

Lifespan estimates for local solutions to the semilinear wave equation in Einstein – de Sitter spacetime

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Abstract

In this paper, we prove some blow-up results for the semilinear wave equation in generalized Einstein-de Sitter spacetime by using an iteration argument and we derive upper bound estimates for the lifespan. In particular, we will focus on the critical cases which require the employment of a slicing procedure in the iterative mechanism. Furthermore, in order to deal with the main critical case, we will introduce a non-autonomous and parameter dependent Cauchy problem for a linear ODE of second order, whose explicit solution will be determined by applying the theory of special functions.

Keywords Semilinear wave equation, Einstein – de Sitter spacetime, power nonlinearity, critical case, lifespan estimates, modified Bessel functions

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1 Introduction

In the last four decades, the proof of the Strauss conjecture concerning the critical exponent of the initial value problem for the semilinear wave equation with power nonlinearity required the effort of many mathematicians worldwide. Nowadays, we know that the critical exponent for the Cauchy problem

$$\begin{cases} v_{tt} - \Delta v = |v|^p & x \in \mathbb{R}^n, t > 0, \\ v(0, x) = \varepsilon v_0(x) & x \in \mathbb{R}^n, \\ v_t(0, x) = \varepsilon v_1(x) & x \in \mathbb{R}^n, \end{cases}$$

is the so – called *Strauss exponent* $p_{\text{Str}}(n)$ (cf. [19, 20, 16, 17, 35, 34, 50, 24, 15, 48, 51]), that is, the positive root of the quadratic equation

$$(n - 1)p^2 - (n + 1)p - 2 = 0.$$

We are also interested in not only the critical exponent but also lifespan, the maximal existence time of the solution, when the global in time existence cannot be expected. See the introduction of [18] for the complete picture of the lifespan estimates for the classical semilinear wave equation with power nonlinearity.

While the situation is completely understood in the Euclidean case with flat metric on \mathbb{R}^n , in the last years several papers have been devoted to study the semilinear wave equation in the spacetime \mathbb{R}_+^{1+n} equipped with different Lorentzian metrics. The semilinear wave equation in Schwarzschild has been investigated in [2, 25, 23, 22] in the 1 + 3 dimensional case. Moreover, the wave (or Klein-Gordon) equation in de Sitter and anti – de Sitter spacetimes have been investigated in the linear and semilinear case in [41, 45, 42, 13, 7, 9] and [8, 44, 46, 47], respectively. Finally, the wave equation in Einstein – de Sitter spacetime has been considered in [10, 11, 14]. In this paper, we shall examine the semilinear wave equation with power nonlinearity in a *generalized Einstein – de Sitter spacetime*. More precisely, let us consider the semilinear equation with singular coefficients

$$\varphi_{tt} - t^{-2k} \Delta \varphi + 2t^{-1} \varphi_t = |\varphi|^p, \tag{1}$$

where $k \in [0, 1)$ and $p > 1$. We call this model the semilinear wave equation in a generalized EdeS spacetime since for $k = 2/3$ and $n = 3$ Equation (1) is the semilinear wave equation in Einstein – de Sitter (EdeS) spacetime with power nonlinearity.

In [14, Theorem 1.3] the authors proved that for

$$1 < p < \max \{p_0(n, k), p_1(n, k)\}$$

a local in time solution to the corresponding Cauchy problem (with initial data prescribed at the initial time $t = 1$) blows up in finite time, provided that the initial data fulfill certain integral sign conditions. Here $p_0(n, k)$ is the positive root of the quadratic equation

$$((1 - k)n + 1)p^2 - ((1 - k)n + 3 + 2k)p - 2(1 - k) = 0, \quad (2)$$

while

$$p_1(n, k) \doteq 1 + \frac{2}{(1 - k)n}. \quad (3)$$

Furthermore, in [14] it is also shown that, for the semilinear wave equation in EdeS spacetime, the blow – up is the effect of the semilinear term. For this reason we shall focus our analysis on the effect of the nonlinear term, prescribing the Cauchy data at the initial time $t = 1$.

Performing the transformation $u = t\varphi$, (1) becomes equivalent to the following semilinear equation for u

$$u_{tt} - t^{-2k}\Delta u = t^{1-p}|u|^p. \quad (4)$$

In this paper, we investigate the blow – up dynamic for (4) and, in particular, we will focus on the upper bound estimates for the lifespan and on the treatment of the critical case $p = \max\{p_0(n, k), p_1(n, k)\}$. More precisely, in the next sections we are going to provide a complete picture of the upper bound estimates for the lifespan of local in time solutions to (4) when $1 < p \leq \max\{p_0(n, k), p_1(n, k)\}$.

In the subcritical case, we employ a Kato – type lemma on the blow – up dynamic for a second order ordinary differential inequality. On the other hand, in the critical case an iteration argument combined with a slicing procedure is applied. More particularly, for $p = p_0(n, k)$ we adapt the approach from [38, 39] to the time – dependent semilinear model (4).

1.1 Notations

Throughout the paper we will employ the following notations: $\phi_k(t) \doteq \frac{t^{1-k}}{1-k}$ denotes a distance function produced by the speed of propagation $a_k(t) = t^{-k}$, while the amplitude of the light cone is given by the function

$$A_k(t) \doteq \int_1^t \tau^{-k} d\tau = \phi_k(t) - \phi_k(1); \quad (5)$$

the ball with radius R around the origin is denoted B_R ; $f \lesssim g$ means that there exists a positive constant C such that $f \leq Cg$ and, similarly, for $f \gtrsim g$; I_ν and K_ν denote the modified Bessel function of first and second kind of order ν , respectively; finally,

$$N(k) \doteq \frac{1 - 2k + \sqrt{4k^2 - 4k + 8}}{2(1 - k)} \quad (6)$$

denotes the threshold for the spatial dimension in determining the dominant exponent between $p_0(n, k)$ and $p_1(n, k)$ (more specifically, $p_0(n, k) > p_1(n, k)$ if and only if $n > N(k)$, while $p_0(n, k) \leq p_1(n, k)$ for $n \leq N(k)$).

1.2 Main results

The main results of this work are the following blow – up results that combined together provide a full picture of the critical case $p = \max\{p_0(n, k), p_1(n, k)\}$ for the Cauchy problem

$$\begin{cases} u_{tt} - t^{-2k}\Delta u = t^{1-p}|u|^p & x \in \mathbb{R}^n, t \in (1, T), \\ u(1, x) = \varepsilon u_0(x) & x \in \mathbb{R}^n, \\ u_t(1, x) = \varepsilon u_1(x) & x \in \mathbb{R}^n, \end{cases} \quad (7)$$

where $p > 1$, $\varepsilon > 0$ is a parameter describing the size of initial data and $k \in [0, 1)$.

Before stating the main results, let us introduce the notion of energy solution to the semilinear Cauchy problem (7).

Definition 1.1. Let $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$. We say that

$$u \in \mathcal{C}([1, T], H^1(\mathbb{R}^n)) \cap \mathcal{C}^1([1, T], L^2(\mathbb{R}^n)) \cap L_{\text{loc}}^p([1, T] \times \mathbb{R}^n)$$

is an energy solution to (7) on $[1, T)$ if u fulfills $u(1, \cdot) = \varepsilon u_0$ in $H^1(\mathbb{R}^n)$ and the integral relation

$$\begin{aligned} & \int_{\mathbb{R}^n} \partial_t u(t, x) \psi(t, x) dx - \varepsilon \int_{\mathbb{R}^n} u_1(x) \psi(1, x) dx - \int_1^t \int_{\mathbb{R}^n} (\partial_t u(s, x) \psi_s(s, x) - s^{-2k} \nabla u(s, x) \cdot \nabla \psi(s, x)) dx ds \\ &= \int_1^t \int_{\mathbb{R}^n} s^{1-p} |u(s, x)|^p \psi(s, x) dx ds \end{aligned} \quad (8)$$

for any $\psi \in \mathcal{C}_0^\infty([1, T) \times \mathbb{R}^n)$ and any $t \in (1, T)$.

We point out that performing a further step of integration by parts in (8), we find the integral relation

$$\begin{aligned} & \int_{\mathbb{R}^n} (\partial_t u(t, x) \psi(t, x) - u(t, x) \psi_s(t, x)) dx - \varepsilon \int_{\mathbb{R}^n} (u_1(x) \psi(1, x) - u_0(x) \psi_s(1, x)) dx \\ &+ \int_1^t \int_{\mathbb{R}^n} u(s, x) (\psi_{ss}(s, x) - s^{-2k} \Delta \psi(s, x)) dx ds = \int_1^t \int_{\mathbb{R}^n} s^{1-p} |u(s, x)|^p \psi(s, x) dx ds \end{aligned} \quad (9)$$

for any $\psi \in \mathcal{C}_0^\infty([1, T) \times \mathbb{R}^n)$ and any $t \in (1, T)$.

Remark 1. Let us stress that if the Cauchy data are compactly supported, say $\text{supp } u_j \subset B_R$ for $j = 0, 1$ and for some $R > 0$, then, for any $t \in (1, T)$ a local solution u to (7) satisfies the support condition

$$\text{supp } u(t, \cdot) \subset B_{R+A_k(t)},$$

where A_k is defined by (5). Therefore, in Definition 1.1 we may also consider test functions which are not compactly supported, namely, $\psi \in \mathcal{C}^\infty([1, T) \times \mathbb{R}^n)$.

Theorem 1.2. *Let $n \in \mathbb{N}^*$ such that $n > N(k)$ and $p = p_0(n, k)$. Let us assume that $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$ are nonnegative, nontrivial and compactly supported functions with supports contained in B_R for some $R > 0$. Let*

$$u \in \mathcal{C}([1, T), H^1(\mathbb{R}^n)) \cap \mathcal{C}^1([1, T), L^2(\mathbb{R}^n)) \cap L_{\text{loc}}^p([1, T) \times \mathbb{R}^n)$$

be an energy solution to (7) according to Definition 1.1 with lifespan $T = T(\varepsilon)$ and satisfying the support condition $\text{supp } u(t, \cdot) \subset B_{A_k(t)+R}$ for any $t \in (1, T)$.

Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(u_0, u_1, n, p, k, R)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the energy solution u blows up in finite time. Moreover, the upper bound estimate for the lifespan

$$T(\varepsilon) \leq \exp\left(C\varepsilon^{-p(p-1)}\right)$$

holds, where the constant $C > 0$ is independent of ε .

Theorem 1.3. *Let $n \in \mathbb{N}^*$ such that $n \leq N(k)$ and $p = p_1(n, k)$. Let us assume that $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$ are nonnegative, nontrivial and compactly supported functions with supports contained in B_R for some $R > 0$. Let*

$$u \in \mathcal{C}([1, T), H^1(\mathbb{R}^n)) \cap \mathcal{C}^1([1, T), L^2(\mathbb{R}^n)) \cap L_{\text{loc}}^p([1, T) \times \mathbb{R}^n)$$

be an energy solution to (7) according to Definition 1.1 with lifespan $T = T(\varepsilon)$ and satisfying the support condition $\text{supp } u(t, \cdot) \subset B_{A_k(t)+R}$ for any $t \in (1, T)$.

Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(u_0, u_1, n, p, k, R)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the energy solution u blows up in finite time. Moreover, the upper bound estimate for the lifespan

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holds, where the constant $C > 0$ is independent of ε .

The remaining part of the paper is organized as follows: in Section 2 we prove Theorem 1.2 by using the approach introduced in [38]; then, in Section 3 we provide a complete overview on upper bound estimates for the subcritical case (cf. Proposition 3.2), while in Section 4 we show the proof of Theorem 1.3; finally, in Appendix A we provide a different proof of Proposition 2.1 in the special case of Einstein – de Sitter spacetime.

2 Semilinear wave equation in EdeS spacetime: 1st critical case

Our goal is to prove a blow – up result in the critical case $p = p_0(n, k)$, where $p_0(n, k)$ is the greatest root of the quadratic equation

$$\left(\frac{n-1}{2} + \frac{2-k}{2(1-k)}\right)p^2 - \left(\frac{n+1}{2} + \frac{2+3k}{2(1-k)}\right)p - 1 = 0. \quad (10)$$

The approach that we will follow is based on the technique introduced in [38] and subsequently applied to different wave models (cf. [39, 31, 32, 21, 3, 4]).

We are going to introduce a time – dependent functional that depends on a local in time solution to (7) and to study its blow – up dynamic. In particular, the blow – up result will be obtained by applying the so – called *slicing procedure* in an iteration argument to show a sequence of lower bound estimates for the above mentioned functional.

The section is organized as follows: in Section 2.1 we determine a pair of auxiliary functions which have a fundamental role in the definition of the time – dependent functional and in the determination of the iteration frame, while in Section 2.2 we establish some fundamental estimates for these functions; then, in Section 2.3 we establish the iteration frame for the functional and, finally, in Section 2.4 we prove the blow – up result by using an iteration procedure.

2.1 Auxiliary functions

In this section, we are going to introduce two auxiliary functions (see ξ_q and η_q below) analogously to the corresponding functions introduced in [38], which represent, in turn, a generalization of the solution to the classical free wave equation given in [51]. Those auxiliary functions are defined by using the remarkable function

$$\varphi(x) \doteq \begin{cases} \int_{\mathbb{S}^{n-1}} e^{x \cdot \omega} d\sigma_\omega & \text{if } n \geq 2, \\ \cosh x & \text{if } n = 1 \end{cases} \quad (11)$$

introduced in [48]. Let us recall briefly the main properties of this function: φ is a positive and smooth function that satisfies $\Delta\varphi = \varphi$ and asymptotically behaves like $|x|^{-\frac{n-1}{2}} e^{|x|}$ as $|x| \rightarrow \infty$ up to a positive multiplicative constant.

In order to introduce the definition of the auxiliary functions, let us begin by determining the solutions $y_j = y_j(t, s; \lambda, k)$, $j \in \{0, 1\}$, of the non-autonomous, parameter-dependent, ordinary Cauchy problems

$$\begin{cases} \partial_t^2 y_j(t, s; \lambda, k) - \lambda^2 t^{-2k} y_j(t, s; \lambda, k) = 0, & t > s, \\ y_j(s, s; \lambda, k) = \delta_{0j}, \\ \partial_t y_j(s, s; \lambda, k) = \delta_{1j}, \end{cases} \quad (12)$$

where δ_{ij} denotes the Kronecker delta, $s \geq 1$ is the initial time and $\lambda > 0$ is a real parameter.

