On Einstein hypersurfaces of a class of regular Sasakian manifolds[*]

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Abstract

We present a non existence result of complete, Einstein hypersurfaces tangent to the Reeb vector field of a regular Sasakian manifold which fibers onto a complex Stein manifold.

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Proposed running head: On Einstein hypersurfaces of Sasakian manifolds.

1 Introduction

In [4], I. Hasegawa established that a Sasakian space form with nonconstant sectional curvature admits no Einstein hypersurfaces. The aim of this note is to prove a new non existence result concerning Einstein hypersurfaces of a relevant class of *regular* Sasakian manifolds:

Theorem. *If* $(M, \varphi, \xi, \eta, g)$ *is a regular Sasakian manifold which fibers onto a complex Stein manifold, then* M *does not admit any complete Einstein hypersurface tangent to* ξ.

We recall that a contact manifold (M, η) is called regular provided the Reeb vector field ξ of the contact form η is, i.e. it determines a regular 1-dimensional foliation on M, so that the space $B = M/\xi$ of maximal integral

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curves of ξ is a manifold. When M carries a Sasakian metric g associated to η, yielding a Sasakian structure (φ, ξ, η, g) (we use the standard terminology and notation according to [2]), since $\mathcal{L}_\xi \varphi = 0$ and $\mathcal{L}_\xi g = 0$, g induces in a natural way a metric g' on M/ξ and φ also descends to an almost complex structure J. Denoting by $\pi : M \to B$ the canonical projection, it turns out by construction that π is a Riemannian submersion with $\ker(d\pi)_x = \mathbb{R}\xi_x$ for every $x \in N$, and

$$d\pi \circ \varphi = J \circ d\pi$$

and, morover (B, J, g') is a Kähler manifold (see for instance [9] and [8]). Hence our assumption on the Sasakian manifold is that B, as a complex manifold, can be realized (up to a biholomorphism) as a closed complex submanifold of some Euclidean space $\mathbb{C}^d$.

For instance, according to a result due to H. Wu ([3], Theorem 4.9), it is known that every simply connected, complete Kähler manifold with non-positive sectional curvature is a Stein manifold; in particular, the Hermitian symmetric spaces of non-compact type are Stein manifolds, hence Takahashi's Sasakian globally φ-symmetric spaces of non-compact type (see [10]) provide a wide class of examples of Sasakian manifolds to which our result applies.

The proof of our result makes use of the natural CR structure of CR codimension 2 which is induced over any smooth hypersurface $N \subset M$, under the assumption that N is everywhere tangent to the Reeb vector field ξ (see for instance [7] and section 2). We establish a basic formula relating the Ricci tensor of N and the trace of a distinguished scalar Levi form of this CR structure (see (3.4)), implying that, in the Einstein case, $\pi(N)$ is a weakly pseudoconcave real hypersurface of B. Hence a non compactness result by D. Hill and M. Nacinovich for weakly pseudoconcave CR submanifolds of Stein manifolds [5] is invoked to get the conclusion.

2 Preliminaries

Let's start by recalling the definitions of CR manifolds, Levi-Tanaka forms and scalar Levi forms. In the following, given a vector bundle E over a smooth differential manifold M, we will denote by $\Gamma(E)$ the $\mathcal{C}^\infty(M)$-module of global smooth sections of E.

Let M be a smooth real manifold of dimension n, and let $m, k \in \mathbb{N}$ such that $2m + k = n$. If HM is a real vector subbundle of rank $2m$ of the tangent bundle TM and $J : HM \to HM$ is a bundle isomorphism such that $J^2 = -Id$, the couple (HM, J) is called a *CR structure* on M if the following properties hold for all $X, Y \in \Gamma(HM)$:

(i) $[JX, JY] - [X, Y] \in \Gamma(HM)$;

(ii) $N_J(X, Y) := [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y] = 0$.

In this case (M, HM, J) is called a *CR manifold of type (m,k)* and m, k are the *CR dimension* and the *CR codimension* of the CR structure, respectively.

Remark 2.1. Let S be a real submanifold of a complex manifold (M, J) and for any $p \in S$ set

$$H_pS := T_pS \cap J(T_pS).$$

Because of the integrability of J, the couple $(HS, J_{|HS})$ canonically defines a CR structure on S if the dimension of H_pS is constant. In this case S is termed a *CR submanifold* of M.
In particular, this condition is always satisfied when S is a real hypersurface of M and hence S is a CR manifold of CR codimension 1.

