non-increasing function of t for all $u \in A^{(\varepsilon)}_{\rho_{**}}$.*
(iv) *$\eta(1, u) \in [I_\varepsilon \le b - \delta]$ if $u \in A^{(\varepsilon)}_{\rho_*} \cap [I_\varepsilon \le b + \delta]$.*
(v) *$\eta(t, u) \in [T_\varepsilon \le \kappa_\varepsilon]$ for all $t \in [0, 1]$ if $u \in [T_\varepsilon \le \kappa_\varepsilon]$.*

The properties (i)–(iv) are standard under the standard Palais-Smale condition. However our concentration-compactness type result Proposition 4.5 ensures a weaker condition; we assume (4.10) in addition to (4.8) and (4.9) and we have (4.23) under the condition (4.22).

We note that $H_\varepsilon(u)$ gives a useful information on deformation. In fact, for $h \in \mathbb{R}^N$ we have

$$\frac{d}{dt}\Big|_{t=0} I_\varepsilon(u(x-\frac{h}{\varepsilon}t)) = \frac{1}{2}\frac{d}{dt}\Big|_{t=0}\int_{\mathbb{R}^N} V(\varepsilon x+ht)u(x)^2 = \frac{1}{2}\int_{\mathbb{R}^N} h\cdot\nabla V(\varepsilon x)u(x)^2$$
$$= \frac{1}{2}h\cdot H_\varepsilon(u).$$

Thus, if $H_\varepsilon(u)\neq 0$, the translation flow:

$$(t,v)\mapsto v\left(x-\frac{h}{\varepsilon}t\right) \quad \text{with } h=-H_\varepsilon(u) \tag{6.2}$$

gives a decreasing flow in a neighborhood of u.

The property (v) means that the set $[T_\varepsilon \le \kappa_\varepsilon]$ is positively invariant for the flow $\eta(t,u)$, i.e.,

$$\eta(t,[T_\varepsilon\le\kappa_\varepsilon])\subset[T_\varepsilon\le\kappa_\varepsilon] \quad \text{for } t\ge 0. \tag{6.3}$$

This property is related to the tail minimizing flow developed in [23]. In [23], we used the tail minimizing flow separately from the deformation flow (the steepest descent flow) for $I_\varepsilon(u)$. Here, extending the idea in [23] we construct a deformation flow for $I_\varepsilon(u)$ which keeps the size $T_\varepsilon(u)$ of the tail $u|_{\mathbb{R}^N\setminus B(\beta(u),4/\sqrt{\varepsilon})}$ small.

Remark 6.2 In [25, 33, 35], we study radially symmetric problems in $\mathbb{R}^N$. A typical example is a nonlinear scalar field equation: $-\Delta u = g(u)$ in $\mathbb{R}^N$. The natural corresponding functional is

$$\mathcal{I}(u)=\frac{1}{2}\|\nabla u\|_2^2-\int_{\mathbb{R}^N} G(u):\ H_r^1(\mathbb{R}^N)\to\mathbb{R}$$

and scaling $\theta\mapsto u(x/e^\theta)$ is important in the arguments in [25, 33, 35]. Precisely Pohozaev functional

$$\mathcal{P}(u)=\frac{N-2}{2}\|\nabla u\|_2^2-N\int_{\mathbb{R}^N} G(u)$$

is characterized as

$$\mathcal{P}(u)=\frac{d}{d\theta}\Big|_{\theta=0}\mathcal{I}(u(x/e^\theta)). \tag{6.4}$$

Thus, if $\mathcal{P}(u) > 0$ ($\mathcal{P}(u) < 0$ resp.), the scaling flow $(\theta, u) \mapsto u(x/e^{-\theta})$ ($u(x/e^{\theta})$ resp.) gives a decreasing flow in a neighborhood of u. In [25, 33, 35], we introduce an augmented functional $\mathcal{J}(\theta, u)$ by

$$\mathcal{J}(\theta, u) = \frac{1}{2} e^{(N-2)\theta} \|\nabla u\|_2^2 - e^{N\theta} \int_{\mathbb{R}^N} G(u),$$

which enjoys the property $\mathcal{J}(\theta, u) = \mathcal{I}(u(x/e^{\theta}))$. We develop a deformation flow for $\mathcal{I}(u)$ through a deformation for $\mathcal{J}(\theta, u)$ in the augmented space $\mathbb{R} \times H_r^1(\mathbb{R}^N)$.

In the following sections, replacing scaling (6.4) to translation (6.2), we give a proof of Proposition 6.1.

6.2 Augmented Functional

To prove Proposition 6.1, we consider the following functional in the augmented space $\mathbb{R}^N \times H^1(\mathbb{R}^N)$:

$$J_\varepsilon(z, u) = \frac{1}{2} \|\nabla u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(\varepsilon x + z) u(x)^2 - \frac{1}{2} \mathcal{D}(u) : \mathbb{R}^N \times H^1(\mathbb{R}^N) \to \mathbb{R}.$$

We note that $J_\varepsilon(z, u) \in C^1(\mathbb{R}^N \times H^1(\mathbb{R}^N), \mathbb{R})$ and

(i) $J_\varepsilon(z, u) = I_\varepsilon(u(x - \frac{z}{\varepsilon}))$.
(ii) $\partial_u J_\varepsilon(z, u)\varphi = I_\varepsilon'(u(x - \frac{z}{\varepsilon}))\varphi(x - \frac{z}{\varepsilon})$.
(iii) $\partial_z J_\varepsilon(z, u) = H_\varepsilon(u(x - \frac{z}{\varepsilon}))$.

Recalling $D = (\partial_z, \partial_u)$, we have

Lemma 6.3 (i) *For $(z, u) \in \mathbb{R}^N \times H^1(\mathbb{R}^N)$, (z, u) is a critical point of J_ε, i.e., $DJ_\varepsilon(z, u) = 0$ if and only if $v(x) = u(x - \frac{z}{\varepsilon})$ satisfies*

$$I_\varepsilon'(v) = 0 \quad \textit{and} \quad H_\varepsilon(v) = 0.$$

(ii) *For $c \in \mathbb{R}$, c is a critical value of J_ε if and only if there exists $v \in H^1(\mathbb{R}^N)$ such that*

$$I_\varepsilon(v) = c, \quad I_\varepsilon'(v) = 0 \quad \textit{and} \quad H_\varepsilon(v) = 0.$$

(iii) *For all $(z, u) \in \mathbb{R}^N \times H^1(\mathbb{R}^N)$*

$$\|DJ_\varepsilon(z, u)\|^2_{(\mathbb{R}^N \times H^1(\mathbb{R}^N))^*} = \left| H_\varepsilon\left(u\left(x - \frac{z}{\varepsilon}\right)\right)\right|^2 + \left\| I_\varepsilon'\left(u\left(x - \frac{z}{\varepsilon}\right)\right)\right\|^2_{(H^1(\mathbb{R}^N))^*}.$$

As in Corollary 2.2, we have

Corollary 6.4 *There exist* $\rho_0 > 0$ *and* $c_0 > 0$ *such that*

$$J_\varepsilon(z,u) \geq c_0\|u\|_{H^1}^2, \quad \partial_u J_\varepsilon(z,u)u \geq c_0\|u\|_{H^1}^2$$

for all $(z,u) \in \mathbb{R}^N \times H^1(\mathbb{R}^N)$ *with* $\|u\|_{H^1} \leq \rho_0$.

To show our Proposition 6.1, we develop a deformation argument in $\mathbb{R}^N \times H^1(\mathbb{R}^N)$ and we construct a flow $\eta(t,u)$ through a flow $\widetilde{\eta}(t,z,u)$ on a product space $\mathbb{R}^N \times H^1(\mathbb{R}^N)$.

We introduce a pseudo-distance $\mathrm{DIST}_\varepsilon(\cdot,\cdot)$ on $\mathbb{R}^N \times H^1(\mathbb{R}^N)$, which is related to $\mathrm{dist}_\varepsilon(\cdot,\cdot)$, by

$$\mathrm{DIST}_\varepsilon((z,u),(z',u')) = \inf_{h\in\mathbb{R}^N}\sqrt{|z'-z-h|^2 + \left\|u\left(x-\frac{h}{\varepsilon}\right)-u'(x)\right\|_{H^1}^2}$$

for $(z,u), (z',u') \in \mathbb{R}^N \times H^1(\mathbb{R}^N)$. We note that

$$\mathrm{DIST}_\varepsilon((z,u),(z',u')) = \mathrm{dist}_\varepsilon\left(u\left(x-\frac{z}{\varepsilon}\right), u'\left(x-\frac{z'}{\varepsilon}\right)\right)$$

and

$$\begin{aligned}\mathrm{DIST}_\varepsilon((z,u),(z',u')) &\leq \mathrm{dist}_{\mathbb{R}^N\times H^1(\mathbb{R}^N)}((z,u),(z',u'))\\ &\equiv \sqrt{|z-z'|^2+\|u-u'\|_{H^1}^2}.\end{aligned}$$

