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WELL-POSEDNESS FOR A SLOW EROSION MODEL

G. M. COCLITE AND E. JANNELLI

Abstract. We improve in two ways the well-posedness results of [2] for a slow erosion model proposed in [11]: firstly we study the asymptotic profile when $\frac{u_0}{1+u_0} \in L^\infty$, where u_0 is the initial datum; moreover, using a compensated compactness based argument we prove the existence of solutions when $\frac{u_0}{1+u_0} \in L^\sigma$, $\sigma \geq 3$.

1. The basic model

This paper is devoted to the analysis of the slow erosion model

$$\partial_t u + \partial_x \left(f(u) E[u] \right) = 0, \qquad t > 0,\ x \in \mathbb{R}, \tag{1.1}$$

where

$$f(u) = \frac{u}{u+1}, \qquad E[u(t,\cdot)](x) = e^{\int_x^\infty f(u(t,\xi))d\xi}.$$

This equation has been studied in [2] and describes the slow erosion limit for a granular flow model proposed in [11]. The function $u+1$ gives the slope of the standing profile of granular matter, that is influenced by the occurrence of small avalanches. The function $f = f(u)$ is the erosion function and has the meaning of the erosion rate per unit length in space covered by the avalanches. A more detailed derivation of the model can be found in [15]. For more general f and a numerical analysis see [15, 1, 7].

We augment (1.1) with the initial condition

$$u(0,x) = u_0(x), \qquad x \in \mathbb{R}, \tag{1.2}$$

and we assume that

$$u_0 \in L^1(\mathbb{R}), \qquad -1 \leq u_0 \leq 0, \qquad f(u_0) \in L^1(\mathbb{R}) \cap L^\sigma(\mathbb{R}), \tag{1.3}$$

for some $1 \leq \sigma \leq \infty$.

We use the following notions of solution for (1.1) and (1.2).

Definition 1.1. *Let $u : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be a function. We say that u is a weak solution of* (1.1) *and* (1.2) *if for any test function $\varphi \in C^\infty(\mathbb{R}^2)$ with compact support we have that*

$$\int_0^\infty \int_{\mathbb{R}} \Big(u \partial_t \varphi + f(u) E[u] \partial_x \varphi \Big) dx dt + \int_{\mathbb{R}} u_0(x) \varphi(0,x) dx = 0. \tag{1.4}$$

Definition 1.2. *Let $u : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be a function. We say that u is an entropy solution of* (1.1) *and* (1.2) *if for any nonnegative test function $\varphi \in C^\infty(\mathbb{R}^2)$ with compact support and any convex entropy $\eta \in C^2(\mathbb{R})$ with entropy flux $q \in C^2(\mathbb{R})$ defined by $q' = \eta' f'$ we have that*

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}} \Big(\eta(u) \partial_t \varphi + q(u) E[u] \partial_x \varphi + (f(u)\eta'(u) - q(u)) f(u) E[u] \varphi \Big) dx dt & \\ + \int_{\mathbb{R}} \eta(u_0(x)) \varphi(0,x) dx \geq 0. & \end{aligned} \tag{1.5}$$

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In [1, 2] the authors studied the well-posedness of the entropy solutions of (1.1) and (1.2) assuming that

$$u_0 \in BV(\mathbb{R}) \tag{1.6}$$

and that (1.3) holds with

$$\sigma = \infty, \tag{1.7}$$

which means

$$-1 < \kappa_0 \leq u_0 \leq 0, \tag{1.8}$$

for some constant κ_0. Using a front tracking algorithm, they proved that the Cauchy problem (1.1) and (1.2) admits a unique entropy solution u such that:

$$u \in L^\infty(0,T;L^1(\mathbb{R})) \cap L^\infty(0,T;BV(\mathbb{R})), \qquad T > 0; \tag{1.9}$$

$$\text{for any } T > 0 \text{ there exists } \kappa_T \text{ s.t. } -1 < \kappa_T \leq u \leq 0 \text{ a.e. in } (0,T) \times \mathbb{R}. \tag{1.10}$$

Moreover, they show that the map $u_0 \mapsto u$ is Lipschitz continuous, in the sense that if u and v are two entropy solutions of (1.1) satisfying (1.3), (1.6), and (1.8) at time $t = 0$, then for any $T > 0$ there exists a constant $L_T > 0$ such that

$$\|u(t,\cdot) - v(t,\cdot)\|_{L^1(\mathbb{R})} \leq L_T \|u(0,\cdot) - v(0,\cdot)\|_{L^1(\mathbb{R})}, \qquad \text{a.e. } 0 < t < T. \tag{1.11}$$

In this paper we consider the following vanishing viscosity approximation of (1.1) and (1.2):

$$\begin{cases} \partial_t u_\varepsilon + \partial_x \left(f(u_\varepsilon) E[u_\varepsilon]\right) = \varepsilon \partial_{xx}^2 u_\varepsilon, & t > 0,\ x \in \mathbb{R}, \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x), & x \in \mathbb{R}, \end{cases} \tag{1.12}$$

where $\varepsilon > 0$ and $u_{0,\varepsilon}$ is a smooth approximation of u_0. The well-posedness of smooth solutions for (1.12) can be proved using the same arguments of [4, 5, 8].

We improve the results of [1, 2] in two ways. We begin by considering their assumptions, namely we require on u_0 (1.3), (1.6), (1.7), and (1.8). The analysis of the BV compactness properties of the solutions of (1.12) allows us to

- give a simpler proof of the existence results of [1, 2] for (1.1) and (1.2);
- prove better pointwise lower bounds on the solution of (1.1) and (1.2) than the ones in [1, 2];
- describe the asymptotic behavior of the solution of (1.1) and (1.2) as $t \to \infty$;
- get hints on the compactness properties of numerical schemes for (1.1) and (1.2).

As a second step, we remove both (1.6) and (1.8), and we assume that (1.3) holds with

$$\sigma \geq 3. \tag{1.13}$$

From a physical point of view when $\sigma < \infty$ the deposition function $u + 1$ can become singular (i.e., can vanish). The fact that in (1.13) we have $\sigma \geq 3$ and not simply $\sigma \geq 1$ is purely technical and is needed to make sense to all the terms in (1.5) under the different choices of η. Under these assumptions, we prove the existence of entropy solutions for (1.1) and (1.2). We bypass the lack of BV bounds on u_ε arguing as in [6, 9, 10] and using the compensated compactness result deduced in [13, 14] for conservation laws with discontinuous fluxes.

Finally, we wish to make an additional comment on the the upper bound $u_0 \leq 0$ on the initial condition in (1.3), that is not considered in [2]. That bound physically says that we have only deposition of material. Mathematically, we use this assumption to simplify the presentation and focus on removing the lower bound $\kappa_0 > -1$, see (1.8); indeed, passing from $-1 \leq u_0 \leq 0$ to the case $-1 \leq u_0$ does not increase difficulty, because f is bounded and Lipshitz continuous in $[0, \infty)$.

The paper is organized as follows. In Section 2, assuming $\sigma = \infty$, we prove the convergence of a vanishing viscosity type approximation and we study the asymptotic behavior of the entropy solutions. In Section 3, assuming $\sigma \geq 3$, we prove that (1.1) and (1.2) admits an entropy solution.

2. The case $\sigma = \infty$

In this section we assume (1.3), (1.6), (1.7), and (1.8).

About the initial condition $u_{0,\varepsilon}$ of (1.12) we assume (here and in the following, as usual, TV means total variation)

$$\begin{aligned}
& u_{0,\varepsilon} \in C^\infty(\mathbb{R}), \qquad \varepsilon > 0,\\
& u_{0,\varepsilon} \to u_0, \quad \text{a.e. in } \mathbb{R} \text{ and in } L^p(\mathbb{R}),\ 1 \le p < \infty \text{ as } \varepsilon \to 0,\\
& \|u_{0,\varepsilon}\|_{L^1(\mathbb{R})} \le \|u_0\|_{L^1(\mathbb{R})}, \quad \|\partial_x u_{0,\varepsilon}\|_{L^1(\mathbb{R})} \le TV(u_0), \quad -1 < \kappa_0 \le u_{0,\varepsilon} \le 0, \qquad \varepsilon > 0,\\
& \varepsilon \left\|\partial_{xx}^2 u_{0,\varepsilon}\right\|_{L^1(\mathbb{R})} \le C, \qquad \varepsilon > 0,
\end{aligned} \tag{2.1}$$

for some constant $C > 0$ independent on ε.

