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The Use of Digital Technologies in Teaching and Assessment

Currently, technologies and mobile technologies are found almost everywhere in our everyday lives. In response to the social and economic changes resulting from technological innovation, various frameworks have been developed in recent years, as catalogues of competencies, knowledge and skills considered necessary to meet personal, societal, and professional demands of the 21st Century. The OECD's "21st Century Competencies" or "21st Century Skills" can be considered overarching goals for educators, business leaders, academics, and governmental agencies to support success in workplaces and society (OECD, 2018). Competencies in mathematics, evident in national curricula in many countries in the world, should be considered in relation to these general OECD competencies. The evolution of mathematical competencies to maintain relevance for the future is an on-going challenge. Schleicher, Director of the OECD Directorate for Education and Skills, notes "Education is no longer just about teaching learners; it is more important to enable them to develop a reliable compass and navigational tools to find their own way in an increasingly complex, volatile and uncertain world."1 (OECD, 2020, p. 8).

However, to be able to learn individually, to set their own goals and to follow their own way, students have to be taught by teachers who know how to support developing these competencies. Nowadays digital competencies are an indispensable part of these general competencies; consequently, there is a need for teachers who have knowledge of the digital competencies for students, as well as adequate digital competencies for teaching. In the following we firstly explain – referring to the OECD framework – "Digital Competencies" for teachers or educators. Secondly, we show expected student competencies based on national and international policy documents. Thirdly, we discuss the potential for developing these competencies in the frame of six examples from authors' research and development knowledge and experiences. Finally, we discuss both the current and future roles of digital technologies for the two areas, *Teaching and Learning* and *Assessment.*

1 Translation of a German quote by the authors.

1. Digital competencies

The "European Framework for the Digital Competence of Educators" (DigCompEdu) (Redecker, 2017) provides a framework "to promote the digital competence of their [i.e., member state] citizens and boost innovation in education" (EU, 2017, Foreword). DigCompEdu emphasises six main areas of the professional activities of educators:

- Area 1: *Professional Engagement*. Using digital technologies for communication, collaboration, and professional development.
- Area 2: *Digital Resources*. Sourcing, creating and sharing digital resources.
- Area 3: *Teaching and Learning*. Managing and orchestrating the use of digital technologies in teaching and learning.
- Area 4: *Assessment*. Using digital technologies and strategies to enhance assessment.
- Area 5: *Empowering Learners*. Using digital technologies to enhance inclusion, personalisation and learners' active engagement.
- Area 6: *Facilitating Learners' Digital Competence*. Enabling learners to creatively and responsibly use digital technologies for information, communication, content creation, wellbeing and problem-solving.

This framework aligns with the educational agenda of the countries involved in our examples and enables exploration of the importance of digital competencies of learners in the future development of mathematics education. Areas 3 and 4 target educators' digital pedagogical competence and the need for educators "to foster efficient, inclusive and innovative teaching and learning strategies" (EU, 2017: p. 16). These two areas can be used to emphasise and distinguish different activities for teaching and learning with digital technologies and are the focus of this chapter. Teaching and Learning are (nearly) always two sides of one coin. In section 2, we give an overview of aims of curricula and competencies, which had to be developed in the schools in the countries of the authors, to offer an overview of different trends.

In section 3, policy and implementation experiences in different countries are discussed, providing concrete examples of technology use in the classroom that support students' competencies and skills. For students' mathematical competencies we refer to Niss and Højgaard (2019), but a more detailed elaboration of these competencies is beyond the scope of this chapter, as teacher/educator competencies are the core of this chapter. Both aspects, teachers' and students' competencies exemplify how research in mathematics education can inform teachers, teacher educators and policymakers in implementation of digital technologies into schools and teacher education in our present and future digital era.

Digital technologies in schools have to support the development of students' competencies in general, however it is the teachers who transforms digital technologies into supportive tools in schools. Moreover, *digital competencies* cannot be developed as isolated goals, they have to be seen in relationship to subject content and, in mathematics, to mathematical competencies (Niss & Højgaard, 2019). It is well known that the use of digital

technologies does not by itself make teaching and learning better, and the selected competencies are intended to support use of digital technology to foster effective mathematics teaching and to promote students' learning.

An *integrated global concept* of the use of digital technologies for learning mathematics in schools is necessary which integrates the interaction of different digital components or resources (e.g., laptops, netbooks and the Internet), the use of classroom materials, teachers pedagogical and content-based knowledge, teacher-student-interactions and partnerships (e.g., teachers, schools, teachers and school administration, schools and universities). Connectivity and interconnectedness will be keys for future developments incorporating digital technologies in schools. The *acceptance* of new technologies and their gainful use require a global concept of teaching and learning.

The use of digital technologies in mathematics can change concepts, teaching and learning (Sinclair, 2020). Changing concepts implies that new mathematical concepts (and tasks) can be included in the curriculum and that this can require, or lead to, new tools; new tools can also lead to new concepts. Not only what is learned and how something is learned, but also the structure and the organisation of lessons, are influenced by digital technologies. Using digital tools (i.e., digital technologies) in teaching and learning mathematics implies, among other things, that teacher roles and teacher activities should adapt in order to capitalise on their potential. Teaching approaches to support and promote students' learning are predominantly the responsibility of the teacher. Curriculum content, whether technology is allowed or forbidden at particular year levels or for particular mathematical tasks, and conditions for assessment are policy decisions made either at a system or school level. Policies can dictate technology expectations in the curriculum (either stated explicitly or implicit in student outcomes) and assessment (e.g., final examinations). Merely using the word technology in curriculum documents is not enough to ensure meaningful integration across school systems. Government policy can provide imperatives for initial teacher education and professional learning of teachers and facilitate hardware and software to be used and implemented in mathematics classrooms. The challenge for policy makers and systems is to ensure that any prescribed curriculum with integrated technology can be enacted in each classroom. Policy related to technology in mathematics education should result in better mathematical outcomes for students, support students to develop a deep understanding of mathematics and foster the ability to identify and use technology for mathematics in different contexts. Research can provide insight into the consequences of curricula and policy decisions, and researchers can work with systems and schools to set the agenda for improving mathematics education through capitalising on the potential of technology for teaching and learning mathematics. Policy considerations provide an opportunity to have discussions about what is important in mathematics curriculum and in teaching mathematics; including decisions about what it means to learn mathematics at school (i.e., what content might be removed or revised, and what new mathematical opportunities can be explored with diverse technologies).

To provide insight into policy decisions in different countries, we look at existing curricula for school mathematics in Australia, Austria, Germany, and Italy and how digital technologies that are integrated into the curricula show connections to the competencies presented above.

2. Digital competencies in (existing) mathematics curricula

In this section, we outline the current state of technology integration and relationship to competencies, as well as the aims of curricula in the countries of the authors, to offer an overview of different trends.

Australia. Education is state-based, however, a national curriculum informs the state-based curricula.[2] The Australian mathematics curriculum at grades 7-10 is organized by content area, with four proficiencies (*fluency*, *understanding*, *problem-solving* and *reasoning)* to be developed across the content areas. ICT competence for learning and doing mathematics is to be developed from the first year of schooling as students.

> investigate, create and communicate mathematical ideas and concepts using fast, automated, interactive and multimodal technologies. They use their ICT capability to perform calculations; draw graphs; collect, manage, analyse and interpret data; share and exchange information and ideas; and investigate and model concepts and relationships (Australian Curriculum Assessment and Reporting Authority, 2023).

A range of technologies is specified for developing mathematical understanding, including dynamic geometry, CAS and spreadsheets. At senior secondary school there is variation about the technology that is allowed or expected for assessments in different states, which influences the use of technology in class. For example, in Victoria and Western Australia students are expected to have a CAS for one mathematics examination in each Year 12 subject.[3] In senior mathematics in Victoria one of three outcomes is related to developing competency with technology in mathematics and there is a long history of technology, including CAS, in assessment (Leigh-Lancaster et al., 2010; Victorian Curriculum and Assessment Authority (VCAA), 2015a.). For contrast, in Queensland, depending on the senior school subject, students are allowed a scientific calculator or non-CAS graphics calculator for examinations.

Austria. In the secondary school curriculum in Austria[4], from Grade 5 (10-year-old students) onwards, there should be possibilities for the planned use of digital technology when working on mathematics problems. It is stressed that digital technologies are to be used to support student-centred, experimental forms of learning and that a critical comparison of inputs and outputs in different programmes and devices should support the critical analysis ability of students.

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[2] https://www.australiancurriculum.edu.au/f-10-curriculum/mathematics/

[3] e.g., https://www.vcaa.vic.edu.au/curriculum/vce/vce-study-designs/mathematicalmethods/Pages/Index.aspx

[4] https://www.ris.bka.gv.at/GeltendeFassung.wxe?Abfrage=Bundesnormen&Gesetzesnummer=10008568

Both the written and the oral school-leaving examination at the end of secondary school is competence-oriented[5]. The Federal Ministry[6] states that there are scientifically useful and less useful definitions of competence orientation and provides two possible definitions of the term competence orientation:

- "Knowledge must be converted into skills. Knowledge and skills must be applied in new situations".
- "Competencies are understood here as longer-term available cognitive abilities and skills that can be developed by learners and enable them to perform certain activities in variable situations and the associated willingness to use these abilities and skills."

