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On a class of supercritical N-Laplacian problems ⋆

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Abstract

We derive a new existence result for a class of N–Laplacian problems where the classical N–Laplacian is replaced by an operator which admits some coefficients depending on the solution itself. Even if such coefficients make the variational approach more difficult, a suitable supercritical growth for the nonlinear term is allowed. Our proof, which exploits the interaction between two different norms, is based on a weak version of the Cerami–Palais–Smale condition and a proper decomposition of the ambient space. Then, a suitable generalization of the Ambrosetti–Rabinowitz Mountain Pass Theorem allows us to establish the existence of at least one nontrivial bounded solution.

Keywords: Supercritical N-Laplacian problem, weak Cerami–Palais–Smale condition, Mountain Pass Theorem, Trudinger-Moser inequality

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1. Introduction

A classical N–Laplacian problem is

$$\begin{cases} -\Delta_N u \;=\; h(u)\, \mathrm{e}^{\alpha|u|^\gamma} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

with Ω open bounded domain in $\mathbb{R}^N$, $N \geq 2$ and $\Delta_N u = \operatorname{div}(|\nabla u|^{N-2}\nabla u)$ standard N–Laplacian operator, where α, $\gamma > 0$ are given constants and $h : \mathbb{R} \to \mathbb{R}$ is a continuous function which has a "subexponential" growth at infinity, i.e., which is so that

$$\lim_{|t|\to+\infty} \frac{h(t)}{\mathrm{e}^{\delta|t|^\gamma}} \;=\; 0 \qquad \forall \delta > 0.$$

It is well known that this problem is governed by the Trudinger–Moser inequality

$$\sup_{\substack{u\in W_0^{1,N}(\Omega)\\ \|u\|_N\leq 1}} \int_\Omega \mathrm{e}^{\alpha_N|u|^{N'}} dx < +\infty, \tag{1.2}$$

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where $W_0^{1,N}(\Omega)$ is the usual Sobolev space with norm $\|u\|_N = \left(\int_\Omega |\nabla u|^N\, dx\right)^{1/N}$, $\alpha_N = N\omega_{N-1}^{1/(N-1)}$, ω_{N-1} is the area of the unit sphere in $\mathbb{R}^N$ and $N' = N/(N-1)$ is the Hölder conjugate of N (see Trudinger [15] and Moser [14]), so (1.1) has a subcritical growth if $\gamma < N'$ while critical if $\gamma = N'$.

Existence of solutions of problem (1.1) in both the subcritical and the critical case has been widely studied in the literature (see, e.g., [1, 2, 5, 6, 7, 8, 9, 10, 16] and references therein). On the other hand, several physical phenomena, for example in the theory of superfluid film and in dissipative quantum mechanics, are described by equations where the principal part is a model of $-\operatorname{div}\left[(A_0(x) + A(x)\,|u|^{ps})\,|\nabla u|^{p-2}\,\nabla u\right]$ (for more details, see [13] and references therein) and some existence results have been obtained for nonlinear terms with a suitable supercritical growth when such a coefficient appears (see [4]).

Thus, in the present paper, we prove the existence of weak bounded solutions to the quasilinear problem

$$\begin{cases} -\operatorname{div}\left[(A_0(x) + A(x)\,|u|^{Ns})\,|\nabla u|^{N-2}\,\nabla u\right] + s\,A(x)\,|u|^{Ns-2}\,u\,|\nabla u|^N \\ \qquad\qquad = h(u)\,\mathrm{e}^{\alpha|u|^\gamma} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $s > 1/N$ is a constant and the coefficients A_0, $A \in L^\infty(\Omega)$ are strictly positive and far away from zero, while the continuous function $h(u)$ has a subexponential growth at infinity and satisfies suitable estimates together with the primitive of the nonlinear term (for the complete statement, see Theorem 2.5).

In our proof, we use a variational approach which is strongly affected by the presence of the coefficient $A_0(x) + A(x)\,|u|^{Ns}$. In fact, in order to find weak solutions of (1.3), the "natural" functional $E(u)$, defined in (2.8), is not smooth in the classical Sobolev space $W_0^{1,N}(\Omega)$ and, in order to overcome such a problem, a suitable variational principle is stated but in the intersection space $X = W_0^{1,N}(\Omega) \cap L^\infty(\Omega)$ (see Proposition 2.6). Unluckily, in such a setting we cannot apply directly an existence result such as the classical Ambrosetti–Rabinowitz theorem. Indeed, X is not reflexive and our functional may not satisfy the Palais–Smale condition, or its Cerami's variant, as some of its Palais–Smale sequences may converge in $W_0^{1,N}(\Omega)$ but not in $L^\infty(\Omega)$. These problems are solved by considering the interplay of two different norms on the Banach space X: we have to weaken the definition of Palais–Smale condition (see Definition 2.1), which can be proved for our functional $E(u)$ in X just making use of the reflexivity of $W_0^{1,N}(\Omega)$ (see Proposition 3.4), and use it for stating a generalized version of the Mountain Pass Theorem (see Theorem 2.2) .

Anyway, even if the term $|u|^{Ns}$ in the coefficient makes the variational approach more difficult, it can allow the nonlinear term to be supercritical as it makes "stronger" the principal part.

Thus, here, the novelty with respect to previous papers, is not only the presence of a coefficient but also that problem (1.3) has weak solutions for $0 < \gamma < (s+1)N'$, which includes also the supercritical range $N' < \gamma < (s+1)N'$.

2. Preliminaries

Firstly, let us point out the abstract setting we deal with.

Let $(X, \|\cdot\|_X)$ be a Banach space with dual space $(X', \|\cdot\|_{X'})$ and consider another Banach space $(W, \|\cdot\|_W)$ such that $X \hookrightarrow W$ continuously, i.e. $X \subset W$ and a constant $\sigma_0 > 0$ exists such that

$$\|u\|_W \;\leq\; \sigma_0\, \|u\|_X \qquad \text{for all } u \in X.$$

Let $J : X \subset W \to \mathbb{R}$ be a C^1 functional and, taking $c \in \mathbb{R}$, we recall that a sequence $(u_n)_n \subset X$ is a Cerami–Palais–Smale sequence for J at level c, briefly $(CPS)_c$–sequence, if

$$\lim_{n\to+\infty} J(u_n) = c \quad \text{ and } \quad \lim_{n\to+\infty} \|dJ(u_n)\|_{X'}(1 + \|u_n\|_X) = 0.$$

As $(CPS)_c$ sequences may exist which are unbounded in $\|\cdot\|_X$ but converge with respect to $\|\cdot\|_W$, we have to weaken the classical Cerami–Palais–Smale condition in a suitable way according to the ideas already developed in previous papers (see, e.g., [3]).

Definition 2.1. The functional J satisfies the *weak Cerami–Palais–Smale condition at level* c ($c \in \mathbb{R}$), briefly $(wCPS)_c$ *condition*, if for every $(CPS)_c$–sequence $(u_n)_n$, a point $u \in X$ exists, such that

(i) $\lim_{n\to+\infty} \|u_n - u\|_W = 0$ (up to subsequences),

(ii) $J(u) = c,\ dJ(u) = 0$.