Definition 2.2. Let (M, HM, J) be a CR manifold of type (m, k). Given a point $x \in M$, the *Levi-Tanaka form of M at x* is the bilinear map

$$L_x : H_xM \times H_xM \to T_xM/H_xM$$

defined by

$$L_x(X, Y) := p_x([\tilde{X}, \tilde{JY}]_x) \quad \forall X, Y \in H_xM, \tag{2.1}$$

where $\tilde{X}, \tilde{Y} \in \Gamma(HM)$ are two arbitrary extensions of X, Y and $p : TM \to TM/HM$ is the canonical projection on the quotient bundle TM/HM.

It is known that L_x is well defined, i.e. the value $p_x([\tilde{X}, \tilde{JY}]_x)$ only depends on the values of $\tilde{X}, \tilde{Y}$ at x, that is on X and Y.
Moreover, according to (i) above, L_x turns to be a vector valued symmetric Hermitian form on the holomorphic tangent space H_xM with respect to the complex structure $J := J_x$, that is

$$L_x(X, Y) = L_x(JX, JY), \quad L_x(X, Y) = L_x(Y, X) \tag{2.2}$$

for all $X, Y \in H_xM$.

Given a point x on the CR manifold (M, HM, J), we will denote by

$$H_x^0 M := \{\omega \in T_x^* M \mid \omega(X) = 0 \quad \forall X \in H_x M\}$$

the annihilator of $H_x M \subset T_x M$. Then we recall the following definition.

Definition 2.3. Let (M, HM, J) be a CR manifold, $x \in M$ and $\omega \in H_x^0 M$. The Hermitian form

$$\mathfrak{L}_\omega : H_x M \times H_x M \to \mathbb{R} \quad \text{s.t.} \quad \mathfrak{L}_\omega(X, Y) := \omega L_x(X, Y) \tag{2.3}$$

is called the *scalar Levi form determined by ω at x*.

The next lemma represents a sort of naturality property of the Levi-Tanaka form with respect a particular class of maps between CR manifolds which preserve the CR structures.

Definition 2.4. Let (M, HM, J) and (N, HN, J') be two CR manifolds. A smooth map $\pi : M \to N$ is called *CR map* if $d\pi(HM) \subset HN$ and $d\pi \circ J = J' \circ d\pi$.

Lemma 2.5. *Let (M, HM, J) and (N, HN, J') be two CR manifolds having the same CR dimension, let $\pi : M \to N$ be a CR map and assume that for every $x \in M$, $(d\pi)_x : H_x M \to H_{\pi(x)} N$ is an isomorphism. Then, given $x \in M$, the following diagram commutes:*

$$\begin{tikzcd}
H_x M \times H_x M \arrow[r, "L_x"] \arrow[d, "\pi_* \times \pi_*"'] & T_x M / H_x M \arrow[d, "\pi_*"] \\
H_y N \times H_y N \arrow[r, "L'_y"'] & T_y N / H_y N
\end{tikzcd}$$

where $y = \pi(x)$, $\pi_* = (d\pi)_x$ and L_x, L'_y are the Levi-Tanaka forms of M and N respectively.

Proof. Let us denote by $p_x : T_x M \to T_x M / H_x M$ and $q_y : T_y N \to T_y N / H_y N$ the canonical projections. As an immediate consequence of the definition of CR map, the differential π_* descends to the quotient and, with abuse of notation, we still denote the quotient map by $\pi_* : T_x M / H_x M \to T_y N / H_y N$. Now consider $X, Y \in H_x M$: according to (2.1), $L'_y(\pi_* X, \pi_* Y) = q_y [Z, J' W]_y$, where $Z, W \in \Gamma(HN)$ are two extensions of $\pi_* X$ and $\pi_* Y$. Since for every $a \in M$, $(d\pi)_a : H_a M \to H_{\pi(a)} N$ is an isomorphism, we can define two extensions $\tilde{X}, \tilde{Y} \in \Gamma(HM)$ of X and Y respectively by putting

$$\tilde{X}_a := (d\pi)_a^{-1}(Z_{\pi(a)}), \quad \tilde{Y}_a := (d\pi)_a^{-1}(W_{\pi(a)}).$$

It turns out that $\tilde{X}$ and $\tilde{Y}$ are π-related to Z and W respectively, and hence $[\tilde{X}, J\tilde{Y}]$ is π-related to $[Z, J'W]$ too, since $d\pi$ commutes with the almost complex structures J and J'. Finally, we have:

$$\begin{aligned} \pi_* L_x(X,Y) &= \pi_*(p_x[\tilde{X}, J\tilde{Y}]_x) \\ &= q_y(\pi_*[\tilde{X}, J\tilde{Y}]_x) \\ &= q_y[Z, J'W]_y \\ &= L'_y(\pi_* X, \pi_* Y). \end{aligned}$$

□

Corollary 2.6. *In the same hypotesis and notation of the previous Lemma, for every* $\psi \in H^0_y N$ *one has that* $\pi^* \mathfrak{L}'_\psi = \mathfrak{L}_{\pi^*\psi}$.

