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Attaching a matrix algebra to affine group schemes: Polynomial identities

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ABSTRACT

We give a definition of polynomial identities for affine group schemes; over a field F of characteristic 0, the identities of an affine group scheme turn out to coincide with the identities of a certain matrix algebra over F. We prove the identities of an algebraic affine group scheme are those of a matrix algebra whose size is related to the rank of the representative algebra of the group scheme. Moreover, we relate the existence of a polynomial identity of an affine group scheme with the existence of an abelian subgroupscheme of finite index, giving a Passman type result for affine group schemes.

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1. Introduction

The main characters of this paper are the so called affine group schemes whose theory started to be developed in the language we use nowadays around 1960's due to the

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contribution of great mathematicians like Alexander Grothendieck, Michel Raynaud and Michel Demazure among all. Speaking about the relevance of group schemes in modern mathematics, we can affirm they arise naturally as symmetries of schemes (intended as locally ringed spaces with local sheaves isomorphic to $Spec(A)$ for some commutative ring A), and they can be seen as a generalization of algebraic groups: all algebraic groups have a group scheme structure whereas group schemes do not necessarily carry some of the peculiar properties of algebraic groups. Furthermore, we can also point out the category of group schemes behaves better than that of group varieties; moreover, there is a well-behaved deformation theory in the group schemes environment. It is worth mentioning group schemes that are not algebraic groups play a significant role in arithmetic geometry and algebraic topology, since they come up in contexts of Galois representations and moduli problems.

We wonder if it could be useful studying group schemes and obtaining information on their structure via their polynomial identities as it happens for group algebras for instance. Actually, the major obstruction is the fact that for group schemes it is not possible to define a fruitful free object. Motivated by this obstruction, given a group scheme $\mathcal{G} : \mathbf{Alg}_F \to \mathbf{Grp}$, that is a representable functor from the category of associative commutative algebras over a field F to the category of groups, we define a polynomial identity of $\mathcal{G}$ in the most natural way: we consider a polynomial $f \in F\langle X\rangle$ and we say it is a polynomial identity for $\mathcal{G}$ if it is a polynomial identity of the group algebra $F[\mathcal{G}(A)]$ for every $A \in \mathbf{Alg}_F$. In the light of the positive solution to Hartley's Conjecture given by the authors in [10], our definition of polynomial identity for group schemes suggests us how to extend to group schemes, and in a natural and compatible way, the definition of group identity, too.

In this paper we aim to give a closer look into those group schemes satisfying a polynomial identity that is the content of Theorem 14 in the text. More precisely, a group scheme $\mathcal{G}$ over a field F of characteristic 0 satisfying the *connecting property* (see Definition 18 in the text) such that the set of the elements of finite conjugacy class of $\mathcal{G}(A)$ determines a subalgebra of A for any $A \in \mathbf{Alg}_F$, satisfies a polynomial identity if and only if it has an abelian subgroupscheme of index bounded by a certain index n and, in this case, the identities of $\mathcal{G}$ are exactly the identities of the full matrix algebra $M_m(F)$ with entries from F for some $m \leq n$. We highlight Passman, in a well celebrated and already classical work, proved a similar result for group algebras. In our opinion, attaching a matrix algebra to an affine group scheme could represent a new point of view for those algebraic geometry themes using group schemes for their outcomes such as Tannakian categories and their role in the (inverse) Riemann-Hilbert problem (see, for instance, [6]). We also think this paper could enable us to make a bridge with the so-called Universal Algebraic Geometry as presented in [18].

We were also able to relate the identities of $\mathcal{G}$ to those of one of its homomorphic images $\mathcal{G}/\mathcal{H}$ provided the identities of $F[\mathcal{G}(A)/\mathcal{H}(A)]$ are identities of $F[\mathcal{G}(A)]$ for any $A \in \mathbf{Alg}_F$. This last result relies with a result on identities of group algebras established by the authors in [19] with the further condition the underlying group is locally finite.

2. Preliminaries

From now on every field is supposed to be of characteristic zero and any algebra is supposed to be associative with unit 1.

Let $X = \{x_1, x_2 \ldots, x_n, \ldots\}$ be a countable set of indeterminates and consider $F\langle X\rangle$ the free algebra freely generated by X over F; let us call *polynomials* its elements. Moreover we say a polynomial $f = f(x_1, \ldots, x_n)$ is *multilinear* if every x_i appears only one time in each summand of f for every $i = 1, \ldots, n$.

