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Asymptotics of quantum channels

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Abstract

We discuss several aspects concerning the asymptotic dynamics of discrete-time semigroups associated with a quantum channel. By using an explicit expression of the asymptotic map, which describes the action of the quantum channel on its attractor manifold, we investigate the role of permutations in the asymptotic dynamics. We show that, in general, they make the asymptotic evolution non-unitary, and they are related to the divisibility of the quantum channel. Also, we derive several results about the asymptotics of faithful and non-faithful channels, and we establish a constructive unfolding theorem for the asymptotic dynamics.

Keywords: quantum channels, open quantum systems, asymptotic dynamics

1. Introduction

Modeling a quantum computer or, more generally, any complex quantum hardware by a sequence of unitary gates is no longer sufficient to obtain efficient devices. Indeed, the unavoidable effects of noise and decoherence due to the coupling of the system with its environment is known to be a key obstacle for quantum information processing [1].

In this respect, dissipation processes can be harnessed in order to protect quantum information in the manifold of steady states of dissipative systems. In particular, the asymptotic dynamics of open quantum systems can be exploited for the realization of unitary operations inside subspaces protected from noise and decoherence [2, 3]. Also, the system evolution at large times may be used in reservoir engineering [4, 5], namely the preparation of a target state by

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relaxation of a system suitably coupled to the environment, or in phase-locking and synchronization of quantum systems [6]. A non-degenerate attractor manifold, i.e. consisting of a unique stationary state towards which the evolution converges, is often not enough in all these applications. Therefore, it is crucial to study the dynamics of dissipative systems at large times under less restrictive assumptions, as it was done in [8–14] for finite-dimensional systems, and [15] for infinite-dimensional ones. In particular, many efforts were devoted to the asymptotics of Markovian continuous dynamics [16–20], described by quantum dynamical semigroups. They were completely characterized by Gorini, Kossakowski, Lindblad and Sudarshan (GKLS) [7, 21], and the manifold of their steady states was analyzed in the following years [22–25].

In this Article we will investigate the asymptotic structure of a quantum channel. The starting point of our analysis is the direct sum decomposition of the attractor manifold, where the asymptotic dynamics takes place, and the action of a quantum channel onto it (theorem 2.2), as obtained by Wolf and Perez-Garcia [10, 11, theorem 2.1]. In spite of the importance of this result and its use in previous works [26, 27], its consequences have not been fully investigated yet in the literature.

In particular, the presence of permutations between the factors of the decomposition generally implies the lack of unitarity of the asymptotic dynamics. For a better understanding of the role of permutations, we will find a sufficient condition and a characterization of their absence, from which the connection with the Markovianity and divisibility of the channel emerges.

Also, we will construct a quantum channel with a given asymptotic dynamics, namely with a fixed attractor subspace and dynamics onto it (theorem 4.1), proving a converse of theorem 2.2. This is an example of extension theorem for quantum channels, in the trail of similar results obtained in the literature [28, 29].

Moreover, we will prove an alternative structure theorem of the asymptotic map (theorem 3.2), which provides a decomposition in terms of unitary and irreducible channels. It yields an alternative decomposition of the asymptotic manifold which may be obtained from [30]. Interestingly, it turns out that the irreducible part represents a classical dynamics which emerges in the asymptotic limit. Also, the unitary part, that encodes the quantum contribution to the asymptotic dynamics, can be used to circumvent decoherence in quantum information tasks [31]. Finally, previous results on the asymptotics of quantum dynamics by Novotný *et al* [8, 9] will be discussed in connection with the structure theorems for a non-faithful channel.

The paper is organized as follows. First, we will recall some general properties of quantum channels (section 2.1) and of infinitely divisible channels (section 2.2), and then we give the structure theorem of the asymptotic map in section 2.3. Afterwards, in section 3 we prove a decomposition of the asymptotic map in terms of unitary and irreducible channels, while in section 4 we will construct a quantum channel unfolding the asymptotic map. Then, in section 5 we will explore in detail the role of permutations in the asymptotic dynamics, characterize their absence, and link them to the infinite divisibility and primitivity of the channel. Finally, in section 6 we discuss several results on the asymptotics of quantum channels and their consistency with the structure theorems. After the concluding remarks (section 7), we prove in the appendices some results needed in the main part of the paper.

2. Asymptotics of open quantum systems

In this section we will set up the notation and recall some known results about the asymptotic dynamics of finite-dimensional open quantum systems.

2.1. Quantum channels

The state of an open quantum system is given by a density operator ρ, i.e. a positive semidefinite operator of unit trace on $\mathcal{H}$, with $\mathcal{H}$ the system Hilbert space, that in the following will be assumed to have a finite dimension d. Let $\mathcal{S}(\mathcal{H})$ be the set of density operators on $\mathcal{H}$. The evolution of the state ρ in the unit time is given by a quantum channel Φ, that is a completely positive trace-preserving map on $\mathcal{B}(\mathcal{H})$, the algebra of bounded operators on $\mathcal{H}$ [32]. The adjoint map $\Phi^\dagger$ of the channel Φ is defined with respect to the Hilbert–Schmidt scalar product $\langle A|B\rangle_{\mathrm{HS}} = \mathrm{tr}(A^\dagger B)$ via

$$\langle A|\Phi(B)\rangle_{\mathrm{HS}} = \langle \Phi^\dagger(A)|B\rangle_{\mathrm{HS}}, \quad A,B \in \mathcal{B}(\mathcal{H}). \tag{1}$$

$\Phi^\dagger$ is a completely positive unital map on $\mathcal{B}(\mathcal{H})$, describing the evolution of system observables, i.e. the dynamics in the Heisenberg picture.

Working in the Schrödinger picture, given a quantum channel Φ, the evolution at time $t = n \in \mathbb{N}$ will be given by the n-fold composition Φ^n of Φ, so we obtain a discrete-time semigroup $(\Phi^n)_{n\in\mathbb{N}}$.

The spectrum $\mathrm{spect}(\Phi)$ of a quantum channel Φ in finite dimensions is the set of its eigenvalues, and it satisfies the following three properties

- $1 \in \mathrm{spect}(\Phi)$,
- $\lambda \in \mathrm{spect}(\Phi) \Rightarrow \bar{\lambda} \in \mathrm{spect}(\Phi)$,
- $\mathrm{spect}(\Phi) \subseteq \{\lambda \in \mathbb{C} \,|\, |\lambda| \leqslant 1\}$.

Incidentally, these properties still hold for the larger class of positive and trace-preserving maps [11]. In particular, the *peripheral spectrum* $\mathrm{spect}_P(\Phi)$ of the channel Φ is defined as the set of eigenvalues with unit modulus, namely

$$\mathrm{spect}_P(\Phi) = \{\lambda \in \mathrm{spect}(\Phi) \,|\, |\lambda| = 1\}. \tag{2}$$

The asymptotic dynamics, obtained in the limit $n \to \infty$, takes place inside the *asymptotic*, *peripheral*, or *attractor subspace* of Φ, defined as

$$\mathrm{Attr}(\Phi) = \bigoplus_{\lambda \in \mathrm{spect}_P(\Phi)} \mathrm{Ker}(\Phi - \lambda \mathbb{1}_{\mathcal{B}(\mathcal{H})}), \tag{3}$$

since the peripheral eigenvalues are semisimple [11]. Equivalently,

$$\mathrm{Attr}(\Phi) = \mathrm{span}\{X \,|\, \Phi(X) = \lambda X \ \text{for some}\ \lambda \in \mathrm{spect}_P(\Phi)\}, \tag{4}$$

i.e. the space spanned by the eigenvectors corresponding to the peripheral eigenvalues.

Physically, the attractor subspace is the space of recurrences Y of the map Φ, namely [11]

$$\forall \epsilon > 0\ \exists n \in \mathbb{N}\ \text{s.t.}\ \|\Phi^n(Y) - Y\|_{\mathrm{HS}} \leqslant \epsilon. \tag{5}$$

Some of its elements are limit cycles

$$\Phi^n(Y) = Y \ \text{for some}\, n \in \mathbb{N}, \tag{6}$$

and, in particular, fixed points

$$\mathrm{Fix}(\Phi) = \{Y \,|\, \Phi(Y) = Y\}. \tag{7}$$

Now, let us introduce the spectral decomposition of a quantum channel Φ

$$\Phi = \sum_{k=1}^{N} (\lambda_k \mathcal{P}_k + \mathcal{N}_k), \tag{8}$$

where $\mathcal{P}_k$ and $\mathcal{N}_k$ are the eigenprojections and eigenilpotents of Φ corresponding to the kth eigenvalue λ_k, $k = 1,\ldots,N$ [33]. In particular, let us denote by $\mathcal{P}$ the eigenprojection onto the fixed point space $\mathrm{Fix}(\Phi)$, corresponding to the eigenvalue $\lambda = 1$. Also, define the *peripheral projection* by

$$\mathcal{P}_P = \sum_{\lambda_k \in \mathrm{spect}_P(\Phi)} \mathcal{P}_k. \tag{9}$$

Evidently, its range is the attractor subspace $\mathrm{Attr}(\Phi)$. Both projections $\mathcal{P}$ and $\mathcal{P}_P$ can be written in terms of the channel Φ as

$$\mathcal{P} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \Phi^k, \tag{10}$$

$$\mathcal{P}_P = \lim_{i\to\infty} \Phi^{n_i}, \tag{11}$$

for some increasing subsequence $(n_i)_{i\in\mathbb{N}}$. From equations (10) and (11) it is clear that both $\mathcal{P}$ and $\mathcal{P}_P$ are quantum channels. We can also introduce the *peripheral channel* Φ_P of Φ:

$$\Phi_P = \mathcal{P}_P\Phi = \Phi\mathcal{P}_P = \sum_{\lambda_k \in \mathrm{spect}_P(\Phi)} \lambda_k \mathcal{P}_k. \tag{12}$$

Note that in equation (12) the eigenilpotents do not appear, since the peripheral eigenvalues are semisimple.