This means that tasks should include a reproduction aspect, a transfer aspect and a reflection or problem-solving aspect. For the subject of mathematics, this means that mathematical competencies have an activity dimension (i.e., what has to be done), a content dimension (i.e., what has to be learned) and a complexity dimension (i.e., from using basic knowledge to using reflective knowledge).

The mathematical competencies should always be extended by the students using digital technologies and demonstrated in exams using them[7]. The minimum requirements for digital technologies for mathematics school-leaving examinations are that the technologies can: A) represent geometric objects, functions and curves; B) transform terms symbolically and solve equations and systems of equations; C) determine derivatives and antiderivatives and perform integral calculations; and D) perform calculations from the field of stochastics. These minimal requirements of technologies for learning mathematics are also emphasised in the upper school mathematics curriculum[8] (grades 9-12) where it is stressed that such technologies are ideally used as tools in modelling, visualising and experimenting; these are strongly in-line with the listed competencies of the EU.

Germany. In Germany, the states are responsible for the education system. However, there are national standards for primary and secondary education that have been accepted by all 16 states and adapted accordingly at state level (Kultusministerkonferenz [KMK][9] 2004, 2012, 2022). These standards are based on the NCTM standards in terms of competencies. Since 2022, "Working with digital media" is a new process competence, which adds to expectations for argumentation, reasoning, modelling, representation, communication and working with mathematical objects. Digital media includes special mathematics tools (e.g., apps and interactive learning environments), as well as general media (e.g., videos and presentations software). Moreover, the

[5] https://www.matura.gv.at/index.php?eID=dumpFile&t=f&f=4826&token=4574fed24b889f914a68a7411172dbce06459c69

[6] https://www.ris.bka.gv.at/GeltendeFassung.wxe?Abfrage=Bundesnormen&Gesetzesnummer=20007845

[7] https://www.matura.gv.at/downloads/download/mathematische-grundkompetenzen-fuer-die-srp-in-mathematik-ahs, https://www.matura.gv.at/index.php?eID=dumpFile&t=f&f=4826&token=4574fed24b889f914a68a7411172dbce06459c69

[8] https://www.ris.bka.gv.at/GeltendeFassung.wxe?Abfrage=Bundesnormen&Gesetzesnummer=10008568

[9] KMK = Kultusministerkonferenz, conference of the ministers of education of the German states.

development of all competencies is to be supported by the meaningful use of digital media, whereby the potential unfolds through special activities. Firstly, *discovering mathematical connections*, especially through interactive explorations in modelling and problem solving. Secondly, promoting *understanding of mathematical relationships*, especially by means of multiple representations. Thirdly, *reducing schematic processes and working* with big(ger) data. Fourthly, by *supporting individual preferences and approaches* when working on tasks, including the use of control and feedback (KMK 2012, p. 12–13).

Concerning the use of digital technologies in tests and examinations, the kind of allowed technology differs from state to state with ongoing discussion about which technologies should be allowed. Some states allow arithmetic calculators only (e.g., Baden-Wuerttemberg), graphing calculators (e.g., North Rhine-Westphalia), or have mandatory CAS (e.g., Thuringia). In some states students have the choice between arithmetic, graphing and CAS calculators (e.g., Lower Saxony or Hesse), whereas in Bavaria students can only choose between arithmetic and CAS calculators. These diverse situations in the German states are symptomatic of how differently the importance of using digital technology is viewed but aim to correspond to national and EU-wide competency-related goals.

Italy. The State is exclusively responsible for the general organisation of the education system (e.g., minimum education standards, staffing, finances).[10]

The National Curriculum Guidelines (NCG) for High School claim that the use of digital resources is instrumental in improving classroom activities and supports students' work. The Guidelines stress that digital tools available today offer suitable contexts for representing and manipulating mathematical objects and that teaching of mathematics should offer numerous opportunities to become familiar with these tools and to understand their methodological value. Moreover, technologies need to be introduced critically, without creating the illusion that it is an automatic means of solving problems and without compromising the necessary acquisition of mental calculation skills.

The National Institute for the Evaluation of the Education System (INVALSI) performs a yearly external assessment of students for five selected grades; the aim is to verify the learning attainments of the students. In secondary school, the tests are computer-based which allows new forms of interaction with the assessment tool. The framework on which these tests are based is strongly linked to the NCG which addresses mathematical competencies from two perspectives: modelling aspects and applications for reading, interpreting reality and solving problems of concrete life, similar to PISA; mathematical topics, characteristic constructs and aspects relating to the development of curricula (i.e., for TIMSS).

[10] https://eacea.ec.europa.eu/national-policies/eurydice/content/italy_en

However, despite the claims of the NCG, the final State examination at the end of upper secondary school for the scientific branch includes a written national test (mathematics or physics or both), mainly consisting of traditional computational exercises; use of CAS or other Dynamic Software is not allowed. For this reason, with respect to the integration of technologies into mathematics classrooms, a crucial aspect remains the real implementation of the NCG and to address relevant digital competencies.

Summary

We outlined in this section that modern technologies can be found everywhere in our digital era leading to and requiring new competencies to be able to fully participate in society. This also implies that students need to acquire competencies that are different from those in the past, including both general competencies and those related to mathematics teaching and learning. These new mathematical competencies are becoming reflected in national and international policy documents and address numerous aims of globally stated directions. In sum, these new mathematical competencies have in common that a progression from *knowing what* through *knowing how* to *knowledge in action* must be considered and technologies are enablers of such transitions.

3. Examples

Competencies and calibrating them to local and culture-based needs are highly important. In this section, we present six examples to illustrate how technologies can enhance the development of students' competencies. However, the development of students' competencies requires teachers' competencies, in particular knowledge and abilities to initiate processes for developing learners' competencies in adequate learning environments. The six examples, focussed on algebra and geometry, show different uses of digital technologies. Some examples are well-accepted and – in some countries – digital tools such as computer algebra systems, dynamic geometry systems and spreadsheets are used widely in mathematics classrooms. We also investigate the future, through consideration of examples utilising up-to-date digital technologies like 3D-printing, adaptive systems, virtual and augmented reality in learning environments. The adequate use of these digital tools and systems requires important students' competencies, like collaborative learning, self-regulated learning or using feedback, but also teachers' competencies like guidance, feedback and planning or assessment strategies. These examples are from authors' research and development knowledge and experiences and give an authentic and realistic view into mathematics classrooms while using digital technologies. The examples could be suited for teaching and learning in a specific country or more widely within any culture. We must acknowledge that it would be possible to offer countless examples demonstrating the development of the variety of competencies, but we found these examples useful to demonstrate some of our points and hope these will inspire readers to develop and generate other appropriate examples for local needs. Furthermore, our used and - in relation to the examples - listed

educators' competencies (see Table 1) can be found in the "21st Century Competencies", in DigCompEdu as well as in all the standards or national curricula mentioned above.

Our six examples are summarised in Table 1, stating the title, main idea, competencies, and policy issues we believe these examples support in the education process. Nevertheless, as described earlier, all competencies are connected and can be developed simultaneously, but offer highlights of competencies developed through these examples.

Table 1: Summary of Examples

Example	**Main idea**	**Educators' Competencies**	**Policy issues**
A - Looking at octahedron using Augmented Reality and 3D-printing	3D – dynamically experienced	Collaborative learning (TL-3) Self-regulated learning (TL-4) Assessment strategies (A-1)	Showing the possibilities of innovative digital technologies like virtual and augmented reality
B - Solving linear or quadratic equations in an adaptive learning environment	Independent, individual, adaptive learning	Guidance (TL-2) Self-regulated learning (TL-4) Feedback and planning (A-3)	Using innovative digital technologies (like adaptive systems) in the frame of traditional contents
C - Determining a bike path	Modelling with digital technologies	Guidance (TL-2) Feedback and planning (A-3)	Developing the interrelationship between the real and the mathematical world via digital technologies
D - Introducing quadratic function through virtual simulation	Covariational reasoning	Collaborative learning (TL-3) Assessment strategies (A-1)	Explaining virtual simulations of real-life-situations as a basis for virtual reality
E- Solving equations using Computer Algebra Systems	New approaches to traditional procedures	Teaching (TL-1) Guidance (TL-2) Analysing evidence (A-2)	Discussing and explaining the interrelationship between hand skills and automatic calculations
F- Discovering Discrete Mathematics	New opportunities to look at sequences	Teaching (TL-1) Analysing evidence (A-2)	Showing the transfer of calculations from continuous to discrete mathematics via digital technologies

All examples are self-contained and structured to give the main ideas and potentials for teaching, learning and assessment. Our examples are connected to some key competencies in the discussion section to further illustrate our argument in this chapter.

3.1 Example A - Looking at octahedron using Augmented Reality and 3D-printing

Content. Knowledge of octahedra properties; performing calculations concerning octahedra; training spatial imagination and reasoning.
Technology. GeoGebra Classic, GeoGebra AR and 3D printing.

Goals. Students can define an octahedron, including its components and properties; recognise the properties/characteristics of octahedra, perform distance, area, and volume calculations, and perform angle calculations.

