NORMALIZED SOLUTIONS TO A SCHRÖDINGER-BOPP-PODOLSKY SYSTEM UNDER NEUMANN BOUNDARY CONDITIONS

DANILO G. AFONSO AND GAETANO SICILIANO

DOI: 10.1142/S0219199721501005

Abstract. In this paper we study a Schrödinger-Bopp-Podolsky system of partial differential equations in a bounded and smooth domain of $\mathbb{R}^3$ with a non constant coupling factor. Under a compatibility condition on the boundary data we deduce existence and multiplicity of solutions by means of the Ljusternik-Schnirelmann theory.

1. Introduction

The Schrödinger-Newton equation consists of a nonlinear coupling of the Schrödinger equation with a gravitational potential of newtonian form, representing the interaction of a particle with its own gravitational field.

In 1998, Benci and Fortunato [2] treated a similar problem, where the Schrödinger equation was coupled with Maxwell's equations. Such coupling represents the interaction of the particle with its own electromagnetic field. The coupling factor is a constant $q \neq 0$. In their paper the authors consider standing waves solutions in the purely electrostatic field and this leads to the so-called Schrödinger-Poisson system. They impose a Dirichlet boundary condition both on the matter field u and the electrostatic field ϕ and employed variational methods and critical point theory to develop a procedure that would become standard to treat other similar problems.

Later, Pisani and Siciliano [10] treated a Schrödinger-Poisson system with Neumann boundary conditions on the scalar field ϕ and considered the case in which the interaction factor responsible for the coupling of the equations is non-constant. This gives rise to important and interesting considerations regarding the geometry of the manifold on which find the solutions.

In this paper we treat a modification of the problem dealt with by Pisani and Siciliano consisting in the addition of a biharmonic term in the equation of the electrostatic potential and imposing appropriate boundary conditions. The problem studied can be interpreted as a coupling of the Schrödinger equation with the Bopp-Podolsky electrodynamics (for more details on this matter, see [3] and the references therein). However here we focus on the mathematical aspects of the problem.

We point out that in the literature there are few papers concerning Schrödinger-Bopp-Podolski systems. Beside [3] we cite here [4,8] where the authors study the problem with a critical nonlinearity, [5] where solutions with *a priori* given interaction energy for the Schrödinger-Bopp-Podolski system are found, [7] where the problem has been addressed in the context of closed 3−dimensional manifolds both in the subcritical and critical case and [11] where the fibering method has been used to prove existence results depending on a parameter and also nonexistence.

2010 *Mathematics Subject Classification.* Primary 35J50, 35J58; Secondary 35Q55, 35Q61.

Key words and phrases. Schrödinger-Bopp-Podolsky system, Krasnoselskii genus, Lagrange multipliers, weak solutions.

D. G. Afonso was supported by CNPq grant 132634/2018-0. G. Siciliano was supported by Fapesp grant 2018/17264-4, CNPq grant 304660/2018-3 and Capes.

Coming back to our problem, the aim here is to study the following system of partial differential equations in a connected, bounded, smooth open set $\Omega \subset \mathbb{R}^3$:

$$-\Delta u + q\phi u - \kappa|u|^{p-2}u = \omega u \quad \text{in } \Omega \tag{1.1}$$

$$\Delta^2\phi - \Delta\phi = qu^2 \quad \text{in } \Omega \tag{1.2}$$

in the unknowns $u, \phi : \Omega \to \mathbb{R}$ and $\omega \in \mathbb{R}$. Here $\kappa \in \mathbb{R}$ and $q : \Omega \to \mathbb{R}$ are given. We assume the following boundary conditions:

$$u = 0 \quad \text{on } \partial\Omega \tag{1.3}$$

$$\frac{\partial\phi}{\partial\mathbf{n}} = h_1 \quad \text{on } \partial\Omega \tag{1.4}$$

$$\frac{\partial\Delta\phi}{\partial\mathbf{n}} = h_2 \quad \text{on } \partial\Omega \tag{1.5}$$

and for simplicity we assume h_1, h_2 continuous. The symbol $\mathbf{n}$ denotes the unit vector normal to $\partial\Omega$ pointing outwards. Since u represents physically the amplitude of the wave function of a particle confined in Ω, we assume the following normalizing condition:

$$\int_\Omega u^2 dx = 1. \tag{1.6}$$

We also assume that the coupling factor q is continuous on $\overline{\Omega}$:

$$q \in C(\overline{\Omega}). \tag{1.7}$$

Our main result is the following:

Theorem 1. *Let $\kappa > 0$, $p \in (2, 10/3)$ and*

$$\alpha := \int_{\partial\Omega} h_2 ds - \int_{\partial\Omega} h_1 ds. \tag{1.8}$$

Assume that $\inf_\Omega q < \alpha < \sup_\Omega q$ and that $|q^{-1}(\alpha)| = 0$. Then there exist infinitely many solutions $(u_n, \omega_n, \phi_n) \in H_0^1(\Omega) \times \mathbb{R} \times H^2(\Omega)$ to the problem (1.1) and (1.2) under conditions (1.3)-(1.7), with

$$\int_\Omega |\nabla u_n|^2 dx \to +\infty.$$

Moreover the ground state solution can be assumed positive.

Our approach is variational, indeed the solutions will be found as critical points of an energy functional restricted to a suitable constraint. In this context by a ground state solution we mean the solution with minimal energy. Moreover as a byproduct of the proof, we obtain that also the energy of these solutions is divergent.