Interestingly, one gets

$$\mathrm{Fix}(\Phi) \subseteq 0 \oplus \mathcal{B}(\mathcal{H}_0), \tag{13}$$

where $\mathcal{H}_0 = \mathrm{supp}\mathcal{P}(\mathbb{I}) = \mathrm{Ran}\mathcal{P}(\mathbb{I})$. Here, $\mathrm{supp}A$ and $\mathrm{Ran}A$ denote the support and the range of the operator A, and 0 acts on $\mathcal{H}_0^\perp$, the orthogonal complement of $\mathcal{H}_0$. Therefore

$$X \in \mathrm{Fix}(\Phi) \quad \Rightarrow \quad \mathrm{supp}X, \quad \mathrm{Ran}X \subseteq \mathcal{H}_0, \tag{14}$$

and, in particular, $\mathcal{P}(\mathbb{I})$ is a maximum-rank fixed point of Φ. By applying [11, proposition 6.9] to $\mathcal{P}_P$, we obtain

$$\mathrm{Attr}(\Phi) \subseteq 0 \oplus \mathcal{B}(\mathcal{H}_0'), \tag{15}$$

where $\mathcal{H}_0' = \mathrm{supp}\mathcal{P}_P(\mathbb{I}) = \mathrm{Ran}\mathcal{P}_P(\mathbb{I}) \supseteq \mathcal{H}_0$. In fact, as proved in appendices A. and B., it turns out that

$$\mathcal{H}_0 = \mathcal{H}_0', \tag{16}$$

even though, in general,

$$\mathcal{P}_P(\mathbb{I}) \neq \mathcal{P}(\mathbb{I}). \tag{17}$$

However, notice that $\mathcal{P}_P(\mathbb{I}) = \mathcal{P}(\mathbb{I})$ for unital channels, i.e. $\Phi(\mathbb{I}) = \mathbb{I}$, or channels with trivial peripheral spectrum, viz. $\mathrm{spect}_P(\Phi) = \{1\}$, such as primitive maps (see definition 3.1).

2.2. Infinitely divisible channels

We now introduce the important class of infinitely divisible quantum channels which in the following will be proved to exhibit a special asymptotic behavior.

Let $(\Phi_t)_{t\in\mathbb{R}^+}$ be a one-parameter family of channels with $\Phi_0 = 1_{\mathcal{B}(\mathcal{H})}$, the identity channel. If the semigroup property $\Phi_{t_1+t_2} = \Phi_{t_1}\Phi_{t_2}$ is satisfied for all $t_1, t_2 \in \mathbb{R}^+$, the family is called a quantum dynamical semigroup, and it is of the form

$$\Phi_t = e^{t\mathcal{L}}, \quad t \in \mathbb{R}^+. \tag{18}$$

Here, $\mathcal{L}$ is a GKLS generator and has the following structure [7, 21]:

$$\mathcal{L}(X) = -i[H,X] + \sum_{k=1}^{d^2-1}\left(A_kXA_k^\dagger - \frac{1}{2}\{A_k^\dagger A_k, X\}\right) = \mathcal{L}_H(X) + \mathcal{L}_D(X), \tag{19}$$

for $X \in \mathcal{B}(\mathcal{H})$. The square (curly) brackets denote the (anti)commutator, $H = H^\dagger$ is the system Hamiltonian, and the noise operators A_k are arbitrary operators on a d-dimensional Hilbert space. Moreover, $\mathcal{L}_H$ and $\mathcal{L}_D$ stand for the Hamiltonian and dissipative parts of the generator $\mathcal{L}$, respectively.

A quantum channel $\Phi = e^{\mathcal{L}}$, with $\mathcal{L}$ of the form (19), is said to be *Markovian*. Note that a Markovian channel Φ is invertible (as a linear map), which is equivalent to say that $0 \notin \mathrm{spect}(\Phi)$.

Another relevant class of channels are the *idempotent* channels, satisfying the condition $\Phi^2 = \Phi$. A paradigmatic example is the contraction channel

$$\Phi(X) = \mathrm{tr}(X)\rho, \quad \rho \in \mathcal{S}(\mathcal{H}), \quad X \in \mathcal{B}(\mathcal{H}). \tag{20}$$

Unlike Markovian channels, idempotent channels are not invertible, except for the trivial identity channel $\mathsf{1}_{\mathcal{B}(\mathcal{H})}$. However, both classes of channels share the following more general property, called infinite divisibility.

Definition 2.1. Let Φ be a quantum channel. Then Φ is said to be infinitely divisible iff for all positive $n \in \mathbb{N}$,

$$\Phi = \Phi_n^n, \tag{21}$$

for some quantum channel Φ_n, called an nth root of Φ.

While Markovian and idempotent channels are infinitely divisible, there are examples of infinitely divisible channels which are neither Markovian nor idempotent. Nevertheless, it can be proved that Φ is infinitely divisible iff

$$\Phi = \mathcal{Q}e^{\mathcal{L}}, \tag{22}$$

where $\mathcal{Q}$ is an idempotent channel, $\mathcal{L}$ an GKLS generator, and $\mathcal{Q}\mathcal{L} = \mathcal{Q}\mathcal{L}\mathcal{Q}$ [34, 35].

2.3. The structure of the asymptotic map

One may ask whether there exists a finer structure for the asymptotic manifold $\mathrm{Attr}(\Phi)$ and for the action of the *asymptotic map*

$$\hat{\Phi}_P = \Phi|_{\mathrm{Attr}(\Phi)} = \Phi_P|_{\mathrm{Attr}(\Phi)} \tag{23}$$

of a quantum channel Φ. The answer is given by the following structure theorem of Wolf and Perez-Garcia [10, theorem 8].

Theorem 2.2 (Asymptotic map). *Let Φ be a quantum channel on $\mathcal{B}(\mathcal{H})$, and $\mathcal{P}$ its eigenprojection corresponding to $\lambda = 1$.*

There exists a decomposition of the Hilbert space $\mathcal{H}$

$$\mathcal{H} = \mathcal{H}_0^\perp \oplus \bigoplus_{k=1}^{M} \mathcal{H}_{k,1} \otimes \mathcal{H}_{k,2}, \tag{24}$$

with $\mathcal{H}_0 = \operatorname{supp}\mathcal{P}(\mathbb{I})$*,* $\mathcal{H}_{k,i}$ *being Hilbert spaces (*$k = 1,\ldots,M$ *and* $i = 1,2$*), and there exist positive definite density operators* ρ_k *on* $\mathcal{H}_{k,2}$ *such that*

$$\operatorname{Attr}(\Phi) = 0 \oplus \bigoplus_{k=1}^{M} \mathcal{B}(\mathcal{H}_{k,1}) \otimes \mathbb{C}\rho_k. \tag{25}$$

Therefore every $X \in \operatorname{Attr}(\Phi)$ *can be written as*

$$X = 0 \oplus \bigoplus_{k=1}^{M} x_k \otimes \rho_k, \tag{26}$$

for some operators $x_k \in \mathcal{B}(\mathcal{H}_{k,1})$*.*

Moreover, there exist unitaries U_k *on* $\mathcal{H}_{k,1}$ *and a permutation* π *on the set* $\{1,\ldots,M\}$ *such that the asymptotic map of the quantum channel* Φ *reads*

$$\hat{\Phi}_P(X) = 0 \oplus \bigoplus_{k=1}^{M} U_k x_{\pi(k)} U_k^\dagger \otimes \rho_k, \tag{27}$$

with $X \in \operatorname{Attr}(\Phi)$ *in the form (26).*

Remark 1. The permutation π acts on subsets of $\{1,\ldots,M\}$, so that the corresponding Hilbert spaces must have the same dimension, i.e.

$$d_{\pi(k)} = d_k, \quad k = 1,\ldots,M, \tag{28}$$

with $d_k = \dim(\mathcal{H}_{k,1})$, consistently with the fact that $\operatorname{Attr}(\Phi)$ is an invariant subspace for Φ

$$\Phi\operatorname{Attr}(\Phi) = \operatorname{Attr}(\Phi). \tag{29}$$

Remark 2. ([36]). If Φ is faithful, i.e. it has a full-rank fixed state, then $\dim(\mathcal{H}_0^\perp) = 0$, so that the zero term in (25) disappears. Consequently,

$$\mathcal{P}_P(X) = \sum_{k=1}^{M} \operatorname{tr}_{k,2}(P_k X P_k) \otimes \rho_k, \tag{30}$$

where P_k is the projection onto $\mathcal{H}_{k,1} \otimes \mathcal{H}_{k,2}$ and $\operatorname{tr}_{k,2}$ denotes the partial trace over $\mathcal{H}_{k,2}$.

Moreover, it can be shown that $\operatorname{Attr}(\Phi^\dagger)$ is a unital $*$-algebra [37] with the following structure [15, theorem 1]

$$\operatorname{Attr}(\Phi^\dagger) = \bigoplus_{k=1}^{M} \mathcal{B}(\mathcal{H}_{k,1}) \otimes \mathbb{C}\mathbb{I}_{k,2}, \tag{31}$$

where $\mathbb{I}_{k,2}$ is the identity on $\mathcal{H}_{k,2}$.

$\operatorname{Attr}(\Phi)$ is an algebra with respect to the $\star$-product [38, 39]

$$A \star B := A\mathcal{P}_P(\mathbb{I})^{-1}B \in \operatorname{Attr}(\Phi), \quad A, B \in \operatorname{Attr}(\Phi). \tag{32}$$

This can be immediately seen if we observe that (30) implies that

$$\mathcal{P}_P(\mathbb{I}) = \bigoplus_{k=1}^{M} m_k \mathbb{I}_{k,1} \otimes \rho_k, \tag{33}$$

where $\mathbb{I}_{k,1}$ is the identity on $\mathcal{H}_{k,1}$ and $m_k = \dim(\mathcal{H}_{k,2})$.

Remark 3. As a consequence of (27), every maximum-rank fixed state σ has the form

$$\sigma = \frac{1}{\sum_{k=1}^{M} \mathrm{tr}(\sigma_k)} \left(0 \oplus \bigoplus_{k=1}^{M} \sigma_k \otimes \rho_k \right), \tag{34}$$

with $\sigma_k > 0$ such that

$$U_k \sigma_{\pi(k)} U_k^{\dagger} = \sigma_k, \quad k = 1, \ldots, M. \tag{35}$$

3. Cyclic structure of the asymptotic map

In this section, we are going to exhibit an interesting alternative form for the asymptotic map $\hat{\Phi}_P$ of a quantum channel Φ. In particular, we are going to show that, in each disjoint cycle of the permutation π, the channel can be written as the tensor product between a unitary channel and an irreducible one, being a permutation between the density matrices ρ_k in the decomposition (25) of the asymptotic manifold. First, let us recall the notion of irreducible and primitive channels [11].

Definition 3.1. Let Φ be a quantum channel. Then

- Φ is *irreducible* if it has a unique full-rank fixed state σ, i.e. $\Phi(\sigma) = \sigma$, with $0 < \sigma \in \mathcal{S}(\mathcal{H})$;
- Φ is called *primitive* if it is irreducible and it has trivial peripheral spectrum, namely $\mathrm{spect}_P(\Phi) = \{1\}$.

Now, we aim at rewriting the map (27) in a form where in each cycle all the unitaries U_k are the same. According to theorem 2.2, given the Hilbert space decomposition (24) and an element $X \in \mathrm{Attr}(\Phi)$ in the form (26), the action of the asymptotic map $\hat{\Phi}_P$ on X is in the form (27). Notice that local unitary transformations (i.e. local basis changes) $V_k : \mathcal{H}_{k,1} \to \mathcal{H}'_{k,1}$, with $k = 1, \ldots, M$, induce a unitary map $V : \mathcal{H} \to \mathcal{H}'$, where

$$\mathcal{H}' = \mathcal{H}_0^{\perp} \oplus \bigoplus_{k=1}^{M} \mathcal{H}'_{k,1} \otimes \mathcal{H}_{k,2} \tag{36}$$

has exactly the same structure of $\mathcal{H}$. We will use this freedom in order to simplify the expression of the asymptotic map $\hat{\Phi}_P$.

