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A note on blow-up results for semilinear wave equations in de Sitter and anti-de Sitter spacetimes



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ABSTRACT

In this work we derive some blow-up results for semilinear wave equations both in de Sitter and anti-de Sitter spacetimes. By requiring suitable conditions on a time-dependent factor in the nonlinear term, we prove the blow-up in finite time of the spatial averages of local in time solutions. In particular, we derive a sequence of lower bound estimates for the spatial average by combining a suitable slicing procedure with an iteration frame for this time-dependent functional.

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1. Introduction

In the first part of this paper, we investigate the blow-up dynamic for local in time solutions to the following semilinear wave equation with damping and mass in de Sitter spacetime

$$\begin{cases} \partial_t^2 u - c^2 e^{-2Ht} \Delta u + b \partial_t u + m^2 u = f(t, u), & x \in \mathbb{R}^n, t \in (0, T), \\ u(0, x) = \varepsilon u_0(x), & x \in \mathbb{R}^n, \\ \partial_t u(0, x) = \varepsilon u_1(x), & x \in \mathbb{R}^n, \end{cases} \quad (1.1)$$

where c, H are positive constants, b, m^2 are nonnegative real parameters, $\varepsilon > 0$ is a parameter describing the size of initial data and $T = T(\varepsilon) \in (0, \infty]$ is the lifespan (maximal existence time) of the weak solution with C^1 regularity in time variable (cf. Definition 1.1). In the literature concerning cosmology, the constant

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H is called Hubble constant, m denotes the mass of a particle and the coefficient b for the damping term is taken equal to the space dimension n (see, for example, (0.6) in [28]).

As nonlinear term we consider a nonlocal term given by the product of three terms: a time-dependent coefficient providing (possible) additional exponential and/or polynomial growth, a p -power nonlinearity, and a power of the spatial L^p norm of the solution. Namely, we set

$$f(t, u) \doteq \Gamma(t) \left(\int_{\mathbb{R}^n} |u(t, y)|^p dy \right)^\beta |u|^p, \quad (1.2)$$

where $p > 1$, $\beta \geq 0$, and $\Gamma(t)$ is a suitable nonnegative function. Our goal in the present paper will be to determine growth conditions on $\Gamma = \Gamma(t)$ (depending on p, β, b, m^2) in a such way that blow-up phenomena for the local solutions to (1.1) occur under suitable sign assumptions for the Cauchy data.

We investigate the case in which the damping term $b\partial_t u$ is dominant over the mass term $m^2 u$, by prescribing a restriction on the size of m^2 . More precisely, we will always work under the following assumption

$$b^2 \geq 4m^2 \quad (1.3)$$

for the coefficients of the lower order terms. Following the nomenclature introduced in [7], we call $b^2 > 4m^2$ the case with *dominant dissipation*, $b^2 = 4m^2$ the case with *balanced dissipation and mass*, and $b^2 < 4m^2$ the case with *dominant mass*. We do not consider the dominant mass case since this case is somehow related to Klein-Gordon equation with real positive mass, which cannot be treated with the approach that we are going to use in the present work. More specifically, we investigate the dynamic of the space average of a local solution to (1.1), by determining a lower bound estimate for this time-dependent functional, where the space average appears also in a nonlinear form in an integral term on the right-hand side of this inequality (the so-called *iteration frame*). For this kind of approach it is essential to work with nonnegative lower bounds and (1.3) ensures us that the time-dependent factors on the right-hand side of the iteration frame have no oscillations and are positive.

We point out that the speed of propagation, namely, the function $a_{\text{dS}}(t) \doteq ce^{-Ht}$, is exponentially decreasing in the previous semilinear wave equation. Moreover, the amplitude of the forward light-cone, provided by

$$A_{\text{dS}}(t) \doteq \int_0^t a_{\text{dS}}(\tau) d\tau = \frac{c}{H} (1 - e^{-Ht})$$

is a bounded function. In other words, by working with smooth solutions, if we assume u_0 and u_1 compactly supported in $B_R \doteq \{x \in \mathbb{R}^n : |x| \leq R\}$, given a local solution u to (1.1), we have that

$$\text{supp } u(t, \cdot) \subset B_{R+A_{\text{dS}}(t)} \quad \text{for any } t \in (0, T). \quad (1.4)$$

For this support condition we used the property of finite speed of propagation or, alternatively, the explicit representation formulas from the series of works by Galstian and Yagdjian [27,28,21,22,24,25]. Therefore, assuming compactly supported Cauchy data, the support of a local in time solution will be contained in an infinite half cylinder (as long as the solution exists). As we will see in the proof of our blow-up results, this property will play a key role when establishing the iteration frame.

We emphasize that the inclusion of the time-dependent factor Γ in (1.2) is made in order to be able to prove the blow-up in finite time for $p > 1$ and $\beta \geq 0$. Indeed, the exponentially decaying speed of propagation and the presence of the mass term both make extremely difficult the occurrence of a blow-up in finite time of the solution. In the massless case (i.e. for $m^2 = 0$), we will be able not to require any additional exponential

growth in the nonlinear term, that is, we may consider the case $\Gamma(t) = 1$ as well. In particular, for $\beta = 0$ we will recover (with a different technique) the result recently proved by Tsutaya-Wakasugi [20] with the test function method. On the contrary, when a mass term is present in the partial differential operator on the left-hand side of (1.1) and we work under the assumption (1.3), then, our method produces a sequence of lower bound estimates too weak that is not enough to prove the blow-up in finite time unless we require additional exponential growth through the factor $\Gamma(t)$.

We emphasize that the local case $\beta = 0$ can be included in our result as well. This corresponds to the usual power nonlinearity with a time-dependent factor.

The nonlinear term in (1.2) has been already considered in the literature for the Klein-Gordon equation in de Sitter spacetime by Yagdjian [21] when (1.3) is satisfied and by Nakamura [12] for a pure imaginary mass (i.e. for $m^2 < 0$ with our notations) both for de Sitter and anti-de Sitter spacetimes. In both these papers a blow-up result is proved by means of a comparison argument for a certain ODE. In our approach we work with the corresponding integral formulation that will allow us to slightly improve the growth condition for $\Gamma(t)$ in comparison to that one in [21, Theorem 1.1].

In the second part of the paper, we investigate what happens if we consider an exponentially increasing speed of propagation, say $a_{\text{AdS}}(t) = ce^{Ht}$ with $c, H > 0$, in place of an exponentially decreasing function as in (1.1). In other words, we are interested to study the following semilinear problem associated with the wave equation in anti-de Sitter spacetime

$$\begin{cases} \partial_t^2 v - c^2 e^{2Ht} \Delta v + b \partial_t v + m^2 v = f(t, v), & x \in \mathbb{R}^n, t \in (0, T), \\ v(0, x) = \varepsilon v_0(x), & x \in \mathbb{R}^n, \\ \partial_t v(0, x) = \varepsilon v_1(x), & x \in \mathbb{R}^n, \end{cases} \quad (1.5)$$

where c, H are positive constants, b, m^2 are nonnegative real parameters satisfying (1.3) and the nonlinear term is defined analogously as in (1.2).

As in the corresponding results for the semilinear wave equation in de Sitter spacetime, we want to examine the growth conditions on the factor Γ that provide local in time solutions that blow up in finite time (under suitable sign conditions for the Cauchy data). To the knowledge of the authors, while (1.1) has been already studied in the literature, the semilinear Cauchy problem in (1.5) has never been investigated from the viewpoint of blow-up results when (1.3) is satisfied and $m^2 \geq 0$. As we are going to explain in Subsection 1.1, in the case of anti-de Sitter spacetime the growth assumptions on Γ depend strongly on the dimension n . On the one hand, for low dimensions the situation is quite similar to the corresponding case with exponentially decreasing speed of propagation. On the other hand, for high dimensions the influence of the nonlinear term is dominant and, in particular, the treatment of a threshold case, which could be considered as a critical case in some sense, is more delicate and requires a more deep analysis of the growth properties of the spatial average of a local solution.

In this last part of the introduction, we recall other known results from the literature on semilinear wave models in de Sitter spacetime and how our result can be framed and understood in relation to these. Over the last decade several results were established for wave models by Yagdjian and Yagdjian-Galstian in de Sitter spacetime [28, 21–26] with normalized constants c, H (meaning $a_{\text{dS}}(t) = e^{-t}$) and in anti-de Sitter spacetime (that is, when the speed of propagation is $a_{\text{AdS}}(t) = e^t$) [27, 29, 30], respectively. Truly remarkable integral representation formulas for the solutions of the linear Cauchy problem associated with Klein-Gordon equations, both with pure imaginary and real positive mass term, are derived in the case of de Sitter spacetime [28, 21] and anti-de Sitter spacetime [29], respectively. These integral representation formulas have been applied, among other things, to study $L^p - L^q$ estimates, the existence of self-similar solutions, blow-up results with nonlocal nonlinear term as in (1.2) and to investigate under which assumptions on the coefficients for the mass term and on the space dimension a Huygens' type principle holds. Afterwards, the Cauchy problem associated with the semilinear wave equation in de Sitter spacetime with power nonlinearity

was studied by Nakamura [10,11] and Ebert-Reissig [7] and several global existence results were established not only in classical energy space but also in Sobolev space on L^2 basis with different regularities (both below and above the regularity of energy solutions). We point out that in [10,11] also a nonlinearity of exponential type is considered besides the power nonlinearity. In spite of the above quoted global existence results for small data solutions with a nonnegative power nonlinearity, it seems that there is a lack of understanding concerning the expression for the critical exponent, due to the absence of a corresponding blow-up counterpart. In this scenario, our results for (1.1) should emphasize how the presence of the mass term does not allow to prove the blow-up in finite time of any local in time solution when $f(u) = |u|^p$. Indeed, when $\beta = 0$ and as $r \rightarrow 0^+$ the method that we are going to employ for studying the blow-up is no longer efficient, meaning that the argument that provides the blow-up of the space average fails, with a unique remarkable exception given by the massless case $m^2 = 0$ (established for the first time in [20], as mentioned above).

1.1. Main results

Before stating our main results, we introduce the class of solutions to (1.1) that we will consider throughout this paper. We emphasize that, even though we will call these solutions weak solutions, we require more regularity than usual distributional solutions. More precisely, we consider the larger class of solutions that can be considered with our approach, and this requires some regularity with respect to the time-variable according to the next definition.

Definition 1.1. Let $u_0, u_1 \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that $\text{supp } u_0, \text{supp } u_1 \subset B_R$ for some $R > 0$. We say that

$$u \in C^1([0, T], L^1_{\text{loc}}(\mathbb{R}^n)) \text{ such that } f(t, u) \in L^1_{\text{loc}}((0, T) \times \mathbb{R}^n),$$

where the definition of the nonlinear term $f(t, u)$ is given in (1.2), is a *weak solution* to (1.1) on $[0, T]$ if u fulfills the support condition (1.4) and the integral identity

$$\begin{aligned} & \int_{\mathbb{R}^n} \partial_t u(t, x) \varphi(t, x) \, dx - \int_{\mathbb{R}^n} u(t, x) \varphi_t(t, x) \, dx + b \int_{\mathbb{R}^n} u(t, x) \varphi(t, x) \, dx \\ & + \int_0^t \int_{\mathbb{R}^n} u(s, x) (\varphi_{ss}(s, x) - c^2 e^{-2Hs} \Delta \varphi(s, x) - b \varphi_s(s, x) + m^2 \varphi(s, x)) \, dx \, ds \\ & = \varepsilon \int_{\mathbb{R}^n} u_1(x) \varphi(0, x) \, dx + \varepsilon \int_{\mathbb{R}^n} u_0(x) (b \varphi(0, x) - \varphi_t(0, x)) \, dx \\ & + \int_0^t \Gamma(s) \left(\int_{\mathbb{R}^n} |u(s, y)|^p \, dy \right)^\beta \int_{\mathbb{R}^n} |u(s, x)|^p \varphi(s, x) \, dx \, ds \end{aligned} \quad (1.6)$$

holds for any $t \in (0, T)$ and any test function $\varphi \in C_0^\infty([0, T] \times \mathbb{R}^n)$.

Considering the following family of time-dependent factors for the nonlinear term in (1.2)

$$\Gamma(t) \doteq \mu e^{rt} (1+t)^\kappa \quad (1.7)$$

depending on the real parameters r, κ , where the multiplicative constant μ is positive, we are interested in describing how the ranges for r, κ affect the blow-up in finite time of local solutions to (1.1). In particular, introducing the threshold values

$$r_{\text{crit}}(b, m^2, \beta, p) \doteq \frac{1}{2} \left(b - \sqrt{b^2 - 4m^2} \right) ((\beta + 1)p - 1), \tag{1.8}$$

$$\kappa_{\text{crit}}(b, m^2, \beta, p) \doteq \begin{cases} -1 & \text{if } b^2 > 4m^2, \\ -1 - (\beta + 1)p & \text{if } b^2 = 4m^2, \end{cases} \tag{1.9}$$

we may distinguish between three different subcases, depending on the range for the parameters r, κ in (1.7):

- the case with *exponential growth* when $r > r_{\text{crit}}(b, m^2, \beta, p)$ and $\kappa \in \mathbb{R}$;
- the case with *polynomial growth* when $r = r_{\text{crit}}(b, m^2, \beta, p)$ and $\kappa > \kappa_{\text{crit}}(b, m^2, \beta, p)$;
- the case with *logarithmic growth* when $r = r_{\text{crit}}(b, m^2, \beta, p)$ and $\kappa = \kappa_{\text{crit}}(b, m^2, \beta, p)$.

Note that the word “growth” in the previous list of subcases does not refer to the growth rate for the function $\Gamma(t)$, rather to the growth of the lower bound for a time-dependent functional related to a local solution u , whose evolution in time will be investigated to prove the blow-up in finite time of u .

We shall see that a suitable iteration argument for U in Theorem 1.2 and for \mathcal{U} in Theorems 1.3 and 1.4 can be used together with a slicing procedure. For the definition of U and \mathcal{U} , respectively, see (2.1) and (2.24) below. In the first case we deal with exponential factors in the lower bounds for U , while in the threshold case $r = r_{\text{crit}}(b, m^2, \beta, p)$, depending on whether $\kappa > \kappa_{\text{crit}}(b, m^2, \beta, p)$ or $\kappa = \kappa_{\text{crit}}(b, m^2, \beta, p)$, we find lower bounds of polynomial or logarithmic type for \mathcal{U} , respectively.

The first result concerns the case with exponential growth.

Theorem 1.2. *Let $n \geq 1$ and $b, m^2 \geq 0$ such that (1.3) is fulfilled. Let us assume $\beta \geq 0, p > 1$ and $r > r_{\text{crit}}(b, m^2, \beta, p)$, where $r_{\text{crit}}(b, m^2, \beta, p)$ is defined in (1.8), and consider*

$$\Gamma(t) \doteq \mu e^{rt}(1+t)^\kappa \tag{1.10}$$

for some $\mu > 0$ and some $\kappa \in \mathbb{R}$ in (1.2).

Let us assume that $u_0, u_1 \in L^1_{\text{loc}}(\mathbb{R}^n)$ are nonnegative, nontrivial and compactly supported functions with supports contained in B_R for some $R > 0$.

Let $u \in C^1([0, T), L^1_{\text{loc}}(\mathbb{R}^n))$ be a weak solution to the Cauchy problem (1.1) according to Definition 1.1 with lifespan $T = T(\varepsilon)$.

Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(n, c, H, b, m^2, \beta, p, \mu, r, \kappa, u_0, u_1, R)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the weak solution u blows up in finite time. Furthermore, the following upper bound estimates for the lifespan hold

$$\theta_{b, m^2, p, \beta, r, \kappa}(T(\varepsilon)) \leq C\varepsilon^{-\left(\frac{r}{(\beta+1)p-1} - \frac{1}{2}(b - \sqrt{b^2 - 4m^2})\right)^{-1}}, \tag{1.11}$$

where the positive constant C is independent of ε and

$$\theta_{b, m^2, p, \beta, r, \kappa}(\tau) \doteq \begin{cases} e^{\tau} \tau^{\frac{\kappa}{r - r_{\text{crit}}(b, m^2, \beta, p)}} & \text{if } b^2 > 4m^2, \\ e^{\tau} \tau^{\frac{(\beta+1)p-1+\kappa}{r - r_{\text{crit}}(b, m^2, \beta, p)}} & \text{if } b^2 = 4m^2. \end{cases} \tag{1.12}$$

Remark 1. In Theorem 1.2 we note that in the case without any additional polynomial growth (or decay) in Γ , that is for $\kappa = 0$, the function in (1.12) is simply the exponential function for $b^2 > 4m^2$. On the other hand, for $\kappa = 0$ and $b^2 = 4m^2$ we still have a polynomial correction in (1.12) (that improves the upper bound estimate).

The second result concerns the case with polynomial growth. In particular, we have a limit value for the coefficient in the exponential term in (1.7), while for the polynomial factor we consider the parameter κ above the threshold value $\kappa_{\text{crit}}(b, m^2, \beta, p)$.

Theorem 1.3. *Let $n \geq 1$ and $b, m^2 \geq 0$ such that (1.3) is fulfilled. Let us assume $\beta \geq 0, p > 1$ and $r = r_{\text{crit}}(b, m^2, \beta, p)$, $\kappa > \kappa_{\text{crit}}(b, m^2, \beta, p)$, where $r_{\text{crit}}(b, m^2, \beta, p)$ and $\kappa_{\text{crit}}(b, m^2, \beta, p)$ are defined in (1.8) and (1.9), respectively, and consider*

$$\Gamma(t) \doteq \mu e^{r_{\text{crit}}(b, m^2, \beta, p)t} (1+t)^\kappa \quad (1.13)$$

for some $\mu > 0$ in (1.2).

Let us assume that $u_0, u_1 \in L^1_{\text{loc}}(\mathbb{R}^n)$ are nonnegative, nontrivial and compactly supported functions with supports contained in B_R for some $R > 0$.

Let $u \in C^1([0, T], L^1_{\text{loc}}(\mathbb{R}^n))$ be a weak solution to the Cauchy problem (1.1) according to Definition 1.1 with lifespan $T = T(\varepsilon)$.

Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(n, c, H, b, m^2, \beta, p, \mu, \kappa, u_0, u_1, R)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the weak solution u blows up in finite time. Furthermore, the following upper bound estimates for the lifespan hold

$$T(\varepsilon) \leq \begin{cases} C\varepsilon^{-\frac{(\beta+1)p-1}{\kappa+1}} & \text{if } b^2 > 4m^2, \\ C\varepsilon^{-\left(\frac{\kappa+2}{(\beta+1)p-1}+1\right)^{-1}} & \text{if } b^2 = 4m^2, \end{cases} \quad (1.14)$$

where the positive constant C is independent of ε .

Remark 2. Let us point out that in the massless case, i.e. for $m^2 = 0$, we have $r_{\text{crit}}(b, 0, \beta, p) = 0$. Therefore, in this special case we obtained the blow-up in finite time of local solutions even without requiring additional exponential or polynomial growth for the nonlinear term (setting $\Gamma(t) = 1$). In particular, for the local case $\beta = 0$, which corresponds to the usual power nonlinearity $|u|^p$, namely, for the semilinear Cauchy problem

$$\begin{cases} \partial_t^2 u - c^2 e^{-2Ht} \Delta u + b \partial_t u = |u|^p, & x \in \mathbb{R}^n, t \in (0, T), \\ u(0, x) = \varepsilon u_0(x), & x \in \mathbb{R}^n, \\ \partial_t u(0, x) = \varepsilon u_1(x), & x \in \mathbb{R}^n, \end{cases} \quad (1.15)$$

we obtained, with a quite different approach, the same blow-up result recently proved in [20]. Moreover, we found the same lifespan estimates as in [20]. Indeed, from Theorem 1.3 in this special case the upper bound estimates for the lifespan are given by

$$T(\varepsilon) \leq \begin{cases} C\varepsilon^{-(p-1)} & \text{if } b > 0, \\ C\varepsilon^{-\frac{p-1}{p+1}} & \text{if } b = 0, \end{cases}$$

where the positive constant C is independent of ε .