Remark 1. *It is easy to see that if $\kappa < 0$ the result holds with $p \in (2, 6)$. For $\kappa = 0$, see [1].*

The paper is organized as follows.

In Section 2 we introduce an auxiliary problem which will be useful in order to deal with homogeneous boundary conditions.

In Section 3 we give some properties of the constraint M on which we will find the solution.

In Section 4 we introduce the energy functional and show that its critical points on M will give solutions of the problem.

In the final Section 5 by implementing the Ljusternick-Schnirelmann theory we prove Theorem 1.

As a matter of notations, we use the letters $c, c', \dots$ to denote positive constant whose value can change from line to line. We use $\|\cdot\|_p$ to denote the standard L^p-norm.

2. An auxiliary problem

Our aim is to define a functional whose critical points will be the weak solutions to the problem. In order to deal with homogeneous boundary conditions, that will permit to write the functional in a simpler form, we make a change of variable.

Consider the following auxiliary problem (where α is defined in (1.8))

$$\Delta^2\chi - \Delta\chi = \alpha/|\Omega| \quad \text{in } \ \Omega, \tag{2.1}$$

$$\frac{\partial\chi}{\partial\mathbf{n}} = h_1 \quad \text{on } \ \partial\Omega, \tag{2.2}$$

$$\frac{\partial\Delta\chi}{\partial\mathbf{n}} = h_2 \quad \text{on } \ \partial\Omega, \tag{2.3}$$

$$\int_\Omega \chi dx = 0. \tag{2.4}$$

It is easy to see it has a unique solution. Indeed, let θ be the unique function satisfying

$$\Delta\theta - \theta = \alpha/|\Omega| \quad \text{in } \ \Omega$$
$$\frac{\partial\theta}{\partial\mathbf{n}} = h_2 \quad \text{on } \ \partial\Omega,$$
$$\int_\Omega \theta dx = \int_{\partial\Omega} h_1 ds$$

and then let χ the unique function which satisfies

$$\Delta\chi = \theta, \quad \text{in } \ \Omega$$
$$\frac{\partial\chi}{\partial\mathbf{n}} = h_1 \quad \text{in } \ \partial\Omega$$
$$\int_\Omega \chi dx = 0,$$

see e.g. [13]. Then it is easy to see that by construction χ satisfies (2.1)-(2.4).

The change of variables we make is

$$\varphi = \phi - \chi - \mu,$$

where

$$\mu = \frac{1}{|\Omega|}\int_\Omega \phi dx.$$

With the new variables $(u, \omega, \varphi, \mu)$ our problem becomes

$$-\Delta u + q(\chi + \varphi)u - \kappa|u|^{p-2}u = \omega u - \mu q u \quad \text{in } \ \Omega, \tag{2.5}$$

$$\Delta^2\varphi - \Delta\varphi = qu^2 - \alpha/|\Omega| \quad \text{in } \ \Omega, \tag{2.6}$$

$$u = 0 \quad \text{on } \ \partial\Omega, \tag{2.7}$$

$$\int_\Omega u^2 dx = 1, \tag{2.8}$$

$$\frac{\partial\varphi}{\partial\mathbf{n}} = 0 \quad \text{on } \ \partial\Omega, \tag{2.9}$$

$$\frac{\partial\Delta\varphi}{\partial\mathbf{n}} = 0 \quad \text{on } \ \partial\Omega, \tag{2.10}$$

$$\int_\Omega \varphi dx = 0. \tag{2.11}$$

Notice that the compatibility condition between (2.6), (2.9) and (2.10) now reads as

$$\int_\Omega qu^2dx = \alpha.$$

Let us define the sets

$$\begin{aligned} S &:= \left\{ u \in H_0^1(\Omega) \ : \ \int_\Omega u^2dx = 1 \right\}, \\ N &:= \left\{ u \in H_0^1(\Omega) \ : \ \int_\Omega qu^2dx = \alpha \right\}, \\ M &:= S \cap N. \end{aligned}$$

Recall that α depends on both the boundary conditions to the original problem.

If the problem has a solution, then of course $M \neq \emptyset$. Hence,

$$q_{\min} \leq \alpha \leq q_{\max} \tag{2.12}$$

where

$$q_{\min} = \inf_\Omega q \quad \text{and} \quad q_{\max} = \sup_\Omega q.$$

Indeed, if $\alpha < q_{\min}$, then

$$\alpha = \int_\Omega qu^2dx \geq q_{\min} > \alpha,$$

which is a contradiction. The case $\alpha > q_{\max}$ is analogous.

From (2.12) we deduce that $q^{-1}(\alpha)$ is not empty, and indeed is its measure that will play a major role.

Suppose $\alpha = q_{\min}$ and $|q^{-1}(\alpha)| = 0$. Then

$$\int_\Omega qu^2dx = \int_{\{x\in\Omega : q(x)>\alpha\}} qu^2dx > \alpha,$$

so $M = \emptyset$. If $\alpha = q_{\max}$ and $|q^{-1}(\alpha)| = 0$ we proceed in an analogous manner to conclude that M is empty and so the problem has no solutions. Therefore, we arrive at the following necessary condition for the existence of solutions: either

$$q_{\min} < \alpha < q_{\max} \tag{2.13}$$

or

$$|q^{-1}(\alpha)| \neq 0. \tag{2.14}$$

3. THE MANIFOLD M

We now state some properties of the set M, referring the reader to [10] for the omitted proofs.