Definition 1. Let $f = f(x_1, \ldots, x_n)$ be a polynomial and A be an algebra. We say f is a polynomial identity for A if $f(a_1, \ldots, a_n) = 0$ for any $a_1, \ldots, a_n \in A$. If an algebra A satisfies a nontrivial polynomial identity, then A is said to be an algebra with polynomial identities (PI-algebra). We will denote the set of polynomial identities of a given algebra A by $Id(A)$ (the T-ideal of A).

It could be useful recalling if A is a subalgebra or a homomorphic image of B, then $Id(B) \subseteq Id(A)$.

The class of PI-algebras contains the class of commutative algebras and finite dimensional algebras, for instance. In particular, we have the following crucial result due to Amitsur and Levitzki (see [3]). We recall the *standard polynomial* of degree n is defined as

$$S_n := \sum_{\sigma \in S_n} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(n)},$$

where S_n denotes the symmetric group on n elements.

Theorem 2. *Let C be a commutative ring. Then $M_n(C)$ satisfies the standard polynomial S_{2n} of degree $2n$.*

The theory of algebras with polynomial identities is now a big research area and it has been investigated since the first half of the 20-th century. For an exhaustive account toward algebras with polynomial identities we refer to the books by Giambruno and Zaicev [11], Drensky [8], Aljadeff, Giambruno, Procesi and Regev [1].

If F is a field of characteristic 0, the ideal of identities of a given algebra over F is determined by its multilinear part. Moreover, it is finitely generated as a T-ideal in the light of the work by Kemer (see [14] and [15]) and generalized to other polynomial identities settings by many authors in a non-associative setting too (see for instance [4], [2], [13], [5], [12]).

We want to recall here the definition of one of the main characters of this paper: the *group algebra*. Actually, if $(G, \cdot)$ is a group and F is a field, we can consider the vector space $F[G]$ over F having the elements of G as a basis. Further, we can naturally attach a product to $F[G]$ starting from the one of $(G, \cdot)$ so that $F[G]$ turns out to be an algebra.

Several authors tried to build a bridge between the theory of PI-algebras and the theory of group algebras. The milestone in this setting is, of course, represented by Passmann's book [17]. In particular, we would like to highlight Theorem 5.3.7 of Passman's book that is a result of deepest importance obtained after several years of research in that direction by a plethora of algebraists. Here are the results.

Theorem 3. *Let F be a field of characteristic 0 and G be a group. Then the group algebra $F[G]$ is a PI-algebra if and only if G has an abelian subgroup of finite index.*

Theorem 4. *Let F be a field of characteristic $p > 0$ and G be a group. Then the group algebra $F[G]$ is a PI-algebra if and only if G has a p-abelian subgroup of finite index.*

In [19] the authors characterized the identities of group algebras $F[G]$ provided F is of characteristic 0.

Theorem 5. *Let F be a field of characteristic 0 and let G be a group such that $F[G]$ is a PI-algebra. Then there exists a positive integer n such that $Id(F[G])$ equals $Id(M_n(F))$; more precisely, n is equal to the maximal dimension of irreducible finite dimensional representations of G over algebraically closed extension of F.*

Unluckily, Theorem 5 has not a counterpart in characteristic $p > 0$. More precisely, we have a counterexample (see pages 4288-4289 of [19]).

We shall denote by $\Delta(G)$ the set of elements of the group G having finite conjugation classes; it is a subgroup of G. The next will be useful in the sequel, too (see [16]).

Theorem 6. *Let G be a group and F be a field so that $F[G]$ is a PI-algebra satisfying a polynomial identity of degree n. Then $\Delta(G)$ has index bounded by $\frac{n}{2}$.*

Remark 7. The previous result turns out to be a necessary and sufficient condition provided $\Delta(G)$ is abelian.

In these notes we aim to generalize the previous results in a functorial setting. Because of this, we want to give a small account concerning affine group schemes. We remand to [20] or Appendix A of [9] for a compendium of this topic.

In what follows $\mathbf{Alg}_F$ will denote the category of associative commutative unital algebras over the field F, **Grp** will denote the category of groups and **AbGrp** will denote the category of abelian groups with their canonical morphisms denoted by *hom*.