The third result concerns the case with logarithmic growth. In this case, we have limit values both for the coefficient of the exponential term and of the polynomial term in (1.7).

Theorem 1.4. *Let $n \geq 1$ and $b, m^2 \geq 0$ such that (1.3) is fulfilled. Let us assume $\beta \geq 0, p > 1$ and $r = r_{\text{crit}}(b, m^2, \beta, p)$, $\kappa = \kappa_{\text{crit}}(b, m^2, \beta, p)$, where $r_{\text{crit}}(b, m^2, \beta, p)$ and $\kappa_{\text{crit}}(b, m^2, \beta, p)$ are defined in (1.8) and (1.9), respectively, and consider*

$$\Gamma(t) \doteq \mu e^{r_{\text{crit}}(b,m^2,\beta,p)t}(1+t)^{\kappa_{\text{crit}}(b,m^2,\beta,p)} \tag{1.16}$$

for some $\mu > 0$ in (1.2).

Let us assume that $u_0, u_1 \in L^1_{\text{loc}}(\mathbb{R}^n)$ are nonnegative, nontrivial and compactly supported functions with supports contained in B_R for some $R > 0$.

Let $u \in C^1([0, T], L^1_{\text{loc}}(\mathbb{R}^n))$ be a weak solution to the Cauchy problem (1.1) according to Definition 1.1 with lifespan $T = T(\varepsilon)$.

Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(n, c, H, b, m^2, \beta, p, \mu, u_0, u_1, R)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the weak solution u blows up in finite time. Furthermore, the following upper bound estimate for the lifespan holds

$$T(\varepsilon) \leq C \exp\left(K\varepsilon^{-((\beta+1)p-1)}\right), \tag{1.17}$$

where the positive constants C, K are independent of ε .

Remark 3. We emphasize that the results from Theorems 1.2, 1.3 and 1.4 correspond to ones from Theorem 1.1 in [21]. In particular, in the case $b^2 > 4m^2$ we improved the limit threshold for the polynomial factor from $\kappa > 2$ to $\kappa \geq -1$, while we found exactly the same result in the case $b^2 = 4m^2$. In addition, we established upper bound estimates for the lifespan depending on the precise growth rate of the function $\Gamma(t)$. We underline that in the above mentioned Yagdjian’s paper the blow-up of the spatial average of a local solution is proved by means of a comparison argument for certain ordinary differential inequalities, that generalize Kato’s lemma (cf. [8] or [18]). In particular, in that paper, applying the dissipative transformation $w(t, x) = e^{\frac{b}{2}t}u(t, x)$ and keeping our notations, the equation in (1.1) is transformed in

$$\partial_t^2 w - c^2 e^{-2Ht} \Delta w - \left(\frac{b^2}{4} - m^2\right) w = e^{-\frac{b}{2}((\beta+1)p-1)t} f(t, w)$$

and, then, a modified Kato’s lemma is applied to study the blow-up of local in time solutions.

The second part of the paper will be devoted to the study of blow-up results for local in time solutions to (1.5), with the time dependent factor Γ chosen as follows:

$$\Gamma(t) \doteq \mu e^{gt}(1+t)^{\varsigma}. \tag{1.18}$$

The amplitude function describing the forward light-cone is given by $A_{\text{AdS}}(t) \doteq cH^{-1}(e^{Ht} - 1)$ for this model.

Before stating the main results for (1.5), also in this case we introduce the class of solutions to (1.5) with which we will work.

Definition 1.5. Let $v_0, v_1 \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that $\text{supp } v_0, \text{supp } v_1 \subset B_R$ for some $R > 0$. We say that

$$v \in C^1([0, T], L^1_{\text{loc}}(\mathbb{R}^n)) \text{ such that } f(t, v) \in L^1_{\text{loc}}((0, T) \times \mathbb{R}^n),$$

where the definition of the nonlinear term $f(t, v)$ is given in (1.2), is a weak solution to (1.5) on $[0, T]$ if v fulfills the support condition

$$\text{supp } v(t, \cdot) \subset B_{R+cH^{-1}(e^{Ht}-1)} \quad \text{for any } t \in (0, T), \tag{1.19}$$

and the integral identity

$$\begin{aligned}
& \int_{\mathbb{R}^n} \partial_t v(t, x) \varphi(t, x) \, dx - \int_{\mathbb{R}^n} v(t, x) \varphi_t(t, x) \, dx + b \int_{\mathbb{R}^n} v(t, x) \varphi(t, x) \, dx \\
& + \int_0^t \int_{\mathbb{R}^n} v(s, x) (\varphi_{ss}(s, x) - c^2 e^{2Hs} \Delta \varphi(s, x) - b \varphi_s(s, x) + m^2 \varphi(s, x)) \, dx \, ds \\
& = \varepsilon \int_{\mathbb{R}^n} v_1(x) \varphi(0, x) \, dx + \varepsilon \int_{\mathbb{R}^n} v_0(x) (b \varphi(0, x) - \varphi_t(0, x)) \, dx \\
& + \int_0^t \Gamma(s) \left(\int_{\mathbb{R}^n} |v(s, y)|^p \, dy \right)^\beta \int_{\mathbb{R}^n} |v(s, x)|^p \varphi(s, x) \, dx \, ds \tag{1.20}
\end{aligned}$$

holds for any $t \in (0, T)$ and any test function $\varphi \in C_0^\infty([0, T] \times \mathbb{R}^n)$.

Differently from what happens in the case of de Sitter spacetime, when we work in anti-de Sitter spacetime it is possible to derive two different threshold values for the parameter ϱ in (1.18). In the next lines we are going to define these two values depending on the range for the space dimension.

We introduce the threshold values

$$\varrho_{\text{crit}}(n, H, b, m^2, \beta, p) \doteq \frac{1}{2} \left(b - \sqrt{b^2 - 4m^2} \right) ((\beta + 1)p - 1) + nH(\beta + 1)(p - 1) \tag{1.21}$$

for $n \leq \frac{\sqrt{b^2 - 4m^2}}{H} + \frac{2}{p}$, and

$$\varrho_{\text{crit}}(n, H, b, m^2, \beta, p) \doteq \frac{1}{2} (b + nH) ((\beta + 1)p - 1) + nH - (n - 1)H(\beta + 1) - \frac{H}{p} \tag{1.22}$$

for $n > \frac{\sqrt{b^2 - 4m^2}}{H} + \frac{2}{p}$.

The reasons that lead to consider two different values for ϱ_{crit} depending on whether n is smaller than/equal to or bigger than $\frac{\sqrt{b^2 - 4m^2}}{H} + \frac{2}{p}$ and the steps towards to this distinction will be clarified in detail in Subsection 3.3. Nevertheless, naively and roughly speaking, we can assert that when (1.21) holds the Cauchy data have a stronger influence in the iteration argument than the nonlinear term, while in (1.22) the situation is reversed.

The next three theorems are the counterpart in anti-de Sitter spacetime of Theorems 1.2-1.4.

Theorem 1.6. *Let $n \geq 1$ and $b, m^2 \geq 0$ such that (1.3) is fulfilled. Let us assume $\beta \geq 0$ and $p > 1$ such that*

$$\frac{n}{2} - \frac{\sqrt{b^2 - 4m^2}}{2H} \leq \frac{1}{p} \tag{1.23}$$

and $\varrho > \varrho_{\text{crit}}(n, H, b, m^2, \beta, p)$, where $\varrho_{\text{crit}}(n, H, b, m^2, \beta, p)$ is defined in (1.21), and consider

$$\Gamma(t) \doteq \mu e^{\varrho t} (1 + t)^\varsigma$$

for some $\mu > 0$ and some $\varsigma \in \mathbb{R}$ in the term $f(t, v)$ given by (1.2).

Let us assume that $v_0, v_1 \in L_{\text{loc}}^1(\mathbb{R}^n)$ are nonnegative, nontrivial and compactly supported functions with supports contained in B_R for some $R > 0$.

Let $v \in C^1([0, T], L_{\text{loc}}^1(\mathbb{R}^n))$ be a weak solution to the Cauchy problem (1.5) according to Definition 1.5 with lifespan $T = T(\varepsilon)$.

Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(n, c, H, b, m^2, \beta, p, \mu, \varrho, \varsigma, v_0, v_1, R)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the weak solution v blows up in finite time. Furthermore, the following upper bound estimates for the lifespan hold

$$\zeta_{n,H,b,m^2,p,\beta,\varrho,\varsigma}(T(\varepsilon)) \leq C\varepsilon^{-\frac{(\beta+1)p-1}{\varrho-\varrho_{\text{crit}}(n,H,b,m^2,\beta,p)}},$$

where the positive constant C is independent of ε and

$$\zeta_{n,H,b,m^2,p,\beta,\varrho,\varsigma}(\tau) \doteq \begin{cases} e^{\tau} \tau^{\frac{\varsigma}{\varrho-\varrho_{\text{crit}}(n,H,b,m^2,\beta,p)}} & \text{if } b^2 > 4m^2, \\ e^{\tau} \tau^{\frac{(\beta+1)p-1+\varsigma}{\varrho-\varrho_{\text{crit}}(n,H,b,m^2,\beta,p)}} & \text{if } b^2 = 4m^2. \end{cases}$$

Theorem 1.7. Let $n \geq 1$ and $b, m^2 \geq 0$ such that (1.3) is fulfilled. Let us assume $\beta \geq 0$ and $p > 1$ satisfying (1.23) and $\varrho = \varrho_{\text{crit}}(n, H, b, m^2, \beta, p)$ and $\varsigma > \kappa_{\text{crit}}(b, m^2, \beta, p)$, where $\varrho_{\text{crit}}(n, H, b, m^2, \beta, p)$ and $\kappa_{\text{crit}}(b, m^2, \beta, p)$ are defined in (1.21) and in (1.9), respectively, and consider

$$\Gamma(t) \doteq \mu e^{\varrho_{\text{crit}}(n,H,b,m^2,\beta,p)t} (1+t)^\varsigma$$

for some $\mu > 0$ in the term $f(t, v)$ given by (1.2).

Let us assume that $v_0, v_1 \in L^1_{\text{loc}}(\mathbb{R}^n)$ are nonnegative, nontrivial and compactly supported functions with supports contained in B_R for some $R > 0$.

Let $v \in C^1([0, T], L^1_{\text{loc}}(\mathbb{R}^n))$ be a weak solution to the Cauchy problem (1.5) according to Definition 1.5 with lifespan $T = T(\varepsilon)$.

Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(n, c, H, b, m^2, \beta, p, \mu, \varsigma, v_0, v_1, R)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the weak solution v blows up in finite time. Furthermore, the following upper bound estimates for the lifespan hold

$$T(\varepsilon) \leq \begin{cases} C\varepsilon^{-\frac{(\beta+1)p-1}{\varsigma+1}} & \text{if } b^2 > 4m^2, \\ C\varepsilon^{-\left(\frac{\varsigma+2}{(\beta+1)p-1}+1\right)^{-1}} & \text{if } b^2 = 4m^2, \end{cases}$$

where the positive constant C is independent of ε .

Theorem 1.8. Let $n \geq 1$ and $b, m^2 \geq 0$ such that (1.3) is fulfilled. Let us assume $\beta \geq 0$ and $p > 1$ satisfying (1.23) and $\varrho = \varrho_{\text{crit}}(n, H, b, m^2, \beta, p)$ and $\varsigma = \kappa_{\text{crit}}(b, m^2, \beta, p)$, where $\varrho_{\text{crit}}(n, H, b, m^2, \beta, p)$ and $\kappa_{\text{crit}}(b, m^2, \beta, p)$ are defined in (1.21) and in (1.9), respectively, and consider

$$\Gamma(t) \doteq \mu e^{\varrho_{\text{crit}}(n,H,b,m^2,\beta,p)t} (1+t)^{\kappa_{\text{crit}}(b,m^2,\beta,p)}$$

for some $\mu > 0$ in the term $f(t, v)$ given by (1.2).

Let us assume that $v_0, v_1 \in L^1_{\text{loc}}(\mathbb{R}^n)$ are nonnegative, nontrivial and compactly supported functions with supports contained in B_R for some $R > 0$.

Let $v \in C^1([0, T], L^1_{\text{loc}}(\mathbb{R}^n))$ be a weak solution to the Cauchy problem (1.5) according to Definition 1.5 with lifespan $T = T(\varepsilon)$.

Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(n, c, H, b, m^2, \beta, p, \mu, v_0, v_1, R)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the weak solution v blows up in finite time. Furthermore, the following upper bound estimate for the lifespan holds

$$T(\varepsilon) \leq C \exp\left(K\varepsilon^{-((\beta+1)p-1)}\right),$$

where the positive constants C, K are independent of ε .

Let us emphasize that in Theorems 1.6, 1.7 and 1.8 we used (1.21) as threshold value for the coefficient ϱ in the exponential term in Γ , since the condition (1.23) on the space dimension ensures us that the value for ϱ_{crit} in (1.21) is smaller than or equal to the one in (1.22). In other words, when (1.23) holds, the wider range for ϱ , provided by the condition $\varrho \geq \varrho_{\text{crit}}$, is obtained by the definition in (1.21) for ϱ_{crit} .

On the contrary, in the next results we assume that ϱ_{crit} is given by (1.22), that is, when the opposite inequality of the one in (1.23) holds.

Theorem 1.9. *Let $n \geq 1$ and $b, m^2 \geq 0$ such that (1.3) is fulfilled. Let us assume $\beta \geq 0$ and $p > 1$ such that*

$$\frac{n}{2} - \frac{\sqrt{b^2 - 4m^2}}{2H} > \frac{1}{p}, \tag{1.24}$$

and $\varrho > \varrho_{\text{crit}}(n, H, b, m^2, \beta, p)$, where $\varrho_{\text{crit}}(n, H, b, m^2, \beta, p)$ is defined in (1.22), and consider

$$\Gamma(t) \doteq \mu e^{\varrho t} (1+t)^\varsigma \tag{1.25}$$

for some $\mu > 0$ and some $\varsigma \in \mathbb{R}$ in the term $f(t, v)$ given by (1.2).

Let us assume that $v_0, v_1 \in L^1_{\text{loc}}(\mathbb{R}^n)$ are nonnegative, nontrivial and compactly supported functions with supports contained in B_R for some $R > 0$.

Let $v \in C^1([0, T], L^1_{\text{loc}}(\mathbb{R}^n))$ be a weak solution to the Cauchy problem (1.5) according to Definition 1.5 with lifespan $T = T(\varepsilon)$.

Then, there exists a positive constant $\varepsilon_0 = \varepsilon_0(n, c, H, b, m^2, \beta, p, \mu, \varrho, \varsigma, v_0, v_1, R)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the weak solution v blows up in finite time. Furthermore, the following upper bound estimates for the lifespan hold

$$\chi_{n, H, b, m^2, p, \beta, \varrho, \varsigma}(T(\varepsilon)) \leq C \varepsilon^{-\frac{(\beta+1)p-1}{\varrho - \varrho_{\text{crit}}(n, H, b, m^2, \beta, p)}}, \tag{1.26}$$

where the positive constant C is independent of ε and

$$\chi_{n, H, b, m^2, p, \beta, \varrho, \varsigma}(\tau) \doteq e^\tau \tau^{\frac{\varsigma}{\varrho - \varrho_{\text{crit}}(n, H, b, m^2, \beta, p)}}. \tag{1.27}$$

Remark 4. Notice that the upper bound estimate for the lifespan (1.26) is formally identical to the one in the statement of Theorem 1.6 in the case $b^2 > 4m^2$. Of course, the difference relies in the different definition for the quantity ϱ_{crit} depending on whether either (1.23) or (1.24) holds.

Remark 5. In [17] we provide a blow-up result for (1.5) in the case (1.24) for the critical case $\varrho = \varrho_{\text{crit}}$ by adapting the approach from [19].

2. Models in de Sitter spacetime

2.1. Derivation of the iteration frame

In order to prove Theorem 1.2, we are going to use an iteration argument to show that the space average of a local solution blows up in finite time. Hence, given u local in time solution to (1.1), we consider the functional

$$U(t) \doteq \int_{\mathbb{R}^n} u(t, x) dx \quad \text{for } t \in [0, T]. \tag{2.1}$$

As first step, we are going to determine an iteration frame for the functional U . As we explained in the introduction, for us an iteration frame is an integral inequality where U appears both on the left-hand side and on the right-hand side (as a nonlinear term in an integral expression). This iteration frame will allow us to establish a sequence of lower bound estimates of exponential type for $r > r_{\text{crit}}(b, m^2, \beta, p)$, through which we will prove the blow-up in finite time of U .

On the other hand, for the proofs of Theorems 1.3 and 1.4 rather than with $U(t)$ we will work with the functional $\mathcal{U}(t)$ given by the product of $U(t)$ with a suitable t -dependent exponential factor $e^{\alpha t}$. From the iteration argument for U we will establish immediately the corresponding one for \mathcal{U} . By working with \mathcal{U} we will be able to balance the effect of the exponential term in (1.7) in a much more simpler way when $r = r_{\text{crit}}(b, m^2, \beta, p)$. As a result of this balance we may apply a very precise slicing procedure in the different settings of Theorems 1.3 and 1.4, depending on whether we work with exponential and/or logarithmic factors and on how many steps are necessary in the slicing procedure.

We point out that in the iteration frame for U (or for \mathcal{U}) it is necessary to deal with unbounded exponential multipliers (see also the series of papers [4–6,2,14,3,16], where iteration frames with unbounded exponential multipliers are employed). For this purpose, we apply a *slicing procedure* while deriving the sequence of lower bound estimates for U . This procedure is a variation of the first slicing procedure introduced in [1] for the treatment of the critical case for the weakly coupled system of semilinear wave equations in the three dimensional case, where this technique is used to handle factors of logarithmic type. Clearly, the choice of the coefficients characterizing the slicing procedure (see the sequence $\{L_j\}_{j \in \mathbb{N}}$ defined below) is done in order to handle exponential factors in the iteration frame. We will also see how the number of exponential multipliers in the iteration frame will influence the number of steps for the slicing procedure (either a 1 step or a 2 steps procedure).