The main result of this section is the following

Theorem 2.1. *Assume* (1.3), (1.6), (1.7), (1.8), *and* (2.1). *Let u be the unique entropy solution of* (1.1) *and* (1.2) *and u_ε the one of* (1.12). *We have that*

$$u_\varepsilon \to u, \quad \text{a.e. in } \mathbb{R} \text{ and in } L^p_{loc}((0,\infty)\times\mathbb{R}),\ 1 \le p < \infty \text{ as } \varepsilon \to 0. \tag{2.2}$$

Moreover, $u \in BV((0,T)\times\mathbb{R})$ for any $T > 0$, and satisfies

$$\frac{-1}{\kappa_1 t + \kappa_2 + \sqrt{(\kappa_1 t + \kappa_2)^2 + 1}} \le u(t,x) \le 0, \qquad \text{a.e. in } (0,\infty)\times\mathbb{R}, \tag{2.3}$$

where

$$\kappa_1 = \frac{e^{-\frac{\|u_0\|_{L^1(\mathbb{R})}}{1+\kappa_0}}}{2}, \qquad \kappa_2 = \frac{\kappa_0^2 - 1}{2\kappa_0}.$$

In particular, u has the following asymptotic behavior:

$$u(t,\cdot) \to 0, \quad \text{a.e. in } \mathbb{R} \text{ and in } L^p_{loc}(\mathbb{R}),\ 1 \le p < \infty \text{ as } t \to \infty. \tag{2.4}$$

In order to prove Theorem 2.1 we need some preliminary lemmas, for all of which we assume the hypotheses of Theorem 2.1.

Lemma 2.1 (L^∞ estimate)**.** *The following inequalities*

$$\kappa_0 \le u_\varepsilon(t,x) \le 0 \tag{2.5}$$

hold for any $\varepsilon > 0$, $t \ge 0$, $x \in \mathbb{R}$. In particular

$$\frac{\kappa_0}{\kappa_0 + 1} \le f(u_\varepsilon) \le 0, \qquad 1 \le f'(u_\varepsilon) \le \frac{1}{(1+\kappa_0)^2}. \tag{2.6}$$

Proof. Consider the initial value problem

$$\begin{cases} \partial_t v + f'(v)E[u_\varepsilon]\partial_x v - f^2(v)E[u_\varepsilon] = \varepsilon \partial_{xx}^2 v, & t > 0,\ x \in \mathbb{R},\\ v(0,x) = u_{0,\varepsilon}(x), & x \in \mathbb{R}. \end{cases} \tag{2.7}$$

We know that u_ε is the unique solution of (2.7), see [4, 5, 8].

Being

$$\begin{aligned}
\partial_t v + f'(v)E[u_\varepsilon]\partial_x v - f^2(v)E[u_\varepsilon] - \varepsilon\partial_{xx}^2 v\Big|_{v\equiv 0} &= 0,\\
\partial_t v + f'(v)E[u_\varepsilon]\partial_x v - f^2(v)E[u_\varepsilon] - \varepsilon\partial_{xx}^2 v\Big|_{v\equiv \kappa_0} &= -f^2(\kappa_0)E[u_\varepsilon] \le 0,
\end{aligned}$$

by (2.1) we get that 0 is a supersolution and κ_0 is a subsolution to (2.7). Therefore, (2.5) follows from the Comparison Principle for Parabolic equations.

Since f is concave and increasing in the interval $(-1, 0]$, (2.5) implies (2.6). □

Lemma 2.2 (L^1 ESTIMATE)**.** *The following inequality*

$$\|u_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})} \le \|u_0\|_{L^1(\mathbb{R})} \tag{2.8}$$

holds for any $\varepsilon > 0$ *and* $t \ge 0$. *Moreover*

$$e^{-\kappa_3} \le E[u_\varepsilon](t,x) \le 1, \qquad t>0,\ x\in\mathbb{R}, \tag{2.9}$$

where

$$\kappa_3 = \frac{\|u_0\|_{L^1(\mathbb{R})}}{1+\kappa_0}.$$

Proof. Since u_ε is nonpositive (see (2.5)) and $f(0)=0$, we have that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} |u_\varepsilon| dx &= \int_{\mathbb{R}} \partial_t u_\varepsilon \mathrm{sign}\,(u_\varepsilon)\, dx = -\int_{\mathbb{R}} \partial_t u_\varepsilon dx \\ &= -\,\varepsilon \int_{\mathbb{R}} \partial_{xx}^2 u_\varepsilon dx + \int_{\mathbb{R}} \partial_x (f(u_\varepsilon)E[u_\varepsilon]) dx = 0.\end{aligned}$$

An integration over $(0,t)$ gives

$$\|u_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})} = \|u_{0,\varepsilon}\|_{L^1(\mathbb{R})}.$$

Therefore, (2.8) follows from (2.1).

By (2.5), (2.8) and the very definition of f, we get

$$\begin{aligned} 0 &\ge \int_x^\infty f(u_\varepsilon(t,\xi))d\xi = \int_x^\infty \frac{u_\varepsilon(t,\xi)}{1+u_\varepsilon(t,\xi)} d\xi \\ &\ge -\int_{\mathbb{R}} \frac{|u_\varepsilon(t,\xi)|}{1+u_\varepsilon(t,\xi)} d\xi \ge -\frac{1}{1+\kappa_0}\int_{\mathbb{R}} |u_\varepsilon(t,\xi)| d\xi \ge -\frac{\|u_0\|_{L^1(\mathbb{R})}}{1+\kappa_0} = -\kappa_3. \end{aligned}$$

Using the definition of the integral operator $E[\cdot]$ we gain (2.9). □

Lemma 2.3 (LOWER BOUND)**.** *The inequality*

$$u_\varepsilon(t,x) \ge \frac{-1}{\kappa_1 t + \kappa_2 + \sqrt{(\kappa_1 t+\kappa_2)^2+1}} \tag{2.10}$$

holds for any $\varepsilon > 0$ *and* $t \ge 0$.

Proof. Consider the function

$$w(t) = \frac{-1}{\kappa_1 t + \kappa_2 + \sqrt{(\kappa_1 t+\kappa_2)^2+1}}$$

which solves

$$w' = 2\kappa_1 \frac{w^2}{w^2+1}, \qquad w(0)=\kappa_0.$$

Using (2.9) and the identity $2\kappa_1 = e^{-\kappa_3}$, we get

$$\begin{aligned} &\partial_t v + f'(v)E[u_\varepsilon]\partial_x v - f^2(v)E[u_\varepsilon] - \varepsilon \partial_{xx}^2 v\Big|_{v\equiv w} \\ &\quad = w' - f^2(w)E[u_\varepsilon] \le w' - e^{-\kappa_3} f^2(w) \\ &\quad = 2\kappa_1 \frac{w^2}{w^2+1} - e^{-\kappa_3}\frac{w^2}{(w+1)^2} = \frac{2e^{-\kappa_3}w^3}{(w+1)^2(w^2+1)} \le 0. \end{aligned}$$

Therefore, by (2.1), w is a subsolution to (2.7). The Comparison Principle for Parabolic equations guarantees that

$$w(t) \le u_\varepsilon(t,x),$$

that is (2.10). □

Lemma 2.4 (BV ESTIMATE IN x). *The inequality*

$$\|\partial_x u_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})} \le TV(u_0) + \frac{\kappa_0^2}{(1+\kappa_0)^3}\|u_0\|_{L^1(\mathbb{R})}\, t \tag{2.11}$$

holds for any $\varepsilon > 0$ and $t \ge 0$.

Proof. Differentiating the equation in (1.12) with respect to x, we get

$$\partial_{tx}^2 u_\varepsilon + \partial_x\left(f'(u_\varepsilon)E[u_\varepsilon]\partial_x u_\varepsilon\right) - 2f(u_\varepsilon)f'(u_\varepsilon)E[u_\varepsilon]\partial_x u_\varepsilon + f^3(u_\varepsilon)E[u_\varepsilon] = \varepsilon\partial_{xxx}^3 u_\varepsilon. \tag{2.12}$$

Thanks to (2.5), (2.6), (2.8), (2.9), and the definition of f, we have that

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}}|\partial_x u_\varepsilon|dx =& \int_{\mathbb{R}} \partial_{tx}^2 u_\varepsilon \mathrm{sign}\left(\partial_x u_\varepsilon\right) dx\\
=&\varepsilon\int_{\mathbb{R}} \partial_{xxx}^3 u_\varepsilon \mathrm{sign}\left(\partial_x u_\varepsilon\right) dx - \int_{\mathbb{R}}\partial_x\left(f'(u_\varepsilon)E[u_\varepsilon]\partial_x u_\varepsilon\right)\mathrm{sign}\left(\partial_x u_\varepsilon\right) dx\\
&+2\int_{\mathbb{R}} f(u_\varepsilon)f'(u_\varepsilon)E[u_\varepsilon]|\partial_x u_\varepsilon|dx - \int_{\mathbb{R}} f^3(u_\varepsilon)E[u_\varepsilon]\mathrm{sign}\left(\partial_x u_\varepsilon\right) dx\\
=&\underbrace{-\varepsilon\int_{\mathbb{R}}(\partial_{xx}^2 u_\varepsilon)^2 d\delta_{\{\partial_x u_\varepsilon=0\}}}_{\le 0} + \underbrace{\int_{\mathbb{R}} f'(u_\varepsilon)E[u_\varepsilon]\partial_x u_\varepsilon \partial_{xx}^2 u_\varepsilon d\delta_{\{\partial_x u_\varepsilon = 0\}}}_{=0}\\
&+\underbrace{2\int_{\mathbb{R}} f(u_\varepsilon)f'(u_\varepsilon)E[u_\varepsilon]|\partial_x u_\varepsilon|dx}_{\le 0} - \int_{\mathbb{R}} f^3(u_\varepsilon)E[u_\varepsilon]\mathrm{sign}\left(\partial_x u_\varepsilon\right) dx\\
\le&\frac{\kappa_0^2}{(1+\kappa_0)^3}\int_{\mathbb{R}}|u_\varepsilon|dx \le \frac{\kappa_0^2}{(1+\kappa_0)^3}\|u_0\|_{L^1(\mathbb{R})},
\end{aligned}$$

where $\delta_{\{\partial_x u_\varepsilon=0\}}$ is the Dirac delta concentrated on the set $\{\partial_x u_\varepsilon = 0\}$. An integration over $(0,t)$ gives

$$\|\partial_x u_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})} \le \|\partial_x u_{0,\varepsilon}\|_{L^1(\mathbb{R})} + \frac{\kappa_0^2}{(1+\kappa_0)^3}\|u_0\|_{L^1(\mathbb{R})}\, t.$$

