The quantum moment problem for a classical random variable and a classification of interacting Fock spaces

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Abstract

The fact that any classical random variable with all moments has a **canonical quantum decomposition** allows to associate to it a family of **quantum moments**. On the other hand a classical random variable may have several inequivalent quantum decompositions, which lead to different quantum moments. Even in the simplest Central Limit Theorems (CLT), i.e. those of Bernoulli type, there are examples in which the corresponding quantum moments converge to the canonical quantum moments of the associated classical random variable, and examples in which this is not the case. This poses the problem to find a constructive criterium that characterizes the quantum moments associated to the canonical quantum decomposition with respect to the other ones. Theorem 3 of the present paper provides such a criterium. Theorem 5 gives a sufficient condition which reduces the problem to the verification of a 4–th moment conditions (see (91)) which is simpler than the verification of the necessary and sufficient conditions of Theorem 3. Theorem 3 naturally leads to a classification of Interacting Fock Spaces (IFS) into three types. We construct examples showing that all these possibilities can effectively take place.

Keywords: Interacting Fock Spaces, quantum decomposition of a classical random variable, Bernoulli type Central Limit Theorems

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1 Statement of the quantum moment problem for a classical real valued random variable

The literature devoted to various non–commutative extensions of the classical moment problem for real or vector valued random variables has a relatively long history (see for example [22], [18], [12] and bibliography in these papers). In a general setting, the problem can be formulated as follows: given a $*$–algebra $\mathcal{A}$ and a self–adjoint set $G \subset \mathcal{A}$ of algebraically free generators (i.e. not satisfying non–trivial polynomial identities), a state φ on $\mathcal{A}$ defines a map

$$\kappa : (g_1, \dots, g_n) \in \bigcup_{n\in\mathbb{N}} G^n \to \kappa_{(g_1,\dots,g_n)} := \varphi(g_1 \cdots g_n) \in \mathbb{C} \tag{1}$$

The problem is to characterize those maps $\kappa : \bigcup_{n\in\mathbb{N}} G^n \to \mathbb{C}$ such that there exists a state φ on $\mathcal{A}$ satisfying (1).
The **quantum moment problem for a classical real valued random variable**, discussed in the present paper, **is different** and, in order to formulate it, we need to introduce some notations.

In the following, all probability measures on $\mathbb{R}$ are referred to the Borel $\sigma-$algebra and supposed to have moments of any order. A classical real valued random variable is identified to its probability distribution and two measures on $\mathbb{R}$ (real valued classical random variables) with the same moments are identified. A real valued classical random variable is called **symmetric** if all its odd moments vanish. We will freely use some basic facts on orthogonal polynomials in one indeterminate and the connection of this theory with that of 1–mode interacting Fock spaces (IFS). For a clear and succinct exposition of these facts, we refer to Hora and Obata's monograph [19].

To any probability measure μ on $\mathbb{R}$ with all moments, it is canonically associated a sequence $\omega := \{\omega_n\}_{n\in\mathbb{N}} =: (\omega_n)$ of positive real numbers satisfying the condition:

$$\omega_0 = 0 \qquad ; \qquad \omega_n = 0 \ \Rightarrow \ \omega_k = 0 \quad , \quad \forall k \geq n \in \mathbb{N} \setminus \{0\} \tag{2}$$

Conversely, by Favard Lemma, any positive sequence (ω_n), satisfying (2) defines a probability measure μ on $\mathbb{R}$ with all moments. In the following any positive sequence (ω_n), satisfying (2) will be called **a principal Jacobi sequence** and we will denote

$$n_\omega := \begin{cases} \min\{k : \omega_k = 0\} \ , & \text{if } \{k : \omega_k = 0\} \neq \emptyset \\ \infty \ , & \text{if } \{k : \omega_k = 0\} = \emptyset \end{cases} \tag{3}$$

Property (2) also characterizes those sequences (λ_n) which define a 1–**mode interacting Fock space** (1–MIFS, see section 2 below for definition, notations and main properties) through the prescription

$$\Gamma(\omega) := \bigoplus_{n\in\mathbb{N}} \left(\mathbb{C} \cdot \tilde{\Phi}_n, \langle \ \cdot \ , \ \cdot \ \rangle_n\right) \quad ; \quad \|\Phi_n\|_n^2 := \prod_{k=1}^{n} \omega_k =: \omega_n! \tag{4}$$

where the sum is orthogonal and no completion is taken, so that $\Gamma(\omega)$ is a pre–Hilbert space (i.e. a vector space endowed with a positive sesqui–linear form) and (Φ_n) is a Hamel basis of $\Gamma(\omega)$.
The orthogonality of the Φ_n, implies that there exists a unique linear operator a^+ on $\Gamma(\omega)$, called the **creation operator**, satisfying

$$a^+\Phi_n := \Phi_{n+1} \qquad ; \qquad \forall n \in \mathbb{N}$$

and condition (2) is equivalent to the existence of an operator, denoted a and called **the annihilation operator**, satisfying the adjointness condition:

$$\langle a\xi, \eta\rangle = \langle \xi_0, a^+\eta\rangle \qquad ; \qquad \forall \xi, \eta \in \Gamma(\omega) \tag{5}$$

and the Fock condition:

$$a\Phi_0 = 0$$

The operator a is uniquely defined by condition (5) up to vectors of zero norm. The decomposition $\Gamma(\omega) = \bigoplus_{n\in\mathbb{N}} \mathbb{C} \cdot a^{+n}\Phi_0$ is called the orthogonal gradation of $\Gamma(\omega)$. Notice that any gradation preserving linear operator $a^0 : \Gamma(\omega) \to \Gamma(\omega)$ is uniquely determined by a sequence of complex numbers (α_n) through the identity

$$a^0\Phi_n = \alpha_n\Phi_n \qquad ; \qquad \forall n \in \mathbb{N}$$

and the α_n are real if a^0 is self–adjoint, in the sense that (5) holds with a^0 replacing both a and a^+.

Definition 1 Let X be a real valued classical random variable with moment sequence $(M_n) \subset \mathbb{R}$. A **quantum decomposition of** X is a quadruple $(\mathcal{H}, L^+, L^0, \Phi)$ such that:
1) $\mathcal{H}$ is a pre–Hilbert space;
2) L^+ and L^0 are linear operators on $\mathcal{H}$ such that L^0 is self–adjoint and L^+ has an adjoint denoted L^-;
3) For each $n \in \mathbb{N}$ and $\varepsilon = (\varepsilon(1), \dots, \varepsilon(n)) \in \{-1, 0, 1\}^n$, the vector Φ is in the domain of $L^{\varepsilon(1)} \dots L^{\varepsilon(n)}$, where hereinafter, if B^+ is an adjointable operator on a pre–Hilbert space $\mathcal{K}$ and B is any adjoint of B^+, for any $\varepsilon \in \{-1, 1\}$, we define

$$B^\varepsilon := \begin{cases} B^+ \ , \ \text{if } \varepsilon = 1 \\ B \ , \ \text{if } \varepsilon = -1 \end{cases} \tag{6}$$

4) For each $n \in \mathbb{N}$

$$M_n = \left\langle \Phi, \left(L^+ + L^0 + L^-\right)^n \Phi \right\rangle \tag{7}$$

The operators L^+, L^-, L^0 are called the CAP operators of the given quantum decomposition.

The **quantum moments**, associated to the quantum decomposition $(\mathcal{H}, L^+, L^0, \Phi)$ of X, are by definition

$$M_n\left(\varepsilon\right) = \left\langle \Phi, L^{\varepsilon(1)} \dots L^{\varepsilon(n)} \Phi \right\rangle \qquad ; \qquad \forall n \in \mathbb{N},\ \varepsilon \in \{-1,0,1\}^n \tag{8}$$

Two quantum decompositions of X are called **isomorphic**, if they have the same quantum moments.

If, in addition $L\Phi = 0$, we speak of a **Fock decomposition of** X.

Remark 1 It is known that a sequence (M_n) of real numbers is the moment sequence of a classical real valued random variable if and only if the kernel $(m,n) \mapsto M(m+n)$ is positive definite. The general quantum moment problem for a classical random variable is stated similarly: given a family of complex numbers $(M_n\left(\varepsilon\right))$ with $n \in \mathbb{N}$ and $\varepsilon \in \{-1,0,1\}^n$ give criteria that guarantee the existence of s a quadruple $(\mathcal{H}, L^+, L^0, \Phi)$ satisfying the conditions of Definition 1.

Denote $G := \{+,0,-\}$ and define an involution on this set by $0 = 0^*$, $(+)^* = -$. The set $\mathcal{G} := \bigcup_{n\in\mathbb{N}} G^n$, $G^0 := \emptyset$ is a $*$–semi–group with involution

$$\delta^* = (\delta_1, \dots, \delta_n)^* \qquad ; \qquad \emptyset^* := \emptyset$$

identity given by $\emptyset$ and composition law

$$(\delta_1, \dots, \delta_n)(\varepsilon_1, \dots, \varepsilon_n) := (\delta_1, \dots, \delta_n, \varepsilon_1, \dots, \varepsilon_n)$$

If $(M_n\left(\varepsilon\right))$ is the family of quantum moments of a quantum decomposition of X, it defines a positive–definite (PD) complex valued normalized function M on $\mathcal{G}$, in the sense that the kernel $K(\delta, \varepsilon) := M(\delta^*\varepsilon)$ is PD and $M(\emptyset) = 1$. Moreover the quantum moments define a sequence of classical moments (namely those of X) through the right hand side of formula (61) (with $\{-1,1\}^{2n}_+$ replaced by $\{-1,0,1\}^{2n}$). These two necessary conditions characterize the families of quantum moments in the sense that, if $(M_n\left(\varepsilon\right))$ is a family satisfying the above two conditions, denoting M the associated

PD function, K the associated kernel, $(\mathcal{H}, \langle \,\cdot\, , \,\cdot\, \rangle, v)$ the Kolmogorov representation of K ($K(\delta, \varepsilon) = \langle v_\delta, v_\varepsilon \rangle$) and π the associated $*$–representation of $\mathcal{G}$ ($\pi(\varepsilon) v_\delta := v_{\varepsilon\delta} \in \mathcal{H}$), and defining

$$\Phi := v_\emptyset \quad ; \quad L_M(\varepsilon) := \pi(\varepsilon) \quad ; \quad \varepsilon \in G$$

one has, for $\varepsilon := (\varepsilon_1, \dots, \varepsilon_n)$:

$$v_\varepsilon = L_M(\varepsilon_1) \dots L_M(\varepsilon_n)\Phi \quad ; \quad L_M(0) = L_m(0)^* \quad ; \quad L_M(+)^* = L_M(-)$$

hence the Φ–moments of the classical random variable

$$X := L_M(+) + L_M(0) + L_M(-)$$

are those defined by the family $(M_n\,(\varepsilon))$.
In conclusion, the above two conditions are necessary and sufficient for a family $(M_n\,(\varepsilon))$ to be the quantum moments of a classical random variable. A family with this property will be called a **quantum moment function**. Moreover the above discussion shows that if $(\mathcal{K}, L^+, L^0, \Psi)$ is another quantum decomposition of a classical random variable X, then denoting $\mathcal{K}_0$ the Ψ–cyclic space of the polynomial algebra in the indeterminates L^+, L^0, L^-, there exists a unitary isomorphism $U : \mathcal{H} \to \mathcal{K}_0$ which intertwines the states and the CAP operators of the two quantum decompositions, i.e.

$$U\Phi = \Psi \quad ; \quad UL_M(\varepsilon)U^* = L(\varepsilon) \qquad ; \qquad \forall \varepsilon \in \{+, 0, -\} \tag{9}$$

where $(\mathcal{H}, L_M(+), L_M(0), \Phi)$ is the decomposition constructed above. This shows that there is a one–to–one correspondence between quantum moment functions that define the same sequence of classical moments and equivalence classes (for the equivalence relation introduced above) of quantum decompositions of the classical random variable defined by this sequence.
Finally the above construction works for any set G with an involution and the classical moment problem corresponds to the case $G = \{0\}$ with identity as involution and with $\mathcal{G}$ identified to $\mathbb{N}$.

In quantum probability the moment problem mainly arises in two contexts. One is when one tries to prove that a linear functional on a $*$–Lie algebra defines a family of quantum moments. Here the main problem is to prove positive–definitness. Another is given by quantum central limit (QCL) theorems based on moments which produce a moment function on a $*$–semi–group $\mathcal{G}$ that depends on the conditions of the theorem (for Bernoulli type CLT $\mathcal{G} := \bigcup_{n\in\mathbb{N}}\{+, 0, -\}^n$) or, in the symmetric case, $\mathcal{G} := \bigcup_{n\in\mathbb{N}}\{+, -\}^n$). In this case one knows that it is a moment function because it is a limit of functions with this property and the problem is to find a concrete functional representation of the limit space given abstractly by the arguments discussed above. This is in general a difficult problem for which at the moment no general criterium is known. In this paper we discuss this problem in the case of a special but important quantum decomposition..

Definition 2 The quantum decomposition of X, defined by Theorem 1 below in the symmetric case (see [8] for the general case), is called **the canonical quantum decomposition**, or simply the **quantum decomposition**, of X. The quantum moments (8), associated to the canonical quantum decomposition of X, are called **canonical quantum moments**.

The **quantum moment problem for a classical symmetric random variable** X can now be stated as follows:
Find necessary and sufficient conditions on a given a family of complex numbers $(M_n(\varepsilon))$ *(*$n \in \mathbb{N}, \varepsilon \in \{+, -\}^n$*) that guarantee the existence of a classical real valued symmetric random variable* X *such that* $(M_n(\varepsilon))$ *is the family of canonical quantum moments of* X *or equivalently that the cyclic representation associated to the moments and the canonical one of* X *are equivalent in the sense of* (9).