Fixed $t \in (0, T)$, we choose a bump function $\varphi \in C^\infty([0, T] \times \mathbb{R}^n)$ that localizes the support of u on the strip $[0, t] \times \mathbb{R}^n$, that is, $\varphi = 1$ on $\{(s, x) \in [0, t] \times \mathbb{R}^n : |x| \leq R + cH^{-1}(1 - e^{-sH})\}$. Hence, using this φ in (1.6), we get

$$\begin{aligned} & \int_{\mathbb{R}^n} \partial_t u(t, x) dx + b \int_{\mathbb{R}^n} u(t, x) dx + m^2 \int_0^t \int_{\mathbb{R}^n} u(s, x) dx ds \\ &= \varepsilon \int_{\mathbb{R}^n} (u_1(x) + bu_0(x)) dx + \int_0^t \Gamma(s) \left(\int_{\mathbb{R}^n} |u(s, x)|^p dx \right)^{\beta+1} ds, \end{aligned}$$

that can be rewritten as

$$U'(t) + bU(t) + m^2 \int_0^t U(s) ds = \varepsilon \int_{\mathbb{R}^n} (u_1(x) + bu_0(x)) dx + \int_0^t \Gamma(s) \left(\int_{\mathbb{R}^n} |u(s, x)|^p dx \right)^{\beta+1} ds.$$

From the previous relation we see that U is twice continuously differentiable and that

$$U''(t) + bU'(t) + m^2U(t) = \Gamma(t) \left(\int_{\mathbb{R}^n} |u(t, x)|^p dx \right)^{\beta+1}. \tag{2.2}$$

Thanks to the assumption (1.3), we may factorize the differential operator on the left-hand side of (2.2) as follows:

$$e^{-\alpha_1 t} \frac{d}{dt} \left(e^{(\alpha_1 - \alpha_2)t} \frac{d}{dt} (e^{\alpha_2 t} U(t)) \right) = U''(t) + (\alpha_1 + \alpha_2)U'(t) + \alpha_1 \alpha_2 U(t), \tag{2.3}$$

where the pair of real parameters (α_1, α_2) satisfies

$$\alpha_1 + \alpha_2 = b, \quad \alpha_1 \alpha_2 = m^2.$$

Clearly, the previous conditions for α_1 and α_2 are symmetric and they are satisfied if $\alpha_{1/2}$ are the roots of the quadratic equation

$$\alpha^2 - b\alpha + m^2 = 0. \quad (2.4)$$

Note that in the balanced case $b^2 = 4m^2$ the previous equation has a double root and $\alpha_1 = \alpha_2 = \frac{b}{2}$ and that in the dominant mass case the roots of (2.4) are complex conjugate, so oscillations appear. Therefore, we may rewrite (2.2) as follows:

$$e^{-\alpha_1 t} \frac{d}{dt} \left(e^{(\alpha_1 - \alpha_2)t} \frac{d}{dt} (e^{\alpha_2 t} U(t)) \right) = \Gamma(t) \left(\int_{\mathbb{R}^n} |u(t, x)|^p dx \right)^{\beta+1}. \quad (2.5)$$

Next, we can use (2.5) to derive the iteration frame for U by assuming nonnegative u_0 and u_1 . Let us begin with the case $b^2 > 4m^2$ (when $\alpha_1 \neq \alpha_2$). Multiplying (2.5) by $e^{\alpha_1 t}$ and integrating over $[0, t]$, we find

$$\int_0^t e^{\alpha_1 \tau} \Gamma(\tau) \left(\int_{\mathbb{R}^n} |u(\tau, x)|^p dx \right)^{\beta+1} d\tau = e^{(\alpha_1 - \alpha_2)t} \frac{d}{dt} (e^{\alpha_2 t} U(t)) - (U'(0) + \alpha_2 U(0)).$$

Analogously, from this last relation we obtain

$$\int_0^t e^{(\alpha_2 - \alpha_1)s} \int_0^s e^{\alpha_1 \tau} \Gamma(\tau) \left(\int_{\mathbb{R}^n} |u(\tau, x)|^p dx \right)^{\beta+1} d\tau ds = e^{\alpha_2 t} U(t) - U(0) + \frac{e^{(\alpha_2 - \alpha_1)t} - 1}{\alpha_1 - \alpha_2} (U'(0) + \alpha_2 U(0)),$$

which implies in turn

$$\begin{aligned} U(t) &= \frac{\alpha_2 e^{-\alpha_1 t} - \alpha_1 e^{-\alpha_2 t}}{\alpha_2 - \alpha_1} U(0) + \frac{e^{-\alpha_1 t} - e^{-\alpha_2 t}}{\alpha_2 - \alpha_1} U'(0) \\ &\quad + e^{-\alpha_2 t} \int_0^t e^{(\alpha_2 - \alpha_1)s} \int_0^s e^{\alpha_1 \tau} \Gamma(\tau) \left(\int_{\mathbb{R}^n} |u(\tau, x)|^p dx \right)^{\beta+1} d\tau ds \\ &= \varepsilon \frac{\alpha_2 e^{-\alpha_1 t} - \alpha_1 e^{-\alpha_2 t}}{\alpha_2 - \alpha_1} \int_{\mathbb{R}^n} u_0(x) dx + \varepsilon \frac{e^{-\alpha_1 t} - e^{-\alpha_2 t}}{\alpha_2 - \alpha_1} \int_{\mathbb{R}^n} u_1(x) dx \\ &\quad + e^{-\alpha_2 t} \int_0^t e^{(\alpha_2 - \alpha_1)s} \int_0^s e^{\alpha_1 \tau} \Gamma(\tau) \left(\int_{\mathbb{R}^n} |u(\tau, x)|^p dx \right)^{\beta+1} d\tau ds. \end{aligned} \quad (2.6)$$

In the limit case $b^2 = 4m^2$, we can proceed similarly obtaining

$$U(t) = e^{-\frac{b}{2}t} U(0) + (U'(0) + \frac{b}{2}U(0)) t e^{-\frac{b}{2}t} + e^{-\frac{b}{2}t} \int_0^t \int_0^s e^{\frac{b}{2}\tau} \Gamma(\tau) \left(\int_{\mathbb{R}^n} |u(\tau, x)|^p dx \right)^{\beta+1} d\tau ds$$

$$\begin{aligned}
 &= \varepsilon \left(1 + \frac{b}{2}t\right) e^{-\frac{b}{2}t} \int_{\mathbb{R}^n} u_0(x) \, dx + \varepsilon t e^{-\frac{b}{2}t} \int_{\mathbb{R}^n} u_1(x) \, dx \\
 &+ e^{-\frac{b}{2}t} \int_0^t \int_0^s e^{\frac{b}{2}\tau} \Gamma(\tau) \left(\int_{\mathbb{R}^n} |u(\tau, x)|^p \, dx \right)^{\beta+1} \, d\tau \, ds. \tag{2.7}
 \end{aligned}$$

Consequently, requiring that u_0 and u_1 are nonnegative functions, then, from the previous identities we obtain immediately that U is a nonnegative functional. Next we determine the iteration frame. Since $\text{supp } u(t, \cdot) \subset B_{R+A_{\text{dS}}(t)}$ for any $t \in (0, T)$, by using Hölder’s inequality we have

$$\begin{aligned}
 0 \leq U(t) &\leq \left(\int_{\mathbb{R}^n} |u(t, x)|^p \, dx \right)^{\frac{1}{p}} \left(\text{meas} (B_{R+A_{\text{dS}}(t)}) \right)^{\frac{1}{p'}} \\
 &\lesssim (R + A_{\text{dS}}(t))^{\frac{n}{p'}} \left(\int_{\mathbb{R}^n} |u(t, x)|^p \, dx \right)^{\frac{1}{p}} \lesssim \left(\int_{\mathbb{R}^n} |u(t, x)|^p \, dx \right)^{\frac{1}{p}},
 \end{aligned}$$

and, hence,

$$\int_{\mathbb{R}^n} |u(t, x)|^p \, dx \gtrsim (U(t))^p.$$

Notice that in the previous step, we took advantage of the fact that the light-cone is contained in an infinite half cylinder.

Thus, from (2.6) and (2.7) we get the iteration frame

$$U(t) \geq C e^{-\alpha_2 t} \int_0^t e^{(\alpha_2 - \alpha_1)s} \int_0^s e^{\alpha_1 \tau} \Gamma(\tau) (U(\tau))^{(\beta+1)p} \, d\tau \, ds, \tag{2.8}$$

where $C = C(n, c, H, p, R) > 0$ is a suitable constant.

Clearly, in order to be able to apply the previous iteration frames to get a sequence of lower bound estimates for U , we need to determine a first lower bound for U . From (2.6) and (2.7), since the Cauchy data are taken nonnegative and nontrivial, we have immediately the lower bound estimates

$$U(t) \geq \begin{cases} K_0 \varepsilon e^{-\left(\frac{b}{2} - \frac{1}{2}\sqrt{b^2 - 4m^2}\right)t} & \text{if } b^2 > 4m^2, \\ K_0 \varepsilon (1 + t) e^{-\frac{b}{2}t} & \text{if } b^2 = 4m^2, \end{cases} \tag{2.9}$$

for any $t \in (0, T)$, where $K_0 = K_0(b, m^2, u_0, u_1)$ is a suitable positive and independent of ε constant.

We emphasize that (2.8) is the iteration frame that will be used in the proof of Theorem 1.2, while for Theorems 1.3 and 1.4 the choice of the time-dependent functional and the corresponding iteration frame will follow directly from (2.8).

In the next three subsections, we will prove these theorems. In each case the growth condition assumed on Γ has a crucial role in determining the key factors in the iteration frame and, consequently, the main features of the associated slicing procedure.

2.2. Case with exponential growth: proof of Theorem 1.2

In this subsection, we prove Theorem 1.2. As anticipated, the time-dependent functional that we consider to prove the blow-up result is the space average U defined in (2.1).

Since the time-dependent factor Γ in (1.2) is given by (1.10) with $r > r_{\text{crit}(b,m^2,\beta,p)}$, the exponential growth of Γ is dominant over the first lower bound for U in (2.9) (which decays exponentially). Therefore, when deriving the sequence of lower bound estimates for U through (2.8), we need to handle exponentially increasing factors both in the τ -integral and in the s -integral. Hence, we apply a 2 steps slicing procedure and the coefficients characterizing the shrinking of the domains of integration on the right-hand side of (2.8) are chosen in order to allow the handling of the unbounded exponential multipliers $e^{(\alpha_1+r)\tau}$ and $e^{(\alpha_2+r)s}$ in the first and in the second integral, respectively.

We may define now the parameters $\{L_j\}_{j \in \mathbb{N}}$ that characterize the slicing procedure:

$$L_j \doteq \prod_{k=0}^j \ell_k \quad \text{for any } j \in \mathbb{N}, \tag{2.10}$$

where the coefficients $\{\ell_k\}_{k \in \mathbb{N}}$ are given by

$$\ell_0 \doteq \max \{ (r + \alpha_1)^{-1}, (r + \alpha_2)^{-1} \}, \quad \ell_k \doteq 1 + ((\beta + 1)p)^{-k/2} \quad \text{for any } k \geq 1.$$

Notice that ℓ_0 is well defined thanks to the condition on r . Moreover, since $\ell_k > 1$ for any $k \geq 1$, the sequence $\{L_j\}_{j \in \mathbb{N}}$ is strictly increasing. Finally, due to the choice of $\{\ell_k\}_{k \geq 1}$, we have that the series $\sum_{k=1}^\infty \ln \ell_k$ is convergent, and this is equivalent to prove the convergence of the following infinite product

$$L \doteq \prod_{j=0}^\infty \ell_j \in \mathbb{R}_+.$$

Our first goal is to prove the following sequence of lower bound estimates for U :

$$U(t) \geq C_j e^{a_j t} (t - L_{2j})^{b_j} (1 + t)^{-\beta_j} \quad \text{for } t \geq L_{2j} \text{ and for any } j \in \mathbb{N}, \tag{2.11}$$

where $\{C_j\}_{j \in \mathbb{N}}$, $\{a_j\}_{j \in \mathbb{N} \setminus \{0\}}$, $\{b_j\}_{j \in \mathbb{N}}$, $\{\beta_j\}_{j \in \mathbb{N}}$ are sequences of nonnegative real numbers to be determined iteratively.

For $j = 0$ (2.11) is given by (2.9) provided that $C_0 \doteq K_0 \varepsilon$, $a_0 \doteq -\frac{b}{2} + \frac{1}{2} \sqrt{b^2 - 4m^2}$, $\beta_0 = 0$, and, finally, $b_0 \doteq 0$ if $b^2 > 4m^2$ and $b_0 \doteq 1$ if $b^2 = 4m^2$. We underline that a_0 is the only term in the sequence $\{a_j\}_j$ that is not positive. Denoting by κ_+ and κ_- the positive and the negative part of κ (i.e., $\kappa_+ \doteq \max\{\kappa, 0\}$ and $\kappa_- \doteq -\min\{\kappa, 0\}$), from (2.8) we get

$$U(t) \geq \mu C (1 + t)^{-\kappa_-} e^{-\alpha_2 t} \int_0^t e^{(\alpha_2 - \alpha_1)s} \int_0^s e^{(\alpha_1 + r)\tau} \tau^{\kappa_+} (U(\tau))^{(\beta + 1)p} d\tau ds. \tag{2.12}$$

We want to prove (2.11) by induction with respect to j . We have already remarked the validity of the base case. Next we prove the induction step. Assuming that (2.11) is satisfied for some $j \geq 0$ we prove it for $j + 1$. Plugging the lower bound estimate (2.11) in (2.12), for $t \geq L_{2j}$ we have

$$U(t) \geq \mu C (1 + t)^{-\kappa_-} e^{-\alpha_2 t} \int_{L_{2j}}^t e^{(\alpha_2 - \alpha_1)s} \int_{L_{2j}}^s e^{(\alpha_1 + r)\tau} \tau^{\kappa_+} (U(\tau))^{(\beta + 1)p} d\tau ds$$

$$\geq \mu C C_j^q (1+t)^{-(\kappa_- + q\beta_j)} e^{-\alpha_2 t} \int_{L_{2j}}^t e^{(\alpha_2 - \alpha_1)s} \int_{L_{2j}}^s e^{(\alpha_1 + r + qa_j)\tau} (\tau - L_{2j})^{\kappa_+ + qb_j} d\tau ds,$$

where from now on, for the sake of brevity, we denote $q \doteq (\beta + 1)p$. For $t \geq L_{2j+1}$ we can shrink the domain of integration in the previous inequality as follows:

$$\begin{aligned} U(t) &\geq \mu C C_j^q (1+t)^{-(\kappa_- + q\beta_j)} e^{-\alpha_2 t} \int_{L_{2j+1}}^t e^{(\alpha_2 - \alpha_1)s} \int_{\frac{L_{2j}s}{L_{2j+1}}}^s e^{(\alpha_1 + r + qa_j)\tau} (\tau - L_{2j})^{\kappa_+ + qb_j} d\tau ds \\ &\geq \frac{\mu C C_j^q}{\ell_{2j+1}^{\kappa_+ + qb_j}} (1+t)^{-(\kappa_- + q\beta_j)} e^{-\alpha_2 t} \int_{L_{2j+1}}^t e^{(\alpha_2 - \alpha_1)s} (s - L_{2j+1})^{\kappa_+ + qb_j} \int_{\frac{s}{\ell_{2j+1}}}^s e^{(\alpha_1 + r + qa_j)\tau} d\tau ds, \end{aligned} \tag{2.13}$$

where in the second step we used the monotonicity of the factor $(\tau - L_{2j})^{\kappa_+ + qb_j}$. Let us show now how we can estimate from below the τ -integral. By a direct computation we have

$$\begin{aligned} \int_{\frac{s}{\ell_{2j+1}}}^s e^{(\alpha_1 + r + qa_j)\tau} d\tau &= (\alpha_1 + r + qa_j)^{-1} e^{(\alpha_1 + r + qa_j)s} \left(1 - e^{-(\alpha_1 + r + qa_j)(1 - 1/\ell_{2j+1})s} \right) \\ &\geq (\alpha_1 + r + qa_j)^{-1} e^{(\alpha_1 + r + qa_j)s} \left(1 - e^{-(\alpha_1 + r + qa_j)(\ell_{2j+1} - 1)L_{2j}} \right) \\ &\geq (\alpha_1 + r + qa_j)^{-1} e^{(\alpha_1 + r + qa_j)s} \left(1 - e^{-(\alpha_1 + r)(\ell_{2j+1} - 1)L_0} \right) \\ &\geq (\alpha_1 + r + qa_j)^{-1} e^{(\alpha_1 + r + qa_j)s} \left(1 - e^{-(\ell_{2j+1} - 1)} \right) \end{aligned}$$

for $s \geq L_{2j+1}$, where in the previous chain of inequalities we used the following properties $\alpha_1 + r > 0$, $a_j \geq 0$, $\ell_{2j+1} > 1$, $L_{2j} \uparrow$ and $L_0 = \ell_0 \geq (\alpha_1 + r)^{-1}$. Then, using the inequality $1 - e^{-y} \geq y - \frac{y^2}{2}$ for any $y \geq 0$ we have

$$\begin{aligned} 1 - e^{-(\ell_{2j+1} - 1)} &\geq (\ell_{2j+1} - 1) \left(1 - \frac{1}{2}(\ell_{2j+1} - 1) \right) = q^{-(2j+1)} (q^{j+1/2} - \frac{1}{2}) \\ &\geq q^{-(2j+1)} (q - \frac{1}{2}). \end{aligned} \tag{2.14}$$

Combining these two last inequalities, from (2.13) we obtain

$$U(t) \geq \frac{\mu C (q - \frac{1}{2}) C_j^q}{\ell_{2j+1}^{\kappa_+ + qb_j} (\alpha_1 + r + qa_j) q^{2j+1}} (1+t)^{-(\kappa_- + q\beta_j)} e^{-\alpha_2 t} \int_{L_{2j+1}}^t e^{(\alpha_2 + r + qa_j)s} (s - L_{2j+1})^{\kappa_+ + qb_j} ds.$$

Until now we applied a first step in the slicing procedure to deal with the τ -integral. Repeating analogous computations after shrinking the domain of integration to $[t/\ell_{2j+2}, t]$ in the s -integral for $t \geq L_{2j+2}$, we arrive at the lower bound estimate

$$U(t) \geq \frac{\mu C (q - \frac{1}{2})^2 C_j^q (\ell_{2j+1} \ell_{2j+2})^{-(\kappa_+ + qb_j)}}{(\alpha_1 + r + qa_j)(\alpha_2 + r + qa_j) q^{4j+3}} e^{(r + qa_j)t} (t - L_{2j+2})^{\kappa_+ + qb_j} (1+t)^{-(\kappa_- + q\beta_j)},$$

which is exactly (2.11) for $j + 1$ provided that

$$C_{j+1} \doteq \frac{\mu C(q - \frac{1}{2})^2 C_j^q (\ell_{2j+1} \ell_{2j+2})^{-(\kappa_+ + qb_j)}}{(\alpha_1 + r + qa_j)(\alpha_2 + r + qa_j) q^{4j+3}}, \tag{2.15}$$

$$a_{j+1} \doteq r + qa_j, \quad b_{j+1} \doteq \kappa_+ + qb_j, \quad \beta_{j+1} \doteq \kappa_- + q\beta_j. \tag{2.16}$$