Therefore, (2.11) follows from (2.1). □

Lemma 2.5 (BV ESTIMATE IN t). *The following inequality*

$$\|\partial_t u_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})} \le \left(C + \frac{TV(u_0)}{(1+\kappa_0)^2} + \frac{|\kappa_0|}{(1+\kappa_0)^2}\|u_0\|_{L^1(\mathbb{R})}\right)e^{\kappa_4 t + \kappa_5 t^2} \tag{2.13}$$

holds for any $\varepsilon > 0$ and $t \ge 0$, where

$$\kappa_4 = -\frac{\kappa_0}{(1+\kappa_0)^3} + \frac{TV(u_0)}{(1+\kappa_0)^4} + \frac{\kappa_0^2\|u_0\|_{L^1(\mathbb{R})}}{(1+\kappa_0)^4}, \qquad \kappa_5 = \frac{\kappa_0^2}{2(1+\kappa_0)^7}\|u_0\|_{L^1(\mathbb{R})}.$$

Proof. Differentiating the equation in (1.12) with respect to t, we get

$$\begin{aligned}
&\partial_{tt}^2 u_\varepsilon + f''(u_\varepsilon)E[u_\varepsilon]\partial_x u_\varepsilon \partial_t u_\varepsilon + f'(u_\varepsilon)E[u_\varepsilon]\partial_{tx}^2 u_\varepsilon\\
&\quad + f'(u_\varepsilon)E[u_\varepsilon]\partial_x u_\varepsilon\int_x^\infty f'(u_\varepsilon(t,\xi))\partial_t u_\varepsilon(t,\xi)d\xi - 2f(u_\varepsilon)f'(u_\varepsilon)E[u_\varepsilon]\partial_t u_\varepsilon\\
&\quad - f^2(u_\varepsilon)E[u_\varepsilon]\int_x^\infty f'(u_\varepsilon(t,\xi))\partial_t u_\varepsilon(t,\xi)d\xi = \varepsilon\partial_{txx}^3 u_\varepsilon.
\end{aligned} \tag{2.14}$$

Thanks to (2.5), (2.6), (2.8), (2.9), (2.11), and the definition of f, we have that

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}}|\partial_t u_\varepsilon|dx =& \int_{\mathbb{R}}\partial_{tt}^2 u_\varepsilon \mathrm{sign}\left(\partial_t u_\varepsilon\right) dx\\
=&\varepsilon\int_{\mathbb{R}}\partial_{txx}^3 u_\varepsilon \mathrm{sign}\left(\partial_t u_\varepsilon\right) dx - \int_{\mathbb{R}} f''(u_\varepsilon)E[u_\varepsilon]\partial_x u_\varepsilon|\partial_t u_\varepsilon|dx
\end{aligned}$$

$$
\begin{aligned}
&- \int_{\mathbb{R}} f'(u_\varepsilon) E[u_\varepsilon] \partial_{tx}^2 u_\varepsilon \mathrm{sign}\,(\partial_t u_\varepsilon)\, dx \\
&- \int_{\mathbb{R}} f'(u_\varepsilon) E[u_\varepsilon] \partial_x u_\varepsilon \mathrm{sign}\,(\partial_t u_\varepsilon) \int_x^\infty f'(u_\varepsilon(t,\xi)) \partial_t u_\varepsilon(t,\xi) d\xi dx \\
&+ 2 \int_{\mathbb{R}} f(u_\varepsilon) f'(u_\varepsilon) E[u_\varepsilon] |\partial_t u_\varepsilon| dx \\
&+ \int_{\mathbb{R}} f^2(u_\varepsilon) E[u_\varepsilon] \mathrm{sign}\,(\partial_t u_\varepsilon) \int_x^\infty f'(u_\varepsilon(t,\xi)) \partial_t u_\varepsilon(t,\xi) d\xi dx \\
= &\underbrace{-\varepsilon \int_{\mathbb{R}} (\partial_{tx}^2 u_\varepsilon)^2 d\delta_{\{\partial_t u_\varepsilon = 0\}}}_{\leq 0} - \int_{\mathbb{R}} \partial_x (f'(u_\varepsilon) |\partial_t u_\varepsilon|) E[u_\varepsilon] dx \\
&- \int_{\mathbb{R}} f'(u_\varepsilon) E[u_\varepsilon] \partial_x u_\varepsilon \mathrm{sign}\,(\partial_t u_\varepsilon) \int_x^\infty f'(u_\varepsilon(t,\xi)) \partial_t u_\varepsilon(t,\xi) d\xi dx \\
&+ 2 \underbrace{\int_{\mathbb{R}} f(u_\varepsilon) f'(u_\varepsilon) E[u_\varepsilon] |\partial_t u_\varepsilon| dx}_{\leq 0} \\
&+ \int_{\mathbb{R}} f^2(u_\varepsilon) E[u_\varepsilon] \mathrm{sign}\,(\partial_t u_\varepsilon) \int_x^\infty f'(u_\varepsilon(t,\xi)) \partial_t u_\varepsilon(t,\xi) d\xi dx \\
\leq &- \int_{\mathbb{R}} f'(u_\varepsilon) f(u_\varepsilon) |\partial_t u_\varepsilon| E[u_\varepsilon] dx \\
&+ \left(\int_{\mathbb{R}} f'(u_\varepsilon) E[u_\varepsilon] |\partial_x u_\varepsilon| dx \right) \left(\int_{\mathbb{R}} f'(u_\varepsilon) |\partial_t u_\varepsilon| dx \right) \\
&+ \left(\int_{\mathbb{R}} f^2(u_\varepsilon) E[u_\varepsilon] dx \right) \left(\int_{\mathbb{R}} f'(u_\varepsilon) |\partial_t u_\varepsilon| dx \right) \\
\leq &- \frac{\kappa_0}{(1+\kappa_0)^3} \int_{\mathbb{R}} |\partial_t u_\varepsilon| dx + \frac{1}{(1+\kappa_0)^4} \left(\int_{\mathbb{R}} |\partial_x u_\varepsilon| dx \right) \left(\int_{\mathbb{R}} |\partial_t u_\varepsilon| dx \right) \\
&+ \frac{\kappa_0^2}{(1+\kappa_0)^4} \left(\int_{\mathbb{R}} |u_\varepsilon| dx \right) \left(\int_{\mathbb{R}} |\partial_t u_\varepsilon| dx \right) \\
\leq &- \frac{\kappa_0}{(1+\kappa_0)^3} \int_{\mathbb{R}} |\partial_t u_\varepsilon| dx \\
&+ \frac{1}{(1+\kappa_0)^4} \left(TV(u_0) + \frac{\kappa_0^2}{(1+\kappa_0)^3} \|u_0\|_{L^1(\mathbb{R})} t \right) \int_{\mathbb{R}} |\partial_t u_\varepsilon| dx \\
&+ \frac{\kappa_0^2 \|u_0\|_{L^1(\mathbb{R})}}{(1+\kappa_0)^4} \int_{\mathbb{R}} |\partial_t u_\varepsilon| dx \\
= &(\kappa_4 + 2\kappa_5 t) \int_{\mathbb{R}} |\partial_t u_\varepsilon| dx,
\end{aligned}
$$

where $\delta_{\{\partial_t u_\varepsilon = 0\}}$ is the Dirac delta concentrated on the set $\{\partial_t u_\varepsilon = 0\}$. The Gronwall Lemma, (1.12), (2.1), and (2.9) give

$$
\begin{aligned}
\|\partial_t u_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})} &\leq \|\partial_t u_\varepsilon(0,\cdot)\|_{L^1(\mathbb{R})} e^{\kappa_4 t + \kappa_5 t^2} \\
&= \left\| \varepsilon \partial_{xx}^2 u_{0,\varepsilon} - f'(u_{0,\varepsilon}) E[u_{0,\varepsilon}] \partial_x u_{0,\varepsilon} + f^2(u_{0,\varepsilon}) E[u_{0,\varepsilon}] \right\|_{L^1(\mathbb{R})} e^{\kappa_4 t + \kappa_5 t^2} \\
&\leq \left(\varepsilon \left\| \partial_{xx}^2 u_{0,\varepsilon} \right\|_{L^1(\mathbb{R})} + \frac{\|\partial_x u_{0,\varepsilon}\|_{L^1(\mathbb{R})}}{(1+\kappa_0)^2} + \frac{|\kappa_0|}{(1+\kappa_0)^2} \|u_{0,\varepsilon}\|_{L^1(\mathbb{R})} \right) e^{\kappa_4 t + \kappa_5 t^2}
\end{aligned}
$$

$$\leq \left(C + \frac{TV(u_0)}{(1+\kappa_0)^2} + \frac{|\kappa_0|}{(1+\kappa_0)^2} \|u_0\|_{L^1(\mathbb{R})} \right) e^{\kappa_4 t + \kappa_5 t^2}.$$

Therefore, (2.13) is proved. □

Now, we are ready for the proof of Theorem 2.1.