A necessary condition for the family $(M_n(\varepsilon))$ to come from a quantum decomposition of a classical symmetric random variable is that this family defines, through the right hand side of formula (61), a family M_{2n} of classical moments (this is condition (C1) of Theorem 3). If this condition is satisfied, the classical random variable X is uniquely determined, therefore the problem is equivalent to the following 2 sub–problems:
1) To express the principal Jacobi sequence of X in terms of its moments.
2) To express the quantum moments of X in terms of its principal Jacobi sequence.
With this information the solution of the problem is obtained by comparing the given quantum moments with those obtained in step 2) above applied to the Jacobi sequence produced in step 1).
Problem 1) is equivalent to prove the converse of the Accardi–Bozeiko formula

[8], that expresses the classical moments in terms of the Jacobi parameters (also in the non–symmetric case), and it is solved in section 3.1 (see formula (53)). Problem 2) is equivalent to an extension of the above mentioned formula to the quantum moments and is solved in section 3 in the symmetric case (see formula (52)). However formula (53), giving the solution of Problem 1), is complex and not easy to use in practical applications. So the problem arises to find a more explicit set of conditions on the family $(M_n(\varepsilon))$ that are equivalent to a positive solution of the problem. This is done in Theorem 3 in section 4 where condition (C1) is assumed but **positive definitness is a consequence of the theorem**. For example condition (C2) in this theorem, which does not depend on the principal Jacobi sequence, can be used as a sufficient condition for a negative answer to the problem.
As already said, in CLT positive definitness is guaranteed a priori and this allows to apply the abstract construction of Remark 1 that guarantees the existence of a quantum decomposition $(\mathcal{H}, L^+, \Phi)$ of X. This allows to simplify condition (C3) of Theorem 3 (see Corollary 2).

In section 2 we recall some basic properties of 1–mode interacting Fock spaces and of the commutation relations canonically associated to them. This gives a general method to construct deformations of the CCR as well as an interpretation of them in terms of classical probability. We prove that all q–deformations considered up to now can be obtained as particular cases of our construction.

In section 5 we recall the definition of a general IFS and of 1–mode–type IFS $\Gamma_I(\mathcal{H})$ over a pre–Hilbert space $\mathcal{H}$ we introduce, for any vector $f \in \mathcal{H}$, the **field operator** by

$$X_f := A_f^+ + A_f \tag{10}$$

With respect to the vacuum state $\langle \Phi, (\,\cdot\,) \Phi \rangle$, each X_f is a classical symmetric operator–random variable with all moments. Therefore the triple $(\Gamma_{I,f}(\mathcal{H}), A_f^+, \Phi)$, where $\Gamma_{I,f}(\mathcal{H})$ is the Φ–cyclic space of the polynomial algebra in $A_f^{\pm}$, is a quantum decomposition of the classical random variable X_f in the sense of Definition 1.
It is clear that, when f varies in $\mathcal{H}$, three situations can arise:
1) For any $f \in \mathcal{H}$, the triple $(\Gamma_I(\mathcal{H}), A_f^+, \Phi)$ is isomorphic to the canonical quantum decomposition of X_f.
2) Statement 1) is true for some, but not all, $f \in \mathcal{H}$.

3) The only $f \in \mathcal{H}$, for which the statement in 1) is true is $f = 0$.

Definition 3 The IFS $(\Gamma_I(\mathcal{H}), A_f^+, \Phi)$ is called:
– of type I, if it satisfies condition 1).
– of type II, if it satisfies condition 2).
– of type III, if it satisfies condition 3).

In Theorem 4, we characterize type I IFS and we prove (see Corollary 3) that this class includes the class of 1–**mode–type IFS** (see (70)).
This class includes the **Jacobi fields** introduced and studied by Berezansky, Kondratev, Lytvynov and several other authors (see [14], [20], [15]) while the usual Boson Fock spaces provides examples of type I symmetric IFS (see [4]) that **are not** 1–**mode–type IFS**.
In section 5.1 we prove the existence of (non–trivial) IFS of type II and, in section 5.2, the same thing is proved for type III.

The present paper was motivated by our paper [1] devoted to Bernoulli–type central limit theorems. In these theorems one considers a $*$–algebra $\mathcal{A}$ and a sequence of operator random variables in $\mathcal{A}$ of the form $\{S_N^+, S_N\}$ (sums of Bernoulli–type random variables), satisfying an algebraic independence condition (see the beginning of section 6).
In the notation (6) let φ be a state on $\mathcal{A}$ such that the limit

$$\varphi\left(S_N^{(\varepsilon(1))} \cdots S_N^{(\varepsilon(n))}\right) \tag{11}$$

exists for all $\varepsilon \in \{-1, 1\}^n$. Then, because of the algebraic independence condition, one can apply the reconstruction theorem of [11] and deduce the existence of a pre–Hilbert space $\mathcal{H}$, two pre–closed mutually adjoint operators a^+, a and a unit vector $\Phi \in \mathcal{H}$, cyclic for the algebra generated by a^+ and a such the limit (11) is equal to

$$\left\langle \Phi, a^{(\varepsilon(1))} \ldots a^{(\varepsilon(m))} \Phi \right\rangle \tag{12}$$

By construction, the triple $(\mathcal{H}, a^+, \Phi)$ is a quantum decomposition, in the sense of Definition 1, of the classical random variable

$$X := \lim_{N \to \infty} \left(S_N^+ + S_N\right) \tag{13}$$

In the paper [1] the problem arose to characterize those sequences $\{S_N^+, S_N\}$ for which this quantum decomposition is isomorphic to the canonical quantum decomposition. Equivalently, those sequences $\{S_N^+, S_N\}$ such that the

φ–moments of the pairs $\{S_N^+, S_N\}$ converge to the quantum moments of X. Theorem 5 solves this problem in a constructive way. In particular of the five conditions given in this theorem, the first three are automatically satisfied in Bernoulli–type central limit theorems. The results of the papers [1], [2], [3] show that, even in the case of Bernoulli CLT, the existence of such an isomorphism strongly depends on the specific form of the embedding (hence of the associated notion of stochastic independence).

The extension of this result to a self–adjoint set $\{S_N^{(j)} : j \in D\}$ (D a finite set) is a difficult open problem in quantum probability and its solution is strongly related to the theory of multi–dimensional orthogonal polynomials [5].

2 1–mode interacting Fock spaces (1–MIFS)

Definition 4 *A sequence of positive numbers $\lambda : n \in \mathbb{N} \to \lambda_m \in \mathbb{R}_+$ such that $\lambda_0 = 1$ and, if $\lambda_m = 0$ for some $m \in \mathbb{N}$, then $\lambda_n = 0$ for every $n \geq m$, will be called an* 1–**MIFS–sequence**. *Such a sequence defines a pre–scalar product on $\mathbb{C}$ by:*

$$\langle x, y\rangle_n := \lambda_n \overline{x} y \qquad ; \qquad \forall x, y \in \mathbb{C} \tag{14}$$

By definition the **1–mode interacting Fock space (1–MIFS) associated to the sequence** (λ_n) *is the pre–Hilbert space*

$$\Gamma\left(\mathbb{C}; \{\lambda_n\}_{n=1}^{\infty}\right) := \mathbb{C} \oplus (\mathbb{C}, \langle\cdot,\cdot\rangle_1) \oplus (\mathbb{C}, \langle\cdot,\cdot\rangle_2) \oplus (\mathbb{C}, \langle\cdot,\cdot\rangle_3) \oplus \ldots =: \bigoplus_{n=0}^{\infty} \mathcal{H}_n \tag{15}$$

characterized by the fact that the sum on the right hand side is orthogonal and meant in the weak sense (i.e. the elements of $\Gamma\left(\mathbb{C}; \{\lambda_n\}_{n=1}^{\infty}\right)$ are **finite** *linear combination of elements of the spaces $\mathcal{H}_n$.*

For $n \in \mathbb{N}^*$, define

$$m_\lambda := \begin{cases} \infty \ , \text{ if } \ \lambda_n \neq 0 \text{ for any } n \in \mathbb{N} \\ \max\{k : \lambda_k \neq 0\} \ , \text{ otherwise} \end{cases} \tag{16}$$

$$\omega_n := \begin{cases} 0 \ , \ \text{if } n = 0 \\ \lambda_n \ , \ \text{if } n = 1 \\ 0 \ , \ \text{if } n > m_\lambda < \infty \\ \frac{\lambda_n}{\lambda_{n-1}} \ , \ \text{otherwise} \end{cases} \tag{17}$$

$$\omega_n! := \prod_{k=1}^{n} \omega_k = \lambda_n \qquad ; \qquad \forall n \in \mathbb{N} \tag{18}$$

Then

$$m_\lambda := \begin{cases} \infty \ , \ \text{if } \omega_n \neq 0 \text{ for any } n \in \mathbb{N} \\ \max \{k : \omega_k \neq 0\} \ , \ \text{otherwise} \end{cases}$$

With these notations, the Hilbert space associated to the pre–Hilbert space (15) is denoted with the same symbol unless confusion may arise:

$$\bigoplus_{h=0}^{m_\lambda} (\mathbb{C}, \langle \cdot, \cdot \rangle_h) \tag{19}$$

Since to give the sequence (λ_n) is equivalent to give the sequence (ω_n) in the following we will use indifferently the notations:

$$\Gamma\left(\mathbb{C}; \{\lambda_n\}_{n=1}^{\infty}\right) = \Gamma(\lambda) = \Gamma(\omega) = \Gamma_\omega(\mathbb{C}) \tag{20}$$

and, unless confusion may arise, we will use the same symbol to denote the pre–Hilbert space (15) and the associated Hilbert space (19). For each $n \in \mathbb{N}$, denote Φ_0 the vector 1 in the space $(\mathbb{C}, \langle \cdot, \cdot \rangle_0)$. Φ_0 is called the **vacuum vector**. (14), (18) imply that

$$\|\Phi_n\|^2 = \lambda_n = \omega_n! \tag{21}$$

The **creation operator** on $\Gamma\left(\mathbb{C}; \{\lambda_n\}_{n=1}^{\infty}\right)$ is defined by linear extension of the map

$$a^+\Phi_n = \begin{cases} \Phi_{n+1} \ , \ \text{if } n \leq m_\lambda \\ 0 \ , \ \text{otherwise} \end{cases} \tag{22}$$

on the algebraic linear span of the (Φ_n). a^+ is well defined because (Φ_n) is a linear orthogonal basis of $\Gamma\left(\mathbb{C}; \{\lambda_n\}_{n=1}^{\infty}\right)$ The properties of λ imply that the adjoint of a^+,in the following denoted a and called the **annihilation operator**, exists and is given by

$$a\Phi_k = \omega_k \Phi_{k-1} \qquad ; \qquad \forall k \in \mathbb{N} \qquad ; \qquad \Phi_{-1} := 0 \tag{23}$$

In fact

$$\langle\Phi_n, a\Phi_k\rangle = \langle a^+\Phi_n, \Phi_k\rangle = \langle\Phi_{n+1}, \Phi_k\rangle = \lambda_k\delta_{n+1,k}$$
$$= \begin{cases} 0\ ,\ \text{if } k \neq n+1 \\ \lambda_{n+1} = \omega_{n+1}\lambda_n\ ,\ \text{if } k = n+1 \end{cases}$$

Remark. Defining

$$\tilde{\Phi}_n := \begin{cases} \frac{\Phi_n}{\sqrt{\lambda_k}}\ ,\ \text{if } k \leq m_\lambda \\ \Phi_n\ ,\ \text{if } k > m_\lambda \end{cases}$$

where unless otherwise specified in the following the symbol $\sqrt{(\ \cdot\ )}$ denotes the positive square root, one finds the action of a^+, a on the normalized basis $(\tilde{\Phi}_n)$ of $\Gamma\left(\mathbb{C}; \{\lambda_n\}_{n=1}^\infty\right)$:

$$a^+\tilde{\Phi}_n = a^+\frac{\Phi_n}{\sqrt{\lambda_n}} = \frac{\Phi_{n+1}}{\sqrt{\lambda_n}} = \sqrt{\omega_{n+1}}\frac{\Phi_{n+1}}{\sqrt{\lambda_n}} = \sqrt{\omega_{n+1}}\tilde{\Phi}_{n+1} \tag{24}$$

$$a\tilde{\Phi}_n = \frac{a\Phi_n}{\sqrt{\lambda_n}} = \omega_n\frac{\Phi_{n-1}}{\sqrt{\lambda_n}} = \sqrt{\omega_n}\frac{\Phi_{n-1}}{\sqrt{\lambda_{n-1}}} = \sqrt{\omega_n}\tilde{\Phi}_{n-1} \tag{25}$$

In the particular IFS considered in physics up to now, one uses this representation (symmetric) rather than the non–normalized one called *monic* in the literature on orthogonal polynomials.

2.1 Commutation relations associated to a 1–MIFS

In this section we show that all the q–deformations of the usual CCR can be framed within the general theory of 1–MIFS.

Lemma 1 Denote Λ the number operator on $\Gamma\left(\mathbb{C}; \{\lambda_n\}_{n=1}^\infty\right)$, defined by

$$\Lambda\Phi_n := n\Phi_n$$

Λ is a pre–Hilbert space operator and the linear operators a, a^+ on $\Gamma\left(\mathbb{C}; \{\lambda_n\}_{n=1}^\infty\right)$ satisfy

$$a^\pm(\mathcal{H}_n) = \mathcal{H}_{n\pm1} \quad ; \quad \mathcal{H}_{-1} = \{0\} \quad ; \quad a = (a^+)^* \tag{26}$$

$$aa^+ = \omega_{\Lambda+1} \qquad ; \qquad a^+a = \omega_\Lambda \tag{27}$$

and (27) implies the following commutation relations:

$$\left[a, a^+\right] = aa^+ - a^+a = \omega_{\Lambda+1} - \omega_\Lambda \tag{28}$$

$$aa^+ - \omega_{\Lambda+1}\omega_\Lambda^{-1}a^+a = P_0 \tag{29}$$

where P_0 denotes the projection onto the vacuum space and ω_Λ^{-1} is by definition zero whenever $\omega_n = 0$. An equivalent form of (29) is

$$aa^+ - Q_\Lambda a^+a = 1 \tag{30}$$

where by definition:

$$Q_\Lambda := (\omega_{\Lambda+1}\omega_\Lambda^{-1} - \omega_\Lambda^{-1}P_0^\perp) \tag{31}$$

Remark. In fact the multiplication table (27) implies several types of commutation (or anti–commutation) relations (see [4]), but in the following we will restrict our attention to the ones listed in Lemma 1.