By applying recursively the previous relations among two consecutive terms from the sequences $\{a_j\}_{j \in \mathbb{N}}$, $\{b_j\}_{j \in \mathbb{N}}$, $\{\beta_j\}_{j \in \mathbb{N}}$ we obtain the explicit representation

$$a_j = r \sum_{k=0}^{j-1} q^k + q^j a_0 = \frac{q^j - 1}{q - 1} r + q^j a_0 = \left(\frac{r}{q - 1} + a_0 \right) q^j - \frac{r}{q - 1}, \tag{2.17}$$

and, in an analogous way,

$$b_j = \left(\frac{\kappa_+}{q - 1} + b_0 \right) q^j - \frac{\kappa_+}{q - 1}, \tag{2.18}$$

$$\beta_j = \frac{\kappa_-}{q - 1} q^j - \frac{\kappa_-}{q - 1}, \tag{2.19}$$

where in the last relation we used $\beta_0 = 0$. The next step is to determine a lower bound for the constant C_j that we can handle more easily. We remark that, since $r > r_{\text{crit}}(b, m^2, \beta, p)$ the quantity $r/(q - 1) + a_0$ is strictly positive. Therefore,

$$\begin{aligned} \alpha_{1/2} + r + qa_j &= \alpha_{1/2} + a_{j+1} < \left(\frac{r}{q-1} + a_0 \right) q^j + \alpha_{1/2} + a_0 \\ &\leq \left(\frac{r}{q-1} + a_0 \right) q^j + \sqrt{b^2 - 4m^2} \leq M_0 q^j \end{aligned} \tag{2.20}$$

for any $j \in \mathbb{N}$, where $M_0 = M_0(b, m^2, r, \beta, p)$ is a suitable positive quantity which is independent of j . Furthermore, we remark that

$$\begin{aligned} \lim_{j \rightarrow \infty} (\ell_{2j+1} \ell_{2j+2})^{b_{j+1}} &= \lim_{j \rightarrow \infty} \exp(b_{j+1} [\ln \ell_{2j+1} + \ln \ell_{2j+2}]) \\ &= \lim_{j \rightarrow \infty} \exp\left(\left(\frac{\kappa_+}{q-1} + b_0 \right) q^{j+1} \left[\ln \left(1 + q^{-(j+1/2)} \right) + \ln \left(1 + q^{-(j+1)} \right) \right] \right) \\ &= \exp\left(\left(\frac{\kappa_+}{q-1} + b_0 \right) (1 + \sqrt{q}) \right), \end{aligned}$$

consequently, there exists a uniform (i.e. independent of j) constant $M_1 = M_1(b, m^2, \kappa, \beta, p) > 0$ such that $(\ell_{2j+1} \ell_{2j+2})^{b_{j+1}} \leq M_1$ for any $j \in \mathbb{N}$. Combining (2.15), (2.16), (2.20) and the previous uniform upper bound, we see that

$$C_{j+1} = \frac{\mu C(q - \frac{1}{2})^2 C_j^q}{(\ell_{2j+1} \ell_{2j+2})^{b_{j+1}} (\alpha_1 + a_{j+1})(\alpha_2 + a_{j+1}) q^{4j+3}} \geq \underbrace{\frac{\mu C(q - \frac{1}{2})^2 q^3}{M_1 M_0^2}}_{D \doteq} q^{-6(j+1)} C_j^q.$$

We can now use the inequality $C_j \geq D q^{-6j} C_{j-1}^q$ to derive a more convenient lower bound for C_j for sufficiently large indexes. Applying the logarithmic function to both sides of the previous inequality and using iteratively the resulting inequality, we find

$$\begin{aligned} \ln C_j &\geq q \ln C_{j-1} - 6j \ln q + \ln D \geq q^2 \ln C_{j-1} - 6(j + (j - 1)q) \ln q + (1 + q) \ln D \\ &\geq \dots \geq q^j \ln C_0 - 6 \left(\sum_{k=0}^{j-1} (j - k) q^k \right) \ln q + \left(\sum_{k=0}^{j-1} q^k \right) \ln D. \end{aligned}$$

Using the following identity

$$\sum_{k=0}^{j-1} (j-k)q^k = \frac{1}{q-1} \left(\frac{q^{j+1} - q}{q-1} - j \right), \tag{2.21}$$

we have

$$\ln C_j \geq q^j \left(\ln C_0 - \frac{6q \ln q}{(q-1)^2} + \frac{\ln D}{q-1} \right) + \frac{6q \ln q}{(q-1)^2} + \frac{6 \ln q}{q-1} j - \frac{\ln D}{q-1}.$$

Let $j_0 = j_0(n, c, H, b, m^2, \mu, r, \kappa, \beta, p, R) \in \mathbb{N}$ be the smallest integer such that $j_0 \geq \frac{\ln D}{6 \ln q} - \frac{q}{q-1}$. Then, for any $j \geq j_0$ it results

$$\ln C_j \geq q^j \left(\ln(K_0 \varepsilon) - \frac{6q \ln q}{(q-1)^2} + \frac{\ln D}{q-1} \right) = q^j \ln(\tilde{D} \varepsilon), \tag{2.22}$$

where $\tilde{D} \doteq K_0 q^{-6q/(q-1)^2} D^{1/(q-1)}$. Hence, recalling that $L_{2j} \uparrow L$, if we combine (2.11), (2.17), (2.18), (2.19) and (2.22) for $t \geq L$ and for any $j \geq j_0$ it holds

$$\begin{aligned} U(t) &\geq \exp \left(q^j \left(\ln(\tilde{D} \varepsilon) + \left(\frac{r}{q-1} + a_0 \right) t + \left(\frac{\kappa_+}{q-1} + b_0 \right) \ln(t-L) - \frac{\kappa_-}{q-1} \ln(1+t) \right) \right) \\ &\quad \times \exp \left(-\frac{r}{q-1} t \right) (t-L)^{-\frac{\kappa_+}{q-1}} (1+t)^{-\frac{\kappa_-}{q-1}}. \end{aligned}$$

Next, using the trivial inequalities $\ln(t-L) \geq \ln t - \ln 2$ and $-\ln(1+t) \geq -\ln t - \ln 2$ for $t \geq \max\{2L, 1\}$ and the identity $\kappa = \kappa_+ - \kappa_-$, from the previous estimate we obtain

$$U(t) \geq \exp \left(q^j \ln \left(\hat{D} \varepsilon \left(\frac{\kappa_+ + (q-1)b_0}{t^{r+(q-1)a_0}} e^t \right)^{\frac{r}{q-1} + a_0} \right) \right) \exp \left(-\frac{r}{q-1} t \right) (t-L)^{-\frac{\kappa_+}{q-1}} (1+t)^{-\frac{\kappa_-}{q-1}},$$

where $\hat{D} \doteq 2^{-((\kappa_+ + \kappa_-)/(q-1) + b_0)} \tilde{D}$. By using the function defined in (1.12), we may rewrite

$$U(t) \geq \exp \left(q^j \ln \left(\hat{D} \varepsilon \left(\theta_{b, m^2, p, \beta, r, \kappa}(t) \right)^{\frac{r}{q-1} + a_0} \right) \right) \exp \left(-\frac{r}{q-1} t \right) (t-L)^{-\frac{\kappa_+}{q-1}} (1+t)^{-\frac{\kappa_-}{q-1}} \tag{2.23}$$

for $t \geq \max\{2L, 1\}$ and for $j \geq j_0$.

From (1.12) we see that $\theta_{b, m^2, p, \beta, r, \kappa}$ is strictly increasing (and hence invertible) for $t \geq \tilde{T}$, where $\tilde{T} = \tilde{T}(b, m^2, p, \beta, r, \kappa)$ is a suitable nonnegative quantity. Note that for $\kappa \geq 0$ if $b^2 > 4m^2$ and $\kappa \geq -(\beta+1)p+1$ if $b^2 = 4m^2$ we can simply take $\tilde{T}(b, m^2, p, \beta, r, \kappa) = 0$. With a slight abuse of notation, in what follows we denote by $\theta_{b, m^2, p, \beta, r, \kappa}^{-1}$ the inverse function of the restriction $\theta_{b, m^2, p, \beta, r, \kappa}|_{[\tilde{T}, \infty)}$.

We remark that the logarithmic factor multiplying q^j in (2.23) is strictly positive if and only if $\hat{D} \varepsilon \left(\theta_{b, m^2, p, \beta, r, \kappa}(t) \right)^{\frac{r}{q-1} + a_0} > 1$. For $t \geq \tilde{T}$, this is equivalent to require

$$t > \theta_{b, m^2, p, \beta, r, \kappa}^{-1} \left(\left(\hat{D} \varepsilon \right)^{-\left(\frac{r}{q-1} + a_0 \right)^{-1}} \right).$$

Since $\lim_{s \rightarrow \infty} \theta_{b, m^2, p, \beta, r, \kappa}^{-1}(s) = \infty$, we may fix $\varepsilon_0 = \varepsilon_0(n, c, H, b, m^2, q, \mu, r, \kappa, u_0, u_1, R) > 0$ sufficiently small so that

$$\theta_{b, m^2, p, \beta, r, \kappa}^{-1} \left(\left(\hat{D} \varepsilon_0 \right)^{-\left(\frac{r}{q-1} + a_0 \right)^{-1}} \right) \geq \max\{2L, 1, \tilde{T}\}.$$

Thus, for any $\varepsilon \in (0, \varepsilon_0]$ and any $t > \theta_{b,m^2,p,\beta,r,\kappa}^{-1} \left((\hat{D}\varepsilon)^{-\left(\frac{r}{q-1} + \alpha_0\right)^{-1}} \right)$ we find that $t \geq \{2L, 1, \tilde{T}\}$ and that the factor multiplying q^j in (2.23) is positive, so, letting $j \rightarrow \infty$ in (2.23) we see that the lower bound for $U(t)$ is not finite. Hence, we proved that U blows up in finite time and, as byproduct of the iteration procedure, we got the upper bound estimate for the lifespan in (1.11). This completes the proof of Theorem 1.2.

Remark 6. In the proof of (2.11) the assumption $r > r_{\text{crit}}(b, m^2, \beta, p)$ allows to define properly ℓ_0 , since $r + \alpha_{1/2} > 0$. However, the crucial point in the previous iteration argument where this assumption on the range for r is used is in the representation (2.17) for a_j . Indeed, the term a_j allows to get a growth of exponential type in the lower bound estimates (2.11). In the next subsections, we consider the limit case $r = r_{\text{crit}}(b, m^2, \beta, p)$ for which the previous argument does no longer hold. A first step will be to introduce a new time-dependent functional related to U and the relative iteration frame. In this new iteration frame we have to deal with just one or no exponential multiplier depending on whether we consider the case $b^2 > 4m^2$ or the case $b^2 = 4m^2$. Therefore, in Theorems 1.3 and 1.4 a significant role will be played by the power κ for the polynomial term.

Remark 7. For $b > 0$ and $m^2 \in \left[0, \frac{b^2}{4}\right]$ we may weaken the sign assumptions on u_0, u_1 in Theorem 1.2. In fact, it is sufficient to suppose that $\int_{\mathbb{R}^n} u_0(x) dx, \int_{\mathbb{R}^n} u_1(x) dx$ are nonnegative and that at least one between them is strictly positive. Indeed, under these assumptions (2.9) keeps to be fulfilled for $b \neq 0$ and this suffices to start the iteration argument.

On the other hand, it is interesting to consider for $b = m^2 = 0$ the case in which the second Cauchy data satisfy $\int_{\mathbb{R}^n} u_1(x) dx = 0$ (and, of course, $\int_{\mathbb{R}^n} u_0(x) dx > 0$). Then, Theorem 1.2 is still valid in the case $b = m^2 = 0$, however, the lifespan estimate in this case is the same one as in the case $b^2 > 4m^2$ in (1.11). This worsening in the upper bound is caused by the fact that the lower bound for U in this case is given by $U(t) \geq K_0 \varepsilon e^{-\left(\frac{b}{2} - \frac{1}{2}\sqrt{b^2 - 4m^2}\right)t}$, i.e., without any additional linearly increasing t -factor on the right-hand side differently from (2.9).

2.3. Case with polynomial growth: proof of Theorem 1.3

In the present subsection, we provide the proof of Theorem 1.3. In this framework, the time-dependent factor Γ in (1.2) is given by (1.13) with $\kappa > \kappa_{\text{crit}}(b, m^2, \beta, p)$.

Let us multiply both sides of (2.8) by $e^{\alpha_1 t}$. Then, introducing the functional

$$U(t) \doteq e^{\alpha_1 t} U(t) \quad \text{for } t \in [0, T), \tag{2.24}$$

we obtain

$$U(t) \geq C e^{(\alpha_1 - \alpha_2)t} \int_0^t e^{(\alpha_2 - \alpha_1)s} \int_0^s e^{-\alpha_1((\beta+1)p-1)\tau} \Gamma(\tau) (U(\tau))^{(\beta+1)p} d\tau ds$$

for $t \geq 0$. Differently from the previous subsection, where the role of α_1 and α_2 are interchangeable, we need to set specific values for α_1 and α_2 . From the previous inequality it is clear that it would be beneficial to fix α_1 in a such a way that in the τ -integral the exponential factor $e^{-\alpha_1((\beta+1)p-1)\tau}$ is balanced by the exponential factor $e^{r_{\text{crit}}(b, m^2, \beta, p)\tau}$ in $\Gamma(\tau)$.

Therefore, hereafter we set $\alpha_1 = \frac{b}{2} - \frac{1}{2}\sqrt{b^2 - 4m^2}$ and $\alpha_2 = \frac{b}{2} + \frac{1}{2}\sqrt{b^2 - 4m^2}$. In particular, with this choice we obtain from the previous inequality the following iteration frame for U

$$U(t) \geq C \mu e^{(\alpha_1 - \alpha_2)t} \int_0^t e^{(\alpha_2 - \alpha_1)s} \int_0^s (1 + \tau)^\kappa (U(\tau))^q d\tau ds \tag{2.25}$$

for $t \geq 0$, where $q = (\beta + 1)p$ as in the previous subsection. Notice that the coefficient $\alpha_2 - \alpha_1$ in the exponential multiplier in the s -integral is positive, due to our choice.

Remark 8. Considering alternatively the functional $\tilde{\mathcal{U}}(t) \doteq e^{\alpha_2 t} U(t)$ and switching the values of α_1 and α_2 with respect to the values we have just fixed, we would have found the iteration frame

$$\tilde{\mathcal{U}}(t) \geq C\mu \int_0^t e^{(\alpha_2 - \alpha_1)s} \int_0^s e^{(\alpha_1 - \alpha_2)\tau} (1 + \tau)^\kappa (\tilde{\mathcal{U}}(\tau))^q d\tau ds$$

for $t \geq 0$. Even though the structure of this iteration frame would require somehow different computations in the induction step (since the slicing procedure has to be carried out in the τ -integral rather than in the s -integral as we will do in the next steps of the proof), the final outcome, meaning the blow-up of u and the upper bound estimate for the lifespan, is exactly the same. In this sense, we can still say that the role of α_1 and α_2 are interchangeable even in this limit case for r .

From (2.9) and (2.24), we get immediately the first lower bound estimates for \mathcal{U} , namely,

$$\mathcal{U}(t) \geq \begin{cases} K_0\varepsilon & \text{if } b^2 > 4m^2, \\ K_0\varepsilon(1 + t) & \text{if } b^2 = 4m^2, \end{cases} \tag{2.26}$$

for $t \geq 0$.

From (2.25) it is clear that when $\alpha_1 = \alpha_2$, that is for $b^2 = 4m^2$, the iteration procedure which we use to establish the sequence of lower bound estimates for \mathcal{U} is quite different. Indeed, depending on whether or not an unbounded exponential multiplier is present in the s -integral we might need to apply the slicing procedure or not. Hence, we will consider separately the cases $b^2 > 4m^2$ and $b^2 = 4m^2$.

2.3.1. Case with polynomial growth: sub-case with dominant damping

In this case $\alpha_1 \neq \alpha_2$ so that $\alpha_2 - \alpha_1 = \sqrt{b^2 - 4m^2} > 0$. Since in the iteration frame for \mathcal{U} given by (2.25) we have the exponential multiplier $e^{(\alpha_2 - \alpha_1)s}$, we have to modify the choice of the parameters $\{L_j\}_{j \in \mathbb{N}}$ characterizing the slicing procedure with respect to Subsection 2.2. Formally, L_j is defined as in (2.10), however, the coefficients $\{\ell_k\}_{k \in \mathbb{N}}$ are given in this case by

$$\begin{aligned} \ell_0 &\doteq (\alpha_2 - \alpha_1)^{-1}, \\ \ell_k &\doteq 1 + q^{-k} \quad \text{for any } k \geq 1. \end{aligned}$$

Since $\ell_k > 1$ for any $k \in \mathbb{N} \setminus \{0\}$, also in this case we have $L_j \uparrow$. Moreover, we keep using the notation $L \doteq \lim_{j \rightarrow \infty} L_j$ (the convergence of this infinite product can be proved exactly as in the previous case).

Let us prove now the following sequence of lower bound estimates for \mathcal{U} :

$$\mathcal{U}(t) \geq C_j(t - L_j)^{b_j}(1 + t)^{-\beta_j} \quad \text{for } t \geq L_j \text{ and for any } j \in \mathbb{N}, \tag{2.27}$$

where $\{C_j\}_{j \in \mathbb{N}}$, $\{b_j\}_{j \in \mathbb{N}}$, $\{\beta_j\}_{j \in \mathbb{N}}$ are sequences of nonnegative real numbers to be determined during the inductive argument. From (2.26), we have that (2.27) for $j = 0$, provided that $C_0 \doteq K_0\varepsilon$, $b_0 \doteq 0$ and $\beta_0 \doteq 0$.