Let us recall that we denote by $\phi_k(t) = \frac{t^{1-k}}{1-k}$ a primitive of the speed of propagation $a(t) = t^{-k}$ for the wave equation in (7). In order to find a system of independent solutions to

$$\frac{d^2 y}{dt^2} - \lambda^2 t^{-2k} y = 0 \quad (13)$$

we perform first a change of variables. Let $\tau = \tau(t; \lambda, k) \doteq \lambda \phi_k(t)$. Since

$$\frac{dy}{dt} = \lambda t^{-k} \frac{dy}{d\tau}, \quad \frac{d^2 y}{dt^2} = \lambda^2 t^{-2k} \frac{d^2 y}{d\tau^2} - \lambda k t^{-k-1} \frac{dy}{d\tau},$$

then, y solves (13) if and only if it solves

$$\tau \frac{d^2 y}{d\tau^2} - \frac{k}{1-k} \frac{dy}{d\tau} - \tau y = 0. \quad (14)$$

Next, we carry out the transformation $y(\tau) = \tau^\nu w(\tau)$ with $\nu \doteq \frac{1}{2(1-k)}$. Therefore, y solves (14) if and only if w solves the modified Bessel equation of order ν

$$\tau^2 \frac{d^2 w}{d\tau^2} + \tau \frac{dw}{d\tau} - (\nu^2 + \tau^2) w = 0, \quad (15)$$

where we applied the straightforward relations

$$\frac{dy}{d\tau} = \nu\tau^{\nu-1}w(\tau) + \tau^\nu \frac{dw}{d\tau}, \quad \frac{d^2y}{d\tau^2} = \nu(\nu-1)\tau^{\nu-2}w + 2\nu\tau^{\nu-1} \frac{dw}{d\tau} + \tau^\nu \frac{d^2w}{d\tau^2}.$$

If we employ as independent solutions to (15) the modified Bessel function of first and second kind of order ν , denoted, respectively, by $I_\nu(\tau)$ and $K_\nu(\tau)$, then, the pair of functions

$$\begin{aligned} V_0(t; \lambda, k) &\doteq \tau^\nu I_\nu(\tau) = (\lambda\phi_k(t))^\nu I_\nu(\lambda\phi_k(t)), \\ V_1(t; \lambda, k) &\doteq \tau^\nu K_\nu(\tau) = (\lambda\phi_k(t))^\nu K_\nu(\lambda\phi_k(t)) \end{aligned}$$

is a basis of the space of solutions to (13).

Proposition 2.1. *The functions*

$$y_0(t, s; \lambda, k) \doteq \lambda(t/s)^{1/2} \phi_k(s) [I_{\nu-1}(\lambda\phi_k(s)) K_\nu(\lambda\phi_k(t)) + K_{\nu-1}(\lambda\phi_k(s)) I_\nu(\lambda\phi_k(t))], \quad (16)$$

$$y_1(t, s; \lambda, k) \doteq (1-k)^{-1}(st)^{1/2} [K_\nu(\lambda\phi_k(s)) I_\nu(\lambda\phi_k(t)) - I_\nu(\lambda\phi_k(s)) K_\nu(\lambda\phi_k(t))], \quad (17)$$

solve the Cauchy problems (12) for $j = 0$ and $j = 1$, respectively, where $\nu = 1/(2(1-k))$, $\phi_k(t) = t^{1-k}/(1-k)$ and I_ν, K_ν denote the modified Bessel function of order ν of the first and second kind, respectively.

Proof. We have seen that V_0, V_1 form a system of independent solutions to (13). Therefore, we may express the solutions of (12) as linear combinations of V_0, V_1 as follows:

$$y_j(t, s; \lambda, k) = a_j(s; \lambda, k)V_0(t; \lambda, k) + b_j(s; \lambda, k)V_1(t; \lambda, k) \quad (18)$$

for suitable coefficients $a_j(s; \lambda, k), b_j(s; \lambda, k)$, $j \in \{0, 1\}$. Using the initial conditions $\partial_t^i y_j(s, s; \lambda, k) = \delta_{ij}$, we find the system

$$\begin{pmatrix} V_0(s; \lambda, k) & V_1(s; \lambda, k) \\ \partial_t V_0(s; \lambda, k) & \partial_t V_1(s; \lambda, k) \end{pmatrix} \begin{pmatrix} a_0(s; \lambda, k) & a_1(s; \lambda, k) \\ b_0(s; \lambda, k) & b_1(s; \lambda, k) \end{pmatrix} = I,$$

where I denotes the identity matrix. So, in order to determine the coefficients in (18), we have to calculate explicitly the inverse matrix

$$\begin{pmatrix} V_0(s; \lambda, k) & V_1(s; \lambda, k) \\ \partial_t V_0(s; \lambda, k) & \partial_t V_1(s; \lambda, k) \end{pmatrix}^{-1} = (\mathcal{W}(V_0, V_1)(s; \lambda, k))^{-1} \begin{pmatrix} \partial_t V_1(s; \lambda, k) & -V_1(s; \lambda, k) \\ -\partial_t V_0(s; \lambda, k) & V_0(s; \lambda, k) \end{pmatrix}, \quad (19)$$

where $\mathcal{W}(V_0, V_1)$ is the Wronskian of V_0, V_1 . Clearly, we need to express in a more suitable way $\mathcal{W}(V_0, V_1)$. Let us calculate the t -derivative of V_0, V_1 . Recalling that $\phi_k(t) = t^{1-k}/(1-k)$ and $\nu = 1/(2(1-k))$, it results

$$\begin{aligned} \partial_t V_0(t; \lambda, k) &= \nu(\lambda\phi_k(t))^{\nu-1} \lambda\phi_k'(t) I_\nu(\lambda\phi_k(t)) + (\lambda\phi_k(t))^\nu I_\nu'(\lambda\phi_k(t)) \lambda\phi_k'(t) \\ &= \frac{1}{2t} (\lambda\phi_k(t))^\nu I_\nu(\lambda\phi_k(t)) + (\lambda\phi_k(t))^\nu (\lambda\phi_k'(t)) I_\nu'(\lambda\phi_k(t)) \end{aligned}$$

and, analogously,

$$\partial_t V_1(t; \lambda, k) = \frac{1}{2t} (\lambda\phi_k(t))^\nu K_\nu(\lambda\phi_k(t)) + (\lambda\phi_k(t))^\nu (\lambda\phi_k'(t)) K_\nu'(\lambda\phi_k(t)).$$

Consequently, we can express $\mathcal{W}(V_0, V_1)$ as follows:

$$\begin{aligned} \mathcal{W}(V_0, V_1)(t; \lambda, k) &= (\lambda\phi_k(t))^{2\nu} (\lambda\phi_k'(t)) [K_\nu'(\lambda\phi_k(t)) I_\nu(\lambda\phi_k(t)) - I_\nu'(\lambda\phi_k(t)) K_\nu(\lambda\phi_k(t))] \\ &= (\lambda\phi_k(t))^{2\nu} (\lambda\phi_k'(t)) \mathcal{W}(I_\nu, K_\nu)(\lambda\phi_k(t)) = -(\lambda\phi_k(t))^{2\nu-1} (\lambda\phi_k'(t)) \\ &= -\lambda^{2\nu} (\phi_k(t))^{2\nu-1} \phi_k'(t) = -c_k^{-1} \lambda^{2\nu}, \end{aligned}$$

where $c_k \doteq (1-k)^{k/(1-k)}$ and in the third equality we used the value of the Wronskian of I_ν, K_ν

$$\mathcal{W}(I_\nu, K_\nu)(z) = I_\nu(z) \partial_z K_\nu(z) - \partial_z I_\nu(z) K_\nu(z) = -\frac{1}{z}.$$

Let us underline that $\mathcal{W}(V_0, V_1)(t; \lambda, k)$ does not actually depend on t , due to the absence of the first order term in (13).

Plugging the previous representation of $\mathcal{W}(V_0, V_1)$ in (19), we get

$$\begin{pmatrix} a_0(s; \lambda, k) & a_1(s; \lambda, k) \\ b_0(s; \lambda, k) & b_1(s; \lambda, k) \end{pmatrix} = -c_k \lambda^{-2\nu} \begin{pmatrix} \partial_t V_1(s; \lambda, k) & -V_1(s; \lambda, k) \\ -\partial_t V_0(s; \lambda, k) & V_0(s; \lambda, k) \end{pmatrix}.$$

Let us begin by proving (16). Using the above representation of $a_0(s; \lambda, k)$, $b_0(s; \lambda, k)$ in (18), we obtain

$$\begin{aligned}
y_0(t, s; \lambda, k) &= c_k \lambda^{-2\nu} \{ \partial_t V_0(s; \lambda, k) V_1(t; \lambda, k) - \partial_t V_1(s; \lambda, k) V_0(t; \lambda, k) \} \\
&= c_k \lambda^{-2\nu} (\lambda \phi_k(s))^\nu (\lambda \phi_k(t))^\nu \left\{ \left[\frac{1}{2s} I_\nu(\lambda \phi_k(s)) + (\lambda \phi_k'(s)) I_\nu'(\lambda \phi_k(s)) \right] K_\nu(\lambda \phi_k(t)) \right. \\
&\quad \left. - \left[\frac{1}{2s} K_\nu(\lambda \phi_k(s)) + (\lambda \phi_k'(s)) K_\nu'(\lambda \phi_k(s)) \right] I_\nu(\lambda \phi_k(t)) \right\} \\
&= c_k (\phi_k(s) \phi_k(t))^\nu (2s)^{-1} \{ I_\nu(\lambda \phi_k(s)) K_\nu(\lambda \phi_k(t)) - K_\nu(\lambda \phi_k(s)) I_\nu(\lambda \phi_k(t)) \} \\
&\quad + c_k \lambda (\phi_k(s) \phi_k(t))^\nu \phi_k'(s) \{ I_\nu'(\lambda \phi_k(s)) K_\nu(\lambda \phi_k(t)) - K_\nu'(\lambda \phi_k(s)) I_\nu(\lambda \phi_k(t)) \}.
\end{aligned}$$

Applying the recursive relations for the derivatives of the modified Bessel functions

$$\begin{aligned}
\frac{\partial I_\nu}{\partial z}(z) &= -\frac{\nu}{z} I_\nu(z) + I_{\nu-1}(z), \\
\frac{\partial K_\nu}{\partial z}(z) &= -\frac{\nu}{z} K_\nu(z) - K_{\nu-1}(z),
\end{aligned}$$

to the last relation, we arrive at

$$\begin{aligned}
y_0(t, s; \lambda, k) &= c_k (\phi_k(s) \phi_k(t))^\nu \underbrace{\left[(2s)^{-1} - \frac{\nu \lambda \phi_k'(s)}{\lambda \phi_k(s)} \right]}_{=0} \{ I_\nu(\lambda \phi_k(s)) K_\nu(\lambda \phi_k(t)) - K_\nu(\lambda \phi_k(s)) I_\nu(\lambda \phi_k(t)) \} \\
&\quad + c_k \lambda (\phi_k(s) \phi_k(t))^\nu \phi_k'(s) \{ I_{\nu-1}(\lambda \phi_k(s)) K_\nu(\lambda \phi_k(t)) + K_{\nu-1}(\lambda \phi_k(s)) I_\nu(\lambda \phi_k(t)) \} \\
&= c_k \lambda (\phi_k(s) \phi_k(t))^\nu \phi_k'(s) \{ I_{\nu-1}(\lambda \phi_k(s)) K_\nu(\lambda \phi_k(t)) + K_{\nu-1}(\lambda \phi_k(s)) I_\nu(\lambda \phi_k(t)) \}. \tag{20}
\end{aligned}$$

Since $c_k (\phi_k(s) \phi_k(t))^\nu \phi_k'(s) = (1-k)^{-1} (st)^{1/2} s^{-k} = (t/s)^{1/2} \phi_k(s)$, (20) yields immediately (16). Let us prove now the representation (17). Plugging the above determined expressions for $a_1(s; \lambda, k)$, $b_1(s; \lambda, k)$ in (18), we have

$$\begin{aligned}
y_1(t, s; \lambda, k) &= c_k \lambda^{-2\nu} \{ V_1(s; \lambda, k) V_0(t; \lambda, k) - V_0(s; \lambda, k) V_1(t; \lambda, k) \} \\
&= c_k \lambda^{-2\nu} (\lambda \phi_k(s))^\nu (\lambda \phi_k(t))^\nu \{ K_\nu(\lambda \phi_k(s)) I_\nu(\lambda \phi_k(t)) - I_\nu(\lambda \phi_k(s)) K_\nu(\lambda \phi_k(t)) \} \\
&= c_k (\phi_k(s) \phi_k(t))^\nu \{ K_\nu(\lambda \phi_k(s)) I_\nu(\lambda \phi_k(t)) - I_\nu(\lambda \phi_k(s)) K_\nu(\lambda \phi_k(t)) \}. \tag{21}
\end{aligned}$$

Thus, using $c_k (\phi_k(s) \phi_k(t))^\nu = (st)^{1/2} / (1-k)$, from (21) follows (17). This concludes the proof. \square

Remark 2. In the special case $k = 2/3$, $y_0(t, s; \lambda, k)$ and $y_1(t, s; \lambda, k)$ can be expressed in terms of elementary functions. Indeed by using the explicit representations

$$\begin{aligned}
I_{\frac{1}{2}}(z) &= \sqrt{\frac{2}{\pi}} \frac{\sinh z}{z^{1/2}}, & I_{\frac{3}{2}}(z) &= \sqrt{\frac{2}{\pi}} \frac{z \cosh z - \sinh z}{z^{3/2}}, \\
K_{\frac{1}{2}}(z) &= \sqrt{\frac{\pi}{2}} \frac{e^{-z}}{z^{1/2}}, & K_{\frac{3}{2}}(z) &= \sqrt{\frac{\pi}{2}} \frac{e^{-z}(z+1)}{z^{3/2}},
\end{aligned}$$

we can derive the following representations

$$y_0(t, s; \lambda, 2/3) = \left(\frac{t}{s} \right)^{1/3} \cosh(3\lambda(t^{1/3} - s^{1/3})) - \frac{1}{3\lambda s^{1/3}} \sinh(3\lambda(t^{1/3} - s^{1/3})), \tag{22}$$

$$y_1(t, s; \lambda, 2/3) = \left[\frac{(st)^{1/3}}{\lambda} - \frac{1}{9\lambda^3} \right] \cosh(3\lambda(t^{1/3} - s^{1/3})) + \frac{1}{3\lambda^2} (t^{1/3} - s^{1/3}) \sinh(3\lambda(t^{1/3} - s^{1/3})). \tag{23}$$

Actually, in this case it is possible to derive the representations of $y_0(t, s; \lambda, 2/3)$, $y_1(t, s; \lambda, 2/3)$ by reducing (13) to a confluent hypergeometric equation instead of a modified Bessel equation. For a detailed proof see Appendix A.

Lemma 2.2. *Let y_0, y_1 be the functions defined in (16) and (17), respectively. Then, the following identities are satisfied for any $t \geq s \geq 1$*

$$\frac{\partial y_1}{\partial s}(t, s; \lambda, k) = -y_0(t, s; \lambda, k), \tag{24}$$

$$\frac{\partial^2 y_1}{\partial s^2}(t, s; \lambda, k) - \lambda^2 s^{-2k} y_1(t, s; \lambda, k) = 0. \tag{25}$$

Remark 3. As the operator $(\partial_t^2 - \lambda^2 t^{-2k})$ is formally self-adjoint, in particular (24) and (25) tell us that y_1 solves also the adjoint problem to (13) with final conditions $(0, -1)$.

