Generalized Kantorovich operators on Bauer simplices and their limit semigroups

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Abstract

[1] In this paper we prove an asymptotic formula for generalized Kantorovich operators associated with the canonical Markov projection on a given Bauer simplex K. That formula involves an operator acting on the subalgebra of all products of affine functions on K. Moreover, we prove that such an operator is closable and its closure is the generator of a Markov semigroup which, in turn, may be represented in terms of iterates of the above mentioned generalized Kantorovich operators.

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Introduction

In [5] the authors introduced and studied a sequence $(C_n)_{n\geq 1}$ of positive linear operators acting on function spaces defined on a convex compact subset K of some locally convex Hausdorff space. Their construction depends on a given Markov operator T on $\mathscr{C}(K)$, a real number $a \geq 0$ and a sequence $(\mu_n)_{n\geq 1}$ of probability Borel measures on K. For particular choices of these parameters and for particular convex compact subsets such as the unit interval or the multidimensional hypercube and simplex, these operators turn into the Kantorovich operators and into several of their wide-ranging generalizations (for more details, see [5] and the references quoted therein).

In [6], in the finite dimensional setting, the authors proved an asymptotic formula for the operators C_n; such an asymptotic formula involves an elliptic second-order differential operator which is the pre-generator of a Markov semigroup; it is also possible to represent such a semigroup in terms of

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suitable iterates of the operators C_n. This, in turn, allows to infer some spatial regularity properties of the solution to the initial-boundary value problems governed by the above mentioned differential operator by means of the relevant ones held by the C_n's.

The main aim of this paper is to extend the results in [6] to the infinite dimensional setting and, more precisely, in the context of Bauer simplices. As it is well known, those mathematical structures play an important role in the theory of integral representations for convex compact sets (see, e.g., [13] or [3, Section 1.5]), and a Markov projection T, which will be used in the paper as one of the parameters in the construction of the sequence $(C_n)_{n\geq 1}$, is naturally associated with them.

In particular, in what follows, we prove an asymptotic formula for the operators C_n associated with the canonical Markov projection T on $\mathscr{C}(K)$, K being a Bauer simplex; that formula involves an operator acting on the subalgebra of all products of affine functions on K. In the finite dimensional setting of the canonical simplex K_d on $\mathbf{R}^d$, $d \geq 1$, this operator coincides with an elliptic second-order differential operator which has been studied in [6] and which is a first order perturbation of the well known Fleming-Viot operator on K_d (see, for example, [1], [2], [9], [10], [11]).

Moreover, coming back to the general case, we prove that the operator in the asymptotic formula is closable and its closure A is the generator of a Markov semigroup which, it turn, may be represented in terms of iterates of the C_n's. That representation allows us to determine some regularity properties for the solution to the abstract Cauchy problems governed by A, by inferring them by means of similar ones shared by the operators C_n.

1 Preliminaries and notations

Throughout this paper we shall fix a locally convex Hausdorff space X and a convex compact subset K of X. The symbol $\partial_e K$ denotes the set of all the *extreme points* of K, i.e., those points $x_0 \in K$ such that $K \setminus \{x_0\}$ is a convex set.

As usual, we denote by $\mathscr{C}(K)$ the space of all real-valued continuous functions on K; $\mathscr{C}(K)$ is endowed with the natural (pointwise) ordering and the sup-norm $\|\cdot\|_\infty$, with respect to which it is a Banach lattice.

From now on, the symbol $\mathbf{1}$ stands for the constant function of constant value 1 on K; moreover, we denote by $A(K)$ the space of all continuous affine functions on K. For every $m \geq 1$, the symbol $P_m(K)$ stands for the linear subspace generated by products of m continuous affine functions on K, namely,

$$P_m(K) := \operatorname{span}\left(\left\{\prod_{i=1}^{m} h_i \;\middle|\; h_1, \dots, h_m \in A(K)\right\}\right).$$

Clearly, $P_m(K) \subset P_{m+1}(K)$ for every $m \geq 1$ and

$$P_\infty(K) := \bigcup_{m \geq 1} P_m(K) \tag{1.1}$$

is a subalgebra of $\mathscr{C}(K)$ which, by the Stone-Weierstrass theorem, is dense in $\mathscr{C}(K)$.

Finally, let B_K be the σ-algebra of all Borel subsets of K and $M^+(K)$ (resp., $M_1^+(K)$) the cone of all regular Borel measures on K (resp., the cone of all regular probability Borel measures on K).

The paper will focus on particular convex compact subsets, namely the Bauer simplices, which, as it is well known, play an important role in the theory of integral representations for convex compact sets. For more details we refer to [13] or [3, Section 1.5 and the references therein]. In what follows, we briefly recall the definition of those mathematical objects.

In particular, once we set

$$G(K) := \{\lambda K + a \ \mid \ \lambda \geq 0, \ a \in X\},$$

K is said to be a *Choquet simplex* if the intersection of two arbitrary elements of $G(K)$ belongs to $G(K)$, provided that it is non-empty.