Let us prove the induction step. Plugging (2.27) in (2.25) we have

$$\begin{aligned} \mathcal{U}(t) &\geq \mu C e^{(\alpha_1 - \alpha_2)t} \int_{L_j}^t e^{(\alpha_2 - \alpha_1)s} \int_{L_j}^s (1 + \tau)^\kappa (\mathcal{U}(\tau))^q d\tau ds \\ &\geq \mu C C_j^q (1 + t)^{-(\kappa_- + qb_j)} e^{(\alpha_1 - \alpha_2)t} \int_{L_j}^t e^{(\alpha_2 - \alpha_1)s} \int_{L_j}^s (\tau - L_j)^{\kappa_+ + b_j q} d\tau ds \\ &= \mu C C_j^q (1 + \kappa_+ + qb_j)^{-1} (1 + t)^{-(\kappa_- + qb_j)} e^{(\alpha_1 - \alpha_2)t} \int_{L_j}^t e^{(\alpha_2 - \alpha_1)s} (s - L_j)^{1 + \kappa_+ + qb_j} ds \end{aligned}$$

for $t \geq L_j$, where κ_+, κ_- denote the positive and the negative part of κ , respectively. For $t \geq L_{j+1}$, it is possible to shrink the domain of integration to $[t/L_{j+1}, t]$ in the last integral, obtaining

$$\begin{aligned} \mathcal{U}(t) &\geq \mu C C_j^q (1 + \kappa_+ + qb_j)^{-1} (1 + t)^{-(\kappa_- + qb_j)} e^{(\alpha_1 - \alpha_2)t} \int_{\frac{L_j t}{L_{j+1}}}^t e^{(\alpha_2 - \alpha_1)s} (s - L_j)^{1 + \kappa_+ + qb_j} ds \\ &\geq \frac{\mu C C_j^q}{(1 + \kappa_+ + qb_j) \ell_{j+1}^{1 + \kappa_+ + qb_j}} (t - L_j)^{1 + \kappa_+ + qb_j} (1 + t)^{-(\kappa_- + qb_j)} e^{(\alpha_1 - \alpha_2)t} \int_{\frac{L_j t}{L_{j+1}}}^t e^{(\alpha_2 - \alpha_1)s} ds \\ &\geq \frac{\mu C (\alpha_2 - \alpha_1)^{-1} C_j^q}{(1 + \kappa_+ + qb_j) \ell_{j+1}^{1 + \kappa_+ + qb_j}} (t - L_j)^{1 + \kappa_+ + qb_j} (1 + t)^{-(\kappa_- + qb_j)} \left(1 - e^{-(\alpha_2 - \alpha_1)(1 - \frac{1}{\ell_{j+1}})t}\right). \end{aligned}$$

Using the estimate

$$\begin{aligned} 1 - e^{-(\alpha_2 - \alpha_1)(1 - \frac{1}{\ell_{j+1}})t} &\geq 1 - e^{-(\alpha_2 - \alpha_1)(\ell_{j+1} - 1)L_j} \geq 1 - e^{-(\alpha_2 - \alpha_1)(\ell_{j+1} - 1)L_0} = 1 - e^{-(\ell_{j+1} - 1)} \\ &\geq (\ell_{j+1} - 1) \left(1 - \frac{1}{2}(\ell_{j+1} - 1)\right) \geq (q - \frac{1}{2})q^{-2(j+1)} \end{aligned} \quad (2.28)$$

for $t \geq L_{j+1}$, we find

$$\mathcal{U}(t) \geq \frac{\mu C (q - \frac{1}{2}) C_j^q}{(\alpha_2 - \alpha_1)(1 + \kappa_+ + qb_j) \ell_{j+1}^{1 + \kappa_+ + qb_j} q^{2(j+1)}} (t - L_j)^{1 + \kappa_+ + qb_j} (1 + t)^{-(\kappa_- + qb_j)},$$

which is exactly (2.27) for $j + 1$, provided that

$$C_{j+1} \doteq \frac{\mu C (q - \frac{1}{2})}{(\alpha_2 - \alpha_1)} (1 + \kappa_+ + qb_j)^{-1} \ell_{j+1}^{-(1 + \kappa_+ + qb_j)} q^{-2(j+1)} C_j^q, \quad (2.29)$$

$$b_{j+1} \doteq 1 + \kappa_+ + qb_j, \quad \beta_{j+1} \doteq \kappa_- + qb_j. \quad (2.30)$$

By employing recursively (2.30) among two consecutive terms from the sequences $\{b_j\}_{j \in \mathbb{N}}$ and $\{\beta_j\}_{j \in \mathbb{N}}$, we get

$$b_j = \frac{1 + \kappa_+}{q - 1} q^j - \frac{1 + \kappa_+}{q - 1}, \quad (2.31)$$

$$\beta_j = \frac{\kappa_-}{q - 1} q^j - \frac{\kappa_-}{q - 1}, \quad (2.32)$$

where we used $b_0 = \beta_0 = 0$. Thanks to (2.30) and (2.31), we have

$$1 + \kappa_+ + qb_j = b_{j+1} \leq \frac{1 + \kappa_+}{q - 1} q^{j+1}. \tag{2.33}$$

Moreover,

$$\lim_{j \rightarrow \infty} \ell_{j+1}^{b_{j+1}} = \lim_{j \rightarrow \infty} \exp(b_{j+1} \ln \ell_{j+1}) = \lim_{j \rightarrow \infty} \exp\left(\frac{1 + \kappa_+}{q - 1} q^{j+1} \ln\left(1 + q^{-(j+1)}\right)\right) = \exp\left(\frac{1 + \kappa_+}{q - 1}\right)$$

implies the existence of a constant $M_2 = M_2(\beta, p, \kappa)$ such that $\ell_{j+1}^{b_{j+1}} \leq M_2$ for any $j \in \mathbb{N}$. Combining this last uniform upper bound with (2.29) and (2.33), we obtain

$$C_{j+1} \doteq \underbrace{\frac{\mu C(q - \frac{1}{2})(q - 1)}{(\alpha_2 - \alpha_1)(1 + \kappa_+)M_2}}_{B \doteq} q^{-3(j+1)} C_j^q$$

for any $j \in \mathbb{N}$. Applying the logarithmic function to both sides of the inequality $C_j \geq Bq^{-3j} C_{j-1}^q$ and, then, using iteratively the resulting inequality, we find

$$\begin{aligned} \ln C_j &\geq q \ln C_{j-1} - 3j \ln q + \ln B \geq q^2 \ln C_{j-1} - 3(j + (j - 1)q) \ln q + (1 + q) \ln B \\ &\geq \dots \geq q^j \ln C_0 - 3 \left(\sum_{k=0}^{j-1} (j - k) q^k \right) \ln q + \left(\sum_{k=0}^{j-1} q^k \right) \ln B \\ &\geq q^j \left(\ln C_0 - \frac{3q \ln q}{(q - 1)^2} + \frac{\ln B}{q - 1} \right) + \frac{3q \ln q}{(q - 1)^2} + \frac{3 \ln q}{q - 1} j - \frac{\ln B}{q - 1}, \end{aligned}$$

where in the last step we used (2.21).

Let $j_1 = j_1(n, c, H, b, m^2, \mu, \kappa, \beta, p, R) \in \mathbb{N}$ be the smallest integer such that $j_1 \geq \frac{\ln B}{3 \ln q} - \frac{q}{q - 1}$. Hence, for any $j \geq j_1$ it holds

$$\ln C_j \geq q^j \left(\ln(K_0 \varepsilon) - \frac{3q \ln q}{(q - 1)^2} + \frac{\ln B}{q - 1} \right) = q^j \ln(\tilde{B} \varepsilon), \tag{2.34}$$

where $\tilde{B} \doteq K_0 q^{-3q/(q-1)^2} B^{1/(q-1)}$. Since $L_j \uparrow L$ as $j \rightarrow \infty$, in particular (2.27) is true for $t \geq L$ and any $j \in \mathbb{N}$. Combining (2.27), (2.31), (2.32) and (2.34), for $t \geq L$ and $j \geq j_1$ we arrive at

$$\mathcal{U}(t) \geq \exp\left(q^j \left(\ln(\tilde{B} \varepsilon) + \frac{1 + \kappa_+}{q - 1} \ln(t - L) - \frac{\kappa_-}{q - 1} \ln(1 + t)\right)\right) (t - L)^{-\frac{1 + \kappa_+}{q - 1}} (1 + t)^{-\frac{\kappa_-}{q - 1}}.$$

Then, for $t \geq \max\{2L, 1\}$ and $j \geq j_1$, by using $\kappa = \kappa_+ - \kappa_-$, from the previous estimate we have

$$\mathcal{U}(t) \geq \exp\left(q^j \ln\left(\hat{B} \varepsilon t^{\frac{1 + \kappa}{q - 1}}\right)\right) (t - L)^{-\frac{1 + \kappa_+}{q - 1}} (1 + t)^{-\frac{\kappa_-}{q - 1}}, \tag{2.35}$$

where $\hat{B} \doteq 2^{-(1 + \kappa_+ + \kappa_-)/(q - 1)} \tilde{B}$.

The logarithmic factor multiplying q^j in (2.35) is strictly positive if and only if $t > (\hat{B} \varepsilon)^{-\frac{q - 1}{1 + \kappa}}$. We set $\varepsilon_0 = \varepsilon_0(n, c, H, b, m^2, \beta, p, \mu, \kappa, u_0, u_1, R) > 0$ sufficiently small so that

$$(\hat{B} \varepsilon_0)^{-\frac{q - 1}{1 + \kappa}} \geq \max\{2L, 1\}.$$

Then, for any $\varepsilon \in (0, \varepsilon_0]$ and any $t > (\hat{B} \varepsilon)^{-\frac{q - 1}{1 + \kappa}}$ we obtain that $t \geq \{2L, 1\}$ and that the factor multiplying q^j in (2.35) is positive, thus, taking the limit as $j \rightarrow \infty$ in (2.35) we have that the lower bound for $\mathcal{U}(t)$ is

not finite. Hence, we proved that \mathcal{U} blows up in finite time as well and, as byproduct of the former iteration procedure, the upper bound estimate for the lifespan in (1.14) has been proven when $b^2 > 4m^2$.

2.3.2. Case with polynomial growth: sub-case with balanced damping and mass

When $b^2 = 4m^2$ in (2.25) the exponential terms disappear.

Indeed, in this special case the iteration frame for \mathcal{U} is given by

$$\mathcal{U}(t) \geq C\mu \int_0^t \int_0^s (1 + \tau)^\kappa (\mathcal{U}(\tau))^q d\tau ds \quad (2.36)$$

for $t \geq 0$. Combining (2.36) with the lower bound estimate for \mathcal{U} in (2.26), from [9, Lemma 1] we see that \mathcal{U} blows up in finite time provided that $\gamma \doteq 2[(q-1) + \kappa + 2] > 0$, that is, for $\kappa > -1 - q = \kappa_{\text{crit}}(b, m^2, \beta, p)$. Moreover, the upper bound estimate $T(\varepsilon) \lesssim \tilde{T}(\varepsilon)$ holds for the lifespan, where \tilde{T} is defined through the relation $\varepsilon \tilde{T}^{\frac{\gamma}{2(q-1)}} = 1$. Consequently, $T(\varepsilon) \lesssim \varepsilon^{-\left(\frac{\kappa+2}{q-1}+1\right)^{-1}}$, which is exactly the upper bound estimate in (1.14) for the case $b^2 = 4m^2$.

Remark 9. We emphasize that for $b > 0$ and $m^2 \in \left[0, \frac{b^2}{4}\right]$ in order to prove Theorem 1.3, concerning the sign assumptions for the Cauchy data it is sufficient to require that $\int_{\mathbb{R}^n} u_0(x) dx, \int_{\mathbb{R}^n} u_1(x) dx$ are nonnegative and that at least one between them is strictly positive (analogously to what we pointed out in Remark 7 for Theorem 1.2). Indeed, under these assumptions (2.26) still holds true if $b > 0$.

On the other hand, if we have $\int_{\mathbb{R}^n} u_1(x) dx = 0$ (and, of course, $\int_{\mathbb{R}^n} u_0(x) dx > 0$), Theorem 1.3 still holds in the case $b = m^2 = 0$, however, the range for κ becomes $\kappa > -2$ and the lifespan estimate is in this case changes to

$$T(\varepsilon) \lesssim \varepsilon^{-\frac{q-1}{\kappa+2}},$$

due to the fact that the lower bound for \mathcal{U} in this case is given by $\mathcal{U}(t) \geq K_0\varepsilon$ (i.e., without any linear increasing t -factor on the right-hand side).

2.4. Case with logarithmic growth: proof of Theorem 1.4

In this section, we prove Theorem 1.4 by showing that the functional \mathcal{U} introduced in Subsection 2.3 blows up even when the Γ factor in (1.2) is given by (1.16) with threshold values both for the exponential factor and for the polynomial factor. The iteration frame is the one given in (2.25). Nonetheless, we will employ it for deriving different kinds of lower bound estimates for \mathcal{U} , depending on whether we work with $b^2 > 4m^2$ or with $b^2 = 4m^2$. We emphasize that in this final case we still need to apply a slicing procedure to handle logarithmic terms in the s -integral. In the dominant damping case (i.e., for $b^2 > 4m^2$) the slicing procedure will enable us to control both the logarithmic factors and the exponential multiplier, while in the balanced case $b^2 = 4m^2$ the exponential term disappears, so the slicing procedure will serve to deal with the logarithmic terms only.

In the first case we work with the same sequence $\{L_j\}_{j \in \mathbb{N}}$ as the one defined in Subsection 2.3. Whilst in the case $b^2 = 4m^2$ we consider a simpler sequence $\{L_j\}_{j \in \mathbb{N}}$ which is analogous to the one introduced for the first time in [1].

2.4.1. Case with logarithmic growth: sub-case with dominant damping

When the damping term is dominant ($b^2 > 4m^2$) and for $\kappa = \kappa_{\text{crit}}(b, m^2, \beta, p) = -1$, the iteration frame in (2.25) can be rewritten as follows:

$$\mathcal{U}(t) \geq \mu C e^{(\alpha_1 - \alpha_2)t} \int_0^t e^{(\alpha_2 - \alpha_1)s} \int_0^s (1 + \tau)^{-1} (\mathcal{U}(\tau))^q d\tau ds \quad \text{for } t \geq 0. \tag{2.37}$$

The next step is to show the following sequence of lower bound estimates for \mathcal{U}

$$\mathcal{U}(t) \geq C_j \left(\ln \left(\frac{t}{L_j} \right) \right)^{d_j} \quad \text{for } t \geq L_j \text{ and any } j \in \mathbb{N}, \tag{2.38}$$

where $\{C_j\}_{j \in \mathbb{N}}$, $\{d_j\}_{j \in \mathbb{N}}$ are sequences of nonnegative real numbers to be determined iteratively, and $\{L_j\}_{j \in \mathbb{N}}$ is defined as in Subsection 2.3 in the case $b^2 > 4m^2$.

Clearly, from (2.26) we get (2.38) with $C_0 \doteq K_0 \varepsilon$ and $d_0 \doteq 0$. Let us proceed now with the induction step. Plugging (2.38) in (2.37), for $t \geq L_j$ we obtain

$$\begin{aligned} \mathcal{U}(t) &\geq \mu C e^{(\alpha_1 - \alpha_2)t} \int_{L_j}^t e^{(\alpha_2 - \alpha_1)s} \int_{L_j}^s (1 + \tau)^{-1} (\mathcal{U}(\tau))^q d\tau ds \\ &\geq \mu C C_j^q \left(1 + \frac{1}{L_j} \right)^{-1} e^{(\alpha_1 - \alpha_2)t} \int_{L_j}^t e^{(\alpha_2 - \alpha_1)s} \int_{L_j}^s \tau^{-1} \left(\ln \left(\frac{\tau}{L_j} \right) \right)^{qd_j} d\tau ds \\ &\geq \mu C C_j^q \left(1 + \frac{1}{\ell_0} \right)^{-1} (1 + qd_j)^{-1} e^{(\alpha_1 - \alpha_2)t} \int_{L_j}^t e^{(\alpha_2 - \alpha_1)s} \left(\ln \left(\frac{s}{L_j} \right) \right)^{1+qd_j} ds, \end{aligned}$$

where in the second inequality we used $1 + \tau \leq (1 + L_j^{-1})\tau$ for $\tau \geq L_j$. For $t \geq L_{j+1}$, by cutting away a slice from the domain of integration, we have

$$\begin{aligned} \mathcal{U}(t) &\geq \mu C C_j^q \left(1 + \frac{1}{\ell_0} \right)^{-1} (1 + qd_j)^{-1} e^{(\alpha_1 - \alpha_2)t} \int_{\frac{L_j t}{L_{j+1}}}^t e^{(\alpha_2 - \alpha_1)s} \left(\ln \left(\frac{s}{L_j} \right) \right)^{1+qd_j} ds \\ &\geq \mu C C_j^q \left(1 + \frac{1}{\ell_0} \right)^{-1} (1 + qd_j)^{-1} \left(\ln \left(\frac{t}{L_{j+1}} \right) \right)^{1+qd_j} e^{(\alpha_1 - \alpha_2)t} \int_{\frac{L_j t}{L_{j+1}}}^t e^{(\alpha_2 - \alpha_1)s} ds \\ &= \mu C (\alpha_2 - \alpha_1)^{-2} (\ell_0 + 1)^{-1} C_j^q (1 + qd_j)^{-1} \left(\ln \left(\frac{t}{L_{j+1}} \right) \right)^{1+qd_j} \left(1 - e^{-(\alpha_2 - \alpha_1)(1 - \frac{1}{\ell_{j+1}})t} \right) \\ &\geq \mu C \left(q - \frac{1}{2} \right) (\alpha_2 - \alpha_1)^{-2} (\ell_0 + 1)^{-1} C_j^q (1 + qd_j)^{-1} q^{-2(j+1)} \left(\ln \left(\frac{t}{L_{j+1}} \right) \right)^{1+qd_j}, \end{aligned}$$

where in the last inequality we used (2.28). The previous chain of inequalities provides exactly (2.38) for $j + 1$ by setting

$$C_{j+1} \doteq \frac{\mu C \left(q - \frac{1}{2} \right) C_j^q}{(\alpha_2 - \alpha_1)^2 (\ell_0 + 1) (1 + qd_j) q^{2(j+1)}}, \tag{2.39}$$

$$d_{j+1} \doteq 1 + qd_j. \tag{2.40}$$

Analogously to what we have done in the previous subsections, we derive first an explicit representation for d_j and then we determine a suitable lower bound for C_j when j is large enough. By using recursively (2.40) and $d_0 = 0$, we get

$$d_j = 1 + qd_{j-1} = \sum_{k=0}^{j-1} q^k + d_0q^j = \frac{q^j - 1}{q - 1}. \tag{2.41}$$

Therefore, (2.40) and (2.41) imply that $(1 + qd_j)^{-1} \geq (q - 1)q^{-(j+1)}$. Consequently,

$$C_{j+1} \doteq \underbrace{\frac{\mu C (q - \frac{1}{2}) (q - 1)}{(\alpha_2 - \alpha_1)^2 (\ell_0 + 1)}}_{E \doteq} q^{-3(j+1)} C_j^q.$$

Similarly as we did in Subsection 2.3, from the inequality $C_j \geq Eq^{-3j}C_{j-1}$ we get

$$\ln C_j \geq q^j \left(\ln C_0 - \frac{3q \ln q}{(q - 1)^2} + \frac{\ln E}{q - 1} \right) + \frac{3q \ln q}{(q - 1)^2} + \frac{3 \ln q}{q - 1} j - \frac{\ln E}{q - 1}.$$

Setting $j_2 = j_2(n, c, H, b, m^2, \mu, \beta, p, R) \in \mathbb{N}$ to be the smallest integer such that $j_2 \geq \frac{\ln E}{3 \ln q} - \frac{q}{q-1}$, for any $j \geq j_2$ we get

$$\ln C_j \geq q^j \left(\ln(K_0 \varepsilon) - \frac{3q \ln q}{(q - 1)^2} + \frac{\ln E}{q - 1} \right) = q^j \ln(\tilde{E} \varepsilon), \tag{2.42}$$

where $\tilde{E} \doteq K_0 q^{-3q/(q-1)^2} E^{1/(q-1)}$. Combining (2.38), (2.41) and (2.42), for $t \geq L = \lim_{j \rightarrow \infty} L_j$ and $j \geq j_2$ we find

$$\begin{aligned} \mathcal{U}(t) &\geq \exp(q^j \ln(\tilde{E} \varepsilon)) \left(\ln \left(\frac{t}{L} \right) \right)^{\frac{q^j - 1}{q - 1}} \\ &= \exp \left(q^j \left(\ln(\tilde{E} \varepsilon) + \frac{1}{q - 1} \ln \left(\ln \left(\frac{t}{L} \right) \right) \right) \right) \left(\ln \left(\frac{t}{L} \right) \right)^{-\frac{1}{q - 1}} \\ &= \exp \left(q^j \ln \left(\tilde{E} \varepsilon \left(\ln \left(\frac{t}{L} \right) \right)^{\frac{1}{q - 1}} \right) \right) \left(\ln \left(\frac{t}{L} \right) \right)^{-\frac{1}{q - 1}}. \end{aligned} \tag{2.43}$$

We remark that the logarithmic factor multiplying q^j in (2.43) is strictly positive if and only if $t > L \exp(\tilde{E} \varepsilon)^{-(q-1)}$.

Now we fix $\varepsilon_0 = \varepsilon_0(n, c, H, b, m^2, \beta, p, \mu, u_0, u_1, R)$ such that $\exp(\tilde{E} \varepsilon_0)^{-(q-1)} > 1$. Then, for any $\varepsilon \in (0, \varepsilon_0]$ and any $t > L \exp(\tilde{E} \varepsilon)^{-(q-1)}$ the right-hand side of (2.43) diverges as $j \rightarrow \infty$, so $\mathcal{U}(t)$ cannot be finite. Summarizing, we proved the blow-up of \mathcal{U} in finite time and the upper bound estimate in (1.17) when $b^2 > 4m^2$.