We set

$$\begin{aligned}\mathcal{N}_\rho^{(\varepsilon)} &= \{(z,u)\in\mathbb{R}^N\times H^1(\mathbb{R}^N) : \mathrm{DIST}_\varepsilon((z,u),\widehat{\mathcal{K}}_b) < \rho\}\\ &= \left\{(z,u)\in\mathbb{R}^N\times H^1(\mathbb{R}^N) : \mathrm{dist}_\varepsilon\left(u\left(x-\frac{z}{\varepsilon}\right), \widehat{K}_b^{(\varepsilon)}\right)) < \rho\right\},\\ \mathcal{A}_\rho^{(\varepsilon)} &= \{(z,u)\in\mathbb{R}^N\times H^1(\mathbb{R}^N) : \mathrm{DIST}_\varepsilon((z,u),\widehat{\mathcal{Z}}_b) < \rho\}\\ &= \left\{(z,u)\in\mathbb{R}^N\times H^1(\mathbb{R}^N) : \mathrm{dist}_\varepsilon\left(u\left(x-\frac{z}{\varepsilon}\right), \widehat{Z}_b^{(\varepsilon)}\right) < \rho\right\}.\end{aligned}$$

Clearly these sets are uniformly bounded with respect to $\varepsilon \in (0,1]$ and we have $\mathcal{N}_\rho^{(\varepsilon)} \subset \mathcal{A}_\rho^{(\varepsilon)}$. From Proposition 4.9 (i), Corollary 5.3 and Lemma 6.3 we have the following

Proposition 6.5 *Let* $0 < \rho_* < \rho_{**}$ *be the numbers satisfying* (4.20). *Then we have*

(i) *There exist* $\nu_0 > 0$ *and* $\delta_0 > 0$ *independent of* ε *such that for* $\varepsilon > 0$ *small*

$$\|DJ_\varepsilon(z,u)\|_{(\mathbb{R}^N\times H^1(\mathbb{R}^N))^*} \geq \nu_0 \tag{6.5}$$

for all $(z,u) \in \mathcal{A}_{\rho_{**}}^{(\varepsilon)} \setminus \overline{\mathcal{N}_{\rho_*}^{(\varepsilon)}}$ *with* $J_\varepsilon(z,u) \in [b-\delta_0, b+\delta_0]$.

(ii) *Suppose that* (6.1) *holds, in other words, it holds that*

$$DJ_\varepsilon(z,u) \neq 0 \quad \text{for all } (z,u) \in \overline{\mathcal{N}^{(\varepsilon)}_{\rho_*}} \text{ with } J_\varepsilon(z,u) \in [b-\delta_0, b+\delta_0]. \tag{6.6}$$

Then there exists $\nu_\varepsilon > 0$ *such that*

$$\|DJ_\varepsilon(z,u)\|_{(\mathbb{R}^N \times H^1(\mathbb{R}^N))^*} \geq \nu_\varepsilon \quad \text{for all } (z,u) \in \mathcal{A}^{(\varepsilon)}_{\rho_{**}} \text{ with } J_\varepsilon(z,u) \in [b-\delta_0, b+\delta_0]. \tag{6.7}$$

We note that we may assume $\nu_\varepsilon < \nu_0$.

6.3 Construction of a Vector Field

In what follows, we will show that the existence of a critical point $(z,u) \in \overline{\mathcal{N}^{(\varepsilon)}_{\rho_*}}$ with $J_\varepsilon(z,u) \in [b-\delta_0, b+\delta_0]$. Arguing indirectly, we assume (6.1) holds. To construct a deformation flow, we find a special vector field $V_{z,u} : \mathcal{A}^{(\varepsilon)}_{\rho_{**}} \to \mathbb{R}^N \times H^1(\mathbb{R}^N)$. Since (6.5) and (6.7) hold by Proposition 6.5, for $(z,u) \in \mathcal{A}^{(\varepsilon)}_{\rho_{**}}$ with $J_\varepsilon(z,u) \in [b-\delta_0, b+\delta_0]$ there exists $(\xi, w) \in \mathbb{R}^N \times H^1(\mathbb{R}^N)$ such that

$$|\xi|^2 + \|w\|^2_{H^1} \leq 1, \tag{6.8}$$

$$DJ_\varepsilon(z,u)(\xi,w) > \nu_0 \quad \text{if } (z,u) \in \mathcal{A}^{(\varepsilon)}_{\rho_{**}} \setminus \overline{\mathcal{N}^{(\varepsilon)}_{\rho_*}}, \tag{6.9}$$

$$DJ_\varepsilon(z,u)(\xi,w) > \nu_\varepsilon \quad \text{if } (z,u) \in \overline{\mathcal{N}^{(\varepsilon)}_{\rho_*}}. \tag{6.10}$$

We compute for $(z,u) \in \mathcal{A}^{(\varepsilon)}_{\rho_{**}}$ and $\ell \geq 0$

$$\partial_u T_\varepsilon(u)(w + \ell u^{(2)}) = \partial_u T_\varepsilon(u) w + \ell \partial_u T_\varepsilon(u) u^{(2)} \geq -C_1 + 2\ell T_\varepsilon(u), \tag{6.11}$$

where $C_1 > 0$ is independent of ε and u. Here we used (5.12) and the boundedness of $\|\partial_u T_\varepsilon(u)\|_{H^1(\mathbb{R}^N)^*}$.

For κ_ε defined in (5.15), we set

$$\ell_\varepsilon \equiv \frac{C_1}{\kappa_\varepsilon} \to \infty \quad \text{as } \varepsilon \to 0. \tag{6.12}$$

Finally we define $V_{z,u} \in \mathbb{R}^N \times H^1(\mathbb{R}^N)$ for $(z,u) \in \mathcal{A}^{(\varepsilon)}_{\rho_{**}}$ with $J_\varepsilon(z,u) \in [b-\delta_0, b+\delta_0]$ by

$$V_{z,u} = \begin{cases} (\xi, w + \ell_\varepsilon u^{(2)}) & \text{if } T_\varepsilon(u) \geq \kappa_\varepsilon, \\ (\xi, w) & \text{if } T_\varepsilon(u) < \kappa_\varepsilon. \end{cases}$$

Then we have

Proposition 6.6 *Suppose that (6.1) holds. Then for* $\varepsilon \in (0, \frac{1}{9R_0^2})$ *and* $(z,u) \in \mathcal{A}_{\rho_{**}}^{(\varepsilon)}$, *we have*

(i) *If* $T_\varepsilon(u) \geq \kappa_\varepsilon$, *then*

$$DT_\varepsilon(u)V_{z,u} > 0.$$

(ii) *For* $(z,u) \in \mathcal{A}_{\rho_{**}}^{(\varepsilon)}$ *with* $J_\varepsilon(z,u) \in [b-\delta_0, b+\delta_0]$,

$$DJ_\varepsilon(z,u)V_{z,u} > \nu_\varepsilon.$$

(iii) *For* $(z,u) \in \mathcal{A}_{\rho_{**}}^{(\varepsilon)} \setminus \overline{\mathcal{N}_{\rho_*}^{(\varepsilon)}}$ *with* $J_\varepsilon(z,u) \in [b-\delta_0, b+\delta_0]$,

$$DJ_\varepsilon(z,u)V_{z,u} > \nu_0.$$

(iv) *There exist* $C, C' > 0$ *such that for* $M_1(u)$, $M_2(u)$ *given in (5.3)*

$$\|DM_1(u)V_{z,u}\|_{H^1} < C, \tag{6.13}$$
$$D(\|M_2(u)\|_{H^1}^2)V_{z,u} > -C'. \tag{6.14}$$

In the above proposition, we write

$$DT_\varepsilon(u) = (0, \partial_u T_\varepsilon(u)), \quad DM_i(u) = (0, \partial_u M_i(u)) \quad \text{for } i = 1, 2.$$

In particular,

$$DT_\varepsilon(u)V_{z,u} = \begin{cases} \partial_u T_\varepsilon(u)(w + \ell_\varepsilon u^{(2)}) & \text{if } T_\varepsilon(u) \geq \kappa_\varepsilon, \\ \partial_u T_\varepsilon(u)w & \text{if } T_\varepsilon(u) < \kappa_\varepsilon. \end{cases}$$

We use similar formulas also for $M_1(u)$ and $\|M_2(u)\|_{H^1}^2$.

Proof First we recall that (6.5), (6.7) hold under (6.1).
(i) By (6.11) and (6.12), we have for $T_\varepsilon(u) \geq \kappa_\varepsilon$

$$DT_\varepsilon(u)V_{z,u} \geq -C_1 + 2\ell_\varepsilon T_\varepsilon(u) \geq -C_1 + 2\ell_\varepsilon\kappa_\varepsilon = C_1 > 0.$$

Thus we have (i).
(ii), (iii) By our choice (5.15) of κ_ε, as in Corollary 5.2 we have $DJ_\varepsilon(z,u)(0,u^{(2)}) \geq 0$ when $T_\varepsilon(u) \geq \kappa_\varepsilon$. Thus (ii) and (iii) follow from (6.9)–(6.10).
(iv) Since

$$\partial_u M_1(u)w = -\zeta'_{1/\sqrt{\varepsilon}}(x - \beta(u))(\beta'(u)w)u + \zeta_{1/\sqrt{\varepsilon}}(x - \beta(u))w,$$

$\|\partial_u M_1(u)w\|_{H^1}$ and $\|\partial_u M_2(u)w\|_{H^1}$ are uniformly bounded by the boundedness of $\|\beta'(u)\|_{H^1(\mathbb{R}^N)^*}$. Thus (6.13) follows from (5.9). As to (6.14), we have from (5.10) and (6.12)

$$\partial_u(\|M_2(u)\|_{H^1}^2)\ell_\varepsilon u^{(2)} \geq -\ell_\varepsilon c_\varepsilon = -C_1.$$