Proof. Let us introduce the pair of independent solutions to (13)

$$\begin{aligned} z_0(t; \lambda, k) &\doteq y_0(t, 1; \lambda, k), \\ z_1(t; \lambda, k) &\doteq y_1(t, 1; \lambda, k). \end{aligned}$$

By standard computations, we may show the representations

$$\begin{aligned} y_0(t, s; \lambda, k) &= z_1'(s; \lambda, k)z_0(t; \lambda, k) - z_0'(s; \lambda, k)z_1(t; \lambda, k), \\ y_1(t, s; \lambda, k) &= z_0(s; \lambda, k)z_1(t; \lambda, k) - z_1(s; \lambda, k)z_0(t; \lambda, k). \end{aligned}$$

In particular, we used that the Wronskian of z_0, z_1 is identically 1. First we prove (24). Differentiating the second one of the previous representations with respect to s and then using the first one, we get immediately

$$\frac{\partial y_1}{\partial s}(t, s; \lambda, k) = z_0'(s; \lambda, k)z_1(t; \lambda, k) - z_1'(s; \lambda, k)z_0(t; \lambda, k) = -y_0(t, s; \lambda, k).$$

Since z_0, z_1 are solutions of (13), then,

$$\begin{aligned} \frac{\partial^2 y_1}{\partial s^2}(t, s; \lambda, k) &= z_0''(s; \lambda, k)z_1(t; \lambda, k) - z_1''(s; \lambda, k)z_0(t; \lambda, k) \\ &= \lambda^2 s^{-2k} z_0(s; \lambda, k)z_1(t; \lambda, k) - \lambda^2 s^{-2k} z_1(s; \lambda, k)z_0(t; \lambda, k) = \lambda^2 s^{-2k} y_1(t, s; \lambda, k). \end{aligned}$$

So, we prove (25) as well. \square

Proposition 2.3. *Let $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$ be functions such that $\text{supp } u_j \subset B_R$ for $j = 0, 1$ and for some $R > 0$ and let $\lambda > 0$ be a parameter. Let u be a local in time energy solution to (7) on $[1, T)$ according to Definition 1.1. Then, the following integral identity is satisfied for any $t \in [1, T)$*

$$\begin{aligned} \int_{\mathbb{R}^n} u(t, x) \varphi_\lambda(x) dx &= \varepsilon y_0(t, 1; \lambda, k) \int_{\mathbb{R}^n} u_0(x) \varphi_\lambda(x) dx + \varepsilon y_1(t, 1; \lambda, k) \int_{\mathbb{R}^n} u_1(x) \varphi_\lambda(x) dx \\ &\quad + \int_1^t s^{1-p} y_1(t, s; \lambda, k) \int_{\mathbb{R}^n} |u(s, x)|^p \varphi_\lambda(x) dx ds, \end{aligned} \quad (26)$$

where $\varphi_\lambda(x) \doteq \varphi(\lambda x)$ and φ is defined by (11).

Proof. Since we assumed u_0, u_1 compactly supported, we may consider a test function $\psi \in \mathcal{C}^\infty([1, T) \times \mathbb{R}^n)$ in Definition 1.1 according to Remark 1. Therefore, we consider $\psi(s, x) = y_1(t, s; \lambda, k) \varphi_\lambda(x)$ (here t, λ can be considered fixed parameters). Hence, ψ satisfies

$$\begin{aligned} \psi(t, x) &= y_1(t, t; \lambda, k) \varphi_\lambda(x) = 0, & \psi_s(t, x) &= \partial_s y_1(t, t; \lambda, k) \varphi_\lambda(x) = -y_0(t, t; \lambda, k) \varphi_\lambda(x) = -\varphi_\lambda(x), \\ \psi(1, x) &= y_1(t, 1; \lambda, k) \varphi_\lambda(x), & \psi_s(1, x) &= \partial_s y_1(t, 1; \lambda, k) \varphi_\lambda(x) = -y_0(t, 1; \lambda, k) \varphi_\lambda(x), \end{aligned}$$

and

$$\psi_{ss}(s, x) - s^{-2k} \Delta \psi(s, x) = (\partial_s^2 y_1(t, s; \lambda, k) - \lambda^2 s^{-2k} y_1(t, s; \lambda, k)) \varphi_\lambda(x) = 0,$$

where we used (24), (25) and the property $\Delta \varphi = \varphi$.

Hence, employing this ψ in (9), we find immediately (26). \square

Proposition 2.4. *Let y_0, y_1 be the functions defined in (16) and (17), respectively. Then, the following estimates are satisfied for any $t \geq s \geq 1$*

$$y_0(t, s; \lambda, k) \geq \cosh(\lambda(\phi_k(t) - \phi_k(s))), \quad (27)$$

$$y_1(t, s; \lambda, k) \geq (st)^{\frac{k}{2}} \frac{\sinh(\lambda(\phi_k(t) - \phi_k(s)))}{\lambda}. \quad (28)$$

Proof. The proof of the inequalities (27) and (28) is based on the following minimum type principle: let $w = w(t, s; \lambda, k)$ be a solution of the Cauchy problem

$$\begin{cases} \partial_t^2 w - \lambda^2 t^{-2k} w = h, & \text{for } t > s \geq 1, \\ w(s) = w_0, \quad \partial_t w(s) = w_1, \end{cases} \quad (29)$$

where $h = h(t, s; \lambda, k)$ is a continuous function; if $h \geq 0$ and $w_0 = w_1 = 0$ (i.e. w is a supersolution of the homogeneous problem with trivial initial conditions), then, $w(t, s; \lambda, k) \geq 0$ for any $t > s$.

In order to prove this minimum principle, we apply the continuous dependence on initial conditions (note that for $t \geq 1$ the function t^{-2k} is smooth). Indeed, if we denote by w_ϵ the solution to (29) with $w_0 = \epsilon > 0$ and $w_1 = 0$, then, w_ϵ solves the integral equation

$$w_\epsilon(t, s; \lambda, k) = \epsilon + \int_s^t \int_s^\tau (\lambda^2 \sigma^{-2k} w_\epsilon(\sigma, s; \lambda, k) + h(\sigma, s; \lambda, k)) d\sigma d\tau.$$

By contradiction, one can prove easily that $w_\epsilon(t, s; \lambda, k) > 0$ for any $t > s$. Hence, by the continuous dependence on initial data, letting $\epsilon \rightarrow 0$, we find that $w(t, s; \lambda, k) \geq 0$ for any $t > s$.

Note that if $w_0, w_1 \geq 0$ and $w_0 + w_1 > 0$, then, the positivity of w follows straightforwardly from the corresponding integral equation via a contradiction argument, rather than working with the family $\{w_\epsilon\}_{\epsilon > 0}$. Nevertheless, in what follows we consider exactly the limit case $w_0 = w_1 = 0$, for this reason the previous digression was necessary.

Let us prove the validity of (28). We denote by $w_1 = w_1(t, s; \lambda, k)$ the function on the right - hand side of (28). It is easy to see that $w_1(s, s; \lambda, k) = 0$ and $\partial_t w_1(s, s; \lambda, k) = 1$. Moreover,

$$\begin{aligned} \partial_t^2 w_1(t, s; \lambda, k) &= \lambda^{-1} s^{\frac{k}{2}} \left[\frac{k}{2} \left(\frac{k}{2} - 1 \right) t^{\frac{k}{2}-2} \sinh(\lambda(\phi_k(t) - \phi_k(s))) + kt^{\frac{k}{2}-1} \cosh(\lambda(\phi_k(t) - \phi_k(s))) \lambda \phi_k'(t) \right. \\ &\quad \left. + t^{\frac{k}{2}} \sinh(\lambda(\phi_k(t) - \phi_k(s))) (\lambda \phi_k'(t))^2 + t^{\frac{k}{2}} \cosh(\lambda(\phi_k(t) - \phi_k(s))) \lambda \phi_k''(t) \right] \\ &= \lambda^{-1} s^{\frac{k}{2}} \left[\frac{k}{2} \left(\frac{k}{2} - 1 \right) t^{\frac{k}{2}-2} \sinh(\lambda(\phi_k(t) - \phi_k(s))) + t^{\frac{k}{2}} \sinh(\lambda(\phi_k(t) - \phi_k(s))) (\lambda t^{-k})^2 \right] \\ &\leq \lambda^{-1} (st)^{\frac{k}{2}} \sinh(\lambda(\phi_k(t) - \phi_k(s))) (\lambda t^{-k})^2 = \lambda^2 t^{-2k} w_1(t, s; \lambda, k). \end{aligned}$$

Therefore, $y_1 - w_1$ is a supersolution of (29) with $h = 0$ and $w_0 = w_1 = 0$. Thus, applying the minimum principle we have that $(y_1 - w_1)(t, s; \lambda, k) \geq 0$ for any $t > s$, that is, we showed (28).

In a completely analogous way, one can prove (27), repeating the previous argument based on the minimum principle with $w_0(t, s; \lambda, k) \doteq \cosh(\lambda(\phi_k(t) - \phi_k(s)))$ in place of $w_1(t, s; \lambda, k)$ and y_0 in place of y_1 , respectively. \square

After the preliminary results that we have proved so far in this section, we can now introduce the definition of the following *auxiliary function*

$$\xi_q(t, s, x; k) \doteq \int_0^{\lambda_0} e^{-\lambda(A_k(t)+R)} \cosh(\lambda(\phi_k(t) - \phi_k(s))) \varphi_\lambda(x) \lambda^q d\lambda, \quad (30)$$

$$\eta_q(t, s, x; k) \doteq (st)^{k/2} \int_0^{\lambda_0} e^{-\lambda(A_k(t)+R)} \frac{\sinh(\lambda(\phi_k(t) - \phi_k(s)))}{\lambda(\phi_k(t) - \phi_k(s))} \varphi_\lambda(x) \lambda^q d\lambda, \quad (31)$$

where $q > -1$, $\lambda_0 > 0$ is a fixed parameter and A_k is defined by (5).

Remark 4. For $k = 0$ the functions ξ_q and η_q coincide with the corresponding ones given in [38], provided of course that we shift the initial time in the Cauchy problem from 0 to 1.

Combining the results from Propositions 2.3 and 2.4, we may finally derive a fundamental inequality, whose role will be crucial in the next sections in order to prove the blow - up result.

Corollary 2.5. *Let $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$ such that $\text{supp } u_j \subset B_R$ for $j = 0, 1$ and for some $R > 0$. Let u be a local in time energy solution to (7) on $[1, T)$ according to Definition 1.1. Let $q > -1$ and let $\xi_q(t, s, x; k), \eta_q(t, s, x; k)$ be the functions defined by (30) and (31), respectively. Then,*

$$\begin{aligned} \int_{\mathbb{R}^n} u(t, x) \xi_q(t, t, x; k) dx &\geq \varepsilon \int_{\mathbb{R}^n} u_0(x) \xi_q(t, 1, x; k) dx + \varepsilon (\phi_k(t) - \phi_k(1)) \int_{\mathbb{R}^n} u_1(x) \eta_q(t, s, x; k) dx \\ &\quad + \int_1^t (\phi_k(t) - \phi_k(s)) s^{1-p} \int_{\mathbb{R}^n} |u(s, x)|^p \eta_q(t, s, x; k) dx ds \end{aligned} \quad (32)$$

for any $t \in [1, T)$.

Proof. Combining (26) and the lower bound estimates (27), (28), we find

$$\begin{aligned} \int_{\mathbb{R}^n} u(t, x) \varphi_\lambda(x) dx &\geq \varepsilon \cosh(\lambda(\phi_k(t) - \phi_k(1))) \int_{\mathbb{R}^n} u_0(x) \varphi_\lambda(x) dx \\ &\quad + \varepsilon t^{\frac{k}{2}} \lambda^{-1} \sinh(\lambda(\phi_k(t) - \phi_k(1))) \int_{\mathbb{R}^n} u_1(x) \varphi_\lambda(x) dx \\ &\quad + \int_1^t s^{1-p} (st)^{\frac{k}{2}} \lambda^{-1} \sinh(\lambda(\phi_k(t) - \phi_k(s))) \int_{\mathbb{R}^n} |u(s, x)|^p \varphi_\lambda(x) dx ds. \end{aligned}$$

Multiplying both sides of the previous identity by $e^{-\lambda(A_k(t)+R)}\lambda^q$, integrating with respect to λ over $[0, \lambda_0]$ and applying Fubini's theorem, we get (32). \square

2.2 Properties of the auxiliary functions

In this section, we determine some lower and upper bound estimates for the auxiliary functions ξ_q, η_q under suitable assumptions on q .

Let us begin with the lower bound estimates

Lemma 2.6. *Let $n \geq 1$ and $\lambda_0 > 0$. If we assume $q > -1$, then, for $t \geq s \geq 1$ and $|x| \leq A_k(s) + R$ the following lower bound estimates hold:*

$$\xi_q(t, s, x; k) \geq B_0 \langle A_k(s) \rangle^{-q-1}; \quad (33)$$

$$\eta_q(t, s, x; k) \geq B_1 (st)^{\frac{k}{2}} \langle A_k(t) \rangle^{-1} \langle A_k(s) \rangle^{-q}. \quad (34)$$

Here B_0, B_1 are positive constants depending only on λ_0, q, R, k and we employ the notation $\langle y \rangle \doteq 3 + |s|$.