2.4.2. Case with logarithmic growth: sub-case with balanced damping and mass

If $b^2 = 4m^2$ and $\kappa = \kappa_{\text{crit}}(b, m^2, \beta, p) = -1 - q$, we may rewrite the iteration frame (2.25) as follows:

$$\mathcal{U}(t) \geq \mu C \int_0^t \int_0^s (1 + \tau)^{-1-q} (\mathcal{U}(\tau))^q d\tau ds \quad \text{for } t \geq 0. \tag{2.44}$$

The next step is to show the following sequence of lower bound estimates for \mathcal{U}

$$\mathcal{U}(t) \geq C_j t \left(\ln \left(\frac{t}{L_j} \right) \right)^{d_j} \quad \text{for } t \geq L_j \text{ and any } j \in \mathbb{N}, \tag{2.45}$$

where $\{C_j\}_{j \in \mathbb{N}}$, $\{d_j\}_{j \in \mathbb{N}}$ are suitable sequences of nonnegative real numbers, and

$$L_j \doteq 2 - 2^{-j} \quad \text{for } j \in \mathbb{N}. \tag{2.46}$$

Remark 10. The choice of the sequence $\{L_j\}_{j \in \mathbb{N}}$ in (2.46) is done in order to handle the logarithmic factors in the s -integral. Notice that in this case no exponential multiplier appears, so the construction of the parameters characterizing the slicing procedure is simpler than in the previous proofs and it is inspired by the one from [1, Section 6].

We begin by observing that (2.45) for $j = 0$ follows from (2.26) with $C_0 \doteq K_0 \varepsilon$ and $d_0 \doteq 0$. Let us proceed with the inductive step. If we plug (2.45) in (2.44), for $t \geq L_j$ it results

$$\begin{aligned} \mathcal{U}(t) &\geq \mu C \left(1 + \frac{1}{L_j} \right)^{-1-q} \int_{L_j}^t \int_{L_j}^s \tau^{-1-q} (\mathcal{U}(\tau))^q d\tau ds \\ &\geq \mu C C_j^q \left(1 + \frac{1}{L_j} \right)^{-1-q} \int_{L_j}^t \int_{L_j}^s \tau^{-1} \left(\ln \left(\frac{\tau}{L_j} \right) \right)^{qd_j} d\tau ds \\ &\geq \mu C C_j^q \left(1 + \frac{1}{L_j} \right)^{-1-q} (1 + qd_j)^{-1} \int_{L_j}^t \left(\ln \left(\frac{s}{L_j} \right) \right)^{1+qd_j} ds. \end{aligned}$$

Hence, using the slicing procedure, for $t \geq L_{j+1}$ we have

$$\begin{aligned} \mathcal{U}(t) &\geq \mu C C_j^q \left(1 + \frac{1}{L_j} \right)^{-1-q} (1 + qd_j)^{-1} \int_{\frac{L_j t}{L_{j+1}}}^t \left(\ln \left(\frac{s}{L_j} \right) \right)^{1+qd_j} ds \\ &\geq \mu C C_j^q \left(1 + \frac{1}{L_j} \right)^{-1-q} (1 + qd_j)^{-1} \left(1 - \frac{L_j}{L_{j+1}} \right) t \left(\ln \left(\frac{t}{L_{j+1}} \right) \right)^{1+qd_j}, \end{aligned}$$

which is (2.45) for $j + 1$ provided that we set

$$C_{j+1} \doteq \mu C (1 + qd_j)^{-1} \left(1 + \frac{1}{L_j} \right)^{-1-q} \left(1 - \frac{L_j}{L_{j+1}} \right) C_j^q, \tag{2.47}$$

$$d_{j+1} \doteq 1 + qd_j. \tag{2.48}$$

In a complete analogous way as in the previous case, we derive the representation (2.41) for d_j even in this case. Hence, using

$$\left(1 + \frac{1}{L_j} \right)^{-1-q} \geq 2^{-1-q}, \quad 1 - \frac{L_j}{L_{j+1}} \geq 2^{-(j+2)}$$

for any $j \in \mathbb{N}$ and (2.41), we arrive at

$$C_{j+1} \geq \underbrace{2^{-2-q} \mu C(q-1)(2q)^{-(j+1)}}_{F \doteq} C_j^q.$$

Repeating similar computations as in the previous proofs, from the inequality $C_j \geq F(2q)^{-j} C_{j-1}^q$, we can derive the following lower bound for $\ln C_j$

$$\ln C_j \geq q^j \left(\ln(K_0 \varepsilon) - \frac{q \ln(2q)}{(q-1)^2} + \frac{\ln F}{q-1} \right) = q^j \ln(\tilde{F} \varepsilon), \tag{2.49}$$

for any $j \geq j_3$, where $j_3 = j_3(n, c, H, b, m^2, \mu, \beta, p, R) \in \mathbb{N}$ is the smallest integer such that $j_3 \geq \frac{\ln F}{\ln(2q)} - \frac{q}{q-1}$ and $\tilde{F} \doteq K_0(2q)^{-q/(q-1)^2} F^{1/(q-1)}$.

Combining (2.45) and (2.49), for $t \geq 2 = \lim_{j \rightarrow \infty} L_j$ and $j \geq j_3$ we have

$$\begin{aligned} \mathcal{U}(t) &\geq \exp(q^j \ln(\tilde{F} \varepsilon)) t \left(\ln \left(\frac{t}{2} \right) \right)^{\frac{q^j - 1}{q-1}} \\ &= \exp \left(q^j \ln \left(\tilde{F} \varepsilon \left(\ln \left(\frac{t}{2} \right) \right)^{\frac{1}{q-1}} \right) \right) t \left(\ln \left(\frac{t}{2} \right) \right)^{-\frac{1}{q-1}}. \end{aligned} \tag{2.50}$$

In (2.50) the logarithmic factor that multiplies q^j is strictly positive if and only if $t > 2 \exp(\tilde{F} \varepsilon)^{-(q-1)}$, therefore, we fix $\varepsilon_0 = \varepsilon_0(n, c, H, b, m^2, \beta, p, \mu, u_0, u_1, R)$ small enough such that $\exp(\tilde{F} \varepsilon_0)^{-(q-1)} > 1$. Thus, for any $\varepsilon \in (0, \varepsilon_0]$ and any $t > 2 \exp(\tilde{F} \varepsilon)^{-(q-1)}$ the right-hand side of (2.50) diverges as $j \rightarrow \infty$ and, in particular, $\mathcal{U}(t)$ cannot be finite. In conclusion, we showed the blow-up of \mathcal{U} in finite time and the upper bound estimate in (1.17) for $b^2 = 4m^2$.

Remark 11. Analogously to what we pointed out in Remarks 7 and 9 for Theorems 1.2 and 1.3, respectively, it is possible to weaken the sign assumptions on the Cauchy data in the statement of Theorem 1.4. More precisely, for $b > 0$ and $m^2 \in \left[0, \frac{b^2}{4}\right]$ assuming that $\int_{\mathbb{R}^n} u_0(x) dx, \int_{\mathbb{R}^n} u_1(x) dx$ are nonnegative and that at least one between them is strictly positive, then, the blow-up result from Theorem 1.4 is still valid.

On the other hand, in the limit case $b = m^2 = 0$ if $\int_{\mathbb{R}^n} u_1(x) dx > 0$, then, it holds the same blow-up result as in Theorem 1.4; while if $\int_{\mathbb{R}^n} u_1(x) dx = 0$ and $\int_{\mathbb{R}^n} u_0(x) dx > 0$ then the result has to be modified accordingly to Remark 9 with the threshold value $\kappa = -1 - q$ replaced by $\kappa = -2$.

3. Models in anti-de Sitter spacetime

3.1. Derivation of the iteration frame

The derivation of the iteration frame for (1.5) can be done in a complete analogous way as we did for (1.1) in Subsection 2.1.

If v is a local in time solution to (1.5), denoting

$$V(t) \doteq \int_{\mathbb{R}^n} v(t, x) dx \quad \text{for } t \in (0, T),$$

then, the iteration frame involves this functional V .

Fixed $t \in (0, T)$ we consider a cutoff function $\varphi \in \mathcal{C}([0, T] \times \mathbb{R}^n)$ that localizes the support of v on the strip $[0, t] \times \mathbb{R}^n$, that is, $\varphi = 1$ on $\{(s, x) \in [0, t] \times \mathbb{R}^n : |x| \leq R + cH^{-1}(e^{sH} - 1)\}$. Consequently, employing this φ in (1.20) and differentiating with respect to t the resulting relation, we have

$$V''(t) + bV'(t) + m^2V(t) = \Gamma(t) \left(\int_{\mathbb{R}^n} |v(t, x)|^p dx \right)^{\beta+1},$$

which is formally identical to (2.2).

By using the same factorization of the operator $\frac{d^2}{dt^2} + b\frac{d}{dt} + m^2$ as in Subsection 2.1, we derive the following representation for V

$$V(t) = \varepsilon V_{\text{lin}}(t) + e^{-\alpha_2 t} \int_0^t e^{(\alpha_2 - \alpha_1)s} \int_0^s e^{\alpha_1 \tau} \Gamma(\tau) \left(\int_{\mathbb{R}^n} |v(\tau, x)|^p dx \right)^{\beta+1} d\tau ds, \tag{3.1}$$

where $\alpha_{1,2}$ are the roots of the quadratic equation $\alpha^2 - b\alpha + m^2 = 0$ and

$$V_{\text{lin}}(t) \doteq \begin{cases} \frac{\alpha_2 e^{-\alpha_1 t} - \alpha_1 e^{-\alpha_2 t}}{\alpha_2 - \alpha_1} \int_{\mathbb{R}^n} v_0(x) dx + \frac{e^{-\alpha_1 t} - e^{-\alpha_2 t}}{\alpha_2 - \alpha_1} \int_{\mathbb{R}^n} v_1(x) dx & \text{if } b^2 > 4m^2, \\ (1 + \frac{b}{2}t) e^{-\frac{b}{2}t} \int_{\mathbb{R}^n} v_0(x) dx + t e^{-\frac{b}{2}t} \int_{\mathbb{R}^n} v_1(x) dx & \text{if } b^2 = 4m^2. \end{cases} \tag{3.2}$$

From (3.1) we derive immediately

$$V(t) \gtrsim e^{-\alpha_2 t} \int_0^t e^{(\alpha_2 - \alpha_1)s} \int_0^s e^{\alpha_1 \tau} \Gamma(\tau) (R + cH^{-1}(e^{H\tau} - 1))^{-n(\beta+1)(p-1)} (V(\tau))^q d\tau ds,$$

where we used Hölder’s inequality and the support condition in (1.19), and $q = (\beta + 1)p$.

Finally, using the inequality $R + \frac{c}{H}(e^{H\tau} - 1) \leq (R + \frac{c}{H}) e^{H\tau}$ for $\tau \geq 0$, from the previous inequality we obtain the iteration frame for V

$$V(t) \geq C e^{-\alpha_2 t} \int_0^t e^{(\alpha_2 - \alpha_1)s} \int_0^s e^{\alpha_1 \tau} \Gamma(\tau) e^{-nH(\beta+1)(p-1)\tau} (V(\tau))^q d\tau ds, \tag{3.3}$$

where $C = C(n, c, H, \beta, p, R)$ is a suitable positive constant.

Furthermore, (3.1) provides us a first lower bound estimate for V as well, namely,

$$V(t) \geq \begin{cases} K_0 \varepsilon e^{-(\frac{b}{2} - \frac{1}{2}\sqrt{b^2 - 4m^2})t} & \text{if } b^2 > 4m^2, \\ K_0 \varepsilon (1 + t) e^{-\frac{b}{2}t} & \text{if } b^2 = 4m^2, \end{cases} \tag{3.4}$$

for any $t \in (0, T)$, where $K_0 = K_0(b, m^2, v_0, v_1)$ is a suitable positive and independent of ε constant.

Notice that such lower bound for V is completely analogous to the one in (2.9) for U , due to the fact that they both follow from the “linear part” of U and V , respectively. Nevertheless, for anti-de Sitter spacetime we can exploit the nonlinear term in order to get an alternative lower bound estimate to start the iteration procedure. In the next subsection we will introduce an auxiliary functional that will allow us to derive this further lower bound for V .

3.2. Lower bound estimate for the nonlinearity

In the present subsection, we investigate the growth properties of the auxiliary functional given by the following weighted space average of v

$$V_0(t) \doteq \int_{\mathbb{R}^n} v(t, x)\Psi(t, x) dx, \tag{3.5}$$

where the weight function $\Psi = \Psi(t, x; c, H, b, m^2)$ is going to be introduced in few lines and it is chosen as a positive solution of the adjoint homogeneous equation, namely,

$$\partial_t^2 \Psi - c^2 e^{2Ht} \Delta \Psi - b \partial_t \Psi + m^2 \Psi = 0. \tag{3.6}$$

We work with a function Ψ with separable variables, namely, we use the following ansatz

$$\Psi(t, x) \doteq \lambda(t; c, H, b, m^2)\Phi(x).$$

As x -dependent function we consider the well-known “eigenfunction” for the Laplace operator

$$\begin{aligned} \Phi(x) &\doteq e^x + e^{-x} && \text{if } n = 1, \\ \Phi(x) &\doteq \int_{\mathbb{S}^{n-1}} e^{x \cdot \omega} d\sigma_\omega && \text{if } n \geq 2. \end{aligned}$$

This function has been introduced for the first time in the study of blow-up results for wave models in [31]. The function Φ is a positive smooth function that satisfies the following crucial properties:

$$\Delta \Phi = \Phi, \tag{3.7}$$

$$\Phi(x) \sim c_n |x|^{-\frac{n-1}{2}} e^x \quad \text{as } |x| \rightarrow \infty, \tag{3.8}$$

for some suitable positive constant c_n . Also, in order to get a solution of (3.6), we have to determine $\lambda = \lambda(t; c, H, b, m^2)$ such that

$$\frac{d^2 \lambda}{dt^2} - b \frac{d\lambda}{dt} + (m^2 - c^2 e^{2Ht}) \lambda = 0. \tag{3.9}$$

For the sake of readability, in what follows we skip the dependence of λ on (c, H, b, m^2) in the notations. Let us perform the change of variables $\tau = \frac{c}{H} e^{Ht}$. Then,

$$\frac{d^2 \lambda}{dt^2} = H^2 \tau^2 \frac{d^2 \lambda}{d\tau^2} + H^2 \tau \frac{d\lambda}{d\tau} \quad \text{and} \quad \frac{d\lambda}{dt} = H \tau \frac{d\lambda}{d\tau}.$$

So, with respect to τ the function λ satisfies the equation

$$\tau^2 \frac{d^2 \lambda}{d\tau^2} + \left(-\frac{b}{H} + 1\right) \tau \frac{d\lambda}{d\tau} + \left(\frac{m^2}{H^2} - \tau^2\right) \lambda = 0. \tag{3.10}$$

Next, we carry out the transformation $\lambda(\tau) = \tau^\rho \eta(\tau)$, with ρ real parameter to be determined. By straightforward computations, we have that λ solves (3.10) if and only if

$$\tau^2 \frac{d^2 \eta}{d\tau^2} + \left(2\rho - \frac{b}{H} + 1\right) \tau \frac{d\eta}{d\tau} + \left(\rho \left(\rho - \frac{b}{H}\right) + \frac{m^2}{H^2} - \tau^2\right) \eta = 0. \tag{3.11}$$

If we choose $\rho \doteq \frac{b}{2H}$, then, (3.11) can be rewritten as the following modified Bessel equation

$$\tau^2 \frac{d^2 \eta}{d\tau^2} + \tau \frac{d\eta}{d\tau} - \left[\frac{1}{4H^2} (b^2 - 4m^2) + \tau^2\right] \eta = 0. \tag{3.12}$$

Setting $\nu \doteq \frac{1}{2H}(b^2 - 4m^2)^{1/2}$, a complete system of independent solutions to (3.12) is given by I_ν and K_ν (modified Bessel functions of the first and second kind, respectively, of order ν). For further details on the properties of I_ν and K_ν that will be used in this subsection, we address the reader to [13, Chapter 10]. Let us recall the asymptotic behavior of I_ν and K_ν for large values

$$I_\nu(\tau) \sim (2\pi\tau)^{-1/2}e^\tau \quad \text{and} \quad K_\nu(\tau) \sim \sqrt{\frac{\pi}{2}}\tau^{-1/2}e^{-\tau} \quad \text{as } \tau \rightarrow \infty.$$

Moreover, for $\nu \geq 0$ the function $I_\nu(\tau)$ has no real zero excluding $\tau = 0$ when $\nu > 0$, and, similarly, for $\nu \geq 0$ the function $K_\nu(\tau)$ has no real zero.

Hereafter, we set (neglecting the unessential multiplicative constant)

$$\lambda(t; c, H, b, m^2) \doteq e^{\frac{b}{2}t} K_\nu\left(\frac{c}{H}e^{Ht}\right). \tag{3.13}$$

By using the previous recalled asymptotic behavior of K_ν for large arguments and the fact that K_ν has no real zero, we may consider the following uniform estimate

$$\lambda_0 e^{\frac{1}{2}(b-H)t} \exp\left(-\frac{c}{H}e^{Ht}\right) \leq \lambda(t) \leq \Lambda_0 e^{\frac{1}{2}(b-H)t} \exp\left(-\frac{c}{H}e^{Ht}\right) \quad \text{for any } t \geq 0, \tag{3.14}$$

for some positive constants $\lambda_0 = \lambda_0(c, H, b, m^2)$, $\Lambda_0 = \Lambda_0(c, H, b, m^2)$.