Proof of Theorem 2.1. Let $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ be such that $\varepsilon_k \to 0$ and let T be any positive time. Since the sequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ is bounded in $L^\infty((0,\infty)\times\mathbb{R}) \cap BV((0,T)\times\mathbb{R})$ (see Lemmas 2.1, 2.4, and 2.5), there exist a function $u \in L^\infty((0,\infty)\times\mathbb{R}) \cap BV((0,T)\times\mathbb{R})$ and a subsequence $\{u_{\varepsilon_{k_h}}\}_{h\in\mathbb{N}}$ such that

$$u_{\varepsilon_{k_h}} \longrightarrow u \qquad \text{in } L^p_{loc}((0,\infty)\times\mathbb{R}),\ 1 \leq p < \infty, \text{ and a.e. in } (0,\infty)\times\mathbb{R}.$$

We claim that u is the unique entropy solution to (1.1) and (1.2). Let $\eta \in C^2(\mathbb{R})$ be a convex entropy with flux q defined by $q' = \eta' f'$. Multiplying (1.12) by $\eta'(u_{\varepsilon_{k_h}})$ we get

$$\begin{aligned}
&\partial_t \eta(u_{\varepsilon_{k_h}}) + \partial_x(q(u_{\varepsilon_{k_h}})E[u_{\varepsilon_{k_h}}]) - (f(u_{\varepsilon_{k_h}})\eta'(u_{\varepsilon_{k_h}}) - q(u_{\varepsilon_{k_h}}))f(u_{\varepsilon_{k_h}})E[u_{\varepsilon_{k_h}}] \\
&\quad = \varepsilon_{k_h}\partial^2_{xx} u_{\varepsilon_{k_h}} \eta'(u_{\varepsilon_{k_h}}) = \varepsilon_{k_h}\partial^2_{xx}\eta(u_{\varepsilon_{k_h}}) \underbrace{-\varepsilon_{k_h}\eta''(u_{\varepsilon_{k_h}})(\partial_x u_{\varepsilon_{k_h}})^2}_{\leq 0} \leq \varepsilon_{k_h}\partial^2_{xx}\eta(u_{\varepsilon_{k_h}}).
\end{aligned}$$

For any nonnegative test function $\varphi \in C^\infty(\mathbb{R}^2)$ with compact support we have that

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}} \Big(\eta(u_{\varepsilon_{k_h}})\partial_t\varphi + q(u_{\varepsilon_{k_h}})E[u_{\varepsilon_{k_h}}]\partial_x\varphi + (f(u_{\varepsilon_{k_h}})\eta'(u_{\varepsilon_{k_h}}) - q(u_{\varepsilon_{k_h}}))f(u_{\varepsilon_{k_h}})E[u_{\varepsilon_{k_h}}]\varphi \Big) dxdt \\
&\quad + \int_{\mathbb{R}} \eta(u_{0,\varepsilon_{k_h}}(x))\varphi(0,x)dx \geq -\varepsilon_{k_h}\int_0^\infty\int_{\mathbb{R}} \eta(u_{\varepsilon_{k_h}})\partial^2_{xx}\varphi dxdt.
\end{aligned}$$

As $h \to \infty$, the Dominated Convergence Theorem gives

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{R}} \Big(\eta(u)\partial_t\varphi + q(u)E[u]\partial_x\varphi + (f(u)\eta'(u) - q(u))f(u)E[u]\varphi \Big) dxdt& \\
+ \int_{\mathbb{R}} \eta(u_0(x))\varphi(0,x)dx &\geq 0,
\end{aligned}$$

proving that u is the unique entropy solution of (1.1) and (1.2).

Thanks to Urysohn Property, (2.2) is proved.

Moreover, (2.3) follows from (2.5) and (2.10). Finally, (2.4) follows from (2.3). □

3. The case $\sigma \geq 3$

In this section we assume that (1.3) holds with

$$\sigma = 3, \tag{3.1}$$

(and *a fortiori* if $\sigma > 3$); therefore now u_0 may attain the value -1 at some point. This case has not been considered in [1, 2].

On the initial condition $u_{0,\varepsilon}$ of (1.12) we assume

$$\begin{aligned}
&u_{0,\varepsilon} \in C^\infty(\mathbb{R}), \qquad \varepsilon > 0, \\
&u_{0,\varepsilon} \to u_0, \quad \text{a.e. in } \mathbb{R} \text{ and in } L^p(\mathbb{R}),\ 1 \leq p < \infty \text{ as } \varepsilon \to 0, \\
&f(u_{0,\varepsilon}) \to f(u_0), \quad \text{a.e. in } \mathbb{R} \text{ and in } L^p(\mathbb{R}), 1 \leq p \leq 3 \text{ as } \varepsilon \to 0, \\
&\|u_{0,\varepsilon}\|_{L^1(\mathbb{R})} \leq \|u_0\|_{L^1(\mathbb{R})}, \quad -1 < -\frac{1}{1+\varepsilon} \leq u_{0,\varepsilon} \leq 0, \qquad \varepsilon > 0, \\
&\|f(u_{0,\varepsilon})\|_{L^p(\mathbb{R})} \leq C, \qquad \varepsilon > 0,\ 1 \leq p \leq 3,
\end{aligned} \tag{3.2}$$

for some constant $C > 0$ independent on ε.

The main result of this section is the following.

Theorem 3.1. *Assume* (1.3)*,* (3.1)*, and* (3.2)*. There exist a sequence* $\{\varepsilon_h\}_{h\in\mathbb{N}} \subset (0,\infty)$*,* $\varepsilon_h \to 0$*, and a function* $u : [0,\infty)\times\mathbb{R} \to \mathbb{R}$ *such that*

$$
\begin{aligned}
&u \text{ is an entropy solution of (1.1) and (1.2)},\\
&-1 \le u(t,x) \le 0, \qquad a.e. \text{ in } (0,\infty)\times\mathbb{R},\\
&f(u) \in L^\infty_{loc}(0,\infty;L^p(\mathbb{R})), \qquad 1 \le p \le 3,\\
&\|u(t,\cdot)\|_{L^1(\mathbb{R})} \le \|u_0\|_{L^1(\mathbb{R})}, \qquad a.e.\ t \ge 0,\\
&u_{\varepsilon_h} \to u, \quad a.e. \text{ in } (0,\infty)\times\mathbb{R} \text{ and in } L^p_{loc}((0,\infty)\times\mathbb{R}),\ 1 \le p < \infty \text{ as } h \to \infty,\\
&f(u_{\varepsilon_h}) \to f(u), \quad a.e. \text{ in } (0,\infty)\times\mathbb{R} \text{ and in } L^p_{loc}((0,\infty)\times\mathbb{R}),\ 1 \le p < 3 \text{ as } h \to \infty,\\
&E[u_{\varepsilon_h}] \to E[u], \quad a.e. \text{ in } (0,\infty)\times\mathbb{R} \text{ and in } L^p_{loc}((0,\infty)\times\mathbb{R}),\ 1 \le p < \infty \text{ as } h \to \infty.
\end{aligned}
\tag{3.3}
$$

Finally, if (1.6) *holds, and*

$$
\|\partial_x u_{0,\varepsilon}\|_{L^1(\mathbb{R})} \le TV(u_0), \qquad \varepsilon > 0,
\tag{3.4}
$$

we have also

$$
\begin{aligned}
&u \in L^\infty_{loc}(0,\infty;BV(\mathbb{R})),\\
&TV(u(t,\cdot)) \le TV(u_0) + Ct, \qquad a.e.\ t > 0.
\end{aligned}
\tag{3.5}
$$

In order to prove Theorem 3.1 we need some preliminary lemmas, for all of which we assume the hypotheses of Theorem 3.1.