Thus (6.14) follows from the boundedness of $\|\partial_u M_2(u)w\|_{H^1}$. □

Proposition 6.7 *Suppose that* (6.1) *holds. Then for* $\varepsilon > 0$ *small, there exists a locally Lipschitz vector field* $W(z,u) : \mathcal{A}_{\rho_{**}}^{(\varepsilon)} \cap \{(z,u) : J_\varepsilon(z,u) \in [b-\delta_0, b+\delta_0]\} \to \mathbb{R}^N \times H^1(\mathbb{R}^N)$ *with the following properties.*

(i) $DT_\varepsilon(u)W(z,u) > 0$ *if* $T_\varepsilon(u) > \kappa_\varepsilon$.
(ii) $DJ_\varepsilon(z,u)W(z,u) > \nu_\varepsilon$ *if* $(z,u) \in \mathcal{A}_{\rho_{**}}^{(\varepsilon)}$ *and* $J_\varepsilon(z,u) \in [b-\delta_0, b+\delta_0]$.
(iii) $DJ_\varepsilon(z,u)W(z,u) > \nu_0$ *if* $(z,u) \in \mathcal{A}_{\rho_{**}}^{(\varepsilon)} \setminus \overline{\mathcal{N}_{\rho_*}^{(\varepsilon)}}$ *and* $J_\varepsilon(z,u) \in [b-\delta_0, b+\delta_0]$.
(iv) $\|DM_1(u)W(z,u)\|_{H^1} \leq C$, $D(\|M_2(u)\|_{H^1}^2)W(z,u) \geq -C'$.

Proof Let $V_{z,u}$ be a vector field given in Proposition 6.6. We remark that for any $(z,u) \in \mathcal{A}_{\rho_{**}}^{(\varepsilon)}$ there exists a small neighborhood $U_{z,u}$ of (z,u) in $\mathbb{R}^N \times H^1(\mathbb{R}^N)$ such that for $(z',u') \in U_{z,u}$

(i) $DT_\varepsilon(u')V_{z,u} > 0$ if $T_\varepsilon(u) > \kappa_\varepsilon$.
(ii) $DJ_\varepsilon(z',u')V_{z,u} > \nu_\varepsilon$ if $(z,u) \in \mathcal{A}_{\rho_{**}}^{(\varepsilon)}$ and $J_\varepsilon(z,u) \in [b-\delta_0, b+\delta_0]$.
(iii) $DJ_\varepsilon(z',u')V_{z,u} > \nu_0$ if $(z,u) \in \mathcal{A}_{\rho_{**}}^{(\varepsilon)} \setminus \overline{\mathcal{N}_{\rho_*}^{(\varepsilon)}}$ and $J_\varepsilon(z,u) \in [b-\delta_0, b+\delta_0]$.
(iv) $\|DM_1(u')V_{z,u}\|_{H^1} < C$, $D(\|M_2(u')\|_{H^1}^2)V_{z,u} > -C'$.

We may choose a neighborhood $U_{z,u}$ of (z,u) so that

$$U_{z,u} \subset \{(z',u') : T_\varepsilon(u') > \kappa_\varepsilon\} \quad \text{if } T_\varepsilon(u) > \kappa_\varepsilon,$$
$$U_{z,u} \subset \mathcal{A}_{\rho_{**}}^{(\varepsilon)} \setminus \overline{\mathcal{N}_{\rho_*}^{(\varepsilon)}} \quad \text{if } (z,u) \in \mathcal{A}_{\rho_{**}}^{(\varepsilon)} \setminus \overline{\mathcal{N}_{\rho_*}^{(\varepsilon)}}.$$

Using a partition of unity, we can construct a locally Lipschitz continuous vector field $W(z,u) : \mathcal{A}_{\rho_{**}}^{(\varepsilon)} \cap \{(z,u) : J_\varepsilon(z,u) \in [b-\delta_0, b+\delta_0]\} \to \mathbb{R}^N \times H^1(\mathbb{R}^N)$ in a standard way. We can easily see that $W(z,u)$ satisfies (i)–(iv). □

We note that $W(z,u)$ is bounded in the following sense:

$$\|W(z,u)\|_{\mathbb{R}^N \times H^1(\mathbb{R}^N)} \leq C(1+\ell_\varepsilon) \tag{6.15}$$

for all (z,u), where $C > 0$ is independent of ε, (z,u).

6.4 Deformation Flow for the Augmented Functional $J_\varepsilon(z,u)$

Using the pseudo-gradient flow $W(z,u)$ obtained in Proposition 6.7, we have

Proposition 6.8 *For* $\varepsilon > 0$ *small, suppose that* (6.1) *holds. Then for any given* $\delta_1 \in (0,\delta_0)$ *there exist* $\delta \in (0,\delta_1)$ *and a continuous map* $\widetilde{\eta}(t,z,u) : [0,1] \times \mathcal{A}_{\rho_{**}}^{(\varepsilon)} \to \mathcal{A}_{\rho_{**}}^{(\varepsilon)}$ *such that*

(i) $\widetilde{\eta}(0,z,u)=(z,u)$ *for all* $(z,u)\in\mathcal{A}^{(\varepsilon)}_{\rho_{**}}$.
(ii) $\widetilde{\eta}(t,z,u)=(z,u)$ *for all* $t\in[0,1]$ *if* $J_\varepsilon(z,u)\notin[b-\delta_1,b+\delta_1]$ *or* $(z,u)\notin\mathcal{A}^{(\varepsilon)}_{\frac{3\rho_{**}+\rho_*}{4}}$.
(iii) $t\mapsto J_\varepsilon(\widetilde{\eta}(t,z,u))$ *is non-increasing on* $[0,1]$ *for all* $(z,u)\in\mathcal{A}^{(\varepsilon)}_{\rho_{**}}$.
(iv) $J_\varepsilon(\widetilde{\eta}(1,z,u))\le b-\delta$ *if* $(z,u)\in\mathcal{A}^{(\varepsilon)}_{\rho_*}$ *satisfies* $J_\varepsilon(z,u)\le b+\delta$.
(v) $T_\varepsilon(\widetilde{\eta}(1,z,u))\le\kappa_\varepsilon$ *if* $T_\varepsilon(u)\le\kappa_\varepsilon$.

For a proof we use notation for $c\in\mathbb{R}$

$$[[J_\varepsilon\le c]]=\{(z,u)\in\mathbb{R}^N\times H^1(\mathbb{R}^N):\ J_\varepsilon(z,u)\le c\}.$$

Proof Let $W(z,u)$ be a locally Lipschitz continuous vector field given in Proposition 6.7. For $\delta\in(0,\frac{1}{2}\delta_1)$ we choose locally Lipschitz continuous functions $\varphi_1:\mathbb{R}\to[0,1]$, $\varphi_2:\mathbb{R}^N\times H^1(\mathbb{R}^N)\to[0,1]$ such that

$$\varphi_1(s)=\begin{cases}1 & \text{for } s\in[b-\delta,b+\delta],\\ 0 & \text{for } s\notin[b-2\delta,b+2\delta],\end{cases}\qquad \varphi_2(z,u)=\begin{cases}1 & \text{for } (z,u)\in\mathcal{A}^{(\varepsilon)}_{\frac{\rho_{**}+\rho_*}{2}},\\ 0 & \text{for } (z,u)\notin\mathcal{A}^{(\varepsilon)}_{\frac{3\rho_{**}+\rho_*}{4}}.\end{cases}$$

We consider the following ODE:

$$\frac{d\widetilde{\eta}}{dt}=-\varphi_1(J_\varepsilon(\widetilde{\eta}))\varphi_2(\widetilde{\eta})W(\widetilde{\eta}),\qquad \widetilde{\eta}(0,z,u)=(z,u). \tag{6.16}$$

First we note that for each $\varepsilon\in(0,1]$ the vector field $W(z,u)$ is locally Lipschitz and uniformly bounded, where the bound depends on ε (c.f. (6.15)), the solution $\widetilde{\eta}(t)=\widetilde{\eta}(t,z,u)$ of (6.16) is extendable as long as $\widetilde{\eta}(t)\in\mathcal{A}^{(\varepsilon)}_{\rho_{**}}$. Moreover the right hand side of (6.16) vanishes in $\mathcal{A}^{(\varepsilon)}_{\rho_{**}}\setminus\mathcal{A}^{(\varepsilon)}_{\frac{3\rho_{**}+\rho_*}{4}}$ and thus $\widetilde{\eta}(t)$ exists for all $t\ge 0$.