Let **Set** be the category of sets and let $\mathcal{C}$ be a category. A functor $F : \mathcal{C} \to \mathbf{Set}$ is said to be *representable* if it is naturally isomorphic to $hom(A, -) : \mathcal{C} \to \mathbf{Set}$ for some fixed $A \in \mathcal{C}$.

Definition 8. We say a covariant functor $\mathcal{G} : \mathbf{Alg}_F \to \mathbf{Grp}$ is an affine group scheme if $\mathcal{G}$ is representable. An affine group scheme $\mathcal{H}$ is said to be a subgroupscheme of $\mathcal{G}$ if for any $A \in \mathbf{Alg}_F$ we have $\mathcal{H}(A)$ is a subgroup of $\mathcal{G}(A)$.

In order to establish if a given functor is representable, it is important the definition of left (right) adjoint.

Definition 9. If $F : \mathcal{C} \to \mathcal{D}$ is a functor, a functor $F' : \mathcal{D} \to \mathcal{C}$ is said to be a left adjoint of F' if for every $A \in \mathcal{C}$ there exists $F'(A) \in \mathcal{D}$ and a morphism $\epsilon_A : F'(F(A)) \to A$ such that for every $B \in \mathcal{D}$ and for every morphism $f : F'(B) \to A$ there exists a unique morphism $f' : B \to F(A)$ such that $\epsilon_A \circ F'(f') = f$.

Actually, if a given category $\mathcal{C}$ admits a coproduct, then the existence of a left (right) adjoint of a functor $F : \mathcal{C} \to \mathcal{D}$ is a necessary and sufficient condition for F being representable. This is the content of the following which is a consequence of the so-called *Yoneda's Lemma.*

Proposition 10. *Let $\mathcal{C}, \mathcal{D}$ be categories. If $\mathcal{C}$ has a coproduct, then a functor $F : \mathcal{C} \to \mathcal{D}$ is representable if and only if F has a left adjoint.*

3. Identities of group schemes

The goal of the section is achieving a Passman type result for affine group schemes. As we mentioned above, it is a kind of a way to see functorial properties of the polynomial identities.

Herein, if $P(G)$ is a certain property of a group G, we say a group scheme $\mathcal{G}$ has the property P if $P(\mathcal{G}(A))$ holds for every $A \in \mathbf{Alg}_F$. For instance, $\mathcal{G}$ is abelian if $\mathcal{G}(A)$ is abelian for every $A \in \mathbf{Alg}_F$ or $\mathcal{G}$ is nilpotent if $\mathcal{G}(A)$ is nilpotent for every $A \in \mathbf{Alg}_F$, etc.

We start with the following.

Definition 11. Let $\mathcal{G}$ be an affine group scheme and let $\mathcal{K}$ be a subgroupscheme of $\mathcal{G}$. We say $\mathcal{K}$ is of bounded index n if for every $A \in \mathbf{Alg}_F$ we have $\mathcal{K}(A)$ is a subgroup of $\mathcal{G}(A)$ of index bounded by n.

Definition 12. Let $\mathcal{G} : \mathbf{Alg}_F \to \mathbf{Grp}$ be an affine group scheme and $f = f(x_1, \dots, x_n) \in F\langle X\rangle$. We say f is a polynomial identity of $\mathcal{G}$ if for any $A \in \mathbf{Alg}_F$, we have f is a polynomial identity of the group algebra $F[\mathcal{G}(A)]$. We say $\mathcal{G}$ satisfies a polynomial identity if there exists a nontrivial $f \in F\langle X\rangle$ such that f is a polynomial identity of $\mathcal{G}$.

Remark the definition of polynomial identity for a group scheme $\mathcal{G}$ jointly to the result of Theorem 5 leads us to say

$$Id(\mathcal{G}) = \cap_{A\in \mathbf{Alg}_F} Id(F[\mathcal{G}(A)]) = \cap_{A\in \mathbf{Alg}_F} Id(M_{n(A)}(F)) = Id(M_n(F)),$$

for some $n \geq 1$.

Problem 13. Find a positive integer n and an algebra $A \in \mathbf{Alg}_F$ such that $Id(\mathcal{G}) = Id(\mathcal{G}(A)) = Id(M_n(F))$.

Remark 14. If $\mathcal{G}$ satisfies a polynomial identity f, then f is a polynomial identity of any subgroupscheme of $\mathcal{G}$. Moreover, isomorphic group schemes do satisfy the same identities.