Proof. Λ is a pre–Hilbert space operator because $\{a^{+n}\Phi\ :\ \}$ is an orthogonal linear basis of $\mathcal{K}$ and $\Lambda\{a^{+n}\Phi\ :\ \} \subset \{a^{+n}\Phi\ :\ \}$. It is clear that (27) implies (28). Let us prove that (28) implies (27). Since $\omega_0 = 0$, (28) implies that, for any $n \in \mathbb{N}^*$

$$aa^+a^{+n}\Phi = \left[a, a^{+(n+1)}\right]\Phi = \sum_{h=0}^{n} a^{+h}\left[a, a^+\right]a^{+(n+1-(h+1))}\Phi \tag{32}$$

$$= \sum_{h=0}^{n} a^{+h}\left(\omega_{\Lambda+1} - \omega_\Lambda\right)a^{+(n-h)}\Phi = \sum_{h=0}^{n}\left(\omega_{n+1-h} - \omega_{n-h}\right)a^{+n}\Phi = \omega_{n+1}a^{+n}\Phi$$

which proves the first identity in (27). The same argument, with n replaced by $n-1$, implies that

$$aa^{+n}\Phi = \left[a, a^{+n}\right]\Phi = \omega_n a^{+(n-1)}\Phi$$

and this implies the second identity in (27). From $aP_0 = 0$, one deduces $a = a(P_0^\perp + P_0) = aP_0^\perp$. Therefore

$$\omega_{\Lambda+1}^{-1}aa^+ = 1 = P_0^\perp + P_0 \qquad ; \qquad \omega_\Lambda^{-1}a^+aP_0^\perp = P_0^\perp$$

$$\omega_{\Lambda+1}^{-1}aa^+ = \omega_\Lambda^{-1}a^+aP_0^\perp + P_0 \iff aa^+ = \omega_{\Lambda+1}\omega_\Lambda^{-1}a^+aP_0^\perp + \omega_{\Lambda+1}P_0$$

$$\iff aa^+ - \omega_{\Lambda+1}\omega_\Lambda^{-1}a^+aP_0^\perp = \omega_1 P_0 \iff aa^+ - \omega_{\Lambda+1}\omega_\Lambda^{-1}a^+a = \omega_1 P_0 = P_0$$

Finally (30) follows from (29) and the identity $\omega_\Lambda^{-1}P_0^\perp a^+a = P_0^\perp$. $\square$

2.1.1 The commutator theorem and uniqueness of the quantum decomposition: 1–dimensional case

Remark. The main implication of the following Lemma is that, for the given a, a^{+}, the properties (27), (28) (29) are mutually equivalent. The equivalence between (27) and (28) is a particular case of a general result, valid also in the multi–dimensional case (see [6]).

Lemma 2 For two linear operators b, b^{+} on $\Gamma\left(\mathbb{C};\{\lambda_n\}_{n=1}^{\infty}\right)$ consider the identities (26), (27), (28) and (29) with $a^{\pm}$ replaced by $b^{\pm}$.
Then (27) and (28) are equivalent and imply (29). If moreover b, b^{+} satisfy (26), then (29) is equivalent to (27), hence to (28).

Proof. It is clear that condition (27) implies both (28) and (29). Let us prove that (28) implies (27). Since $\omega_0 = 0$, (28) implies that, for any $n \in \mathbb{N}^*$

$$bb^{+}b^{+n}\Phi = \left[b, b^{+(n+1)}\right]\Phi = \sum_{h=0}^{n} b^{+h}\left[b, b^{+}\right] b^{+(n+1-(h+1))}\Phi \tag{33}$$

$$= \sum_{h=0}^{n} b^{+h}\left(\omega_{\Lambda+1} - \omega_{\Lambda}\right) b^{+(n-h)}\Phi = \sum_{h=0}^{n}\left(\omega_{n+1-h} - \omega_{n-h}\right) b^{+n}\Phi = \omega_{n+1} b^{+n}\Phi$$

which proves the first identity in (27). The same argument, with n replaced by $n-1$, implies that

$$bb^{+n}\Phi = \left[b, b^{+n}\right]\Phi = \sum_{h=0}^{n-1} b^{+h}\left[b, b^{+}\right] b^{+(n-(h+1))}\Phi = \sum_{h=0}^{n-1} b^{+h}\left(\omega_{\Lambda+1} - \omega_{\Lambda}\right) b^{+(n-1-h)}\Phi$$

$$= \sum_{h=0}^{n-1}\left(\omega_{n-h} - \omega_{n-1-h}\right) b^{+(n-1)}\Phi = \omega_n b^{+(n-1)}\Phi$$

and this implies the second identity in (27). $\square$

2.1.2 Generalized commutation relations

Motivated by the results in the previous section, we illustrate in this section the relationships between general principal Jacobi sequences $\{\omega_n\}_{n=1}^{\infty}$ and commutation relations of the type

$$aa^{+} - Q_{\Lambda} a^{+} a = P_{\Lambda} \tag{34}$$

in three particular cases which emerged from different lines of research in quantum probability. Notice that, due to (27), the commutation relation (34) is equivalent to the identity

$$P_\Lambda = \omega_{\Lambda+1} - Q_\Lambda \omega_\Lambda \tag{35}$$

Moreover, since $\omega_1 = 1$, if (34) holds then

$$P_\Lambda \Phi = \Phi \tag{36}$$

2.2 Example (1). The q–deformed 1–MIFS

A short historical outline of the development of q–deformations in quantum probability was done in the Introduction of [2] and will not be repeated here.

Lemma 3 With above notations and definitions, for any $q \geq -1$ and any 1–MIFS, the following two statements are equivalent:

$$\omega_n = \begin{cases} n \ , & \text{if } \ q = 1 \text{ (Gauss–Poisson case)} \\ \sum_{k=0}^{n-1} q^k = \frac{1-q^n}{1-q}, & \text{if } \ q > -1 \ , \ q \neq 1 \\ \delta_{1,n} \ , & \text{if } q = -1 \text{ (Fermi–Boolean case)} \end{cases} \tag{37}$$

$$aa^+ - qa^+a = 1 \tag{38}$$

Proof. If (37) holds, then (38) clearly holds for $q = 1$ and, if $q \neq 1$, then

$$\omega_{n+1} - q\omega_n = \frac{1-q^{n+1}}{1-q} - q\frac{1-q^n}{1-q} = \frac{1-q^{n+1}-q+q^{n+1}}{1-q} = \frac{1-q}{1-q} = 1$$

If (38) holds and $q = -1$, then for any $n \in \mathbb{N}$

$$\begin{aligned} \lambda_n = \left\langle a^{+n}\Phi, a^{+n}\Phi \right\rangle_n &= \left\langle a^{+n}\Phi, \left(aa^+ + a^+a\right) a^{+n}\Phi \right\rangle_n \\ &= \lambda_{n+1} + \omega_n\lambda_n = \lambda_n \left(\omega_{n+1} + \omega_n\right) \end{aligned} \tag{39}$$

Taking $n = 0$ in (39) and using $\lambda_0 = 1$, one finds

$$1 = \|\Phi\|^2 = \lambda_0 \left(\omega_1 + \omega_0\right) = \omega_1$$

Therefore $\omega_n = 0$ for $n \geq 2$. Hence (37) holds for $q = -1$.
If (38) holds for $q > -1$, then using (27) one finds

$$1 = aa^+ - qa^+a = \omega_{\Lambda+1} - q\omega_\Lambda \iff 1 + q\omega_\Lambda = \omega_{\Lambda+1}$$

and this, together with the initial condition $\omega_1 = 1$ implies that $\omega_n = \sum_{k=0}^{n-1} q^k$. Therefore (37) holds also for $q > -1$, $q \neq 1$. $\square$

Remark. The identity $\langle a^2 a^{+2} \rangle = 1 + q$ shows that $q \geq -1$ is a necessary condition for the positivity of the scalar product.

2.3 Example (2). The Parthasarathy–Shürmann q^Λ – deformation

The 1–MIFS defined by the sequence (40) below is the 1–mode version of the q^Λ–deformation of the Boson Fock Brownian motion, introduced by Parthasarathy in his quantum decomposition of the Azema martingale [21], which has been the main model for Shürmann's extension of the theory of Levy processes to q–deformed $*$–bi–algebras [23]. In [3] it is proved that this class of 1–MIFS arises from Bernoulli central limit theorems with symmetric q–embeddings.

Lemma 4 With above notations and definitions, for any $x \in \mathbb{C} \setminus \{0\}$, and any 1–MIFS, the following three statements are equivalent:

$$\omega_n = n \cdot |x|^{2n} \quad ; \quad \lambda_n := \omega_n! = n! \cdot |x|^{n(n+1)} \quad ; \quad \forall n \geq 1 \tag{40}$$

$$aa^+ - |x|^2 a^+ a = |x|^{2\Lambda} \tag{41}$$

Proof. The two statements in (40) are equivalent because

$$\frac{\lambda_n}{\lambda_{n-1}} = \frac{n! \cdot |x|^{n(n+1)}}{(n-1)! \cdot |x|^{(n-1)n}} = n \cdot |x|^{2n} = \omega_n \quad ; \quad \forall n \geq 1$$

Suppose that one of the two statements in (40) holds. Then

$$aa^+ = \omega_{\Lambda+1} = (\Lambda + 1) \cdot |x|^{2(\Lambda+1)} \quad ; \quad a^+ a = \omega_\Lambda = \Lambda \cdot |x|^{2\Lambda}$$

Consequently (41) holds because,

$$aa^+ - |x|^2 a^+ a = \omega_{\Lambda+1} - |x|^2 \omega_\Lambda = (\Lambda + 1) |x|^{2(\Lambda+1)} - \Lambda |x|^2 |x|^{2\Lambda}$$
$$= (\Lambda + 1) |x|^{2(\Lambda+1)} - \Lambda |x|^{2(\Lambda+1)} = |x|^{2(\Lambda+1)}$$

Conversely, if (41) holds, then

$$\omega_{\Lambda+1} = |x|^2 \omega_\Lambda + |x|^{2(\Lambda+1)}$$

and since

$(n+1)|x|^{2(n+1)} = |x|^2(n|x|^{2n})+|x|^{2(n+1)}$ and for $n = 0,$ one has $n|x|^{2n} = |x|^2$

(40) holds because the two sequences (ω_n) and $n|x|^{2n}$ satisfy the same induction relation with the same initial condition. $\square$

Remark. The commutation relations (41) can be written in the form (30) with the operator Q_Λ given by

$$Q_\Lambda\xi := \begin{cases} \xi, \text{ if } \xi \in \mathbb{C}\cdot\Phi \\ \left(\Lambda\,|x|^{2\Lambda}\right)^{-1}\left((\Lambda+1)\,|x|^{2(\Lambda+1)} - 1\right)\xi, \quad \text{if } \xi \in (\mathbb{C}\cdot\Phi)^\perp \end{cases}$$

i.e.

$$Q\Lambda = 1 \oplus \left(\Lambda\,|x|^{2(\Lambda-1)}\right)^{-1}\left((\Lambda+1)\,|x|^{2\Lambda} - 1\right)$$

Remark. I With the notation $q := |x|^2$, (41) becomes

$$aa^+ - qa^+a =: \partial\omega_\Lambda = q^{\Lambda+1} \tag{42}$$

Equation (42) is the 1–mode version of the **left q–quantum Brownian motion** (in the sense of Schürmann [23], see [3] for a description that emphasizes this fact).

2.4 Example (3). q–deformations arising from CLT's

The sequence

$$\lambda_n := \prod_{m=1}^{n}\left|\sum_{k=0}^{m-1}x^k\right|^2 \quad ; \quad \lambda_k := 0\ ,\ \forall k \geq n \text{ if } \lambda_n = 0\ ;\ \forall n \geq 1\ ,\ x \in \mathbb{C} \tag{43}$$

comes out in the q–deformed central limit theorems discussed in the paper [1], where the problem discussed in the present paper first emerged in a particular case.

Since the case $x = 0$ is included in the first example (i.e. $aa^+ = 1$), we take $x \neq 0$. For $m \in \mathbb{N}$, define

$$\mathcal{K}_m := \left\{x\ :\ \sum_{k=0}^{m}x^k = 0\right\} \quad ; \quad m_x := \min\left\{m : x \in \mathcal{K}_m \text{ and } x \notin \bigcup_{h=1}^{m-1}\mathcal{K}_h\right\}$$

$$\omega_n := \omega_n(x) := \begin{cases} 0, & \text{if } m_x < \infty \text{ and } n > m_x \\ \left|\sum_{k=0}^{n-1} x^k\right|^2 & \text{otherwise} \end{cases} \tag{44}$$

In this case (34) holds with $P = 1$ and one has:

$$Qa^{+n}\Phi = \left|\sum_{k=0}^{n-1} x^k\right|^{-2} \left(\left|\sum_{k=0}^{n} x^k\right|^2 - 1\right) a^{+n}\Phi \qquad ; \qquad \forall n \le m_x$$

i.e.

$$Q = c\mathbf{1} \oplus \left|\left(1 - x^\Lambda\right)^2\right|^{-1} \left(\left|1 - x^{\Lambda+1}\right|^2 - |1 - x|^2\right)$$

with c arbitrary element of $\mathbb{C}$.

3 The canonical quantum moments of a classical symmetric random variable

In case of a general real valued random variable X with all moments, a formula expressing the classical moments of X in terms of its Jacobi sequences was proved in [8]. In this section we extend this formula to the quantum moments of a **symmetric** classical random variable.