We compute

$$\begin{aligned}\frac{d}{dt}J_\varepsilon(\widetilde{\eta})&=DJ_\varepsilon(\widetilde{\eta})\frac{d\widetilde{\eta}}{dt}=-\varphi_1(J_\varepsilon(\widetilde{\eta}))\varphi_2(\widetilde{\eta})DJ_\varepsilon(\widetilde{\eta})W(\widetilde{\eta}),\\ \frac{d}{dt}T_\varepsilon(\widetilde{\eta})&=-\varphi_1(J_\varepsilon(\widetilde{\eta}))\varphi_2(\widetilde{\eta})DT_\varepsilon(\widetilde{\eta})W(\widetilde{\eta}).\end{aligned}$$

Thus, we have from Proposition 6.7 that

$$\frac{d}{dt}J_\varepsilon(\widetilde{\eta})\le 0\quad\text{on } \mathcal{A}^{(\varepsilon)}_{\rho_{**}}, \tag{6.17}$$

$$\frac{d}{dt}J_\varepsilon(\widetilde{\eta})\le-\nu_\varepsilon\quad\text{if } \widetilde{\eta}\in\mathcal{A}^{(\varepsilon)}_{\frac{\rho_{**}+\rho_*}{2}} \text{ and } J_\varepsilon(\widetilde{\eta})\in[b-\delta,b+\delta], \tag{6.18}$$

$$\frac{d}{dt}J_\varepsilon(\widetilde{\eta})\le-\nu_0\quad\text{if } \widetilde{\eta}\in\mathcal{A}^{(\varepsilon)}_{\frac{\rho_{**}+\rho_*}{2}}\setminus\overline{\mathcal{N}^{(\varepsilon)}_{\rho_*}} \text{ and } J_\varepsilon(\widetilde{\eta})\in[b-\delta,b+\delta], \tag{6.19}$$

$$\frac{d}{dt}T_\varepsilon(\widetilde{\eta})\le 0\quad\text{if } T_\varepsilon(\widetilde{\eta})\ge\kappa_\varepsilon. \tag{6.20}$$

The properties (i)–(iii) and (v) follow from the definition (6.16) and the properties (6.17) and (6.20). To complete the proof, we need to show (iv).

We suppose $(z,u) \in \mathcal{A}^{(\varepsilon)}_{\rho_*} \cap [[J_\varepsilon \le b+\delta]]$ and we show for some $\bar{t}_\varepsilon > 0$

$$\widetilde{\eta}(\bar{t}_\varepsilon, z, u) \in [[J_\varepsilon \le b-\delta]]. \tag{6.21}$$

Arguing indirectly, we assume that $\widetilde{\eta}(t) \in [[J_\varepsilon > b-\delta]]$ for all $t \ge 0$. If $\widetilde{\eta}(t) = \widetilde{\eta}(t,z,u)$ satisfies

$$\widetilde{\eta}(t_0) \in \partial\mathcal{A}^{(\varepsilon)}_{\frac{\rho_{**}+\rho_*}{2}} \quad \text{for some } t_0 > 0, \tag{6.22}$$

then we can find an interval $[s_{z,u}, t_{z,u}]$ such that

$$\widetilde{\eta}(t) \in \mathcal{A}^{(\varepsilon)}_{\frac{\rho_{**}+\rho_*}{2}} \setminus \mathcal{A}^{(\varepsilon)}_{\rho_*} \quad \text{for } t \in (s_{z,u}, t_{z,u}), \tag{6.23}$$

$$\widetilde{\eta}(s_{z,u}) \in \partial\mathcal{A}^{(\varepsilon)}_{\rho_*}, \qquad \widetilde{\eta}(t_{z,u}) \in \partial\mathcal{A}^{(\varepsilon)}_{\frac{\rho_{**}+\rho_*}{2}}. \tag{6.24}$$

The following Lemma 6.9 shows that for some $\tau_0 > 0$ independent of ε, (z,u)

$$t_{z,u} - s_{z,u} \ge \tau_0. \tag{6.25}$$

Thus by (6.19),

$$J_\varepsilon(\widetilde{\eta}(t_{z,u})) \le J_\varepsilon(\widetilde{\eta}(s_{z,u})) - \nu_0\tau_0 \le b+\delta-\nu_0\tau_0.$$

Choosing $\delta < \frac{1}{3}\nu_0\tau_0$, we have

$$J_\varepsilon(\widetilde{\eta}(t_{z,u})) \le b-2\delta, \tag{6.26}$$

which is in contradiction. Thus (6.22) cannot occur and we have $\widetilde{\eta}(t) \in \mathcal{A}^{(\varepsilon)}_{\frac{\rho_{**}+\rho_*}{2}}$ for all $t \ge 0$. By (6.18), setting $\bar{t}_\varepsilon = \frac{2\delta}{\nu_\varepsilon} > 0$, we have (6.21) and (iv) holds. □

The following lemma is a key of the proof of Proposition 6.8. We remark that

$$\mathrm{dist}_{\mathbb{R}^N\times H^1(\mathbb{R}^N)}(\mathcal{A}^{(\varepsilon)}_{\rho_*}, \partial\mathcal{A}^{(\varepsilon)}_{\frac{\rho_{**}+\rho_*}{2}}) \ge \mathrm{DIST}_\varepsilon(\mathcal{A}^{(\varepsilon)}_{\rho_*}, \partial\mathcal{A}^{(\varepsilon)}_{\frac{\rho_{**}+\rho_*}{2}}) \ge \frac{1}{2}(\rho_{**}-\rho_*).$$

However, since $\ell_\varepsilon \to \infty$ as $\varepsilon \to 0$, $\|\frac{d\widetilde{\eta}}{dt}\|_{\mathbb{R}^N\times H^1(\mathbb{R}^N)} = \|W(\widetilde{\eta})\|_{\mathbb{R}^N\times H^1(\mathbb{R}^N)}$ is not uniformly bounded by (6.15). Thus (6.25) does not follow from (6.23)–(6.24). In the following lemma, (iv) of Proposition 6.7 plays a role.

Lemma 6.9 *There exists $\tau_0 > 0$ independent of $\varepsilon > 0$ such that if $\widetilde{\eta}(t) = \widetilde{\eta}(t,z,u)$ satisfies (6.23)–(6.24), then (6.25) holds.*

Proof By Proposition 6.7 (iv), we have for $\widetilde{\eta}(t) = \widetilde{\eta}(t, z, u)$

$$\begin{aligned}
&\|\frac{d}{dt} M_1(\widetilde{\eta}(t))\|_{H^1} \le \varphi_1(J_\varepsilon(\widetilde{\eta}))\varphi_2(\widetilde{\eta})\|DM_1(\widetilde{\eta})W(\widetilde{\eta})\|_{H^1} \le C, \\
&\frac{d}{dt}\left(\|M_2(\widetilde{\eta}(t))\|_{H^1}^2\right) = -\varphi_1(J_\varepsilon(\widetilde{\eta}))\varphi_2(\widetilde{\eta})D(\|M_2(\widetilde{\eta})\|_{H^1}^2)W(\widetilde{\eta}) \le C'.
\end{aligned}$$

Thus, for $t \in [s_{z,u}, s_{z,u} + \tau]$ we have

$$\begin{aligned}
&\|P_2\widetilde{\eta}(t) - P_2\widetilde{\eta}(s_{z,u})\|_{H^1} \\
&\quad \le \|M_1(P_2\widetilde{\eta}(t)) - M_1(P_2\widetilde{\eta}(s_{z,u}))\|_{H^1} + \|M_2(P_2\widetilde{\eta}(t)) - M_2(P_2\widetilde{\eta}(s_{z,u}))\|_{H^1} \\
&\quad \le C(t - s_{z,u}) + \|M_2(P_2\widetilde{\eta}(s_{z,u}))\|_{H^1} + \|M_2(P_2\widetilde{\eta}(t))\|_{H^1} \\
&\quad \le C(t - s_{z,u}) + \|M_2(P_2\widetilde{\eta}(s_{z,u}))\|_{H^1} + \left(\|M_2(P_2\widetilde{\eta}(s_{z,u}))\|_{H^1}^2 + C'(t - s_{z,u})\right)^{1/2} \\
&\quad \le C\tau + \|M_2(P_2\widetilde{\eta}(s_{z,u}))\|_{H^1} + \left(\|M_2(P_2\widetilde{\eta}(s_{z,u}))\|_{H^1}^2 + C'\tau\right)^{1/2}.
\end{aligned} \tag{6.27}$$

On the other hand we have

$$\|M_2(P_2\widetilde{\eta}(s_{z,u}))\|_{H^1} \le 3\rho_* + d_\varepsilon, \tag{6.28}$$

where

$$d_\varepsilon = \sup_{\omega \in \widehat{S}_b, |y| \le R_0} \|(1 - \zeta_{1/\sqrt{\varepsilon}}(x))\omega(x - y)\|_{H^1} \to 0 \quad \text{as } \varepsilon \to 0.$$

In fact, writing $\widetilde{\eta}(s_{z,u}) = (z', u') \in \mathcal{A}_{\rho_*}^{(\varepsilon)}$, we have for some $(\xi_0, \omega_0) \in \overline{\Omega} \times \widehat{S}_b$

$$\mathrm{dist}_\varepsilon\left(u'\left(x - \frac{z'}{\varepsilon}\right), \omega_0\left(x - \frac{\xi_0}{\varepsilon}\right)\right) < \rho_*.$$

Thus, there exists $h \in \mathbb{R}^N$ such that

$$|h|^2 + \left\|u'\left(x - \frac{z'}{\varepsilon}\right) - \omega_0\left(x - \frac{\xi_0 + h}{\varepsilon}\right)\right\|_{H^1}^2 < {\rho_*}^2.$$