By using the uniform estimate (3.14), we can now derive a lower bound for the functional V_0 . As λ and Φ are nonnegative functions, also Ψ is nonnegative. Therefore, plugging Ψ in (1.20), we get

$$\begin{aligned} 0 &\leq \int_0^t \Gamma(s) \left(\int_{\mathbb{R}^n} |v(s, y)|^p dy \right)^\beta \int_{\mathbb{R}^n} |v(s, x)|^p \Psi(s, x) dx ds \\ &= \int_0^t \int_{\mathbb{R}^n} v(s, x) (\Psi_{ss}(s, x) - c^2 e^{2Hs} \Delta \Psi(s, x) - b \Psi_s(s, x) + m^2 \Psi(s, x)) dx ds \\ &\quad + \int_{\mathbb{R}^n} (\partial_t v(s, x) \Psi(s, x) - v(s, x) \Psi_s(s, x) + b v(s, x) \Psi(s, x)) dx \Big|_{s=0}^{s=t} \\ &= V'_0(s) + bV_0(s) - 2 \frac{\lambda'(s)}{\lambda(s)} V_0(s) \Big|_{s=0}^{s=t}, \end{aligned}$$

where in the last step we used (3.6) and the trivial relations

$$V'_0(s) = \int_{\mathbb{R}^n} (\partial_t v(s, x) \Psi(s, x) + v(s, x) \Psi_s(s, x)) dx, \quad \frac{\lambda'(s)}{\lambda(s)} V_0(s) = \int_{\mathbb{R}^n} v(s, x) \Psi_s(s, x) dx.$$

Let us remark that

$$\frac{d}{ds} \left(\frac{e^{bs}}{\lambda^2(s)} V_0(s) \right) = \frac{e^{bs}}{\lambda^2(s)} \left[V'_0(s) + \left(b - 2 \frac{\lambda'(s)}{\lambda(s)} \right) V_0(s) \right].$$

Hence, the previous inequality implies

$$\begin{aligned} \lambda^2(t) e^{-bt} \frac{d}{dt} \left(\frac{e^{bt}}{\lambda^2(t)} V_0(t) \right) &\geq V_0'(0) + bV_0(0) - 2 \frac{\lambda'(0)}{\lambda(0)} V_0(0) \\ &= \varepsilon \underbrace{\int_{\mathbb{R}^n} [\lambda(0)v_1(x) + (b\lambda(0) - \lambda'(0))v_0(x)] \Phi(x) dx}_{\doteq \mathcal{I}[v_0, v_1]}. \end{aligned} \tag{3.15}$$

Using the recursive relation

$$\frac{\partial K_\nu}{\partial z}(z) = -K_{\nu+1}(z) + \frac{\nu}{z} K_\nu(z)$$

(cf. [13, Section 10.29]), it follows

$$\begin{aligned} \lambda'(t) &= \frac{b}{2} \lambda(t) + c e^{(\frac{b}{2}+H)t} K'_\nu \left(\frac{c}{H} e^{Ht} \right) \\ &= \frac{b}{2} \lambda(t) + c e^{(\frac{b}{2}+H)t} \left[-K_{\nu+1} \left(\frac{c}{H} e^{Ht} \right) + \frac{\nu H}{c} e^{-Ht} K_\nu \left(\frac{c}{H} e^{Ht} \right) \right] \\ &= \left(\frac{b}{2} + \frac{1}{2} \sqrt{b^2 - 4m^2} \right) \lambda(t) - c e^{(\frac{b}{2}+H)t} K_{\nu+1} \left(\frac{c}{H} e^{Ht} \right) \end{aligned}$$

which implies in turn

$$b\lambda(t) - \lambda'(t) = \left(\frac{b}{2} - \frac{1}{2} \sqrt{b^2 - 4m^2} \right) \lambda(t) + c e^{(\frac{b}{2}+H)t} K_{\nu+1} \left(\frac{c}{H} e^{Ht} \right) \geq 0.$$

Therefore, assuming v_0, v_1 nonnegative and nontrivial, in particular, we have that $\mathcal{I}[v_0, v_1] > 0$. From (3.15), we find

$$\frac{d}{dt} \left(\frac{e^{bt}}{\lambda^2(t)} V_0(t) \right) \geq \varepsilon \mathcal{I}[v_0, v_1] \frac{e^{bt}}{\lambda^2(t)},$$

from which it follows

$$V_0(t) \geq \frac{V_0(0)}{\lambda^2(0)} e^{-bt} \lambda^2(t) + \varepsilon \mathcal{I}[v_0, v_1] e^{-bt} \lambda^2(t) \int_0^t \frac{e^{bs}}{\lambda^2(s)} ds. \tag{3.16}$$

By (3.14), it results

$$\frac{V_0(0)}{\lambda^2(0)} e^{-bt} \lambda^2(t) \geq \varepsilon \frac{\lambda_0^2}{\lambda(0)} \left(\int_{\mathbb{R}^n} v_0(x) \Phi(x) dx \right) e^{-Ht} \exp \left(-\frac{2c}{H} e^{Ht} \right)$$

and

$$\begin{aligned} e^{-bt} \lambda^2(t) \int_0^t \frac{e^{bs}}{\lambda^2(s)} ds &\geq \frac{\lambda_0^2}{\Lambda_0^2} e^{-Ht} \exp \left(-\frac{2c}{H} e^{Ht} \right) \int_0^t e^{Hs} \exp \left(\frac{2c}{H} e^{Hs} \right) ds \\ &= \frac{\lambda_0^2}{2c\Lambda_0^2} e^{-Ht} \exp \left(-\frac{2c}{H} e^{Ht} \right) \left[\exp \left(\frac{2c}{H} e^{Hs} \right) \right]_{s=0}^{s=t} \\ &= \frac{\lambda_0^2}{2c\Lambda_0^2} e^{-Ht} \left(1 - \exp \left(-\frac{2c}{H} e^{Ht} \right) \right). \end{aligned}$$

Using the first of the two previous estimates for t in a neighborhood of 0 and the second one for t away from zero, we derive the following lower bound estimate

$$V_0(t) \gtrsim \varepsilon e^{-Ht} \quad \text{for } t \in [0, T]. \tag{3.17}$$

The next step is to derive a lower bound estimate for $\|v(t, \cdot)\|_{L^p(\mathbb{R}^n)}^p$ by using the previous lower bound for V_0 . By Hölder’s inequality and using the support condition (1.19), for $t \geq 0$ it follows

$$V_0(t) \leq \|v(t, \cdot)\|_{L^p(\mathbb{R}^n)} \|\Psi(t, \cdot)\|_{L^{p'}(B_{R+A_{\text{AdS}}(t)})}.$$

Consequently, if we get an upper bound for $\|\Psi(t, \cdot)\|_{L^{p'}(B_{R+A_{\text{AdS}}(t)})}$, then, we may obtain the desired estimate by using

$$\int_{\mathbb{R}^n} |v(t, x)|^p dx \geq (V_0(t))^p \|\Psi(t, \cdot)\|_{L^{p'}(B_{R+A_{\text{AdS}}(t)})}^{-p}. \tag{3.18}$$

Repeating the same computations as in [15, Section 3], we get

$$\begin{aligned} \|\Psi(t, \cdot)\|_{L^{p'}(B_{R+A_{\text{AdS}}(t)})} &= \left(\int_{B_{R+A_{\text{AdS}}(t)}} (\Psi(t, x))^{p'} dx \right)^{1/p'} \leq \lambda(t) \left(\int_{B_{R+A_{\text{AdS}}(t)}} (\Phi(x))^{p'} dx \right)^{1/p'} \\ &\lesssim \lambda(t) \exp(R + A_{\text{AdS}}(t)) (R + A_{\text{AdS}}(t))^{(n-1)(\frac{1}{2} - \frac{1}{p})}. \end{aligned}$$

Thus, from (3.14) for $t \geq 0$ we get

$$\begin{aligned} \|\Psi(t, \cdot)\|_{L^{p'}(B_{R+A_{\text{AdS}}(t)})} &\lesssim \Lambda_0 e^{\frac{1}{2}(b-H)t} \exp\left(-\frac{c}{H}e^{Ht}\right) \exp(R + A_{\text{AdS}}(t)) (R + A_{\text{AdS}}(t))^{(n-1)(\frac{1}{2} - \frac{1}{p})} \\ &\lesssim e^{\frac{1}{2}(b-H)t} \left(R + \frac{c}{H} (e^{Ht} - 1)\right)^{(n-1)(\frac{1}{2} - \frac{1}{p})}. \end{aligned} \tag{3.19}$$

Combining (3.17), (3.18) and (3.19), we conclude

$$\int_{\mathbb{R}^n} |v(t, x)|^p dx \gtrsim \varepsilon^p e^{-\frac{1}{2}(b+H)pt} \left(R + \frac{c}{H} (e^{Ht} - 1)\right)^{(n-1)(1 - \frac{p}{2})}, \tag{3.20}$$

for $t \geq 0$.

Finally, plugging (3.20) in (3.1), for $t \geq t_0 > 0$ we get

$$V(t) \geq K_1 \varepsilon^{(\beta+1)p} e^{[\varrho - (b+H)(\beta+1)\frac{p}{2} + (n-1)H(\beta+1)(1 - \frac{p}{2})]t} t^\varsigma, \tag{3.21}$$

where the constant $K_1 = K_1(c, H, b, m^2, v_0, v_1, p, \beta, \mu, \varrho, \varsigma, t_0) > 0$ is independent of ε and t_0 will be fixed time by time depending on the slicing procedure that we will apply in each case.

Remark 12. In (3.13) we considered a modified Bessel function of the second kind of order ν . Nonetheless, if we had chosen a modified Bessel function of the first kind of order ν instead, namely, defining $\lambda(t) \doteq e^{\frac{b}{2}t} I_\nu\left(\frac{c}{H}e^{Ht}\right)$ in place of (3.13), the final outcome (meaning the lower bound for the space integral of the power nonlinearity) would have been the same. More in detail, the lower bound for V_0 in (3.17) would have been $V_0(t) \gtrsim \varepsilon e^{-tH} \exp(2cH^{-1}e^{Ht})$. In particular, this better lower bound for V_0 would have been a consequence of the asymptotic behavior for large argument of the factor involving I_ν in this alternative

definition of λ . Moreover, this lower bound would have followed from the estimate of the first term in (3.16), since the contribute from the integral term in (3.16) would have been weaker in this case. Nevertheless, the better behavior of λ would have influenced the estimate (3.19) as well, providing eventually exactly the same estimate for $\|v(t, \cdot)\|_{L^p(\mathbb{R}^n)}^p$ as the one in (3.20).

Remark 13. Clearly even in the case of de Sitter spacetime we could have considered a weighted functional analogous to the one in (3.5). However, the lower bound estimates corresponding to the one in (3.21) in this case would have been exactly the same as we have plugged (2.9) in (2.8). In other words, we would have start the sequence of lower bound estimates in (2.11) from $j = 1$ rather than from $j = 0$, but obviously the final outcome would have been unaltered.

3.3. Comparison of the first lower bound

In order to prove Theorems 1.6 - 1.9 it is crucial to understand which first lower bound estimate for V between (3.4) and (3.21) has the dominant role.

We remark that (3.4) and (3.21) cannot be directly compared in their current forms due to the different orders for ε in their right-hand sides. Then, we plug (3.4) in (3.3) obtaining

$$V(t) \gtrsim \varepsilon^{(\beta+1)p} (t - L_2)^{\varsigma_+ + qb_0} t^{-\varsigma_-} e^{(\varrho - nH(\beta+1)(p-1) - \frac{1}{2}(b - \sqrt{b^2 - 4m^2})(\beta+1)p)t} \quad (3.22)$$

for $t \geq L_2$ (see next section for the derivation of this inequality and the definition of L_2). Now, we can compare the multiplicative coefficient in the exponential term in (3.21) and in (3.22). In particular, we have that the coefficient in the exponential term in (3.21) is dominant over the one in (3.22) provided that

$$\varrho - nH(\beta + 1)(p - 1) - \frac{1}{2} \left(b - \sqrt{b^2 - 4m^2} \right) q < \varrho - (b + H)\frac{q}{2} + (n - 1)H(\beta + 1) \left(1 - \frac{p}{2} \right)$$

and, by straightforward computations, we have that the previous inequality is equivalent to (1.24). Notice that (1.24) is exactly the condition that provides a $\varrho_{\text{crit}}(n, H, b, m^2, \beta, p)$ given by (1.22). Hence, when (1.24) holds we shall use (3.21) as first lower bound estimate for V , while when (1.23) holds, we shall use (3.4). Notice that even in the limit case $\frac{n}{2} - \frac{\sqrt{b^2 - 4m^2}}{2H} = \frac{1}{p}$ we use (3.4), since in the case $b^2 = 4m^2$ a slight improvement of polynomial type is included, whilst this does not happen in (3.21). So far we discussed which among (3.4) and (3.21) is better to start the iteration argument depending on the values of n, H, b, m^2 and p . Clearly, the reasons behind the actual definition of ϱ_{crit} either as in (1.21) or as in (1.22) will be clarified in the proofs of Theorem 1.6 and Theorem 1.9, respectively, by the corresponding iteration arguments.

We point out that (1.24) is never satisfied for $n \leq N_0$, where

$$N_0 = N_0(H, b, m^2) \doteq \frac{\sqrt{b^2 - 4m^2}}{H} \quad (3.23)$$

since the left-hand side of (1.24) is negative for n in this range, while for $n \geq 2 + N_0$ the condition in (1.24) is always fulfilled since the left-hand side is greater than or equal to 1. Finally, for $n \in (N_0, 2 + N_0)$ we have that (1.24) is true if and only if

$$p > \tilde{p}(n, H, b, m^2) \doteq \frac{2H}{nH - \sqrt{b^2 - 4m^2}}.$$

Summarizing, for $n \leq N_0$ or $n \in (N_0, 2 + N_0)$ and $1 < p \leq \tilde{p}$ we will use (3.4) to star the iteration argument, while for $n \in (N_0, 2 + N_0)$ and $p > \tilde{p}$ or $n \geq 2 + N_0$ we will employ (3.21) as staring point for the iteration procedure.

Finally, we emphasize that the previous conditions for the employment of either (3.4) or (3.21) (which correspond to (1.23) and (1.24), respectively) are completely independent of β .

3.4. Proof of Theorems 1.6, 1.7 and 1.8

As we explained in Subsection 3.3, for $n \leq N_0$ or $n \in (N_0, 2 + N_0)$ and $1 < p \leq \tilde{p}$ (that is, when (1.23) holds) we employ (3.4) as first lower bound estimate for V . By plugging the explicit expression for the factor Γ into (3.3), we find

$$V(t) \geq C\mu(1+t)^{-\varsigma_-} e^{-\alpha_2 t} \int_0^t e^{(\alpha_2 - \alpha_1)s} \int_0^s e^{[\alpha_1 + \varrho - nH(\beta+1)(p-1)]\tau} \tau^{\varsigma_+} (V(\tau))^q d\tau ds \tag{3.24}$$

for $t \geq 0$, where ς_+, ς_- denote the positive and the negative part of ς , respectively.

We recognize that the iteration frame in (3.24) is formally identical to the one in (2.12) for the functional U . Moreover, the lower bound estimates for U and V in (2.9) and (3.4) are completely analogous (up to the multiplicative constants that depend on the Cauchy data) and

$$\varrho_{\text{crit}}(n, H, b, m^2, \beta, p) - nH(\beta + 1)(p - 1) = r_{\text{crit}}(b, m^2, \beta, p).$$

Hence, the proof of Theorem 1.6 is completely similar to the one of Theorem 1.2, provided that we consider as first parameter that characterizes the slicing procedure

$$\ell_0 \doteq \max \{ (\alpha_1 + \varrho - nH(\beta + 1)(p - 1))^{-1}, (\alpha_2 + \varrho - nH(\beta + 1)(p - 1))^{-1} \}$$

and then

$$L_j \doteq \ell_0 \prod_{k=1}^j \left(1 + q^{-\frac{k}{2}} \right) \quad \text{for any } j \in \mathbb{N}^*.$$

On the other hand, Theorems 1.7 and 1.8 can be proved analogously as Theorems 1.3 and 1.4 by working with the functional $\mathcal{V}(t) \doteq e^{\alpha_1 t} V(t)$. Indeed, setting $\alpha_1 = \frac{1}{2}(b - \sqrt{b^2 - 4m^2})$ and proceeding as in Subsection 2.3, from (3.3) we get

$$\mathcal{V}(t) \geq C\mu e^{(\alpha_1 - \alpha_2)t} \int_0^t e^{(\alpha_2 - \alpha_1)s} \int_0^s (1 + \tau)^\varsigma (\mathcal{V}(\tau))^q d\tau ds, \tag{3.25}$$

for $t \geq 0$. So, \mathcal{V} has an iteration frame formally identical to the one for \mathcal{U} in (2.25). Besides, \mathcal{V} satisfies a completely analogous lower bound estimate as the one in (2.26) for \mathcal{U} . By following the same approaches as for the proofs in Subsections 2.3 and 2.4 we conclude the validity of Theorems 1.7 and 1.8.

3.5. Proof of Theorem 1.9

According to Subsection 3.3, for $n \in (N_0, 2 + N_0)$ and $p > \tilde{p}$ or $n \geq 2 + N_0$ (that is, when (1.24) holds) it is appropriate to consider (3.21) as first lower bound estimate for V .

Our first goal is to prove the sequence of lower bound estimates

$$V(t) \geq C_j e^{a_j t} e^{-\gamma_j t} (t - L_{2j})^{b_j} (1 + t)^{-\beta_j} \quad \text{for } t \geq L_{2j} \text{ and for any } j \in \mathbb{N}, \tag{3.26}$$

where $\{C_j\}_{j \in \mathbb{N}}$, $\{a_j\}_{j \in \mathbb{N}}$, $\{\gamma_j\}_{j \in \mathbb{N}}$, $\{b_j\}_{j \in \mathbb{N}}$, $\{\beta_j\}_{j \in \mathbb{N}}$ are sequences of nonnegative real numbers to be determined iteratively and the parameters characterizing the slicing procedure $\{L_j\}_{j \in \mathbb{N}}$ are given formally by (2.10), however, with the following modifications in the definition of the parameters $\{\ell_k\}_{k \in \mathbb{N}}$

$$\ell_0 \doteq \max \{ (A_0 + \alpha_1)^{-1}, (A_0 + \alpha_2)^{-1} \}, \quad \ell_k \doteq 1 + ((\beta + 1)p)^{-\frac{k}{2}} \quad \text{for } k \geq 1,$$

where

$$A_0 \doteq \varrho - \varrho_{\text{crit}}(n, H, b, m^2, \beta, p) + \frac{1}{2}(b + nH)(q - 1) + H(\beta + 1).$$

Moreover, setting

$$A_1 \doteq nH(q - 1) + \frac{H}{p},$$

we can represent the part of the coefficient for the exponential term in the τ -integral in (3.24) given by

$$\begin{aligned} \varrho - nH(\beta + 1)(p - 1) &= \varrho - \varrho_{\text{crit}} + \frac{1}{2}(b + nH)(q - 1) + H(\beta + 1) - nH(q - 1) - \frac{H}{p} \\ &= A_0 - A_1 \end{aligned} \tag{3.27}$$

as difference between the two positive quantities A_0, A_1 . As we will see in the iteration frame, the splitting (3.27) is the reason for the previous choice of ℓ_0 .

Clearly, (3.21) implies the validity of (3.26) for $j = 0$ provided that

$$C_0 \doteq K_1 \varepsilon^q, \quad a_0 \doteq \varrho - \varrho_{\text{crit}} + \frac{nH}{2}, \quad \gamma_0 \doteq \frac{b}{2} + \frac{H}{p}, \quad b_0 \doteq \varsigma_+, \quad \beta_0 \doteq \varsigma_-,$$

where we applied the decomposition $a_0 - \gamma_0 = \varrho - (b + H)\frac{q}{2} + (n - 1)H(\beta + 1) \left(1 - \frac{p}{2}\right)$ for the coefficient of the exponential term in (3.21).

Plugging (1.25) in (3.3) and using (3.27), we get the following iteration frame

$$\begin{aligned} V(t) &\geq C\mu(1+t)^{-\varsigma_-} e^{-\alpha_2 t} \int_0^t e^{(\alpha_2 - \alpha_1)s} \int_0^s e^{[\alpha_1 + \varrho - nH(\beta + 1)(p - 1)]\tau} \tau^{\varsigma_+} (V(\tau))^q d\tau ds \\ &\geq C\mu(1+t)^{-\varsigma_-} e^{-(\alpha_2 + A_1)t} \int_0^t e^{(\alpha_2 - \alpha_1)s} \int_0^s e^{(\alpha_1 + A_0)\tau} \tau^{\varsigma_+} (V(\tau))^q d\tau ds \end{aligned} \tag{3.28}$$

for $t \geq 0$.