Any operator of the asymptotic manifold $X \in \mathrm{Attr}(\Phi)$ is transformed by V into

$$X' = VXV^{\dagger} = 0 \oplus \bigoplus_{k=1}^{M} x'_k \otimes \rho_k, \tag{37}$$

where

$$x'_k = V_k x_k V_k^{\dagger}. \tag{38}$$

Correspondingly, the action of the asymptotic map on X' becomes

$$\hat{\Phi}'_P(X') = V\hat{\Phi}_P(X)V^{\dagger} = V\hat{\Phi}_P(V^{\dagger}X'V)V^{\dagger}, \tag{39}$$

that is

$$\hat{\Phi}'_P(X') = 0 \oplus \bigoplus_{k=1}^{M} (V_k U_k V_{\pi(k)}^{\dagger}) x'_{\pi(k)} (V_{\pi(k)} U_k^{\dagger} V_k^{\dagger}) \otimes \rho_k = 0 \oplus \bigoplus_{k=1}^{M} U'_k x'_{\pi(k)} U_k'^{\dagger} \otimes \rho_k. \tag{40}$$

As a result, changing the local basis will change the operators U_k as

$$U_k \mapsto U_k' = V_k U_k V_{\pi(k)}^\dagger. \tag{41}$$

In general a permutation π is the product of disjoint cycles. Assume first that $\pi = (k_1, \dots, k_M)$ is a cyclic permutation of length M, that is

$$\pi(k_j) = k_{j+1} \mod M. \tag{42}$$

In such a case all Hilbert spaces are isomorphic,

$$\mathcal{H}_{1,1}' \simeq \mathcal{H}_{2,1}' \simeq \cdots \simeq \mathcal{H}_{M,1}' \simeq \mathcal{H}_1, \tag{43}$$

and we can write

$$\mathcal{H}' = \mathcal{H}_0^\perp \oplus \bigoplus_{k=1}^{M} \mathcal{H}_1 \otimes \mathcal{H}_{k,2}. \tag{44}$$

Now, we require that the unitaries U_k' be all equal:

$$U_{k_1}' = U_{k_2}' = \cdots = U_{k_M}' = U, \tag{45}$$

which means that

$$U = V_{k_j} U_{k_j} V_{k_{j+1}}^\dagger \mod M. \tag{46}$$

By writing $V_{k_{j+1}}$ in terms of V_{k_j}, one gets

$$V_{k_1} = (U^\dagger)^M V_{k_1} (U_{k_1} U_{k_2} \cdots U_{k_M}), \tag{47}$$

or explicitly

$$U = V_{k_1} (U_{k_1} U_{k_2} \cdots U_{k_M})^{1/M} V_{k_1}^\dagger. \tag{48}$$

Therefore, up to a local unitary V_{k_1} on $\mathcal{H}_{k_1,1} \simeq \mathcal{H}_1$, the transformed unitary U turns out to be a geometric mean of the original unitaries $U_{k_j}, j = 1, \dots, M$, along the cycle.

Thus, by changing basis and dropping the primes in (40), the expression (27) for the asymptotic map becomes

$$\hat{\Phi}_P(X) = 0 \oplus \bigoplus_{k=1}^{M} U x_{\pi(k)} U^\dagger \otimes \rho_k. \tag{49}$$

Moreover, we can decompose the Hilbert space $\mathcal{H}$ as

$$\mathcal{H} = \mathcal{H}_0^\perp \oplus \bigoplus_{k=1}^{M} \mathcal{H}_1 \otimes \mathcal{H}_{k,2} = \mathcal{H}_0^\perp \oplus \mathcal{H}_1 \otimes \left(\bigoplus_{k=1}^{M} \mathcal{H}_{k,2} \right). \tag{50}$$

With respect to the decomposition (50), the attractor subspace $\mathrm{Attr}(\Phi)$ becomes

$$\mathrm{Attr}(\Phi) = 0 \oplus \mathcal{B}(\mathcal{H}_1) \otimes \left(\bigoplus_{k=1}^{M} \mathbb{C} \rho_k \right), \tag{51}$$

and the action of $\hat{\Phi}_P$ on an element $X \in \mathrm{Attr}(\Phi)$,

$$X = 0 \oplus \sum_{k=1}^{M} x_k \otimes \hat{\rho}_k, \tag{52}$$

with $\hat{\rho}_k = (0 \oplus 0 \oplus \cdots \oplus \rho_k \oplus 0 \oplus \cdots \oplus 0)$, is given by

$$\hat{\Phi}_P(X) = 0 \oplus \sum_{k=1}^{M} U x_{\pi(k)} U^\dagger \otimes \hat{\rho}_k = 0 \oplus \sum_{k=1}^{M} U x_k U^\dagger \otimes \hat{\rho}_{\pi^{-1}(k)}. \tag{53}$$

Now, it is easy to check that one gets $\hat{\rho}_{\pi^{-1}(k)} = \Phi_\pi(\hat{\rho}_k)$, where Φ_π is the channel on $\mathcal{B}(\bigoplus_k \mathcal{H}_{k,2})$ acting as

$$\Phi_\pi(Z) = \bigoplus_{k=1}^{M} \mathrm{tr}(Q_{\pi(k)} Z) \rho_k, \qquad Z \in \mathcal{B}\left(\bigoplus_{k=1}^{M} \mathcal{H}_{k,2}\right), \tag{54}$$

with Q_k being the projection onto $\mathcal{H}_{k,2}$.

Therefore, the asymptotic map can be written as

$$\hat{\Phi}_P(X) = 0 \oplus (\Phi_U \otimes \Phi_\pi)(X_0), \qquad X = 0 \oplus X_0 \in \mathrm{Attr}(\Phi), \tag{55}$$

with

$$\Phi_U(Y) = U Y U^\dagger, \quad Y \in \mathcal{B}(\mathcal{H}_1), \tag{56}$$

being a unitary channel.

The above decomposition, obtained for a permutation consisting of a single M-cycle, generalizes in a natural way to an arbitrary permutation with L disjoint cycles $\pi = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_L$. We have thus proved the following structure theorem for the asymptotic dynamics, which is the main result of this section.

Theorem 3.2 (Cyclic decomposition). *Let Φ be a quantum channel on $\mathcal{B}(\mathcal{H})$.*

Then, there exists a decomposition of the Hilbert space $\mathcal{H}$,

$$\mathcal{H} = \mathcal{H}_0^\perp \oplus \bigoplus_{\ell=1}^{L} \left(\mathcal{H}_1^{(\ell)} \otimes \left(\bigoplus_{k=1}^{m_\ell} \mathcal{H}_{k,2}^{(\ell)} \right) \right), \tag{57}$$

such that the asymptotic manifold of Φ has the form

$$\mathrm{Attr}(\Phi) = 0 \oplus \bigoplus_{\ell=1}^{L} \left(\mathcal{B}(\mathcal{H}_1^{(\ell)}) \otimes \left(\bigoplus_{k=1}^{m_\ell} \mathbb{C} \rho_k^{(\ell)} \right) \right), \tag{58}$$

with $\rho_k^{(\ell)}$ being positive definite density operators.

The asymptotic map of Φ, $\hat{\Phi}_P : \mathrm{Attr}(\Phi) \to \mathrm{Attr}(\Phi)$ is given by

$$\hat{\Phi}_P(X) = 0 \oplus \bigoplus_{\ell=1}^{L} (\Phi_{U_\ell} \otimes \Phi_{\pi_\ell})(X_0), \quad X = 0 \oplus X_0 \in \mathrm{Attr}(\Phi), \tag{59}$$

where

$$\Phi_{U_\ell}(Y) = U_\ell Y {U_\ell}^\dagger, \quad Y \in \mathcal{B}(\mathcal{H}_1^{(\ell)}), \tag{60}$$

are unitary channels, and

$$\Phi_{\pi_\ell}(Z) = \bigoplus_{k=1}^{m_\ell} \mathrm{tr}(Q_{\pi_\ell(k)}^{(\ell)} Z) \rho_k^{(\ell)}, \qquad Z \in \mathcal{B}\left(\bigoplus_{k=1}^{m_\ell} \mathcal{H}_{k,2}^{(\ell)}\right), \tag{61}$$

are irreducible channels with π_ℓ being an m_ℓ-cycle, and $Q_k^{(\ell)}$ being the projection onto $\mathcal{H}_{k,2}^{(\ell)}$, for $\ell = 1, \ldots, L$.

Observe that the decomposition (59) can also be obtained from [30, theorem 2]. In particular, the map Φ_{π_ℓ} corresponds to the peripheral channel of the irreducible map $\mathcal{E}_\ell$ appearing there, up to the identification of $\mathrm{Attr}(\mathcal{E}_\ell)$ with $\bigoplus_{k=1}^{m_\ell} \mathbb{C}\rho_k^{(\ell)}$, and π_ℓ with the permutation appearing in the expression (27) for the asymptotic map of $\mathcal{E}_\ell$.

Thus, the attractor subspace reads

$$\mathrm{Attr}(\Phi) = 0 \oplus \bigoplus_{\ell=1}^{L} \mathcal{B}(\mathcal{H}_1^{(\ell)}) \otimes \mathrm{Attr}(\Phi_{\pi_\ell}). \tag{62}$$

Notice that, in the absence of permutations $\pi = \mathrm{id}$, (62) reduces to the decomposition (27). A sufficient condition and a characterization for $\pi = \mathrm{id}$ will be discussed in section 5.

Armed with this new expression for the asymptotic map $\hat{\Phi}_P$, we can decompose the fixed point space $\mathrm{Fix}(\Phi)$ of Φ as

$$\mathrm{Fix}(\Phi) = 0 \oplus \bigoplus_{\ell=1}^{L} \mathrm{Fix}(\Phi_{U_\ell}) \otimes \mathbb{C}\tilde{\rho}_\ell. \tag{63}$$

Here

$$\tilde{\rho}_\ell = \frac{1}{m_\ell} \bigoplus_{k=1}^{m_\ell} \rho_k^{(\ell)} \tag{64}$$

is the unique (full-rank) fixed point of the irreducible channel Φ_{π_ℓ} and

$$\mathrm{Fix}(\Phi_{U_\ell}) = \{U_\ell\}' = \left\{X \in \mathcal{B}(\mathcal{H}_1^{(\ell)}) : [X, U_\ell] = 0\right\}, \tag{65}$$

where the square brackets denote the commutator. Note that (63) is the Schrödinger analogue of the structure of the fixed points discussed in [15, theorem 2] for faithful quantum channels on the space of trace-class operators $\mathcal{T}(\mathcal{H})$.