If J satisfies the $(wCPS)_c$ condition at each level $c \in I$, I real interval, we say that J satisfies the $(wCPS)$ condition in I.

Definition 2.1 allows us to state the following generalization of the Ambrosetti–Rabinowitz Mountain Pass Theorem (see [4, Theorem 2.3]).

Theorem 2.2. *Let $J \in \mathcal{C}^1(X, \mathbb{R})$ be such that $J(0) = 0$ and the $(wCPS)$ condition holds in $\mathbb{R}_+$. Moreover, assume that there exist a continuous map $\ell : X \to \mathbb{R}$, some constants r_0, $\varrho_0 > 0$, and a point $\bar{u} \in X$ such that*

(i) $\ell(0) = 0$ *and* $\ell(u) \geq \|u\|_W$ *for all* $u \in X$*;*

(ii) $u \in X, \quad \ell(u) = r_0 \quad \Longrightarrow \quad J(u) \geq \varrho_0$*;*

(iii) $\|\bar{u}\|_W > r_0$ *and* $J(\bar{u}) < \varrho_0$*.*

Then, J has a Mountain Pass critical point $u^* \in X$ such that $J(u^*) \geq \varrho_0$.

Now, we are able to give the set of hypotheses we need for our problem. In particular, from now on we assume that $\Omega \subset \mathbb{R}^N$ is an open bounded domain, $N \geq 2$, and we denote by:

- $L^r(\Omega)$ the Lebesgue space with norm $|u|_r = \left(\int_\Omega |u|^r dx\right)^{1/r}$ if $1 \leq r < +\infty$;
- $L^\infty(\Omega)$ the space of Lebesgue–measurable and essentially bounded functions $u : \Omega \to \mathbb{R}$ with norm $|u|_\infty = \operatorname*{ess\,sup}_\Omega |u|$;

- $W_0^{1,N}(\Omega)$ the classical Sobolev space with norm

$$\|u\|_N := |\nabla u|_N = \left(\int_\Omega |\nabla u|^N dx\right)^{\frac{1}{N}};$$

- $|D|$ the usual N–dimensional Lebesgue measure of a measurable set $D \subset \mathbb{R}^N$;

- c_i any strictly positive constant arising during computations.

Furthermore, as pointed out in Section 1, we denote by:

- ω_{N-1} the area of the unit sphere in $\mathbb{R}^N$;

- $\alpha_N = N\omega_{N-1}^{1/(N-1)}$ the best constant for the Trudinger–Moser inequality;

- $N' = N/(N-1)$ the Hölder conjugate of N.

In problem (1.3), let us suppose that the exponents s, α and γ are such that

$$s > \frac{1}{N} \tag{2.1}$$

and also

$$\alpha > 0, \qquad 0 < \gamma < (s+1)N'. \tag{2.2}$$

Moreover, assume that the coefficients $A_0 : \Omega \to \mathbb{R}$ and $A : \Omega \to \mathbb{R}$ are such that:

(h_1) A_0, $A \in L^\infty(\Omega)$ and a constant $\alpha_0 > 0$ exists such that

$$A_0(x) \geq \alpha_0 \quad \text{ and } \quad A(x) \geq \alpha_0 \qquad \text{for a.a. } x \in \Omega.$$

On the other hand, for function $h : \mathbb{R} \to \mathbb{R}$ and the related primitive

$$G(t) = \int_0^t h(v)\mathrm{e}^{\alpha|v|^\gamma} dv, \tag{2.3}$$

which is well defined if $h(t)$ is continuous and $\gamma \geq 0$, we consider the following conditions:

(h_2) $h \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ is such that

$$\lim_{|t|\to+\infty} \frac{h(t)}{\mathrm{e}^{\delta|t|^\gamma}} = 0 \quad \forall \delta > 0, \tag{2.4}$$

and some constants $\bar{\delta}$, σ_1, $\sigma_2 > 0$ and a power $0 \leq q < N(s+1)$ exist such that

$$N(s+1)(1+\bar{\delta})G(t) \; - \; t\, h(t)\, \mathrm{e}^{\alpha|t|^\gamma} \; \leq \; \sigma_1|t|^q + \sigma_2 \quad \text{ for all } t \in \mathbb{R}; \tag{2.5}$$

(g_1) some constants σ_3, $\sigma_4 > 0$ and a power $\tau > N(s+1)$ exist such that

$$G(t) \geq \sigma_3|t|^\tau - \sigma_4 \quad \text{ for all } t \in \mathbb{R};$$

(g_2) some constants σ, $\nu > 0$ exist such that

$$G(t) \leq \left(\frac{\lambda_1\ \alpha_0}{N} - \sigma\right)|t|^N \quad \text{ if } |t| \leq \nu,$$

where λ_1 denotes the first eigenvalue of $-\Delta_N$ in $W_0^{1,N}(\Omega)$ and α_0 is as in (h_1).

Remark 2.3. If conditions (2.1) and (h_1) are verified, then

$$\mathcal{A}(x,t,\xi) = \frac{1}{N}\,(A_0(x) + A(x)|t|^{Ns})|\xi|^N$$

is a $\mathcal{C}^1$–Carathéodory function on $\Omega \times \mathbb{R} \times \mathbb{R}^N$ such that for a.e. $x \in \Omega$ and all $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$ we have that

$$\frac{\partial \mathcal{A}}{\partial t}(x,t,\xi) = sA(x)|t|^{Ns-2}t|\xi|^N \quad \text{and} \quad \nabla_\xi \mathcal{A}(x,t,\xi)\cdot\xi = N\mathcal{A}(x,t,\xi)$$

with also

$$\nabla_\xi \mathcal{A}(x,t,\xi)\cdot\xi \geq \alpha_0(1+|t|^{Ns})|\xi|^N \quad \text{and} \quad \frac{\partial \mathcal{A}}{\partial t}(x,t,\xi)\; t = sA(x)|t|^{Ns}|\xi|^N \geq 0.$$

Hence, the hypotheses (H_0)–(H_5) and (H_7) in [4] are all satisfied.

Example 2.4. Possible examples of function $h(t)$ which satisfy all the previous assumptions are

$$h_r(t) \;=\; \begin{cases} |t|^{r-2}t & \text{if } r > N \\ \beta_0|t|^{N-2}t & \text{if } r = N \text{ but with } 0 < \beta_0 < \lambda_1\alpha_0. \end{cases}$$

In fact, taking $G_r(t)$ as in (2.3) with $h(t) = h_r(t)$, in both cases assumption (h_2) is satisfied as $\gamma > 0$ and direct computations imply that

$$\lim_{|t|\to+\infty} \frac{G_r(t)}{th_r(t)\mathrm{e}^{\alpha|t|^\gamma}} = 0,$$

while (g_1) follows from well known properties of the exponential map and (g_2) is a direct consequence of

$$\lim_{|t|\to 0} \frac{G_r(t)}{|t|^N} = 0 \quad \text{if } r > N, \qquad \lim_{|t|\to 0} \frac{G_N(t)}{|t|^N} = \frac{\beta_0}{N} \quad \text{with } \beta_0 < \lambda_1\alpha_0.$$

Now, we are ready to state our main existence result.