We first note that M is symmetric with respect to the origin: if $u \in M$ then $-u \in M$. This follows trivially from the definition of M. We also note that M is weakly closed in $H_0^1(\Omega)$.

Now, we want to show that under condition (2.13) the set M is not empty. For this, we introduce the following notation.

Let $A \subset \Omega$ be an open subset and define

$$S_A := \left\{ u \in H_0^1(A) \ : \ \int_A u^2dx = 1 \right\}$$

and

$$g_A : u \in S_A \mapsto \int_A qu^2dx \in \mathbb{R}.$$

It is immediately seen that

$$g_A(S_A) \subset [\inf_A q, \sup_A q].$$

Lemma 1. *The following inclusion holds:*

$$(\inf_A q, \sup_A q) \subset \overline{g_A(S_A)}.$$

We can conclude the following:

Proposition 1. *Let $A \subset \Omega$ be an open subset. If $\alpha \in (\inf_A q, \sup_A q)$ then there exists $u \in H_0^1(A)$ such that*

$$\int_A u^2 dx = 1 \qquad \textit{and} \qquad \int_A q u^2 dx = \alpha.$$

In particular by taking $A = \Omega$ we get

Theorem 2. *Assume that $\inf_\Omega q < \alpha < \sup_\Omega q$. Then M is not empty.*

Let us recall the definition of genus of Krasnoselki. Given A a closed and symmetric subset of some Banach space, with $0 \notin A$, the *genus* of A is denoted as $\gamma(A)$ and defined as the least integer k for which there exists a continuous and even map $h : A \to \mathbb{R}^k \setminus \{0\}$. By definition it is $\gamma(\emptyset) = 0$ and if it is not possible to construct continuous odd maps from A to any $\mathbb{R}^k \setminus \{0\}$, it is set $\gamma(A) = +\infty$.

It is known that the genus is a topological invariant (by odd homeomorphism) and that the genus of the sphere in $\mathbb{R}^N$ is N.

The next result says that M has subsets of arbitrarily large genus.

Theorem 3. *Let $u_1, \dots, u_k \in M$ be functions with disjoint supports and let*

$$V_k = \langle u_1, \dots, u_k \rangle$$

be the space spanned by $u_1, \dots, u_k$. Then $M \cap V_k$ is the $(k-1)$-dimensional sphere, hence $\gamma(M \cap V_k) = k$.

Now, it is natural if one raises the question of whether there exists such functions with disjoint supports for arbitrary k. The answer is positive:

Theorem 4. *If (2.13) holds then for every $k \geq 2$ there exist k functions $u_1, \dots, u_k \in M$ with disjoint supports. Hence $\gamma(M) = +\infty$.*

Let

$$G_1 : u \in H_0^1(\Omega) \mapsto \int_\Omega u^2 dx - 1 \in \mathbb{R},$$
$$G_2 : u \in H_0^1(\Omega) \mapsto \int_\Omega q u^2 dx - \alpha \in \mathbb{R}$$

and

$$G = (G_1, G_2).$$

Then

$$M = \Big\{ u \in H_0^1(\Omega) \ : \ G_1(u) = G_2(u) = 0 \Big\} = G^{-1}(0).$$

We note that G is of class C^1.

Let us show, for the reader convenience, that $G_1'(u)$ and $G_2'(u)$ are linearly independent, so G will be a submersion and M a submanifold of codimension 2.

Proposition 2. *Assume M is not empty. The differentials $G_1'(u)$ and $G_2'(u)$ are linearly independent for every $u \in M$ if and only if*

$$|q^{-1}(\alpha)| = 0. \tag{3.1}$$

Proof. First, assume (3.1). We will show that $G_1'(u)$ and $G_2'(u)$ are linearly independent, for all $u \in M$. Suppose that there are $a, b \in \mathbb{R}$ such that

$$aG_1'(u) + bG_2'(u) = 0 \quad \text{in } H^{-1}(\Omega)$$

for some $u \in M$. Evaluating this expression in u, we find that $a + b\alpha = 0$. Then

$$aG_1'(u)[v] + bG_2'(u)[v] = b\left(-\alpha \int_\Omega uvdx + \int_\Omega quvdx\right) = 0 \quad \forall v \in H_0^1(\Omega),$$

that is,

$$b\int_\Omega (q-\alpha)uvdx = 0 \quad \forall v \in H_0^1(\Omega).$$

If $b \neq 0$ then we would have $(q-\alpha)u = 0$ a.e., and hence, in view of (3.1), $u = 0$, a contradiction. Thus $G_1'(u)$ and $G_2'(u)$ are linearly independent for all $u \in M$.

Now, suppose (3.1) is not satisfied. Then $q^{-1}(\alpha)$ has not empty interior, hence there is some test function u with support in $q^{-1}(\alpha)$ such that $||u||_2 = 1$. It is immediately seen that $u \in M$ because $qu = \alpha u$ and hence

$$G_2'(u) = \alpha G_1'(u),$$

which completes the proof. ☐

4. Variational setting

We now proceed to study the variational framework of the problem. Our aim is to construct a functional whose critical points will be the weak solutions of the problem.

Following [6], let

$$V = \left\{\xi \in H^2(\Omega) \ : \ \frac{\partial \xi}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega\right\}.$$

We remark that V is a closed subspace of $H^2(\Omega)$. Indeed, let $\{v_n\} \subset V$ such that $v_n \to v$ in V. Then $0 = \gamma_1(v_n) \to \gamma_1(v)$ and hence $\gamma_1(v) = 0$, where γ_1 denotes the trace operator which for smooth functions gives the directional derivative in the direction of the exterior normal on the boundary. Being a closed subspace, V inherits the Hilbert space structure of $H^2(\Omega)$.