In order to provide a set of examples in a finite dimensional context, we recall that p points $x_1, \ldots, x_p \in \mathbf{R}^d$, $d \geq 1$, are said to be *affinely independent* if for every $\lambda_1, \ldots, \lambda_p \in \mathbf{R}$ satisfying $\sum\limits_{i=1}^{p} \lambda_i x_i = 0$ and $\sum\limits_{i=1}^{p} \lambda_i = 0$, we get $\lambda_1 = \ldots = \lambda_p = 0$.

Accordingly, the Choquet simplices in $\mathbf{R}^d$ are convex hulls of any $d+1$ affinely independent points, where by *convex hull* of a subset B of a vector space X we mean the smallest convex subset of X containing B.

Therefore, the set

$$K_d := \left\{ (x_1, \ldots, x_d) \in \mathbf{R}^d \ \mid \ x_i \geq 0 \text{ for every } i = 1, \ldots, d \text{ and } \sum_{i=1}^{d} x_i \leq 1 \right\} \tag{1.2}$$

is a Choquet simplex in $\mathbf{R}^d$, being the convex hull of $\{v_0, \ldots, v_d\}$, where

$$v_0 := (0, \ldots, 0),\ v_1 := (1, 0, \ldots, 0), \ldots, v_d := (0, \ldots, 0, 1), \tag{1.3}$$

and it is called the *canonical simplex of* $\mathbf{R}^d$.

Further, a *Bauer simplex* is a Choquet simplex for which $\partial_e K$ is closed.

Thus, K_d is a Bauer simplex in $\mathbf{R}^d$, since $\partial_e K_d = \{v_0, \ldots, v_d\}$.

If K is a Bauer simplex, then there always exists a (unique) Markov projection $T : \mathscr{C}(K) \to \mathscr{C}(K)$, i.e., a positive linear operator on $\mathscr{C}(K)$ such that $T \circ T = T$ and $T(\mathbf{1}) = \mathbf{1}$, such that $T(\mathscr{C}(K)) = A(K)$, as the following result shows (see [7] and [15]).

Theorem 1.1. *Given a convex compact subset K of a locally convex Hausdorff space, the following statements are equivalent:*

(a) K is a Bauer simplex.

(b) For every $x \in K$ there exists a (unique) $\tilde{\mu}_x \in M_1^+(K)$ such that $\tilde{\mu}_x(K \setminus \overline{\partial_e K}) = 0$ and

$$\int_K h \, d\tilde{\mu}_x = h(x) \qquad \text{for every } h \in A(K).$$

(c) Every continuous function $f : \partial_e K \to \mathbf{R}$ can be continuously extended to a (unique) function $\tilde{f} \in A(K)$.

(d) There exists a (unique) positive projection $T : \mathscr{C}(K) \to \mathscr{C}(K)$ such that $T(\mathscr{C}(K)) = A(K)$.

Moreover, if one of these statements holds true, then, for every $f \in \mathscr{C}(K)$ and $x \in K$,

$$T(f)(x) = \int_K f \, d\tilde{\mu}_x. \tag{1.4}$$

Given a Bauer simplex K, the positive projection $T : \mathscr{C}(K) \to \mathscr{C}(K)$ given by (1.4) is referred to as the *canonical positive projection associated with* K. Thus, for every $f \in \mathscr{C}(K)$, $T(f)$ is the unique continuous affine function on K which coincides with f on $\partial_e K$.

In the case $K = K_d$, $d \geq 1$, the canonical projection is given by

$$T_d(f)(x) := \left(1 - \sum_{i=1}^{d} x_i\right) f(v_0) + \sum_{i=1}^{d} x_i f(v_i) \tag{1.5}$$

($f \in \mathscr{C}(K_d), x = (x_1, \ldots, x_d) \in K_d, v_0, \ldots, v_d$ as in (1.3)).

In particular, for $d = 1$,

$$T_1(f)(x) := (1 - x) f(0) + x f(1) \tag{1.6}$$

($f \in \mathscr{C}([0,1]), 0 \leq x \leq 1$).

Remark 1.2. Since the canonical projection T satisfies statements (b) and (d) in Theorem 1.1, it is obvious that

$$T(h) = h \qquad \text{for every } h \in A(K) \tag{1.7}$$

and

$$T(P_2(K)) \subset A(K) \text{ i.e., } T(h_1 h_2) \in A(K) \text{ for every } h_1, h_2 \in A(K). \tag{1.8}$$

As a matter of fact, if K is a convex compact subset of some Hausdorff locally convex space X and T is a Markov operator on $\mathscr{C}(K)$ for which (1.7) and (1.8) occur, then necessarily K is a Bauer simplex and T is the canonical projection associated with K (for a proof, see [4, Theorem 4.3.3]).

The aim of this paper is to discuss in the context of Bauer simplices some further properties of a sequence of positive linear operators on $\mathscr{C}(K)$ that were introduced and studied in [5] and [6].