By Proposition 3.3, we have $|\beta(u') - \frac{\xi_0 - z' + h}{\varepsilon}| \le R_0$. Since $P_2\widetilde{\eta}(s_{z,u}) = u'$, we have by (2.2)

$$\begin{aligned}
\|M_2(P_2\widetilde{\eta}(s_{z,u}))\|_{H^1} &= \|M_2(u')\|_{H^1} = \|(1-\zeta_{1/\sqrt{\varepsilon}}(x-\beta(u'))u'(x)\|_{H^1} \\
&\le \left\|(1-\zeta_{1/\sqrt{\varepsilon}}(x-\beta(u')))\left(u'(x)-\omega_0\left(x-\frac{\xi_0-z+h}{\varepsilon}\right)\right)\right\|_{H^1} \\
&\quad + \left\|(1-\zeta_{\frac{1}{\sqrt{\varepsilon}}}(x-\beta(u')))\omega_0\left(x-\frac{\xi_0-z'+h}{\varepsilon}\right)\right\|_{H^1} \\
&\le 3\left\|u'(x)-\omega_0\left(x-\frac{\xi_0-z'+h}{\varepsilon}\right)\right\|_{H^1} + d_\varepsilon \le 3\rho_* + d_\varepsilon.
\end{aligned}$$

Thus we have (6.28). By (6.27),

$$\|P_2\widetilde{\eta}(t) - P_2\widetilde{\eta}(s_{z,u})\|_{H^1} \le C\tau + (3\rho_* + d_\varepsilon) + ((3\rho_* + d_\varepsilon)^2 + C'\tau)^{1/2} \quad \text{for } t \in [s_{z,u}, s_{z,u}+\tau].$$

Since $|P_1W(z,u)| \le 1$ for all (z,u), we have $|P_1\widetilde{\eta}(t) - P_1\widetilde{\eta}(s_{z,u})| \le \tau$. Thus there exists $\tau_0 > 0$ such that for $\varepsilon > 0$ small

$$\begin{aligned}
\mathrm{DIST}_\varepsilon(\widetilde{\eta}(t), \widetilde{\eta}(s_{z,u})) &\le \|\widetilde{\eta}(t) - \widetilde{\eta}(s_{z,u})\|_{\mathbb{R}^N\times H^1(\mathbb{R}^N)} \\
&\le \left(|P_1\widetilde{\eta}(t) - P_1\widetilde{\eta}(s_{z,u})|^2 + \|P_2\widetilde{\eta}(t) - P_2\widetilde{\eta}(s_{z,u})\|_{H^1}^2\right)^{1/2} \\
&< 7\rho_* \quad \text{for } t \in [s_{z,u}, s_{z,u}+\tau_0],
\end{aligned}$$

which implies

$$\begin{aligned}
\mathrm{DIST}_\varepsilon(\widetilde{\eta}(t), \widehat{\mathcal{Z}}_b) &\le \mathrm{DIST}_\varepsilon(\widetilde{\eta}(t), \widetilde{\eta}(s_{z,u})) + \mathrm{DIST}_\varepsilon(\widetilde{\eta}(s_{z,u}), \widehat{\mathcal{Z}}_b) \\
&< 7\rho_* + \rho_* < \frac{\rho_{**}+\rho_*}{2} \quad \text{for } t \in [s_{z,u}, s_{z,u}+\tau_0].
\end{aligned}$$

Here we used (4.20). Thus we have $\widetilde{\eta}(t) \in \mathcal{A}^{(\varepsilon)}_{\frac{\rho_*+\rho_{**}}{2}}$ for $t \in [s_{z,u}, s_{z,u}+\tau_0]$ and the proof of Lemma 6.9 is completed. □

End of the proof of Proposition 6.1 We define $\pi_\varepsilon : \mathbb{R}^N \times H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$ by

$$\pi_\varepsilon(z,u)(x) = u\left(x - \frac{z}{\varepsilon}\right).$$

For the flow $\widetilde{\eta}(t,z,u)$ obtained in Proposition 6.8, set

$$\eta(t,u) = \pi_\varepsilon(\widetilde{\eta}(t,0,u)).$$

Noting $T_\varepsilon(\pi_\varepsilon(z,u)) = T_\varepsilon(u)$, it is easily observed that $\eta(t,u)$ has the desired properties. □

7 Existence of Critical Points

In this section we complete a proof of Theorem 1.2. We argue 2 setting (MP) and (LM) separately.

7.1 Existence Under the Condition (MP)

First we consider (1.1) under the assumptions (f1)–(f4), (V1)–(V4) and (MP). Let $V_0 > 0$ be the number given in (MP) and let $b = E_{V_0}$.

Proposition 7.1 *Assume (f1)–(f4), (V1)–(V4) and (MP) and let* $b = E_{V_0}$. *For any* $\rho_* > 0$ *and* $\overline{\delta} > 0$ *there exists* $\varepsilon_0 = \varepsilon_0(\rho_*, \overline{\delta}) > 0$ *such that for* $\varepsilon \in (0, \varepsilon_0]$, $I_\varepsilon(u)$ *has a critical point u in* $\overline{N_{\rho_*}^{(\varepsilon)}} \cap [T_\varepsilon \le \kappa_\varepsilon]$ *with* $I_\varepsilon(u) \in [b - \overline{\delta}, b + \overline{\delta}]$.

Proof of Proposition 7.1 Let e_1, e_2, Λ be given in (MP). We may choose $\rho_* > 0$ smaller if necessary and choose $\rho_{**} > 0$ so that (4.20) holds.

Let $\omega_0(x)$ be a least energy solution of $L'_{V_0}(u) = 0$. We choose $s_0 \in (0, \frac{1}{2})$ such that

$$\left\| \omega_0\left(\frac{x}{s}\right) - \omega_0(x) \right\|_{H^1} < \frac{\rho_*}{3} \quad \text{for all } s \in [1 - s_0, 1 + s_0], \tag{7.1}$$

$$L_{V(e_i)}\left(\omega_0\left(\frac{x}{s}\right)\right) < b \quad \text{for all } s \in [1 - s_0, 1 + s_0] \text{ and } i = 0, 1. \tag{7.2}$$

Since $L_{V(e_i)}(\omega_0(\frac{x}{s})) < L_{V_0}(\omega_0(\frac{x}{s})) \le b$, (7.2) holds for small $s_0 \in (0, \frac{1}{2})$. We may assume that $\overline{\delta} > 0$ satisfies

$$\max_{s\in[1-s_0,1+s_0]} L_{V(e_i)}\left(\omega_0\left(\frac{x}{s}\right)\right) < b - 2\overline{\delta} \quad \text{for } s \in [1 - s_0, 1 + s_0] \text{ and } i = 0, 1, \tag{7.3}$$

$$L_{V_0}\left(\omega_0\left(\frac{x}{s}\right)\right) < b - 2\overline{\delta} \quad \text{for } s = 1 \pm s_0. \tag{7.4}$$

Arguing indirectly and noting Corollary 5.2, we assume that (6.1) holds. Applying Proposition 6.1, there are $\delta \in (0, \overline{\delta})$ and $\eta(t, u) \in C([0, 1] \times A_{\rho_{**}}^{(\varepsilon)}, A_{\rho_{**}}^{(\varepsilon)})$ such that (i)–(v) of Proposition 6.1 hold.

Step 1: Choice of an initial path $\gamma_\varepsilon(s, \xi) : [1 - s_0, 1 + s_0] \times [0, 1] \to H^1(\mathbb{R}^N)$

For $c(\xi) \in \Lambda$, we set

$$\gamma_{0\varepsilon}(c; s, \xi)(x) = \omega_0\left(\frac{x - c(\xi)/\varepsilon}{s}\right) : [1 - s_0, 1 + s_0] \times [0, 1] \to H^1(\mathbb{R}^N).$$

By the choice (5.15) of κ_ε, we have

$$\gamma_{0\varepsilon}(c; s, \xi) \in [T_\varepsilon \le \kappa_\varepsilon], \tag{7.5}$$

$$\gamma_{0\varepsilon}(c; s, \xi) \in A_{\rho_*}^{(\varepsilon)} \quad \text{for all } (s, \xi) \in [1 - s_0, 1 + s_0] \times [0, 1]. \tag{7.6}$$

In fact, $\omega_0(x - c(\xi)/\varepsilon) \in \widehat{Z}_b^{(\varepsilon)}$ and (7.1) imply (7.6).
We also have

$$\begin{aligned} I_\varepsilon(\gamma_{0\varepsilon}(c; s, \xi)) &\to L\left(c(\xi), \omega_0\left(\frac{x}{s}\right)\right) \\ &= L_{V_0}\left(\omega_0\left(\frac{x}{s}\right)\right) + \frac{1}{2}(V(c(\xi)) - V_0)\left\|\omega_0\left(\frac{x}{s}\right)\right\|_2^2 \end{aligned} \tag{7.7}$$

as $\varepsilon \to 0$ uniformly in $[1 - s_0, 1 + s_0] \times [0, 1]$.