Theorem 1 For a sequence of positive real numbers $M = \{M_{2n}\}_{n=1}^{\infty}$, the following statements are equivalent:
1) there exists a real valued symmetric classical random variable X, with moment sequence (M_{2n});
2) there exists a principal Jacobi sequence (ω_n) such that the vacuum moment sequence of the operator $a^+ + a$, in the 1–MIFS $(\Gamma(\omega), a^+, \Phi_0)$, coincides with (M_{2n}).
3) there exists a unique principal Jacobi sequence (ω_n) such that 2) holds.

Proof. The implication 1) $\Rightarrow$ 3) was proved in [8]. That 3) $\Rightarrow$ 2) is clear. That 2) $\Rightarrow$ 3) follows from the Accardi–Bozeiko formula [8] that implies that the vacuum moments of the symmetric random variable $X := a^+ + a$ are uniquely determined by the sequence (ω_n). $\square$

Recall that, because of the Fock prescription and of the fact that we are restricting our attention to the 1–mode case, the quantum moments are determined by non–crossing pair partitions and among them the only ones that give a non–zero contribution to the mixed moments (8) are those, whose corresponding to those $\varepsilon = (\varepsilon(1), \dots, \varepsilon(2n)) \in \{-1,1\}^{2n}$ belonging to the set

$$\{-1,1\}^{2n}_{+} := \left\{ \varepsilon \in \{-1,1\}^{2n} \ : \ \sum_{h=1}^{2n} \varepsilon_h = 0 \text{ and } \forall k \in \{1, \dots, 2n\} \ , \ \sum_{h=k}^{2n} \varepsilon_h \geq 0 \right\}$$

in the symmetric case, and to the set

$$\{-1,0,1\}^{2n,k}_{+} :=$$

$$\left\{ \varepsilon \in \{-1,0,1\}^{2n+k} \ : \ \left|\varepsilon^{-1}(0)\right| = k \ , \ \sum_{h=1}^{2n+k} \varepsilon_h = 0 \text{ and } \forall m, \in \{1, \dots, 2n\} \ , \ \sum_{h=m}^{2n+k} \varepsilon_h \geq 0 \right\}$$

in the non symmetric case. Here and in the following, for each m and each set I, we identify the set I^m with the set of functions $\varepsilon : k \in \{1, \dots, m\} \to \varepsilon_k \in I$. We use the notations

$$PP(2n) := \{\text{Pair partitions of } \{1, \dots, 2n\}\}$$

$$NCPP(2n) := \{\text{Non–crossing pair partitions of } \{1, \dots, 2n\}\}$$

If $\theta = \{(l_h, r_h)\}_{h=1}^{n} \in PP\,(2n)$, and $k \in \{1, \dots, n\}$, the **depth of the pair** (l_k, r_k) **in** θ is defined by (see [8])

$$d_\theta(l_k, r_k) := 1 + |\{h \in \{1, \dots, n\} \ : \ l_h \leq l_k < r_k \leq r_h\}| \tag{45}$$

$$= |\{\text{pairs} \in \theta \text{ containing } (l_k, r_k)\}|$$

Remark. For completeness we show how to extend the notion of depth to pair partitions with singletons. Denote

$$PPS(2n,k) := \{\text{Pair partitions of } \{1, \dots, 2n+k\} \text{ with } k \text{ singletons}\}$$

$$NCPPS(2n,k) := \{\text{Non–crossing pair partitions of } \{1, \dots, 2n+k\} \text{ with } k \text{ singletons}\}$$

To any $\theta \in PPS(2n,k)$ one uniquely associates a $\theta_0 \in PP(2n)$ with the following prescription:
remove all singletons from θ and re–label in increasing order the remaining indexes with initial condition $\ell_1 = 1$, where ℓ_1 is the first left index in θ. Clearly, this correspondence is surjective and, if $\theta \in NCPP(2n,k)$, then

$\theta_0 \in NCPP(2n)$. Denoting by $\nu : \{1, \ldots, 2n+k\} \to \{1, \ldots, 2n\}$ the above–mentioned relabeling, the definition depth of a pair is extended to $PPS(2n, k)$ ($k \in \mathbb{N}$) defining, for $\theta \in PPS(2n, k)$ and $(l_k, r_k) \in \theta$:

$$d_\theta(l_k, r_k) := d_{\theta_0}(l_{\nu_k}, r_{\nu_k}) \tag{46}$$

Similarly, if $\theta \in PPS(2n, k)$ is any pair partition associated to an $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{2n+k})$ and s_m is a singleton in $PPS(2n, k)$, the **depth of** s_m is defined by

$$d_\theta(s_m) := \sum_{h=m+1}^{2n+k} \varepsilon_h \ (\geq 0) \tag{47}$$

where **the singletons are ordered so that s_m is the m–th singleton in θ starting from left**. If ε labels a product of CAP operators, $d_\theta(s_m)$ is simply the index of the space of the gradation where $a^0_{s_m}$ acts. The fact that we use the same symbol for depth of pairs and of singletons should not give rise to confusions because the former is a function of 2 variables and the latter of 1.

It is known (see [10]) that the set $\{-1, 1\}^{2n}_+$ is in one–to–one correspondence with $NCPP(2n)$ and, once given the indexes of the singletons $\{s_m\}_{m=1}^k$, an element of

$$\{-1, 0, 1\}^{2n,k}_+ :=:= \Big\{\varepsilon := (\varepsilon_{\sigma_1}, \ldots, \varepsilon_{\sigma_{2n+k}}) \ :$$
$$(\varepsilon_1, \ldots, \varepsilon_{2n+k}) := (\varepsilon_1, \ldots, \varepsilon_{2n}, 0, \ldots, 0) \ , \ (\varepsilon_1, \ldots, \varepsilon_{2n}) \in \{-1, 1\}^{2n}_+ \ , \ \sigma \in \mathcal{S}_{2n+k}\Big\}$$

defines a unique NCPP of the remaining indexes. This gives a unique identification

$$\{-1, 0, 1\}^{2n,k}_+ \equiv NCPPS\,(2n, k)$$

Remark. As said before, for any $\epsilon \in \{-1, 1\}^{2n}_+$ there is exactly one $\{(l_h, r_h)\}_{h=1}^n \in NCPP(2n)$ such that $\epsilon_{l_h} = -1$ and $\epsilon_{r_h} = 1$ for any $h \in \{1, \ldots, n\}$. However the set

$$PP(2n; \epsilon) := \{\{(l_h, r_h)\}_{h=1}^n \in PP(2n) \ : \ \epsilon_{l_h} = -1 \text{ and } \epsilon_{r_h} = 1 \ , \ \forall h \in \{1, \ldots, n\}\}$$

could consist of more elements. As shown in Lemma 2.2 of [1], the cardinality of this set is given by

$$|PP\,(2n, \varepsilon)| = \prod_{h=1}^{n} (2k - l_h) \tag{48}$$

The explicit form of the pair–depth–function on $NCPP\,(2n)$, hence on $NCPPS\,(2n, k)$ for any $k \in \mathbb{N}$, is given by the following Lemma.

Lemma 5 Given $n \in \mathbb{N}$ and $\theta = \{(l_h, r_h)\}_{h=1}^{n} \in NCPP\,(2n)$, for any $k \in \{1, \ldots, n\}$ one has

$$d_\theta(l_k, r_k) = 2k - l_k \tag{49}$$

Proof. The non–crossing property implies that

$$d_\theta(l_k, r_k) = |\{h : l_h \le l_k < r_k \le r_h\}|$$

$$= |\{l_k + 1, \ldots, 2n\}| - 2\,|\{j \in \{1, \ldots, n\} \ : \ l_j > l_k\}|$$

$$= 2n - l_k - 2\,(n - k) = 2k - l_k$$

$\square$

Lemma 6 On the 1–MIFS with Jacobi sequence (ω_n), for any $\varepsilon \in \{-1, 1\}_{+}^{2n}$, denoting $\{(l_h, r_h)\}_{h=1}^{n}$ the unique non–crossing pair partition of $\{1, \ldots, 2n\}$ corresponding to ε, one has

$$a^{\varepsilon(1)} \ldots a^{\varepsilon(2n)}\Phi = \prod_{k=1}^{n} \omega_{2k-l_k}\Phi \tag{50}$$

Proof. In a 1–MIFS one has

$$aa^{+n}\Phi = \omega_n a^{+(n-1)}\Phi \qquad , \qquad \forall n \tag{51}$$

therefore, if $n = 1$, (50) is trivial. Suppose that (50) holds for n. Then, since the l_h are strictly increasing, for any $\varepsilon \in \{-1, 1\}_{+}^{2(n+1)}$, the corresponding pair partition (l_h, r_h) satisfies $\varepsilon\,(l_{n+1}) = -1$, $r_{n+1} = l_{n+1} + 1$ and $\varepsilon\,(k) = 1$ for any $k \in \{l_{n+1} + 1, \ldots, 2n + 2\}$. Therefore (51) implies that

$$\begin{aligned} a^{\varepsilon(1)} \ldots a^{\varepsilon(2n+2)}\Phi &= a^{\varepsilon(1)} \ldots a^{\varepsilon(l_{n+1}-1)} aa^{+(2n+2-l_{n+1})}\Phi \\ &= \omega_{2(n+1)-l_{n+1}} a^{\varepsilon(1)} \ldots a^{\varepsilon(l_{n+1}-1)} a^{+(2n+1-l_{n+1})}\Phi \end{aligned}$$

Denote

$$\varepsilon'\,(k) := \begin{cases} \varepsilon\,(k)\,, & \text{if } k \le l_{n+1} - 1 \\ 1, & \text{if } l_{n+1} \le k \le 2n \end{cases}$$

Notice that, since l_{n+1} is the index of the last annihilator, ε' can be obtained by ε by removing $\varepsilon_{l_{n+1}} = -1$ and any $\varepsilon_{l_{n+1}} = +1$, with $j > l_{n+1}$ and re–labeling the indexes starting from $l_{n+1} + 1$. from this it follows that:

- $\varepsilon' \in \{-1, 1\}_{+}^{2n}$;

- the corresponding NCPP $\{(l'_h, r'_h)\}_{h=1}^n$ verifies $l'_h = l_h$ for any $h \in \{1, \dots, n\}$;
- $a^{\varepsilon(1)} \dots a^{\varepsilon(l_{n+1}-1)} a^{+(2n+1-l_{n+1})}\Phi = a^{\varepsilon'(1)} \dots a^{\varepsilon'(2n)}\Phi$.

The induction assumption then implies

$$\begin{aligned} a^{\varepsilon(1)} \dots a^{\varepsilon(2n+2)}\Phi &= \omega_{2(n+1)-l_{n+1}} a^{\varepsilon(1)} \dots a^{\varepsilon(l_{n+1}-1)} a^{+(2n+1-l_{n+1})}\Phi \\ &= \omega_{2(n+1)-l_{n+1}} a^{\varepsilon'(1)} \dots a^{\varepsilon'(2n)}\Phi \\ &= \omega_{2(n+1)-l_{n+1}} \prod_{k=1}^{n} \omega_{2k-l_k}\Phi = \prod_{k=1}^{n+1} \omega_{2k-l_k}\Phi \end{aligned}$$

and this proves the statement. □

Corollary 1 The quantum moments of a classical, symmetric, real valued random variable X with principal Jacobi sequence (ω_n), are given by

$$\langle \Phi, a^{\varepsilon(1)} \dots a^{\varepsilon(2n)}\Phi \rangle = \prod_{k=1}^{n} \omega_{2k-l_k} \quad ; \quad \forall \varepsilon \in \{-1,1\}_+^{2n} \ , \ \forall n \tag{52}$$

where $\{(l_h, r_h)\}_{h=1}^n$ denotes the unique non–crossing pair partition of $\{1, \dots, 2n\}$ corresponding to ε.

Proof. (52) follows by taking the scalar product of both sides of (50) with the vacuum vector. □

3.1 The converse of the Accardi–Bozeiko formula: symmetric case

In this section we prove an inductive form of the converse of the Accardi–Bozeiko formula in the case of **symmetric** random variables.
As an application of this theorem we obtain a characterization of the moment sequence of a classical symmetric random variable which does not involve the positive definiteness of the kernel $(m, n) \mapsto M(2(m+n))$.

Theorem 2 For any n and any $\varepsilon \in \{-1,1\}_+^{2n}$, denote by $\{l_1^\varepsilon, \dots, l_n^\varepsilon\}$ the set $\{k : \varepsilon(k) = -1\}$ with the order $l_1^\varepsilon < \dots < l_n^\varepsilon$ and

$$\varepsilon_{n,1} := \Big(\overbrace{-1, \dots, -1}^{n}, \overbrace{1, \dots, 1}^{n}\Big) =: (-^n, +^n)$$

For any sequence $M = \{M_{2n}\}_{n=1}^{\infty} \subset \mathbb{R}$, define inductively the sequence $\{\omega_n(M)\}_{n=1}^{\infty}$ as follows:

$$\omega_n(M) := \begin{cases} M_2 \ , \text{ if } \quad n = 1 \\ 0 \ , \text{ if } \quad n > 1 \text{ and } \omega_{n-1}(M)! = 0 \\ \frac{1}{\omega_{n-1}(M)!}\Big(M_{2n} - \sum_{\varepsilon_{n,1} \neq \varepsilon \in \{-1,1\}_+^{2n}} \prod_{k=1}^{n} \omega_{2k-l_k^{\varepsilon}}(M)\Big) \\ \ , \text{ if } n > 1 \text{ and } \omega_{n-1}(M)! \neq 0 \end{cases} \tag{53}$$

where, by definition, a sum over an empty index set is zero. Then the following statements are equivalent:
1) (M_{2n}) is the moment sequence of a classical symmetric random variable;
2) For any $n \in \mathbb{N}$

$$\omega_n(M) \geq 0 \tag{54}$$

Remark. Notice that $\varepsilon_{n,1}$ is the unique ε such that the quantum moment (52) is equal to $\omega_n!$.