Main Ideas. In mathematics, students' ability to understand certain concepts, special orientations, and relationships must be developed in geometry (Suarsana et al., 2018). Use of analogue and digital teaching aids, such as wooden or cardboard models, or digital models, significantly contributes to the visualisation of teaching content and students' knowledge in the field of geometry (Korenova & Veress-Bagyi, 2018). Various digital technologies and software can display and represent mathematical content, however, Knill and Slavkovsky (2013) stress that manipulation with a physical model is still unmatched. They also indicate that 3D modelling and printing (3DMP) can contribute to successful mathematical learning through combining digital and physical environments. Researchers have found that 3DMP is a satisfactory learning tool for different topics such as geometry (Dilling & Witzke, 2020; Ng, 2017; Panorkou & Pratt, 2016) and provides great potential for mathematics teaching (El Bedewy et al., 2021; Haas et al., 2021; Ulbrich et al., 2020).

In this example, secondary school students (13-16 years old) had to construct an octahedron in a 3D geometry software package (e.g., GeoGebra[11] or Tinkercad[12]), calculate its area and volume, as well as determine the angles of vertices and length of edges using the 3D model of the octahedron. Additionally, sssssssssstudents used experiences gained during 3D modelling, investigating and 3D printing, as well as the 3D printed octahedron, for checking the accuracy of their calculations.
For students to better understand the structure of the octahedron, a net of its sides was constructed using the GeoGebra learning platform. It was desirable for students to form a net of the octahedron so that they can clearly understand its structure (Figure 1).

[11] https://www.geogebra.org/
[12] https://www.tinkercad.com/

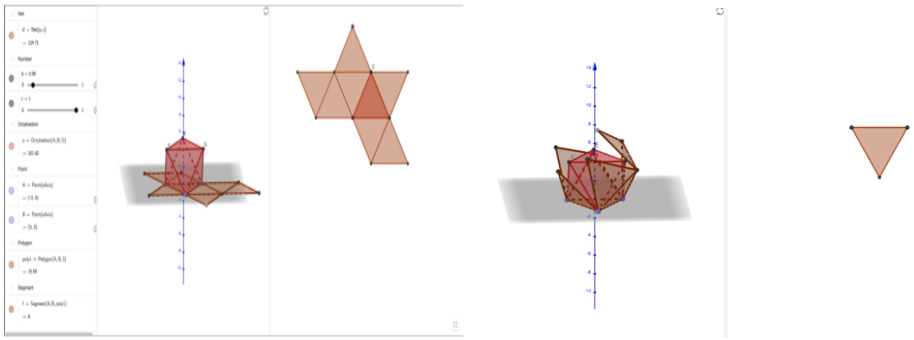


Figure 1: Model of octahedron designed in GeoGebra

To support visualisation in the classroom, the model made by the students can be downloaded as an STL file and printed on a 3D printer. To start the learning process concerning octahedra and using 3D software and 3D printing, two approaches are possible: A) from the definition of the octahedron to the 3D construct of the octahedron or B) from the 3D construct of the octahedron to the definition of the octahedron.

Approach *A*. For students to construct, 3D print, and study an octahedron using 3D software, the following definition of an octahedron is provided to students:

The octahedron is one of the five Platonic solids, more precisely a regular polyhedron with

- 8 congruent equilateral triangles as side faces,
- 12 edges of equal length and
- 6 corners, in each of which four side faces meet.

Thus, the octahedron is a double pyramid with a square base.

Students use this information to construct an octahedron using 3D software such as GeoGebra; this requires knowledge and competencies regarding vector calculus in $\mathbb{R}^3$. As this mathematical topic is only dealt with in upper secondary schools, this approach can only be chosen by students in grade 10 or higher. To avoid students taking shortcuts in this approach, only limited functionality (e.g., drawing points, lines and planes in $\mathbb{R}^3$, measuring lengths in $\mathbb{R}^3$, ...) should be provided in the 3D software. In this approach, 3D software can be used both to construct the octahedron and to check that the construction and definition of the octahedron fit together.

Approach B. In Approach B, students are given a 3D construct of an octahedron in 3D software, told that it is an octahedron and asked to investigate properties and characteristics. According to the age and school level of the students, different information concerning the octahedron and limited functionalities in the 3D software should be provided. As teachers can strongly control and guide this approach by providing information and functionalities of the 3D software, this approach is suitable for both lower and upper-secondary students.

Potentials for teaching, learning and assessment. This example could be used to foster the development of students' mathematical constructions and 3D reasoning skills. When teaching construction and 3D modelling in mathematics, there is a single didactic cycle that begins with students being introduced to a real-life problem.

For example, the teacher can instruct students to create 3D objects (e.g., artistic objects, jewellery, pen holders, key rings, etc.) that will contain octahedra of different areas and volumes. In this way, the teacher can encourage creativity and the application of students' knowledge of the octahedron.

3.2 Example B - Solving linear or quadratic equations in an adaptive learning environment

Content. Solving quadratic equations

Technology. GeoGebra Classic and GeoGebra Classroom

Goals. The aim of the presented tasks elaborated by the team of Eva-Maria Infanger and Alexander Huber, is for students to acquire mathematical knowledge and competencies in solving linear (lower secondary) or quadratic (upper secondary) equations and to test the knowledge or competencies acquired in this process. The tasks aim to facilitate students' independent learning by providing both the tasks and tailored feedback through a digital system. A digital or computer-based learning environment that provides students with tailored feedback and tasks should, among other things, contribute to facilitating mathematics learning outside of classrooms, through provision of high-quality support for students' learning processes.

Main Ideas. Learning mathematics in digital or computer-based learning environments means, among other things, that students are more self-reliant and thus can and must organise and control their learning processes. To progress in such self-directed learning environments, it is essential that students receive feedback on their learning process from a digital or computer-based learning environment or programme (Kopp & Mandl, 2014). In defining feedback, we follow Hattie and Timperley (2007) in understanding feedback as information regarding students' performance or understanding that is provided when students aim to minimise discrepancies between what they already can understand or are able to do and the defined goals of learning processes. According to Derr (2021), formative assessment consists of exercises to structure the learning process, activate students or stimulate reflection. In this process, learners receive feedback on how their current learning level has developed compared to an intended learning goal. Different forms of feedback are needed for a digital or computer-based learning environment to support such learning processes. Feedback can range from simple forms such as *knowledge-of-correct-response*, which checks whether an answer is correct or incorrect and presents the correct solution if incorrect. At the other end of feedback are extensive forms such as *attribute-isolation*, which verifies the answer and provides hints about the task concept and strategies (Mason & Bruning, 2001; Narciss, 2006). Our tasks use a feedback algorithm based on Narciss & Huth (2004) (see Figure 2).

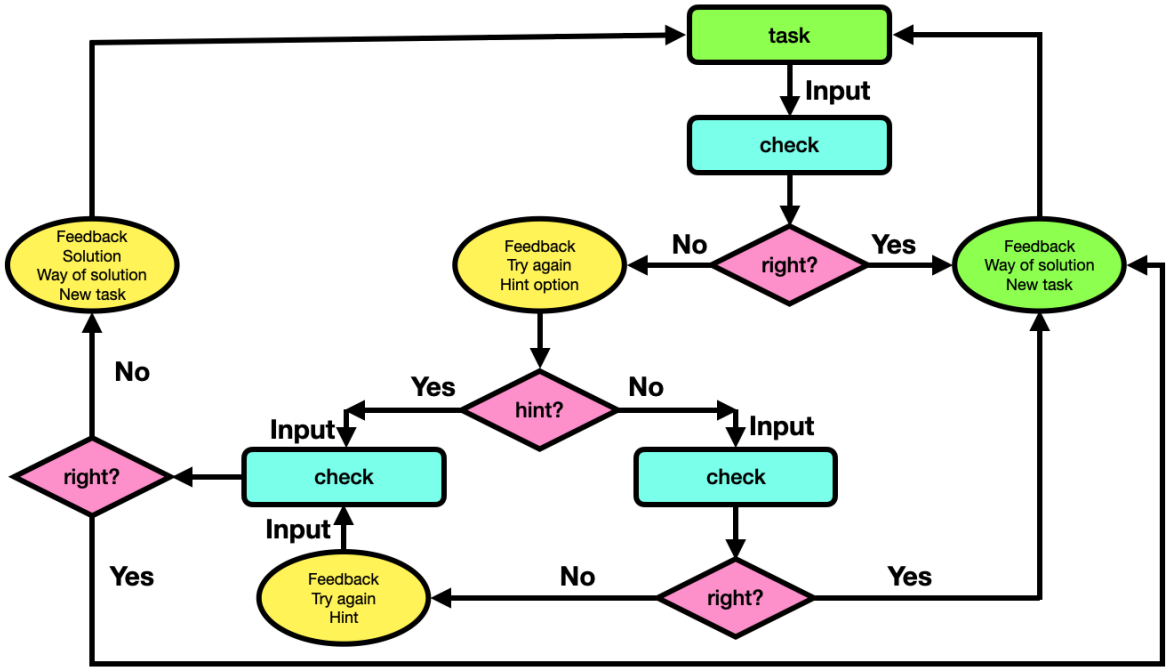


Figure 2: Feedback algorithm in our GeoGebra learning environment following Narciss & Huth (2004)

The feedback algorithm's use is illustrated by one task for each of linear and quadratic equations. There is an assumption that students cannot solve the respective task so that the entire feedback algorithm can be presented. The first step is to present a task to students, then have them enter their answers in the input field and the learning environment or programme checks their answers. Students receive feedback on whether the entered answer is correct or incorrect by pressing the check button. If the answer is correct, students receive a new task that is more difficult than the first task; if incorrect students can check their considerations or calculations, enter a new answer and have the new answer checked. If the answer is correct, students receive a new task with a similar difficulty level as the first task; if incorrect students receive a hint, enter a new answer and have this checked (see Figure 3). If this new answer is correct, students receive another task with a slightly easier difficulty than the first task; if incorrect students are provided with the correct answer and the way to solve the problem. Afterwards, students receive a further task with a significantly easier difficulty than the first task.