Notice now the 'if' part of Theorem 3 is still valid for affine group schemes.

Proposition 15. *Let $\mathcal{G}$ be an affine group scheme. Then $\mathcal{G}$ satisfies a polynomial identity if there exists an abelian subgroupscheme of $\mathcal{G}$ of bounded index.*

Proof. Suppose $\mathcal{G}$ has an abelian subgroupscheme $\mathcal{K}$ of bounded index, to say, n. Then for every $A \in \mathbf{Alg}_F$ we have $\mathcal{K}(A)$ is an abelian subgroup of $\mathcal{G}(A)$ of index $n(A)$; hence $F[\mathcal{G}(A)]$ is a PI-algebra by Theorem 3. By Theorem 5, we have for any $A \in \mathbf{Alg}_F$, $Id(F[\mathcal{G}(A)]) = Id(M_{n'(A)}(F))$ for some $n'(A) \leq n(A)$ but, of course, $Id(M_n(A)) \subseteq Id(M_{n'(A)}(A))$. Now take $f \in Id(M_n(A))$, then f is an identity of $\mathcal{G}$. $\square$

Remark 16. Remark if $\mathcal{K}(A)$ is a subgroup of $\mathcal{G}(A)$ of index $n(A)$, then by Lemma 5.1.10 of [17] we have $F[\mathcal{G}(A)] \subseteq M_{n(A)}(F[\mathcal{K}(A)])$ that means

$$Id(M_{n(A)}(F)) = Id(M_{n(A)}(F[\mathcal{K}(A)])) \subseteq Id(F[\mathcal{G}(A)]),$$

where the first equality follows from the fact over a field of characteristic 0, $Id(R) = Id(R \otimes_F C)$ where C is any commutative F-algebra and R is any F-algebra.

It is clear the sole existence of a polynomial identity for a group scheme is not enough to ensure the converse of Proposition 15. In fact, we need to construct not only an abelian subgroup of $\mathcal{G}(A)$ of finite index for any $A \in \mathbf{Alg}_F$: those subgroups must be linked so that they could form a covariant functor.

Remark 17. If $\mathcal{G}$ is a group scheme and $\phi : A \to B$ is an algebra homomorphism, then $\mathcal{G}(\phi) : \mathcal{G}(A) \to \mathcal{G}(B)$ is a group homomorphism inducing an algebra homomorphism $\mathcal{G}(\phi) : F[\mathcal{G}(A)] \to F[\mathcal{G}(B)]$. By Theorem 3, there exist $\mathcal{H}(A)$ and $\mathcal{H}(B)$ subgroups of finite index of $\mathcal{G}(A)$ and $\mathcal{G}(B)$ respectively. Let us set $n(A) := [\mathcal{G}(A) : \mathcal{H}(A)]$ and $n(B) := [\mathcal{G}(B) : \mathcal{H}(B)]$; we get the algebra $F[\mathcal{G}(A)]$ can be embedded in $M_{n(A)}(F[\mathcal{H}(A)])$ and $F[\mathcal{G}(B)]$ can be embedded in $M_{n(B)}(F[\mathcal{H}(B)])$. Moreover, if $G(\phi)_{|\mathcal{H}(A)}$ is a group homomorphism, then $\mathcal{H} : \mathbf{Alg}_F \to \mathbf{Grp}$ is a well defined subgroupscheme of $\mathcal{G}$ and it is abelian and of bounded index.

Indeed, satisfying the hypotheses of the previous remark is not that easy even in the case of finite or solvable or nilpotent group schemes. Because of this, in the sequel, we are going to capture concrete cases in which Theorem 3 fits (more or less) naturally. As well as in the classical theory of polynomial identities for group algebras, the subgroup of elements of finite conjugacy class of $\Delta(\mathcal{G}(A))$ plays a crucial role. We need the following definition.

Definition 18. We say a group scheme $\mathcal{G}$ satisfies the connecting property if for any $G \in \mathbf{AbGrp}$ there exists a copy of G in $\Delta(\mathcal{G}(F[G]))$.

Now we can state the following.