Proof. We follow the main ideas of the proof of Lemma 3.1 in [38]. Since

$$\langle |x| \rangle^{-\frac{n-1}{2}} e^{|x|} \lesssim \varphi(x) \lesssim \langle |x| \rangle^{-\frac{n-1}{2}} e^{|x|} \quad (35)$$

holds for any $x \in \mathbb{R}^n$, we can find a constant $B = B(\lambda_0, R, k) > 0$ independent of λ and s such that

$$B \leq \inf_{\lambda \in \left[\frac{\lambda_0}{\langle A_k(s) \rangle}, \frac{2\lambda_0}{\langle A_k(s) \rangle} \right]} \inf_{|x| \leq A_k(s) + R} e^{-\lambda(A_k(s)+R)} \varphi_\lambda(x).$$

Let us begin with (33). Shrinking the domain of integration in (30) to $\left[\frac{\lambda_0}{\langle A_k(s) \rangle}, \frac{2\lambda_0}{\langle A_k(s) \rangle} \right]$ and applying the previous inequality, we get

$$\begin{aligned} \xi_q(t, s, x; k) &\geq \int_{\lambda_0/\langle A_k(s) \rangle}^{2\lambda_0/\langle A_k(s) \rangle} e^{-\lambda(A_k(t)-A_k(s))} \cosh(\lambda(\phi_k(t) - \phi_k(s))) e^{-\lambda(A_k(s)+R)} \varphi_\lambda(x) \lambda^q d\lambda \\ &\geq B \int_{\lambda_0/\langle A_k(s) \rangle}^{2\lambda_0/\langle A_k(s) \rangle} e^{-\lambda(A_k(t)-A_k(s))} \cosh(\lambda(\phi_k(t) - \phi_k(s))) \lambda^q d\lambda \\ &= B/2 \int_{\lambda_0/\langle A_k(s) \rangle}^{2\lambda_0/\langle A_k(s) \rangle} \left(1 + e^{-2\lambda(\phi_k(t)-\phi_k(s))} \right) \lambda^q d\lambda \\ &\geq B/2 \int_{\lambda_0/\langle A_k(s) \rangle}^{2\lambda_0/\langle A_k(s) \rangle} \lambda^q d\lambda = \frac{B(2^{q+1} - 1)\lambda_0^{q+1}}{2(q+1)} \langle A_k(s) \rangle^{-q-1}. \end{aligned}$$

We prove now (34). Repeating similar steps as before, we arrive at

$$\begin{aligned} \eta_q(t, s, x; k) &\geq (st)^{\frac{k}{2}} \int_{\lambda_0/\langle A_k(s) \rangle}^{2\lambda_0/\langle A_k(s) \rangle} e^{-\lambda(A_k(t)-A_k(s))} \frac{\sinh(\lambda(\phi_k(t) - \phi_k(s)))}{\lambda(\phi_k(t) - \phi_k(s))} e^{-\lambda(A_k(s)+R)} \varphi_\lambda(x) \lambda^q d\lambda \\ &\geq \frac{B}{2} (st)^{\frac{k}{2}} \int_{\lambda_0/\langle A_k(s) \rangle}^{2\lambda_0/\langle A_k(s) \rangle} \frac{1 - e^{-2\lambda(\phi_k(t)-\phi_k(s))}}{\phi_k(t) - \phi_k(s)} \lambda^{q-1} d\lambda \\ &\geq \frac{B}{2} (st)^{\frac{k}{2}} \frac{1 - e^{-2\lambda_0 \frac{\phi_k(t)-\phi_k(s)}{\langle A_k(s) \rangle}}}{\phi_k(t) - \phi_k(s)} \int_{\lambda_0/\langle A_k(s) \rangle}^{2\lambda_0/\langle A_k(s) \rangle} \lambda^{q-1} d\lambda \\ &= \frac{B(2^q - 1)\lambda_0^q}{2q} (st)^{\frac{k}{2}} \langle A_k(s) \rangle^{-q} \frac{1 - e^{-2\lambda_0 \frac{\phi_k(t)-\phi_k(s)}{\langle A_k(s) \rangle}}}{\phi_k(t) - \phi_k(s)}. \end{aligned}$$

The previous inequality implies (34), provided that

$$\frac{1 - e^{-2\lambda_0 \frac{\phi_k(t)-\phi_k(s)}{\langle A_k(s) \rangle}}}{\phi_k(t) - \phi_k(s)} \gtrsim \langle A_k(t) \rangle^{-1}$$

holds. Let us prove this last inequality. For $\phi_k(t) - \phi_k(s) \geq \frac{1}{2\lambda_0} \langle A_k(s) \rangle$, we have

$$1 - e^{-2\lambda_0 \frac{\phi_k(t)-\phi_k(s)}{\langle A_k(s) \rangle}} \geq 1 - e^{-1}$$

and, consequently,

$$\frac{1 - e^{-2\lambda_0 \frac{\phi_k(t) - \phi_k(s)}{\langle A_k(s) \rangle}}}{\phi_k(t) - \phi_k(s)} \gtrsim (\phi_k(t) - \phi_k(s))^{-1} \geq A_k(t)^{-1} \geq \langle A_k(t) \rangle^{-1}.$$

On the other hand, in the case $\phi_k(t) - \phi_k(s) \leq \frac{1}{2\lambda_0} \langle A_k(s) \rangle$, employing the inequality $1 - e^{-\sigma} \geq \sigma/2$ for $\sigma \in [0, 1]$, we find immediately

$$\frac{1 - e^{-2\lambda_0 \frac{\phi_k(t) - \phi_k(s)}{\langle A_k(s) \rangle}}}{\phi_k(t) - \phi_k(s)} \geq \frac{\lambda_0}{\langle A_k(s) \rangle} \geq \frac{\lambda_0}{\langle A_k(t) \rangle}.$$

So, also the proof of (34) is completed. \square

Next we prove an upper bound estimate in the special case $s = t$.

Lemma 2.7. *Let $n \geq 1$ and $\lambda_0 > 0$. If we assume $q > (n - 3)/2$, then, for $t \geq 1$ and $|x| \leq A_k(t) + R$ the following upper bound estimate holds:*

$$\xi_q(t, t, x; k) \leq B_2 \langle A_k(t) \rangle^{-\frac{n-1}{2}} \langle A_k(t) - |x| \rangle^{\frac{n-3}{2}-q}. \quad (36)$$

Here B_2 is a positive constant depending only on λ_0, q, R, k and $\langle y \rangle$ denotes the same function as in the statement of Lemma 2.6.

Proof. We follow the proof of Lemma 3.1 (iii) in [38]. Applying (35), we get

$$\xi_q(t, t, x; k) = \int_0^{\lambda_0} e^{-\lambda(A_k(t)+R)} \varphi_\lambda(x) \lambda^q d\lambda \lesssim \int_0^{\lambda_0} \langle \lambda |x| \rangle^{-\frac{n-1}{2}} e^{-\lambda(A_k(t)+R-|x|)} \lambda^q d\lambda.$$

Let us consider separately two different cases. If $|x| \leq (A_k(t) + R)/2$, then,

$$\begin{aligned} \xi_q(t, t, x; k) &\lesssim \int_0^{\lambda_0} e^{-\lambda(A_k(t)+R-|x|)} \lambda^q d\lambda \lesssim \int_0^{\lambda_0} e^{-\lambda(A_k(t)+R)/2} \lambda^q d\lambda \\ &\lesssim (A_k(t) + R)^{-q-1} \int_0^\infty e^{-\mu/2} \mu^q d\mu \lesssim (A_k(t) + R)^{-q-1} \lesssim \langle A_k(t) \rangle^{-q-1} \\ &\lesssim \langle A_k(t) \rangle^{-\frac{n-1}{2}} \langle A_k(t) - |x| \rangle^{\frac{n-3}{2}-q}. \end{aligned}$$

In particular, in the last estimate we used the inequality $\langle A_k(t) - |x| \rangle \lesssim \langle A_k(t) \rangle$, which follows trivially from $|A_k(t) - |x|| \leq A_k(t)$ for $|x| \leq A_k(t)$ and from $\langle A_k(t) - |x| \rangle \lesssim 1$ for $A_k(t) \leq |x| \leq (A_k(t) + R)/2$.

On the other hand, for $|x| \geq (A_k(t) + R)/2$, we may estimate

$$\begin{aligned} \xi_q(t, t, x; k) &\lesssim (A_k(t) + R)^{-\frac{n-1}{2}} \int_0^{\lambda_0} e^{-\lambda(A_k(t)+R-|x|)} \lambda^{q-\frac{n-1}{2}} d\lambda \\ &\lesssim \langle A_k(t) \rangle^{-\frac{n-1}{2}} (A_k(t) + R - |x|)^{-q+\frac{n-3}{2}} \int_0^\infty e^{-\mu} \mu^{q-\frac{n-1}{2}} d\mu \\ &\lesssim \langle A_k(t) \rangle^{-\frac{n-1}{2}} (A_k(t) + R - |x|)^{-q+\frac{n-3}{2}}. \end{aligned} \quad (37)$$

When $(A_k(t) + R)/2 \leq |x| \leq A_k(t)$, thanks to the inequality $A_k(t) + R - |x| \gtrsim \langle A_k(t) - |x| \rangle$, from (37) it follows easily (36); while for $A_k(t) \leq |x| \leq A_k(t) + R$, as $\langle A_k(t) - |x| \rangle \approx 1$, the estimate

$$\begin{aligned} \xi_q(t, t, x; k) &\lesssim (A_k(t) + R)^{-\frac{n-1}{2}} \int_0^{\lambda_0} e^{-\lambda(A_k(t)+R-|x|)} \lambda^{q-\frac{n-1}{2}} d\lambda \\ &\lesssim \langle A_k(t) \rangle^{-\frac{n-1}{2}} \int_0^{\lambda_0} \lambda^{q-\frac{n-1}{2}} d\lambda \lesssim \langle A_k(t) \rangle^{-\frac{n-1}{2}} \end{aligned}$$

is sufficient to conclude (36). This completes the proof. \square

2.3 Derivation of the iteration frame

In this section, we introduce the time – dependent functional whose dynamic is studied in order to prove the blow – up result. Hence, we derive the iteration frame for this functional and a first lower bound estimate of logarithmic type.

Let us introduce the functional

$$\mathcal{U}(t) \doteq t^{-\frac{k}{2}} \int_{\mathbb{R}^n} u(t, x) \xi_q(t, t, x; k) dx \quad (38)$$

for $t \geq 1$ and for some $q > (n-3)/2$. From (32), (33) and (34), it follows

$$\mathcal{U}(t) \gtrsim B_0 \varepsilon t^{-\frac{k}{2}} \int_{\mathbb{R}^n} u_0(x) dx + B_1 \varepsilon \frac{A_k(t)}{\langle A_k(t) \rangle} \int_{\mathbb{R}^n} u_1(x) dx.$$

If we assume that u_0, u_1 are both nonnegative and nontrivial, then, we find that

$$\mathcal{U}(t) \gtrsim \varepsilon \quad (39)$$

for any $t \in [1, T)$, where the unexpressed multiplicative constant depends on u_0, u_1 .

In the next proposition, we derive the iteration frame for the functional \mathcal{U} .

Proposition 2.8. *Suppose that the assumptions in Corollary 2.5 are satisfied and let $q = (n-1)/2 - 1/p$. If \mathcal{U} is defined by (38), then, there exists a positive constant $C = C(n, p, R, k)$ such that*

$$\mathcal{U}(t) \geq C \langle A_k(t) \rangle^{-1} \int_1^t \frac{\phi_k(t) - \phi_k(s)}{s} (\log \langle A_k(s) \rangle)^{-(p-1)} (\mathcal{U}(s))^p ds \quad (40)$$

for any $t \in (1, T)$.

Proof. By the definition of the functional (38), applying Hölder's inequality we get

$$s^{\frac{k}{2}} \mathcal{U}(s) \leq \left(\int_{\mathbb{R}^n} |u(s, x)|^p \eta_q(t, s, x; k) dx \right)^{1/p} \left(\int_{B_{R+A_k(s)}} \frac{(\xi_q(s, s, x; k))^{p'}}{(\eta_q(t, s, x; k))^{p'/p}} dx \right)^{1/p'}$$

where $1/p + 1/p' = 1$. Therefore,

$$\int_{\mathbb{R}^n} |u(s, x)|^p \eta_q(t, s, x; k) dx \geq (s^{\frac{k}{2}} \mathcal{U}(s))^p \left(\int_{B_{R+A_k(s)}} \frac{(\xi_q(s, s, x; k))^{p/(p-1)}}{(\eta_q(t, s, x; k))^{1/(p-1)}} dx \right)^{-(p-1)}. \quad (41)$$

Let us determine now an upper bound estimates for the integral on the right – hand side of (41). By using (36) and (34), we obtain

$$\begin{aligned} & \int_{B_{R+A_k(s)}} \frac{(\xi_q(s, s, x; k))^{p/(p-1)}}{(\eta_q(t, s, x; k))^{1/(p-1)}} dx \\ & \leq B_1^{-\frac{1}{p-1}} B_2^{\frac{p}{p-1}} \langle A_k(s) \rangle^{-\frac{n-1}{2} \frac{p}{p-1}} (st)^{-\frac{k}{2(p-1)}} \langle A_k(t) \rangle^{\frac{1}{p-1}} \langle A_k(s) \rangle^{\frac{q}{p-1}} \int_{B_{R+A_k(s)}} \langle A_k(s) - |x| \rangle^{\frac{n-3}{2} - q} \frac{p}{p-1} dx \\ & \leq B_1^{-\frac{1}{p-1}} B_2^{\frac{p}{p-1}} (st)^{-\frac{k}{2(p-1)}} \langle A_k(t) \rangle^{\frac{1}{p-1}} \langle A_k(s) \rangle^{\frac{1}{p-1} (-\frac{n-1}{2} p + \frac{n-1}{2} - \frac{1}{p})} \int_{B_{R+A_k(s)}} \langle A_k(s) - |x| \rangle^{-1} dx \\ & \leq B_1^{-\frac{1}{p-1}} B_2^{\frac{p}{p-1}} (st)^{-\frac{k}{2(p-1)}} \langle A_k(t) \rangle^{\frac{1}{p-1}} \langle A_k(s) \rangle^{\frac{1}{p-1} (-\frac{n-1}{2} p + \frac{n-1}{2} - \frac{1}{p}) + n-1} \log \langle A_k(s) \rangle, \end{aligned}$$

where in the second step we used $q = (n-1)/2 - 1/p$ to get exactly -1 as power of the function in the integral. Hence, from (41) we get

$$\begin{aligned} \int_{\mathbb{R}^n} |u(s, x)|^p \eta_q(t, s, x; k) dx & \gtrsim (s^{\frac{k}{2}} \mathcal{U}(s))^p (st)^{\frac{k}{2}} \langle A_k(t) \rangle^{-1} \langle A_k(s) \rangle^{\frac{n-1}{2}(p-1) + \frac{1}{p} - (n-1)(p-1)} (\log \langle A_k(s) \rangle)^{-(p-1)} \\ & \gtrsim t^{\frac{k}{2}} \langle A_k(t) \rangle^{-1} s^{\frac{k}{2}(p+1)} \langle A_k(s) \rangle^{-\frac{n-1}{2}(p-1) + \frac{1}{p}} (\log \langle A_k(s) \rangle)^{-(p-1)} (\mathcal{U}(s))^p. \end{aligned}$$