We remark that the scalar Levi forms $\mathfrak{L}_\omega$ are symmetric and hence it makes sense to consider their index $i(\mathfrak{L}_\omega)$, defined as the minimum between the number of positive and negative eigenvalues of $\mathfrak{L}_\omega$.
More specifically, we recall the following terminology from CR geometry; see for instance [6].

Definition 2.7. Let (M, HM, J) be a CR manifold of type (m,k) and let $x \in M$.
M is called *pseudoconvex* at x if $\mathfrak{L}_\omega$ is positive definite for some $\omega \in H^0_x M$.
If there exists a global section $\omega \in \Gamma(H^0 M)$ such that $\mathfrak{L}_\omega$ is positive definite at each point $x \in M$, M is called *strongly pseudoconvex.*
M is said *pseudoconcave* at x if $i(\mathfrak{L}_\omega) > 0$ for every $\omega \in H^0_x M$, $\omega \neq 0$.
M is said *weakly pseudoconcave* at x if $\mathfrak{L}_\omega = 0$ or $i(\mathfrak{L}_\omega) > 0$ for every $\omega \in H^0_x M$.

In this regard we recall that a Sasakian manifold $(M, \varphi, \xi, \eta, g)$, as defined in [2], is a particular kind of strongly pseudoconvex CR manifold of hypersurface type, i.e. of CR codimension 1. We shall refer to [2] for the notation and basic facts concerning Sasakian geometry. We only remark that in this case the CR structure is given by the contact distribution $\mathcal{D} = \ker \eta = \langle \xi \rangle^\perp$ and the almost complex structure is $J = \varphi_{|\mathcal{D}}$. Therefore, for any $x \in M$, $H^0_x M$ is spanned by η_x and, up to scaling, we have only one scalar Levi form $\mathfrak{L}_{\eta_x}$. Moreover, since M is a contact metric manifold, the identity

$$d\eta(X,Y) = g(X, \varphi Y)$$

yields that

$$\mathfrak{L}_{\eta_x} = 2 g_{x|H_x M \times H_x M}.$$

We end this section by recalling the definition of Stein manifold (for more information, see for instance [3]) and a theorem due to Hill and Nacinovich [5, 6], which provides a basic restriction to the topology of CR weakly pseudoconcave submanifolds of a Stein manifold.

Definition 2.8. A *Stein manifold* is a closed complex submanifold of $\mathbb{C}^d$, for some $d \geq 1$.

Theorem 2.9. *Every weakly pseudoconcave CR submanifold of a Stein manifold cannot be compact.*

3 Main result

Let $(M, \varphi, \xi, \eta, g)$ be a Sasakian manifold and let N be a hypersurface of M, tangent to the Reeb vector field ξ. At each point $x \in N$, let us consider the linear subspace of T_xN defined by

$$H_xN := \{X \in T_xN \mid X \perp \xi_x \text{ and } \varphi X \in T_xN\}.$$

Observe that, if $\nu \in T_xN^{\perp}$ is a unit normal vector at x, then we have the following orthogonal decomposition:

$$T_xN = \langle \xi_x \rangle \oplus \langle \varphi\nu \rangle \oplus H_xN.$$

It follows that HN is a subbundle of TN with constant rank and in [7] M. Munteanu proved that the couple $(HN, \varphi_{|HN})$ defines a CR structure of CR codimension 2 on N. We remark that he assumes the orientability of N, but this is unnecessary for our aim and the result holds true even if N is not orientable.

The CR structure $(HN, \varphi_{|HN})$ on the hypersurface N allows us to consider, for every unit normal vector ν, the scalar Levi form $\mathfrak{L}_\omega$ attached to the covector

$$\omega(X) = g_x(X, \varphi\nu) \quad \forall X \in T_xN. \tag{3.1}$$

We shall denote this scalar Levi form with the symbol $\mathfrak{L}_\nu$ and in the following proposition we establish the relationship between $\mathfrak{L}_\nu$ and the second fundamental form of the hypersurface N.