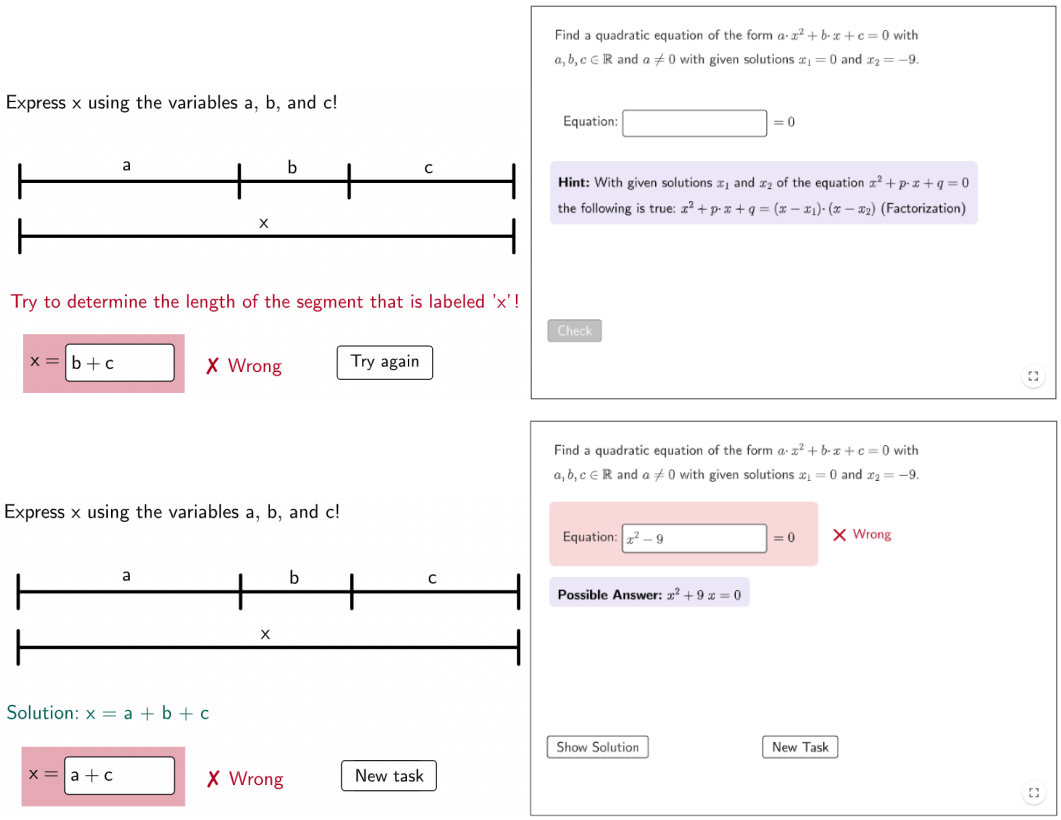


Figure 3: Adaptive learning environment hints for solving tasks on linear or quadratic equations

Potentials for teaching, learning and assessment. Possible further development of such adaptive digital learning environments could be that not exclusively pure mathematical operations, knowledge and competencies are practised and tested, but additionally, extra-curricular applications of mathematics are identified by digital learning environments and integrated into the learning process. Another further development of the adaptive digital learning environment presented above could be to extend the current single-user approach to a multi-user approach and then provide tailored feedback for the group and the respective group members. Similarly, further development of the adaptive digital learning environment could be that adaptive digital assessment environments are developed and, most importantly, approved by the relevant authorities as legitimate means of formally testing and grading student competencies.

3.3 Example C - Determining a bike path

Content. Combining quadratic function with a smooth joint.

Technology. CAS (TI-Nspire).

Goals. The determination of a group of functions that, when considered together can form a path to follow.

Main ideas. Students are provided with a section of a map and must determine an appropriate cycling path along a river. The example is based on part of *Splining a pathway*[13] (VCAA, 2015b.). This task, set in a real-life context, presents opportunities for technology to support procedural work, enabling an emphasis on planning a solution and interpretation of results. Students are presented with the following task:

Consider the section of the Yarra River shown in Figure 4b. Construct a path made up of smoothly joined quadratic functions. Determine a second path, made up of smoothly joined functions, that better fits the curve of the river and justify why your new path is a better fit.

Technology enables students to add axes behind the map (e.g., Figure 4b). Figure 4a shows an approach where a general quadratic function is defined; *f(x*) can be used in subsequent steps. A general third point (2, m) supports graphical exploration later. The ability to generate a slider and explore the effect of changing values for *m* enables consideration of where the 'best' third point might be. Students use a point (*n, m*) where n and m are sliders; CAS can support working with general cases. One line of syntax is used to substitute coordinates into the defined function f(x) and to solve the resultant simultaneous equations for *a, b* and *c* in terms of the parameter, *m*.

Using CAS to support procedural work enables a focus on the solution at a macro level, by 'defining the function, substituting and solving'; this approach can be applied to other problems. Classroom discourse about solutions at a macro level can promote consideration of features of well communicated written solutions, with Ball and Stacey (2003) highlighting the importance of the plan being evident in a student's written solution. Discussion of alternative approaches, including pen-and-paper, can promote procedural fluency.

Figure 4a shows *g(x)*, which is a quadratic function *f* (x) with *a, b* and c (in terms of *m*) substituted. Students explore a range of graphs quickly, by changing the value for m. Strategic selection of an appropriate graph will form part of the path that fits the river (Figure 4b). CAS supports solving both symbolically and graphically.

[13] Victorian Curriculum and Assessment Authority: Splining a pathway https://www.vcaa.vic.edu.au/Documents/vce/adviceforteachers/mathsmethods/Samp_App_task_3_splining.doc

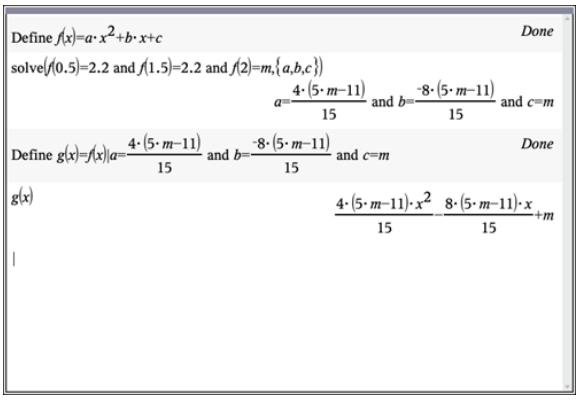


Figure 4a: Quadratic function in terms of *m*

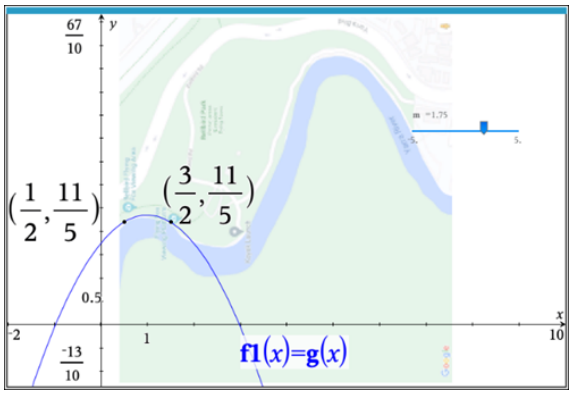


Figure 4b: Slider used to find part of path (Map accessed from Google Maps)

Students decide where a second parabola should join the graph of g to create a smooth join, using understanding of gradient and technology features. A similar approach is used, with an additional step of determining the gradient at a selected point where the two curves meet (Figures 5a fand 5b).

Figure 5c demonstrates use of technology to restrict the domain for each function to model the bike path. The ability to visualise results supports checking of reasoning when solving. Choosing labels on the graph enables students to determine how to make reasoning apparent. Figure 5d shows selected results, relevant to the particular stage of the solution.