Recall that

$$\varphi = \phi - \chi - \mu$$

where

$$\mu = \frac{1}{|\Omega|}\int_\Omega \phi dx.$$

In this way, we have $\overline{\varphi} = 0$, where from now on, given a function f, we denote with $\overline{f}$ its average in Ω. Consider then the following natural decomposition of V:

$$V = \widetilde{V} \oplus \mathbb{R} \tag{4.1}$$

where

$$\widetilde{V} = \{\eta \in V \ : \ \overline{\eta} = 0\}.$$

On $\widetilde{V}$ we have the equivalent norm

$$||\eta||_{\widetilde{V}} = \left(||\nabla\eta||_2^2 + ||\Delta\eta||_2^2\right)^{1/2}.$$

Consider the functional $F : H_0^1(\Omega) \times H^2(\Omega)$ defined below:

$$\begin{aligned} F(u,\varphi) =& \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega q(\varphi+\chi)u^2 dx - \frac{\kappa}{p}\int_\Omega |u|^p dx \\ & - \frac{1}{4}\int_\Omega (\Delta\varphi)^2 dx - \frac{1}{4}\int_\Omega |\nabla\varphi|^2 dx - \frac{\alpha}{2|\Omega|}\int_\Omega \varphi dx. \end{aligned}$$

It is easy to see that this functional is of class C^1 and that given $u \in H_0^1(\Omega)$ and $\varphi \in H^2(\Omega)$ we have

$$\begin{aligned} F'_u(u,\varphi)[v] &= \int_\Omega \nabla u \nabla v dx + \int_\Omega q(\varphi+\chi)uv dx - \kappa\int_\Omega |u|^{p-2}uv dx \\ F'_\varphi(u,\varphi)[\xi] &= \frac{1}{2}\int_\Omega q\xi u^2 dx - \frac{1}{2}\int_\Omega \Delta\varphi\Delta\xi dx - \frac{1}{2}\int_\Omega \nabla\varphi\nabla\xi dx - \frac{\alpha}{2|\Omega|}\int_\Omega \xi dx \end{aligned}$$

for every $v \in H_0^1(\Omega)$ and $\xi \in H^2(\Omega)$.

Then, $(u,\varphi,\omega,\mu) \in H_0^1(\Omega) \times H^2(\Omega) \times \mathbb{R} \times \mathbb{R}$ is a weak solution to (2.5)-(2.11) if and only if

$$\begin{aligned} & (u,\varphi) \in M \times \widetilde{V}, \\ & \forall v \in H_0^1(\Omega): \ F'_u(u,\varphi)[v] = \omega\int_\Omega uv dx - \mu\int_\Omega quv dx, \\ & \forall \xi \in V: \ F'_\varphi(u,\varphi)[\xi] = 0. \end{aligned} \tag{4.2}$$

Theorem 5. *Let $(u,\varphi) \in H_0^1(\Omega) \times H^2(\Omega)$. Then there exist $\omega,\mu \in \mathbb{R}$ such that (u,φ,ω,μ) is a solution to (2.5)-(2.11) if and only if (u,φ) is a critical point of F constrained on $M \times \widetilde{V}$, in which case the real constants ω,μ are the two Lagrange multipliers with respect to F'_u.*

Proof. Indeed (u,φ) is a critical point of F constrained on $M \times \widetilde{V}$ if and only if

$$\begin{aligned} & \forall v \in T_uM: \ F'_u(u,\varphi)[v] = 0, \\ & \forall \xi \in \widetilde{V}: \ F'_\varphi(u,\varphi)[\xi] = 0. \end{aligned}$$

Note that the tangent space to $\widetilde{V}$ at φ is $\widetilde{V}$ itself.

Then a weak solution, according to (4.2) and the Lagrange multipliers rule, is a constrained critical point.

Suppose on the contrary that (u,φ) is a constrained critical point. Then, again by the Lagrange multipliers rule, we have that there exists $\omega,\mu \in \mathbb{R}$ such that

$$\forall v \in H_0^1(\Omega) : F'_u(u,\varphi)[v] = \omega\int_\Omega uv dx - \mu\int_\Omega quv dx.$$

It remains to prove that $F'_\varphi(u,\varphi)[\xi] = 0$ for all $\xi \in V$. But this follows by the decomposition (4.1), noticing that $F'_\varphi(u,\varphi)[r] = 0$ for every constant $r \in \mathbb{R}$. Then (4.2) is satisfied and this concludes the proof. □

The functional F constrained on $M \times \widetilde{V}$ is unbounded from above and from below. This issue has been addressed by Benci and Fortunato [2] and in many subsequent papers. Their standard reduction argument goes as follows:

(i) For every fixed $u \in H_0^1(\Omega)$ there exists a unique $\Phi(u)$ such that $F'_\varphi(u,\Phi(u)) = 0$.
(ii) The map $u \mapsto \Phi(u)$ is of class C^1.
(iii) The graph of Φ is a manifold, and we are reduced to study the functional $J(u) = F(u,\Phi(u))$, possibly constrained.