Theorem 2.5. *Let us suppose that the exponents s, α and γ are such that* (2.1) *and* (2.2) *hold and assume that the hypotheses (h_1), (h_2), (g_1) and (g_2) are satisfied, too. Then, problem* (1.3) *possesses at least one bounded nontrivial weak solution.*

Our aim is investigating the existence of weak solutions of problem (1.3) by means of the abstract variational tools introduced at the beginning of this section and which involve two different Banach spaces. Thus, as first Banach space we consider

$$X \;=\; W_0^{1,N}(\Omega) \cap L^\infty(\Omega) \tag{2.6}$$

endowed with the intersection norm

$$\|u\|_X = \|u\|_N + |u|_\infty, \quad u \in X,$$

while as second Banach space we consider

$$W = W_0^{1,N}(\Omega) \quad \text{ with } \|u\|_W = \|u\|_N.$$

Clearly, it is $X \hookrightarrow W_0^{1,N}(\Omega)$ continuously with $\|u\|_N \;\leq\; \|u\|_X$ for all $u \in X$. Moreover, since

$$|\nabla(|u|^s u)|^N \;=\; (s+1)^N\, |u|^{Ns}\, |\nabla u|^N \quad \text{a.e. in } \Omega, \tag{2.7}$$

for every $u \in X$ we have that

$$\int_\Omega |\nabla(|u|^s u)|^N dx \;=\; (s+1)^N \int_\Omega |u|^{Ns}\, |\nabla u|^N dx \le (s+1)^N\, |u|_\infty^{Ns}\, \|u\|_N^N \le (s+1)^N\, \|u\|_X^{N(s+1)}$$

which implies

$$\||u|^s u\|_N \;\le\; (s+1)\, \|u\|_X^{s+1}$$

and $|u|^s\, u \in W_0^{1,N}(\Omega)$, too.

So, if conditions (2.1) and (h_1) are verified and if $\gamma > 0$ and $h(t)$ is a continuous map, then the functional

$$E(u) = \frac{1}{N}\int_\Omega (A_0(x) + A(x)|u|^{Ns})|\nabla u|^N dx \;-\; \int_\Omega G(u)dx \tag{2.8}$$

is well defined for all $u \in X$ with $G(u)$ as in (2.3). Or better, if taking any $u, v \in X$, we consider the Gateaux derivative of the functional E in u along the direction v given by

$$\begin{aligned}\langle dE(u), v\rangle = &\int_\Omega (A_0(x) + A(x)|u|^{Ns})|\nabla u|^{N-2}\nabla u \cdot \nabla v\; dx\\ &+ s\int_\Omega A(x)|u|^{Ns-2}u\; v\; |\nabla u|^N dx \;-\; \int_\Omega h(u)\mathrm{e}^{\alpha|u|^\gamma} v\; dx,\end{aligned} \tag{2.9}$$

the following regularity result holds.

Proposition 2.6. *Assume that* (2.1) *and* (h_1) *are verified and let* $\gamma > 0$ *and* $h \in \mathcal{C}(\mathbb{R},\mathbb{R})$*. Then, if* $(u_n)_n \subset X$*,* $u \in X$ *are such that*

$$\|u_n - u\|_W \to 0, \quad u_n \to u \textit{ a.e. in } \Omega \qquad \textit{if } n \to +\infty$$

$$\textit{and } M > 0 \textit{ exists so that } |u_n|_\infty \le M \textit{ for all } n \in \mathbb{N},$$

then

$$E(u_n) \to E(u) \quad \textit{and} \quad \|dE(u_n) - dE(u)\|_{X'} \to 0 \quad \textit{if} \;\; n \to +\infty.$$

Hence, $E : X \to \mathbb{R}$ *is a* $\mathcal{C}^1$ *functional with Fréchet derivative defined as in* (2.9).

Proof. The proof follows from [4, Proposition 3.2] by means of Remark (2.3) and the Lebesgue's Dominated Convergence Theorem. □

3. The weak Cerami–Palais–Smale condition

The main purpose of this section is showing that the functional $E(u)$, defined as in (2.8), complies with Definition 2.1 on the Banach space X in (2.6). To this aim, we introduce some crucial lemmas.

Firstly, we note that from the classical Trudinger–Moser inequality (1.2) and equality (2.7) in X we obtain a variant of Trudinger–Moser inequality but dealing with the Banach space X:

$$K_N := \sup_{\substack{u\in X\\ \||u|^s u\|_N \le 1}} \int_\Omega \mathrm{e}^{\alpha_N |u|^{(s+1)N'}} dx \;<\; +\infty. \tag{3.1}$$

Such an estimate allows us to obtain the following lemma which is useful in our setting.

Lemma 3.1. *If power γ is such that (2.2) holds, then*

$$K_\beta := \sup_{\substack{u\in X\\ \||u|^s u\|_N\le 1}} \int_\Omega \mathrm{e}^{\beta|u|^\gamma} dx < +\infty \quad \textit{for every } \beta>0. \tag{3.2}$$

Moreover, if $(u_n)_n \subset X$ is such that

$$\||u_n|^s u_n\|_N \le c_* \quad \textit{for all } n\in\mathbb{N} \tag{3.3}$$

for some constant $c_*>0$, then for every $\beta>0$ a constant $k^*_\beta = k^*_\beta(c^*)>0$ exists such that

$$\int_\Omega \mathrm{e}^{\beta|u_n|^\gamma} dx \ \le\ k^*_\beta \quad \textit{for all } n\in\mathbb{N}. \tag{3.4}$$

Proof. Let $\beta>0$ be fixed. Then, since (2.2) implies $p = \frac{(s+1)N'}{\gamma} > 1$, from the Young inequality applied to such a power, for any $\varepsilon>0$ a constant $C_\varepsilon = C_\varepsilon(\beta)>0$ exists such that

$$\beta|t|^\gamma \le \varepsilon|t|^{(s+1)N'} + C_\varepsilon \quad \text{for all } t\in\mathbb{R}. \tag{3.5}$$

Thus, if $u\in X$ is such that $\||u|^s u\|_N \le 1$, taking $\varepsilon = \alpha_N$ in (3.5), from (3.1) we obtain that

$$\int_\Omega \mathrm{e}^{\beta|u|^\gamma} dx \ \le\ \int_\Omega \mathrm{e}^{\alpha_N|u|^{(s+1)N'}+C_{\alpha_N}} dx \ \le\ \mathrm{e}^{C_{\alpha_N}}\ K_N$$

which implies (3.2).