Lemma 3.1 (L^∞ AND L^1 ESTIMATE)**.** *The following inequalities*

$$
\begin{aligned}
&-\frac{1}{1+\varepsilon} \le u_\varepsilon(t,x) \le 0,\\
&f(u_\varepsilon(t,x)) \le 0, \qquad f'(u_\varepsilon(t,x)) \ge 1,\\
&\|u_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})} \le \|u_0\|_{L^1(\mathbb{R})},\\
&0 \le E[u_\varepsilon](t,x) \le 1,
\end{aligned}
\tag{3.6}
$$

hold for any $\varepsilon > 0$*,* $t \ge 0$*,* $x \in \mathbb{R}$*.*

Proof. Quite similar to the proofs of Lemmas 2.1 and 2.2. □

Lemma 3.2. *Let* $\eta \in C^2((-1,0])$ *be a convex nonnegative entropy with entropy flux*

$$
q(\xi) = \int_0^\xi f'(s)\eta'(s)ds, \qquad -1 < \xi \le 0.
\tag{3.7}
$$

Then

$$
f(u_\varepsilon)\eta'(u_\varepsilon) - q(u_\varepsilon) \ge 0.
\tag{3.8}
$$

Moreover, if

$$
\eta(u_{0,\varepsilon}) \in L^1(\mathbb{R}),
\tag{3.9}
$$

we have

$$
\begin{aligned}
\|\eta(u_\varepsilon(t,\cdot))\|_{L^1(\mathbb{R})} &+ \int_0^t\int_{\mathbb{R}} (f(u_\varepsilon)\eta'(u_\varepsilon) - q(u_\varepsilon))|f(u_\varepsilon)|E[u_\varepsilon]dsdx\\
&+ \varepsilon\int_0^t\int_{\mathbb{R}} \eta''(u_\varepsilon)(\partial_x u_\varepsilon)^2 dxds = \|\eta(u_{0,\varepsilon})\|_{L^1(\mathbb{R})}.
\end{aligned}
\tag{3.10}
$$

Proof. Observe that, for $\xi \in (-1,0]$,

$$
\partial_\xi(f(\xi)\eta'(\xi) - q(\xi)) = f'(\xi)\eta'(\xi) + f(\xi)\eta''(\xi) - f'(\xi)\eta'(\xi) = f(\xi)\eta''(\xi) \le 0.
$$

Therefore, using the first inequality of (3.6), we have

$$
f(u_\varepsilon)\eta'(u_\varepsilon) - q(u_\varepsilon) \ge f(0)\eta'(0) - q(0) = 0,
$$

that gives (3.8).

Multiplying the equation in (1.12) by $\eta'(u_\varepsilon)$ we get

$$\partial_t\eta(u_\varepsilon)+\partial_x(q(u_\varepsilon)E[u_\varepsilon])-(f(u_\varepsilon)\eta'(u_\varepsilon)-q(u_\varepsilon))f(u_\varepsilon)E[u_\varepsilon]=\varepsilon\partial^2_{xx}\eta(u_\varepsilon)-\varepsilon\eta''(u_\varepsilon)(\partial_x u_\varepsilon)^2. \tag{3.11}$$

From (3.11) we obtain

$$\partial_t\eta(u_\varepsilon)+\partial_x(q(u_\varepsilon)E[u_\varepsilon])=\underbrace{(f(u_\varepsilon)\eta'(u_\varepsilon)-q(u_\varepsilon))}_{\geq 0}\underbrace{f(u_\varepsilon)}_{\leq 0}\underbrace{E[u_\varepsilon]}_{\geq 0}+\varepsilon\partial^2_{xx}\eta(u_\varepsilon)\underbrace{-\varepsilon\eta''(u_\varepsilon)(\partial_x u_\varepsilon)^2}_{\leq 0}.$$

Integrating over $(0,t)\times\mathbb{R}$

$$\begin{aligned}\int_{\mathbb{R}}\eta(u_\varepsilon(t,x))dx+\int_0^t\int_{\mathbb{R}}(f(u_\varepsilon)\eta'(u_\varepsilon)-q(u_\varepsilon))|f(u_\varepsilon)|E[u_\varepsilon]dsdx+\varepsilon\int_0^t\int_{\mathbb{R}}\eta''(u_\varepsilon)(\partial_x u_\varepsilon)^2dsdx\\ =\int_{\mathbb{R}}\eta(u_{0,\varepsilon})dx.\end{aligned}$$

Now (3.10) follows from (3.9). □

Lemma 3.3. *The following inequalities*

$$\|f(u_\varepsilon(t,\cdot))\|_{L^1(\mathbb{R})}+\int_0^t\int_{\mathbb{R}}|(f(u_\varepsilon)f'(u_\varepsilon)+Q(u_\varepsilon))f(u_\varepsilon)|E[u_\varepsilon]dxds +\varepsilon\int_0^t\int_{\mathbb{R}}|f''(u_\varepsilon)|(\partial_x u_\varepsilon)^2dxds\leq C, \tag{3.12}$$

$$\|Q(u_\varepsilon(t,\cdot))\|_{L^1(\mathbb{R})}\leq C, \tag{3.13}$$

$$\|f(u_\varepsilon(t,\cdot))\|_{L^3(\mathbb{R})}\leq C, \tag{3.14}$$

$$\left\|f'(u_\varepsilon(t,\cdot))-1\right\|_{L^1(\mathbb{R})}\leq C, \tag{3.15}$$

$$\varepsilon\int_0^t\int_{\mathbb{R}}|f'(u_\varepsilon)|(\partial_x u_\varepsilon)^2dsdx\leq C, \tag{3.16}$$

hold for any $\varepsilon>0$, $t\geq 0$ *and some constant* $C>0$ *independent on* t *and* ε, *where*

$$Q(\xi)=\frac{1}{3}\frac{1}{(1+\xi)^3}-\frac{1}{3},\qquad -1<\xi\leq 0. \tag{3.17}$$

Proof. Estimate (3.12) follows from Lemma 3.2 by choosing

$$\eta(\xi)=-f(\xi)=\frac{-\xi}{1+\xi},\qquad q(\xi)=Q(\xi),\qquad -1<\xi\leq 0.$$

Indeed, for $\xi\in(-1,0]$,

$$\begin{aligned}\eta(\xi)&=-f(\xi)\geq 0,\\ \eta'(\xi)&=-f'(\xi)=-\frac{1}{(1+\xi)^2},\\ \eta''(\xi)&=-f''(\xi)=\frac{2}{(1+\xi)^3}\geq 0.\end{aligned}$$

Moreover, thanks to (3.2),

$$\int_{\mathbb{R}}|\eta(u_{0,\varepsilon})|\,dx=\int_{\mathbb{R}}|f(u_{0,\varepsilon})|dx=\|f(u_{0,\varepsilon})\|_{L^1(\mathbb{R})}\leq C,$$

that guarantees (3.9).

Estimate (3.13) follows from Lemma 3.2 by choosing

$$\eta(\xi)=Q(\xi),\qquad q(\xi)=\int_0^\xi f'(s)Q'(s)ds,\qquad -1<\xi\leq 0.$$

Indeed, for $\xi \in (-1,0]$,

$$\begin{aligned}\eta(\xi) =&Q(\xi) \geq 0,\\ \eta'(\xi) =&Q'(\xi) = -\frac{1}{(1+\xi)^4},\\ \eta''(\xi) =&Q''(\xi) = \frac{4}{(1+\xi)^5} \geq 0.\end{aligned}$$