In addition, as already observed in [14, section 5.3], we can obtain from (59) the structure of the peripheral eigenvalues and eigenvectors of Φ. In particular, the structure of the peripheral spectrum is consistent with [10, theorem 9].

As discussed above, theorem 3.2 implies that the asymptotic map of any quantum channel can be decomposed in terms of unitary and irreducible channels. Physically, the former ones represent a coherent quantum evolution undergoing at large times after the dissipation took place, whereas the latter ones are related to a (somewhat surprising) residual classical dynamics in the asymptotic limit. As a result, the decomposition (59) of the asymptotic map allows one to factorize the classical and quantum contributions to the large-time evolution of an open quantum system.

4. Unfolding an asymptotic map

In section 2.3, we discussed the expression of the asymptotic map $\hat{\Phi}_P$ of a quantum channel Φ, that describes the asymptotic dynamics of the quantum system under investigation. An alternative decomposition (58) of the attractor manifold $\mathrm{Attr}(\Phi)$ and its corresponding asymptotic map (59) was then given in section 3.

A fundamental question to be addressed is whether, given an arbitrary subspace $\mathcal{K}$ decomposed as in (25) and an arbitrary map $\Phi_{\mathcal{K}}$ of the form (27) acting on it, there always exists a quantum channel on the whole algebra $\mathcal{B}(\mathcal{H})$ whose attractor subspace is $\mathcal{K}$ and whose asymptotic map is $\Phi_{\mathcal{K}}$. The answer is affirmative, as shown in the following theorem.

Theorem 4.1 (Unfolding an asymptotic dynamics). *Let $\mathcal{H}$ be a d-dimensional Hilbert space of the form*

$$\mathcal{H} = \mathcal{H}_0^\perp \oplus \mathcal{H}_0, \qquad \mathcal{H}_0 = \bigoplus_{k=1}^{M} \mathcal{H}_{k,1} \otimes \mathcal{H}_{k,2}, \tag{66}$$

with $\dim \mathcal{H}_{k,1} = d_k$*. Let* ρ_k *be full-rank density operators over* $\mathcal{H}_{k,2}$, $k = 1, \ldots, M$*, and consider the subspace* $\mathcal{K} \subset \mathcal{B}(\mathcal{H})$ *given by*

$$\mathcal{K} = 0 \oplus \bigoplus_{k=1}^{M} \mathcal{B}(\mathcal{H}_{k,1}) \otimes \mathbb{C}\rho_k. \tag{67}$$

Let π *be a permutation of* $\{1, \ldots, M\}$ *such that* $d_k = d_{\pi(k)}$*, and* U_k *be unitaries in* $\mathcal{B}(\mathcal{H}_{k,1})$*. Consider the map* $\Phi_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}$ *given by*

$$X = 0 \oplus \bigoplus_{k=1}^{M} x_k \otimes \rho_k \mapsto \Phi_{\mathcal{K}}(X) = 0 \oplus \bigoplus_{k=1}^{M} U_k x_{\pi(k)} U_k^\dagger \otimes \rho_k. \tag{68}$$

Then, there exists a quantum channel Φ_{E} *on* $\mathcal{B}(\mathcal{H})$ *such that*

i. $\mathrm{Attr}(\Phi_{\mathrm{E}}) = \mathcal{K}$,
ii. $(\widehat{\Phi_E})_P = \Phi_{\mathrm{E}}|_{\mathcal{K}} = \Phi_{\mathcal{K}}$.

Remark 4. This result is along the lines of a series of extension theorems of completely positive maps [40], such as the classic result by Arveson [28] and the more recent theorem by Jencova [29]. It is worth mentioning that the proof of theorem 4.1 is constructive and provides an explicit form for the quantum channel Φ_E which unfolds the given asymptotic dynamics.

Proof. We will prove the statement by providing an explicit example of a quantum channel Φ_E satisfying the conditions *i.* and *ii.*. First, let us define the pinching channel

$$\begin{aligned} \Phi_{\mathrm{pinch}} : \mathcal{B}(\mathcal{H}) &\to \mathcal{B}(\mathcal{H}_0^\perp) \oplus \bigoplus_{k=1}^{M} \mathcal{B}(\mathcal{H}_{k,1} \otimes \mathcal{H}_{k,2}), \\ Z &\mapsto Q^\perp Z Q^\perp \oplus \bigoplus_{k=1}^{M} P_k Z P_k, \end{aligned} \tag{69}$$

where P_k are the projections onto $\mathcal{H}_{k,1} \otimes \mathcal{H}_{k,2}$, while $Q^\perp$ is the projection onto $\mathcal{H}_0^\perp$. Clearly, Φ_{pinch} is a quantum channel.

Let us post-compose Φ_{pinch} with the sum of two maps. The first one is a variant of the one-state contraction channel

$$\begin{aligned} \Phi_0^\perp : \mathcal{B}(\mathcal{H}_0^\perp) \oplus \bigoplus_{k=1}^{M} \mathcal{B}(\mathcal{H}_{k,1} \otimes \mathcal{H}_{k,2}) &\to 0 \oplus \mathcal{B}(\mathcal{H}_0), \\ Z_0 \oplus \bigoplus_{k=1}^{M} Z_k &\mapsto 0 \oplus \mathrm{tr}(Z_0)\sigma, \end{aligned} \tag{70}$$

with $\sigma \in \mathcal{S}(\mathcal{H}_0)$. It is immediate to check that $\Phi_0^\perp$ is a nilpotent completely positive map of degree 2. [More generally, we could replace $\mathrm{tr}(Z_0)\sigma$ with $\Theta(Z_0)$, with $\Theta : \mathcal{B}(\mathcal{H}_0^\perp) \to \mathcal{B}(\mathcal{H}_0)$ being an arbitrary quantum channel].

The second map has the form

$$\Phi_0 : \mathcal{B}(\mathcal{H}_0^\perp) \oplus \bigoplus_{k=1}^{M} \mathcal{B}(\mathcal{H}_{k,1} \otimes \mathcal{H}_{k,2}) \to 0 \oplus \bigoplus_{k=1}^{M} \mathcal{B}(\mathcal{H}_{k,1} \otimes \mathcal{H}_{k,2}),$$
$$Z_0 \oplus \bigoplus_{k=1}^{M} Z_k \mapsto 0 \oplus \bigoplus_{k=1}^{M} U_k \mathrm{tr}_{\pi(k),2}(Z_{\pi(k)}) U_k^\dagger \otimes \rho_k, \tag{71}$$

where $\mathrm{tr}_{\pi(k),2}$ is the partial trace over $\mathcal{H}_{\pi(k),2}$.

We claim that the map

$$\Phi_{\mathrm{E}} = (\Phi_0 + \Phi_0^\perp)\Phi_{\mathrm{pinch}}, \tag{72}$$

is indeed the sought extension.

We need to check that Φ_{E} is a quantum channel, and that conditions *i*. and *ii*. hold. Clearly, Φ_{E} is trace-preserving by construction. For what concerns complete positivity, we only need to prove that Φ_0 satisfies this property. The proof is straightforward but lengthy, and it is given in appendix C.

Furthermore, observe that

$$(\Phi_{\mathrm{E}})^n(\mathcal{B}(\mathcal{H})) = \mathcal{K}, \quad \forall n \geqslant 2, \tag{73}$$

implying $\mathrm{Attr}(\Phi_{\mathrm{E}}) = \mathcal{K}$ by (11). Condition *ii*. follows easily from the definition (72) of Φ_{E}. □

Remark 5. Observe that, if $\mathcal{H}_0^\perp = 0$, then we have $\Phi_0^\perp = 0$ by (70) and the map (72) reduces to the faithful channel $\Phi_{\mathrm{E}} = \Phi_0\Phi_{\mathrm{pinch}}$, coinciding with its peripheral channel. Incidentally, this channel appears in the proof of [10, theorem 9] in the particular case $m_k = 1$, $k = 1, \ldots, M$, i.e. $\rho_k = 1$.

Therefore, we can conclude that any evolution described by equation (68) and acting on a subspace of the form (67) can always be regarded as the asymptotic dynamics of an open quantum system. From a physical perspective, the explicit construction of the channel with given asymptotic dynamics can be exploited in order to engineer specific quantum information processing in decoherence-free subspaces, obtained as asymptotic manifolds of a quantum channel.

5. On the role of permutations

One of the most intriguing features of the asymptotic dynamics (27) and (59) is the occurrence of permutations between the factors of the decomposition (24) and (57), which in general prevents the asymptotic map $\hat{\Phi}_P$ to be a unitary channel and, indeed, Hilbert–Schmidt unitary. For the sake of clarity, we give the following definition.

Definition 5.1. Let Φ be a quantum channel. Then, we say that its asymptotic map

$$\hat{\Phi}_P : \mathrm{Attr}(\Phi) \to \mathrm{Attr}(\Phi) \tag{74}$$

is unitary iff

$$\hat{\Phi}_P(X) = UXU^\dagger, \tag{75}$$

with $X \in \mathrm{Attr}(\Phi)$, and $U \in \mathcal{B}(\mathcal{H})$ being unitary, $UU^\dagger = U^\dagger U = \mathbb{I}$.

Obviously,

$$\pi = \mathrm{id} \quad \Rightarrow \quad \hat{\Phi}_P(X) = UXU^\dagger \quad \Rightarrow \quad \hat{\Phi}_P \hat{\Phi}_P^\dagger = \hat{\Phi}_P^\dagger \hat{\Phi}_P = \mathsf{1}_P, \tag{76}$$

where $\mathsf{1}_P$ is the identity on $\mathrm{Attr}(\Phi)$. When the last equality in (76) holds, we say that the asymptotic map is Hilbert–Schmidt unitary. Here, the adjoint $\hat{\Phi}_P^\dagger$ is defined via the Hilbert–Schmidt inner product,

$$\langle X | \hat{\Phi}_P(Y) \rangle_{\mathrm{HS}} = \langle \hat{\Phi}_P^\dagger(X) | Y \rangle_{\mathrm{HS}}, \quad X, Y \in \mathrm{Attr}(\Phi), \tag{77}$$

and can be easily shown to read

$$\hat{\Phi}_P^\dagger(X) = 0 \oplus \bigoplus_{k=1}^{M} U^\dagger_{\pi^{-1}(k)} x_{\pi^{-1}(k)} U_{\pi^{-1}(k)} \otimes \frac{\mathrm{tr}(\rho^2_{\pi^{-1}(k)})}{\mathrm{tr}(\rho_k^2)} \rho_k, \quad X \in \mathrm{Attr}(\Phi). \tag{78}$$

In words, the absence of permutations implies that the asymptotic map is unitary and thus Hilbert–Schmidt unitary.