Proposition 3.1. *Let $(M, \varphi, \xi, \eta, g)$ be a Sasakian manifold and let $N \subset M$ be a hypersurface, tangent to ξ, with second fundamental form α. Let ν be a unit normal vector at some point $x \in N$. Then one has:*

$$\mathfrak{L}_\nu(X, X) = g_x(\alpha(X, X) + \alpha(\varphi X, \varphi X), \nu) \tag{3.2}$$

for every $X \in H_xN$.

Proof. First we recall that Sasakian manifolds are characterized by means of the following identity, involving the covariant derivatives of φ with respect to the Levi-Civita connection (see [2]):

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X. \tag{3.3}$$

Now, fix $x \in N$, $X \in H_xN$ and consider a smooth section in $\Gamma(HN)$ which extends X and a local normal vector field extending ν. Then φX is again tangent to N. Using the fact that X, φX and $\varphi\nu$ are all orthogonal to ξ and identity (3.3), we get:

$$\begin{aligned} & \mathfrak{L}_\nu(X,X) = \\ = \;& g_x([X,\varphi X],\varphi\nu) = \\ = \;& g_x(\nabla_X \varphi X,\varphi\nu) - g_x(\nabla_{\varphi X} X,\varphi\nu) = \\ = \;& g_x(\varphi\nabla_X X,\varphi\nu) + g_x(\varphi\nabla_{\varphi X} X,\nu) = \\ = \;& g_x(\nabla_X X,\nu) + g_x(\nabla_{\varphi X}\varphi X,\nu) = \\ = \;& g_x(\alpha(X,X) + \alpha(\varphi X,\varphi X),\nu). \end{aligned}$$

□

We shall use this formula to establish an identity relating the trace (with respect to g) of $\mathfrak{L}_\nu$ and the Ricci tensor field of N.
Hereinafter we will denote with an overline the relevant geometric entities of the hypersurface N (Levi-Civita connection, curvature, etc.).

Proposition 3.2. *Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be a Sasakian manifold and let $N \subset M$ be a hypersurface tangent to ξ. Let $x \in N$ and let $\nu \in T_xN^\perp$ be a unit normal vector. Then one has:*

$$\overline{\mathrm{Ric}}(\xi,\varphi\nu) = \frac{1}{2}\mathrm{tr}(\mathfrak{L}_\nu). \tag{3.4}$$

Proof. By a well known property of Sasakian manifolds (see [2]), for every $X \in \mathfrak{X}(M)$,

$$R(\xi,X)X = g(X,X)\xi - \eta(X)X. \tag{3.5}$$

Then, given $X \in \Gamma(HN)$, since ξ and X are normal to $\varphi\nu$, it follows that

$$R(\xi,X,\varphi\nu,X) = g(R(\xi,X)X,\varphi\nu) = 0. \tag{3.6}$$

Since $\varphi X = -\nabla_X \xi$ is still tangent to N, we also deduce that the normal component of $\nabla_X \xi$ vanishes, i.e. $\alpha(X, \xi) = 0$. Moreover,

$$\alpha(\xi, \varphi\nu) = g(\nabla_{\varphi\nu}\xi, \nu)\nu = g(-\varphi^2\nu, \nu)\nu = \nu. \tag{3.7}$$

Therefore, by using the Gauss formula, for every $X \in \Gamma(HN)$ we have that

$$\overline{R}(\xi, X, \varphi\nu, X) = g(\alpha(X, X), \nu). \tag{3.8}$$

Thus, fixed a local orthonormal frame of TN of type $\{\xi, \varphi\nu, E_i, \varphi E_i\}_{i=1,\dots,n-1}$, with $E_i, \varphi E_i \in \Gamma(HN)$, from (3.8) and (3.2) we get:

$$\begin{aligned}
\overline{\mathrm{Ric}}(\xi, \varphi\nu) &= \sum_{i=1}^{n-1} \left[\overline{R}(\xi, E_i, \varphi\nu, E_i) + \overline{R}(\xi, \varphi E_i, \varphi\nu, \varphi E_i)\right] \\
&= \sum_{i=1}^{n-1} g(\alpha(E_i, E_i) + \alpha(\varphi E_i, \varphi E_i), \nu) \\
&= \sum_{i=1}^{n-1} \mathfrak{L}_\nu(E_i, E_i) = \frac{1}{2}\mathrm{tr}(\mathfrak{L}_\nu),
\end{aligned}$$

where the last equality follows from the fact that $\mathfrak{L}_\nu$ is Hermitian and symmetric. □

Now we come to the proof of our main result.