Moreover, thanks to (3.2), for every given $-1<\delta<0$

$$\begin{aligned}\int_{\mathbb{R}} |\eta(u_{0,\varepsilon})|\,dx \leq&\frac{1}{3}\int_{\mathbb{R}}\left(\frac{3|u_{0,\varepsilon}|}{(1+u_{0,\varepsilon})^3}+\frac{3(u_{0,\varepsilon})^2}{(1+u_{0,\varepsilon})^3}+\frac{|u_{0,\varepsilon}|^3}{(1+u_{0,\varepsilon})^3}\right)dx\\ \leq&\frac{1}{3}\int_{\{u_{0,\varepsilon}<\delta\}}\left(\frac{3|u_{0,\varepsilon}|}{(1+u_{0,\varepsilon})^3}+\frac{3(u_{0,\varepsilon})^2}{(1+u_{0,\varepsilon})^3}+\frac{|u_{0,\varepsilon}|^3}{(1+u_{0,\varepsilon})^3}\right)dx\\ &+\frac{1}{3}\int_{\{u_{0,\varepsilon}\geq\delta\}}\left(\frac{3|u_{0,\varepsilon}|}{(1+u_{0,\varepsilon})^3}+\frac{3(u_{0,\varepsilon})^2}{(1+u_{0,\varepsilon})^3}+\frac{|u_{0,\varepsilon}|^3}{(1+u_{0,\varepsilon})^3}\right)dx\\ \leq&\frac{1}{3}\int_{\mathbb{R}}\left(\frac{3}{\delta^2}\frac{|u_{0,\varepsilon}|^3}{(1+u_{0,\varepsilon})^3}+\frac{3|u_{0,\varepsilon}|}{(1+\delta)^3}+\frac{3}{|\delta|}\frac{|u_{0,\varepsilon}|^3}{(1+u_{0,\varepsilon})^3}+\frac{3|u_{0,\varepsilon}|^2}{(1+\delta)^3}+\frac{|u_{0,\varepsilon}|^3}{(1+u_{0,\varepsilon})^3}\right)dx\\ \leq&\frac{1}{3}\left(\left(\frac{3}{\delta^2}+\frac{3}{|\delta|}+1\right)\|f(u_{0,\varepsilon})\|^3_{L^3(\mathbb{R})}+\frac{3\,\|u_{0,\varepsilon}\|_{L^1(\mathbb{R})}}{(1+\delta)^3}+\frac{\|u_{0,\varepsilon}\|^2_{L^2(\mathbb{R})}}{(1+\delta)^3}\right)\\ \leq&\frac{1}{3}\left(\left(\frac{3}{\delta^2}+\frac{3}{|\delta|}+1\right)\|f(u_{0,\varepsilon})\|^3_{L^3(\mathbb{R})}+\frac{3\,\|u_{0,\varepsilon}\|_{L^1(\mathbb{R})}}{(1+\delta)^3}+\frac{3\,\|u_{0,\varepsilon}\|^2_{L^1(\mathbb{R})}}{(1+\delta)^3}\right)\\ \leq&\frac{1}{3}\left(\left(\frac{3}{\delta^2}+\frac{3}{|\delta|}+1\right)C^3+\frac{3\,\|u_0\|_{L^1(\mathbb{R})}}{(1+\delta)^3}+\frac{3\,\|u_0\|^2_{L^1(\mathbb{R})}}{(1+\delta)^3}\right),\end{aligned}$$

that guarantees (3.9).

Estimate (3.14) follows follows from Lemma 3.2 by choosing

$$\eta(\xi) = -f^3(\xi), \qquad q(\xi) = -3\int_0^{\xi}(f'(s))^2(f(s))^2ds, \qquad -1<\xi\leq 0.$$

Indeed, for $\xi \in (-1,0]$,

$$\begin{aligned}\eta(\xi) =&-f(\xi)\geq 0,\\ \eta'(\xi) =&-3f'(\xi)f^2(\xi),\\ \eta''(\xi) =&-3f''(\xi)f^2(\xi)-6(f'(\xi))^2f(\xi)\geq 0.\end{aligned}$$

Moreover, thanks to (3.2), for every given $-1<\delta<0$

$$\int_{\mathbb{R}}|\eta(u_{0,\varepsilon})|\,dx = \int_{\mathbb{R}}|f(u_{0,\varepsilon})|^3dx = \|f(u_{0,\varepsilon})\|^3_{L^3(\mathbb{R})}\leq C,$$

that guarantees (3.9).

On the other hand (3.15) follows from Lemma 3.2 by choosing

$$\eta(\xi) = f'(\xi)-1, \qquad q(\xi) = \int_0^{\xi} f'(s)f''(s)ds, \qquad -1<\xi\leq 0.$$

Indeed, for $\xi \in (-1,0]$,

$$\begin{aligned}\eta(\xi) =&f'(\xi)-1 = \frac{1}{(1+\xi)^2}-1\geq 0,\\ \eta'(\xi) =&f''(\xi) = -\frac{2}{(1+\xi)^3},\end{aligned}$$

$$\eta''(\xi) = f'''(\xi) = \frac{6}{(1+\xi)^4} \ge 0.$$

Moreover, thanks to (3.2), for every given $-1 < \delta < 0$

$$\begin{aligned}\int_{\mathbb{R}} |\eta(u_{0,\varepsilon})| dx =& \int_{\mathbb{R}} \left(\frac{1}{(1+u_{0,\varepsilon})^2} - 1\right) dx = -\int_{\mathbb{R}}\int_0^{u_{0,\varepsilon}} \frac{2}{(1+y)^3} dy dx \\ \le& \int_{\mathbb{R}} \frac{2|u_{0,\varepsilon}|}{(1+u_{0,\varepsilon})^3} dx = \int_{\{u_{0,\varepsilon}<\delta\}} \frac{2|u_{0,\varepsilon}|}{(1+u_{0,\varepsilon})^3} dx + \int_{\{u_{0,\varepsilon}\ge\delta\}} \frac{2|u_{0,\varepsilon}|}{(1+u_{0,\varepsilon})^3} dx \\ \le& \frac{2}{\delta^2} \int_{\{u_{0,\varepsilon}<\delta\}} \frac{|u_{0,\varepsilon}|^3}{(1+u_{0,\varepsilon})^3} dx + \int_{\{u_{0,\varepsilon}\ge\delta\}} \frac{2|u_{0,\varepsilon}|}{(1+\delta)^3} dx \\ \le& \frac{2}{\delta^2} \|f(u_{0,\varepsilon})\|^3_{L^3(\mathbb{R})} + \frac{2\|u_{0,\varepsilon}\|_{L^1(\mathbb{R})}}{(1+\delta)^3} \le \frac{2}{\delta^2} C^3 + \frac{2\|u_0\|_{L^1(\mathbb{R})}}{(1+\delta)^3},\end{aligned}$$

that guarantees (3.9).

Finally, estimate (3.16) follows from Lemma 3.2 by choosing

$$\eta(\xi) = -\log(1+\xi), \qquad q(\xi) = \int_0^\xi f'(s)\eta'(s)ds, \qquad -1 < \xi \le 0.$$

Indeed, for $\xi \in (-1, 0]$,

$$\begin{aligned}\eta(\xi) =& -\log(1+\xi) \ge 0, \\ \eta'(\xi) =& -\frac{1}{1+\xi}, \\ \eta''(\xi) =& f'(\xi) = \frac{1}{(1+\xi)^2} \ge 0.\end{aligned}$$

Moreover, thanks to (3.2),

$$\int_{\mathbb{R}} |\eta(u_{0,\varepsilon})|\, dx = -\int_{\mathbb{R}} \log(1+u_{0,\varepsilon}) dx = \int_{\mathbb{R}}\int_{u_{0,\varepsilon}}^0 \frac{dy}{1+y} dx \le \int_{\mathbb{R}} \frac{|u_{0,\varepsilon}|}{1+u_{0,\varepsilon}} dx = \|f(u_{0,\varepsilon})\|_{L^1(\mathbb{R})} \le C,$$

that guarantees (3.9). □

Lemma 3.4 (BV estimates on $E[u_\varepsilon]$)**.** *The family $\{E[u_\varepsilon]\}_{\varepsilon>0}$ is bounded in $W^{1,1}_{loc}((0,\infty)\times\mathbb{R})$.*

Proof. Observe that

$$\partial_x E[u_\varepsilon] = -f(u_\varepsilon)E[u_\varepsilon],$$

therefore thanks to Lemma 3.1 and (3.12) we have

$$\{\partial_x E[u_\varepsilon]\}_{\varepsilon>0} \text{ is bounded in } L^1((0,T)\times\mathbb{R}) \text{ for any } T>0. \tag{3.18}$$

Moreover, using (1.12),

$$\begin{aligned}\partial_t E[u_\varepsilon] =& E[u_\varepsilon]\int_x^\infty f'(u_\varepsilon(t,\xi))\partial_t u_\varepsilon(t,\xi) d\xi \\ =& -E[u_\varepsilon]\int_x^\infty f'(u_\varepsilon)\partial_x(f(u_\varepsilon)E[u_\varepsilon])d\xi + \varepsilon E[u_\varepsilon]\int_x^\infty f'(u_\varepsilon)\partial^2_{xx}u_\varepsilon d\xi \\ =& E[u_\varepsilon]\int_x^\infty \partial_x(Q(u_\varepsilon)E[u_\varepsilon])d\xi + E[u_\varepsilon]\int_x^\infty (f(u_\varepsilon)f'(u_\varepsilon) + Q(u_\varepsilon))f(u_\varepsilon)E[u_\varepsilon]d\xi \\ &+ \varepsilon E[u_\varepsilon]\int_x^\infty \partial_x(f'(u_\varepsilon)\partial_x u_\varepsilon)d\xi - \varepsilon E[u_\varepsilon]\int_x^\infty f''(u_\varepsilon)(\partial_x u_\varepsilon)^2 d\xi \\ =& -Q(u_\varepsilon)E[u_\varepsilon]^2 + E[u_\varepsilon]\int_x^\infty (f(u_\varepsilon)f'(u_\varepsilon) + Q(u_\varepsilon))f(u_\varepsilon)E[u_\varepsilon]d\xi \\ &- \varepsilon E[u_\varepsilon]f'(u_\varepsilon)\partial_x u_\varepsilon - \varepsilon E[u_\varepsilon]\int_x^\infty f''(u_\varepsilon)(\partial_x u_\varepsilon)^2 d\xi,\end{aligned}$$

where Q is defined in (3.17). By Lemma 3.1

$$\begin{aligned}|\partial_t E[u_\varepsilon]| \le & |Q(u_\varepsilon)| + \int_{\mathbb{R}} |(f(u_\varepsilon)f'(u_\varepsilon) + Q(u_\varepsilon))f(u_\varepsilon)|E[u_\varepsilon]dx \\ & + \varepsilon f'(u_\varepsilon)|\partial_x u_\varepsilon| + \varepsilon \int_{\mathbb{R}} |f''(u_\varepsilon)|(\partial_x u_\varepsilon)^2 dx.\end{aligned}$$