Now, let $(u_n)_n\subset X$ be such that (3.3) holds and, without loss of generality, we assume that $u_n\neq 0$ for all $n\in\mathbb{N}$, so that we can put

$$w_n = \frac{|u_n|^s u_n}{\||u_n|^s u_n\|_N}$$

with $\|w_n\|_N = 1$. Thus, denoting

$$\varepsilon_N := \frac{\alpha_N}{c_*^{N'}}$$

and taking $\varepsilon=\varepsilon_N$ in (3.5), from (3.3) and direct computations it follows that

$$\int_\Omega \mathrm{e}^{\beta|u_n|^\gamma} dx \ \le\ \int_\Omega \mathrm{e}^{\varepsilon_N \||u_n|^s u_n\|_N^{N'}\ |w_n|^{N'}+C_{\varepsilon_N}} dx \le \mathrm{e}^{C_{\varepsilon_N}} \int_\Omega \mathrm{e}^{\alpha_N|w_n|^{N'}} dx$$

which, together with (1.2), implies (3.4). □

Now, in order to consider the supercritical growth, we need the following application of the Rellich Embedding Theorem.

Lemma 3.2. *Taking $p=N$ and $s>0$, let $(u_n)_n\subset X$ be a sequence such that*

$$\left(\int_\Omega (1+|u_n|^{Ns})\ |\nabla u_n|^N dx\right)_n \quad \textit{is bounded.}$$

Then, $u\in W_0^{1,N}(\Omega)$ exists such that $|u|^s u\in W_0^{1,N}(\Omega)$, too, and, up to subsequences, if $n\to+\infty$ we have

$$u_n \rightharpoonup u \ \textit{weakly in } W_0^{1,N}(\Omega), \tag{3.6}$$

$$|u_n|^s u_n \rightharpoonup |u|^s u \ \textit{weakly in } W_0^{1,N}(\Omega), \tag{3.7}$$

$$u_n \to u \ \textit{a.e. in } \Omega, \tag{3.8}$$

$$u_n \to u \ \textit{strongly in } L^r(\Omega) \textit{ for } 1\le r<+\infty. \tag{3.9}$$

Proof. For the proof, it is enough reasoning as in [4, Lemma 3.8] but with $p^* = +\infty$. $\square$

The last technical lemma we recall is necessary for proving the boundedness of the weak limit in $W_0^{1,N}(\Omega)$ of a (CPS)-sequence (for the proof, see [11, Theorem II.5.1]).

Lemma 3.3. *Take $v \in W_0^{1,N}(\Omega)$ and assume that $L_0 > 0$ and $k_0 \in \mathbb{N}$ exist such that*

$$\int_{\Omega_{\tilde{k}}^+} |\nabla v|^N dx \leq L_0 \left(\int_{\Omega_{\tilde{k}}^+} (v - \tilde{k})^r dx \right)^{N/r} + L_0 \sum_{i=1}^{m} \tilde{k}^{l_i} |\Omega_{\tilde{k}}^+|^{\varepsilon_i} \quad \text{for all } \tilde{k} \geq k_0,$$

with $\Omega_{\tilde{k}}^+ = \{x \in \Omega : v(x) > \tilde{k}\}$, where r, m, l_i, ε_i ($i \in \{1, \dots, m\}$), are such that

$$1 \leq r < +\infty, \quad \varepsilon_i > 0, \quad N \leq l_i < +\infty.$$

Then, $\operatorname{ess\,sup}_\Omega v$ *is bounded from above by a positive constant which can be chosen so that it depends only on $|\Omega|$, N, r, L_0, k_0, m, ε_i, l_i and $|v|_{L^1(\Omega_{k_0}^+)}$.*

Now, we are able to prove that our functional $E(u)$ satisfies the weak Cerami–Palais–Smale condition.

Proposition 3.4. *Under assumptions* (2.1), (2.2), (h_1), (h_2) *and* (g_1), *the functional $E : X \to \mathbb{R}$ as in* (2.8) *satisfies the* $(wCPS)$ *condition in* $\mathbb{R}$.

Proof. Let $c \in \mathbb{R}$ and let $(u_n)_n \subset X$ be a $(CPS)_c$ sequence, i.e.

$$E(u_n) \to c \quad \text{and} \quad \|dE(u_n)\|_{X'}(1 + \|u_n\|_X) \to 0 \quad \text{as } n \to +\infty. \tag{3.10}$$

For simplicity, our proof is divided in several steps; more precisely, we will prove that:

1. $(u_n)_n$ is bounded in $W_0^{1,N}(\Omega)$, or to be more precise, that
$$\left(\int_\Omega (1 + |u_n|^{Ns}) |\nabla u_n|^N dx \right)_n \quad \text{is bounded;} \tag{3.11}$$
hence, also $(\||u_n|^s u_n\|_N)_n$ is bounded in $W_0^{1,N}(\Omega)$ and $u \in W_0^{1,N}(\Omega)$ exists such that $|u|^s u \in W_0^{1,N}(\Omega)$, too, and, up to subsequences, the limits (3.6)–(3.9) hold;
2. $u \in L^\infty(\Omega)$;
3. if $k > |u|_\infty + 1$ is large enough, then $(T_k(u_n))_n$ is a Palais–Smale sequence at level c, that is
$$\|dE(T_k(u_n))\|_{X'} \to 0 \tag{3.12}$$
and
$$E(T_k(u_n)) \to c, \tag{3.13}$$
where $T_k : \mathbb{R} \to \mathbb{R}$ is defined as
$$T_k t := \begin{cases} t & \text{if } |t| \leq k \\ k \frac{t}{|t|} & \text{if } |t| > k \end{cases}; \tag{3.14}$$

4. $\|T_k u_n - u\|_N \to 0$, and then $\|u_n - u\|_N \to 0$, too;
5. $E(u) = c$ and $dE(u) = 0$.

We use the notation $(\varepsilon_n)_n$ for any infinitesimal sequence depending only on $(u_n)_n$ and $(\varepsilon_{k,n})_n$ when the infinitesimal sequence depends also on a constant k.

Step 1. Taking $\bar{\delta} > 0$ as in (2.5), from (2.8), (2.9), (3.10), (h_1), (2.5) and direct computations it follows that

$$\begin{aligned} N(s+1)(1+\bar{\delta})c + \varepsilon_n \; &= N(s+1)(1+\bar{\delta})E(u_n) - \langle dE(u_n), u_n\rangle \\ &\geq (s+1)\bar{\delta}\ \alpha_0 \int_\Omega (1+|u_n|^{Ns})|\nabla u_n|^N dx \\ &\quad + \int_\Omega \Big(u_n h(u_n) \mathrm{e}^{\alpha |u_n|^\gamma} - N(s+1)(1+\bar{\delta})G(u_n)\Big) dx \\ &\geq (s+1)\bar{\delta}\ \alpha_0 \int_\Omega (1+|u_n|^{Ns})|\nabla u_n|^N dx - \sigma_1 \int_\Omega |u_n|^q dx \ - \ \sigma_2 |\Omega|. \end{aligned} \tag{3.15}$$

Then, if without loss of generality we suppose $q \geq 1+s$, from the Sobolev Embedding Theorem and direct computations we have that

$$\int_\Omega |u_n|^q dx \leq c_1 \||u_n|^s u_n\|_N^{\frac{q}{s+1}};$$

hence, since $q < N(s+1)$, from (3.15) we conclude that (3.11) is satisfied and then, from Lemma 3.2, a function $u \in W_0^{1,N}(\Omega)$ exists which satisfies all the requirements in *Step 1.*