From now on, fix $a \geq 0$ and a sequence $(\mu_n)_{n\geq 1}$ in $M_1^+(K)$. Then, for every $n \geq 1$, we consider the positive linear operator C_n defined by setting

$$C_n(f)(x)=\int_K \cdots \int_K f\left(\frac{x_1+\ldots+x_n+ax_{n+1}}{n+a}\right) d\tilde{\mu}_x(x_1)\cdots d\tilde{\mu}_x(x_n) d\mu_n(x_{n+1}) \tag{1.9}$$

for every $x \in K$ and for every $f \in \mathscr{C}(K)$, where $(\tilde{\mu}_x)_{x\in K}$ is the continuous selection of probability Borel measures associated with the canonical projection T by means of (1.4).

Moreover, introducing the auxiliary continuous function

$$I_n(f)(x) := \int_K f\left(\frac{n}{n+a}x + \frac{a}{n+a}t\right) d\mu_n(t) \qquad (f \in \mathscr{C}(K),\, x \in K)$$

then, for every $n \geq 1$,

$$C_n(f) = B_n(I_n(f)), \tag{1.10}$$

where $(B_n)_{n\geq 1}$ is the sequence of the canonical Bernstein-Schnabl operators associated with the Bauer simplex K (see [4, Subsection 3.1.2]), which are defined, for every $f \in \mathscr{C}(K)$, $x \in K$ and $n \geq 1$, as

$$B_n(f)(x) := \int_K \cdots \int_K f\left(\frac{x_1+\ldots+x_n}{n}\right) d\tilde{\mu}_x(x_1)\cdots d\tilde{\mu}_x(x_n). \tag{1.11}$$

In particular, if $a = 0$, the operators C_n correspond to the B_n's. Moreover, $C_n(f) \in \mathscr{C}(K)$ for any $n \geq 1$ and the C_n's are positive linear operators on $\mathscr{C}(K)$; hence, each C_n is continuous and $\|C_n\| = 1$, since $C_n(\mathbf{1}) = \mathbf{1}$.

Finally, the sequence $(C_n)_{n\geq 1}$ is an approximation process on $\mathscr{C}(K)$, i.e., for every $f \in \mathscr{C}(K)$,

$$\lim_{n\to\infty} C_n(f) = f \quad \text{uniformly on } \ K \tag{1.12}$$

(see [5, Theorem 4.2]). In particular, in [5, Section 4] the authors investigated also some quantitative estimates of the rate of convergence in (1.12), both in the finite and infinite dimensional setting.

We end the section with a few examples; many others might be found in [5, Examples 3.1].

Assume $K = [0, 1]$ and consider the Markov projection T_1 on $\mathscr{C}([0, 1])$ defined by (1.6). The Bernstein-Schnabl operators (1.11) associated with T_1 are the classical Bernstein operators

$$B_n(f)(x) := \sum_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k} f\left(\frac{k}{n}\right) \tag{1.13}$$

$(n \geq 1, f \in \mathscr{C}([0,1]), x \in [0,1])$. Fix $a \geq 0$ and $(\mu_n)_{n\geq 1}$ in $M_1^+([0,1])$; then, from (1.9) and (1.10) we get

$$C_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \int_0^1 f\left(\frac{k+as}{n+a}\right) d\mu_n(s) \tag{1.14}$$

$(n \geq 1,\ f \in \mathscr{C}(K)$,$0 \leq x \leq 1)$.

In particular, for $a = 1$ and all the μ_n equal to the Borel-Lebesgue measure λ_1 on $[0,1]$, the operators in (1.14) turn into the classical Kantorovich operators on $[0,1]$ ([12]; see also [3, Subsection 5.3.7]). Moreover, as already remarked, if $a = 0$ we obtain the Bernstein operators (1.13); thus, by means of (1.14), a link between those fundamental approximation processes by means of the continuous parameter $a \in [0,1]$ is achieved.

In the setting of the d-dimensional simplex K_d (see (1.2)), let $a \geq 0$ be fixed and consider a sequence $(\mu_n)_{n\geq 1}$ in $M_1^+(K_d)$; by means of (1.10) and [4, (3.1.18)], the sequence $(C_n)_{n\geq 1}$ associated with the canonical projection T_d defined by (1.5) is given by

$$C_n(f)(x) := \sum_{\substack{h=(h_1,\dots,h_d)\in\{0,\dots,n\}^d \\ h_1+\dots+h_d\leq n}} P^*_{n,h}(x) \int_{K_d} f\left(\frac{h_1+as_1}{n+a}, \dots, \frac{h_d+as_d}{n+a}\right) d\mu_n(s_1,\dots,s_d) \tag{1.15}$$

$(n \geq 1, f \in \mathscr{C}(K_d)$ and $x = (x_1,\dots,x_d) \in K_d)$, where, for every $n \geq 1$, $h = (h_1,\dots,h_d) \in \{0,\dots,n\}^d$, $|h| := h_1+\dots+h_d \leq n$ and $x = (x_1,\dots,x_d) \in K_d$,

$$P^*_{n,h}(x) := \frac{n!}{h_1!\dots h_d!(n-h_1-\dots-h_d)!} x_1^{h_1} \dots x_d^{h_d} \left(1 - \sum_{i=1}^{d} x_i\right)^{n-\sum_{i=1}^{d} h_i}.$$

2 Asymptotic formulae

In this section, we present an asymptotic formula for the operators C_n defined by (1.9). To this end, we need some additional notation and preliminaries.