Thus, choosing $c(\xi) \in \Lambda$ such that $\max_{\xi \in [0,1]} V(c(\xi))$ is very close to V_0, from (7.3), (7.4) and (7.7) we have for sufficiently small $\varepsilon > 0$

$$\gamma_{0\varepsilon}(c; s, \xi) \in [I_\varepsilon \le b - \overline{\delta}] \quad \text{for } (s, \xi) \in \partial([1 - s_0, 1 + s_0] \times [0, 1]), \tag{7.8}$$

$$\gamma_{0\varepsilon}(c; s, \xi) \in [I_\varepsilon \le b + \delta] \quad \text{for } (s, \xi) \in [1 - s_0, 1 + s_0] \times [0, 1]. \tag{7.9}$$

Let $\eta(t, u) : [0, 1] \times A_{\rho_{**}}^{(\varepsilon)} \to A_{\rho_{**}}^{(\varepsilon)}$ be a deformation given in Proposition 6.1 and we set

$$\gamma_\varepsilon(s, \xi) = \eta(1, \gamma_{0\varepsilon}(c; s, \xi)). \tag{7.10}$$

By (7.8) and the property (ii) of Proposition 6.1,

$$\gamma_\varepsilon(s, \xi) = \gamma_{0\varepsilon}(c; s, \xi) = \omega_0\left(\frac{x - c(\xi)/\varepsilon}{s}\right) \quad \text{for } (s, \xi) \in \partial([1 - s_0, 1 + s_0] \times [0, 1]). \tag{7.11}$$

By (7.9) and the properties (iv), (v) of Proposition 6.1, we have for $(s, \xi) \in [1 - s_0, 1 + s_0] \times [0, 1]$

$$\gamma_\varepsilon(s, \xi) \in [I_\varepsilon \le b - \delta] \cap [T_\varepsilon \le \kappa_\varepsilon]. \tag{7.12}$$

Next we will show under (7.5)–(7.6) and (7.11) that $\gamma_\varepsilon(s, \xi)$ satisfies

$$\liminf_{\varepsilon \to 0} \max_{(s,\xi) \in [1-s_0, 1+s_0] \times [0,1]} I_\varepsilon(\gamma_\varepsilon(s, \xi)) \ge b. \tag{7.13}$$

We note that (7.13) is incompatible with (7.12) and it shows the existence of a critical point in $\overline{N_{\rho_*}^{(\varepsilon)}} \cap [T_\varepsilon \le \kappa_\varepsilon]$.

We remark that under (MP) there exists a small neighborhood $\Omega' (\supset \Omega)$ of Ω with the following properties:

(1) For $\varepsilon > 0$ small,

$$\varepsilon \beta(\gamma_\varepsilon(s, \xi)) \in \Omega' \quad \text{for all } (s, \xi) \in [1 - s_0, 1 + s_0] \times [0, 1].$$

(2) Set

$$W = \{x \in \Omega' : V(x) < V_0\},$$

then e_0 and e_1 belong to different components of W.

Since $\gamma_\varepsilon(s, z) \in \mathcal{A}^{(\varepsilon)}_{\rho_{**}}$ for all (s, z), we have

$$\mathrm{dist}_{\mathbb{R}^N}(\varepsilon\beta(\gamma_\varepsilon(s, \xi)), \overline{\Omega}) \le \varepsilon R_0 + \rho_{**}$$

and (1) follows.

We denote by W_- the component of W, to which e_0 belongs, and we set

$$W_+ = W \setminus W_-, \qquad W_0 = \{x \in \Omega' : V(x) \ge V_0\}.$$

We also introduce a signed distance function $d_0(x)$ on Ω' by

$$d_0(x) = \begin{cases} -\frac{\mathrm{dist}(x, W_0)}{\mathrm{dist}(e_0, W_0)} & \text{if } x \in W_-, \\ \frac{\mathrm{dist}(x, W_0)}{\mathrm{dist}(e_1, W_0)} & \text{if } x \in W_+, \\ 0 & \text{if } x \in W_0. \end{cases}$$

For $P_a(u)$ defined in (3.2), we set $a = V_0$ and consider

$$F_\varepsilon(u) = (P_{V_0}(u), d_0(\varepsilon\beta(u))) : A^{(\varepsilon)}_{\rho_{**}} \to \mathbb{R} \times \mathbb{R}.$$

Then we have

Step 2: For $\gamma_\varepsilon(s, \xi)$ *defined in* (7.10),

$$\deg(F_\varepsilon(\gamma_\varepsilon(s, \xi)), [1 - s_0, 1 + s_0] \times [0, 1], (0, 0)) = -1. \tag{7.14}$$

In particular, there exists $(s_\varepsilon, \xi_\varepsilon) \in [1 - s_0, 1 + s_0] \times [0, 1]$ *such that*

$$P_{V_0}(\gamma_\varepsilon(s_\varepsilon, \xi_\varepsilon)) = 0 \quad \text{and} \quad V(\varepsilon\beta(\gamma_\varepsilon(s_\varepsilon, \xi_\varepsilon))) \ge V_0. \tag{7.15}$$

In fact, for $(s, \xi) \in \partial([1 - s_0, 1 + s_0] \times [0, 1])$, we have by (7.11)

$$\begin{aligned} F_\varepsilon(\gamma_\varepsilon(s, \xi)) &= F_\varepsilon\left(\omega_0\left(\frac{x - c(\xi)/\varepsilon}{s}\right)\right) \\ &= \left(P_{V_0}\left(\omega_0\left(\frac{x}{s}\right)\right), d_0\left(\varepsilon\beta\left(\omega_0\left(\frac{x - c(\xi)/\varepsilon}{s}\right)\right)\right)\right) \\ &= \left(P_{V_0}\left(\omega_0\left(\frac{x}{s}\right)\right), d_0(c(\xi) + o(1))\right). \end{aligned}$$

By Proposition 3.2 we have

$$P_{V_0}\left(\omega_0\left(\frac{x}{s}\right)\right)\begin{cases} > 0 & \text{for } s = 1 - s_0, \\ < 0 & \text{for } s = 1 + s_0, \end{cases} \qquad d_0(c(\xi))\begin{cases} > 0 & \text{for } \xi = 0, \\ < 0 & \text{for } \xi = 1, \end{cases}$$

and thus we have (7.14). Since $d_0(y) = 0$ implies $V(y) \geq V_0$, (7.14) implies the existence of $(s_\varepsilon, \xi_\varepsilon)$ with (7.15).
Step 3: $I_\varepsilon(\gamma_\varepsilon(s_\varepsilon, \xi_\varepsilon)) \geq b + o(1)$ *as* $\varepsilon \to 0$.
We write $w_\varepsilon = \gamma_\varepsilon(s_\varepsilon, \xi_\varepsilon)$. Since $w_\varepsilon \in A^{(\varepsilon)}_{\rho_{**}} \cap [T_\varepsilon \leq \kappa_\varepsilon]$, it follows from Proposition 5.4

$$\begin{aligned} I_\varepsilon(w_\varepsilon) &\geq L(V(\varepsilon\beta(w_\varepsilon)), w_\varepsilon) - c_\varepsilon - \frac{1}{2}\overline{V}\kappa_\varepsilon \\ &\geq L_{V_0}(w_\varepsilon) + \frac{1}{2}(V(\varepsilon\beta(w_\varepsilon)) - V_0)\|w_\varepsilon\|_2^2 - c_\varepsilon - \frac{1}{2}\overline{V}\kappa_\varepsilon. \end{aligned}$$

By (7.15), we have

$$I_\varepsilon(w_\varepsilon) \geq L_{V_0}(w_\varepsilon) - c_\varepsilon - \frac{1}{2}\overline{V}\kappa_\varepsilon.$$

By (3.3), it follows from $P_{V_0}(w_\varepsilon) = 0$ that $L_{V_0}(w_\varepsilon) \geq E_{V_0} = b$. Thus we have Step 3.
Step 4: Conclusion.
(7.12) and (7.13) are incompatible and thus (6.1) does not hold. Thus we have the conclusion of Proposition 7.1. □

7.2 Existence Under the Condition (LM)

In this section we consider (1.1) under the assumptions (f1)–(f4), (V1)–(V4) and (LM). Let $V_0 > 0$ be the maximum in Ω and let $b = E_{V_0}$. We have

Proposition 7.2 *Assume (f1)–(f4), (V1)–(V4) and (LM) and let* $b = E_{V_0}$*. For any* $\rho_* > 0$ *and* $\overline{\delta} > 0$ *there exists* $\varepsilon_0 = \varepsilon_0(\rho_*, \overline{\delta}) > 0$ *such that for* $\varepsilon \in (0, \varepsilon_0]$, $I_\varepsilon(u)$ *has a critical point u in* $\overline{N^{(\varepsilon)}_{\rho_*}} \cap [T_\varepsilon \leq \kappa_\varepsilon]$ *satisfying* $I_\varepsilon(u) \in [b - \overline{\delta}, b + \overline{\delta}]$.

Proof of Proposition 7.2 Let $\omega_0(x)$ be a least energy solution corresponding to $b = E_{V_0}$. We choose $s_0 \in (0, \frac{1}{2})$ satisfying (7.1) and set $\gamma_{0\varepsilon}(s, \xi) : [1 - s_0, 1 + s_0] \times \overline{\Omega} \to H^1(\mathbb{R}^N)$ by

$$\gamma_{0\varepsilon}(s, \xi)(x) = \omega_0\left(\frac{x - \xi/\varepsilon}{s}\right).$$

We note that

$$I_\varepsilon(\gamma_{0\varepsilon}(s,\xi)) = \frac{1}{2}\left\|\nabla\left(\omega_0\left(\frac{x}{s}\right)\right)\right\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x+\xi)\omega_0\left(\frac{x}{s}\right)^2 - \frac{1}{2}\mathcal{D}\left(\omega_0\left(\frac{x}{s}\right)\right)$$
$$\to L\left(\xi,\omega_0\left(\frac{x}{s}\right)\right) = L_{V_0}\left(\omega_0\left(\frac{x}{s}\right)\right) - \frac{1}{2}(V_0 - V(\xi))\left\|\omega_0\left(\frac{x}{s}\right)\right\|_2^2$$

as $\varepsilon \to 0$ uniformly in $(s,\xi) \in [1-s_0, 1+s_0] \times \overline{\Omega}$.