Let T, $R > 0$ be given. We have that

$$\begin{aligned}\int_0^T \int_{-R}^R |\partial_t E[u_\varepsilon]| dxdt \le & \int_0^T \|Q(u_\varepsilon(t,\cdot))\|_{L^1(\mathbb{R})} dt \\ & + 2R \int_0^T \int_{\mathbb{R}} |(f(u_\varepsilon)f'(u_\varepsilon) + Q(u_\varepsilon))f(u_\varepsilon)|E[u_\varepsilon]dxdt \\ & + \frac{\varepsilon}{2} \int_0^T \left\| f'(u_\varepsilon(t,\cdot)) - 1 \right\|_{L^1(\mathbb{R})} dt + \varepsilon RT \\ & + \frac{\varepsilon}{2} \int_0^T \int_{\mathbb{R}} f'(u_\varepsilon)(\partial_x u_\varepsilon)^2 dxdt \\ & + 2R\varepsilon \int_0^T \int_{\mathbb{R}} |f''(u_\varepsilon)|(\partial_x u_\varepsilon)^2 dxdt,\end{aligned}$$

therefore from Lemma 3.3 we deduce

$$\{\partial_t E[u_\varepsilon]\}_{\varepsilon>0} \text{ is bounded in } L^1((0,T)\times(-R,R)) \text{ for any } T,\ R>0. \tag{3.19}$$

Now our claim follows from (3.18), (3.19), and the last condition in (3.6). □

Lemma 3.5. *There exist a function $\mathcal{E}$ and a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0,\infty)$, $\varepsilon_k \to 0$, such that*

$$\begin{aligned}& 0 \le \mathcal{E} \le 1, \qquad \mathcal{E} \in BV_{loc}((0,\infty)\times\mathbb{R}), \\ & E[u_{\varepsilon_k}] \to \mathcal{E}, \quad \textit{a.e. in } (0,\infty)\times\mathbb{R} \textit{ and in } L^p_{loc}((0,\infty)\times\mathbb{R}),\ 1\le p<\infty.\end{aligned} \tag{3.20}$$

Proof. Direct consequence of Lemma 3.4 and the Helly Theorem [3, Theorem 2.4]. □

Before stating our next lemma we recall two well-know results which will play a key role in what follows.

Proposition 3.1 (see [13, Theorem 5], [14])**.** *Let $\{u_\nu\}_{\nu>0}$ be a family of functions defined on $(0,\infty)\times\mathbb{R}$. If $\{u_\nu\}_{\nu\in\mathbb{N}}$ lies in a bounded set of $L^1_{loc}((0,\infty)\times\mathbb{R})$ and for any constant $c\in\mathbb{R}$ the family*

$$\{\partial_t |u_\nu - c| + \partial_x(\text{sign}\,(u_\nu - c)\,(f(u_\nu) - f(c))\mathcal{E})\}_{\nu>0}$$

lies in a compact set of $H^{-1}_{loc}((0,\infty)\times\mathbb{R})$, then there exist a sequence $\{\nu_n\}_{n\in\mathbb{N}} \subset (0,\infty)$, $\nu_n \to 0$, and a map $v \in L^\infty((0,\infty)\times\mathbb{R})$ such that

$$v_{\nu_n} \to v \quad \textit{a.e. and in } L^p_{loc}((0,\infty)\times\mathbb{R}),\ 1\le p<\infty.$$

Proposition 3.2 (see [12])**.** *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N\ge 2$. Suppose the sequence $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ of distributions is bounded in $W^{-1,\infty}(\Omega)$. Suppose also that*

$$\mathcal{L}_n = \mathcal{L}_{1,n} + \mathcal{L}_{2,n},$$

where $\{\mathcal{L}_{1,n}\}_{n\in\mathbb{N}}$ lies in a compact subset of $H^{-1}_{\mathrm{loc}}(\Omega)$ and $\{\mathcal{L}_{2,n}\}_{n\in\mathbb{N}}$ lies in a bounded subset of $\mathcal{M}_{loc}(\Omega)$. Then $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ lies in a compact subset of $H^{-1}_{loc}(\Omega)$.

Lemma 3.6. *There exist a function u and a subsequence $\{\varepsilon_h\}_{h\in\mathbb{N}} \subset (0,\infty)$, $\varepsilon_h \to 0$, such that*

$$\begin{aligned}& -1 \le u \le 0, \qquad \sup_{t>0} \|u(t,\cdot)\|_{L^1(\mathbb{R})} \le \|u_0\|_{L^1(\mathbb{R})}, \\ & u_{\varepsilon_h} \to u, \quad \textit{a.e. in } (0,\infty)\times\mathbb{R} \textit{ and in } L^p_{loc}((0,\infty)\times\mathbb{R}),\ 1\le p<\infty, \\ & f(u_{\varepsilon_h}) \to f(u), \quad \textit{a.e. in } (0,\infty)\times\mathbb{R} \textit{ and in } L^p_{loc}((0,\infty)\times\mathbb{R}),\ 1\le p<3, \\ & \mathcal{E} = E[u].\end{aligned} \tag{3.21}$$

Proof. Let $c \in \mathbb{R}$ be fixed. We claim that the family

$$\{\partial_t |u_{\varepsilon_k} - c| + \partial_x(\text{sign}(u_{\varepsilon_k} - c)(f(u_{\varepsilon_k}) - f(u))\mathcal{E})\}_{\varepsilon>0}$$

is compact in $H^{-1}_{\text{loc}}(\mathbb{R}^N \times (0,\infty))$. For the sake of notational simplicity we introduce the following notations:

$$\begin{aligned}
\eta_0(\xi) &= |\xi - c| - |c|,\\
q_0(\xi) &= \text{sign}(\xi - c)\big(f(\xi) - f(c)\big) + \text{sign}(-c) f(c).
\end{aligned}$$

Let us remark that

$$\begin{aligned}
&\eta_0(0) = q_0(0) = 0,\\
&\partial_t |u_{\varepsilon_k} - c| + \partial_x(\text{sign}(u_{\varepsilon_k} - c)\big(f(u_{\varepsilon_k}) - f(c)\big)\mathcal{E}) = \partial_t\eta_0(u_{\varepsilon_k}) + \partial_x(q_0(u_{\varepsilon_k})\mathcal{E}).
\end{aligned} \tag{3.22}$$

Let $\{(\eta_\varepsilon, q_\varepsilon)\}_{\varepsilon>0}$ be a family of maps such that

$$\begin{aligned}
&\eta_\varepsilon \in C^2([-1,0]), \quad q_\varepsilon \in C^2([-1,0]),\\
&q'_\varepsilon = f'\eta'_\varepsilon, \quad \eta''_\varepsilon \geq 0\\
&\|\eta_\varepsilon - \eta_0\|_{L^\infty(-1,0)} \leq \varepsilon^3, \quad \|\eta'_\varepsilon - \eta'_0\|_{L^1(-1,0)} \leq \varepsilon^3,\\
&\|\eta'_\varepsilon\|_{L^\infty(-1,0)} \leq 1, \quad \eta_\varepsilon(0) = q_\varepsilon(0) = 0,
\end{aligned} \tag{3.23}$$

for any $\varepsilon > 0$. By (1.12)

$$\begin{aligned}
&\partial_t\eta_0(u_{\varepsilon_k}) + \partial_x(q_0(u_{\varepsilon_k})\mathcal{E})\\
&\quad= \partial_t\eta_{\varepsilon_k}(u_{\varepsilon_k}) + \partial_x(q_{\varepsilon_k}(u_{\varepsilon_k})E[u_{\varepsilon_k}]) + \partial_t(\eta_0(u_{\varepsilon_k}) - \eta_{\varepsilon_k}(u_{\varepsilon_k}))\\
&\qquad + \partial_x((q_0(u_{\varepsilon_k}) - q_{\varepsilon_k}(u_{\varepsilon_k}))\mathcal{E}) + \partial_x(q_{\varepsilon_k}(u_{\varepsilon_k})(\mathcal{E} - E[u_{\varepsilon_k}]))\\
&\quad= \underbrace{(f(u_{\varepsilon_k})\eta'_{\varepsilon_k}(u_{\varepsilon_k}) - q_{\varepsilon_k}(u_{\varepsilon_k}))f(u_{\varepsilon_k})E[u_{\varepsilon_k}]}_{I_1} + \underbrace{\varepsilon_k\partial^2_{xx}\eta_{\varepsilon_k}(u_{\varepsilon_k})}_{I_2}\\
&\qquad - \underbrace{\varepsilon_k\eta''_{\varepsilon_k}(u_{\varepsilon_k})(\partial_x u_{\varepsilon_k})^2}_{I_3} + \underbrace{\partial_t(\eta_0(u_{\varepsilon_k}) - \eta_{\varepsilon_k}(u_{\varepsilon_k}))}_{I_4}\\
&\qquad + \underbrace{\partial_x((q_0(u_{\varepsilon_k}) - q_{\varepsilon_k}(u_{\varepsilon_k}))\mathcal{E})}_{I_5} + \underbrace{\partial_x(q_{\varepsilon_k}(u_{\varepsilon_k})(\mathcal{E} - E[u_{\varepsilon_k}]))}_{I_6}.
\end{aligned}$$