Proof. The only measure for which $M_{2n} = 0$ for some n is the δ at 0, for which the statement is trivial. Therefore we can restrict our attention to the case of a strictly positive sequence (M_{2n}).
The implication 1) ⇒ 2) follows immediately from the Accardi–Bozeiko formula for symmetric random variables combined with the identity (52).
To prove that 2) ⇒ 1), notice that, if (54) holds, then $(\omega_n(M))$ is a principal Jacobi sequence because of (53). Therefore it defines a unique 1–MIFS. If $a^{\pm}$ are the creation and annihilation operators of this IFS, $a^+ + a$ is a symmetric random variable whose principal Jacobi sequence is $(\omega_n(M))$. But then, by the converse of the Accardi–Bozeiko formula, its moment sequence is defined by (53), hence it coincides with (M_{2n}). □

4 Solution of the canonical quantum moment problem for a classical symmetric random variable

Theorem 3 For a family of real numbers $\left\{M_{2n}(\varepsilon) : \varepsilon \in \{-1,1\}_+^{2n}\right\}_{n=1}^{\infty}$, the following statements are equivalent:
1) there exists a principal Jacobi sequence (ω_n) such that

$$M_{2n}(\varepsilon) = \left\langle \Phi, a^{\varepsilon(1)} \dots a^{\varepsilon(2n)} \Phi \right\rangle \qquad ; \qquad \forall n \in \mathbb{N},\ \varepsilon \in \{-1,1\}^{2n} \tag{55}$$

holds on the 1–MIFS $\Gamma(\omega)$ (see (20));

2) there exists a unique principal Jacobi sequence satisfying 1);

3) The family $\{M_{2n}(\varepsilon)\}$ satisfies the following properties:
(C1): the sequence

$$M_{2n} := \sum_{\varepsilon\in\{-1,1\}_+^{2n}} M_{2n}(\varepsilon) \qquad ; \qquad \forall n \tag{56}$$

is the moment sequence of a classical symmetric real valued random variable.
(C2): for any $n \in \mathbb{N}$ and $\varepsilon \in \{-1,1\}_+^{2n}$ such that, for some $p < n$, $\sum_{k=1}^{2p} \varepsilon(k) = 0$, one has

$$M_{2n}(\varepsilon) = M_p(\varepsilon_p) \cdot M_{n-p}(\varepsilon^p) \tag{57}$$

with

$$\varepsilon_p := \varepsilon|_{\{1,\dots,2p\}} \qquad ; \qquad \varepsilon^p(k) := \varepsilon(k+2p) \ , \ \forall k \in \{1,\dots,2n-2p\}$$

(C3): for any $n \geq 2$ and $\varepsilon \in \{-1,1\}_+^{2n}$, denoting $(l_k^\varepsilon, r_k^\varepsilon)_{k=1}^n$ the unique non–crossing pair partition associated to ε, and defining $\rho\varepsilon \in \{-1,1\}_+^{2(n-1)}$ by

$$\rho\varepsilon(k) := \begin{cases} \varepsilon(k), & \text{if } k < l_n^\varepsilon \\ \varepsilon(k+2), & \text{if } k \in \{l_n^\varepsilon, \dots, 2n-2\} \end{cases}$$

then

$$M_{2n}(\varepsilon) = \omega_{2n-l_n^\varepsilon}(M)\, M_{2n}(\rho\varepsilon) \tag{58}$$

where the $\omega_k(M)$ are defined by (53) Moreover, if any of these 3 properties holds, then also

$$\omega_n = \omega_n(M) \qquad , \qquad \forall n \tag{59}$$

holds, where (ω_n) is the principal Jacobi sequence defined in item 1) above.

Proof. The equivalence 1) $\iff$ 2) follows from the uniqueness of the quantum decomposition.
Assuming that 1) (or 2)) holds, then (ω_n) must be a principal Jacobi sequence and the M_{2n}, defined by (62) are the moment sequence of the classical symmetric real valued random variable $X := a^+ + a$. Thus (C1) is a necessary

condition. (C2) follows from the *factorization principle*, valid in any IFS (see [9], section 25).
Finally, by Theorem 2, 1) implies (64) i.e., for each n, $\omega_n = \omega_n(M)$.
Therefore by the same argument as in the proof of Lemma 6 (C3) holds.

Conversely, suppose that (C1), (C2), (C3) hold. The same argument as in the first part of the proof implies that we can only consider the case $M_{2n} > 0$ for all $n \in \mathbb{N}$. In this case (C1) and Theorem 2 guarantees that $(\omega_n(M))$ is a principal Jacobi sequence, so the 1–MIFS $\Gamma(\omega_n(M))$ is well defined and (1) holds on this 1–MIFS. Now we show that (61) holds on the 1–MIFS $\Gamma(\{\omega_n(M)\}_{n=1}^{\infty})$.
First of all, recall from Lemma 5 that on the 1–MIFS $\Gamma(\omega_n(M))$, one has

$$\left\langle \tilde{\Phi}_0, a^{\varepsilon(1)} \dots a^{\varepsilon(2n)} \tilde{\Phi}_0 \right\rangle = \prod_{k=1}^{n} \omega_{2k-l_k^{\varepsilon}}(M) \qquad ; \qquad \forall n \in \mathbb{N} \text{ and } \varepsilon \in \{-1,1\}_{+}^{2n}$$

So one needs only to show that for any $n \in \mathbb{N}$ and $\varepsilon \in \{-1,1\}_{+}^{2n}$

$$M_{2n}(\varepsilon) = \prod_{k=1}^{n} \omega_{2k-l_k^{\varepsilon}}(M) =: \Omega_n(\varepsilon)$$

For $n = 1$, by definition, one has $\Omega_1(\varepsilon) := \omega_1(M) := M_2 = M_2(\varepsilon)$.
For $n = 2$, (C2) and (C3) imply that

$$M_2((-1,1,-1,1)) = M_2((-1,1))\, M_2((-1,1)) = \omega_1^2(M) = \Omega_n((-1,1,-1,1))$$

$$M_2((-1,-1,1,1)) \overset{(C3)}{=} \omega_{4-2}(M)\, M_2((-1,1)) = \omega_2(M)\,\omega_1(M) = \Omega_n((-1,-1,1,1))$$

Suppose that for any $n \leq m$ and $\varepsilon \in \{-1,1\}_{+}^{2n}$,

$$M_{2n}(\varepsilon) = \Omega_n(\varepsilon)$$

then (C3) implies that, for any $\varepsilon \in \{-1,1\}_{+}^{2(m+1)}$,

$$M_{m+1}(\varepsilon) = \omega_{2(m+1)-l_{m+1}^{\varepsilon}}(M)\, M_m(\rho\varepsilon) \tag{60}$$

Clearly $\rho\varepsilon \in \{-1,1\}_{+}^{2m}$ and $l_k^{\rho\varepsilon} = l_k^{\varepsilon}$ for any $k \in \{1, \dots, m\}$ and the induction assumption implies that

$$M_m(\rho\varepsilon) = \Omega_m(\rho\varepsilon) = \prod_{k=1}^{m} \omega_{2k-l_k^{\rho\varepsilon}}(M) = \prod_{k=1}^{m} \omega_{2k-l_k^{\varepsilon}}(M)$$

and therefore (60) gives

$$M_{m+1}(\varepsilon) = \omega_{2(m+1)-l^{\varepsilon}_{m+1}}(M)\, M_m(\rho\varepsilon) = \omega_{2(m+1)-l^{\varepsilon}_{m+1}} \prod_{k=1}^{m} \omega_{2k-l^{\varepsilon}_k}(M) = \Omega_{m+1}(\varepsilon)$$

Thus the statement follows by induction. □

Corollary 2 For a family of real numbers $\left\{M_{2n}(\varepsilon) : \varepsilon \in \{-1,1\}^{2n}_{+}\right\}_{n=1}^{\infty}$ satisfying the positive definitness condition of Remark 1 with $\{-1,0,1\}$ replaced by $\{-1,1\}$, the following statements are equivalent:
1) there exists a principal Jacobi sequence (ω_n) such that the equality

$$M_{2n}(\varepsilon) = \left\langle \Phi, a^{\varepsilon(1)} \dots a^{\varepsilon(2n)} \Phi \right\rangle \qquad ; \qquad \forall n \in \mathbb{N},\ \varepsilon \in \{-1,1\}^{2n} \tag{61}$$

holds on the 1–MIFS $\Gamma(\omega)$ (see (20));

2) there exists a unique principal Jacobi sequence satisfying 1);

3) The family $\{M_{2n}(\varepsilon)\}$ satisfies the following properties:
(C1): the sequence

$$M_{2n} := \sum_{\varepsilon \in \{-1,1\}^{2n}_{+}} M_{2n}(\varepsilon) \qquad ; \qquad \forall n \tag{62}$$

is the moment sequence of a classical symmetric real valued random variable.
(C3)': for any $n \geq 2$ and $\varepsilon \in \{-1,1\}^{2n}_{+}$,

$$M_{2n}(\varepsilon_1, \dots, \varepsilon_{n-1}, -, +^{n}) = \omega_n(M) M_{2(n-1)}\left(\varepsilon_1, \dots, \varepsilon_{n-1}, +^{(n-1)}\right) \tag{63}$$

where the $\omega_k(M)$ are defined by (53). Moreover, if any of these 3 properties holds, then also

$$\omega_n = \omega_n(M) \qquad ; \qquad \forall n \tag{64}$$

holds, where (ω_n) is the principal Jacobi sequence defined in item 1) above.

Proof. The conditions are clearly necessary. To prove sufficiency denote X the moment unique classical real valued symmetric random variable whose moments are the M_{2n} and notice that condition (C1) and positive definitness guarantee the existence of a quantum decomposition $(\mathcal{H}, L^{+}, \Phi)$ of X whose quantum moments are the $\{M_{2n}(\varepsilon)\}$. If $m+n$ is even and $m \neq n$, $(-^m, +^n) \notin \{-1,1\}_+^{m+n}$. Therefore

$$\langle L^{+n}\Phi, L^{+n}\Phi\rangle = M_{m+n}(-^m, +^n) = 0$$

hence the vectors $L^{+n}\Phi$ are mutually orthogonal. Since all non–zero moments have the form $M_{2n}(\varepsilon_1, \ldots, \varepsilon_{n-1}, -, +^n)$ for some $n \in \mathbb{N}$ and since we know from condition (C3)' that

$$M_{2n}(\varepsilon_1, \ldots, \varepsilon_{n-1}, -, +^n) = \omega_n(M) M_{2(n-1)}\left(\varepsilon_1, \ldots, \varepsilon_{n-1}, +^{(n-1)}\right)$$

Therefore by the same argument as in the proof of Lemma 6

$$M_{2n}(\varepsilon_1, \ldots, \varepsilon_{2n}) = \prod_{k=1}^{n} \omega_{2k-l_k} \qquad ; \qquad \forall \varepsilon \in \{-1,1\}_+^{2n} \ , \ \forall n$$

where $(l_k^\varepsilon, r_k^\varepsilon)_{k=1}^n$ is the unique non–crossing pair partition associated to ε. Therefore $(\mathcal{H}, L^+, \Phi)$ is isomorphic to the canonical quantum decomposition of X.

4.0.1 Minimality of the conditions (C1), (C2), (C3)

(C3) is not a consequence of (C1) and (C2). To prove this consider the annihilation and creation operators A, A^+ with test function 1 on the monotone Fock space over $\mathbf{L}^2([0,1])$ (see section 5.2 below). Then the family of operators

$$M_{2n}(\varepsilon) := \left\langle \Phi, A^{\varepsilon(1)} \ldots A^{\varepsilon(2n)}\Phi \right\rangle \qquad ; \qquad \forall n \ \text{ and } \ \varepsilon \in \{-1,1\}_+^{2n}$$

$\left\{M_{2n}(\varepsilon) : \varepsilon \in \{-1,1\}_+^{2n}\right\}_{n=1}^\infty$ verifies trivially (C1) and (C2), if we choose (M_{2n}) to be the moment sequence of the arc–sine distribution over $(-1,1)$ and (ω_n) the corresponding principal Jacobi sequence. In particular one has,

$$M_2 = \left\langle \Phi, AA^+\Phi \right\rangle = 1 \quad ; \quad M_2((-1,-1,1,1)) = \frac{1}{2}$$

$$M_2((-1,1,-1,1)) = \left\langle \Phi, AA^+AA^+\Phi \right\rangle = 1$$

hence

$$\omega_1(M) := M_2 = 1$$

$$M_2 = M_2((-1,-1,1,1)) + M_2((-1,1,-1,1)) = \frac{3}{2}$$

$$\omega_2(M) := \frac{1}{\omega_1(M)}\left(M_2 - \omega_1^2(M)\right) = \frac{1}{2}$$

For $n = 3$ and $\varepsilon = (-1,-1,1,-1,1,1) \in \{-1,1\}_+^{2\times 3}$, one has $l_3^\varepsilon = 4$, $\rho\varepsilon = (-1,-1,1,1)$. Therefore, in the canonical quantum decomposition:

$$\left\langle \Phi, aaa^+aa^+a^+\Phi \right\rangle = \omega_{2n-l_n^\varepsilon}\left(M\right) M_{2n}\left(\rho\varepsilon\right) = \omega_2\left(M\right) M_2\left((-1,-1,1,1)\right) = \frac{1}{4}$$

But, in the IFS–quantum decomposition:

$$M_{2n}\left(\varepsilon\right) = M_6\left((-1,-1,1,-1,1,1)\right) = \left\langle \Phi, AAA^+AA^+A^+\Phi \right\rangle = \frac{1}{3}$$

Therefore (73) is false, hence (C3) cannot hold.