However, notice that the converse of the implications (76) does not hold in general. Indeed, consider the qubit channel [14, section 3.1]:

$$\Phi(X) = \frac{1}{2}(\sigma_1 X \sigma_1 + \sigma_3 X \sigma_3), \quad X \in \mathcal{B}(\mathbb{C}^2), \tag{79}$$

where $\sigma_j, j = 1, 2, 3$ are the Pauli matrices. The peripheral eigenvalues and eigenvectors read

$$\Phi^\dagger(\mathbb{I}) = \Phi(\mathbb{I}) = \mathbb{I}, \qquad \Phi^\dagger(\sigma_2) = \Phi(\sigma_2) = -\sigma_2, \tag{80}$$

so that

$$\mathrm{Attr}(\Phi) = \{\alpha \mathbb{I} + \beta \sigma_2 \,|\, \alpha, \beta \in \mathbb{C}\}. \tag{81}$$

In the basis in which σ_2 is diagonal, $X \in \mathrm{Attr}(\Phi)$ has the form

$$X = \begin{pmatrix} \alpha + \beta & 0 \\ 0 & \alpha - \beta \end{pmatrix}, \tag{82}$$

and the action of the asymptotic map $\hat{\Phi}_P$ on it will be given by

$$\hat{\Phi}_P(X) = \begin{pmatrix} \alpha - \beta & 0 \\ 0 & \alpha + \beta \end{pmatrix}. \tag{83}$$

Thus, $\hat{\Phi}_P$ is just the flip of the two diagonal coefficients of X. However, $\hat{\Phi}_P$ is a unitary channel and, in particular,

$$\hat{\Phi}_P(X) = \sigma_3 X \sigma_3, \quad X \in \mathrm{Attr}(\Phi). \tag{84}$$

So, to sum up,

$$\hat{\Phi}_P(X) = UXU^\dagger \quad \not\Rightarrow \quad \pi = \mathrm{id}, \tag{85}$$

implying that

$$\hat{\Phi}_P^\dagger = \hat{\Phi}_P^{-1} \quad \not\Rightarrow \quad \pi = \mathrm{id}. \tag{86}$$

We conclude that equation (75) and, consequently, the Hilbert–Schmidt unitarity of $\hat{\Phi}_P$ are not sufficient conditions for the absence of permutations. Instead, a characterization is given in the following lemma.

Lemma 5.2. *Let Φ be a quantum channel, with asymptotic map $\hat{\Phi}_P$. Then $\hat{\Phi}_P$ has no permutations iff, for all positive $n \in \mathbb{N}$, $\hat{\Phi}_P$ is the* n*-fold composition of the asymptotic map of some quantum channel Φ_n. Explicitly*

$$\pi = \mathrm{id} \quad \textit{iff} \quad \hat{\Phi}_P = (\hat{\Phi}_{n,P})^n \quad \forall n \in \mathbb{N}, \quad n > 0, \tag{87}$$

where $\hat{\Phi}_{n,P}$ is the asymptotic map of a quantum channel Φ_n with attractor subspace $\mathrm{Attr}(\Phi_n) = \mathrm{Attr}(\Phi)$.

Proof. The $\Rightarrow$ implication is immediate, since we can simply set $\forall n \geqslant 1$

$$\hat{\Phi}_{n,P}(X) = 0 \oplus \bigoplus_{k=1}^{M} U_k^{1/n} x_k (U_k^{1/n})^\dagger \otimes \rho_k, \quad X \in \mathrm{Attr}(\Phi), \tag{88}$$

which can be thought as the asymptotic maps of channels Φ_n constructed through the extension theorem 4.1.

With regards to the other implication $\Leftarrow$, we can explicitly write the maps $\hat{\Phi}_{n,P}$ as

$$\hat{\Phi}_{n,P}(X) = 0 \oplus \bigoplus_{k=1}^{M} U_{k,n} x_{\pi_n(k)} U_{k,n}^\dagger \otimes \rho_k, \quad n \geqslant 1 \tag{89}$$

for some unitaries $U_{k,n}$ and permutations π_n on $\{1, \dots, M\}$. Therefore, our assumption reads

$$U_{k,n} U_{\pi_n(k),n} \dots U_{\pi_n^{n-1}(k),n} x_{\pi_n^n(k)} U_{\pi_n^{n-1}(k),n}^\dagger \dots U_{\pi_n(k),n}^\dagger U_{k,n}^\dagger = U_k x_{\pi(k)} U_k^\dagger, \tag{90}$$

for $k = 1, \dots, M$, $n \geqslant 1$. In other words,

$$\widetilde{U}_{k,n} x_{\pi_n^n(k)} \widetilde{U}_{k,n}^\dagger = x_{\pi(k)}, \quad k = 1, \dots, M, \quad n \geqslant 1, \tag{91}$$

where

$$\widetilde{U}_{k,n} = U_k^\dagger \prod_{l=0}^{n-1} U_{\pi_n^l(k),n}, \tag{92}$$

with $\pi_n^0 = \mathrm{id}$.

We claim that this implies that

$$\pi = \pi_n^n, \tag{93}$$

for all $n \geqslant 1$. Indeed, by supposing that there exists some $\bar{n} \geqslant 1$ such that $\pi \neq \pi_{\bar{n}}^{\bar{n}}$ we will arrive at a contradiction.

When $M = 2$, we have two cases: $\pi = \mathrm{id}$ and $\pi_{\bar{n}}^{\bar{n}}(k) = k + 1 \mod 2$, i.e. the transposition, and vice versa. In the first case condition (91) yields

$$\begin{aligned} \widetilde{U}_{1,\bar{n}} x_2 \widetilde{U}_{1,\bar{n}}^\dagger &= x_1, \\ \widetilde{U}_{2,\bar{n}} x_1 \widetilde{U}_{2,\bar{n}}^\dagger &= x_2. \end{aligned} \tag{94}$$

Since the elements x_1 and x_2 can be freely chosen in $\mathcal{B}(\mathcal{H}_{1,1})$ and $\mathcal{B}(\mathcal{H}_{2,1})$, respectively, equation (94) cannot be satisfied in general (take for instance $x_1 = 0$ and $x_2 \neq 0$). In the second case the system (91) is the same up to the exchange $1 \leftrightarrow 2$.

The generalization to $M > 2$ is straightforward. Indeed, suppose that $\pi(k_0) \neq \pi_{\bar{n}}^{\bar{n}}(k_0)$ for some $\bar{n} \geqslant 1$ and $k_0 \in \{1, \dots, M\}$. Then, equation (91) becomes

$$\widetilde{U}_{k_0,\bar{n}} x_{\pi_{\bar{n}}^{\bar{n}}(k_0)} \widetilde{U}_{k_0,\bar{n}}^\dagger = x_{\pi(k_0)}. \tag{95}$$

Again, since $x_{\pi(k_0)}$ and $x_{\pi_{\bar{n}}^{\bar{n}}(k_0)}$ are arbitrary, the previous equation cannot be satisfied unless $\pi(k_0) = \pi_{\bar{n}}^{\bar{n}}(k_0)$. Therefore, equation (91) becomes

$$\widetilde{U}_{k,n} x_{\pi(k)} \widetilde{U}_{k,n}^{\dagger} = x_{\pi(k)}, \quad k = 1, \ldots, M, \quad n \geqslant 1. \tag{96}$$

This means that $\widetilde{U}_{k,n}$ is an element of $\mathcal{B}(\mathcal{H}_{k,1})$ which commutes with all operators in $\mathcal{B}(\mathcal{H}_{k,1})$. Since the commutant of $\mathcal{B}(\mathcal{H}_{k,1})$ is trivial, $\widetilde{U}_{k,n}$ will be in the form

$$\widetilde{U}_{k,n} = e^{i\phi_{k,n}} \mathbb{I}_{k,1}, \quad \text{for some } \phi_{k,n} \in \mathbb{R}. \tag{97}$$

By choosing n to be the least common multiple of $1, \ldots, M$, it will be a multiple of the lengths of the cycles of the permutation π_n, so that $\pi = \mathrm{id}$. □

Now, we use the previous lemma to prove the main result of this section.

Theorem 5.3. *Let Φ be an infinitely divisible quantum channel. Then its asymptotic dynamics has no permutations, $\pi = \mathrm{id}$, and thus is unitary.*

Proof. According to lemma 5.2 it is sufficient to prove that

$$\hat{\Phi}_P = (\hat{\Phi}_{n,P})^n, \quad \forall n \geqslant 1, \tag{98}$$

where $\Phi_n^n = \Phi$, as in definition 2.1. First, observe that

$$\mathrm{Attr}(\Phi) = \mathrm{Attr}(\Phi_n), \tag{99}$$

$$\mathrm{spect}(\Phi) = \mathrm{spect}(\Phi_n)^n = \{\lambda^n \mid \lambda \in \mathrm{spect}(\Phi_n)\}, \tag{100}$$

as it can be seen from the Jordan decompositions of Φ and Φ_n (see appendix D). So, if $\mathcal{P}_{P,n}$ is the peripheral projection of Φ_n, we can compute

$$\Phi_P = \mathcal{P}_P \Phi_n^n = \mathcal{P}_{P,n} \Phi_n^n = (\mathcal{P}_{P,n} \Phi_n)^n = \Phi_{n,P}^n, \tag{101}$$

where we used $[\mathcal{P}_{P,n}, \Phi_n] = 0$, by (11). The statement follows by restricting both sides of (101) to the attractor subspace $\mathrm{Attr}(\Phi) = \mathrm{Attr}(\Phi_n)$. □

An immediate corollary to theorem 5.3 is the following [36].

Corollary 5.4. *Let Φ be a quantum channel with asymptotic map $\hat{\Phi}_P$.*

(i) If $\Phi = e^{\mathcal{L}}$, with $\mathcal{L}$ being a GKLS generator (Markovian channel), then $\hat{\Phi}_P(X) = UXU^{\dagger}$, with U unitary,
(ii) If $\Phi^2 = \Phi$ (idempotent channel), then $\hat{\Phi}_P(X) = UXU^{\dagger}$, with U unitary.

The first implication is consistent with the dynamics at large times of quantum dynamical semigroups [18, section IV], marking a difference with the discrete-time semigroups at the level of the asymptotics (see also [12, section 5.2]). Ultimately, the absence of permutations in the expression (27) for $\hat{\Phi}_P$ is linked to the divisibility of the channel.

The above results are also consistent with [36, section 4] where it was shown that the unitary Tomita–Takesaki modular dynamics on the attractor subspace $\mathrm{Attr}(\Phi)$ is the asymptotic map of Φ^M, with M denoting the least common multiple of the lengths of the cycles of the permutation. In such way, the original discrete-time dynamics $(\Phi^n)_{n\in\mathbb{N}}$ associated with Φ was coarse-grained, becoming ‘sufficiently’ Markovian for the disappearance of permutations.

There are other classes of channels Φ for which no permutations appear in their asymptotic dynamics $\hat{\Phi}_P$. For example, this is clearly true for primitive channels, see definition 3.1.

Proposition 5.5. *Let Φ be a quantum channel. If Φ is primitive, then its asymptotic dynamics has no permutations,* $\pi = \mathrm{id}$.