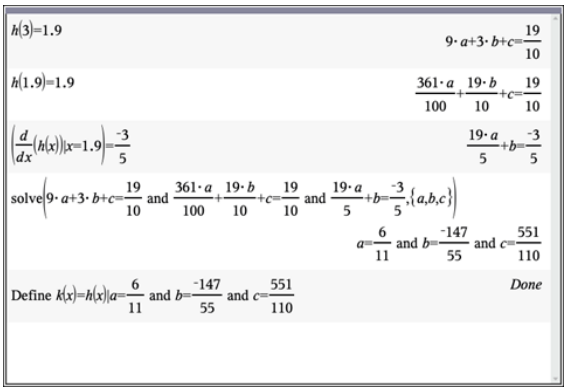


Figure 5a: Determining smooth join

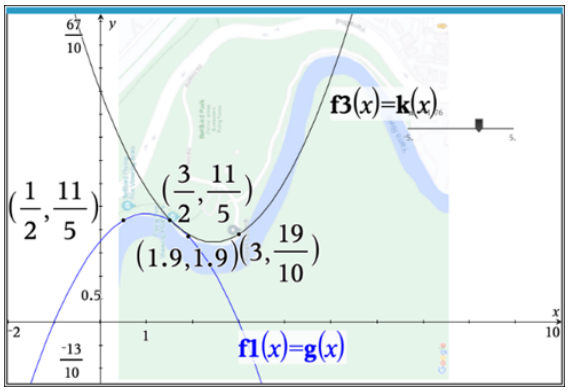


Figure 5b: Displaying smooth join

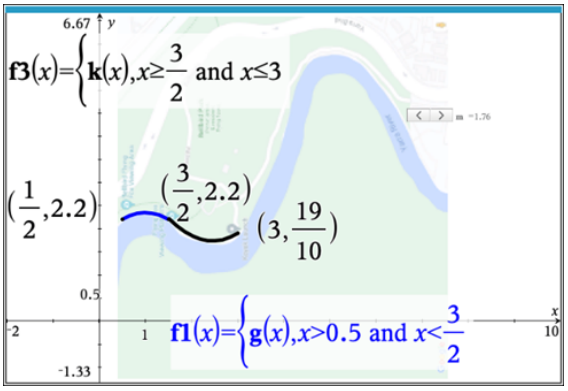


Figure 5c: Restricting domain

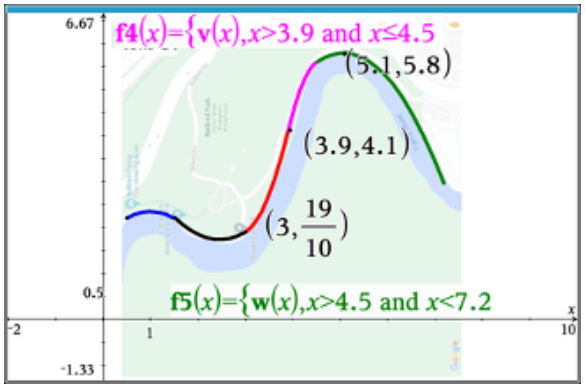


Figure 5d: Choosing results to display and determining path using quadratics

Students may note the need to use a number of short segments to follow the curve of the river. In the second part, where a range of functions can be used, students may use fewer functions.

In justifying whether a second path, constructed from a range of functions, is 'better' students can determine criteria and apply these, e.g., by determining the 2nd derivative of the functions where curves join smoothly.

Potentials for teaching, learning and assessment. Communication is a key goal in this task, students must communicate with technology to produce desired results and interpret technology displays to communicate a solution. Knowledge of syntax, including the use of menus, multiple representations and moving between mathematical notation and technological representations (Pierce et al., 2011) is essential to communicate with technology, interpreting technology results and communicating with peers and the teacher (Ball & Barzel, 2018).

The task is designed to be used as an assessment task, where students work individually. Informal assessment of work can be through observation of graphs, functions used and students' interpretation of technology results. Formal assessment could include an assessment of correctness of the functions produced by students, the extent to which the functions follow the curve of the river and are smoothly joined. Teachers could assess the extent to which the reasoning is accompanied by appropriate graphs and calculations; noting that different correct solutions reflecting different levels of sophistication are possible.

3.4 Example D - Introducing quadratic function through virtual simulation

Content. Function concept, quadratic function.

Technology. Dynamic virtual simulation (e.g., with GeoGebra) which allows display and manipulation of multiple representations of functions.

Goals. Covariational reasoning.

Main ideas. An important perspective in the development of the concept of function is covariation (Thompson & Carlson, 2017; Arzarello, 2019), according to which the variation of one variable is related in some way to the variation of the other. This example shows how students can be guided to consider the covariational perspective of the quadratic function by investigating the motion of a ball on an inclined plane (Swidan et al., 2019; Swidan & Faggiano, 2021; Swidan et al., 2022). For quadratic function students need to understand and connect its different representations (e.g., algebraically, the quadratic function is a second-degree polynomial, while numerically, the quadratic function can be represented by a discrete set of ordered pairs of numbers).

Technology introduces students to a real-world phenomenon and supports exploration, through different representations, to foster investigation of the dynamics of covariation to develop the meaning of quadratic function.

The example is based on a dynamic virtual simulation of a ball moving on an inclined plane, simultaneously generating a table in the spreadsheet and numerically showing the values of the distances moved by the ball (Figure 6). The simulation allows some variation (i.e., inclination of the plane) but not others (i.e., the x-difference). Students interactively explore the motion of the ball on the inclined plane and describe the motion mathematically.

Figure 6: The dynamic virtual simulation of the motion of a ball on the inclined plane

In the first task the angle of plane inclination is constant and allows students to recognize that the differences between two consecutive natural numbers in the third column (second differences) are constant and not zero. Hence, the distance-time relationship is not linear as the first differences (values in third column) are not the same.

In the second task a further representation is given (Figure 7) and students change the plane inclination and generate a table of values. The aim of this interactive exploration is to answer the following questions: 1) How does plane inclination affect the motion of the ball? 2) Find an equation that describes the motion of the ball.

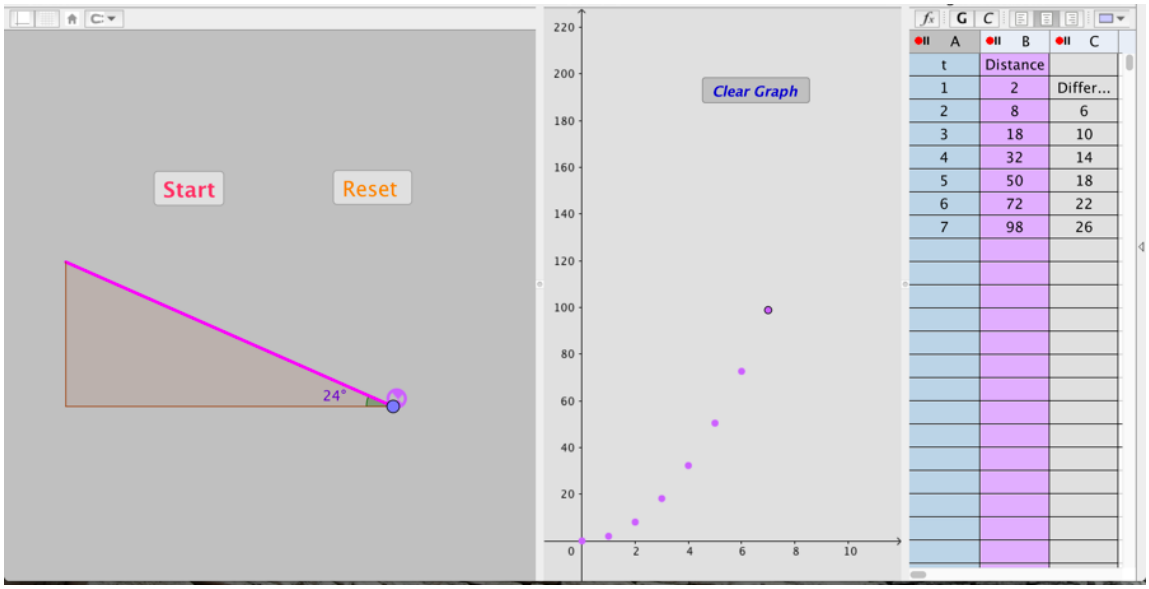


Figure 7: Multiple representations in the dynamic virtual environment

Students discover that for many values of the inclinations, the ratio between the distances and the squares of the times is always constant. Moreover, the graph, which signifies the distance-time relation of the rolling ball on the inclined plane, can prompt students to identify the dilation of the graph and associate it with the physical phenomenon: the coefficient can be found by taking the value of the second difference, which is constant, and dividing it by two.

Potentials for teaching, learning and assessment. Dynamic multiple representations of the real-world phenomenon allow students to explore the mathematical content, namely representations of quadratic functions and the covariation aspect of a function. Technology helps students to understand graphs as representing a continuum of states of covarying quantities: students' engagement with tasks requires them to track two sources of information simultaneously; this helps students view graphs as composed of points, each of which records the simultaneous state of two quantities that covary continuously (Saldanha & Thompson, 1998). The dynamic virtual environment, together with the designed tasks, has the potential to foster students' construction of the meaning of the quadratic functions and the understanding of function as co-variation.

Engaging students in coordinating real-world phenomena (e.g., a moving object along inclined plane using Augmented Reality) with its mathematical representations (e.g., plotting points of graph, ordered pairs in a table of values) through visual-kinaesthetic activities develops multiple meanings of the covariational reasoning.

Students' feedback on related problems can be used to assess students' learning, both in terms of processes and products: for instance, students can write their personal and/or group answer to the questions "How can you mathematically describe the motion of the ball on the inclined plane?" or "What is the equation that describes the movement?" or "What changes and what remains unchanged when the inclination of the plane is varied?".
This assessment can provide information about the students' ability to accomplish given tasks.
However, this way of teaching in a laboratory context, where students are encouraged to work in small groups and discuss their findings with peers, requires an approach beyond summative assessment, towards formative assessment. Throughout group work and collective discussions, the teacher assesses students' learning and identifies learning needs and difficulties, while students develop a stronger understanding of their learning strengths and weaknesses, taking greater responsibility over their own learning progress.

Use of these types of activities for formative assessment requires teacher awareness of potential of the dynamic virtual simulation (Swidan & Faggiano, 2019) and the ability to guide the collective discussion using appropriate strategies (e.g., through questioning or re-voicing) .