As an application of Corollary 3, we construct a module generalization of the q–Fock space (see section 2.2).
Let $\mathcal{H}$ be a pre–Hilbert space with pre–scalar product $(\cdot,\cdot)$ and $\{q_n\}_{n=2}^\infty \subset [-1,1]$ be a sequence such that $q_1 := 1$ and $q_{n+m} = -1$ for any m whenever $q_n = -1$. For any $n \geq 2$, one introduces:

- the linear operator $\lambda_n : \mathcal{H}^{\otimes n} \longmapsto \mathcal{H}^{\otimes n}$ defined by

  $$\lambda_n\left(f_1 \otimes \ldots \otimes f_n\right) := \sum_{\sigma\in\mathcal{S}_n} q_n^{|\sigma|} f_{\sigma(1)} \otimes \ldots \otimes f_{\sigma(n)} \qquad ; \qquad \forall\left\{f_1,\ldots,f_n\right\} \subset \mathcal{H}$$

- the (positive) bilinear form $\langle\cdot,\cdot\rangle_n$ on $\mathcal{H}^{\otimes n} \times \mathcal{H}^{\otimes n}$ by

  $$\begin{aligned}\left\langle f_1 \otimes \ldots \otimes f_n, g_1 \otimes \ldots \otimes g_n\right\rangle_n &:= \left(f_1 \otimes \ldots \otimes f_n, \lambda_n g_1 \otimes \ldots \otimes g_n\right)_n \\ &= \sum_{\sigma\in\mathcal{S}_n} q_n^{|\sigma|} \prod_{k=1} \left(f_k, g_{\sigma(1)}\right) \qquad ; \qquad \forall\left\{f_1, g_1, \ldots, f_n, g_n\right\} \subset \mathcal{H} \qquad (65)\end{aligned}$$

  where $(\cdot,\cdot)_n$ is the standard scalar product on $\mathcal{H}^{\otimes n}$,

- The Hilbert space $\mathcal{H}_n := \mathcal{H}^{\otimes n} / \langle\cdot,\cdot\rangle_n$.

Clearly, $\mathbb{C} \oplus \mathcal{H} \oplus \bigoplus_{k=2}^{\infty} \mathcal{H}_k$ is a generalization of the q–Fock space, and will be called the $\{q_n\}_{n=2}^\infty$–IFS. It is a 1-mode type IFS with

$$\omega_n := \begin{cases} \left(\sum_{\sigma\in\mathcal{S}_{n-1}} q_{n-1}^{|\sigma|}\right)^{-1} \sum_{\sigma\in\mathcal{S}_n} q_n^{|\sigma|}, & \text{if } n \geq 2 \\ 1, & \text{if } n = 1 \end{cases}$$

So (73) holds on the 1–mode type IFS $\Gamma\left(\left\{\|f\|^2 \omega_n\right\}_{n=1}^\infty\right)$ for any $f \in \mathcal{H}$.

5 Characterization of quantum decompositions of symmetric classical random variables in IFSs

We recall from [7] the definition of IFS. Intuitively and IFS is the most general pre–Hilbert space where it makes sense to speak of creation and annihilation operators.
All constructions used in the following, like direct sums and tensor products, are algebraic. For any pair of pre–Hilbert spaces $(H, \langle \cdot , \cdot \rangle_H)$, $(K, \langle \cdot , \cdot \rangle_K)$, $\mathcal{L}_a((H, \langle \cdot , \cdot \rangle_H), (K, \langle \cdot , \cdot \rangle_K))$, or simply when no confusion is possible $\mathcal{L}_a(H, K)$, denotes the space of all adjointable pre–Hilbert space maps $A : H \to K$, such that there exists a linear map $A^* : K \to H$ satisfying

$$\langle f, Ag \rangle_K = \langle A^* f, g \rangle_H , \qquad \forall g \in H \ , \ \forall f \in K.$$

If $H = K$ $\mathcal{L}(K, \langle \cdot , \cdot \rangle_K)$ has a natural structure of $*$–algebra and we simply write $\mathcal{L}_a(K)$.

Definition 5 Let V be a vector space. An **interacting Fock space on** V is a pair:

$$\left\{ (H_n, \langle \cdot , \cdot \rangle_n)_{n \in \mathbb{N}}, a^+ \right\} \tag{66}$$

such that:
– $\left(H_n, \langle \cdot , \cdot \rangle_n\right)_{n \in \mathbb{N}}$ is a sequence of pre–Hilbert spaces with

$$H_0 =: \mathbb{C} \cdot \Phi_0 \ , \quad \|\Phi_0\| = 1,$$

Φ_0 is called the **vacuum or Fock vector**;
– denoting $\langle \cdot , \cdot \rangle$ the unique pre–Hilbert space scalar product on the vector space direct sum of the family $\left(H_n\right)_{n \in \mathbb{N}}$ which makes this direct sum

$$H := \bigoplus_{n \in \mathbb{N}} (H_n, \langle \cdot , \cdot \rangle_n) \tag{67}$$

an orthogonal sum and whose restriction on each H_n^0 coincides with $\langle \cdot , \cdot \rangle_n$, the linear operator

$$a^+ : V \to \mathcal{L}_a \left((H_n, \langle \cdot , \cdot \rangle_n)_{n \in \mathbb{N}} \right)$$

satisfies the following conditions:

$$H_{n+1} = \text{lin-span } \left\{ a^{+}(V)H_n \right\} , \qquad \forall n \in \mathbb{N}. \tag{68}$$

For each $v \in V$, $a^{+}(v)$ has an adjoint denoted $a^{-}(v)$ (or simply a_v) so that

$$a(v)\Phi_0 = 0 \text{ Fock prescription} , \qquad \forall v \in V \tag{69}$$

The operators $a^{+}(v)$ $(f \in V)$ are called **creators** and their adjoints $a(v)$ **annihilators**. The spaces $\big(H_n\big)_{n\in\mathbb{N}}$ are called the n–**particle spaces**, if $n = 0$ one speaks of the vacuum space.

An IFS over the pre–Hilbert space $\mathcal{H}$,

$$\Gamma_I(\mathcal{H}) := \big((\mathcal{H}_n, \langle \,\cdot\, , \,\cdot\, \rangle_n)_{n\in\mathbb{N}}, \mathcal{H}, A^{+}, \Phi\big)$$

is said to be of 1–**mode type, associated to the principal Jacobi sequence** (ω_n) if the tensor representation of $\Gamma_I(\mathcal{H})$ (see [5]) has the form:

$$\mathcal{H}_n := \mathcal{H}^{\otimes n} \quad ; \quad \langle F, G\rangle_n := (F, G)_n \prod_{k=1}^{n} \omega_k \quad ; \quad \forall n \ , \ \forall F, G \in \mathcal{H}^{\otimes n} \tag{70}$$

where $\otimes$ denotes algebraic tensor product, $(\cdot, \cdot)_n := \langle \cdot, \cdot \rangle_{\mathcal{H}}^{\otimes n}$ and $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ denotes the scalar product on $\mathcal{H}$. For such spaces we use the notation $\Gamma_I(\mathcal{H}, (\omega_n))$.

Practically all known quantum decompositions of classical random variables are realized in some IFS space. In this section we study the quantum moment problem in this class of spaces. Let be given an IFS

$$\Gamma_I(\mathcal{H}) := \big((\mathcal{H}_n, \langle \,\cdot\, , \,\cdot\, \rangle_n)_{n\in\mathbb{N}}, \mathcal{H}, A^{+}, \Phi\big)$$

over the pre–Hilbert space $\mathcal{H}$ in its tensor representation (i.e. $\mathcal{H}_{n+1} = \mathcal{H} \otimes \mathcal{H}_n$ see [7]), with vacuum vector Φ. By definition the family of pre–scalar products $\{\langle \cdot, \cdot \rangle_n\}_{n=1}^{\infty}$ satisfies the consistency condition: for any n and $f \in \mathcal{H}$,

$$\langle F, F\rangle_n = 0 \ \Rightarrow \ \langle f \otimes F, f \otimes F\rangle_{n+1} = 0 \qquad ; \qquad \forall F \in \mathcal{H}_n$$

This implies that the creation $A^{+}(f)$ with test–vector f is well defined by $A^{+}(f)F := f \otimes F$ $(f \in \mathcal{H},\ F \in \mathcal{H}_n)$ and has an adjoint, denoted $A(\,\cdot\,)$

and called annihilation) operator. For any test–vector $f \in \mathcal{H}$ one defines the **field operator**

$$X_f := A_f^+ + A_f \tag{71}$$

With respect to the vacuum state $\langle \Phi, (\ \cdot\ )\Phi \rangle$, X_f is a classical symmetric random variable with all moments. Therefore the triple $(\Gamma_I(\mathcal{H}), A^+(f), \Phi)$ is a quantum decomposition of X_f, in the sense of Definition 1. Any such a decomposition will be called an **IFS–quantum decomposition**.

Denote $(\Gamma(\omega), a^+, \Phi_0)$ the IFS of the canonical quantum decomposition of X_f, in the sense of Definition 2. This exists because X_f has moments of all orders.

In the present section, we study the following problem:

under which conditions is a quantum decomposition $(\Gamma_{I,f}(\mathcal{H}), A^+(f), \Phi)$ **of X_f isomorphic to its canonical quantum decomposition**?

In other words, by definition of quantum decomposition, one has

$$\left\langle \Phi, \left(A_f + A_f^+\right)^{2n} \Phi \right\rangle = \left\langle \Phi, \left(a + a^+\right)^{2n} \Phi \right\rangle \qquad ; \qquad \forall n \tag{72}$$

and we are interested in knowing under which conditions for all n and for all $\varepsilon \in \{-1, 1\}_+^{2n}$:

$$\left\langle \Phi, A^{\varepsilon(1)}(f) \ldots A^{\varepsilon(2n)}(f)\, \Phi \right\rangle = \left\langle \Phi, a^{\varepsilon(1)} \ldots a^{\varepsilon(2n)} \Phi \right\rangle \tag{73}$$

The following Theorem provides a necessary an sufficient condition to guarantee the validity of (73).

Theorem 4 In the above notations and assumptions, for any fixed $f \in \mathcal{H}$, the following statements are equivalent:

1) The IFS of the quantum decomposition $(\Gamma_I(\mathcal{H}), A^+(f), \Phi)$, of X_f, is isomorphic to the canonical quantum decomposition of X_f.

2) There exists a principal Jacobi sequence $(\omega_{f,n})$ such that

$$A_f\left(A_f^{+n}\Phi\right) = \omega_{f,n} A_f^{+(n-1)}\Phi \qquad ; \qquad \forall n \in \mathbb{N} \tag{74}$$

Remark. Notice that, if a sequence $(\omega_{f,n})$ satisfies (74), then it is necessarily a principal Jacobi sequence. In fact the $\omega_{f,n}$ are clearly positive and, if $\omega_{f,n} = 0$, then for $m \geq n$ one has, for some positive number c_{m-n}:

$$\omega_{f,m} = \|A_f^{+m}\Phi\|^2 = \langle A_f^{+m}\Phi, A_f^{+m}\Phi \rangle = \langle A_f^{+n}\Phi, A_f^{(m-n)} A_f^{+(m-n)} A_f^{+n}\Phi \rangle$$

$$= c_{m-n}\langle A_f^{+n}\Phi, A_f^{+n}\Phi\rangle = c_{m-n}\omega_{f,n}\|A_f^{+(n-1)}\Phi\|^2 = 0$$

Proof. If 2) is verified, then the same arguments used in the proof of Lemma 6 lead to the identity

$$\left\langle \Phi, A^{\varepsilon(1)}(f)\dots A^{\varepsilon(2n)}(f)\,\Phi\right\rangle = \prod_{k=1}^{n} \omega_{f,2k-l_k^\varepsilon} \tag{75}$$

Therefore (73) holds because of Corollary 1.

Conversely, assume that 1) holds and let $(\omega_{f,n})$ be the principal Jacobi sequence of X_f. Since $A_f A_f^+$ is gradation preserving, for each $k \in \mathbb{N}^*$, there exists a scalar c_k such that

$$A_f A_f^+ \left(A_f^+\right)^{k-1}\Phi = c_k \left(A_f^+\right)^{k-1}\Phi \tag{76}$$

The identity

$$\omega_{f,1} = \left\langle a^+\Phi, a^+\Phi\right\rangle = \left\langle \Phi, aa^+\Phi\right\rangle = \left\langle \Phi, A(f)\,A_f^+\Phi\right\rangle$$

then implies that $A_f A_f^+\Phi = \omega_{f,1}\Phi$. Suppose by induction that, for any $k \le n-1$, $c_k = \omega_{f,k}$. If, for some $k \le n-1$, $c_k = \omega_{f,k} = 0$, then $c_n = \omega_{f,n}$ because both c_n and $\omega_{f,n}$ are principal Jacobi sequences. Then one can suppose that, for any $k \le n-1$, $c_k = \omega_{f,k} \neq 0$. Hence

$$\prod_{k=1}^{n}\omega_{f,k} = \left\langle \left(a^+\right)^n\Phi, \left(a^+\right)^n\Phi\right\rangle = \left\langle \left(A_f^+\right)^n\Phi, \left(A_f^+\right)^n\Phi\right\rangle = \left\langle \left(A_f^+\right)^{n-1}\Phi, A_f A_f^+\left(A_f^+\right)^{n-1}\Phi\right\rangle$$

$$= c_n\left\langle \left(A_f^+\right)^{n-1}\Phi, \left(A_f^+\right)^{n-1}\Phi\right\rangle = c_n\left\langle \left(a^+\right)^{n-1}\Phi, \left(a^+\right)^{n-1}\Phi\right\rangle = c_n\prod_{k=1}^{n-1}\omega_{f,k}$$

and this implies that

$$c_n = \omega_{f,n} \qquad ; \qquad \forall n \tag{77}$$

From (76) and (77) , (73) follows easily by induction. □

Corollary 3 Let (ω_n) be a principal Jacobi sequence and let $\mathcal{H}$ be any non–zero pre–Hilbert space. Then, for any field operator X_f, on the 1–mode type IFS $\Gamma_I(\mathcal{H}, (\omega_n))$:

1) the quantum decomposition (71) of X_f is isomorphic to its canonical quantum decomposition;
2) the principal Jacobi sequence of X_f is $\omega_{f,n} := \omega_n \|f\|^2$.
In particular, any 1–mode type IFS is of type I in the sense of Definition 3.