However the method sketched above fails in our situation, for two reasons. First, we see that $F'_\varphi(u,\varphi)=0$ with $\varphi\in\widetilde{V}$ is just

$$\begin{aligned}
\Delta^2\varphi-\Delta\varphi-qu^2+\alpha/|\Omega|&=0 \quad \text{in } \ \Omega,\\
\frac{\partial\varphi}{\partial\mathbf{n}}&=0 \quad \text{on } \ \partial\Omega,\\
\frac{\partial\Delta\varphi}{\partial\mathbf{n}}&=0 \quad \text{on } \ \partial\Omega,\\
\int_\Omega\varphi dx&=0.
\end{aligned}$$

The problem above has not a unique solution for any fixed u: this happens, due to the compatibility condition, if and only if $u\in N$. Moreover, since N is not a manifold (unless $\alpha\neq 0$) we cannot require the map $\Phi:u\mapsto\Phi(u)$ to be of class C^1 in N. We shall then extend such a map Φ.

Proposition 3. *For every $w\in L^{6/5}(\Omega)$ there exists a unique $L(w)\in\widetilde{V}$ solution of*

$$\begin{aligned}
\Delta^2\varphi-\Delta\varphi-w+\overline{w}&=0 \quad \textit{in } \ \Omega,\\
\frac{\partial\varphi}{\partial\mathbf{n}}&=0 \quad \textit{on } \ \partial\Omega,\\
\frac{\partial\Delta\varphi}{\partial\mathbf{n}}&=0 \quad \textit{on } \ \partial\Omega,\\
\int_\Omega\varphi dx&=0.
\end{aligned}$$

The map $L:L^{6/5}(\Omega)\longrightarrow\widetilde{V}$ is linear and continuous, hence of class C^∞.

Proof. The weak solutions to the problem are functions φ in the Hilbert space $\widetilde{V}$ such that

$$\int_\Omega\Delta\varphi\Delta v dx+\int_\Omega\nabla\varphi\nabla v dx-\int_\Omega wv dx=0 \qquad \forall v\in\widetilde{V}.$$

So the result follows by applying the Riesz Theorem since the bilinear form $b:\widetilde{V}\times\widetilde{V}\longrightarrow\mathbb{R}$ given by

$$b(\varphi,v)=\int_\Omega\Delta\varphi\Delta v dx+\int_\Omega\nabla\varphi\nabla v dx.$$

is just the scalar product in $\widetilde{V}$. □

The following proposition follows from well-known properties of Nemytsky operators.

Proposition 4. *The map*

$$u\in L^6(\Omega)\mapsto qu^2\in L^{6/5}(\Omega)$$

is of class C^1.

As a consequence of the previous propositions, we can define the following map:

$$\Phi:u\in H_0^1(\Omega)\mapsto L(qu^2)\in\widetilde{V}.$$

It is clear that

$$\Phi(u)=\Phi(-u)=\Phi(|u|).$$

Moreover, for every $(u,\varphi)\in H_0^1(\Omega)\times\widetilde{V}$ we have that $\varphi=\Phi(u)$ if and only if for every $\eta\in\widetilde{V}$

$$\int_\Omega\Delta\varphi\Delta\eta dx+\int_\Omega\nabla\varphi\nabla\eta dx=\int_\Omega qu^2\eta dx.$$

Taking $\eta = \Phi(u)$ we have in particular the important relation

$$\int_\Omega (\Delta\Phi(u))^2 dx + \int_\Omega |\nabla\Phi(u)|^2 dx = \int_\Omega q u^2 \Phi(u) dx. \tag{4.3}$$

The right hand side above is the *interaction energy* term. Then we infer

$$\begin{aligned} ||\Phi(u)||^2_{\widetilde{V}} &\leq ||q||_\infty \int_\Omega u^2\Phi(u)dx \\ &\leq c||u||_4^2||\Phi(u)||_2 \\ &\leq c||\nabla u||_2^2||\nabla\Phi(u)||_2 \\ &\leq c||\nabla u||_2^2||\Phi(u)||_{\widetilde{V}} \end{aligned}$$

and hence

$$||\Phi(u)||_{\widetilde{V}} \leq c||\nabla u||_2^2, \tag{4.4}$$

that is, Φ is bounded on bounded sets. We have

Lemma 2. *If $u_n \rightharpoonup u$ in $H_0^1(\Omega)$ then*

$$\int_\Omega q u_n^2 \Phi(u_n) dx \to \int_\Omega q u^2 \Phi(u) dx.$$

Moreover the map Φ is compact.

Proof. Let $u_n \rightharpoonup u$ in $H_0^1(\Omega)$ and define $B_n, B : \widetilde{V} \longrightarrow \mathbb{R}$ by

$$B_n(\eta) := \int_\Omega q u_n^2 \eta dx, \qquad B(\eta) := \int_\Omega q u^2 \eta dx.$$

Such operators are continuous due to the Hölder's inequality. For example:

$$\left|\int_\Omega q u^2 \eta dx\right| \leq ||q||_\infty ||u||_4^2 ||\eta||_2 \leq c||\nabla\eta||_2 \leq c||\eta||_{\widetilde{V}}$$

(where here c depends on u).

Due to the compact embedding of $H_0^1(\Omega)$ into $L^p(\Omega)$ for $p \in [1, 6)$, we get $u_n^2 \to u^2$ in $L^{6/5}(\Omega)$ and then

$$\begin{aligned} |B_n(\eta) - B(\eta)| &\leq ||q||_\infty ||u_n^2 - u^2||_{6/5}||\eta||_6 \\ &\leq c||q||_\infty||u_n^2 - u^2||_{6/5}||\eta||_{\widetilde{V}}. \end{aligned}$$

Hence

$$||B_n - B|| \leq \sup_{\eta\neq 0} \frac{c||u_n^2 - u^2||_{6/5}||\eta||_{\widetilde{V}}}{||\eta||_{\widetilde{V}}} \to 0,$$

namely $B_n \to B$ as operators in $\widetilde{V}$.