If we combine the previous lower bound estimate and (32), we have

$$\begin{aligned} \mathcal{U}(t) & \geq t^{-\frac{k}{2}} \int_1^t (\phi_k(t) - \phi_k(s)) s^{1-p} \int_{\mathbb{R}^n} |u(s, x)|^p \eta_q(t, s, x; k) dx ds \\ & \gtrsim \langle A_k(t) \rangle^{-1} \int_1^t (\phi_k(t) - \phi_k(s)) s^{1-p + \frac{k}{2}(p+1)} \langle A_k(s) \rangle^{-\frac{n-1}{2}(p-1) + \frac{1}{p}} (\log \langle A_k(s) \rangle)^{-(p-1)} (\mathcal{U}(s))^p ds \\ & \gtrsim \langle A_k(t) \rangle^{-1} \int_1^t (\phi_k(t) - \phi_k(s)) \langle A_k(s) \rangle^{\frac{1-p}{1-k} + \frac{k(p+1)}{2(1-k)} - \frac{n-1}{2}(p-1) + \frac{1}{p}} (\log \langle A_k(s) \rangle)^{-(p-1)} (\mathcal{U}(s))^p ds \\ & \gtrsim \langle A_k(t) \rangle^{-1} \int_1^t (\phi_k(t) - \phi_k(s)) \langle A_k(s) \rangle^{-\left(\frac{n-1}{2} + \frac{2-k}{2(1-k)}\right)p + \left(\frac{n-1}{2} + \frac{2+k}{2(1-k)}\right) + \frac{1}{p}} (\log \langle A_k(s) \rangle)^{-(p-1)} (\mathcal{U}(s))^p ds, \end{aligned}$$

where in third step we used $s = (1 - k)^{\frac{1}{1-k}} (A_k(s) + \phi_k(1))^{\frac{1}{1-k}} \approx \langle A_k(s) \rangle^{\frac{1}{1-k}}$ for $s \geq 1$. Since $p = p_0(n, k)$ from (10) it follows

$$-\left(\frac{n-1}{2} + \frac{2-k}{2(1-k)}\right)p + \left(\frac{n-1}{2} + \frac{2+k}{2(1-k)}\right) + \frac{1}{p} = -1 - \frac{k}{1-k} = -\frac{1}{1-k}, \quad (42)$$

then, plugging (42) in the last lower bound estimate for $\mathcal{U}(t)$ we find

$$\begin{aligned} \mathcal{U}(t) &\gtrsim \langle A_k(t) \rangle^{-1} \int_1^t (\phi_k(t) - \phi_k(s)) \langle A_k(s) \rangle^{-\frac{1}{1-k}} (\log \langle A_k(s) \rangle)^{-(p-1)} (\mathcal{U}(s))^p ds \\ &\gtrsim \langle A_k(t) \rangle^{-1} \int_1^t \frac{\phi_k(t) - \phi_k(s)}{s} (\log \langle A_k(s) \rangle)^{-(p-1)} (\mathcal{U}(s))^p ds, \end{aligned}$$

which is precisely (40). This completes the proof. \square

Lemma 2.9. *Suppose that the assumptions in Corollary 2.5 are satisfied. Then, there exists a positive constant $K = K(u_0, u_1, n, p, R, k)$ such that the lower bound estimate*

$$\int_{\mathbb{R}^n} |u(t, x)|^p dx \geq K \varepsilon^p \langle A_k(t) \rangle^{(n-1)(1-\frac{p}{2}) + \frac{kp}{2(1-k)}} \quad (43)$$

holds for any $t \in (1, T)$.

Proof. We adapt the proof of Lemma 5.1 in [38] to our model. Let us fix $q > (n-3)/2 + 1/p'$. Combining (38), (39) and Hölder's inequality, it results

$$\varepsilon t^{\frac{k}{2}} \lesssim t^{\frac{k}{2}} \mathcal{U}(t) = \int_{\mathbb{R}^n} u(t, x) \xi_q(t, t, x; k) dx \leq \left(\int_{\mathbb{R}^n} |u(t, x)|^p dx \right)^{1/p} \left(\int_{B_{R+A_k(t)}} (\xi_q(t, t, x; k))^{p'} dx \right)^{1/p'}.$$

Hence,

$$\int_{\mathbb{R}^n} |u(t, x)|^p dx \gtrsim \varepsilon^p t^{\frac{kp}{2}} \left(\int_{B_{R+A_k(t)}} (\xi_q(t, t, x; k))^{p'} dx \right)^{-(p-1)}. \quad (44)$$

Let us determine an upper bound estimates for the integral of $\xi_q(t, t, x; k)^{p'}$. By using (36), we have

$$\begin{aligned} \int_{B_{R+A_k(t)}} (\xi_q(t, t, x; k))^{p'} dx &\lesssim \langle A_k(t) \rangle^{-\frac{n-1}{2}p'} \int_{B_{R+A_k(t)}} \langle A_k(t) - |x| \rangle^{(n-3)p'/2 - p'q} dx \\ &\lesssim \langle A_k(t) \rangle^{-\frac{n-1}{2}p'} \int_0^{R+A_k(t)} r^{n-1} \langle A_k(t) - r \rangle^{(n-3)p'/2 - p'q} dr \\ &\lesssim \langle A_k(t) \rangle^{-\frac{n-1}{2}p' + n-1} \int_0^{R+A_k(t)} \langle A_k(t) - r \rangle^{(n-3)p'/2 - p'q} dr. \end{aligned}$$

Performing the change of variable $A_k(t) - r = \varrho$, one gets

$$\begin{aligned} \int_{B_{R+A_k(t)}} (\xi_q(t, t, x; k))^{p'} dx &\lesssim \langle A_k(t) \rangle^{-\frac{n-1}{2}p' + n-1} \int_{-R}^{A_k(t)} (3 + |\varrho|)^{(n-3)p'/2 - p'q} d\varrho \\ &\lesssim \langle A_k(t) \rangle^{-\frac{n-1}{2}p' + n-1} \end{aligned}$$

because of $(n-3)p'/2 - p'q < -1$.

Combining this upper bound estimates for the integral of $\xi_q(t, t, x; k)^{p'}$, (44) and using $t \approx \langle A_k(t) \rangle^{\frac{1}{1-k}}$ for $t \geq 1$, we arrive at (43). The proof is over. \square

In Proposition 2.8, we derive the iteration frame for \mathcal{U} . In the next result, we shall prove a first lower bound estimate for \mathcal{U} , which shall be the base case of the inductive argument in Section 2.4.

Proposition 2.10. *Suppose that the assumptions in Corollary 2.5 are satisfied and let $q = (n-1)/2 - 1/p$. Let \mathcal{U} be defined by (38). Then, for $t \geq 3/2$ the functional $\mathcal{U}(t)$ fulfills*

$$\mathcal{U}(t) \geq M \varepsilon^p \log\left(\frac{2t}{3}\right), \quad (45)$$

where the positive constant M depends on u_0, u_1, n, p, R, k .

Proof. From (32) we know that

$$\mathcal{U}(t) \geq t^{-\frac{k}{2}} \int_1^t (\phi_k(t) - \phi_k(s)) s^{1-p} \int_{\mathbb{R}^n} |u(s, x)|^p \eta_q(t, s, x; k) dx ds.$$

Consequently, applying (34) first and then (43), we find

$$\begin{aligned} \mathcal{U}(t) &\geq B_1 \langle A_k(t) \rangle^{-1} \int_1^t (\phi_k(t) - \phi_k(s)) s^{1-p+\frac{k}{2}} \langle A_k(s) \rangle^{-q} \int_{\mathbb{R}^n} |u(s, x)|^p dx ds \\ &\geq B_1 K \varepsilon^p \langle A_k(t) \rangle^{-1} \int_1^t (\phi_k(t) - \phi_k(s)) s^{1-p+\frac{k}{2}} \langle A_k(s) \rangle^{-q+(n-1)(1-\frac{p}{2})+\frac{kp}{2(1-k)}} ds \\ &\gtrsim \varepsilon^p \langle A_k(t) \rangle^{-1} \int_1^t (\phi_k(t) - \phi_k(s)) \langle A_k(s) \rangle^{(1-p+\frac{k}{2})\frac{1}{1-k}-\frac{n-1}{2}+\frac{1}{p}+(n-1)(1-\frac{p}{2})+\frac{kp}{2(1-k)}} ds \\ &\gtrsim \varepsilon^p \langle A_k(t) \rangle^{-1} \int_1^t (\phi_k(t) - \phi_k(s)) \langle A_k(s) \rangle^{-\left(\frac{n-1}{2}+\frac{2-k}{2(1-k)}\right)p+\left(\frac{n-1}{2}+\frac{2+k}{2(1-k)}\right)+\frac{1}{p}} ds \\ &\gtrsim \varepsilon^p \langle A_k(t) \rangle^{-1} \int_1^t (\phi_k(t) - \phi_k(s)) \langle A_k(s) \rangle^{-\frac{1}{1-k}} ds \gtrsim \varepsilon^p \langle A_k(t) \rangle^{-1} \int_1^t \frac{\phi_k(t) - \phi_k(s)}{s} ds. \end{aligned}$$

We estimate now the integral in the right – hand side of the previous chain of inequalities. Integration by parts leads to

$$\begin{aligned} \int_1^t \frac{\phi_k(t) - \phi_k(s)}{s} ds &= (\phi_k(t) - \phi_k(s)) \log s \Big|_{s=1}^{s=t} + \int_1^t \phi_k'(s) \log s ds \\ &= \int_1^t s^{-k} \log s ds \geq t^{-k} \int_1^t \log s ds. \end{aligned}$$

Therefore, for $t \geq 3/2$

$$\begin{aligned} \mathcal{U}(t) &\gtrsim \varepsilon^p \langle A_k(t) \rangle^{-1} t^{-k} \int_1^t \log s ds \geq \varepsilon^p \langle A_k(t) \rangle^{-1} t^{-k} \int_{2t/3}^t \log s ds \geq (1/3) \varepsilon^p \langle A_k(t) \rangle^{-1} t^{1-k} \log(2t/3) \\ &\gtrsim \varepsilon^p \log(2t/3), \end{aligned}$$

where in the last line we employed $t \approx \langle A_k(t) \rangle^{\frac{1}{1-k}}$ for $t \geq 1$. Also, the proof is complete. \square

2.4 Iteration argument

In this section we prove the blow – up result. More specifically, we are going to show a sequence of lower bound estimates for \mathcal{U} and from these lower bound estimates we conclude that for t over a certain ε – dependent threshold the functional $\mathcal{U}(t)$ may not be finite.

Our goal is to show the validity of the sequence of lower bound estimates

$$\mathcal{U}(t) \geq C_j (\log \langle A_k(t) \rangle)^{-\beta_j} \left(\log \left(\frac{t}{\ell_j} \right) \right)^{\alpha_j} \quad \text{for } t \geq \ell_j \quad (46)$$

for any $j \in \mathbb{N}$, where the bounded sequence of parameters characterizing the slicing procedure is $\{\ell_j\}_{j \in \mathbb{N}}$ with $\ell_j \doteq 2 - 2^{-(j+1)}$ and $\{C_j\}_{j \in \mathbb{N}}, \{\alpha_j\}_{j \in \mathbb{N}}, \{\beta_j\}_{j \in \mathbb{N}}$ are sequences of positive numbers that we will determine throughout the iteration argument.

In order to show (46), we apply an inductive argument. As we have already said, the crucial idea here is to apply a slicing procedure for the domain of integration in the iteration frame (40), in order to increase the power of the second logarithmic term in (46) step by step. This idea was introduced for the first time in [1] and since then it has been applied successfully to study the blow – up dynamic of semilinear wave models in critical cases, overcoming the difficulties in the application of Kato's lemma for critical cases.

Since (46) is true in the base case $j = 0$, provided that $C_0 \doteq M \varepsilon^p$ and $\alpha_0 \doteq 1, \beta_0 = 0$ (cf. Proposition 2.10), it remains to prove the inductive step. We assume (46) true for $j \geq 0$ and we have to prove it for $j + 1$. Plugging (46) in (40), for $t \geq \ell_{j+1}$ we get

$$\begin{aligned} \mathcal{U}(t) &\geq C C_j^p \langle A_k(t) \rangle^{-1} \int_{\ell_j}^t \frac{\phi_k(t) - \phi_k(s)}{s} (\log \langle A_k(s) \rangle)^{-(p-1)-\beta_j p} \left(\log \left(\frac{s}{\ell_j} \right) \right)^{\alpha_j p} ds \\ &\geq C C_j^p \langle A_k(t) \rangle^{-1} (\log \langle A_k(t) \rangle)^{-(p-1)-\beta_j p} \int_{\ell_j}^t \frac{\phi_k(t) - \phi_k(s)}{s} \left(\log \left(\frac{s}{\ell_j} \right) \right)^{\alpha_j p} ds. \end{aligned}$$

Using integration by parts, we find

$$\begin{aligned}
& \int_{\ell_j}^t \frac{\phi_k(t) - \phi_k(s)}{s} \left(\log \left(\frac{s}{\ell_j} \right) \right)^{\alpha_j p} ds \\
&= (\phi_k(t) - \phi_k(s)) (\alpha_j p + 1)^{-1} \left(\log \left(\frac{s}{\ell_j} \right) \right)^{\alpha_j p + 1} \Big|_{s=\ell_j}^{s=t} + (\alpha_j p + 1)^{-1} \int_{\ell_j}^t \phi_k'(s) \left(\log \left(\frac{s}{\ell_j} \right) \right)^{\alpha_j p + 1} ds \\
&= (\alpha_j p + 1)^{-1} \int_{\ell_j}^t s^{-k} \left(\log \left(\frac{s}{\ell_j} \right) \right)^{\alpha_j p + 1} ds \geq (\alpha_j p + 1)^{-1} t^{-k} \int_{\ell_j}^t \left(\log \left(\frac{s}{\ell_j} \right) \right)^{\alpha_j p + 1} ds \\
&\geq (\alpha_j p + 1)^{-1} t^{-k} \int_{\frac{\ell_j t}{\ell_{j+1}}}^t \left(\log \left(\frac{s}{\ell_j} \right) \right)^{\alpha_j p + 1} ds \geq (\alpha_j p + 1)^{-1} t^{1-k} \left(1 - \frac{\ell_j}{\ell_{j+1}} \right) \left(\log \left(\frac{t}{\ell_{j+1}} \right) \right)^{\alpha_j p + 1} \\
&\geq (\alpha_j p + 1)^{-1} 2^{-(j+3)} \gamma_k \langle A_k(t) \rangle \left(\log \left(\frac{t}{\ell_{j+1}} \right) \right)^{\alpha_j p + 1},
\end{aligned}$$

where in the last step we applied $1 - \ell_j/\ell_{j+1} > 2^{-(j+3)}$ and $t^{1-k} \geq \gamma_k \langle A_k(t) \rangle$ for $t \geq 1$ with

$$\gamma_k \doteq \begin{cases} 1/3 & \text{if } k \in [0, 2/3], \\ (1-k) & \text{if } k \in [2/3, 1]. \end{cases}$$

Therefore,

$$\mathcal{U}(t) \geq C \gamma_k 2^{-(j+3)} (\alpha_j p + 1)^{-1} C_j^p (\log \langle A_k(t) \rangle)^{-(p-1) - \beta_j p} \left(\log \left(\frac{t}{\ell_{j+1}} \right) \right)^{\alpha_j p + 1}$$

for $t \geq \ell_{j+1}$, that is, we proved (46) for $j+1$, provided that

$$C_{j+1} \doteq C \gamma_k 2^{-(j+3)} (\alpha_j p + 1)^{-1} C_j^p, \quad \alpha_{j+1} \doteq 1 + p \alpha_j, \quad \beta_{j+1} \doteq p - 1 + p \beta_j.$$

Next we establish a lower bound estimate for C_j . For this purpose, we provide first an explicit representation of the exponents α_j and β_j . Employing recursively the relations $\alpha_j = 1 + p \alpha_{j-1}$ and $\beta_j = (p-1) + p \beta_{j-1}$ and the initial exponents $\alpha_0 = 1$, $\beta_0 = 0$, we obtain

$$\alpha_j = \alpha_0 p^j + \sum_{k=0}^{j-1} p^k = \frac{p^{j+1} - 1}{p-1} \quad \text{and} \quad \beta_j = p^j \beta_0 + (p-1) \sum_{k=0}^{j-1} p^k = p^j - 1. \quad (47)$$

In particular, $\alpha_{j-1} p + 1 = \alpha_j \leq p^{j+1}/(p-1)$ implies that

$$C_j \geq D (2p)^{-j} C_{j-1}^p \quad (48)$$

for any $j \geq 1$, where $D \doteq 2^{-2} C \gamma_k (p-1)/p$. Applying the logarithmic function to both sides of (48) and using iteratively the resulting inequality, we find

$$\begin{aligned}
\log C_j &\geq p \log C_{j-1} - j \log(2p) + \log D \\
&\geq \dots \geq p^j \log C_0 - \left(\sum_{k=0}^{j-1} (j-k) p^k \right) \log(2p) + \left(\sum_{k=0}^{j-1} p^k \right) \log D \\
&= p^j \left(\log M \varepsilon^p - \frac{p \log(2p)}{(p-1)^2} + \frac{\log D}{p-1} \right) + \left(\frac{j}{p-1} + \frac{p}{(p-1)^2} \right) \log(2p) - \frac{\log D}{p-1},
\end{aligned}$$

where we used the identity

$$\sum_{k=0}^{j-1} (j-k) p^k = \frac{1}{p-1} \left(\frac{p^{j+1} - p}{p-1} - j \right). \quad (49)$$

Let us define $j_0 = j_0(n, p, k)$ as the smallest nonnegative integer such that

$$j_0 \geq \frac{\log D}{\log(2p)} - \frac{p}{p-1}.$$

Hence, for any $j \geq j_0$ we have the estimate

$$\log C_j \geq p^j \left(\log M \varepsilon^p - \frac{p \log(2p)}{(p-1)^2} + \frac{\log D}{p-1} \right) = p^j \log(E \varepsilon^p), \quad (50)$$

where $E \doteq M(2p)^{-p/(p-1)^2} D^{1/(p-1)}$.