Proof. For any $f \in \mathcal{H}$, (70) implies that

$$A_f\left(f^{\otimes n}\right) = \omega_n \|f\|^2 f^{\otimes(n-1)} \qquad ; \qquad \forall n$$

Therefore condition (73) of Theorem 4 holds on $\Gamma_I(\mathcal{H}, (\omega_n))$ with
$\omega_{f,n} := \omega_n \|f\|^2$. $\square$

As an application of Corollary 3, we construct a module generalization of the q–Fock space (see section 2.2).
Let $\mathcal{H}$ be a pre–Hilbert space with pre–scalar product $(\cdot,\cdot)$ and $\{q_n\}_{n=2}^{\infty} \subset [-1,1]$ be a sequence such that $q_1 := 1$ and $q_{n+m} = -1$ for any m whenever $q_n = -1$. For any $n \geq 2$, one introduces:

- the linear operator $\lambda_n : \mathcal{H}^{\otimes n} \longmapsto \mathcal{H}^{\otimes n}$ defined by

$$\lambda_n\left(f_1 \otimes \ldots \otimes f_n\right) := \sum_{\sigma \in \mathcal{S}_n} q_n^{|\sigma|} f_{\sigma(1)} \otimes \ldots \otimes f_{\sigma(n)} \qquad ; \qquad \forall \{f_1, \ldots, f_n\} \subset \mathcal{H}$$

- the (positive) bilinear form $\langle\cdot,\cdot\rangle_n$ on $\mathcal{H}^{\otimes n} \times \mathcal{H}^{\otimes n}$ by

$$\begin{aligned}\langle f_1 \otimes \ldots \otimes f_n, g_1 \otimes \ldots \otimes g_n\rangle_n &:=:= \left(f_1 \otimes \ldots \otimes f_n, \lambda_n g_1 \otimes \ldots \otimes g_n\right)_n \\ &= \sum_{\sigma \in \mathcal{S}_n} q_n^{|\sigma|} \prod_{k=1} \left(f_k, g_{\sigma(1)}\right) \qquad ; \qquad \forall \{f_1, g_1, \ldots, f_n, g_n\} \subset \mathcal{H} \qquad (78)\end{aligned}$$

  where $(\cdot,\cdot)_n$ is the standard scalar product on $\mathcal{H}^{\otimes n}$,

- The Hilbert space $\mathcal{H}_n := \mathcal{H}^{\otimes n} / \langle\cdot,\cdot\rangle_n$.

Clearly, $\mathbb{C} \oplus \mathcal{H} \oplus \bigoplus_{k=2}^{\infty} \mathcal{H}_k$ is a generalization of the q–Fock space, and will be called the $\{q_n\}_{n=2}^{\infty}$–IFS. It is a 1-mode type IFS with

$$\omega_n := \begin{cases} \left(\sum_{\sigma \in \mathcal{S}_{n-1}} q_{n-1}^{|\sigma|}\right)^{-1} \sum_{\sigma \in \mathcal{S}_n} q_n^{|\sigma|}, & \text{if } n \geq 2 \\ 1, & \text{if } n = 1 \end{cases}$$

5.1 An example of type II IFS

Let $X \in \mathbb{M}_2 := \mathbb{M}_2(\mathbb{C})$ be a positive definite 2×2 matrix and denote $1 \in \mathbb{M}_2$ the identity matrix. We introduce the following operators:

$$\lambda_1 := 1 \in \mathbb{M}_2$$

$$\lambda_2 := 1 \otimes X \in \mathbb{M}_2^{\otimes 2} \equiv \mathbb{M}_{2^2}$$

$$\begin{aligned} \lambda_n &:= 0 \in \mathbb{M}_2^{\otimes 2n} \equiv \mathbb{M}_{2^n} \qquad ; \qquad \forall n \geq 3 \\ \Gamma_I\left(\mathbb{C}^2, \lambda\right) &:= \text{ the IFS over } \mathbb{C}^2 \text{ with interaction } \lambda := \{\lambda_n\}_{n=1}^{\infty} \\ &= \mathbb{C} \oplus \mathbb{C}^2 \oplus \left(\mathbb{C}^2\right)^{\otimes 2} \end{aligned} \tag{79}$$

The scalar products on $\mathbb{C}$ and $\mathbb{C}^2$ are the Euclidean ones, while the scalar product on $(\mathbb{C}^2)^{\otimes 2}$ is given by

$$\langle f_1 \otimes g_1, f_2 \otimes g_2 \rangle_2 := \langle f_1, f_2 \rangle \langle g_1, X g_2 \rangle \quad ; \quad \forall \{f_1, g_1, f_2, g_2\} \subset \mathbb{C}^2$$

For $f_1, f_2, f_3 \in \mathbb{C}^2$, the creation operators are defined by

$$\Phi := (1,0,0) \overset{A^+(f_1)}{\longrightarrow} (0, f_1, 0) \overset{A^+(f_2)}{\longrightarrow} (0,0,f_2 \otimes f_1) \overset{A^+(f_3)}{\longrightarrow} 0 \tag{80}$$

and the family $\{\lambda_n\}_{n=1}^{\infty}$, satisfies the consistency condition that guarantees the existence of their adjoints, i.e. the annihilation operators

$$A(f) := \left(A^+(f)\right)^* \qquad ; \qquad \forall f \in \mathbb{C}^2$$

In the following we will use indifferently the notations $A^{\pm}(f)$ or $A_f^{\pm}$. Explicitly, $A(f)$ is given by

$$A(f)\Phi = 0 \quad ; \quad A(f)g = A(f)A^+(g)\Phi = \langle f, g \rangle \Phi \tag{81}$$

$$\begin{aligned} A(f)(g_1 \otimes g_2) = A(f)A^+(g_1)A^+(g_2)\Phi &= \langle f, g_1 \rangle A^+(Xg_2)\Phi \\ &= \langle f, g_1 \rangle X g_2 \end{aligned} \tag{82}$$

(80), (81) and (82) imply that

$$A^{\varepsilon(1)}(f_1) \dots A^{\varepsilon(n)}(f_n)\Phi = 0 \tag{83}$$

for any $\varepsilon \in \{-1, 1\}^n$ such that either

$$\left\{ p \in \{1, \dots, n\} \ : \ \sum_{k=p}^{n} \varepsilon(k) < 0 \right\} \neq \emptyset$$

or

$$\left\{p \in \{1, \ldots, n\} : \sum_{k=p}^{n} \varepsilon(k) > 2\right\} \neq \emptyset$$

Lemma 7 *Denote* $\langle\cdot\rangle$ *the vacuum expectation in the above defined IFS, and let* $f \in \mathbb{C}^2$ *be a unit–norm vector. Then the vacuum quantum moments of the field operator* $A^+(f) + A^-(f)$ *coincide with the canonical ones if an only if*

$$\langle f, X^m f\rangle = \langle f, Xf\rangle^m \qquad ; \qquad \forall m \in \mathbb{N} \tag{84}$$

and in this case the associated principal Jacobi sequence is

$$\omega_1 := 1 \quad ; \quad \omega_2 := \langle f, Xf\rangle \quad ; \quad \omega_n := 0 \ \text{ for any } \ n \geq 3 \tag{85}$$

Proof. Denote (ω_n) the vacuum principal Jacobi sequence of $A^+(f)+A^-(f)$ and $(\Gamma(\mathbb{C}, \omega), a^+, \Phi)$ the associated 1–MIFS. Suppose that the vacuum quantum moments of $A^+(f) + A^-(f)$ coincide with the canonical ones:

$$\left\langle A^{\varepsilon(1)}(f) \ldots A^{\varepsilon(2n)}(f)\right\rangle = \left\langle a^{\varepsilon(1)} \ldots a^{\varepsilon(2n)}\right\rangle \quad ; \quad \forall n \in \mathbb{N}, \varepsilon \in \{-1, 1\}^{2n} \tag{86}$$

Then one must have

$$\omega_1 = \left\langle \Phi, aa^+\Phi\right\rangle = \left\langle \Phi, A(f) A^+(f) \Phi\right\rangle = \|f\|^2 = 1$$

$$\begin{aligned}\omega_1\omega_2 = \omega_2 &= \left\langle a^2 a^{+2}\right\rangle = \left\langle A(f) A(f) A^+(f) A^+(f)\right\rangle \\ &= \|f\|^2 \left\langle A(f) A^+(Xf)\right\rangle = \|f\|^2 \langle f, Xf\rangle = \langle f, Xf\rangle\end{aligned}$$

$$\omega_{n+1} = \langle \Phi, a^{-1}(a^{+1})^{n+1}\Phi\rangle = \left\langle A(f) (A^+(f))^{n+1}\right\rangle = 0 \qquad ; \qquad \forall n \geq 2$$

Therefore (85) is a necessary condition for (86). Furthermore (81) and (82) imply that

$$\left\langle A(f) \left(A(f) A^+(f)\right)^n A^+(f)\right\rangle = \left\langle A(f) \left(A(f) A^+(f)\right)^{n-1} A(f) A^+(f) A^+(f)\right\rangle$$

$$= \langle f, f\rangle \left\langle A(f) \left(A(f) A^+(f)\right)^{n-1} A^+(Xf)\right\rangle$$

$$= \left\langle A(f) \left(A(f) A^+(f)\right)^{n-2} A(f) A^+(f) A^+(Xf)\right\rangle = \left\langle A(f) \left(A(f) A^+(f)\right)^{n-2} A^+\left(X^2 f\right)\right\rangle$$

$$= \ldots = \left\langle A(f) A^+(X^n f)\right\rangle = \langle f, X^n f\rangle \tag{87}$$

On the other hand,

$$\begin{aligned}\left\langle A\left(f\right)\left(A\left(f\right)A^{+}\left(f\right)\right)^{n}A^{+}\left(f\right)\right\rangle &= \left\langle a\left(aa^{+}\right)^{n}a^{+}\right\rangle \\ &= \omega_2^n\left\langle aa^{+}\right\rangle = \omega_2^n = \langle f, Xf\rangle^n \qquad (88)\end{aligned}$$

Therefore also (84) is a necessary condition for (86).

Conversely, suppose that (84) and (85) hold and let us prove that (86) holds. Surely (86) holds for $n = 0, 1, 2$ because of (85). Suppose that (86) holds for all $m \leq n$ and let us prove that it holds for $n+1 \geq 3$. For any $\varepsilon \in \{-1, 1\}_{+}^{2n+2}$, denote

$$p_\varepsilon := \min\left\{p \in \{1, \ldots, n+1\} \ : \ \sum_{k=1}^{2p}\varepsilon\left(k\right) = 0\right\}$$

If $p_\varepsilon < n+1$, the factorization principle of IFS gives

$$\left\langle B^{\varepsilon(1)}\cdots B^{\varepsilon(2n+2)}\right\rangle = \left\langle B^{\varepsilon(1)}\cdots B^{\varepsilon(2p_\varepsilon)}\right\rangle\left\langle B^{\varepsilon(2p_\varepsilon+1)}\cdots B^{\varepsilon(2n+2)}\right\rangle \qquad (89)$$

for both $B^{\pm} = a^{\pm}$ and $B^{\pm} = A^{\pm}\left(f\right)$. Therefore, by the induction assumption

$$\begin{aligned}&\left\langle A^{\varepsilon(1)}\left(f_1\right)\cdots A^{\varepsilon(2n+2)}\left(f_{2n+2}\right)\right\rangle = \\ &= \left\langle A^{\varepsilon(1)}\left(f_1\right)\cdots A^{\varepsilon(2p_\varepsilon)}\left(f_{2p_\varepsilon}\right)\right\rangle\left\langle A^{\varepsilon(2p_\varepsilon+1)}\left(f_{2p_\varepsilon+1}\right)\cdots A^{\varepsilon(2n+2)}\left(f_{2n+2}\right)\right\rangle \\ &= \left\langle a^{\varepsilon(1)}\cdots a^{\varepsilon(2p_\varepsilon)}\right\rangle\left\langle a^{\varepsilon(2p_\varepsilon+1)}\cdots a^{\varepsilon(2n+2)}\right\rangle = \left\langle a^{\varepsilon(1)}\cdots a^{\varepsilon(2n+2)}\right\rangle\end{aligned}$$

Now let us consider the case $p_\varepsilon = n+1$. We can restrict to the case in which $\varepsilon\left(1\right) = -1$ and $\varepsilon\left(2n+2\right) = +1$. Otherwise the identity (86) is reduced to $0 = 0$, that is identically satisfied. In this case, the definition of p_ε implies that

$$0 = \sum_{k=1}^{2n+2}\varepsilon\left(k\right) = -1 + \sum_{k=2}^{2n+1}\varepsilon\left(k\right) + 1 = \sum_{k=2}^{2n+1}\varepsilon\left(k\right)$$

But, since we are in the case $p_\varepsilon = n+1$, then also $\varepsilon\left(2n+1\right) = +1$.
If there exists a $p \in \{1, \ldots, 2n-1\}$ such that $\sum_{k=p}^{2n+2}\varepsilon\left(k\right) > 2$, then

$$A^{\varepsilon(p)}\left(f\right)\ldots A^{\varepsilon(2n+2)}\left(f\right)\Phi = 0 \qquad (90)$$

But, due to the fact that $\omega_n = 0$ for $n \geq 3$, one has also

$$a^{\varepsilon(p)}\ldots a^{\varepsilon(2n+2)}\Phi = 0 \qquad (91)$$

Therefore also in this case, the identity (86) is reduced to $0 = 0$. Therefore one must have

$$\varepsilon(2n) = -1 \quad ; \quad \sum_{k=p}^{2n+2} \varepsilon(k) \leq 2 \iff \sum_{k=p}^{2n} \varepsilon(k) \leq 0 \quad ; \quad \forall p \in \{1, \ldots, 2n-1\}$$

But if $\sum_{k=p}^{2n} \varepsilon(k) < 0$, the moment $\langle \Phi, A^{\varepsilon(p)}(f) \ldots A^{\varepsilon(2n+2)}(f) \Phi \rangle = 0$, hence it coincides with $\langle \Phi, a^{\varepsilon(p)}(f) \ldots a^{\varepsilon(2n+2)}(f) \Phi \rangle$. Therefore the only possible non–zero configuration is

$$(\varepsilon(2k), \varepsilon(2k+1) = (-1, +1) \qquad ; \qquad \forall k = 1, \ldots, n$$

corresponding to the expectation value

$$\left\langle A(f) \left(A(f) A^+(f)\right)^n A^+(f) \right\rangle$$

which has already been shown, in (87), to be equal to $\langle f, X^n f \rangle$.
On the other hand, we have seen in (88) that, on any 1–MIFS with $\omega_2 = \langle f, Xf \rangle$, one has

$$\left\langle a \left(aa^+\right)^n a^+ \right\rangle = \omega_2^n = \langle f, Xf \rangle^n$$