On the other hand, we have that $\Phi(u_n) \rightharpoonup \Phi(u)$ in $\widetilde{V}$. Indeed, let $g \in \widetilde{V}'$. Then there is some $v_g \in \widetilde{V}$ such that

$$g(\Phi(u_n)) = \int_\Omega \nabla\Phi(u_n)\nabla v_g dx + \int_\Omega \Delta\Phi(u)\Delta v_g dx = \int_\Omega q u_n^2 v_g dx.$$

But then

$$\begin{aligned} g(\Phi(u_n)) - g(\Phi(u)) &= \int_\Omega q(u_n^2 - u^2)v_g dx \\ &\leq ||q||_\infty||u_n^2 - u^2||_2||v_g||_2 \to 0 \end{aligned}$$

since $u_n^2 \to u^2$ in $L^2(\Omega)$ as well.

We then conclude that

$$\int_\Omega q u_n^2 \Phi(u_n) dx \to \int_\Omega q u^2 \Phi(u) dx$$

and by (4.3) that $\|\Phi(u_n)\|_{\widetilde{V}} \to \|\Phi(u)\|_{\widetilde{V}}$. Consequently $\Phi(u_n) \to \Phi(u)$ in $\widetilde{V}$. □

Note that for every $u \in N$ we have that $F'_\varphi(u, \Phi(u)) = 0$. Indeed, $\Phi(u)$ is the unique solution to the problem in Proposition 3 with $w = qu^2$.

We now define the reduced functional of a single variable:

$$\begin{array}{rccl} J & : & H_0^1(\Omega) & \longrightarrow & \mathbb{R} \\ & & u & \longmapsto & F(u, \Phi(u)) \end{array}$$

With the notation $\varphi_u := \Phi(u)$ the functionl J is explicitly given by (recall (4.3))

$$\begin{aligned} J(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega q\varphi_u u^2 dx + \frac{1}{2}\int_\Omega q\chi u^2 dx - \frac{\kappa}{p}\int_\Omega |u|^p dx \\ &\quad - \frac{1}{4}\int_\Omega (\Delta\varphi_u)^2 dx - \frac{1}{4}\int_\Omega |\nabla\varphi_u|^2 dx - \frac{\alpha}{2|\Omega|}\int_\Omega \varphi_u dx \\ &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{4}\int_\Omega (\Delta\varphi_u)^2 dx + \frac{1}{4}\int_\Omega |\nabla\varphi_u|^2 dx + \int_\Omega q\chi u^2 dx \\ &\quad - \frac{\kappa}{p}\int_\Omega |u|^p dx. \end{aligned}$$

We note that J is of class C^1 on $H_0^1(\Omega)$ and even. Moreover, for every $u \in M$ we have that

$$J'(u)[v] = F'_u(u, \varphi_u)[v] + F'_\varphi(u, \varphi_u)[\Phi'(u)[v]] = F'_u(u, \varphi_u)[v] \quad \forall v \in H_0^1(\Omega)$$

and hence we deduce the following

Theorem 6. *The pair $(u, \varphi) \in M \times \widetilde{V}$ is a critical point of F constrained on $M \times \widetilde{V}$ if and only if u is a critical point of $J|_M$ and $\varphi = \Phi(u)$.*

5. Proof of the main result

The next lemma will be useful.

Lemma 3. *Let D be a regular domain of $\mathbb{R}^N$ and*

$$1 \le s \le N,$$

$$s < p < s^* = \frac{Ns}{N-s}$$

and

$$0 < r \le N\left(1 - \frac{p}{s^*}\right).$$

Then there exists a constant $C > 0$ such that for every $u \in W^{1,s}(D)$ it holds that

$$||u||_p^p \le C||u||_{W^{1,s}}^{p-r}||u||_s^r$$

Proof. See [9, Lemma 3.1]. □

Remark 2. *If D is bounded, then the conclusion of the lemma is true also in the case $p \in [1, s]$ with $r < p$. Also, if D is bounded and $u \in W_0^{1,p}(D)$, then, by Poincaré inequality,*

$$||u||_p^p \le C||\nabla u||_s^{p-r}||u||_s^r.$$

The following lemma gives the existence of solutions to our modified problem.

Lemma 4. *The functional J on M is weakly lower semicontinuous and coercive. In particular, it has a minimum $u \in M$, and it can be assumed positive.*

Proof. We have

$$\begin{aligned} J(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{4}\int_\Omega (\Delta\varphi_u)^2 dx + \frac{1}{4}\int_\Omega |\nabla\varphi_u|^2 dx + \int_\Omega q\chi u^2 dx - \frac{\kappa}{p}\int_\Omega |u|^p dx \\ &\geq \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \|q\|_\infty \|\chi\|_\infty - \frac{\kappa}{p}\int_\Omega |u|^p dx. \end{aligned}$$

Finally, we apply Lemma 3 with $s = 2$ and $N = 3$. Since $p \in (2, 10/3)$ it holds that

$$p - 2 < 3\left(1 - \frac{p}{6}\right) < 2$$

and we can choose

$$p - 2 < r < 3\left(1 - \frac{p}{6}\right),$$

so that by the Lemma it follows that

$$\frac{\kappa}{p}\int_\Omega |u|^p dx \leq c||\nabla u||_2^{p-r}.$$

Hence,

$$J(u) \geq \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \|q\|_\infty \|\chi\|_\infty - c'||\nabla u||_2^{p-r}$$

and thus J is coercive and bounded from below on M.