3.5 Example E - Solving equations using Modular Mathematics Systems (MMS)

Content. Solving, working with and interpreting equations.

Technology. The MMS[14] Geogebra

Goals: Students learn how to solve equations with MMS in different representations and interpret the solutions of equations, obtained with MMS, in multiple linked representations

Main Ideas. Equations are central objects in mathematics and the solving of equations is a key competency for students in secondary classes (Arcavi et al., 2017). MMS, especially the computer algebra systems (CAS), can be used to solve (many) equations by only 'pressing one button', but they also give reasons to think mathematically more deeply about equations and their solutions.

Solving of equations with a MMS. The solving process can be represented by an Input-Operation-Output model (Figure 8) in which the "operation" is either unknown to the user (black box) or it encapsulates known algorithms in a one-step procedure (white box).

Figure 8: Input – Output model for the object-view

[14]) A MMS is a mutual linked combination of a computer algebra system (CAS), a dynamic geometry system, a spreadsheet and function plotter

As an example, Figure 9 shows the solution of quadratic equations.

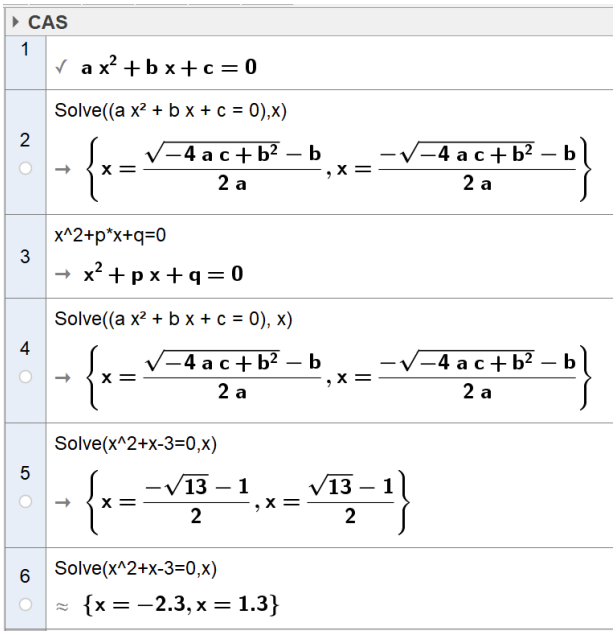


Figure 9: Solving of quadratic equations with a MMS (CAS)

However, an MMS (CAS) can also be used as a *teaching-learning system* that shows transformations carried out step-by-step while solving an equation (Figure 10). It can be used to explain a one-step result or to control calculations done by hand. An example for a learning app is *Photomath*[15] or the assessment system STACK (Sangwin, 2013).

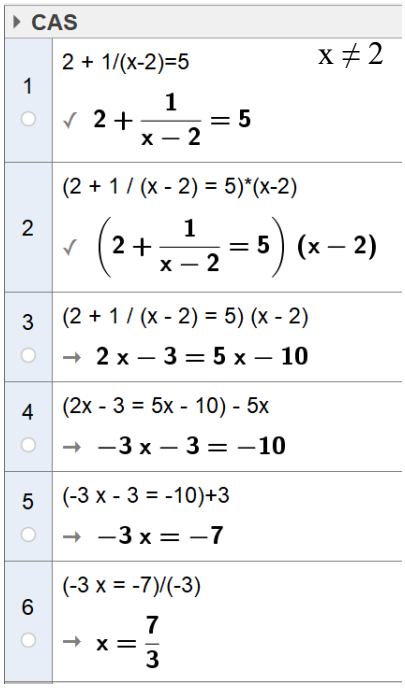


Figure 10: Stepwise solution

[15] https://photomath.app/

Moreover, an MMS can also be used to explain the step-by-step execution of (equivalence) transformations for an equation in a *learning environment.* For this, algebraic operations are applied to an equation as a whole, with the algebraic transformations visualized graphically (Figure 11). The static representation in Figure 10 has to be seen in a dynamic step-by-step presentation to show the invariance of the x-coordinate of the intersection point of the graphs and hence the consistency of the solution of each transformed equation.

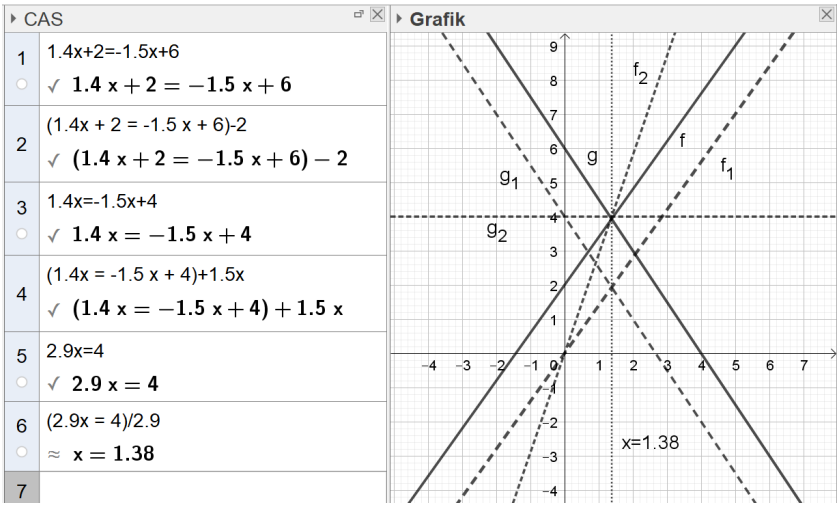


Figure 11: Step-by-step transformations of a linear equation

In this way an MMS can also solve more complex equations. The numeric solution of an equation (e.g., $x^3 - 3x + 1 = 0$) can be obtained with GeoGebra in one step. Concerning the graphic solution, the graph of the function $f(x) = x^3 - 3x + 1$ shows the existence of three zeros, hence three solutions of the equation (Figure 12). Moreover**,** the solution of the equation $x^3 - 2x + 1 = 0$ can be solved on the symbolic level. The reason is the existence of the integer solution x = 1. The system (GeoGebra) is able to do the factorization of the left-hand side of the equation gives a linear and a quadratic polynomial factor (Figure 13). Results of CAS calculations challenge students to think about the mathematical background.

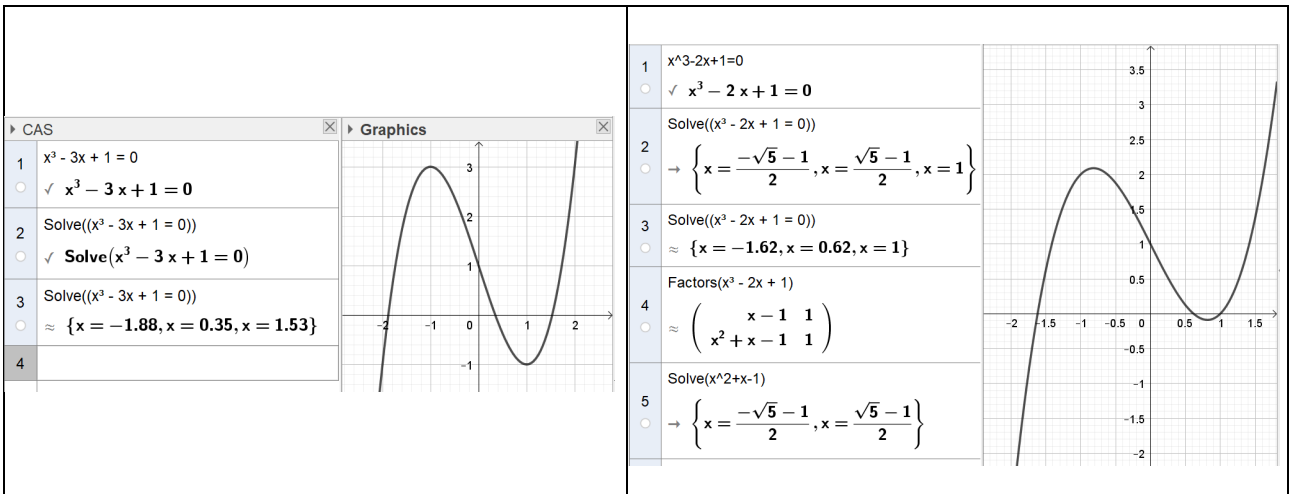


Figure 12: Numerical and graphical solution of the equation $x^3 - 3x + 1 = 0$	**Figure 13:** Symbolic**,** numeric and graphic solution of the equation $x^3 - 2x + 1 = 0$

Special equations of higher degree (particularly equations of degree 3 and, for example, trigonometric and exponential equations) can be solved–quite often–numerically, sometimes symbolically, and nearly always graphically.

Solving (more complex) equations with an MMS is not only a "pressing a button"-activity; there is a basic knowledge of the solution variety of the considered equations required. Furthermore, one needs strategies for the handling of representations and the changing of representations if an approach used by a student did not lead to a successful solution (Weigand, 2017).

Potentials for teaching, learning and assessment. Solving of equations is important in nearly all kinds of mathematics assessment at any school level. The possibility of solving equations on different representation levels opens the possibility for integrating new kinds of problems, especially more authentic real-life problems. Moreover, it is now possible to solve equations on a numeric, graphic and symbolic level, including equations that traditionally could not be solved in school mathematics.