Therefore the identity

$$\left\langle A(f) \left(A(f) A^+(f)\right)^n A^+(f) \right\rangle = \left\langle a \left(aa^+\right)^n a^+ \right\rangle$$

is equivalent to (84), which holds by assumption. In conclusion, if (84) and (85) hold, then (86) holds for any n. $\square$

Now we use the above result to show that X can be chosen so that the IFS (79) is of type II. Take $X = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Then $X^m = \begin{pmatrix} 1 & 0 \\ 0 & 2^m \end{pmatrix}$ for any $m \in \mathbb{N}$. For any $s, t \in \mathbb{C}$ such that $|s|^2 + |t|^2 = 1$, denote by $f := \begin{pmatrix} s \\ t \end{pmatrix}$ and

$$\langle f, X^m f \rangle = |s|^2 + 2^m |t|^2 \qquad ; \qquad \langle f, Xf \rangle^m = \left(|s|^2 + 2|t|^2\right)^m$$

Let us show that condition (84) of Lemma 7 holds if and only if $st = 0$. Denote $x := |t|^2$. Then $x \in [0, 1]$ and

$$g_m(x) := \langle f, Xf \rangle^m - \langle f, X^m f \rangle$$

$$= (1+x)^m - (1 + (2^m - 1)\,x) = \sum_{k=1}^{m} \binom{m}{k} \left(x^k - x\right) \qquad (92)$$

Condition (84) is then equivalent to $g_m(x) = 0$ and (92) implies that $g_m(0) = g_m(1) = 0$. On the other hand, for any $m \geq 2$ and $x \in (0,1)$, one has

$$g_m(x) = x \sum_{k=1}^{m} \binom{m}{k} \left(x^{k-1} - 1\right) = x \sum_{k=2}^{m} \binom{m}{k} \left(x^{k-1} - 1\right) < 0$$

i.e. $x \in (0,1)$ cannot be a root of g_m whenever $m \geq 2$.
In conclusion: the conditions of Lemma 7 are satisfied by the two vectors

$$f_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad ; \qquad f_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

but by no other vector. This shows that the IFS defined by (79) and (80) is type II.

5.2 An example of type III IFS: the monotone Fock space over $\mathbf{L}^2([0,1])$

The monotone Fock space over $\mathbf{H} := \mathbf{L}^2([0,1])$ is defined, in its tensor decomposition, by

$$\mathbf{H}_n := \mathbf{H}^{\odot n} \qquad ; \qquad A_f^+ F := f \odot F \qquad ; \qquad f \in \mathbf{H}\,,\ F \in \mathbf{H}_n\,,\ n \in \mathbb{N} \quad (93)$$

and the scalar product on each $\mathbf{H}_n$ is uniquely determined by the condition that, for any n and for any $f_0, g_0, g_1, \ldots, g_n \in \mathbf{L}^2([0,1])$,

$$A_{g_0} A_{g_0}^+ A_{g_1}^+ \ldots A_{g_n}^+ \Phi = \int_0^1 \left(\overline{f} g\right)(t)\, A^+\left(g_1 \chi_{(t,1]}\right) A^+(g_2) \ldots A^+(g_n)\, \Phi dt \quad (94)$$

$$= \int_0^1 \left(\overline{f} g\right)(t)\, A^+\left(g_1 \chi_{(t,1]}\right) A^+\left(g_2 \chi_{(t,1]}\right) \ldots A^+\left(g_n \chi_{(t,1]}\right) \Phi dt$$

Lemma 8 *The monotone Fock space over* $\mathbf{H} := \mathbf{L}^2([0,1])$ *is a Type III IFS.*

Proof. By definition for all $s \in [0,1]$:

$$\left(A_f \left(f^{\otimes 2}\right)\right)(s) = \int_0^1 |f|^2 (t) \left(A^+\left(f \chi_{(t,1]}\right) \Phi\right)(s)\, dt \qquad (95)$$

$$= f(s) \int_0^s |f|^2 (t)\, dt$$

Therefore

$$A_f \left(f^{\otimes 2}\right) = 0 \cdot f \iff f(s) \int_0^s |f|^2 (t)\, dt = 0 \qquad ; \qquad \text{for a.a. } s \in [0,1]$$

Therefore also

$$0 = \int_0^1 |f|^2 (s)\, ds \int_0^s |f|^2 (t)\, dt = \frac{1}{2} \left(\int_0^1 |f|^2 (s)\, ds \right)^2$$

so that $f = 0$ a.e.. If $A_f \left(f^{\otimes 2}\right) = cf$ with $c \neq 0$, (95) gives

$$f(s) = c^{-1} f(s) \int_0^s |f|^2 (t)\, dt \qquad ; \qquad \forall s \in [0,1]$$

hence

$$|f|^2 (s) = c^{-1} |f|^2 (s) \int_0^s |f|^2 (t)\, dt \qquad ; \qquad \forall s \in [0,1]$$

Therefore, for all $n \in \mathbb{N}$ and $s \in [0,1]$:

$$\begin{aligned} |f|^2 (s) &= c^{-1} |f|^2 (s) \int_0^s |f|^2 (t_1)\, dt_1 = c^{-1} |f|^2 (s) \int_0^s c^{-1} |f|^2 (t_1)\, dt_1 \int_0^{t_1} |f|^2 (t_2)\, dt_2 \\ &= \ldots = c^{-n} |f|^2 (s) \int_0^s |f|^2 (t_1)\, dt_1 \int_0^{t_1} |f|^2 (t_2)\, dt_2 \int_0^{t_2} \ldots \int_0^{t_{n-1}} |f|^2 (t_n)\, dt_n \\ &= |f|^2 (s) \frac{c^{-n}}{n!} \left(\int_0^s |f|^2 (t)\, dt \right)^n \leq |f|^2 (s) \frac{c^{-n}}{n!} \left(\int_0^1 |f|^2 (t)\, dt \right)^n \end{aligned}$$

hence again $f = 0$ a.e.. Thus $A_f A_f^{+2} \Phi = c A_f^{+} \Phi$ if and only if $f = 0$ a.e.. The same argument used in the Remark after Theorem 4 allows then to conclude that (73) can be satisfied if and only if $f = 0$.

6 1–MIFSs arising from Fock type limit theorems

Let be given:
– a $*$–algebra $\mathcal{A}$ with identity 1;

– a state φ on $\mathcal{A}$;
– a sequence $\{S_N\}_{N=1}^{\infty}$ in $\mathcal{A}$ satisfying the following **algebraic independence condition**:
for any $n \in \mathbb{N}$, denoting $\mathcal{A}_n$ the algebra generated by $\{S_m, S_m^+, 1\}$, one has

$$\mathcal{A}_n \cap \mathcal{A}_m = \mathbb{C} \cdot 1 \qquad ; \qquad \forall m \neq n$$

We will use the following notations:

$$\{-1,1\}_-^{2m} :=$$

$$\left\{ \varepsilon \in \{-1,1\}^{2m} : \exists k \in \{1,\dots,n\} \text{ such that either } \sum_{h=k}^{2m} \varepsilon(h) < 0 \text{ or } \sum_{h=h}^{k} \varepsilon(h) > 0 \right\}$$

$$\{-1,1\}_+^{2n} := \{-1,1\}^{2n} \setminus \{-1,1\}_-^{2n} \qquad ; \qquad n \in \mathbb{N}$$

Theorem 5 *Suppose that the following conditions are satisfied.*
*1) (***uniform boundedness***) for any $n \in \mathbb{N}$ there is $C_n \geq 0$ such that*

$$\left| \varphi\left(S_N^{(\varepsilon(1))} \cdots S_N^{(\varepsilon(n))} \right) \right| \leq C_n \quad ; \quad \forall N \ ; \ \forall \varepsilon \in \{-1,1\}^n \tag{96}$$

*2) (***symmetry***) for any $n \in \mathbb{N}$*

$$\lim_{N\to\infty} \varphi\left(S_N^{(\varepsilon(1))} \cdots S_N^{(\varepsilon(2n-1))} \right) = 0 \quad ; \quad \forall \varepsilon \in \{-1,1\}^{2n-1} \tag{97}$$

*3) (***strong Fock condition***) for any $n \in \mathbb{N}$*

$$\lim_{N\to\infty} \varphi\left(S_N^{(\varepsilon(1))} \cdots S_N^{(\varepsilon(2n))} \right) = 0 \quad ; \quad \forall \varepsilon \in \{-1,1\}_-^{2n} \tag{98}$$

*4) (***existence of n–particle vectors***) there exists a 1–MIFS–sequence $\{\lambda_n\}_{n=1}^{\infty}$ such that*

$$\lim_{N\to\infty} \varphi\left(S_N^n S_N^{+n} \right) = \lambda_n \quad ; \quad \forall n \in \mathbb{N} \tag{99}$$

5) (4–**th moment condition***)*

$$\lim_{N\to\infty} \varphi\left(S_N^n S_N^+ S_N S_N^{+n} \right) = \frac{\lambda_n^2}{\lambda_{n-1}} \quad ; \quad \forall n \in \mathbb{N} \tag{100}$$

where the right hand side of (100) is understood to be zero if $\lambda_{n-1} = 0$.

Then for any $m \in \mathbb{N}$ *and* $\varepsilon \in \{-1,1\}^m$

$$\lim_{N\to\infty}\varphi\left(S_N^{(\varepsilon(1))}\cdots S_N^{(\varepsilon(m))}\right)=\left\langle\Phi, a^{(\varepsilon(1))}\dots a^{(\varepsilon(m))}\Phi\right\rangle \tag{101}$$

where a^+ *(resp.* a*) is the creation (resp. annihilation) operator on the* 1*–MIFS* $\Gamma\left(\mathbb{C},\{\lambda_n\}_{n=1}^{\infty}\right)$ *(see Definition (4)).*

Proof. For any $n,m \in \mathbb{N}$, the uniform boundedness condition (96), the existence of n–particle vectors (99) and the commutator property (100) imply that, in the notation (17):

$$\begin{aligned}
&\left|\varphi\left(S_N^{(\varepsilon(1))}\cdots S_N^{(\varepsilon(m))}\cdot S_N\cdot S_N^{+n}\right)-\omega_n\varphi\left(S_N^{(\varepsilon(1))}\cdots S_N^{(\varepsilon(m))}\cdot S_N^{+(n-1)}\right)\right|\\
&=\left|\varphi\left(S_N^{(\varepsilon(1))}\cdots S_N^{(\varepsilon(m))}\cdot\left(S_NS_N^{+}-\omega_n\right)S_N^{+(n-1)}\right)\right|\leq\\
&\leq\left|\varphi\left(\left|S_N^{(\varepsilon(1))}\cdots S_N^{(\varepsilon(m))}\right|^2\right)\right|^{1/2}\left|\varphi\left(\left|\left(S_NS_N^{+}-\omega_n\right)S_N^{+(n-1)}\right|^2\right)\right|^{1/2}\\
&\leq C_{2m}^{1/2}\left|\varphi\left(S_N^nS_N^{+}S_NS_N^{+n}\right)+\omega_n^2\varphi\left(S_N^{n-1}S_N^{+(n-1)}\right)-2\omega_n\varphi\left(S_N^nS_N^{+n}\right)\right|^{1/2}\longrightarrow\\
&\longrightarrow C_{2m}^{1/2}\left|\omega_n\lambda_n+\omega_n^2\lambda_{n-1}-2\omega_n\lambda_n\right|^{1/2}=0
\end{aligned}\tag{102}$$

Because of (97) and (98), to prove (101), it is sufficient to consider the case m even, say $m=2n\in\mathbb{N}$ and $\varepsilon\in\{-1,1\}_{+}^{2n}$. Given such an ε, from (102) it follows that, if $\varepsilon(l_n)$ is the first $-$–sign from the right, then

$$\lim_{N\to\infty}\varphi\left(S_N^{(\varepsilon(1))}\cdots S_N^{(\varepsilon(2n))}\right)=\lim_{N\to\infty}\varphi\left(S_N^{(\varepsilon(1))}\cdots S_N^{+(2n-l_n)}\right)=$$

$$=\omega_{2n-l_n}\lim_{N\to\infty}\varphi\left(S_N^{(\varepsilon(1))}\cdots S_N^{+(2n-l_n-1)}\right)$$

On the other hand, one knows for any $n\geq 1$

$$a\left(a^+\right)^n\Phi=\omega_n\left(a^+\right)^{n-1}\Phi\quad;\quad\forall n\in\mathbb{N}$$

Therefore, in the above notations

$$\langle\Phi, a^{(\varepsilon(1))}\dots a^{(\varepsilon(2n))}\Phi\rangle$$

$$=\langle\Phi, a^{(\varepsilon(1))}\dots a^{(\varepsilon(l_n-1))}a\left(a^+\right)^{2n-l_n}\Phi\rangle=\omega_{2n-l_n}\langle\Phi, a^{(\varepsilon(1))}\dots a^{(\varepsilon(l_n-1))}\left(a^+\right)^{2n-l_n-1}\Phi\rangle$$

Therefore the two expressions

$$\lim_{N\to\infty} \varphi\left(S_N^{(\varepsilon(1))} \cdots S_N^{+(2n)}\right) \quad \text{and} \quad \langle \Phi, a^{(\varepsilon(1))} \ldots a^{(\varepsilon(2n))} \Phi \rangle$$

satisfy the same induction relation. Since (99) implies that

$$\lim_{N\to\infty} \varphi\left(S_N^- S_N^+\right) = \lambda_1 = \langle \Phi, aa^+ \Phi \rangle$$

(101) follows by induction.

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