First of all, for every $m \geq 1$ and $h_1,\dots,h_m \in A(K)$ we set

$$\Theta_T(h_1,\dots,h_m) := \begin{cases} 0 & \text{if } m = 1; \\ T(h_1h_2) - h_1h_2 & \text{if } m = 2; \\ \sum\limits_{1\leq i<j\leq m} (T(h_ih_j) - h_ih_j) \prod\limits_{\substack{r=1 \\ r\neq i,j}}^{m} h_r & \text{if } m \geq 3. \end{cases} \tag{2.1}$$

In [4, Theorem 4.2.4] a useful asymptotic formula for Bernstein-Schnabl operators (1.11) which involves the operator (2.1) was presented. It runs as follows.

Theorem 2.1. *For every $m \geq 1$ and $h_1, \ldots, h_m \in A(K)$,*

$$\lim_{n\to\infty} n \left(B_n \left(\prod_{j=1}^{m} h_j \right) - \prod_{j=1}^{m} h_j \right) = \Theta_T(h_1, \ldots, h_m) \qquad \textit{uniformly on } K.$$

From Theorem 2.1 it also follows that for every $u \in P_\infty(K)$ (see (1.1)) there exists $\lim\limits_{n\to\infty} n(B_n(u)-u)$ in $\mathscr{C}(K)$ and hence we can consider the linear operator $L_T : P_\infty(K) \to \mathscr{C}(K)$ defined by

$$L_T(u) := \lim_{n\to\infty} n(B_n(u) - u) \qquad (u \in P_\infty(K)). \tag{2.2}$$

Thus, if $h_1, \ldots, h_m \in A(K)$, $m \geq 1$, then

$$L_T \left(\prod_{j=1}^{m} h_j \right) = \Theta_T(h_1, \ldots, h_m).$$

We pass to present an asymptotic formula for the sequence $(C_n)_{n\geq 1}$.
To this end, we need to evaluate the C_n's on products of affine functions.
First of all, for every $m, q \geq 1$, $1 \leq q \leq m$, set

$$N_m(q) := \{(i_1, \ldots, i_q) \in \{1, \ldots, m\}^q \mid i_r \neq i_s \text{ for } r \neq s\} \tag{2.3}$$

and

$$\begin{aligned} \widetilde{N}_m := \{((i_1, \ldots, i_q), (j_1, \ldots, j_{m-q})) \in N_m(q) \times N_m(m-q) \mid i_h \neq j_k \\ \text{for every } h = 1, \ldots, q, \text{ and } k = 1, \ldots, m-q\}. \end{aligned} \tag{2.4}$$

The following result holds true (for a proof see [6, Lemma 1.2]).

Lemma 2.2. *Let $h_1, \ldots, h_m \in A(K)$, $m \geq 1$. Then, for every $n \geq 1$,*

$$\begin{aligned} C_n \left(\prod_{j=1}^{m} h_j \right) = \frac{1}{(n+a)^m} \Bigg[\left(a^m \int_K \prod_{j=1}^{m} h_j \, d\mu_n \right) \mathbf{1} + n^m B_n \left(\prod_{j=1}^{m} h_j \right) \\ + \sum_{q=1}^{m-1} a^q n^{m-q} \sum_{((i_1,\ldots,i_q),(j_1,\ldots,j_{m-q}))\in \widetilde{N}_m} \left(\int_K h_{i_1} \cdots h_{i_q} \, d\mu_n \right) B_n \left(h_{j_1} \cdots h_{j_{m-q}} \right) \Bigg], \end{aligned} \tag{2.5}$$

where the operators B_n are defined by (1.11).

Since $M_1^+(K)$ is weakly compact (see [8]), unless replacing $(\mu_n)_{n\geq 1}$ with a subsequence, we can assume that it converges weakly to some $\mu \in M_1^+(K)$, i.e.,

$$\lim_{n\to\infty} \int_K f \, d\mu_n = \int_K f \, d\mu \quad \text{for every } f \in \mathscr{C}(K). \tag{2.6}$$

We denote by $b \in K$ the barycenter of μ, so that

$$\int_K h\,d\mu = h(b) \qquad \text{for every } h \in A(K)$$

(see, e.g., [3, p. 55]).