Let us prove the induction step: assuming (3.26) satisfied for some $j \geq 0$, we prove it for $j + 1$. Plugging (3.26) in (3.28), we get

$$V(t) \geq \frac{\mu C C_j^q}{(1+t)^{\varsigma_- + qb_j}} e^{-(\alpha_2 + A_1 + q\gamma_j)t} \int_{L_{2j}}^t e^{(\alpha_2 - \alpha_1)s} \int_{L_{2j}}^s e^{(\alpha_1 + A_0 + a_j q)\tau} (\tau - L_{2j})^{\varsigma_+ + qb_j} d\tau ds \tag{3.29}$$

for $t \geq L_{2j}$. Let us apply a slicing procedure to the τ -integral. For $t \geq L_{2j+1}$ we may estimate

$$\begin{aligned} & \int_{L_{2j}}^t e^{(\alpha_2 - \alpha_1)s} \int_{L_{2j}}^s e^{(\alpha_1 + A_0 + a_j q)\tau} (\tau - L_{2j})^{\varsigma_+ + qb_j} d\tau ds \\ & \geq \int_{L_{2j+1}}^t e^{(\alpha_2 - \alpha_1)s} \int_{\frac{L_{2j}s}{L_{2j+1}}}^s e^{(\alpha_1 + A_0 + a_j q)\tau} (\tau - L_{2j})^{\varsigma_+ + qb_j} d\tau ds \\ & \geq \ell_{2j+1}^{-(\varsigma_+ + qb_j)} \int_{L_{2j+1}}^t e^{(\alpha_2 - \alpha_1)s} (s - L_{2j+1})^{\varsigma_+ + qb_j} \int_{\frac{L_{2j}s}{L_{2j+1}}}^s e^{(\alpha_1 + A_0 + a_j q)\tau} d\tau ds \\ & = \frac{\ell_{2j+1}^{-(\varsigma_+ + qb_j)}}{A_0 + \alpha_1 + qa_j} \int_{L_{2j+1}}^t (s - L_{2j+1})^{\varsigma_+ + qb_j} e^{(\alpha_2 + A_0 + a_j q)s} \left(1 - e^{-(\alpha_1 + A_0 + a_j q)(1 - 1/\ell_{2j+1})s}\right) ds. \end{aligned}$$

For $s \geq L_{2j+1}$ we can estimate

$$\begin{aligned} 1 - e^{-(\alpha_1 + A_0 + a_j q)(1 - 1/\ell_{2j+1})s} & \geq 1 - e^{-(\alpha_1 + A_0 + a_j q)(\ell_{2j+1} - 1)L_{2j}} \geq 1 - e^{-(\alpha_1 + A_0 + a_j q)(\ell_{2j+1} - 1)\ell_0} \\ & \geq 1 - e^{-(\alpha_1 + A_0)(\ell_{2j+1} - 1)\ell_0} \geq 1 - e^{-(\ell_{2j+1} - 1)} \geq q^{-(2j+1)} \left(q - \frac{1}{2}\right), \end{aligned}$$

where in the last step we used (2.14), therefore, for $t \geq L_{2j+1}$ we have

$$\begin{aligned} & \int_{L_{2j}}^t e^{(\alpha_2 - \alpha_1)s} \int_{L_{2j}}^s e^{(\alpha_1 + A_0 + a_j q)\tau} (\tau - L_{2j})^{\varsigma_+ + qb_j} d\tau ds \\ & \geq \frac{\left(q - \frac{1}{2}\right) \ell_{2j+1}^{-(\varsigma_+ + qb_j)} q^{-(2j+1)}}{A_0 + \alpha_1 + qa_j} \int_{L_{2j+1}}^t (s - L_{2j+1})^{\varsigma_+ + qb_j} e^{(\alpha_2 + A_0 + a_j q)s} ds. \end{aligned}$$

Repeating a similar estimate for the s -integral, for $t \geq L_{2j+2}$ we obtain

$$\begin{aligned} & \int_{L_{2j}}^t e^{(\alpha_2 - \alpha_1)s} \int_{L_{2j}}^s e^{(\alpha_1 + A_0 + a_j q)\tau} (\tau - L_{2j})^{\varsigma_+ + qb_j} d\tau ds \\ & \geq \frac{\left(q - \frac{1}{2}\right)^2 (\ell_{2j+1} \ell_{2j+2})^{-(\varsigma_+ + qb_j)} q^{-(4j+3)}}{(A_0 + \alpha_1 + qa_j)(A_0 + \alpha_2 + qa_j)} (t - L_{2j+2})^{\varsigma_+ + qb_j} e^{(\alpha_2 + A_0 + a_j q)t}. \end{aligned}$$

Combining the previous inequality and (3.29), for $t \geq L_{2j+2}$ we find

$$V(t) \geq \frac{\mu C C_j^q \left(q - \frac{1}{2}\right)^2 (\ell_{2j+1} \ell_{2j+2})^{-(\varsigma_+ + qb_j)}}{(A_0 + \alpha_1 + qa_j)(A_0 + \alpha_2 + qa_j) q^{4j+3}} e^{[(A_0 + qa_j) - (A_1 + q\gamma_j)]t} \frac{(t - L_{2j+2})^{\varsigma_+ + qb_j}}{(1 + t)^{\varsigma_- + q\beta_j}}$$

which is (3.26) for $j + 1$ provided that

$$\begin{aligned} C_{j+1} & \doteq \frac{\mu C \left(q - \frac{1}{2}\right)^2 C_j^q}{(A_0 + \alpha_1 + qa_j)(A_0 + \alpha_2 + qa_j) (\ell_{2j+1} \ell_{2j+2})^{\varsigma_+ + qb_j} q^{4j+3}}, \\ a_{j+1} & \doteq A_0 + qa_j, \quad \gamma_{j+1} \doteq A_1 + q\gamma_j, \\ b_{j+1} & \doteq \varsigma_+ + qb_j, \quad \beta_{j+1} \doteq \varsigma_- + q\beta_j. \end{aligned}$$

By using iteratively the previous relations, we may express $a_{j+1}, \gamma_{j+1}, b_{j+1}, \beta_{j+1}$ as follows:

$$\begin{aligned} a_{j+1} &= \left(a_0 + \frac{A_0}{q-1}\right) q^{j+1} - \frac{A_0}{q-1}, & \gamma_{j+1} &= \left(\gamma_0 + \frac{A_1}{q-1}\right) q^{j+1} - \frac{A_1}{q-1}, \\ b_{j+1} &= \frac{q^{j+2}-1}{q-1} \varsigma_+, & \beta_{j+1} &= \frac{q^{j+2}-1}{q-1} \varsigma_-. \end{aligned} \quad (3.30)$$

Therefore,

$$\begin{aligned} A_0 + \alpha_{1/2} + qa_j &= \alpha_{1/2} + a_{j+1} \leq \alpha_{1/2} + \left(a_0 + \frac{A_0}{q-1}\right) q^{j+1} \\ &\leq \left(\frac{1}{2}(b + \sqrt{b^2 - 4m^2}) + a_0 + \frac{A_0}{q-1}\right) q^{j+1} \doteq M_0 q^{j+1} \end{aligned}$$

implies $(A_0 + \alpha_1 + qa_j)^{-1}(A_0 + \alpha_2 + qa_j)^{-1} \geq M_0^{-2} q^{-2(j+1)}$. Moreover, repeating similar considerations as those in the proof of Theorem 1.2, we find that there exists a constant $M_1 = M_1(\varsigma, \beta, p) > 0$ such that $(\ell_{2j+1} \ell_{2j+2})^{b_{j+1}} \leq M_1$ holds for any $j \in \mathbb{N}$. Combining the previous estimates, we have

$$C_{j+1} \geq \frac{\mu C q (q - \frac{1}{2})^2}{M_0^2 M_1} q^{-6(j+1)} C_j^q \doteq G q^{-6(j+1)} C_j^q$$

for any $j \in \mathbb{N}$. From the inequality $C_j \geq G q^{-6j} C_{j-1}^q$, repeating analogous intermediate steps as those in the proof of Theorem 1.2, we can find a $j_4 = j_4(n, c, H, b, m^2, \mu, \varrho, \varsigma, \beta, p, R) \in \mathbb{N}$ such that for $j \geq j_4$ we have

$$\ln C_j \geq q^j \left(\ln(K_1 \varepsilon^q) - \frac{6q \ln q}{(q-1)^2} + \frac{\ln G}{q-1} \right) = q^j \ln(\tilde{G} \varepsilon^q), \quad (3.31)$$

where $\tilde{G} \doteq K_1 q^{-6q/(q-1)^2} G^{1/(q-1)}$.

Next, we combine (3.26), (3.30) and (3.31), obtaining for $t \geq L \doteq \lim_{j \rightarrow \infty} L_j$ and $j \geq j_4$

$$\begin{aligned} V(t) &\geq \exp\left(q^j \ln(\tilde{G} \varepsilon^q)\right) e^{(a_j - \gamma_j)t} (t-L)^{b_j} (1+t)^{-\beta_j} \\ &= \exp\left(q^j \left(\ln(\tilde{G} \varepsilon^q) + \left(a_0 - \gamma_0 + \frac{A_0 - A_1}{q-1}\right)t + \frac{q\varsigma_+}{q-1} \ln(t-L) - \frac{q\varsigma_-}{q-1} \ln(1+t)\right)\right) \\ &\quad \times e^{\frac{A_1 - A_0}{q-1} t} (t-L)^{-\frac{\varsigma_+}{q-1}} (1+t)^{\frac{\varsigma_-}{q-1}}. \end{aligned}$$

Thus, for $t \geq \max\{2L, 1\}$ and $j \geq j_4$, by $\ln(t-L) \geq \ln t - \ln 2$ and $-\ln(1+t) \geq -\ln t - \ln 2$ we have

$$V(t) \geq \exp\left(q^j \left(\ln\left(\tilde{G} \varepsilon^q e^{(a_0 - \gamma_0 + \frac{A_0 - A_1}{q-1})t} t^{\frac{\varsigma q}{q-1}}\right)\right)\right) e^{\frac{A_1 - A_0}{q-1} t} (t-L)^{-\frac{\varsigma_+}{q-1}} (1+t)^{\frac{\varsigma_-}{q-1}},$$

where $\hat{G} \doteq 2^{-(\varsigma_+ + \varsigma_-) \frac{q}{q-1}} \tilde{G}$. By (1.22) and (3.27), we find

$$\begin{aligned} a_0 - \gamma_0 + \frac{A_0 - A_1}{q-1} &= \frac{q}{q-1} (\varrho - \varrho_{\text{crit}}) + \frac{nH}{2} - \frac{b}{2} - \frac{H}{p} + \frac{1}{q-1} \left(\frac{1}{2}(b + nH)(q-1) + H(\beta+1) - nH(q-1) - \frac{H}{p}\right) \\ &= \frac{q}{q-1} (\varrho - \varrho_{\text{crit}}). \end{aligned}$$

Hence, we may rewrite the previous estimate as follows

$$V(t) \geq \exp\left(q^j \left(\ln\left(\hat{G} \varepsilon^q (\chi_{n, H, b, m^2, p, \beta, \varrho, \varsigma}(t))^{\frac{q}{q-1} (\varrho - \varrho_{\text{crit}})}\right)\right)\right) e^{\frac{A_1 - A_0}{q-1} t} (t-L)^{-\frac{\varsigma_+}{q-1}} (1+t)^{\frac{\varsigma_-}{q-1}} \quad (3.32)$$

for $t \geq \max\{2L, 1\}$ and $j \geq j_4$, where $\chi_{n,H,b,m^2,p,\beta,\varrho,\varsigma}$ is defined in (1.27).

By (1.27) it follows that $\chi_{n,H,b,m^2,p,\beta,\varrho,\varsigma}$ is a strictly increasing function for $t \geq \bar{T}$, for some $\bar{T} = \bar{T}(n, H, b, m^2, \beta, p, \varrho, \varsigma) \geq 0$. Clearly, for $\varsigma \geq 0$ it results $\bar{T}(n, H, b, m^2, \beta, p, \varrho, \varsigma) = 0$. With a slight abuse of notation, in the next lines we use the notation $\chi_{n,H,b,m^2,p,\beta,\varrho,\varsigma}^{-1}$ for the inverse function of the restriction $\chi_{n,H,b,m^2,p,\beta,\varrho,\varsigma}|_{[\bar{T}, \infty)}$.

The logarithmic factor multiplying q^j in the lower bound for V in (3.32) is strictly positive if and only if $\hat{G}\varepsilon^q (\chi_{n,H,b,m^2,p,\beta,\varrho,\varsigma}(t))^{\frac{q}{q-1}(\varrho - \varrho_{\text{crit}})} > 1$. For $t \geq \bar{T}$, this condition can be rewritten as

$$t > \chi_{n,H,b,m^2,p,\beta,\varrho,\varsigma}^{-1} \left((\hat{G}^{\frac{1}{q}} \varepsilon)^{-\frac{q-1}{\varrho - \varrho_{\text{crit}}}} \right).$$

Due to $\lim_{s \rightarrow \infty} \chi_{n,H,b,m^2,p,\beta,\varrho,\varsigma}^{-1}(s) = \infty$, we can fix $\varepsilon_0 = \varepsilon_0(n, c, H, b, m^2, q, \mu, \varrho, \varsigma, u_0, u_1, R) > 0$ small enough that satisfies

$$\chi_{n,H,b,m^2,p,\beta,\varrho,\varsigma}^{-1} \left((\hat{G}^{\frac{1}{q}} \varepsilon_0)^{-\frac{q-1}{\varrho - \varrho_{\text{crit}}}} \right) \geq \max\{2L, 1, \bar{T}\}.$$

Hence, for any $\varepsilon \in (0, \varepsilon_0]$ and any $t > \chi_{n,H,b,m^2,p,\beta,\varrho,\varsigma}^{-1} \left((\hat{G}^{\frac{1}{q}} \varepsilon)^{-(q-1)/(\varrho - \varrho_{\text{crit}})} \right)$ we have that $t \geq \{2L, 1, \bar{T}\}$ and that the factor multiplying q^j in (3.32) is positive, therefore, letting $j \rightarrow \infty$ in (3.32) we find that the lower bound for $V(t)$ is not finite. Summarizing, we showed that V blows up in finite time and we obtained the upper bound estimate for the lifespan in (1.26).

4. Final remarks

In the present paper, we derived a hierarchy of blow-up results for the semilinear models (1.1) and (1.5) prescribing different levels of assumptions on the function $\Gamma(t)$.

For the semilinear Cauchy problem in de Sitter spacetime our results in Theorems 1.2-1.4 refined the results from [21, Theorem 1.1] (cf. Remark 3). On the other hand, for the semilinear Cauchy problem in anti-de Sitter spacetime, to the best of our knowledge, the results from Theorems 1.6-1.9 are completely new.

Let us make some final remarks on generalizations of the obtained results and conjectures on the models that we have treated.

4.1. Semilinear wave equation with a summable speed of propagation

We point out explicitly that all results we proved for the semilinear wave equation in de Sitter spacetime (namely, Theorems 1.2, 1.3 and 1.4) can be naturally extended to the following semilinear Cauchy problem

$$\begin{cases} \partial_t^2 u - a^2(t)\Delta u + b\partial_t u + m^2 u = f(t, u), & x \in \mathbb{R}^n, t \in (0, T), \\ u(0, x) = \varepsilon u_0(x), & x \in \mathbb{R}^n, \\ \partial_t u(0, x) = \varepsilon u_1(x), & x \in \mathbb{R}^n, \end{cases}$$

where $a \in L^1([0, \infty))$ is a nonnegative function and $f(t, u)$ is given by (1.2). This is straightforward consequence of the fact that $A(t) = \int_0^t a(\tau) d\tau$ is a bounded function (see also Theorem 1 in [20]).

Nevertheless, in the previous sections we consider just the case $a = a_{\text{dS}}$, since we expect that our results cannot be improved by working with the spatial average of a local solution as we explained in Remark 13.

4.2. Dominant mass case

In the case with dominant mass ($b^2 < 4m^2$), our approach is unfruitful due to the conjugate complex roots $\alpha_{1/2}$ in (2.3). For the average of a local solution u to (1.1), we obtain the representation

$$U(t) = \varepsilon y_0(t; b, m^2) \int_{\mathbb{R}^n} u_0(x) dx + \varepsilon y_1(t; b, m^2) \int_{\mathbb{R}^n} u_1(x) dx + \int_0^t y_1(t - \tau; b, m^2) \Gamma(\tau) \left(\int_{\mathbb{R}^n} |u(\tau, x)|^p dx \right)^{\beta+1} d\tau,$$

where

$$y_0(t; b, m^2) \doteq e^{-\frac{b}{2}t} \cos \left(\sqrt{m^2 - \frac{b^2}{4}}t \right) + \frac{b}{2} \frac{e^{-\frac{b}{2}t} \sin \left(\sqrt{m^2 - \frac{b^2}{4}}t \right)}{\sqrt{m^2 - \frac{b^2}{4}}},$$

$$y_1(t; b, m^2) \doteq \frac{e^{-\frac{b}{2}t} \sin \left(\sqrt{m^2 - \frac{b^2}{4}}t \right)}{\sqrt{m^2 - \frac{b^2}{4}}}.$$

Analogously, for a local solution v to (1.5). We see that the damped oscillations from the time factors y_0, y_1 prevent us to work with a nonnegative U making impossible to establish an iteration frame and, consequently, a sequence of lower bound estimates for U .

4.3. Critical exponent for anti-de Sitter with a nonlocal nonlinearity

In the main results from Subsection 1.1 we analyzed how prescribing certain conditions on Γ it is possible to prove blow-up results for local solutions to (1.1) and (1.5). Let us now change our perspective in the following sense: is it possible to prove blow-up results either for (1.1) or for (1.5) without requiring additional exponential/polynomial growth for the nonlinear term though the factor Γ ? As we explained in the introduction, for the model (1.1) this is not possible unless $m^2 = 0$. Similarly, for (1.5) when (1.23) holds, due to the fact that $\varrho_{\text{crit}} > 0$, Theorems 1.6, 1.7 and 1.8 do not provide a blow-up result unless $\varrho > 0$. On the other hand, when (1.24) holds it is possible to find some $p > 1$ such that $\varrho_{\text{crit}} \leq 0$. Indeed, from straightforward computations we see that the condition $\varrho_{\text{crit}} \leq 0$ can be rewritten as the following quadratic equation for p

$$(b + nH)(\beta + 1)(p - 1)^2 + [2(b + H)\beta + (b + nH) + H](p - 1) - ((n - 2)H - b)\beta \leq 0. \tag{4.1}$$

In the nonlocal case $\beta > 0$, the term $-((n - 2)H - b)\beta$ in (4.1) can be negative, more precisely this happens for $n > 2 + \frac{b}{H}$. Therefore, for $n > 2 + \frac{b}{H}$, thanks to Descartes' rule of signs, there exists $p_0 = p_0(n, H, b, \beta) > 1$ such that for $1 < p \leq p_0(n, H, b, \beta)$ the quadratic equation (4.1) is satisfied. Since $2 + \frac{b}{H} \geq N_0$, where N_0 is defined in (3.23), in particular we are in the case in which (1.24) is always fulfilled for any $p > 1$, hence, Theorem 1.9 provides a blow-up result for $1 < p < p_0(n, H, b, \beta)$ for local in time solutions to (1.5) when $n > 2 + \frac{b}{H}$.

In other words, when $n > 2 + \frac{b}{H}$ and $\beta > 0$, considering $\Gamma(t) = 1$, that is, when the nonlinearity in (1.5) is given by

$$f(v) = \left(\int_{\mathbb{R}^n} |v(t, y)|^p dy \right)^\beta |v|^p,$$

we found as candidate to be the critical exponent for the semilinear Cauchy problem (1.5) the positive root $p_0(n, H, b, \beta)$ of the quadratic equation

$$(b + nH)(\beta + 1)(p - 1)^2 + [2(b + H)\beta + (b + nH) + H](p - 1) - ((n - 2)H - b)\beta = 0.$$

In [17] we will prove a blow-up result for local solutions to (1.5) for $\varrho = \varrho_{\text{crit}}$ when (1.24) holds and ϱ_{crit} is given by (1.22). In particular, for $n > 2 + \frac{b}{H}$ and $\beta > 0$ this result will show the blow-up of local solutions even in the threshold case $p = p_0(n, H, b, \beta)$.

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