Remark 6. More generally, $\pi = \mathrm{id}$ also for quantum channels with one-dimensional attractor subspace.

Note that the properties of primitivity and infinite divisibility are independent, although both conditions guarantee $\pi = \mathrm{id}$. More precisely,

$$\Phi \text{ infinitely divisible} \quad \not\Rightarrow \quad \Phi \text{ primitive}, \tag{102}$$

as confirmed by the one-state contraction channel (20) with ρ non-invertible. Conversely,

$$\Phi \text{ primitive} \quad \not\Rightarrow \quad \Phi \text{ infinitely divisible}, \tag{103}$$

as it may be proved by taking into account the indivisible unital qubit channel [35]

$$\Phi(X) = \frac{1}{3}(X^T + \mathrm{tr}(X)\mathbb{I}), \quad X \in \mathcal{B}(\mathbb{C}^2), \tag{104}$$

where X^T denotes the transpose of X. It has the non-degenerate eigenvalue $\lambda = 1$ and no other peripheral eigenvalues, namely it is primitive. Thus, from (102) and (103), we can conclude that infinite divisibility and primitivity are not necessary conditions for $\pi = \mathrm{id}$.

Now, after characterizing and interpreting the absence of permutations in equation (27), we conclude this section with a last proposition which characterizes the unitarity and the Hilbert–Schmidt unitarity of an asymptotic map $\hat{\Phi}_P$.

Proposition 5.6. *Let Φ be a quantum channel with asymptotic map $\hat{\Phi}_P$ of the form (27). Then,*

(i) $\hat{\Phi}_P^\dagger = \hat{\Phi}_P^{-1}$, *namely* $\hat{\Phi}_P$ *is Hilbert–Schmidt unitary iff* $\|\rho_k\|_{\mathrm{HS}} = \|\rho_{\pi(k)}\|_{\mathrm{HS}}$, *for all* $k = 1, \ldots, M$.

(ii) $\hat{\Phi}_P(X) = UXU^\dagger$, $X \in \mathrm{Attr}(\Phi)$ *iff* $\rho_k = V_k \rho_{\pi(k)} V_k^\dagger, k = 1, \ldots M$ *for some unitaries* $V_k : \mathcal{H}_{\pi(k),2} \to \mathcal{H}_{k,2}$.

The proof of the first characterization readily follows from the definition of Hilbert–Schmidt unitarity (see also section 6), whereas the assertion *(ii)* with proof may be found in [36, theorem 3].

Remark 7. From this result it is clear that $\pi = \mathrm{id}$ is not necessary for $\hat{\Phi}_P$ to be either a unitary channel or Hilbert–Schmidt unitary, as already discussed previously, see the example of the qubit channel (79).

Remark 8. As it can be seen by using theorem 4.1

$$\|\rho_k\|_{\mathrm{HS}} = \|\rho_{\pi(k)}\|_{\mathrm{HS}} \quad \not\Rightarrow \quad \rho_k = V_k \rho_{\pi(k)} V_k^\dagger, \tag{105}$$

so that

$$\hat{\Phi}_P^\dagger = \hat{\Phi}_P^{-1} \quad \not\Rightarrow \quad \hat{\Phi}_P(X) = UXU^\dagger. \tag{106}$$

6. Asymptotics of faithful and non-faithful channels

The aim of this section is twofold. First, we will discuss several results of Novotný *et al* [8, 9] by looking in detail at the non-faithful case. Then, we will connect them to the structure theorems.

Let us start by recalling the following theorem, that expresses the irreversibility of a non-unitary open-system dynamics.

Theorem 6.1 ([11]). *Let Φ be a quantum channel on $\mathcal{B}(\mathcal{H})$. Then $\Phi^\dagger = \Phi^{-1}$ iff $\Phi(X) = UXU^\dagger$, with U unitary, i.e. Φ is a unitary channel.*

Remark 9. As already discussed in the previous section (see equation (106)), this is not true in general for the asymptotic map $\hat{\Phi}_P$.

More generally, given a quantum channel Φ and a subset $\mathcal{M} \subseteq \mathcal{S}(\mathcal{H})$, we can look for another quantum channel Φ' such that

$$\Phi'\Phi(X) = X, \quad X \in \mathcal{M}. \tag{107}$$

If such channel exists, then Φ is said to be sufficient (or reversible) with respect to the subset $\mathcal{M}$, and Φ' is called the recovery map of Φ for $\mathcal{M}$ [41]. Interestingly, this concept finds applications in quantum error correcting codes [42].

For instance, at least for faithful channels with full-rank fixed state σ, it turns out that [9]

$$\Phi\Phi^\ddagger(X) = \Phi^\ddagger\Phi(X) = X, \quad X \in \mathrm{Attr}(\Phi), \tag{108}$$

where $\Phi^\ddagger$ is the adjoint map of Φ with respect to the scalar product

$$\begin{aligned}\langle A|B\rangle_{1/2} &= \langle A|\sigma^{-1/2}B\sigma^{-1/2}\rangle_{\mathrm{HS}} = \mathrm{tr}(A^\dagger\sigma^{-1/2}B\sigma^{-1/2}) = \mathrm{tr}(\sigma^{-1/2}A^\dagger\sigma^{-1/2}B) \\ &= \mathrm{tr}(\sigma^{-1}\sigma^{1/2}A^\dagger\sigma^{-1/2}B) = \mathrm{tr}(\sigma^{-1}A^\ddagger B), \quad A,B \in \mathcal{B}(\mathcal{H}),\end{aligned} \tag{109}$$

i.e. a generalized Hilbert–Schmidt scalar product involving a modified adjoint operation on $\mathcal{B}(\mathcal{H})$ defined by

$$B^\ddagger = \sigma^{1/2}B^\dagger\sigma^{-1/2}, \quad B \in \mathcal{B}(\mathcal{H}). \tag{110}$$

The map $\Phi^\ddagger$ explicitly reads

$$\Phi^\ddagger(X) = \sum_{k=1}^{N} \sigma^{1/2}A_k^\dagger\sigma^{-1/2}X\sigma^{-1/2}A_k\sigma^{1/2}, \quad X \in \mathcal{B}(\mathcal{H}), \tag{111}$$

with $\{A_k\}_{k=1}^N$ being a system of Kraus operators of Φ. It is clear that $\Phi^\ddagger$ is a quantum channel. Note that $\Phi^\ddagger$ is a particular example of Petz's recovery map [43, 44].

Furthermore, it is worthwhile to observe that

$$\mathrm{Attr}(\Phi) = \mathrm{Attr}(\Phi^\ddagger), \tag{112}$$

as a consequence of a one-to-one correspondence between eigenspaces corresponding to peripheral eigenvalues of Φ and $\Phi^\dagger$ stated in the following proposition [9, section 4.3].

Proposition 6.2. *Let Φ be a faithful quantum channel with full-rank fixed state σ. Then, if λ is a peripheral eigenvalue, i.e. $|\lambda| = 1$,*

$$X_\lambda \in \mathrm{Ker}(\Phi - \lambda \mathbf{1}_{\mathcal{B}(\mathcal{H})}) \quad \Leftrightarrow \quad \sigma^{-1/2}X_\lambda\sigma^{-1/2} \in \mathrm{Ker}(\Phi^\dagger - \bar{\lambda}\mathbf{1}_{\mathcal{B}(\mathcal{H})}). \tag{113}$$

The following simple computation

$$\Phi^\ddagger(X_\lambda) = \sum_{k=1}^{N} \sigma^{1/2}A_k^\dagger\sigma^{-1/2}X_\lambda\sigma^{-1/2}A_k\sigma^{1/2} = \bar{\lambda}\sigma^{1/2}\sigma^{-1/2}X_\lambda\sigma^{-1/2}\sigma^{1/2} = \bar{\lambda}X_\lambda, \tag{114}$$

thanks to proposition 6.2 and the Kraus representation of $\Phi^\dagger$,

$$\Phi^\dagger(X) = \sum_{k=1}^{N} A_k^\dagger X A_k, \quad X \in \mathcal{B}(\mathcal{H}), \tag{115}$$

yields (112).

Now, let us consider the extension of the result (108) to the non-faithful case that was discussed in [8]. See also [14]. We look at it in detail and consider the fine print. Consider the map $\tilde{\Phi} : \mathcal{B}(\mathcal{H}_0) \to \mathcal{B}(\mathcal{H}_0)$ defined as

$$\Phi(X) = 0 \oplus \tilde{\Phi}(\tilde{X}), \quad X = 0 \oplus \tilde{X} \in 0 \oplus \mathcal{B}(\mathcal{H}_0). \tag{116}$$

Note that $\tilde{\Phi}$ is a map between $\mathcal{B}(\mathcal{H}_0)$ and $\mathcal{B}(\mathcal{H}_0)$ because $0 \oplus \mathcal{B}(\mathcal{H}_0)$ is invariant under Φ.

It turns out that $\tilde{\Phi}$ is a faithful quantum channel, which can be called the induced faithful channel of Φ. Incidentally, one can repeat the same procedure by considering the support of any fixed state ρ. From (116) we have

$$\mathrm{Attr}(\Phi) = 0 \oplus \mathrm{Attr}(\tilde{\Phi}), \tag{117}$$

consistently with (15). Now, we can extend to the non-faithful case the definition of $\Phi^{\ddagger}$ by inserting a maximum-rank fixed state σ in (111) and intending σ^{-1} as a generalized inverse of σ, satisfying

$$\sigma^{-1}|_{\mathcal{H}_0^{\perp}} = 0. \tag{118}$$

The map $\Phi^{\ddagger}$ is a quantum operation, i.e. a trace non-increasing completely positive map, and, again, we can define the induced faithful channel $\widetilde{\Phi^{\ddagger}}$ of $\Phi^{\ddagger}$ via

$$\Phi^{\ddagger}(X) = 0 \oplus \widetilde{\Phi^{\ddagger}}(\tilde{X}) = 0 \oplus \tilde{\Phi}^{\ddagger}(\tilde{X}), \quad X = 0 \oplus \tilde{X} \in 0 \oplus \mathcal{B}(\mathcal{H}_0), \tag{119}$$

where the last step readily follows from the definitions. Ultimately, the attractor subspace reads

$$\mathrm{Attr}(\Phi^{\ddagger}) = 0 \oplus \mathrm{Attr}(\widetilde{\Phi^{\ddagger}}) = 0 \oplus \mathrm{Attr}(\tilde{\Phi}^{\ddagger}) = 0 \oplus \mathrm{Attr}(\tilde{\Phi}) = \mathrm{Attr}(\Phi), \tag{120}$$

and (108) can be generalized to arbitrary channels.