Combining (46), (47) and (50), we arrive at

$$\begin{aligned} \mathcal{U}(t) &\geq \exp\left(p^j \log(E\varepsilon^p)\right) (\log\langle A_k(t)\rangle)^{-\beta_j} \left(\log\left(\frac{t}{2}\right)\right)^{\alpha_j} \\ &= \exp\left(p^j \log(E\varepsilon^p)\right) (\log\langle A_k(t)\rangle)^{-p^j+1} \left(\log\left(\frac{t}{2}\right)\right)^{(p^{j+1}-1)/(p-1)} \\ &= \exp\left(p^j \log\left(E\varepsilon^p (\log\langle A_k(t)\rangle)^{-1} \left(\log\left(\frac{t}{2}\right)\right)^{p/(p-1)}\right)\right) \log\langle A_k(t)\rangle \left(\log\left(\frac{t}{2}\right)\right)^{-1/(p-1)} \end{aligned}$$

for $t \geq 2$ and any $j \geq j_0$. Since for $t \geq t_0(k) \doteq \max\{4, \gamma_k^{-1/k}\}$ the inequalities

$$\log\langle A_k(t)\rangle \leq (1-k) \log t - \log \gamma_k \leq \log t \quad \text{and} \quad \log\left(\frac{t}{2}\right) \geq 2^{-1} \log t$$

hold true, then,

$$\mathcal{U}(t) \geq \exp\left(p^j \log\left(2^{-p/(p-1)} E\varepsilon^p (\log t)^{1/(p-1)}\right)\right) \log\langle A_k(t)\rangle \left(\log\left(\frac{t}{2}\right)\right)^{-1/(p-1)} \quad (51)$$

for $t \geq t_0$ and any $j \geq j_0$. Let us denote $J(t, \varepsilon) \doteq 2^{-p/(p-1)} E\varepsilon^p (\log t)^{1/(p-1)}$.

If we choose $\varepsilon_0 = \varepsilon_0(n, p, k, \lambda_0, R, u_0, u_1)$ sufficiently small so that

$$\exp\left(2^p E^{1-p} \varepsilon_0^{-p(p-1)}\right) \geq t_0,$$

then, for any $\varepsilon \in (0, \varepsilon_0]$ and for $t > \exp\left(2^p E^{1-p} \varepsilon^{-p(p-1)}\right)$ we get $t \geq t_0$ and $J(t, \varepsilon) > 1$. Consequently, for any $\varepsilon \in (0, \varepsilon_0]$ and for $t > \exp\left(2^p E^{1-p} \varepsilon^{-p(p-1)}\right)$ letting $j \rightarrow \infty$ in (51) it results that the lower bound for $\mathcal{U}(t)$ blows up; hence, $\mathcal{U}(t)$ is not finite as well. Also, we showed that \mathcal{U} blows up in finite time and, moreover, we proved the upper bound estimate for the lifespan

$$T(\varepsilon) \leq \exp\left(2^p E^{1-p} \varepsilon^{-p(p-1)}\right).$$

Therefore, we completed the proof of Theorem 1.2.

3 Semilinear wave equation in EdeS spacetime: subcritical case

As byproduct of the approach developed in Section 2, we derive in this section the upper bound estimates for the lifespan of local in time solutions in the subcritical case $1 < p < \max\{p_0(n, k), p_1(n, k)\}$. Our main tool will be the generalization of Kato's lemma containing the upper bound estimates for the lifespan proved in [36], whose statement is recalled below for the ease of the reader.

Lemma 3.1. *Let $p > 1$, $a > 0$, $q > 0$ satisfy*

$$M \doteq \frac{p-1}{2}a - \frac{q}{2} + 1 > 0.$$

Assume that $F \in \mathcal{C}^2([\tau, T])$ satisfies

$$F(t) \geq At^a \quad \text{for } t \geq T_0 \geq \tau, \quad (52)$$

$$F''(t) \geq B(t+R)^{-q}|F(t)|^p \quad \text{for } t \geq \tau, \quad (53)$$

$$F(\tau) \geq 0, \quad F'(\tau) > 0, \quad (54)$$

where A, B, R, T_0 are positive constants. Then, there exists a positive constant $C_0 = C_0(p, a, q, B, \tau)$ such that

$$T < 2^{\frac{2}{M}} T_1 \quad (55)$$

holds, provided that

$$T_1 \doteq \max\left\{T_0, \frac{F(\tau)}{F'(\tau)}, R\right\} \geq C_0 A^{-\frac{p-1}{2M}}. \quad (56)$$

As we are going to apply this generalization of Kato's lemma, we will find some estimates already obtained in [14] in the treatment of the subcritical case, although the proofs that lead to these estimates are different.

Let us assume that u_0, u_1 are compactly supported with supports in B_R for some $R > 0$, nonnegative and nontrivial functions. Let u be a solution on $[1, T)$ of (7) according to Definition 1.1 such that

$$\text{supp } u(t, \cdot) \subset B_{R+A_k(t)}$$

for any $t \in (1, T)$, where $T = T(\varepsilon)$ is the lifespan of u .

Hence, we introduce as time – dependent functional the spatial average of u

$$U(t) \doteq \int_{\mathbb{R}^n} u(t, x) \, dx. \quad (57)$$

Choosing a test function ψ such that $\psi = 1$ on $\{(s, x) \in [1, t] \times \mathbb{R}^n : |x| \leq R + A_k(s)\}$ in (8), we get

$$U'(t) = U'(1) + \int_1^t \int_{\mathbb{R}^n} s^{1-p} |u(s, x)|^p \, dx \, ds.$$

Also, differentiating the previous identity with respect to t , it results

$$U''(t) = t^{1-p} \int_{\mathbb{R}^n} |u(t, x)|^p \, dx. \quad (58)$$

By using the support condition for u and Hölder's inequality, from the above inequality we obtain

$$\begin{aligned} U''(t) &\gtrsim t^{1-p} (R + A_k(t))^{-n(p-1)} |U(t)|^p \\ &\gtrsim (R + t)^{-((1-k)n+1)(p-1)} |U(t)|^p \end{aligned} \quad (59)$$

for any $t \in (1, T)$.

Let us derive now two estimates from below for U . On the one hand, thanks to the convexity of U , we have immediately

$$U(t) \geq U(1) + (t-1)U'(1) \gtrsim \varepsilon t \quad (60)$$

for any $t \in (1, T)$, where we used that u_0, u_1 are nonnegative and nontrivial in the unexpressed multiplicative constant. Plugging (60) in (59) and integrating twice, we get

$$U(t) \gtrsim \varepsilon^p t^{-((1-k)n+1)(p-1)+p+2} \quad (61)$$

for any $t \in [T_0, T)$, where $T_0 > 1$. The first lower bound estimate for U in (61) has been obtained from the convexity of U . On the other hand, from Lemma 2.9 and (58), integrating twice, we find a second lower bound estimate for U , that is,

$$U(t) \gtrsim \varepsilon^p t^{(1-k)(n-1)(1-\frac{p}{2}) + \frac{kp}{2} + 1 - p + 2} \quad (62)$$

for any $t \in [T_0, T)$.

Next we apply Lemma 3.1 to the functional U . Since u_0, u_1 are nonnegative and nontrivial we have $U(1), U'(1) > 0$, so (54) is fulfilled. Moreover, (59) corresponds to (53) with $q \doteq ((1-k)n+1)(p-1)$. Finally, combining (61) and (62) we have (52) with $a = \max\{a_1, a_2\}$, where

$$\begin{aligned} a_1 &\doteq -((1-k)n+1)(p-1) + p + 2, \\ a_2 &\doteq (1-k)(n-1)(1-\frac{p}{2}) + \frac{kp}{2} + 1 - p + 2 \end{aligned}$$

and $A \approx \varepsilon^p$. According to this choice we have two possible value for the quantity M in Lemma 3.1: either we use (61), that is, $a = a_1$ and, consequently,

$$M_1 \doteq \frac{p-1}{2} a_1 - \frac{q}{2} + 1 = \frac{p}{2} [-(1-k)n(p-1) + 2]$$

or we use (62), that is, $a = a_2$ and, then,

$$M_2 \doteq \frac{p-1}{2} a_2 - \frac{q}{2} + 1 = \frac{1}{2} \left\{ - \left[(1-k) \frac{p-1}{2} + 1 - \frac{k}{2} \right] p^2 + \left[(1-k) \frac{n+1}{2} + 1 + \frac{3k}{2} \right] p + 1 - k \right\}.$$

Therefore, for $M \doteq \max\{M_1, M_2\} > 0$ Lemma 3.1 provides a blow – up result and the upper bound estimate for the lifespan

$$T \lesssim \varepsilon^{-\frac{p(p-1)}{2M}}.$$

Let us make the condition $M > 0$ more explicit. The condition $M_1 > 0$ is equivalent to $p < p_1(n, k)$, while the condition $M_2 > 0$ is equivalent to $p < p_0(n, k)$. Hence, Lemma 3.1 implies the validity of a blow – up

result for (7) in the subcritical case $1 < p < \max\{p_0(n, k), p_1(n, k)\}$ (exactly as in [14]) and the upper bound estimates for the lifespan

$$T(\varepsilon) \lesssim \begin{cases} \varepsilon^{-\left(\frac{2}{p-1} - (1-k)n\right)^{-1}} & \text{if } p < p_1(n, k), \\ \varepsilon^{-\frac{p(p-1)}{\theta(p, n, k)}} & \text{if } p < p_0(n, k), \end{cases} \quad (63)$$

where

$$\theta(p, n, k) \doteq 1 - k + \left[(1-k)\frac{n+1}{2} + 1 + \frac{3k}{2}\right]p - \left[(1-k)\frac{n-1}{2} + 1 - \frac{k}{2}\right]p^2. \quad (64)$$

Furthermore, we point out that $a > 1$ (so, in particular, $a > 0$ as it is required in the assumptions of Lemma 3.1) if and only if $1 < p < \max\{p_1(n, k), p_2(n, k)\}$, where

$$p_2(n, k) \doteq 2 + \frac{2k}{(1-k)n + 1}.$$

We want to show now that the condition $a > 1$ is always fulfilled whenever $M > 0$ holds. For this purpose, we shall determine how to order the exponents p_0, p_1, p_2 . Since $p_0(n, k)$ is defined through (2), the inequality $p_0(n, k) > p_1(n, k)$ holds if and only if

$$((1-k)n + 1)p_1(n, k)^2 - ((1-k)n + 3 + 2k)p_1(n, k) - 2(1-k) < 0.$$

By straightforward computations it follows that the last inequality is fulfilled if and only if $n > N(k)$, where $N(k)$ is defined in (6). Similarly, $p_0(n, k) > p_2(n, k)$ if and only if $n < N(k)$. Summarizing,

$$\begin{aligned} p_2(n, k) < p_0(n, k) < p_1(n, k) & \quad \text{if } n < N(k), \\ p_0(n, k) = p_1(n, k) = p_2(n, k) & \quad \text{if } n = N(k), \\ p_1(n, k) < p_0(n, k) < p_2(n, k) & \quad \text{if } n > N(k). \end{aligned} \quad (65)$$

Consequently, for $n \geq N(k)$ the critical condition is $p = p_0(n, k)$, so if $p < p_0(n, k)$, in particular, the condition $p < p_2(n, k)$ is fulfilled (that is, $M_2 > 0$ implies $a_2 > 1$). On the other hand, for $n < N(k)$ it holds $p_2(n, k) < p_1(n, k)$ and the condition $M_1 > 0$ and $a_1 > 1$ are both equivalent to $p < p_1(n, k)$ (the critical condition is $p = p_1(n, k)$ in this case). Therefore, we actually proved that $M > 0$ implies $a > 1$.

Remark 5. In [14] the condition in the subcritical case on p under which a blow-up result holds for (7) is written in a slightly different but equivalent way. Indeed, combining [14, Equation (1.9)] with (65), we see immediately that the condition for p in [14, Theorem 1.3] is satisfied if and only if $1 < p < \max\{p_1(n, k), p_0(n, k)\}$.

Finally, we want to compare the upper bound estimates for the lifespan in (63). Clearly, the estimates

$$T(\varepsilon) \lesssim \begin{cases} \varepsilon^{-\left(\frac{2}{p-1} - (1-k)n\right)^{-1}} & \text{if } n < N(k) \text{ and } p \in [p_0(n, k), p_1(n, k)], \\ \varepsilon^{-\frac{p(p-1)}{\theta(p, n, k)}} & \text{if } n > N(k) \text{ and } p \in [p_1(n, k), p_0(n, k)], \end{cases}$$

cannot be improved because it holds either $p \geq p_0(n, k)$ or $p \geq p_1(n, k)$. Note that $p_2(n, k)$ plays no role in the determination of the upper bound estimate for the lifespan.