Theorem 3.3. *If $(M, \varphi, \xi, \eta, g)$ is a regular Sasakian manifold which fibers onto a complex Stein manifold, then M does not admit any complete Einstein hypersurface tangent to ξ.*

Proof. Assume by contradiction that M admits a complete Einstein hypersurface N tangent to ξ, with Einstein constant c.
Let $\overline{\nabla}$ be the Levi-Civita connection of N. Since $\nabla_\xi \xi = 0$, from the Gauss equation we deduce that $\overline{\nabla}_\xi \xi = 0$. Moreover, since ξ is a Killing vector field on N, the operator $A_\xi := -\overline{\nabla}\xi$ is skew-symmetric and hence

$$\overline{\mathrm{Ric}}(\xi, \xi) = -\mathrm{div}(A_\xi \xi) - \mathrm{tr}(A_\xi^2) = -\mathrm{tr}(A_\xi^2) \geq 0.$$

It follows that

$$c = cg(\xi, \xi) = \overline{\mathrm{Ric}}(\xi, \xi) \geq 0.$$

If $c = 0$, then $A_\xi = 0$, i.e. ξ is $\overline{\nabla}$-parallel and this leads to a contradiction. Indeed, if we consider $X \in \Gamma(HN)$, with $X \neq 0$, from $\overline{\nabla}_X \xi = 0$ and the Gauss equation we would get

$$-\varphi X = \nabla_X \xi = \alpha(X, \xi),$$

where $-\varphi X \in \Gamma(HN)$ is non zero and tangent to N, while $\alpha(X, \xi)$ is normal. Therefore $c > 0$ and, because of completeness of N, Myers' theorem ensures that N is compact.
Moreover, from Proposition 3.2 we have:

$$\operatorname{tr}(\mathfrak{L}_\nu) = 2\overline{\mathrm{Ric}}(\xi, \varphi\nu) = 2cg(\xi, \varphi\nu) = 0. \tag{3.9}$$

Now, let $\pi : M \to M/\xi$ be the canonical projection, where $(M/\xi, J, g')$ is a Stein manifold; π is a Riemannian submersion whose fibers are 1-dimensional submanifolds of M tangent to ξ and

$$d\pi \circ \varphi = J \circ d\pi. \tag{3.10}$$

Since at every $x \in M$, $\ker(d\pi)_x = \mathbb{R}\xi_x$, we have that $\pi_{|N} : N \to M/\xi$ has constant rank. Hence, according to Theorem 3.5.18 in [1], $S := \pi(N)$ is a smooth hypersurface of M/ξ and it carries a CR structure (defined as in Remark 2.1), having the same CR dimension of N. Moreover, (3.10) implies that $\pi : N \to S$ is a CR map, such that at every point $x \in N$ the differential $(d\pi)_x : H_xN \to H_{\pi(x)}S$ is an isomorphism.
Fix a point $y = \pi(x) \in S$, with $x \in N$; if $\psi \in H^0_yS$, then $\pi^*\psi$ belongs to the vector space H^0_xN, which is spanned by ω and η, with ω as in (3.1). Actually, if $\pi^*\psi = \alpha\omega + \beta\eta$, for some numbers α, β, evaluating at ξ we obtain $\beta = 0$ and hence $\pi^*\psi = \alpha\omega$. Using Corollary 2.6 we get

$$\pi^*\mathfrak{L}'_\psi = \mathfrak{L}_{\pi^*\psi} = \alpha\mathfrak{L}_\nu$$

and by (3.9) we conclude that $\operatorname{tr}(\mathfrak{L}'_\psi) = 0$, so that $\mathfrak{L}'_\psi = 0$ or $i(\mathfrak{L}'_\psi) > 0$. Therefore S is a compact weakly pseudoconcave CR hypersurface of the complex Stein manifold M/ξ, thus contradicting Theorem 2.9. ☐

With just a small change in the previous proof, we also get the following result.

Theorem 3.4. *If $(M, \varphi, \xi, \eta, g)$ is a regular Sasakian manifold which fibers on a complex Stein manifold, then M cannot admit any compact hypersurface N, tangent to ξ and such that at any point of N ξ is an eigenvector of the Ricci operator $\overline{Q}$ of N.*

Proof. It suffices to note that if $\overline{Q}\xi = \alpha\xi$ along N, with $\alpha \in \mathcal{C}^\infty(N)$, then one has

$$\mathrm{tr}(\mathfrak{L}_\nu) = 2\overline{\mathrm{Ric}}(\xi, \varphi\nu) = 2g(\overline{Q}\xi, \varphi\nu) = 0.$$

Hence the proof ends with the same argument of the previous one. □

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