Thanks to Lemma 3.2 and (3.23) we have

$$\begin{aligned}
\|I_1\|_{L^1((0,\infty)\times\mathbb{R})} &\leq \|\eta_{\varepsilon_k}(u_{0,\varepsilon_k})\|_{L^1(\mathbb{R})} \leq \|u_{0,\varepsilon_k}\|_{L^1(\mathbb{R})},\\
{\varepsilon_k}^2\int_0^\infty\int_{\mathbb{R}} |\eta'_{\varepsilon_k}(u_{\varepsilon_k})|^2(\partial_x u_{\varepsilon_k})^2 dxdt &\leq {\varepsilon_k}^2\int_0^\infty\int_{\mathbb{R}}(\partial_x u_{\varepsilon_k})^2 dxdt \leq \frac{\varepsilon_k}{2}\left\|u^2_{0,\varepsilon_k}\right\|_{L^1(\mathbb{R})} \to 0,\\
\varepsilon_k\int_0^\infty\int_{\mathbb{R}}\eta''_{\varepsilon_k}(u_{\varepsilon_k})(\partial_x u_{\varepsilon_k})^2 dxdt &\leq \|\eta_{\varepsilon_k}(u_{0,\varepsilon_k})\|_{L^1(\mathbb{R})} \leq \|u_{0,\varepsilon_k}\|_{L^1(\mathbb{R})},
\end{aligned}$$

therefore

$$\begin{aligned}
&I_1 \text{ is bounded in } L^1((0,\infty)\times\mathbb{R}),\\
&I_2 \to 0 \text{ in } H^{-1}((0,\infty)\times\mathbb{R}),\\
&I_3 \text{ is bounded in } L^1((0,\infty)\times\mathbb{R}).
\end{aligned}$$

Thanks to Lemma 3.1, (3.20) and (3.23) we have

$$\begin{aligned}
\|\eta_0(u_{\varepsilon_k}) - \eta_{\varepsilon_k}(u_{\varepsilon_k})\|_{L^\infty((0,\infty)\times\mathbb{R})} &\leq \|\eta_0 - \eta_{\varepsilon_k}\|_{L^\infty(-1,0)} \leq {\varepsilon_k}^3 \to 0,\\
\|(q_0(u_{\varepsilon_k}) - q_{\varepsilon_k}(u_{\varepsilon_k}))\mathcal{E}\|_{L^\infty((0,\infty)\times\mathbb{R})} &\leq \|q_0 - q_{\varepsilon_k}\|_{L^\infty(-\frac{1}{1+\varepsilon_k},0)}
\end{aligned}$$

$$\leq \|f'\|_{L^\infty(-\frac{1}{1+\varepsilon_k},0)} \|\eta'_{\varepsilon_k} - \eta'_0\|_{L^1(-\frac{1}{1+\varepsilon_k},0)}$$
$$\leq \frac{(1+\varepsilon_k)^2}{\varepsilon_k{}^2}\varepsilon_k{}^3 \to 0,$$

hence

$$I_4 \to 0 \text{ in } H^{-1}_{loc}((0,\infty)\times\mathbb{R}),$$
$$I_5 \to 0 \text{ in } H^{-1}_{loc}((0,\infty)\times\mathbb{R}).$$

Finally, (3.23) gives for $\xi \in (-1,0]$

$$|q_{\varepsilon_k}(\xi)| \leq \left|\int_0^\xi f'(s)|\eta'_{\varepsilon_k}(s)|ds\right| \leq \left|\int_0^\xi f'(s)ds\right| \leq |f(\xi)|.$$

By (3.14), Lemmas 3.2 and 3.5, for any $K \subset\subset (0,T)\times\mathbb{R}$, we get

$$\begin{aligned}\|q_{\varepsilon_k}(u_{\varepsilon_k})(\mathcal{E}-E[u_{\varepsilon_k}])\|_{L^2(K)} &\leq \|q_{\varepsilon_k}(u_{\varepsilon_k})\|_{L^3(K)}\|\mathcal{E}-E[u_{\varepsilon_k}]\|_{L^6(K)}\\ &\leq T^{1/3}\|f(u_{0,\varepsilon_k})\|_{L^3(\mathbb{R})}\|\mathcal{E}-E[u_{\varepsilon_k}]\|_{L^6(K)}\\ &\leq T^{1/3}C\|\mathcal{E}-E[u_{\varepsilon_k}]\|_{L^6(K)} \to 0,\end{aligned}$$

therefore

$$I_6 \to 0 \text{ in } H^{-1}((0,\infty)\times\mathbb{R}).$$

Now our claim follows from Propositions 3.1 and 3.2. □

Proof of Theorem 3.1. We have to prove that the function u of Lemma 3.6 is an entropy solution of (1.1) and (1.2). Let $\eta \in C^2(\mathbb{R})$ be a convex entropy with flux q defined by $q' = \eta' f'$. Thanks to Lemmas 3.5 and 3.6 we have

$$\begin{aligned}\eta(u_{\varepsilon_h}) &\to \eta(u), && \text{a.e. and in } L^1_{loc}((0,\infty)\times\mathbb{R}),\\ q(u_{\varepsilon_h}) &\to q(u), && \text{a.e. and in } L^1_{loc}((0,\infty)\times\mathbb{R}),\\ (f(u_{\varepsilon_h})\eta'(u_{\varepsilon_h}) - q(u_{\varepsilon_h}))f(u_{\varepsilon_h})E[u_{\varepsilon_h}] &\to (f(u)\eta'(u)-q(u))f(u)E[u], && \text{a.e. and in } L^1_{loc}((0,\infty)\times\mathbb{R}).\end{aligned}$$

Therefore, we can prove (3.3) arguing as in the proof of Theorem 2.1.

Let us turn to (3.5). Differentiating the equation in (1.12) with respect to x, we get (2.12). Thanks to (3.6) and (3.14), we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}}|\partial_x u_\varepsilon|dx =& \int_{\mathbb{R}} \partial^2_{tx}u_\varepsilon \mathrm{sign}\,(\partial_x u_\varepsilon)\,dx\\ =&\varepsilon\int_{\mathbb{R}}\partial^3_{xxx}u_\varepsilon \mathrm{sign}\,(\partial_x u_\varepsilon)\,dx - \int_{\mathbb{R}}\partial_x\left(f'(u_\varepsilon)E[u_\varepsilon]\partial_x u_\varepsilon\right)\mathrm{sign}\,(\partial_x u_\varepsilon)\,dx\\ &+2\int_{\mathbb{R}} f(u_\varepsilon)f'(u_\varepsilon)E[u_\varepsilon]|\partial_x u_\varepsilon|dx - \int_{\mathbb{R}} f^3(u_\varepsilon)E[u_\varepsilon]\mathrm{sign}\,(\partial_x u_\varepsilon)\,dx\\ =&\underbrace{-\varepsilon\int_{\mathbb{R}}(\partial^2_{xx}u_\varepsilon)^2 d\delta_{\{\partial_x u_\varepsilon=0\}}}_{\leq 0} + \underbrace{\int_{\mathbb{R}} f'(u_\varepsilon)E[u_\varepsilon]\partial_x u_\varepsilon \partial^2_{xx}u_\varepsilon d\delta_{\{\partial_x u_\varepsilon=0\}}}_{=0}\\ &+\underbrace{2\int_{\mathbb{R}} f(u_\varepsilon)f'(u_\varepsilon)E[u_\varepsilon]|\partial_x u_\varepsilon|dx}_{\leq 0} - \int_{\mathbb{R}} f^3(u_\varepsilon)E[u_\varepsilon]\mathrm{sign}\,(\partial_x u_\varepsilon)\,dx\\ \leq& \int_{\mathbb{R}}|f(u_\varepsilon)|^3 dx = \|f(u_\varepsilon(t,\cdot))\|^3_{L^3(\mathbb{R})} \leq C^3,\end{aligned}$$

where $\delta_{\{\partial_x u_\varepsilon=0\}}$ is the Dirac delta concentrated on the set $\{\partial_x u_\varepsilon = 0\}$. An integration over $(0,t)$ and (3.4) give

$$\|\partial_x u_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})} \leq \|\partial_x u_{0,\varepsilon}\|_{L^1(\mathbb{R})} + tC^3 \leq TV(u_0) + tC^3.$$

Therefore, (3.5) follows from (3.3). □

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