Finally, for every $m \geq 1$ and $h_1, \ldots, h_m \in A(K)$, we set

$$B(h_1, \ldots, h_m) := \begin{cases} a\,(h_1(b)\mathbf{1} - h_1) & \text{if } m = 1; \\ a \sum_{i=1}^{m} \Big((h_i(b) - h_i) \prod_{\substack{j=1 \\ j \neq i}}^{m} h_j \Big) & \text{if } m \geq 2. \end{cases} \tag{2.7}$$

From (2.7) it easily follows that $B(\mathbf{1}) = 0$ and that, for every $h_1, \ldots, h_m \in A(K)$,

$$\sum_{i=1}^{m} (h_i(b) - h_i) \prod_{\substack{j=1 \\ j \neq i}}^{m} h_j = \sum_{i=1}^{m} h_i(b) \prod_{\substack{j=1 \\ j \neq i}}^{m} h_j - m \prod_{j=1}^{m} h_j. \tag{2.8}$$

We are now ready to state an asymptotic formula for the operators C_n.

Theorem 2.3. *Fix $m \geq 1$ and $h_1, \ldots, h_m \in A(K)$; then*

$$\lim_{n \to \infty} n \left(C_n \left(\prod_{i=1}^{m} h_i \right) - \prod_{i=1}^{m} h_i \right) = \Theta_T(h_1, \ldots h_m) + B(h_1, \ldots, h_m)$$

uniformly on K (see (2.1) *and* (2.7)*).*

Proof. The result is straightforward for $m = 1$, taking (2.7), (2.6), (2.1), (2.5) and Theorem 2.1 into account.

Let us assume that $m \geq 2$. Then, by means of Lemma 2.2 (see also (2.3) and (2.4)), we get that

$$\begin{aligned} n \left(C_n \left(\prod_{i=1}^{m} h_i \right) - \prod_{i=1}^{m} h_i \right) &= \frac{n^{m+1}}{(n+a)^m} \left[B_n \left(\prod_{i=1}^{m} h_i \right) - \prod_{i=1}^{m} h_i \right] \\ &+ \left(\frac{n^{m+1}}{(n+a)^m} - n \right) \prod_{i=1}^{m} h_i + \left(\frac{n a^m}{(n+a)^m} \int_K \prod_{i=1}^{m} h_i \, d\mu_n \right) \mathbf{1} \\ &+ \frac{a\, n^m}{(n+a)^m} \sum_{i=1}^{m} \int_K h_i \, d\mu_n \;\; B_n \left(\prod_{\substack{j=1 \\ j \neq i}}^{m} h_j \right) \\ &+ \frac{n}{(n+a)^m} \sum_{q=2}^{m-1} a^q n^{m-q} \sum_{((i_1,\ldots,i_q),(j_1,\ldots,j_{m-q})) \in \widetilde{N}_m} \left(\int_K h_{i_1} \cdots h_{i_q} \; d\mu_n \right) B_n \big(h_{j_1} \cdots h_{j_{m-q}} \big); \end{aligned}$$

taking Theorem 2.1, (2.8), and (2.6) into account, the result easily follows, since $(B_n)_{n \geq 1}$ is. and approximation process on $\mathscr{C}(K)$ (see [4, Theorem 3.2.1]). □

From Theorem 2.3 it also follows that, for every $u \in P_\infty(K)$, there exists $\lim_{n\to\infty} n(C_n(u)-u)$ in $\mathscr{C}(K)$ and hence we can consider the linear operator $N_T : P_\infty(K) \longrightarrow \mathscr{C}(K)$ defined by

$$N_T(u) := \lim_{n\to\infty} n(C_n(u)-u) \qquad (u \in P_\infty(K)). \tag{2.9}$$

Therefore, for every $h_1, \dots, h_m \in A(K)$, $m \geq 1$,

$$N_T\left(\prod_{j=1}^{m} h_j\right) = \Theta_T(h_1, \dots, h_m) + B(h_1, \dots, h_m). \tag{2.10}$$

In other words, considering the linear operator $\widetilde{B} := N_T - L_T$ from $P_\infty(K)$ into $\mathscr{C}(K)$ (see (2.2)), we have

$$\widetilde{B}\left(\prod_{i=1}^{m} h_i\right) = B(h_1, \dots, h_m)$$

$(h_1, \dots, h_m \in A(K), m \geq 1)$ and N_T can be viewed as a particular additive perturbation of L_T, namely

$$N_T = L_T + \widetilde{B}.$$

Remark 2.4. Theorem 2.3 holds true under the more general assumptions that K is an arbitrary convex compact subset on a locally convex space X and T is a Markov operator on $\mathscr{C}(K)$ such that $T(h) = h$ for every $h \in A(K)$.