Now, let $\{u_n\} \subset M$ such that $u \rightharpoonup u$. Since M is weakly closed, $u \in M$. By Lemma 2 we know that

$$\frac{1}{4}\int_\Omega (\Delta\varphi_{u_n})^2 dx + \frac{1}{4}\int_\Omega |\nabla\varphi_{u_n}|^2 dx \to \frac{1}{4}\int_\Omega (\Delta\varphi_u)^2 dx + \frac{1}{4}\int_\Omega |\nabla\varphi_u|^2 dx.$$

We also know that $u_n^2 \to u^2$ in $L^{6/5}(\Omega)$ so

$$\begin{aligned} \int_\Omega q\chi(u_n^2 - u^2) dx &\leq c\int_\Omega u_n^2 - u^2 dx \\ &\leq c||u_n - u||_{6/5} \to 0. \end{aligned}$$

Finally, the first and last terms are the norms of u in $H_0^1(\Omega)$ and $L^p(\Omega)$ (up to constants), so they are weakly lower semicontinuous.

Thus J is weakly lower semicontinuous and the existence of the minimum follows by standard results. Note that $J(u) = J(|u|)$ so the minimum may be assumed to be positive. □

We will use a deformation argument to show that there are infinitely many solutions. A crucial point is that the functional satisfies the Palais-Smale condition. We recall that in general, it is said that the C^1 functional $\mathcal{I}$ satisfies the Palais-Smale condition on the manifold $\mathcal{M}$, if any sequence $\{u_n\} \subset \mathcal{M}$ such that $\big\{\mathcal{I}(u_n)\big\}$ is bounded and $\mathcal{I}(u_n) \to 0$ in the tangent bundle, admits a convergent subsequence to an element $u \in \mathcal{M}$.

Proposition 5. *The functional J satisfies the Palais-Smale condition on M.*

Proof. Let $\{u_n\} \subset M$ be such that

$$\{J(u_n)\} \text{ is bounded}$$

and

$$J|_M'(u_n) \to 0. \tag{5.1}$$

By (5.1) there exists two sequences of real numbers $\{\lambda_n)\}, \{\beta_n\}$ and a sequence $\{v_n\} \subset H^{-1}$ such that $v_n \to 0$ and

$$-\Delta u_n + q(\varphi_n + \chi)u_n - \kappa|u_n|^{p-2}u_n = \lambda_n u_n + \beta_n q u_n + v_n \tag{5.2}$$

where $\varphi_n := \varphi_{u_n}$.

Since J is coercive and $\{J(u_n)\}$ is bounded, we know that $\{u_n\}$ is bounded in $H^1_0(\Omega)$. Hence there exists $u \in H^1_0(\Omega)$ such that $u_n \rightharpoonup u$, up to a subsequence. By the compact embeddings and Lemma 2 we know that

$$u_n \to u \quad \text{in} \quad L^p(\Omega), \quad \varphi_n \to \varphi_u \quad \text{in} \quad H^2(\Omega). \tag{5.3}$$

Also, since M is weakly closed, we know that $u \in M$. It only remains to show that $u_n \to u$ in $H^1_0(\Omega)$.

By (5.2) we have that

$$\frac{1}{2}\int_\Omega |\nabla u_n|^2 dx + \frac{1}{2}\int_\Omega q(\varphi_n + \chi)u_n^2 dx - \frac{\kappa}{p}\int_\Omega |u_n|^p dx - \langle v_n, u_n\rangle = \lambda_n + \alpha\beta_n. \tag{5.4}$$

By (5.3) we infer

$$\begin{aligned}\left|\int_\Omega \Big(q(\varphi_n + \chi)u_n^2 - q(\varphi_u + \chi)u^2\Big)\, dx\right| &\leq c\int_\Omega |\varphi_n + \chi|\, |u_n^2 - u^2| dx + \int_\Omega u^2\, |\varphi_n - \varphi|\, dx \\ &= o_n(1)\end{aligned}$$

where we are denoting with $o_n(1)$ a vanishing sequence. Then the right-hand side of (5.4) is bounded and we can assume that

$$\lambda_n + \alpha\beta_n = \xi + o_n(1)$$

with $\xi \in \mathbb{R}$. Then (5.2) becomes

$$-\Delta u_n + q(\varphi_n + \chi)u_n - \kappa|u_n|^{p-2}u_n - v_n = (\xi + o(1))u_n - \beta_n(q - \alpha)u_n. \tag{5.5}$$

Now, since $u \in M$ we know that $||u||_2 = 1$. This, together with the assumption $|q^{-1}(\alpha)| = 0$ implies that $(q - \alpha)u$ is not identically zero. Then there exists a test function $w \in C^\infty_0(\Omega)$ such that

$$\int_\Omega (q - \alpha)uw dx \neq 0.$$

Evaluating (5.5) on this w we get

$$\begin{aligned}&\int_\Omega \nabla u_n \nabla w dx + \int_\Omega q(\varphi_n + \chi)u_n w dx - \kappa\int_\Omega |u_n|^{p-2}u_n w dx \\ &\qquad - \langle v_n, w\rangle - (\lambda + o_n(1))\int_\Omega u_n w dx = \beta_n\int_\Omega (q - \alpha)u_n w dx\end{aligned} \tag{5.6}$$

and using again (5.3) we see that every term in the left-hand side converges. Also, by the weak convergence of $\{u_n\}$,

$$\int_\Omega (q - \alpha)u_n w dx \to \int_\Omega (q - \alpha)uw dx.$$

This implies, coming back to (5.6), that $\{\beta_n\}$ is bounded, which in turn implies that $\{\lambda_n\}$ is bounded.