However, in the case $1 < p < \min\{p_0(n, k), p_1(n, k)\}$ it is not clear which of the upper bounds in (63) is better. Of course, in this case we have to compare a_1 and a_2 . A straightforward computation shows that $a_1 \geq a_2$ if and only if

$$((1-k)n - 1)p \leq 2(1-k). \quad (66)$$

If $n \leq \tilde{N}(k) \doteq 1/(1-k)$, then, the previous inequality is always true. On the other hand, for $n > \tilde{N}(k)$ we may introduce the further exponent

$$p_3(n, k) \doteq \frac{2(1-k)}{(1-k)n - 1}.$$

It turns out that $p_3(n, k) > 1$ if and only if $\tilde{N}(k) < n < \hat{N}(k) \doteq 2 + 1/(1-k)$. Moreover, for $n > \tilde{N}(k)$ the inequalities $p_1(n, k) < p_3(n, k)$ and $p_0(n, k) < p_3(n, k)$ are both satisfied if and only if $n < N(k)$.

In order to clarify the upper bound estimates in (63), we shall consider five different subcases depending on the range for the spatial dimension n .

Case $n \leq \tilde{N}(k)$

In this case, (66) is always satisfied as the left-hand side is nonpositive. So, $a_1 \geq a_2$. Therefore, for any $1 < p < p_1(n, k)$ the following upper bound estimate holds

$$T(\varepsilon) \lesssim \varepsilon^{-\left(\frac{2}{p-1} - (1-k)n\right)^{-1}}. \quad (67)$$

Case $\tilde{N}(k) < n < N(k)$

In this case, (66) is satisfied for $p \leq p_3$. Hence, by the ordering $1 < p_0(n, k) < p_1(n, k) < p_3(n, k)$, we get that $a_1 > a_2$ for exponents satisfying $1 < p < p_1(n, k)$. Also, even in this case (67) is a better estimates than $T(\varepsilon) \lesssim \varepsilon^{-\frac{p(p-1)}{\theta(p, n, k)}}$.

Case $n = N(k)$

In this limit case, $p_0(n, k) = p_1(n, k) = p_3(n, k)$. So, for $1 < p < p_1(n, k) = p_3(n, k)$ it holds $a_1 > a_2$ and as in the previous case (67) is the best estimate.

Case $N(k) < n < \hat{N}(k)$

In this case, it results $1 < p_3(n, k) < p_1(n, k) < p_0(n, k)$. So, for $1 < p \leq p_3(n, k)$ it holds $a = a_1$, while for $p_3(n, k) < p < p_0(n, k)$ we have $a = a_2$. Therefore,

$$T(\varepsilon) \lesssim \begin{cases} \varepsilon^{-\left(\frac{2}{p-1} - (1-k)n\right)^{-1}} & \text{if } p \in (1, p_3(n, k)], \\ \varepsilon^{-\frac{p(p-1)}{\theta(p, n, k)}} & \text{if } p \in (p_3(n, k), p_0(n, k)). \end{cases}$$

Case $n \geq \hat{N}(k)$

In this case, $p_3(n, k) \leq 1$ and $1 < p_1(n, k) < p_0(n, k)$ so (66) is never satisfied for $p > 1$. Hence, $a_2 > a_1$ for any $1 < p < p_0(n, k)$, that is,

$$T(\varepsilon) \lesssim \varepsilon^{-\frac{p(p-1)}{\theta(p, n, k)}}$$

is a better estimate than (67).

3.1 Lifespan estimates in the subcritical case

Summarizing, what we established in the above subcases, we proved the following proposition, that completes [14, Theorem 1.3] with the estimate for the lifespan while Theorem 1.2 and Theorem 1.3 deal with the critical case that was not discussed in [14].

Proposition 3.2. *Let $n \geq 1$ and $1 < p < \max\{p_0(n, k), p_1(n, k)\}$. Let us assume that $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$ are nonnegative, nontrivial and compactly supported functions with supports contained in B_R for some $R > 0$. Let*

$$u \in \mathcal{C}([1, T], H^1(\mathbb{R}^n)) \cap \mathcal{C}^1([1, T], L^2(\mathbb{R}^n)) \cap L_{\text{loc}}^p([1, T] \times \mathbb{R}^n)$$

be an energy solution to (7) according to Definition 1.1 with lifespan $T = T(\varepsilon)$ and fulfilling the support condition $\text{supp } u(t, \cdot) \subset B_{A_k(t)+R}$ for any $t \in (1, T)$. Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(u_0, u_1, n, p, k, R)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the energy solution u blows up in finite time. Furthermore, the upper bound estimates for the lifespan

$$T(\varepsilon) \leq \begin{cases} C\varepsilon^{-\left(\frac{2}{p-1} - (1-k)n\right)^{-1}} & \text{if } n \leq N(k) \text{ and } p \in (1, p_1(n, k)), \\ C\varepsilon^{-\left(\frac{2}{p-1} - (1-k)n\right)^{-1}} & \text{if } n \in (N(k), \hat{N}(k)) \text{ and } p \in (1, p_3(n, k)], \\ C\varepsilon^{-\frac{p(p-1)}{\theta(p, n, k)}} & \text{if } n \in (N(k), \hat{N}(k)) \text{ and } p \in (p_3(n, k), p_0(n, k)), \\ C\varepsilon^{-\frac{p(p-1)}{\theta(p, n, k)}} & \text{if } n \geq \hat{N}(k) \text{ and } p \in (1, p_0(n, k)), \end{cases}$$

hold, where the constant $C > 0$ is independent of ε and $\theta(p, n, k)$ is defined by (64).

4 Semilinear wave equation in EdeS spacetime: 2nd critical case

In Section 3 we derived the upper bound for the lifespan in the subcritical case, while in Section 2 we studied the critical case $p = p_0(n, k)$. We have already remarked that $p = p_0(n, k)$ is the critical case when $n > N(k)$. Therefore, it remains to consider the critical case $p = p_1(n, k)$ when $n \leq N(k)$. In this section, we are going to prove a blow – up result even in this critical case $p = p_1(n, k)$ and to derive the corresponding upper bound estimate for the lifespan. Even in this critical case, our approach will be based on a basic iteration argument combined with the slicing procedure we already applied in Section 2.

As time – depending functional we will use the same one employed in Section 3, namely the function U defined in (57). Then, since $p = p_1(n, k)$ is equivalent to the condition

$$((1 - k)n + 1)(p - 1) = p + 1, \quad (68)$$

we may rewrite (59) as

$$U(t) \geq C \int_1^t \int_1^s (R + \tau)^{-(p+1)} (U(\tau))^p d\tau ds \quad (69)$$

for any $t \in (1, T)$ and for a suitable positive constant $C > 0$. Let us point out that (69) will be the iteration frame in the iteration procedure for the critical case $p = p_1(n, k)$.

We know that $U(t) \geq K\varepsilon t$ for any $t \in (1, T)$, where K is a suitable positive constant, provided that u_0, u_1 are nonnegative, nontrivial and compactly supported (cf. the estimate in (60)). Therefore,

$$\begin{aligned} U(t) &\geq CK^p \varepsilon^p \int_1^t \int_1^s (R + \tau)^{-(p+1)} \tau^p d\tau ds \geq CK^p (R + 1)^{-(p+1)} \varepsilon^p \int_1^t \int_1^s \tau^{-1} d\tau ds \\ &= CK^p (R + 1)^{-(p+1)} \varepsilon^p \int_1^t \log s ds \geq CK^p (R + 1)^{-(p+1)} \varepsilon^p \int_{2t/3}^t \log s ds \\ &\geq 3^{-1} CK^p (R + 1)^{-(p+1)} \varepsilon^p t \log \left(\frac{2t}{3} \right) \end{aligned} \quad (70)$$

for $t \geq \ell_0 = 3/2$, where we used $R + \tau \leq (R + 1)\tau$ for $\tau \geq 1$.

Hence, by using recursively (69), we are going to prove now the sequence of lower bound estimates

$$U(t) \geq K_j t \left(\log \left(\frac{t}{\ell_j} \right) \right)^{\sigma_j} \quad \text{for } t \geq \ell_j \quad (71)$$

for any $j \in \mathbb{N}$, where the sequence of parameters $\{\ell_j\}_{j \in \mathbb{N}}$ is defined as in Section 2.3, i.e. $\ell_j = 2 - 2^{-(j+1)}$, and $\{K_j\}_{j \in \mathbb{N}}, \{\sigma_j\}_{j \in \mathbb{N}}$ are sequences of positive reals that we shall determine afterwards.

We remark that for $j = 0$ (71) holds true thanks to (70), provided that $K_0 = (CK^p (R + 1)^{-(p+1)} \varepsilon^p)/3$ and $\sigma_0 = 1$. Next we are going to prove (71) by using an inductive argument. Assumed the validity of (71) for some $j \geq 0$ we have to prove (71) for $j + 1$. For this purpose, we plug (71) in (69), thus, after shrinking the domain of integration, we have

$$\begin{aligned} U(t) &\geq CK_j^p \int_{\ell_j}^t \int_{\ell_j}^s (R + \tau)^{-(p+1)} \tau^p \left(\log \left(\frac{\tau}{\ell_j} \right) \right)^{\sigma_j p} d\tau ds \\ &\geq C(R + 1)^{-(p+1)} K_j^p \int_{\ell_j}^t \int_{\ell_j}^s \tau^{-1} \left(\log \left(\frac{\tau}{\ell_j} \right) \right)^{\sigma_j p} d\tau ds \\ &= C(R + 1)^{-(p+1)} K_j^p (\sigma_j p + 1)^{-1} \int_{\ell_j}^t \left(\log \left(\frac{s}{\ell_j} \right) \right)^{\sigma_j p + 1} ds \end{aligned}$$

for $t \geq \ell_{j+1}$. If we shrink the domain of integration to $[(\ell_j/\ell_{j+1})t, t]$ in the last s – integral we get

$$\begin{aligned} U(t) &\geq C(R + 1)^{-(p+1)} K_j^p (\sigma_j p + 1)^{-1} \int_{\frac{\ell_j t}{\ell_{j+1}}}^t \left(\log \left(\frac{s}{\ell_j} \right) \right)^{\sigma_j p + 1} ds \\ &\geq C(R + 1)^{-(p+1)} K_j^p (\sigma_j p + 1)^{-1} \left(1 - \frac{\ell_j}{\ell_{j+1}} \right) t \left(\log \left(\frac{t}{\ell_{j+1}} \right) \right)^{\sigma_j p + 1} \\ &\geq C(R + 1)^{-(p+1)} 2^{-(j+3)} K_j^p (\sigma_j p + 1)^{-1} t \left(\log \left(\frac{t}{\ell_{j+1}} \right) \right)^{\sigma_j p + 1} \end{aligned}$$

for $t \geq \ell_{j+1}$, where in the last step we applied the inequality $1 - \ell_j/\ell_{j+1} > 2^{-(j+3)}$. Also, we proved (71) for $j + 1$ provided that

$$K_{j+1} \doteq C(R + 1)^{-(p+1)} 2^{-(j+3)} (\sigma_j p + 1)^{-1} K_j^p \quad \text{and} \quad \sigma_{j+1} \doteq \sigma_j p + 1.$$

Next we determine a lower bound estimate for K_j . First we find the value of the exponent σ_j . Applying iteratively the relation $\sigma_j = 1 + p\sigma_{j-1}$ and the initial exponent $\sigma_0 = 1$, we get

$$\sigma_j = \sigma_0 p^j + \sum_{k=0}^{j-1} p^k = \frac{p^{j+1} - 1}{p - 1}. \quad (72)$$

In particular, $\sigma_{j-1}p + 1 = \sigma_j \leq p^{j+1}/(p-1)$ implies that

$$K_j \geq L (2p)^{-j} K_{j-1}^p \quad (73)$$

for any $j \geq 1$, where $L \doteq 2^{-2}C(R+1)^{-(p+1)}(p-1)/p$. Applying the logarithmic function to both sides of (73) and reusing the resulting inequality in an iterative way, we arrive at

$$\begin{aligned} \log K_j &\geq p \log K_{j-1} - j \log(2p) + \log L \\ &\geq \dots \geq p^j \log K_0 - \left(\sum_{k=0}^{j-1} (j-k)p^k \right) \log(2p) + \left(\sum_{k=0}^{j-1} p^k \right) \log L \\ &= p^j \left(\log \left(3^{-1}CK^p(R+1)^{-(p+1)}\varepsilon^p \right) - \frac{p \log(2p)}{(p-1)^2} + \frac{\log L}{p-1} \right) + \left(\frac{j}{p-1} + \frac{p}{(p-1)^2} \right) \log(2p) - \frac{\log L}{p-1}, \end{aligned}$$

where we applied again the identity (49). Let us define $j_1 = j_1(n, p, k)$ as the smallest nonnegative integer such that

$$j_1 \geq \frac{\log L}{\log(2p)} - \frac{p}{p-1}.$$

Hence, for any $j \geq j_1$ the estimate

$$\log K_j \geq p^j \left(\log \left(3^{-1}CK^p(R+1)^{-(p+1)}\varepsilon^p \right) - \frac{p \log(2p)}{(p-1)^2} + \frac{\log L}{p-1} \right) = p^j \log(N\varepsilon^p) \quad (74)$$

holds, where $N \doteq 3^{-1}CK^p(R+1)^{-(p+1)}(2p)^{-p/(p-1)^2}L^{1/(p-1)}$.

Combining (71), (72) and (74), we arrive at

$$\begin{aligned} U(t) &\geq \exp \left(p^j \log(N\varepsilon^p) \right) t \left(\log \left(\frac{t}{\ell_j} \right) \right)^{\sigma_j} \\ &\geq \exp \left(p^j \log(N\varepsilon^p) \right) t \left(\frac{1}{2} \log t \right)^{(p^{j+1}-1)/(p-1)} \\ &= \exp \left(p^j \log \left(2^{-p/(p-1)}N\varepsilon^p (\log t)^{p/(p-1)} \right) \right) t \left(\frac{1}{2} \log t \right)^{-1/(p-1)} \end{aligned}$$

for $t \geq 4$ and for any $j \geq j_1$, where we applied the inequality $\log(t/\ell_j) \geq \log(t/2) \geq (1/2) \log t$ for all $t \geq 4$. If we denote $H(t, \varepsilon) \doteq 2^{-p/(p-1)}N\varepsilon^p (\log t)^{p/(p-1)}$, the last estimate may be rewritten as

$$U(t) \geq \exp \left(p^j \log H(t, \varepsilon) \right) t \left(\frac{1}{2} \log t \right)^{-1/(p-1)} \quad (75)$$

for $t \geq 4$ and any $j \geq j_1$.