Step 2. Arguing by contradiction, assume that $u \notin L^\infty(\Omega)$; hence, either

$$\operatorname*{ess\,sup}_\Omega u = +\infty \tag{3.16}$$

or

$$\operatorname*{ess\,sup}_\Omega (-u) = +\infty. \tag{3.17}$$

For example, suppose that (3.16) holds and for any fixed $\tilde{k} \in \mathbb{N}$ we define the function

$$R_{\tilde{k}}^+ t = \begin{cases} 0 & \text{if } t \leq \tilde{k} \\ t - \tilde{k} & \text{if } t > \tilde{k} \end{cases}. \tag{3.18}$$

Now, taking any $\delta > 0$ such that the limit in (2.4) is verified, a radius $R_1 > 0$ exists such that

$$|h(t)| \leq \mathrm{e}^{\delta |t|^\gamma} \quad \text{if } |t| \geq R_1. \tag{3.19}$$

Thus, taking any integer $k \geq R_1$, from (3.16) we have that

$$|\Omega_k^+| > 0, \qquad \text{with} \quad \Omega_k^+ \ := \ \{x \in \Omega :\ u(x) > k\}. \tag{3.20}$$

So, if we take $\tilde{k} = k^{s+1}$ in definition (3.18), from (3.7) it follows that

$$R_{k^{s+1}}^+(|u_n|^s u_n) \rightharpoonup R_{k^{s+1}}^+(|u|^s u) \quad \text{weakly in } W_0^{1,N}(\Omega)$$

which implies, by the sequentially weakly lower semicontinuity of $\|\cdot\|_N$, that

$$\int_{\Omega_k^+} |\nabla u^{s+1}|^N dx \;\leq\; \liminf_{n\to+\infty} \int_{\Omega_{n,k}^+} |\nabla u_n^{s+1}|^N dx, \tag{3.21}$$

where the set Ω_k^+ defined in (3.20) is such that

$$\Omega_k^+ = \{x \in \Omega : \; |u|^s u > k^{s+1}\} \tag{3.22}$$

and also

$$\Omega_{n,k}^+ := \{x \in \Omega : \; u_n(x) > k\} \quad \text{is such that} \quad \Omega_{n,k}^+ = \{x \in \Omega : \; |u_n|^s u_n > k^{s+1}\}.$$

On the other hand, from definition (3.18) with $\tilde{k} = k$, we have that $(R_k^+ u_n)_n \subset X$ and $\|R_k^+ u_n\|_X \leq \|u_n\|_X$, while from (3.9) we obtain

$$R_k^+ u_n \;\to\; R_k^+ u \qquad \text{strongly in } L^2(\Omega); \tag{3.23}$$

hence, (3.10) gives

$$|\langle dE(u_n), R_k^+ u_n \rangle| \to 0. \tag{3.24}$$

From (3.24) together with (3.20), it follows that $n_k \in \mathbb{N}$ exists such that

$$|\langle dE(u_n), R_k^+ u_n \rangle| \leq |\Omega_k^+| \qquad \text{for all } n \geq n_k. \tag{3.25}$$

We note that, from (2.3), (2.9), definition (3.18) and hypothesis (h_1) we have that (2.7) and direct computations give

$$\begin{aligned}
\langle dE(u_n), R_k^+ u_n \rangle \;&=\; \int_{\Omega_{n,k}^+} (A_0(x) + A(x)|u_n|^{Ns}) |\nabla u_n|^N dx \\
&\quad + s \int_{\Omega_{n,k}^+} A(x)|u_n|^{Ns-2} u_n (u_n - k) |\nabla u_n|^N dx \;-\; \int_\Omega h(u_n) \mathrm{e}^{\alpha |u_n|^\gamma} R_k^+ u_n dx \\
&\geq \alpha_0 \int_{\Omega_{n,k}^+} |u_n|^{Ns} |\nabla u|^N dx \;-\; \int_\Omega h(u_n) \mathrm{e}^{\alpha |u_n|^\gamma} R_k^+ u_n dx \\
&= \frac{\alpha_0}{(s+1)^N} \int_{\Omega_{n,k}^+} |\nabla (u_n^{s+1})|^N dx \;-\; \int_\Omega h(u_n) \mathrm{e}^{\alpha |u_n|^\gamma} R_k^+ u_n dx,
\end{aligned}$$

which, together with (3.25), gives

$$\int_{\Omega_{n,k}^+} |\nabla (u_n^{s+1})|^N dx \;\leq\; \frac{(s+1)^N}{\alpha_0} \left(|\Omega_k^+| + \int_\Omega h(u_n) \mathrm{e}^{\alpha |u_n|^\gamma} R_k^+ u_n dx \right) \qquad \text{for all } n \geq n_k. \tag{3.26}$$

From definition (3.18) with $\tilde{k} = k \geq R_1$ which implies $R_k^+ u_n = 0$ a.e. in $\Omega \setminus \Omega_{n,k}^+$, we have that estimate (3.19) and Cauchy–Schwarz inequality imply

$$\begin{aligned}
\left| \int_\Omega h(u_n) \mathrm{e}^{\alpha |u_n|^\gamma} R_k^+ u_n dx \right| &\leq \int_{\Omega_{n,k}^+} |h(u_n)| \, |R_k^+ u_n| \, \mathrm{e}^{\alpha |u_n|^\gamma} dx \;\leq\; \int_{\Omega_{n,k}^+} \mathrm{e}^{(\alpha+\delta)|u_n|^\gamma} \, |R_k^+ u_n| dx \\
&\leq \left(\int_{\Omega_{n,k}^+} |R_k^+ u_n|^2 dx \right)^{1/2} \left(\int_\Omega \mathrm{e}^{2(\alpha+\delta)|u_n|^\gamma} dx \right)^{1/2} \;\leq\; c_2 \left(\int_\Omega |R_k^+ u_n|^2 dx \right)^{1/2},
\end{aligned}$$

as $\left(\int_\Omega \mathrm{e}^{2(\alpha+\delta)|u_n|^\gamma}dx\right)_n$ is bounded by the statement in *Step 1* and Lemma 3.1 with $\beta = 2(\alpha+\delta)$. Thus, back to (3.26), we obtain

$$\int_{\Omega^+_{n,k}} |\nabla(u_n^{s+1})|^N dx \ \le\ c_3 \left(|\Omega_k^+| + \left(\int_\Omega |R_k^+ u_n|^2 dx\right)^{1/2}\right) \qquad \text{for all } n \ge n_k. \tag{3.27}$$

Passing to the limit in (3.27), from (3.21) and (3.23) we have that

$$\int_{\Omega^+_k} |\nabla(u^{s+1})|^N dx \le c_3 \left(|\Omega_k^+| + \left(\int_\Omega |R_k^+ u_n|^2 dx\right)^{1/2}\right), \tag{3.28}$$

where again from definition (3.18) with $\tilde{k} = k$ and direct computations it results

$$\left(\int_\Omega |R_k^+ u_n|^2 dx\right)^{1/2} \ \le\ \left(\int_{\Omega_k^+} |u|^2 dx\right)^{1/2} \ +\ k\ |\Omega_k^+|^{1/2}. \tag{3.29}$$