Applying (5.5) to $u_n - u$ we get

$$\int_\Omega \nabla u_n \nabla (u_n - u)dx + \int_\Omega q(\varphi_n + \chi)u_n(u_n - u)dx - \kappa \int_\Omega |u_n|^{p-2}u_n(u_n - u) \\ - \langle v_n, u_n - u\rangle = (\lambda + o(1)) \int_\Omega u_n(u_n - u)dx + \beta_n \int_\Omega (q - \alpha)u_n(u_n - u)dx. \tag{5.7}$$

Since (again by (5.3)) we have

$$\int_\Omega q(\varphi_n + \chi)u_n(u_n - u)dx \to 0, \quad \langle v_n, u_n - u\rangle \to 0,$$
$$\int_\Omega |u_n|^{p-2}u_n(u_n - u)dx \to 0, \quad (\lambda + o(1)) \int_\Omega u_n(u_n - u)dx \to 0$$

and

$$\beta_n \int_\Omega (q - \alpha)u_n(u_n - u)dx \to 0,$$

we conclude by (5.7) that $||\nabla u_n||_2 \to ||\nabla u||_2$ and so $u_n \to u$ in $H^1_0(\Omega)$. ☐

Now we can give the proof of Theorem 1.

By Theorem 3, M has compact, symmetric subsets of genus k for every $k \in \mathbb{N}$.

Let us recall now a classical result in critical point theory. We give the proof for the reader convenience.

Lemma 5. *For any $b \in \mathbb{R}$ the sublevel*

$$J^b = \{u \in M \ : \ J(u) \le b\}$$

has finite genus.

Proof. We argue by contradiction. Suppose that

$$D = \Big\{b \in \mathbb{R} \ : \ \gamma(J^b) = \infty\Big\} \ne \emptyset.$$

Since $J|_M$ is bounded from below, then D is bounded from below. Then

$$-\infty < \bar b = \inf D < \infty.$$

Moreover, since $J|_M$ satisfies the Palais-Smale condition, the set

$$Z = \Big\{u \in M \ : \ J(u) = \bar b, \ J|'_M(u) = 0\Big\}$$

is compact. Hence there exists a closed symmetric neighborhood U_Z of Z such that $\gamma(U_Z) < \infty$. By the Deformation Lemma, there exists an $\varepsilon > 0$ such that $J^{\bar b - \varepsilon}$ includes a deformation retract of $J^{\bar b + \varepsilon} \setminus U_Z$. Then, by the properties of the genus,

$$\gamma(J^{\bar b + \varepsilon}) \le \gamma(J^{\bar b + \varepsilon} \setminus U_Z) + \gamma(U_Z) \le \gamma(J^{\bar b - \varepsilon}) + \gamma(U_Z) < \infty,$$

a contradiction. ☐

Let $n \in \mathbb{N}$. By Lemma 5 there exists some $k \in \mathbb{N}$ depending on n such that

$$\gamma(J^n) = k.$$

Let

$$A_{k+1} = \Big\{A \subset M \ : \ A = -A, \overline{A} = A, \gamma(A) = k + 1\Big\}$$

that we know is not empty by Theorem 3.

By the monotonicity property of the genus, any $A \in A_{k+1}$ is not contained in J^n, then $\sup_A J > n$ and therefore

$$c_n = \inf_{A \in A_{k+1}} \sup_{u \in A} J(u) \geq n.$$

Well known results (see e.g. Szulkin [12]) say that c_n are critical levels for $J|_M$ and then there is a sequence $\{u_n\}$ of critical points such that

$$J(u_n) = c_n \to +\infty$$

The critical points give rise to Lagrange multipliers ω_n, μ_n and then, recalling the decomposition $\varphi = \phi - \chi - \mu$, to solutions $(u_n, \omega_n, \phi_n) \in H_0^1(\Omega) \times \mathbb{R} \times H^2(\Omega)$ of the original problem.

We show that $||\nabla u_n||_2 \to +\infty$. Since

$$\int_\Omega q\chi u_n^2 dx \leq ||q\chi||_\infty,$$

and by (4.4) it is

$$||\varphi_n||_{\widetilde{V}} = \int_\Omega (\Delta\varphi_n)^2 dx + \int_\Omega |\nabla\varphi_n|^2 dx \leq c||\nabla u_n||_2^2,$$

we see that

$$|J(u_n)| \leq (1+c)||\nabla u_n||^2 + c'||\nabla u_n||_2^p + ||q\chi||_\infty$$

and then $\{u_n\}$ can not be bounded.

This concludes the proof of Theorem 1.

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(D. G. Afonso)
Departamento de Matemática
Instituto de Matemática e Estatística
Universidade de São Paulo
Rua do Matão 1010, 05508-090 São Paulo, SP, Brazil
E-mail address: d.g.afonso@mat.unb.br

(G. Siciliano)
Departamento de Matemática
Instituto de Matemática e Estatística
Universidade de São Paulo
Rua do Matão 1010, 05508-090 São Paulo, SP, Brazil
E-mail address: sicilian@ime.usp.br