Now, let us turn our attention to the relation between the result (108) and the structure of the asymptotic map given by theorem 2.2. To this purpose, let us define the double adjoint of the asymptotic map $\hat{\Phi}_P$, $\hat{\Phi}_P^{\ddagger} : \mathrm{Attr}(\Phi) \to \mathrm{Attr}(\Phi)$ by

$$\langle A|\hat{\Phi}_P(B)\rangle_{1/2} = \langle \hat{\Phi}_P^{\ddagger}(A)|B\rangle_{1/2}, \quad A, B \in \mathrm{Attr}(\Phi), \tag{121}$$

from which it is possible to obtain, using (27) and (35),

$$\hat{\Phi}_P^{\ddagger}(X) = 0 \oplus \bigoplus_{k=1}^{M} U_{\pi^{-1}(k)}^{\dagger} x_{\pi^{-1}(k)} U_{\pi^{-1}(k)} \otimes \rho_k, \quad X \in \mathrm{Attr}(\Phi). \tag{122}$$

Then, it is immediate to check that

$$\hat{\Phi}_P^{\ddagger}\hat{\Phi}_P = \hat{\Phi}_P\hat{\Phi}_P^{\ddagger} = \mathbb{1}_P. \tag{123}$$

So we can conclude that the asymptotic map $\hat{\Phi}_P$ is a unitary operator on the attractor subspace $\mathrm{Attr}(\Phi)$ with respect to the modified scalar product (109), depending on the maximum-rank fixed state σ of Φ.

This is in line with the fact that $\hat{\Phi}_P$ is not generally Hilbert–Schmidt unitary and, consequently, a unitary channel, as explained in the previous section. In addition, by comparing the expressions (78) and (122) for $\hat{\Phi}_P^{\dagger}$ and $\hat{\Phi}_P^{\ddagger}$ respectively, we obtain the characterization *(i)* of proposition 5.6.

Also, observe that

$$\widehat{(\Phi^{\ddagger})}_P = \hat{\Phi}_P^{\ddagger}, \tag{124}$$

or, equivalently,

$$\Phi^{\ddagger}\mathrm{Attr}(\Phi) = \mathrm{Attr}(\Phi), \tag{125}$$

as a consequence of (120) and (29).

Now, let us prove that (123) is equivalent to (108). Suppose that (108) holds. Then

$$\langle \Phi^{\ddagger}\Phi(X)|Y\rangle_{1/2} = \langle \hat{\Phi}_P(X)|\hat{\Phi}_P(Y)\rangle_{1/2} = \langle X|Y\rangle_{1/2}, \quad X, Y \in \mathrm{Attr}(\Phi), \tag{126}$$

which gives (123), if one takes into account the invertibility of $\hat{\Phi}_P$. The converse implication follows immediately from (124)

$$\Phi^{\ddagger}\Phi(X) = (\widehat{\Phi^{\ddagger}})_P\hat{\Phi}_P(X) = \hat{\Phi}_P^{\ddagger}\hat{\Phi}_P(X) = X, \quad X \in \mathrm{Attr}(\Phi), \tag{127}$$

and analogously for the other equality. To sum up, as anticipated, theorem 2.2 implies (108).

Moreover, from theorems 2.2 and 6.2, it follows that, in the faithful case,

$$\mathrm{Attr}(\Phi^{\dagger}) = \sigma^{-1/2}\mathrm{Attr}(\Phi)\sigma^{-1/2} = \bigoplus_{k=1}^{M}\mathcal{B}(\mathcal{H}_{k,1})\otimes\mathbb{I}_{k,2}, \tag{128}$$

as observed in remark 2.

Incidentally, notice that in the non-faithful case $\mathrm{Attr}(\Phi^{\dagger})$ is an algebra with respect to the star product [45–47]

$$A \star B = \mathcal{P}_P^{\dagger}(AB), \tag{129}$$

where $\mathcal{P}_P^{\dagger}$ is the peripheral projection of $\Phi^{\dagger}$ as a consequence of (11). The product (129) reduces to the composition one if Φ is faithful, as it can be readily seen by taking the adjoint of (30). For related observations see [14, 38, 39, proposition 3].

Proposition 6.2 may be proved from theorem 2.2. The adjoint map $\Phi^{\dagger}$ of a faithful channel Φ acts on its attractor subspace $\mathrm{Attr}(\Phi^{\dagger})$ as follows

$$\Phi^{\dagger}(X) = \bigoplus_{k=1}^{M} U_{\pi^{-1}(k)}^{\dagger}x_{\pi^{-1}(k)}U_{\pi^{-1}(k)}\otimes\mathbb{I}_{k,2}, \tag{130}$$

where $X \in \mathrm{Attr}(\Phi^{\dagger})$ is of the form (31). Now, the eigenvalue equation for Φ

$$\Phi(X) = \lambda X, \quad |\lambda| = 1, \tag{131}$$

namely the condition

$$U_k x_{\pi(k)} U_k^{\dagger} = \lambda x_k, \quad k = 1, \ldots, M, \tag{132}$$

is equivalent to require that

$$U_{\pi^{-1}(k)}^{\dagger}\sigma_{\pi^{-1}(k)}^{-1/2}x_{\pi^{-1}(k)}\sigma_{\pi^{-1}(k)}^{-1/2}U_{\pi^{-1}(k)} = \bar{\lambda}\sigma_k^{-1/2}x_k\sigma_k^{-1/2}, \quad k = 1, \ldots, M, \tag{133}$$

thanks to (34). In conclusion, (133) is an explicit way of writing

$$\Phi^{\dagger}(\sigma^{-1/2}X\sigma^{-1/2}) = \bar{\lambda}\sigma^{-1/2}X\sigma^{-1/2}, \tag{134}$$

i.e. the statement. Finally, note that similar bijective mappings between the eigenspaces $\mathrm{Ker}(\Phi - \lambda 1_{\mathcal{B}(\mathcal{H})})$ and $\mathrm{Ker}(\Phi^{\dagger} - \bar{\lambda}1_{\mathcal{B}(\mathcal{H})})$, established in [8, theorem 3.1] and [9, theorem 2] (see also [48, lemma 3]), may be proved in the same way by means of theorem 2.2.

7. Conclusions

In this Article we have explored several aspects of the asymptotic dynamics of quantum channels of finite-dimensional systems. The starting point of our findings is a structure theorem that provides an expression for the asymptotic map. It is the restriction of the quantum channel generating the discrete-time dynamics to the attractor subspace, where the evolution takes place at large times.

Four main goals were achieved in this paper. First, a structure theorem of the asymptotic map in terms of disjoint cyclic permutations was given (section 3). Second, a quantum channel unfolding a given asymptotic map was constructed, thus finding the converse to the structure theorem (section 4). Third, the role of permutations in the asymptotic dynamics was understood and characterized (section 5). Fourth, several properties of the asymptotics of quantum channels were derived from the structure theorems of the asymptotic map (section 6).

Regarding the first goal, the structure theorem 3.2 provides a decomposition of any asymptotic evolution of an open system in terms of unitary and irreducible channels. Physically, such building blocks respectively represent the purely quantum and classical contributions to the large-time dynamics. Also, the decomposition (62) of the attractor subspace $\mathrm{Attr}(\Phi)$ clearly identifies the decoherence-free subspaces [2] of the dynamics, where a purely unitary evolution occurs. For this reason, they can play a crucial role in quantum computing [31] and quantum error correction [14] tasks.

Also, besides its mathematical interest, the unfolding theorem discussed in section 4 can be useful for quantum information processing, by explicitly designing a quantum channel extending a given asymptotic map.

For what concerns the third goal, the asymptotic dynamics is not purely unitary, since partial permutations between the factors of the attractor manifold may occur. However, the absence of permutations does not characterize the unitarity of the asymptotic map, but a weaker divisibility property, as stated in proposition 5.2. An almost immediate consequence of this result is that the asymptotic dynamics of an infinitely divisible channel, such as a Markovian or an idempotent channel, has no permutations. Again, understanding the presence/absence of permutations in the large-time limit of a given quantum dynamics is relevant for applications in quantum information processing.

Therefore, given a discrete-time semigroup $(\Phi^n)_{n\in\mathbb{N}}$ associated with a quantum channel Φ, the asymptotic map will be unitary if we look at the coarse-grained dynamics generated by the quantum channel Φ^M, with M being the least common multiple of the lengths of the cycles of the permutation [36]. To conclude, the occurrence of permutations is related to long-term non-Markovian effects, making the asymptotic dynamics generally non-unitary. Notice also that permutations can take place only in the presence of a degenerate attractor subspace (see remark 6). It is also worthwhile to observe that, as revealed by equation (123), permutations do not even guarantee the Hilbert–Schmidt unitarity of the asymptotic map, physically implying that the purity of the asymptotic states is not generally conserved under the dynamics.

These results shed light on the crucial role of the asymptotic dynamics of a quantum channel in quantum technology applications. With this respect it would be interesting to better understand its relation with entanglement, a fundamental resource in quantum information processes, and in particular its role in the class of eventually entanglement breaking [27] and entanglement saving [26] channels. Such maps generalize entanglement breaking channels, deeply studied in the last twenty years [49–51] because of their detrimental effects in quantum communication protocols [52, 53].

Data availability statement

No new data were created or analyzed in this study.

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Appendix A. Proof of equation (16)

We want to prove that

$$\operatorname{Ran}\mathcal{P}_P(\mathbb{I}) = \operatorname{supp}\mathcal{P}_P(\mathbb{I}) = \mathcal{H}_0. \tag{A.1}$$

We start by proving that

$$Q^\perp \mathcal{P}_P(\mathbb{I}) Q^\perp = 0, \tag{A.2}$$

where $Q^\perp$ is the projection onto the orthogonal complement $\mathcal{H}_0^\perp$ of $\mathcal{H}_0$. Since $Q^\perp \mathcal{P}_P(\mathbb{I}) Q^\perp \geqslant 0$, it is sufficient to prove that

$$\operatorname{tr}(Q^\perp \mathcal{P}_P(\mathbb{I}) Q^\perp) = 0. \tag{A.3}$$

We know that, by using (10),

$$Q^\perp \mathcal{P}(\mathbb{I}) Q^\perp = 0 \quad \Leftrightarrow \quad \lim_{N\to\infty} \frac{1}{N} Q^\perp \sum_{n=1}^{N} \Phi^n(\mathbb{I}) Q^\perp = 0, \tag{A.4}$$

which implies that

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{tr}\left(Q^\perp \sum_{n=1}^{N} \Phi^n(\mathbb{I}) Q^\perp \right) = 0. \tag{A.5}$$