3.6 Example F - Discovering Discrete Mathematics

Content. Discrete mathematics – Growth processes – especially recursively defined sequences and their graphs

Technology. Spreadsheets or MMS (like GeoGebra)

Goals. Introduction to different kinds of discrete growth processes

Main Ideas. *Discrete mathematics* is a name for various mathematical areas whose common core are finite – or discrete – sets and relations or structures. These include combinatorics, graph theory, combinatorial optimization, complexity theory, coding theory, and cryptography. Basic elements of discrete mathematics are *sequences*. A sequence is a function f with the natural numbers as domain: f: k → a_k with k ∈ $\mathbb{N}$ and a_k ∈ $\mathbb{R}$.

Sequences are *objects* with numerous interesting properties, but they are also *tools* for algorithms, modelling and step-by-step, iterative and recursive thinking and working. In the following we concentrate on sequences as tools for modelling growth processes. Discrete growth processes can be described by sequences.

Linear and exponential growth. A process with constant growth or constant change d = $a_{k+1} - a_k$, k =1, ..., n-1, n ∈ $\mathbb{N}$, in equal (time) steps is called *linear growth***.** A process with a constant rate of growth A = $\frac{a_{k+1}}{a_k}$, k = 1, ..., n–1, n ∈ $\mathbb{N}$, in equal (time) steps is called *exponential growth*, like radioactive decay or the worldwide population

growth in recent centuries. It can be expressed with recursive defined sequences: $a_{k+1} = A \cdot a_k$, $a_k \in \mathbb{R}_o^+$, $n \in \mathbb{N}$, k= 1, ..., n, with a special initial value a_1. Thus, the rate of change[16] is proportional to the term a_k:

$$\Delta a_k = a_{k+1} - a_k = A \cdot a_k - a_k = (A - 1) \cdot a_k, \text{ respectively } \Delta a_k \sim a_k.$$

Digital technologies are tools to represent the described processes on a numerical and graphical level and to make it possible to operate with them even when they cannot easily be transformed into explicit equations.

Limited and logistical growth. Exponential growth allows the growth beyond all limits. Such unlimited growth is not possible in real situations in our world. The growth rate therefore decreases the closer you get to this limit. Two special growth processes are the *discrete limited* and the *discrete logistical growth.*

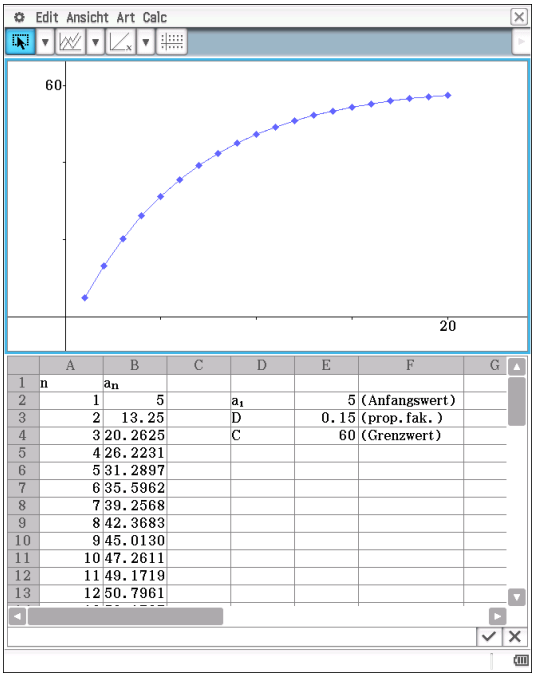
For *discrete limited growth* (Figure 14), the rate of change is proportional to the difference of a limit C and the value after the k-th step:

$$\Delta a_k = a_{k+1} - a_k \sim C - a_k,\ n \in \mathbb{N},\ k = 1, \ldots, n,\ C \in \mathbb{R},\ a_k \in \mathbb{R}_o^+.$$

The following applies to *discrete logistical growth* (Fig. 15):

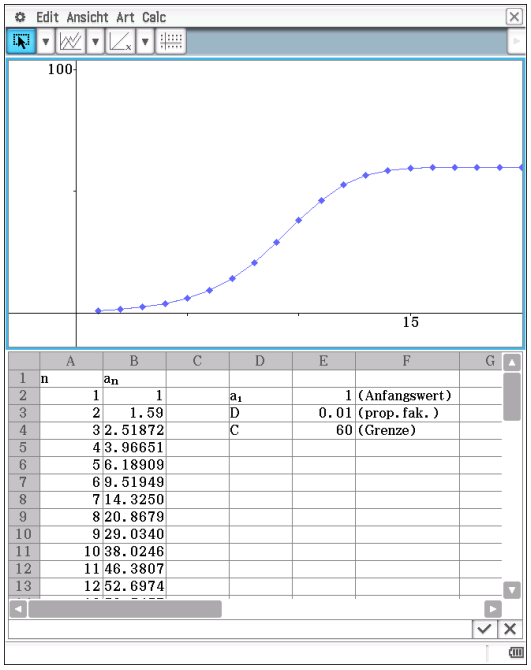
$$\Delta a_k \sim a_k \text{ and } \Delta a_k \sim C - a_k,$$

$$\text{Thus } \Delta a_k \sim a_k \cdot (C - a_k).$$



	A	B	C	D	E	F	G
1	n	a_n					
2	1	5		a_1	5	(Anfangswert)	
3	2	13.25		D	0.15	(prop.fak.)	
4	3	20.2625		C	60	(Grenzwert)	
5	4	26.2231					
6	5	31.2897					
7	6	35.5962					
8	7	39.2568					
9	8	42.3683					
10	9	45.0130					
11	10	47.2611					
12	11	49.1719					
13	12	50.7961					

Figure 14: Limited growth as a result: $a_{k+1} = a_k + D(C - a_k)$(with the Casio-ClassPad)



	A	B	C	D	E	F	G
1	n	a_n					
2	1	1		a_1	1	(Anfangswert)	
3	2	1.59		D	0.01	(prop.fak.)	
4	3	2.51872		C	60	(Grenze)	
5	4	3.96651					
6	5	6.18909					
7	6	9.51949					
8	7	14.3250					
9	8	20.8679					
10	9	29.0340					
11	10	38.0246					
12	11	46.3807					
13	12	52.6974					

Figure 15: Logistical growth as a result: $a_{k+1} = a_k + D \cdot a_k(C - a_k)$

[16] The denominator of the ratio is 1.

Examples of these growths can be found in biology with bacteria, plants and animal populations, and also in economic situations. The related parameters *D* and *C* in Figure 14 and Figure 15 can be interpreted in relation to the situation being explored. The effects of changing the parameters on the growth processes, in particular on the graphs and the numerical results, can be explored experimentally. It is also possible to construct special growth processes with given properties.

Potentials for teaching, learning and assessment. With the increased power of computer technology - computers are discrete machines - also the discrete perspective of real-life, as well as mathematics problems, one consequence is that discrete mathematics continues to be more important alongside classical continuous calculus. Moreover, digital technologies are tools which allow operations with (recursively defined) sequences on an algorithmic level; this was not possible - or only possible with time-consuming effort - by hand. Concerning assessment, modelling of discrete real-life situations could be used for formative and summative assessment (e.g., doing inter- and extrapolation problems or looking for regression curves of given (discrete) data).

4. Discussion

This article elaborates two areas, *Teaching and Learning* and *Assessment* from the DigCompEdu framework (EU, 2017) through six examples that highlight possible uses of digital technologies and illustrate relevant competencies. These examples attempt to illuminate learners' and teachers' experience (and understanding) of mathematics and assessment (and how they might change) through integration of innovative, as well as commonly used, technologies.

Following is a summary of the relevant competencies linked to our examples (Table 2) and discussion of issues and considerations evident through relevant examples and related literature.

Table 2: Examples and associated competencies (TL: Teaching and Learning; A: Assessment)

Competency	Example					
	A	B	C	D	E	F
Teaching (TL-1)					X	X
Guidance (TL-2)		X	X		X	
Collaborative learning (TL-3)	X			X		
Self-regulated learning (TL-4)	X	X				

Assessment strategies (A-1)	X			X		
Analysing evidence (A-2)					X	X
Feedback and planning (A-3)		X	X			

Teaching

Teaching indicates that teachers should carefully plan the implementation of digital technologies into their teaching practices in order to orchestrate their digital teaching strategies as well as experiment with innovative technology-assisted pedagogies (EU, 2017). Example E offers new opportunities for teaching mathematics with technology; these would not have been possible without the application of CAS software. However, teachers must rethink their teaching to utilise the benefits of the emergent opportunities. Similarly, Example F offers new possibilities for integrating areas of knowledge which are important to contemporary mathematics, in this case, discrete mathematics, but were not easily accessible before appropriate technologies became available. The software and related teaching approaches, outlined in Example F, highlight new curriculum directions that are possible in technology- supported school mathematics. Accordingly, constructing interrelationships between the use of tools and the learning aims (e.g., Clark-Wilson et al., 2014), play an increasingly important role in the future of technology-assisted education.