Thus there exists $\overline{\delta} > 0$ such that

$$\max_{(s,\xi)\in[1-s_0,1+s_0]\times\overline{\Omega}} L\left(\xi,\omega_0\left(\frac{x}{s}\right)\right) \le b,$$
$$\max_{(s,\xi)\in\partial([1-s_0,1+s_0]\times\overline{\Omega})} L\left(\xi,\omega_0\left(\frac{x}{s}\right)\right) \le b - 2\overline{\delta}.$$

Moreover for any $\delta \in (0,\overline{\delta})$ we have for sufficiently small $\varepsilon > 0$

$$\max_{(s,\xi)\in[1-s_0,1+s_0]\times\overline{\Omega}} I_\varepsilon(\gamma_{0\varepsilon}(s,\xi)) \le b + \delta,$$
$$\max_{(s,\xi)\in\partial([1-s_0,1+s_0]\times\overline{\Omega})} I_\varepsilon(\gamma_{0\varepsilon}(s,\xi)) \le b - \overline{\delta}.$$

We also note that $\gamma_{0\varepsilon}(s,\xi) \in [T_\varepsilon \le \kappa_\varepsilon]$ for all $(s,\xi) \in [1-s_0, 1+s_0] \times \overline{\Omega}$. We define $F_\varepsilon : A^{(\varepsilon)}_{\rho_{**}} \to \mathbb{R} \times \mathbb{R}^N$ by

$$F_\varepsilon(u) = (P_{V_0}(u), \varepsilon\beta(u)).$$

Arguing as in the proof of Proposition 7.1, we can prove Proposition 7.2. □

7.3 End of the Proof of Theorem 1.2

Finally we derive our Theorem 1.2 from Propositions 7.1 and 7.2.

End of the proof of Theorem 1.2 Let V_0 be the critical value given by (MP) or (LM). Since $V(x) \in C^N(\mathbb{R}^N,\mathbb{R})$, by the Sard Theorem there exists a sequence $(\alpha_n)_{n=1}^\infty \subset (0,\infty)$ such that

(1) $\alpha_1 > \alpha_2 > \cdots > \alpha_n > \alpha_{n+1} > \cdots$;
(2) $\alpha_n \to 0$ as $n \to \infty$;
(3) $V_0 - \alpha_n$ is a regular value of $V(x)$.

We set

$$\Omega_n = \{x \in \Omega :\ V(x) > V_0 - \alpha_n\}.$$

We can see that (V1)–(V4) and (MP) or (LM) hold in Ω_n for large n (See Sect. 4.1). Thus we can apply the arguments in previous sections in Ω_n and, replacing Ω with

Ω_n, we prove Propositions 7.1 or 7.2 for Ω_n. That is, for any $\rho_* > 0$ and $\overline{\delta} > 0$ there exists $\varepsilon_0(n, \rho_*, \overline{\delta}) > 0$ such that for $\varepsilon \in (0, \varepsilon_0(n, \rho_*, \overline{\delta})]$, $I_\varepsilon(u)$ has a critical point u_ε in $N^{(\varepsilon)}_{n,\rho_*}$ with $I_\varepsilon(u_\varepsilon) \in [b - \overline{\delta}, b + \overline{\delta}]$. Precisely,

$$\begin{aligned}
\widehat{\mathcal{K}}_{b,n} &= \{(\xi, \omega) : \xi \in \Omega_n,\ DL(\xi, \omega) = 0,\ L(\xi, \omega) = b,\ \|\omega\|_{L^2(Q)} = \max_{n\in\mathbb{Z}^N} \|\omega\|_{L^2(n+Q)}\} \\
&= \{(\xi, \omega) : \xi \in \Omega,\ V(\xi) \in [V_0 - \alpha_n, V_0],\ \nabla V(\xi) = 0,\ L'_{V(\xi)}(\omega) = 0, \\
&\qquad L_{V(\xi)}(\omega) = b,\ \|\omega\|_{L^2(Q)} = \max_{n\in\mathbb{Z}^N} \|\omega\|_{L^2(n+Q)}\}, \\
\widehat{K}^{(\varepsilon)}_{b,n} &= \left\{\omega\left(x - \frac{\xi}{\varepsilon}\right) : (\xi, \omega) \in \widehat{\mathcal{K}}_{b,n}\right\}, \\
N^{(\varepsilon)}_{n,\rho_*} &= \{u \in H^1(\mathbb{R}^N) : \mathrm{dist}_\varepsilon(u, \widehat{K}^{(\varepsilon)}_{b,n}) < \rho_*\}.
\end{aligned}$$

We note that $\widehat{\mathcal{K}}_{b,n}$ shrinks to the following $\widehat{\mathcal{K}}_{b,\infty}$ as $n \to \infty$:

$$\begin{aligned}
\widehat{\mathcal{K}}_{b,\infty} &= \{(\xi, \omega) : \xi \in \Omega,\ V(\xi) = V_0,\ DL(\xi, \omega) = 0,\ L(\xi, \omega) = b, \\
&\qquad \|\omega\|_{L^2(Q)} = \max_{n\in\mathbb{Z}^N} \|\omega\|_{L^2(n+Q)}\} \\
&= \mathrm{Crit}_{V_0} \times \{\omega \in H^1(\mathbb{R}^N) : L'_{V_0}(\omega) = 0,\ L_{V_0}(\omega) = b, \\
&\qquad \|\omega\|_{L^2(Q)} = \max_{n\in\mathbb{Z}^N} \|\omega\|_{L^2(n+Q)}\}.
\end{aligned}$$

That is, $\mathrm{dist}_{\mathbb{R}^N\times H^1(\mathbb{R}^N)}(\widehat{\mathcal{K}}_{b,n}, \widehat{\mathcal{K}}_{b,\infty}) \to 0$ as $n \to \infty$. Now we can complete the proof of Theorem 1.2. We choose sequences $(\rho_{*n})_{n=1}^\infty$, $(\overline{\delta}_n)_{n=1}^\infty$ with $\rho_{*n} \to 0$, $\overline{\delta}_n \to 0$ as $n \to \infty$. Then there exists $\varepsilon_n = \varepsilon_0(n, \rho_{*n}, \overline{\delta}_n) > 0$ such that for $\varepsilon \in (0, \varepsilon_n]$, $I_\varepsilon(u)$ has a critical point $u_{n\varepsilon} \in N^{(\varepsilon)}_{n,\rho_{*n}}$ with $I_\varepsilon(u_{n\varepsilon}) \in [b - \overline{\delta}_n, b + \overline{\delta}_n]$. We may assume $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_n > \varepsilon_{n+1} > \cdots$ and $\varepsilon_n \to 0$ as $n \to \infty$. Finally we set

$$u_\varepsilon(x) = u_{n\varepsilon}(x) \quad \text{for } \varepsilon \in (\varepsilon_{n-1}, \varepsilon_n].$$

We observe that $(u_\varepsilon)_{\varepsilon\in(0,\varepsilon_1]}$ is the desired family of solutions. □

Proof of Theorem 1.3 Under the assumptions (V1') and (V1"), $V(x)$ has finitely many critical points in Ω. So there exists $\alpha > 0$ such that there are no critical values of $V\ |_\Omega$ in $[V_0 - \alpha, V_0 + \alpha]\backslash\{V_0\}$. Replacing Ω with

$$\{x \in \Omega : V(x) \in (V_0 - \alpha, V_0 + \alpha)\}$$

and arguing as in Sect. 4.1, we may assume that $x \in \Omega$ and $\nabla V(x) = 0$ imply $V(x) = V_0$. Thus for $b = E_{V_0}$

$$\widehat{\mathcal{K}}_b = \mathrm{Crit}_{V_0} \times \mathcal{C}_b,$$

where $\mathcal{C}_b$ is a set of least energy solutions of $L_{V_0}(u) = 0$, that is,

$$\mathcal{C}_b = \{\omega \in H^1(\mathbb{R}^N) : L_{V_0}(\omega) = b,\ L'_{V_0}(\omega) = 0,\ \|\omega\|_{L^2(Q)} = \max_{n\in\mathbb{N}^N} \|\omega\|_{L^2(n+Q)}\}.$$

Thus $N_\rho^{(\varepsilon)}$ is a ρ-neighborhood of

$$\widehat{K}_b^{(\varepsilon)} = \left\{ \omega\left(x - \frac{\xi}{\varepsilon} \right) : \xi \in \mathrm{Crit}_{V_0},\ \omega \in \mathcal{C}_b \right\}.$$

By the arguments in the proof of Propositions 7.1 and 7.2, for any ρ_* and $\overline{\delta} > 0$ there exists $\varepsilon_0 = \varepsilon_0(\rho_*, \overline{\delta}) > 0$ such that for $\varepsilon \in (0, \varepsilon_0]$, $I_\varepsilon(u)$ has a critical point in $N_{\rho_*}^{(\varepsilon)}$.

Taking sequences $(\rho_{*n})_{n=1}^\infty$, $(\overline{\delta}_n)_{n=1}^\infty$ with $\rho_{*n} \to 0$, $\overline{\delta}_n \to 0$ and arguing as in the proof of Theorem 1.2, we complete the proof of Theorem 1.3. □

7.4 Potential $V(x)$ of Class C^1

In previous sections we consider the situation where the set of critical values $\{V(x) : x \in \Omega,\ \nabla V(x) = 0\}$ is of measure 0, which is ensured by Sard Theorem for $V(x) \in C^N(\mathbb{R}^N, \mathbb{R})$. In this section we assume just $V(x) \in C^1(\mathbb{R}^N, \mathbb{R})$. Then the set of critical values may not be of measure 0.