Therefore, as in Ω_k^+ it is $1 \le k < u$, then $u^2 \le u^{2N(s+1)}$ in Ω_k^+ and from (3.29) it follows that (3.28) turns into

$$\int_{\Omega^+_k} |\nabla(u^{s+1})|^N dx \ \le\ c_3 \left(|\Omega_k^+| +\ k\ |\Omega_k^+|^{1/2}\ + \left(\int_{\Omega_k^+} (u^{s+1})^{2N} dx\right)^{1/2}\right). \tag{3.30}$$

At last, if we set $v = |u|^s u$, with $v \in W_0^{1,N}(\Omega)$, since (3.22) implies $\Omega_k^+ = \{x \in \Omega : v(x) > k^{s+1}\}$, then from (3.30) we obtain that

$$\int_{\Omega^+_k} |\nabla v|^N dx \ \le\ c_3 \left(|\Omega_k^+| +\ k\ |\Omega_k^+|^{1/2}\ + \left(\int_{\Omega_k^+} v^{2N} dx\right)^{1/2}\right),$$

where, by direct computations, it is

$$\begin{aligned}
\left(\int_{\Omega_k^+} v^{2N} dx\right)^{1/2} &\le\ c_4 \left(\int_{\Omega_k^+} (v-k^{s+1})^{2N} dx + k^{2N(s+1)}|\Omega_k^+|\right)^{1/2}\\
&\le\ c_4 \left(\left(\int_{\Omega_k^+} (v-k^{s+1})^{2N}\right)^{N/2N} + k^{(s+1)N}\ |\Omega_k^+|^{1/2}\right).
\end{aligned}$$

Hence, summing up, since $1 \le k \le k^{(s+1)N}$, we obtain

$$\int_{\Omega^+_k} |\nabla v|^N dx \ \le\ c_5 \left(\left(\int_{\Omega_k^+} (v-k^{s+1})^{2N}\right)^{N/2N} + k^{(s+1)N}|\Omega_k^+| + k^{(s+1)N}|\Omega_k^+|^{1/2}\right),$$

and then Lemma 3.3 applies to function v but taking any $\tilde{k} \ge R_1^{s+1}$ and then in all the previous computations k such that $\tilde{k} = k^{s+1}$, and $r = 2N$, $m = 2$, $l_1 = l_2 = N$, $\varepsilon_1 = 1$ and $\varepsilon_2 = 1/2$.

Similar arguments allow us to exclude also (3.17) and then it has to be $u \in L^\infty(\Omega)$.

Step 3. The proof of this step is essentially as in *Step 3* of [3, Proposition 4.6], but, for completeness and also for pointing out the different assumptions we need, here we give some details.

Firstly, consider $R_1 > 0$ so that (3.19) holds, and from assumption (g_1) a radius $R_2 > 0$ exists such that

$$G(t) \ge 0 \quad \text{if } |t| \ge R_2. \tag{3.31}$$

Then, fix any integer k such that $k \geq \max\{R_1,\ R_2,\ |u|_\infty\} + 1$ and define the "remainder" of the truncation function in (3.14) as

$$R_k t = t - T_k t = \begin{cases} 0 & \text{if } |t| \leq k \\ t - k\frac{t}{|t|} & \text{if } |t| > k \end{cases}. \tag{3.32}$$

For the choice of k we have that it is

$$T_k u = u \quad \text{and} \quad R_k u = 0 \qquad \text{a.e. in } \Omega,$$

then from (3.6), (3.8) and (3.9) it follows that $T_k u_n \rightharpoonup u$ weakly in $W_0^{1,N}(\Omega)$,

$$R_k u_n \to 0 \quad \text{strongly in } L^2(\Omega), \tag{3.33}$$

$$|\Omega_{n,k}| \to 0 \qquad \text{with} \quad \Omega_{n,k} := \{x \in \Omega : \ |u_n(x)| > k\}\,. \tag{3.34}$$

Furthermore, by definition, it is $\|R_k u_n\|_X \leq \|u_n\|_X$, then (3.10) implies

$$\langle dE(u_n), R_k u_n\rangle \to 0$$

which gives

$$\begin{aligned} \varepsilon_n \ &= \langle dE(u_n), R_k u_n\rangle \ = \ \int_{\Omega_{n,k}} (A_0(x) + A(x)|u_n|^{Ns})|\nabla u_n|^N dx \\ &+ \ s \int_{\Omega_{n,k}} A(x)|u_n|^{Ns}\left(1 - \frac{k}{|u_n|}\right)\ |\nabla u_n|^N dx \ - \ \int_\Omega h(u_n)\mathrm{e}^{\alpha|u_n|^\gamma} R_k u_n dx, \end{aligned}$$

where from the definition in (3.34) and assumption (h_1) it is

$$\int_{\Omega_{n,k}} A(x)|u_n|^{Ns}\left(1 - \frac{k}{|u_n|}\right)\ |\nabla u_n|^N dx \ \geq \ 0,$$

while from definition (3.32), estimate (3.19) and Cauchy–Schwarz inequality we obtain that

$$\begin{aligned} \left|\int_\Omega h(u_n)\mathrm{e}^{\alpha|u_n|^\gamma} R_k u_n dx\right| &\leq \int_{\Omega_{n,k}} |h(u_n)|\mathrm{e}^{\alpha|u_n|^\gamma}|R_k u_n| dx \ \leq \ \int_{\Omega_{n,k}} \mathrm{e}^{(\delta+\alpha)|u_n|^\gamma}|R_k u_n| dx \\ &\leq \ \left(\int_\Omega \mathrm{e}^{2(\delta+\alpha)|u_n|^\gamma} dx\right)^{\frac{1}{2}} \ \left(\int_\Omega |R_k u_n|^2 dx\right)^{\frac{1}{2}} \ \leq \ c_6 \left(\int_\Omega |R_k u_n|^2 dx\right)^{\frac{1}{2}} \end{aligned}$$

by the statement in *Step 1* and Lemma 3.1 with $\beta = 2(\alpha + \delta)$. Thus, summing up, from (3.33) and (h_1) we have that

$$\varepsilon_n \ \geq \ \int_{\Omega_{n,k}} (A_0(x) + A(x)|u_n|^{Ns})|\nabla u_n|^N dx \ \geq \ \alpha_0 \ \int_{\Omega_{n,k}} (1 + |u_n|^{Ns})|\nabla u_n|^N dx \ \geq \ 0$$

which implies

$$\int_{\Omega_{n,k}} (A_0(x) + A(x)|u_n|^{Ns})|\nabla u_n|^N dx \ \to \ 0, \tag{3.35}$$

and also

$$\int_{\Omega_{n,k}} |\nabla u_n|^N dx \ \to \ 0, \qquad \text{i.e.,} \qquad \|R_k u_n\|_N \to 0, \tag{3.36}$$