That being said, compute for a given $n \in \mathbb{N}$,

$$\begin{aligned}
\operatorname{tr}(Q^\perp \Phi^{n+1}(\mathbb{I}) Q^\perp) &= \operatorname{tr}(Q^\perp \Phi((Q+Q^\perp)\Phi^n(\mathbb{I})(Q+Q^\perp))Q^\perp) \\
&= \operatorname{tr}(Q^\perp \Phi(Q\Phi^n(\mathbb{I})Q^\perp + Q^\perp\Phi^n(\mathbb{I})Q + Q^\perp\Phi^n(\mathbb{I})Q^\perp)Q^\perp) + \operatorname{tr}(Q^\perp\Phi(Q\Phi^n(\mathbb{I})Q)Q^\perp) \\
&\leqslant \operatorname{tr}(Q\Phi^n(\mathbb{I})Q^\perp + Q^\perp\Phi^n(\mathbb{I})Q) + \operatorname{tr}(Q^\perp\Phi^n(\mathbb{I})Q^\perp) = \operatorname{tr}(Q^\perp\Phi^n(\mathbb{I})Q^\perp),
\end{aligned} \tag{A.6}$$

where the inequality arises from the fact that the map $\Phi_{Q^\perp}(X) = Q^\perp X Q^\perp$ is trace non-increasing and $0 \oplus \mathcal{B}(\mathcal{H}_0)$ is an invariant subspace for Φ (see equation (116)). So we can write

$$\frac{1}{n_i} \operatorname{tr}\left(Q^\perp \sum_{n=1}^{n_i} \Phi^n(\mathbb{I}) Q^\perp \right) \geqslant \frac{1}{n_i} n_i \operatorname{tr}(Q^\perp \Phi^{n_i}(\mathbb{I}) Q^\perp) \geqslant 0, \tag{A.7}$$

for the increasing subsequence $\{n_i\}_{i\in\mathbb{N}}$ in (11), yielding (A.3) by squeeze theorem and, consequently, (A.2). Finally, from equation (A.2) we have, for $\phi \in \mathcal{H}$

$$\langle\phi|Q^\perp \mathcal{P}_P(\mathbb{I}) Q^\perp|\phi\rangle = ||\mathcal{P}_P(\mathbb{I})^{1/2} Q^\perp \phi||^2 = 0, \tag{A.8}$$

implying $\mathcal{P}_P(\mathbb{I})^{1/2} Q^\perp = 0$. Then, by post-composing with $\mathcal{P}_P(\mathbb{I})^{1/2}$, we get $\mathcal{P}_P(\mathbb{I}) Q^\perp = 0$, and, by taking the Hermitian conjugate, $Q^\perp \mathcal{P}_P(\mathbb{I}) = 0$, i.e. (A.1).

Appendix B. Proof of equation (17)

By (8) it is straightforward to prove that the equality $\mathcal{P}_P(\mathbb{I}) = \mathcal{P}(\mathbb{I})$ is equivalent to

$$\Phi(\mathcal{P}_P(\mathbb{I})) = \mathcal{P}_P(\mathbb{I}), \tag{A.9}$$

which explicitly reads, in the faithful case,

$$\bigoplus_{k=1}^{M} m_{\pi(k)} \mathbb{I}_{k,1} \otimes \rho_k = \bigoplus_{k=1}^{M} m_k \mathbb{I}_{k,1} \otimes \rho_k \quad \Leftrightarrow \quad m_{\pi(k)} = m_k, \quad k = 1, \ldots, M. \tag{A.10}$$

However, it is always possile to construct a channel for which (A.10) does not hold through theorem 4.1.

Appendix C. Complete positivity of the map Φ_0 in (71)

By definition of complete positivity, let us consider a D-dimensional Hilbert space $\mathcal{H}'$, and check that the map

$$\Phi_0 \otimes \mathbf{1}_{\mathcal{B}(\mathcal{H}')} : \mathcal{A} \otimes \mathcal{B}(\mathcal{H}') \mapsto \mathcal{C} \otimes \mathcal{B}(\mathcal{H}') \tag{A.11}$$

is positive, with $\mathcal{A}$ and $\mathcal{C}$ denoting the algebras

$$\mathcal{A} = \mathcal{B}(\mathcal{H}_0^{\perp}) \oplus \bigoplus_{k=1}^{M} \mathcal{B}(\mathcal{H}_{k,1} \otimes \mathcal{H}_{k,2}), \tag{A.12}$$

$$\mathcal{C} = 0 \oplus \bigoplus_{k=1}^{M} \mathcal{B}(\mathcal{H}_{k,1} \otimes \mathcal{H}_{k,2}). \tag{A.13}$$

Let us take a positive operator Z over $\mathcal{A} \otimes \mathcal{B}(\mathcal{H}')$, and write it explicitly as

$$Z = \sum_{\alpha,\beta=1}^{D} Z_{\alpha\beta} \otimes |\alpha\rangle\langle\beta|, \tag{A.14}$$

with $Z_{\alpha\beta} \in \mathcal{A}$ and $\{|\alpha\rangle\}_{\alpha=1}^{D}$ an orthonormal basis of $\mathcal{H}'$. After expressing the vector state $|\phi\rangle \in \mathcal{H} \otimes \mathcal{H}'$ as

$$|\phi\rangle = \sum_{\alpha=1}^{D} |\phi_\alpha\rangle \otimes |\alpha\rangle = \sum_{\alpha=1}^{D} \bigoplus_{k=0}^{M} |\phi_\alpha^{(k)}\rangle \otimes |\alpha\rangle, \tag{A.15}$$

for some $|\phi_\alpha\rangle = \bigoplus_{k=0}^{M} |\phi_\alpha^{(k)}\rangle \in \mathcal{H}$, the positivity of Z reads

$$\langle\phi|Z|\phi\rangle = \sum_{\alpha,\beta=1}^{D} \langle\phi_\alpha|Z_{\alpha\beta}|\phi_\beta\rangle = \sum_{\alpha,\beta=1}^{D} \sum_{k=0}^{M} \langle\phi_\alpha^{(k)}|Z_{\alpha\beta}^{(k)}|\phi_\beta^{(k)}\rangle \geqslant 0. \tag{A.16}$$

Analogously

$$\begin{aligned} \langle\phi|(\Phi_0 \otimes \mathbf{1}_{\mathcal{B}(\mathcal{H}')})(Z)|\phi\rangle &= \sum_{\alpha,\beta=1}^{D} \langle\phi_\alpha|\Phi_0(Z_{\alpha\beta})|\phi_\beta\rangle \\ &= \sum_{\alpha,\beta=1}^{D} \sum_{k=1}^{M} \langle\phi_\alpha^{(k)}|U_k \mathrm{tr}_{\pi(k),2}(Z_{\alpha\beta}^{(\pi(k))}) U_k^{\dagger} \otimes \rho_k|\phi_\beta^{(k)}\rangle. \end{aligned} \tag{A.17}$$

Now, let

$$\rho_k = \sum_{\ell=1}^{m_k} \lambda_\ell^{(k)} |\ell^{(k)}\rangle\langle\ell^{(k)}|, \tag{A.18}$$

$$|\phi_\alpha^{(k)}\rangle = \sum_{\ell=1}^{m_k} |\phi_{\alpha\ell}^{(k)}\rangle \otimes |\ell^{(k)}\rangle \tag{A.19}$$

be the spectral decomposition of ρ_k and the tensor product decomposition of $|\phi_\alpha^{(k)}\rangle$ in terms of the basis $|\ell^{(k)}\rangle$ of eigenvectors of ρ_k. Therefore, the generic term in the last sum in (A.17) becomes

$$\begin{aligned}
&\langle\phi_\alpha^{(k)}|U_k \mathrm{tr}_{\pi(k),2}(Z_{\alpha\beta}^{(\pi(k))})U_k^\dagger \otimes \rho_k|\phi_\beta^{(k)}\rangle \\
&\quad = \sum_{\ell,\ell'=1}^{m_k} \langle\phi_{\alpha\ell}^{(k)} \otimes \ell^{(k)}|U_k \mathrm{tr}_{\pi(k),2}(Z_{\alpha\beta}^{(\pi(k))})U_k^\dagger \otimes \rho_k|\phi_{\beta\ell'}^{(k)} \otimes \ell'^{(k)}\rangle \\
&\quad = \sum_{\ell=1}^{m_k} \lambda_\ell^{(k)} \langle\phi_{\alpha\ell}^{(k)}|U_k \mathrm{tr}_{\pi(k),2}(Z_{\alpha\beta}^{(\pi(k))})U_k^\dagger|\phi_{\beta\ell}^{(k)}\rangle \\
&\quad = \sum_{\ell,\ell'=1}^{m_k} \langle\psi_{\alpha\ell}^{(k)} \otimes \ell'^{(k)}|Z_{\alpha\beta}^{(\pi(k))}|\psi_{\beta\ell}^{(k)} \otimes \ell'^{(k)}\rangle,
\end{aligned} \tag{A.20}$$

with

$$|\psi_{\alpha\ell}^{(k)}\rangle = \sqrt{\lambda_\ell^{(k)}}\, U_k^\dagger |\phi_{\alpha\ell}^{(k)}\rangle. \tag{A.21}$$

Therefore,

$$\langle\phi|(\Phi_0 \otimes \mathbf{1}_{\mathcal{B}(\mathcal{H}')})(Z)|\phi\rangle = \sum_{\ell,\ell'=1}^{m_k} \sum_{k=1}^{M} \sum_{\alpha,\beta=1}^{D} \langle\psi_{\alpha\ell}^{(k)} \otimes \ell'^{(k)}|Z_{\alpha\beta}^{(\pi(k))}|\psi_{\beta\ell}^{(k)} \otimes \ell'^{(k)}\rangle \geqslant 0, \tag{A.22}$$

as a consequence of (A.16).

Appendix D. Proof of equations (99)–(100)

Let us write the Jordan decompositions of Φ and Φ_n

$$\Phi = \sum_{k=1}^{N} (\lambda_k \mathcal{P}_k + \mathcal{N}_k), \tag{A.23}$$

$$\Phi_n = \sum_{k=1}^{N'} (\lambda_{k,n} \mathcal{P}_{k,n} + \mathcal{N}_{k,n}), \tag{A.24}$$

where $\mathcal{P}_k$ and $\mathcal{N}_k$ ($\mathcal{P}_{k,n}$ and $\mathcal{N}_{k,n}$) are the eigenprojection and eigennilpotent of Φ (Φ_n) corresponding to the kth eigenvalue λ_k ($\lambda_{k,n}$). Since $\Phi = \Phi_n^n$, we obtain

$$\sum_{k=1}^{N} (\lambda_k \mathcal{P}_k + \mathcal{N}_k) = \sum_{k=1}^{N'} \left(\lambda_{k,n}^n \mathcal{P}_{k,n} + \sum_{m=0}^{n-1} \binom{n}{m} \lambda_{k,n}^m \mathcal{N}_{k,n}^{n-m} \right). \tag{A.25}$$

It is straightforward to prove that the right-hand side of (A.25) is an alternative Jordan decomposition of Φ, so from the uniqueness of the Jordan decomposition, we have

$$\lambda_k = \lambda_{k,n}^n, \tag{A.26}$$

$$\mathcal{P}_P = \mathcal{P}_{P,n}, \tag{A.27}$$

implying equations (99) and (100) of the paper.

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