We have the following weaker result.

Theorem 7.3 *Assume (f1)–(f4) and (V1'), (V2), (V3). Moreover suppose (LM) or (MP). Moreover assume (V4) in Sect. 4.1 for a constant V_0 appeared in (LM) or (MP). Then (1.1) has a family of solutions, which concentrates in Ω. That is, there exists $\varepsilon_0 > 0$ and a family $(u_\varepsilon)_{\varepsilon \in (0,\varepsilon_0]}$ of solutions of (1.2) with the following property: for any sequence $(\varepsilon_j)_{j=1}^\infty \subset (0, \varepsilon_0]$ with $\varepsilon_j \to 0$ after extracting a subsequence — still we denote it by ε_j — there exist $(x_j)_{j=1}^\infty \subset \mathbb{R}^N$, $x_0 \in \Omega$ and a non-trivial solution $\omega_0(x) \in H^1(\mathbb{R}^N)$ of the limit problem $-\Delta u + V(x_0)u = (I_\alpha * F(u))F'(u)$ in $\mathbb{R}^N$ such that*

$$\varepsilon_j x_j \to x_0, \quad u_j(x + x_j) \to \omega_0(x) \text{ strongly in } H^1(\mathbb{R}^N) \text{ as } j \to \infty.$$

Moreover, (x_0, ω_0) satisfies for $b = E_{V_0}$

$$\nabla V(x_0) = 0, \quad V(x_0) \le V_0, \quad \partial L(x_0, \omega_0) = 0, \quad L(x_0, \omega_0) = b.$$

In Theorem 7.3, the concentration point x_0 is a critical point of $V(x)$ in Ω but its critical level may be lower than V_0 in general.

Proof of Theorem 7.3 For $V_0 > 0$ given in (LM) or (MP) and let $b = E_{V_0} > 0$ be a least energy level for the limit functional $L_{V_0}(u)$. As in the previous sections, we set

$$\widehat{\mathcal{K}}_b = \{(\xi, \omega) \in \Omega \times H^1(\mathbb{R}^N) :\ DL(\xi, \omega) = 0,\ L(\xi, \omega) = b,\\ \|\omega\|_{L^2(Q)} = \max_{n \in \mathbb{Z}^N} \|\omega\|_{L^2(n+Q)}\}.$$

Then, following the proofs of Proposition 7.1 and 7.2, let $0 < \rho_* < \rho_{**}$ be the numbers satisfying (4.20). For any $\overline{\delta} > 0$ there exists $\varepsilon_0(\rho_*, \overline{\delta}) > 0$ such that for $\varepsilon \in (0, \varepsilon_0]$, $I_\varepsilon(u)$ has a critical point u in $\overline{\mathcal{N}_{\rho_*}^{(\varepsilon)}}$ satisfying $I_\varepsilon(u) \in [b - \overline{\delta}, b + \overline{\delta}]$.

Choosing sequences $(\rho_{*n})_{n=1}^{\infty}$, $(\overline{\delta}_n)_{n=1}^{\infty}$ with $\rho_{*n} \to 0$, $\overline{\delta}_n \to 0$, we complete the proof of Theorem 7.3. □

8 Concentration at a Local Minimum

In Sects. 1, 2, 3, 4, 5, and 6, we develop a deformation theory under our new version of Palais-Smale condition (see Proposition 4.5), i.e., if $(\varepsilon_j)_{j=1}^{\infty} \subset (0, 1]$ and $u_j \in A_{\rho_3}^{(\varepsilon_j)}$ satisfy as $j \to \infty$

$$\varepsilon_j \to 0, \quad I_{\varepsilon_j}(u_j) \to b, \quad I'_{\varepsilon_j}(u_j) \to 0 \text{ in } (H^1(\mathbb{R}^N))^*, \tag{8.1}$$

$$H_{\varepsilon_j}(u_j) \to 0, \tag{8.2}$$

then

$$\mathrm{dist}_{\varepsilon_j}(u_j, \widehat{K}_b^{(\varepsilon_j)}) \to 0. \tag{8.3}$$

And our deformation flow $\eta(t, u)$ is constructed through a deformation in the augmented space $\mathbb{R}^N \times H^1(\mathbb{R}^N)$. When a stronger version of Palais-Smale condition, i.e., if (8.3) holds under (8.1) (without (8.2)), we can construct the desired flow directly as a deformation in $H^1(\mathbb{R}^N)$.

We note that for the functional $I_\varepsilon(u)$ corresponding to the nonlinear Choquard equation (1.2) under the conditions (f1)–(f4), (V2) and

($\widetilde{\mathrm{V}1}$) $V \in C(\mathbb{R}^N, \mathbb{R})$;

($\widetilde{\mathrm{LM}}$) There exists a bounded connected open set $\Omega \subset \mathbb{R}^N$ such that

$$V_0 \equiv \inf_{x\in\Omega} V(x) < \inf_{x\in\partial\Omega} V(x),$$

the compactness (8.3) holds under (8.1). This fact is essentially given in Proposition 4.1 in [23].

In fact, if (8.3) holds under (8.1) and if

$$I'_\varepsilon(u) \neq 0 \quad \text{for all } u \in N_{\rho_*}^{(\varepsilon)} \text{ with } I_\varepsilon(u) \in [b - \delta_0, b + \delta_0],$$

then for any $\rho_*, \rho_{**} > 0$ with (4.20) and for $\varepsilon > 0$ small there exist constants $\nu_\varepsilon > 0$ depending on ε and $\nu_0 > 0$ independent of ε and a locally Lipschitz continuous vector field

$$W(u) : A_{\rho_{**}}^{(\varepsilon)} \to H^1(\mathbb{R}^N)$$

such that

(i) For $T_\varepsilon(u) : \widehat{S}_{b,\rho_\varepsilon} \to \mathbb{R}$ defined (5.1),

$$T'_\varepsilon(u)W(u) > 0 \quad \text{if } T_\varepsilon(u) \geq \kappa_\varepsilon.$$

(ii) For $u \in A^{(\varepsilon)}_{\rho_{**}}$ with $I_\varepsilon(u) \in [b-\delta_0, b+\delta_0]$

$$I'_\varepsilon(u)W(u) \geq \nu_\varepsilon.$$

(iii) For $u \in A^{(\varepsilon)}_{\rho_{**}} \setminus \overline{N^{(\varepsilon)}_{\rho_*}}$ with $I_\varepsilon(u) \in [b-\delta_0, b+\delta_0]$

$$I'_\varepsilon(u)W(u) \geq \nu_0.$$

(iv) There exist $C, C' > 0$ such that for $M_1(u)$, $M_2(u)$ given in (5.3)

$$\|M'_1(u)W(u)\|_{H^1} < C,$$
$$\partial_u(\|M_2(u)\|^2_{H^1})W(u) > -C'.$$

Here we use the arguments in Sects. 5 and 6. We obtain a deformation flow $\eta(t,u)$: $[0,1] \times A^{(\varepsilon)}_{\rho_{**}} \to A^{(\varepsilon)}_{\rho_{**}}$ with the properties (i)–(v) in Proposition 6.1 as a solution of ODE in $H^1(\mathbb{R}^N)$:

$$\frac{d\eta}{dt} = -\varphi_1(I_\varepsilon(\eta))\varphi_2(\eta)W(\eta), \qquad \eta(0,u) = u,$$

where $\varphi_1(s) : \mathbb{R} \to [0,1]$, $\varphi_2(u) : H^1(\mathbb{R}^N) \to [0,1]$ are suitable cut-off functions. Thus we have the following result.

Theorem 8.1 (Theorem 1.1 of [23]). *Assume the conditions (f1)–(f4) and ($\widetilde{V1}$), (V2), ($\widetilde{LM}$). Then* (1.1) *has at least one positive solution concentrating in* Ω.

Remark 8.2 In [23], we study the existence of solutions of (1.1) concentrating in a potential well Ω, i.e., under ($\widetilde{\mathrm{LM}}$) using 2 flows; one flow is the standard gradient flow corresponding to $-I'_\varepsilon(u)$ and the other is the tail minimizing flow. We can give a simplified proof to the result in [23] using our deformation flow $\eta(t,u)$, which keeps the size $T_\varepsilon(u)$ of tail of functions small and we can show the existence of critical points using just one flow $\eta(t,u)$. We note that in [23] we also study the multiplicity of solutions using cup length of the critical set $K = \{x \in \Omega : V(x) = V_0\}$.

Remark 8.3 Our deformation argument can be applied to various singular perturbation problems. For example, it is applicable to the following nonlinear Schrödinger equations:

$$-\varepsilon^2\Delta u + V(x)u = g(u) \quad \text{in } \mathbb{R}^N, \tag{8.4}$$

where $N \geq 2$, $g(\xi) \in C(\mathbb{R},\mathbb{R})$.

We can use our new deformation argument to improve results in [8] slightly and to simplify the proofs and arguments (c.f. [26]). In [8], Byeon and the second author studied (8.4) under the assumption $g(\xi) \in C^1(\mathbb{R},\mathbb{R})$, which is used to solve elliptic problems (1.6) outside of a large ball uniquely. By virtue of our new deformation flow obtained in Proposition 6.1, which keeps the H^1-energy small outside a ball, we don't need to solve the elliptic problems outside of a ball uniquely and we can relax the regularity assumption on g to the class C^0.

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