In particular, in such a case, if additionally K is a convex compact subset of $\mathbf{R}^d$ with non-empty interior $\mathrm{int}(K)$, according to [6, Theorem 2.2], the operator N_T defined in (2.9) on $P_m(K)$ coincides with the elliptic second-order differential operator $(V_T, \mathscr{C}^2(K))$, defined by setting, for every $u \in \mathscr{C}^2(K)$ and $x = (x_1, \dots, x_d) \in K$,

$$V_T(u)(x) := \frac{1}{2}\sum_{i,j=1}^{d} \left(T(pr_i pr_j)(x) - x_i x_j\right) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + a\sum_{i=1}^{d}(b_i - x_i)\frac{\partial u}{\partial x_i}(x), \tag{2.11}$$

where, for every $i = 1, \dots, d$, with the symbol pr_i we denote the i^{th} coordinate function on K, i.e., $pr_i(x) := x_i$ for every $x = (x_1, \dots, x_d) \in K$ and $\mathscr{C}^2(K)$ stands for the space of all real-valued (continuous) functions on K which are twice-continuously differentiable on the interior $\mathrm{int}(K)$ of K and whose partial derivatives up to the order 2 can be continuously extended to K.

If K_d is the d-dimensional simplex and T_d is the canonical Markov projection on $\mathscr{C}(K_d)$ given by (1.5), then

$$V_{T_d}(u)(x) = \sum_{i=1}^{d} \frac{x_i(1-x_i)}{2}\frac{\partial^2 u}{\partial x_i^2}(x) - \sum_{1\leq i<j\leq d} x_i x_j \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + a\sum_{i=1}^{d}(b_i - x_i)\frac{\partial u}{\partial x_i}(x)$$

($u \in \mathscr{C}^2(K_d), x = (x_1, \dots, x_d) \in K_d$), where $b = (b_1, \dots, b_d) \in K_d$ and $a \geq 0$.

This class of operators is often referred to as Fleming-Viot operators and it is of interest in many mathematical models in population dynamics (see, for more details, [1], [2], [9], [10], [11] and [4, Subsection 2.3.4]).

In particular, if $K = [0, 1]$ and consider the Markov operator T_1 on $\mathscr{C}([0, 1])$ defined by (1.6), then

$$V_{T_1}(u)(x) = \frac{x(1-x)}{2} u''(x) + a(b-x)u'(x)$$

($u \in \mathscr{C}^2([0, 1]), 0 \leq x \leq 1$).

However, in [6, Section 3], the authors stated more general results in the context of the unit interval.

3 The associated Markov semigroup

The main aim of this section is to show that the operator $(N_T, P_\infty(K))$ (see (2.9)) is closable and its closure $(A, D(A))$ is the generator of a Markov semigroup on $\mathscr{C}(K)$ which in turn may be approximated by suitable iterates of the operators C_n.

These results allow us to represent the solutions to the abstract Cauchy problems governed by A in terms of the C_n's and to deduce some spatial regularity properties of the relevant solutions. For unexplained terminology concerning semigroup theory, we refer, e.g., to [4, Chapter 2].

Theorem 3.1. *Let K be a Bauer simplex of some locally convex Hausdorff space and T the canonical projection on $\mathscr{C}(K)$ associated with K (see* (1.4)*). Moreover, consider $a \geq 0$, a sequence $(\mu_n)_{n\geq 1}$ of probability Borel measures on K and the relevant sequence $(C_n)_{n\geq 1}$ defined by* (1.9)*. Then the operator $(N_T, P_\infty(K))$ (see* (2.9)*) is closable and its closure $(A, D(A))$ generates a Markov semigroup $(T(t))_{t\geq 0}$ on $\mathscr{C}(K)$. Moreover*

(a) if $t \geq 0$ and if $(k(n))_{n\geq 1}$ is a sequence of positive integers such that $\lim_{n\to\infty} k(n)/n = t$, then

$$\lim_{n\to\infty} C_n^{k(n)}(f) = T(t)(f) \quad \textit{uniformly on } K$$

for every $f \in \mathscr{C}(K)$, where each $C_n^{k(n)}$ denotes the iterate of C_n of order $k(n)$.

(b) $(A, D(A))$ coincides with the closure of the linear operator $Z : D(Z) \to \mathscr{C}(K)$ defined by

$$Z(f) := \lim_{n\to\infty} n(C_n(f) - f)$$

for every $f \in D(Z)$, where

$$D(Z) := \left\{ g \in \mathscr{C}(K) \ \middle|\ \lim_{n\to\infty} n(C_n(g) - g) \ \textit{exists in } \mathscr{C}(K) \right\}.$$

(c) $P_\infty(K)$ is a core for $(A, D(A))$.

Proof. By applying Theorem 2.3 and the subsequent formula (2.9), we get that $P_\infty(K) \subset D(Z)$ and $Z = N_T$ on $P_\infty(K)$. We pass to prove that, if $\lambda > 0$, then the range $(\lambda I - Z)(D(Z))$ of $\lambda I - Z$ is dense in $\mathscr{C}(K)$. To this end, since $P_\infty(K)$ is dense in $\mathscr{C}(K)$, it suffices to show that

$$\overline{(\lambda I - Z)(P_\infty(K))} = \mathscr{C}(K) \tag{1}$$

with respect to $\|\cdot\|_\infty$.