and

$$\int_{\Omega_{n,k}} |u_n|^{Ns} |\nabla u_n|^N dx \ \to \ 0. \tag{3.37}$$

Now, in order to prove (3.12), take $v \in X$ such that $\|v\|_X = 1$, whence, $|v|_\infty \le 1$, $\|v\|_N \le 1$. Direct computations and definition (3.14) allow us to prove that

$$\begin{aligned}\langle dE(T_k u_n), v\rangle \ &= \langle dE(u_n), v\rangle - \int_{\Omega_{n,k}} (A_0(x) + A(x)|u_n|^{Ns})|\nabla u_n|^{N-2}\nabla u_n \cdot \nabla v \ dx \\ &\quad - s\int_{\Omega_{n,k}} A(x)|u_n|^{Ns-2}u_n \ v \ |\nabla u_n|^N dx + \int_{\Omega_{n,k}} h(u_n) v \mathrm{e}^{\alpha|u_n|^\gamma} dx - \mathrm{e}^{\alpha k^\gamma} \int_{\Omega_{n,k}} h(u_n) v \ dx,\end{aligned}$$

where from (3.10) we have that

$$|\langle dE(u_n), v\rangle| \le \|dE(u_n)\|_{X'} = \varepsilon_n \quad \text{uniformly with respect to } v \in X \text{ such that } \|v\|_X = 1,$$

while, since $|v|_\infty \le 1$, from (3.19), Cauchy–Schwarz inequality and, again, Lemma 3.1 we obtain

$$\begin{aligned}\left|\int_{\Omega_{n,k}} h(u_n) v \mathrm{e}^{\alpha|u_n|^\gamma} dx\right| &\le \int_{\Omega_{n,k}} |h(u_n)| \mathrm{e}^{\alpha|u_n|^\gamma} dx \ \le \ \int_{\Omega_{n,k}} \mathrm{e}^{(\delta+\alpha)|u_n|^\gamma} dx \\ &\le \ |\Omega_{n,k}|^{1/2} \left(\int_\Omega \mathrm{e}^{2(\delta+\alpha)|u_n|^\gamma} dx\right)^{1/2} \ \le \ c_7 |\Omega_{n,k}|^{1/2}\end{aligned}$$

and also from (h_2) it results

$$\left|\int_{\Omega_{n,k}} h(u_n) v \ dx\right| \ \le \ c_8 |\Omega_{n,k}|.$$

Moreover, since for a.e. $x \in \Omega_{n,k}$ it is $1 \le k \le |u_n|$, being $|v|_\infty \le 1$, from (h_1) it follows that

$$\left|\int_{\Omega_{n,k}} A(x)|u_n|^{Ns-2}u_n \ v \ |\nabla u_n|^N dx\right| \ \le \ |A|_\infty \int_{\Omega_{n,k}} |u_n|^{Ns} \ |\nabla u_n|^N dx,$$

while, being $\|v\|_N \le 1$, from (h_1) and Hölder inequality we have that

$$\begin{aligned}\left|\int_{\Omega_{n,k}} A_0(x) \ |\nabla u_n|^{N-2}\nabla u_n \cdot \nabla v \ dx\right| \ &\le \ |A_0|_\infty \int_{\Omega_{n,k}} |\nabla u_n|^{N-1} |\nabla v| \ dx \\ &\le \ |A_0|_\infty \left(\int_{\Omega_{n,k}} |\nabla u_n|^N dx\right)^{\frac{N-1}{N}};\end{aligned}$$

hence, summing up, from (3.34), (3.36) and (3.37) we obtain that

$$|\langle dE(T_k u_n), v\rangle| \ \le \ \varepsilon_{k,n} + \left|\int_{\Omega_{n,k}} A(x)|u_n|^{Ns}|\nabla u_n|^{N-2}\nabla u_n \cdot \nabla v \ dx\right|$$

with $\varepsilon_{k,n} \to 0$ uniformly with respect to $v \in X$ such that $\|v\|_X = 1$.

At last, reasoning as in the proof of *Step 3* in [3, Proposition 4.6], by means of the test functions $\varphi^+_{k,n} = v R^+_k u_n$, with $R^+_k t$ as in (3.18) with $\tilde{k} = k$, and $\varphi^-_{k,n} = v R^-_k u_n$, where we define

$$R^-_k t \ = \ \begin{cases} 0 & \text{if } t \ge -k \\ t + k & \text{if } t < -k \end{cases},$$

and with careful estimates based on (3.19) and Lemma 3.1, we are able to prove also that

$$\left|\int_{\Omega_{n,k}} A(x)|u_n|^{Ns}|\nabla u_n|^{N-2}\nabla u_n\cdot\nabla v\ dx\right| \ \le\ \varepsilon_{k,n},$$

with $\varepsilon_{k,n}\to 0$ uniformly with respect to $v\in X$ such that $\|v\|_X=1$, which completes the proof of (3.12).

Now, in order to prove (3.13), from direct computations we have that

$$E(T_k u_n)=E(u_n)-\ \frac{1}{N}\ \int_{\Omega_{n,k}}(A_0(x)+A(x)|u_n|^{Ns})|\nabla u_n|^N dx+\int_{\Omega_{n,k}} G(u_n)dx-\int_{\Omega_{n,k}} G(T_k u_n)dx,$$

where (3.35) implies

$$E(T_k u_n)=E(u_n)+\varepsilon_{k,n}+\int_{\Omega_{n,k}} G(u_n)dx-\int_{\Omega_{n,k}} G(T_k u_n)dx. \tag{3.38}$$

We note that $|T_k u_n|_\infty\le k$ for all $n\in\mathbb{N}$ and the continuity of $G(t)$ implies that

$$\left|\int_{\Omega_{n,k}} G(T_k u_n)dx\right| \ \le\ c_9\ |\Omega_{n,k}|,$$

while, being $k>R_2$, from (3.31) and assumption (2.5) it follows that

$$0\ \le\ G(u_n)\ \le\ c_{10}\ \left(u_n\ h(u_n)\ \mathrm{e}^{\alpha|u_n|^\gamma}+\sigma_1|u_n|^q+\sigma_2\right)\qquad\text{for a.e. } x\in\Omega_{n,k},$$

where, again, from (3.19), Cauchy–Schwarz inequality, Lemma 3.1, (3.9) and direct computations we obtain

$$\int_{\Omega_{n,k}}|u_n|\ |h(u_n)|\mathrm{e}^{\alpha|u_n|^\gamma}dx\le\ \left(\int_\Omega \mathrm{e}^{2(\delta+\alpha)|u_n|^\gamma}dx\right)^{1/2}\left(\int_{\Omega_{n,k}}|u_n|^2dx\right)^{1/2}\ \le\ c_{11}|\Omega_{n,k}|^{1/2}$$

and also

$$\int_{\Omega_{n,k}}|u_n|^q dx\ \le\ c_{12}\ |\Omega_{n,k}|^{1/2}.$$

Thus, (3.13) follows by using all the previous estimates in (3.38) together with (3.10) and (3.34).

Step 4. It is enough arguing as in the proof of the corresponding step in [3, Proposition 4.6].