Guidance

Guidance, as teachers' competence, requires that the use of technology fosters interactions among all participants in the learning process, as well as provide learners with tailored feedback both within and outside of particular learning situations (EU, 2017). In addition, the use of digital technologies offers new and personalised support for students, which can orchestrated and monitored by teachers. In Example B, the newly developed adaptive learning environment enables learners to receive individual guidance from the software, both inside and outside the classroom environment, to meet learning goals determined by the teacher. Through the system, students receive tailored feedback on their solutions and hints, as well as appropriate problems for their mathematical development. Example E highlights that students can solve difficult problems with digital tools and experiment with new ways of learning that were not possible in pen-and-paper environments. Analogously, as described in Faggiano et al. (2017), students have to set up a relationship between their mathematical knowledge and the possibilities of digital technologies under the guidance of teachers to support development of competencies.

Collaborative learning

Collaboration is highly important for effective technology-enhanced teaching and learning. The use of digital technologies should play an important role in collaborative tasks by clearly enhancing communication, cooperation and collaborative knowledge creation (EU, 2017). Example A enables students to observe and envision 3D geometry from different perspectives and develop a collaborative design with their peers. The complexity of these tasks requires sufficient mathematical knowledge and know-how of the different software applications. The complexity of the task also requires diverse juggling between content, technology and skills, which necessitates the cooperation of students both physically and virtually. We can also observe in Example D that the co-variation, as in the previous example, requires multiple representations of mathematical objects. Moreover, the guidance of teachers is an important element in such a collaborative environment to reconcile students' knowledge and skills. Research has identified the importance of collaborative activities in technology-enhanced environments (e.g., Haas et al. 2021; El Bedewy et al. 2021; Ulbrich et al, 2021). Furthermore, collaborative skills are becoming crucial not only in schools, but at workplaces and in other areas of students' lives, and teacher practices must reflect these needs.

Self-regulated learning

Self-regulated learning enables students to plan, monitor and reflect on their own learning, provide evidence of progress, share insights and come up with creative solutions (EU, 2017). However, self-regulated learning needs to be monitored by teachers and in conjunction with teachers' aims to progress students' mathematical understanding. Example A demonstrates that use of augmented reality tools and 3D printing in technology-enhanced learning environments can support self-regulated learning by changing perspectives on (mathematical) objects and offering new approaches for inquiry learning. Use of visual aids can contribute to understanding of complex concepts and the augmented reality tools and 3D printing build on this long history of supporting mathematical development. This change of perspectives and having the concrete object at hand could foster students' reflections on their own learning outcomes; at the same time, students using these technologies have the potential to develop more creative solutions to a task provided by teachers (Korenova & Veress-Bagyi, 2018). By using augmented reality and 3D printing to foster students' creativity, content to be learned becomes visualised, which is a vital aspect of learning mathematical topics such as geometry (Korenova & Veress-Bagyi, 2018). In Example B, the adaptive learning environment shows opportunities to enable students to plan, monitor and reflect on their learning. To be able to learn in such an independent way, students might need feedback on their learning progress, which can be provided by a digital or computer-based learning environment or programme (Kopp & Mandl, 2014); in a classroom students may also discuss their individual work with peers, or their teacher, which also supports them to reflect on their learning. Furthermore, adaptive learning environments can provide students with tasks at an appropriate level of difficulty (Narciss & Huth, 2004) to ensure students are working at their ZPD.

Assessment strategies

Digital technologies could be and should be used for both formative and summative assessments to further develop the diversity and suitability of assessment formats and approaches (EU, 2017). Example A, with the use of AR and 3D Printing technologies, can enhance both types of assessment formats. Such visual technologies provide students with visual or physical feedback on their ideas and mathematical working. Having this aspect of assessment in mind, using AR or 3D printing could be a good way of assessing students' ideas, as well as activating students and stimulating their reflections on mathematical concepts for both kinds of assessments (Derr, 2021). Rather than assessment feedback being provided by technology, it can be provided through peer or teacher assessment. Example D illustrates how digital technologies can offer opportunities to focus on conceptual understanding through the integration of different representations. While interacting with technologies students produce different kinds of signs (speeches, sketches, gestures, etc.) that can constitute the basis on which teachers can assess their progress and offer feedback to foster a shared meaning in students' constructions (Bartolini Bussi & Mariotti, 2008).

Analysing evidence

This competence enables teachers to generate, select, critically analyse and interpret performance and progress, in order to inform their teaching and students' learning (EU, 2017). With Example E we showed how digital technologies can be used to assess students' understanding of solving equations by giving meaning to procedures. Construction of equations, interpreting solutions, and understanding various representations are gaining importance in contemporary mathematics compared to the traditional practice of calculating solutions. Discrete mathematics is increasingly being introduced into school curricula through the emergence of appropriate technologies that enable the solution of real-world-problems with discrete mathematical approaches, as demonstrated in Example F. Importantly, such teaks and approaches can constitute the basis of various assessment situations, especially in related topics such as regression, extrapolation, and interpolation analyses. Accordingly, digital technologies can offer new opportunities to assess students' work and understanding, as they are focused on more conceptual mathematical activities rather than simply procedural calculation (FASMED[17], Clark-Wilson et al., 2021).

Feedback and Planning

Digital technologies can provide targeted and timely feedback to students and enable the adaptation of novel teaching strategies, providing targeted support, based on the evidence generated through these technologies. Teachers should be able to empower students (and often parents) to understand the output provided by technologies and to utilise them for their decision-making (EU, 2017). In Example B, an adaptive learning environment could provide students with the kind of feedback to support individual learning. The feedback

[17] https://research.ncl.ac.uk/fasmed/

provided by the system could close the gap between students' actual levels of competencies and their desired levels of competencies (Hattie & Timperley, 2007). In this context, an adaptive learning environment could provide low-level feedback (e.g., knowledge-of-correct-response, which checks whether an input is correct or incorrect) or more sophisticated feedback (e.g., attribute-isolation, which verifies the answer and provides hints about the task concept and strategies) (Mason & Bruning, 2001; Narciss, 2006). Example C showed how fitting functions on pictures offer continuous feedback for students and this enables learning by educated trial and error; it also demonstrated that teachers need to reflect on such guidance for students to support students to successfully work on these tasks. This approach enables investigative learning, offerings opportunities for both formal and formative assessment, but requires careful planning (Ball & Barzel, 2018). However, it is important that both teachers and students know how to manage and communicate with the system and move between different representations of concepts (Pierce et al., 2011). Automatic feedback could be beneficial, but teachers must be well trained to integrate them valuably into their teaching practices.

In this section, we highlighted a sub-section of competencies connected to our examples showing some trends of educational and research directions. Achievement of these competencies can be supported through policy imperatives that provide infrastructure for teachers and students (e.g., hardware and software) and curriculum guidance related to capitalise on technology opportunities for mathematics. Assessment competencies can be supported through opportunities for technology enabled assessment: technology for assessment (i.e., technology incorporated into online assessments) and assessment with technology (e.g., mathematics-focussed technology employed for offline assessment). In the upcoming, conclusion section, we will outline some directions, we believe, researchers and policymakers should consider to further advance mathematics education.

5. Conclusions

Teaching and assessment are closely related in learning environments, especially in mathematics classrooms, and in many systems examination problems will set standards. In addition, assessment is closely tied to use of digital technologies in the classroom, and the interrelationship between technology and assessment are important when, as noted by Jankvist et al. (2021), teachers prefer to work with the technology in their classrooms that can also be used in exams. Above all, teaching and assessment have to be oriented towards goals and general competencies and this chapter concentrated on the competencies of the European Framework for the Digital Competence of Educators (DigCompEdu), in particular *Teaching and Learning* and *Assessment*. Referring to the discussion in Section 3 and the relation to the examples in Section 2, we summarise and generalise some main results in a way that could be taken as recommendations for teachers, teacher educators and policymakers in relation to the implementation of digital technologies in the up-coming digital era.

- *Teaching*: Teachers should have the opportunity to carefully plan the implementation of digital technologies into their teaching and should be supported with permanent and ongoing professional development, especially in peer-to-peer settings.
- *Guidance* as teachers' competence requires a re-definition and a re-orientation as students (and teachers) work with different kinds of digital technologies. The feedback of technical systems has to be added to the (traditional) interrelationship between students and teachers and must be constructively complemented by the feedback of the teacher.
- *Collaborative learning*: Digital technologies are catalysts for collaborative learning. Communication and cooperation are helpful for finding strategies in problem solving situations and they are necessary to discuss and interpret results with digital technologies, often in-the-moment as part of the problem-solving process.
- *Self-regulated learning*: Digital learning environments can enable students to learn on their own and at their own pace. However, it is necessary that teachers monitor and correct, if necessary, students' learning processes. This is a special challenge for teachers, because of the necessary individual supervision of students.
- *Assessment strategies*: The use of digital technologies offer new possibilities for formative and summative assessments (e.g., portfolios, project work and reports, teamwork, software solutions). These alternatives should be taken especially seriously in formative assessment settings.
- *Analysing evidence*: Referring to the dynamic, interactive and multiple representations of digital technologies, there are opportunities for different approaches to assess and evaluate students' understanding of mathematical concepts. This should be seen as a new opportunity for assessment.
- *Feedback and Planning*: Teachers should be competent to integrate their feedback into the feedback of digital technologies. Strategies on how to react also to (un-)expected feedback should be a part of any learning process.

The recommendations above highlight that integrating technology into teaching, learning and assessment is a complex undertaking that requires many more considerations than just learning how to operate the technology. Both teachers and students will need support and some teachers will need to rethink their pedagogical practices to consider changes to the didactic contract, what constitutes effective and informative assessment and fundamentally what it means for students to learn, understand, and apply mathematics.

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