Consider a continuous linear functional $\nu : \mathscr{C}(K) \to \mathbf{R}$ such that $\nu = 0$ on $(\lambda I - Z)(P_\infty(K))$. By a consequence of Hahn-Banach theorem, (1) will be proved once we show that $\nu = 0$ and, to this end, it suffices to prove that $\nu = 0$ on $P_\infty(K)$.

Indeed, by means of (2.10), (2.1) and (2.7),

$$\nu(\mathbf{1}) = \frac{1}{\lambda}\nu(N_T(\mathbf{1})) = \frac{1}{\lambda}\nu(\Theta_T(\mathbf{1}) + B(\mathbf{1})) = 0.$$

If $m = 1$ and $h_1 \in A(K)$, then

$$\nu(h_1) = \frac{1}{\lambda}(\nu(\Theta_T(h_1)) + \nu(B(h_1)) = \frac{ah_1(b)}{\lambda}\ \nu(\mathbf{1}) - \frac{a}{\lambda}\nu(h_1)$$

so that, also in this case, $\nu(h_1) = 0$.

Assume now that $m = 2$ and consider $h_1, h_2 \in A(K)$; then, taking (2.7), (2.1) and (2.8) into account, we have that

$$\nu(h_1h_2) = \frac{1}{\lambda}\left(\nu(T(h_1h_2) - h_1h_2)\right) + \frac{ah_1(b)}{\lambda}\ \nu(h_2) + \frac{ah_2(b)}{\lambda}\ \nu(h_1) - \frac{2a}{\lambda}\nu(h_1h_2)$$

and therefore $\nu(h_1h_2) = 0$, thanks to (1.8).

Let us finally fix $m > 2$ and suppose that $\mu = 0$ on $P_m(K)$; we shall prove that $\nu = 0$ on $P_{m+1}(K)$. To this end, consider $h_1, \ldots, h_{m+1} \in A(K)$ and set $f = \prod_{i=1}^{m+1} h_i$. We preliminarily observe that $\sum_{1\le i<j\le m+1} T(h_ih_j) \prod_{\substack{k=1\\k\ne i,j}}^{m+1} h_k \in P_m(K)$ (see (1.8)), so that, by virtue of (2.1),

$$\begin{aligned}
\nu(f) &= \frac{1}{\lambda}(\nu(\Theta_T(h_1, \ldots, h_{m+1})) + \nu(B(h_1, \ldots, h_{m+1}))) \\
&= \frac{1}{\lambda}\nu\left(\sum_{1\le i<j\le m+1} T(h_ih_j)\prod_{\substack{k=1\\k\ne i,j}}^{m+1} h_k - \binom{m+1}{2} f\right) \\
&\quad + \frac{a}{\lambda}\sum_{i=1}^{m+1} h_i(b)\ \nu\left(\prod_{\substack{j=1\\j\ne i}}^{m+1} h_j\right) - \frac{(m+1)a}{\lambda}\nu(f).
\end{aligned}$$

Accordingly, $\nu(f) = 0$; hence, by induction, $\nu = 0$ on each $P_m(K)$, $m \geq 1$, and thus $\nu = 0$ on $P_\infty(K)$.

By virtue of a theorem due to Trotter (see, for example, [4, Theorem 2.2.1]), there exists a contractive C_0-semigroup $(T(t))_{t\geq 0}$ on $\mathscr{C}(K)$, whose generator $(A, D(A))$ is the closure of $(Z, D(Z))$, such that, for every $t \geq 0$ and $f \in \mathscr{C}(K)$,

$$T(t)(f) = \lim_{n\to\infty} C_n^{k(n)}(f) \tag{2}$$

uniformly on K, for every sequence $(k(n))_{n\geq 1}$ of positive integers such that $\lim\limits_{n\to\infty} k(n)/n = t$. From the approximation formula (2) it also follows that each $T(t)$ is positive. $T(t)(\mathbf{1}) = \mathbf{1}$ $(t \geq 0)$, and consequently $(T(t))_{t\geq 0}$ is a Markov semigroup.

Moreover, $A = Z$ on $P_\infty(K)$. Consequently, it follows that

$$\overline{(I - A)(P_\infty(K))} = \overline{(I - Z)(P_\infty(K))} = \mathscr{C}(K)$$

with respect to $\|\cdot\|_\infty$ and thus $P_\infty(K)$ is a core for $(A, D(A))$. $\square$

As a consequence of the previous theorem, let us now consider the abstract Cauchy problem associated with $(A, D(A))$

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = A(u(\cdot,t))(x) & x \in K, \quad t \geq 0, \\ \\ u(x,0) = u_0(x) & u_0 \in D(A), \quad x \in K. \end{cases} \tag{3.1}$$

As $(A, D(A))$ is the generator of a C_0-semigroup, then (3.1) admits a unique solution $u : K \times [0, +\infty[\to \mathbf{R}$ given by $u(x,t) = T(t)(u_0)(x)$ for every $x \in K$ and $t \geq 0$ (see, e.g., [14, Chapter A-II]). Hence, by Theorem 3.1, it is possible to approximate such a solution by means of iterates of the operators C_n, i.e.,

$$u(x,t) = T(t)(u_0)(x) = \lim_{n\to\infty} C_n^{[nt]}(u_0)(x), \tag{3.2}$$

the limit being uniform with respect to $x \in K$, where $[nt]$ denotes the integer part of $[nt]$. This latter allows us to infer some spatial regularity properties for the solution $u(\cdot,t)$ $(t \geq 0)$, as the following results show.