Step 5. The proof follows from the previous steps by applying Proposition 2.6 to the uniformly bounded sequence $(T_k u_n)_n$ as we obtain that

$$E(T_k u_n)\to E(u),\quad \|dE(T_k u_n)-dE(u)\|_{X'}\to 0$$

and (3.12) and (3.13) hold. $\square$

4. The Mountain Pass geometry

In this section we show that the functional E defined in (2.8) satisfies the Mountain Pass geometry which is required for applying the abstract Theorem 2.2.

Firstly, let us recall that $\lambda_1>0$, the first eigenvalue of $-\Delta_N$ in $W_0^{1,N}(\Omega)$, is achieved by a unique (up to constants) function $\varphi_1\in W_0^{1,N}(\Omega)$ such that

$$\varphi_1>0,\quad \int_\Omega|\varphi_1|^N dx=1\quad\text{and}\quad\int_\Omega|\nabla\varphi_1|^N dx=\lambda_1 \tag{4.1}$$

(see, e.g., [12]); furthermore, it is also $\varphi_1 \in L^\infty(\Omega)$, hence $\varphi_1 \in X$, and

$$\int_\Omega |u|^N dx \le \frac{1}{\lambda_1}\int_\Omega |\nabla u|^N dx \quad \text{for all } u \in W_0^{1,N}(\Omega). \tag{4.2}$$

Now, we define

$$\ell_s(u) \;:=\; \left(\int_\Omega (1+|u|^{Ns})|\nabla u|^N dx\right)^{\frac{1}{N}} \qquad \text{for all } u \in X. \tag{4.3}$$

Clearly, the map $\ell_s : X \to \mathbb{R}$ is continuous with $\ell_s(0) = 0$ and from (2.7) it is such that

$$\ell_s(u) \ge \|u\|_N, \qquad \ell_s(u) \ge \frac{1}{s+1}\||u|^s u\|_N \quad \text{for all } u \in X. \tag{4.4}$$

Proposition 4.1. *If the hypotheses of Theorem 2.5 hold, then a constant $r_0 > 0$ exists such that*

$$\inf_{\ell_s(u)=r_0} E(u) > 0.$$

Proof. From assumption (2.4) a radius $R_1 > 0$ exists such that (3.19) holds. Without loss of generaly, we can suppose $R_1 \ge \max\{\nu, 1\}$ with ν so that condition (g_2) holds. Then, by applying the estimates (2.5) and (3.19) and choosing a power $\beta_1 \ge 0$ so that $q + \beta_1 \ge N+1$, we have that

$$G(t) \;\le\; c_1\ |t|\ |h(t)|\ \mathrm{e}^{\alpha|t|^\gamma} + c_2|t|^q + c_3 \;\le\; c_1\ |t|^{N+1}\mathrm{e}^{(\delta+\alpha)|t|^\gamma} + c_2|t|^{q+\beta_1} + c_3|t|^{N+1} \quad \text{if } |t| \ge R_1, \tag{4.5}$$

while the continuity of $G(t)$ and direct computations imply that

$$G(t) \;\le\; \max_{|t|\le R_1}|G(t)| \;\le\; c_4\ |t|^{N+1} \quad \text{if } \nu \le |t| \le R_1. \tag{4.6}$$

Hence, without loss of generality assuming $\frac{\lambda_1\alpha_0}{N} - \sigma > 0$ in (g_2), summing up estimates (4.5), (4.6) and hypothesis (g_2) it follows that

$$G(t) \;\le\; \left(\frac{\lambda_1\alpha_0}{N} - \sigma\right)\ |t|^N + c_1\ |t|^{N+1}\mathrm{e}^{(\delta+\alpha)|t|^\gamma} + c_2|t|^{q+\beta_1} + c_5|t|^{N+1} \quad \text{for all } t \in \mathbb{R}$$

which implies

$$\int_\Omega G(u)dx \;\le\; \left(\frac{\lambda_1\alpha_0}{N} - \sigma\right)\int_\Omega |u|^N dx + c_1\int_\Omega |u|^{N+1}\mathrm{e}^{(\delta+\alpha)|u|^\gamma}dx + c_2\int_\Omega |u|^{q+\beta_1}dx + c_5\int_\Omega |u|^{N+1}dx \tag{4.7}$$

for all $u \in X$.

We note that, for all $u \in X$ such that $\ell_s(u) \le \frac{1}{s+1}$, from Cauchy–Schwarz inequality, (3.2) with $\beta = 2(\delta+\alpha)$, Sobolev Embedding Theorem and (4.4), we obtain

$$\int_\Omega |u|^{N+1}\mathrm{e}^{(\delta+\alpha)|u|^\gamma}dx \le \left(\int_\Omega |u|^{2(N+1)}dx\right)^{1/2}\left(\int_\Omega \mathrm{e}^{2(\delta+\alpha)|u|^\gamma}dx\right)^{1/2} \le c_6\|u\|_N^{N+1},$$

so, by using (4.2) and (4.4) in (4.7), from direct computations it follows that

$$\int_\Omega G(u)dx \;\le\; \left(\frac{\alpha_0}{N} - \frac{\sigma}{\lambda_1}\right)\ [\ell_s(u)]^N + c_7\ [\ell_s(u)]^{N+1} + c_8[\ell_s(u)]^{q+\beta_1}. \tag{4.8}$$

Thus, combining (4.8) with definitions (2.8) and (4.3) and hypothesis (h_1), and taking $\ell_s(u) = r_0$ with $r_0 \le \frac{1}{s+1}$, we have that

$$E(u) \ge \frac{\sigma}{\lambda_1}r_0^N - c_7\ r_0^{N+1} - c_8 r_0^{q+\beta_1}$$

and, as $q + \beta_1 > N$, the desired result follows from taking $r_0 > 0$ sufficiently small. $\square$

Proposition 4.2. *If $\varphi_1 \in X$ is as in* (4.1)*, then we have that*

$$E(t\varphi_1) \to -\infty \qquad \textit{as} \;\; t \to +\infty.$$

Proof. From (2.8), hypotheses (h_1) and (g_1), the properties of φ_1 in (4.1) and also (2.7), for all $t > 0$ we have that

$$\begin{aligned} E(t\varphi_1) &= \frac{1}{N}\int_\Omega (A_0(x) + A(x)|t\varphi_1|^{Ns})|\nabla t\varphi_1|^N dx - \int_\Omega G(t\varphi_1)dx \\ &\le c_1 t^N \lambda_1 + c_2 t^{N(s+1)} \||\varphi_1|^s\varphi_1\|_N^N - \sigma_3 t^\tau \int_\Omega \varphi_1^\tau dx + c_3. \end{aligned}$$

Thus, since by assumption $\tau > N(s+1)$, as $t \to +\infty$ we obtain the desired result. □

Theorem 2.5. By considering the function $\ell_s(u)$ as in (4.3), from Propositions 3.4, 4.1 and 4.2 we have that Theorem 2.2 applies to functional E in (2.8) and a mountain pass nontrivial critical point of E on X exists. □

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