Let us fix $\varepsilon_0 = \varepsilon_0(n, p, k, R, u_0, u_1)$ so that

$$\exp \left(2N^{-(1-p)/p}\varepsilon_0^{-(p-1)} \right) \geq 4.$$

Then, for any $\varepsilon \in (0, \varepsilon_0]$ and for $t > \exp \left(2N^{-(1-p)/p}\varepsilon^{-(p-1)} \right)$ we get $t \geq 4$ and $H(t, \varepsilon) > 1$. Thus, for any $\varepsilon \in (0, \varepsilon_0]$ and for $t > \exp \left(2N^{-(1-p)/p}\varepsilon^{-(p-1)} \right)$ as $j \rightarrow \infty$ in (75) we find that the lower bound for $U(t)$ blows up and, consequently, $U(t)$ cannot be finite too. Summarizing, we proved that U blows up in finite time and, besides, we showed the upper bound estimate for the lifespan

$$T(\varepsilon) \leq \exp \left(2N^{-(1-p)/p}\varepsilon^{-(p-1)} \right).$$

Altogether, we established Theorem 1.3 in the critical case $p = p_1(n, k)$.

Remark 6. Combining the results from Theorems 1.2 and 1.3 and Proposition 3.2, we a full picture of the upper bound estimates for the lifespan of local in time solutions to (7) whenever $1 < p \leq \max\{p_0(n, k), p_1(n, k)\}$, of course, under suitable sign, size and support assumptions for the initial data.

5 Final remarks

Let us compare our results with the corresponding ones for the semilinear wave equation in the flat case. First, we point out that due to the presence of the term t^{1-p} in the semilinear term in (4), we have a competition between the two exponents p_0, p_1 to be the critical exponent. This for the classical semilinear wave equation with power nonlinearity does not happen since $p_{\text{Str}}(n) \geq \frac{n+1}{n-1}$ for any $n \geq 2$. However, a similar situation it has been observed when lower order terms with time - dependent coefficients in the

scale – invariant case are present, with a competition between a shift of Fujita exponent and a shift Strauss exponent (cf. [6, 5, 26, 30, 33, 28, 29]). On the other hand, the presence of the exponent p_3 for dimensions $n \in (N(k), \widehat{N}(k))$ to distinguish among two different upper bounds for the lifespan depending on the range for p is exactly what happens for the semilinear wave equation in spatial dimensions $n = 2$ (see [36, 18]). Moreover, the situation for (7) when $n \leq N(k)$ is completely analogous to what happens for the semilinear wave equation when $n = 1$, see [49] for the Euclidean case.

After the completion of the final version of this work, we found out the existence of the paper [37], where a more general model is considered. We point out that the approach we used in the critical case is completely different, and that we slightly improved their result in the special case of the semilinear wave equation in the generalized Einstein – de Sitter spacetime, by removing the assumption on the size of the support of the Cauchy data (cf. [37, Theorem 2.3]).

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A Alternative proof of Proposition 2.1 in the special case $k = 2/3$

In this appendix we determine the representation of the solutions $\{y_j(t, s; \lambda)\}_{j \in \{0,1\}}$ to the Cauchy problems

$$\begin{cases} \partial_t^2 y_j(t, s; \lambda) - \lambda^2 t^{-\frac{4}{3}} y_j(t, s; \lambda) = 0, & t > s \geq 1, \\ y_j(s, s; \lambda) = \delta_{0j}, \\ \partial_t y_j(s, s; \lambda) = \delta_{1j}, \end{cases} \quad (76)$$

where $\lambda > 0$ is a parameter and δ_{ij} denotes the Kronecker delta. Let us introduce the change of variables $z = z(t; \lambda) \doteq -2\lambda\phi(t)$, where for the sake of brevity we denote simply $\phi(t) \equiv \phi_{\frac{2}{3}}(t) = 3t^{1/3}$. Furthermore, we perform the transformation $y(t, \lambda) = w(z) e^{-\frac{z}{2}}$. A straightforward computation shows that

$$\begin{aligned} \partial_t y(t, \lambda) &= [w'(z) - \frac{1}{2}w(z)] e^{-\frac{z}{2}} \frac{\partial z}{\partial t}, \\ \partial_t^2 y(t, \lambda) &= [w''(z) - w'(z) + \frac{1}{4}w(z)] e^{-\frac{z}{2}} \left(\frac{\partial z}{\partial t}\right)^2 + [w'(z) - \frac{1}{2}w(z)] e^{-\frac{z}{2}} \frac{\partial^2 z}{\partial t^2}. \end{aligned}$$

Consequently, y solves the equation

$$\frac{d^2 y}{dt^2} - \lambda^2 t^{-\frac{4}{3}} y = 0 \quad (77)$$

if and only if z is a solution of the confluent hypergeometric equation

$$zw''(z) - (z+2)w'(z) + w(z) = 0, \quad (78)$$

where we used $\frac{\partial^2 z}{\partial t^2} = 4\lambda t^{-4/3}(\phi(t))^{-1}$ and $\left(\frac{\partial z}{\partial t}\right)^2 = 4\lambda^2 t^{-4/3}$. According to [27, Equation 13.2.32, p. 324], a fundamental pair of solutions to (78) is given by $z^3 M(z; 2, 4)$ and $z+2$. Here $M(z; a, c)$ denotes Kummer’s function

$$M(z; a, c) \doteq \sum_{h=0}^{\infty} \frac{(a)_h}{(c)_h h!} z^h,$$

where $(b)_h$ denotes the Pochhammer symbol (rising factorial) and is defined by $(b)_h = 1$ for $h = 0$ and $(b)_h = b(b+1) \cdots (b+h-1)$ for $h \geq 1$.

Lemma A.1. *For any $z \in \mathbb{R}$ the following identity holds*

$$z^3 M(z; 2, 4) = 6(e^z(z-2) + z+2). \quad (79)$$

Proof. In order to prove (79) we are going to consider the corresponding Taylor series expansions. Let us denote $f(z) \doteq 6(e^z(z-2) + z+2)$. Since

$$f'(z) = 6(e^z(z-1) + 1), \quad f''(z) = 6e^z z,$$

then, $f(0) = f'(0) = f''(0) = 0$.

Moreover, one can prove recursively that $f^{(2+h)}(z) = 6e^z(z+h)$ for any $h \geq 0$. Therefore,

$$f(z) = \sum_{h=0}^{\infty} \frac{f^{(h)}(0)}{h!} z^h = z^3 \sum_{h=0}^{\infty} \frac{f^{(h+3)}(0)}{(h+3)!} z^h = z^3 \sum_{h=0}^{\infty} \frac{6(h+1)}{(h+3)!} z^h.$$

We remark that

$$\frac{(2)_h}{(4)_h h!} = \frac{(h+1)!}{(1/6)(h+3)! h!} = \frac{6(h+1)}{(h+3)!}$$

for any $h \in \mathbb{N}$, because of $(2)_h = (h+1)!$ and $(4)_h = (1/6)(h+3)!$. Hence,

$$f(z) = z^3 \sum_{h=0}^{\infty} \frac{6(h+1)}{(h+3)!} z^h = z^3 \sum_{h=0}^{\infty} \frac{(2)_h}{(4)_h h!} z^h = z^3 M(z; 2, 4),$$

that is, we proved (79). \square

According to our previous remark, by (79) it follows that $6(e^z(z-2) + z+2)$ and $z+2$ are a fundamental system of solutions for (78). For the sake of simplicity we may consider $\{g_1, g_2\}$, where $g_1(z) \doteq e^z(z-2)$ and $g_2(z) \doteq z+2$ as a basis of the solution space for (77). We point out that $\{g_1, g_2\}$ is clearly a fundamental system of solutions as

$$\mathcal{W}(g_1, g_2)(z) = g_1(z)g_2'(z) - g_2(z)g_1'(z) = -z^2 e^z.$$

Thus, the pair of functions

$$\begin{aligned} \tilde{V}_0(t, \lambda) &\doteq -\frac{1}{2}e^{-\frac{z}{2}}g_1(z) = e^{\frac{z}{2}}\left(-\frac{z}{2} + 1\right) = e^{-\lambda\phi(t)}(\lambda\phi(t) + 1), \\ \tilde{V}_1(t, \lambda) &\doteq -\frac{1}{2}e^{-\frac{z}{2}}g_2(z) = e^{-\frac{z}{2}}\left(-\frac{z}{2} - 1\right) = e^{\lambda\phi(t)}(\lambda\phi(t) - 1) \end{aligned}$$

form a system of fundamental solutions to (77).

Finally, we prove the representations (22) and (23) by using $\{\tilde{V}_0, \tilde{V}_1\}$ as fundamental system of solutions to (77).

Proposition A.2. *Let $y_0(t, s; \lambda, 2/3)$ and $y_1(t, s; \lambda, 2/3)$ be the functions defined in (22) and (23), respectively. Then, $y_0(t, s; \lambda, 2/3)$ and $y_1(t, s; \lambda, 2/3)$ solve the Cauchy problem (76) for $j = 0$ and $j = 1$, respectively.*

Proof. We know that \tilde{V}_0, \tilde{V}_1 form a system of independent solutions to (77). Also, we can write the solutions $y_j(t, s; \lambda)$, $j = 0, 1$ of (76) as linear combinations of \tilde{V}_0, \tilde{V}_1 in the following way

$$y_j(t, s; \lambda) = a_j(s; \lambda)\tilde{V}_0(t; \lambda) + b_j(s; \lambda)\tilde{V}_1(t; \lambda) \quad (80)$$

for suitable coefficients $a_j(s; \lambda)$ and $b_j(s; \lambda)$, $j = 0, 1$.

The application of the initial conditions $\partial_t^i y_j(s, s; \lambda) = \delta_{ij}$ yields the system

$$\begin{pmatrix} \tilde{V}_0(s; \lambda) & \tilde{V}_1(s; \lambda) \\ \partial_t \tilde{V}_0(s; \lambda) & \partial_t \tilde{V}_1(s; \lambda) \end{pmatrix} \begin{pmatrix} a_0(s; \lambda) & a_1(s; \lambda) \\ b_0(s; \lambda) & b_1(s; \lambda) \end{pmatrix} = I,$$

where I denotes the identity matrix. Therefore,

$$\begin{aligned} \begin{pmatrix} a_0(s; \lambda) & a_1(s; \lambda) \\ b_0(s; \lambda) & b_1(s; \lambda) \end{pmatrix} &= \begin{pmatrix} \tilde{V}_0(s; \lambda) & \tilde{V}_1(s; \lambda) \\ \partial_t \tilde{V}_0(s; \lambda) & \partial_t \tilde{V}_1(s; \lambda) \end{pmatrix}^{-1} \\ &= \left(\mathcal{W}(\tilde{V}_0, \tilde{V}_1)(s; \lambda)\right)^{-1} \begin{pmatrix} \partial_t \tilde{V}_1(s; \lambda) & -\tilde{V}_1(s; \lambda) \\ -\partial_t \tilde{V}_0(s; \lambda) & \tilde{V}_0(s; \lambda) \end{pmatrix}. \end{aligned} \quad (81)$$

The Wronskian $\mathcal{W}(\tilde{V}_0, \tilde{V}_1)$ is given by

$$\mathcal{W}(\tilde{V}_0, \tilde{V}_1)(t; \lambda) = \tilde{V}_0(t; \lambda)\partial_t \tilde{V}_1(t; \lambda) - \tilde{V}_1(t; \lambda)\partial_t \tilde{V}_0(t; \lambda) = 2\lambda^3(\phi(t))^2\phi'(t) = 18\lambda^3,$$

where we employed

$$\begin{aligned} \partial_t \tilde{V}_0(t; \lambda) &= -\lambda^2\phi(t)\phi'(t)e^{-\lambda\phi(t)}, \\ \partial_t \tilde{V}_1(t; \lambda) &= \lambda^2\phi(t)\phi'(t)e^{\lambda\phi(t)}. \end{aligned}$$

Plugging the previous representation of $\mathcal{W}(\tilde{V}_0, \tilde{V}_1)$ in (81), we find

$$\begin{pmatrix} a_0(s; \lambda) & a_1(s; \lambda) \\ b_0(s; \lambda) & b_1(s; \lambda) \end{pmatrix} = \frac{1}{18\lambda^3} \begin{pmatrix} \partial_t \tilde{V}_1(s; \lambda) & -\tilde{V}_1(s; \lambda) \\ -\partial_t \tilde{V}_0(s; \lambda) & \tilde{V}_0(s; \lambda) \end{pmatrix}.$$

Let us begin by proving that $y_0(t, s; \lambda) = y_0(t, s; \lambda, 2/3)$. Employing the above representation of the coefficients $a_0(s; \lambda), b_0(s; \lambda)$ in (80), we have

$$\begin{aligned} y_0(t, s; \lambda) &= (18\lambda^3)^{-1} \{ \partial_t \tilde{V}_1(s; \lambda) \tilde{V}_0(t; \lambda) - \partial_t \tilde{V}_0(s; \lambda) \tilde{V}_1(t; \lambda) \} \\ &= (18\lambda^3)^{-1} \lambda^2 \phi(s) \phi'(s) \{ e^{-\lambda(\phi(t) - \phi(s))} (\lambda\phi(t) + 1) + e^{\lambda(\phi(t) - \phi(s))} (\lambda\phi(t) - 1) \} \\ &= 3^{-2} \phi(s) \phi'(s) \phi(t) \cosh(\lambda(\phi(t) - \phi(s))) - 3^{-2} \lambda^{-1} \phi(s) \phi'(s) \sinh(\lambda(\phi(t) - \phi(s))) \\ &= (t/s)^{1/3} \cosh(\lambda(\phi(t) - \phi(s))) - 1/(3\lambda s^{1/3}) \sinh(\lambda(\phi(t) - \phi(s))) = y_0(t, s; \lambda, 2/3). \end{aligned}$$

Analogously, plugging the previously determined expressions for $a_1(s; \lambda), b_1(s; \lambda)$ in (80), we have

$$\begin{aligned} y_1(t, s; \lambda) &= (18\lambda^3)^{-1} \{ \tilde{V}_0(s; \lambda) \tilde{V}_1(t; \lambda) - \tilde{V}_1(s; \lambda) \tilde{V}_0(t; \lambda) \} \\ &= (18\lambda^3)^{-1} \{ (\lambda\phi(s) + 1)(\lambda\phi(t) - 1)e^{\lambda(\phi(t) - \phi(s))} - (\lambda\phi(s) - 1)(\lambda\phi(t) + 1)e^{-\lambda(\phi(t) - \phi(s))} \} \\ &= (18\lambda^3)^{-1} (\lambda^2 \phi(t) \phi(s) - 1)(e^{\lambda(\phi(t) - \phi(s))} - e^{-\lambda(\phi(t) - \phi(s))}) \\ &\quad + (18\lambda^3)^{-1} \lambda(\phi(t) - \phi(s))(e^{\lambda(\phi(t) - \phi(s))} + e^{-\lambda(\phi(t) - \phi(s))}) \\ &= \left((st)^{1/3} / \lambda - 1/(9\lambda^3) \right) \sinh(\lambda(\phi(t) - \phi(s))) + (1/9\lambda^2) (\phi(t) - \phi(s)) \cosh(\lambda(\phi(t) - \phi(s))) \\ &= y_1(t, s; \lambda, 2/3). \end{aligned}$$

The proof is complete. □

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