From now on, assume that K is metrizable and denote by ρ the metric on K which induces its topology.

The ρ-modulus of continuity of a given $f \in \mathscr{C}(K)$ with respect to $\delta > 0$ is then defined by

$$\omega_\rho(f,\delta) := \sup\{|f(x) - f(y)| \mid x, y \in K, \rho(x,y) \leq \delta\}.$$

Assume that

$$\omega_\rho(f, t\delta) \leq (1+t)\omega_\rho(f,\delta) \tag{3.3}$$

for every $f \in \mathscr{C}(K)$, $\delta, t > 0$.

Furthermore, for any $M \geq 0$ and $0 < \alpha \leq 1$, denote by

$$\mathrm{Lip}(M, \alpha) := \{f \in \mathscr{C}(K) \mid |f(x) - f(y)| \leq M\rho(x, y)^\alpha \text{ for every } x, y \in K\}$$

the space of all Hölder continuous functions with exponent α and constant M. In particular, $\mathrm{Lip}(M, 1)$ is the space of all Lipschitz continuous functions with constant M. From now on we suppose that there exists $c \geq 1$ such that

$$T(\mathrm{Lip}(1, 1)) \subset \mathrm{Lip}(1, 1), \tag{3.4}$$

or, equivalently,

$$T(\mathrm{Lip}(M, 1)) \subset \mathrm{Lip}(M, 1),$$

for every $M \geq 0$.

For instance, the Markov operators T_1 (see (1.6)) and T_d (see (1.5)) satisfy condition (3.4), by considering on $[0, 1]$ the usual metric and on K_d the l_1-metric, i.e., the metric generated by the l_1-norm (see [4, p. 124]).

Then the following result holds.

Proposition 3.2. *Under assumptions* (3.3) *and* (3.4)*, if* $u_0 \in \mathrm{Lip}(M, 1)$ *for some* $M \geq 0$*, then* $u(\cdot, t) \in \mathrm{Lip}(M, 1)$ *for every* $t \geq 0$*.*

Proof. Under the above assumptions, by means of [5, Proposition 6.1], it follows that $C_n(\mathrm{Lip}(M, 1)) \subset \mathrm{Lip}(M, 1)$ for every $M \geq 0$ and $n \geq 1$; that completes the proof, since formula (3.2) holds and since $\mathrm{Lip}(M, 1)$ is closed under the uniform norm. □

We now present sufficient conditions in order that $u(\cdot, t)$ $(t \geq 0)$ is a convex function, provided that the initial datum $u_0 \in D(A)$ of (3.1) is convex too. To this end, for a given $f \in \mathscr{C}(K)$ and $x, y \in K$, we set

$$\Delta(f; x, y) := B_2(f)(x) + B_2(f)(y) - 2 \iint\limits_{K^2} f\left(\frac{s+t}{2}\right) d\tilde{\mu}_x^T(s) d\tilde{\mu}_x^T(t)$$

where the operator B_2 is defined as in (1.11).

Then, as a a consequence of [5, Theorem 6.4], we can state the following result.

Theorem 3.3. *Suppose that* T *satisfies the following assumptions:*

(c_1) *T maps continuous convex functions into (continuous) convex functions;*

(c_2) *$\Delta(f; x, y) \geq 0$ for every convex function $f \in \mathscr{C}(K)$ and for every $x, y \in K$.*

If $u_0 \in D(A)$ is convex, then $u(\cdot, t)$ is convex for every $t \geq 0$.

Remark 3.4. In [4, Remark 3.4.4 and Examples 3.4.5-3.4.11] there are several examples of settings where conditions (c_1) and (c_2) are satisfied. This is the case, in particular, when $K = [0, 1]$ and $T = T_1$ (see (1.6)).

We also point out that, if $K = K_d$, $d \geq 1$, then the generalized Kantorovich operators (1.15) map axially convex functions, i.e., continuous functions which are convex on segments parallel to segments joining two vertices of the simplex, into axially convex functions ([5, Corollary 6.6]). Therefore if $u_0 \in D(A)$ is axially convex, then $u(\cdot, t)$ is axially convex for every $t \geq 0$ (see [6, Corollary 3.7]).

We finally remark that the finite dimensional case was studied in its full generality in [6]. In such a case, under suitable assumptions on T, the abstract Cauchy problem (3.1) turns on $P_m(K)$ into a initial-boundary value problem governed by the differential operator (2.11).

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