Proposition 19. *Let $\mathcal{G}$ be an affine group scheme such that for every $A \in \mathbf{Alg}_F$ we have $\Delta(\mathcal{G}(A))$ is abelian and for every morphism $\varphi : F[G] \to A$ of commutaive algebras, there exists a morphism $\epsilon_A : F[\Delta(\mathcal{G}(A))] \to A$ with the property $\epsilon_A \circ \mathcal{G}(\varphi) = \varphi$ and for every morphism ϕ in $\mathbf{Alg}_F$ we have $\mathcal{G}(\phi)$ preserves elements of finite conjugacy class. Suppose further $\mathcal{G}$ satisfies the connecting property for abelian subgroupschemes. If $\mathcal{G}$ satisfies a polynomial identity, then there exists an abelian subgroupscheme of $\mathcal{G}$ of bounded index.*

Proof. Let f be a polynomial identity of $\mathcal{G}$, then for every $A \in \mathbf{Alg}_F$ we have f is a polynomial identity of $F[\mathcal{G}(A)]$ and, of course, $Id(F[\mathcal{G}(A)]) = Id(M_{n(A)}(F))$ for some $n(A)$ by Theorem 5. Now consider the sequence $\{n(A)\}_{A\in\mathbf{Alg}_F}$ that is bounded, then we can take its maximal element, say n. By Theorem 2, for every $A \in \mathbf{Alg}_F$, $F[\mathcal{G}(A)]$ satisfies an identity of degree $2n$, then by Theorem 6 we get $\Delta(\mathcal{G}(A))$ is an abelian subgroup of $\mathcal{G}(A)$ of index bounded by n. Hence let us set $\mathcal{K} : \mathbf{Alg}_F \to \mathbf{Grp}$ such that for every $A \in \mathbf{Alg}_F$ we have $\mathcal{K}(A) = \Delta(\mathcal{G}(A))$. We need to prove $\mathcal{K}$ is a representable functor. First we see $\mathcal{K}$ is covariant. For every $A, B \in \mathbf{Alg}_F$ and for every $\phi : A \to B$, we have $\mathcal{G}(\phi) : \mathcal{G}(A) \to \mathcal{G}(B)$ is a group homomorphism as well as its restriction to $\Delta(\mathcal{G}(A))$, namely $\mathcal{G}(\phi)'$, by hypothesis. We define $\mathcal{K}(\phi)(h) = \mathcal{G}(\phi)'(h)$ for any $h \in \Delta(\mathcal{G}(A))$, then $\mathcal{K}(\phi) : \Delta(\mathcal{G}(A)) \to \Delta(\mathcal{G}(B))$ is a group homomorphism. Moreover, $\mathcal{K}(id_A) = id_{\mathcal{K}(A)}$ and $\mathcal{K}(\phi \circ \psi) = \mathcal{K}(\phi) \circ \mathcal{K}(\psi)$ for all $A, B, C \in \mathbf{Alg}_F$ and for all morphisms $\phi : A \to B$, $\psi : B \to C$ and this proves $\mathcal{K}$ is a covariant functor.

Let us set $\mathcal{K}' : \mathbf{AbGrp} \to \mathbf{Alg}_F$ being the functor sending G to $F[G]$ for any $G \in \mathbf{Grp}$; $\mathcal{K}'$ will map any group homomorphism $\phi : G_1 \to G_2$ to $\mathcal{K}'(\phi) : F[G_1] \to F[G_2]$ such that $\mathcal{K}'(\phi)(g_1) = \phi(g_1)$ for any $g_1 \in G_1$. We will show $\mathcal{K}'$ is a left adjoint of $\mathcal{K}$, then we shall conclude $\mathcal{K}$ is an affine subgroupscheme of $\mathcal{G}$ via Proposition 10 and the proof will follow. Let $G \in \mathbf{AbGrp}$, $A \in \mathbf{Alg}_F$ and $g : \mathcal{K}'(G) \to A$; g induces naturally a homomorphism of groups $f : G \to \mathcal{K}(A)$ such that $\epsilon_A \circ \mathcal{K}'(f) = g$ because of the connecting property of $\mathcal{G}$, where $f = \mathcal{K}(g)_{|G}$ up to an embedding of G in $\Delta(\mathcal{G}(F[G]))$. This means $\mathcal{K}'$ is a left adjoint functor of $\mathcal{K}$ and we are done. □

The content of Propositions 15 and 19 can be summarized as follows.

Theorem 20. *Let $\mathcal{G}$ be an affine group scheme such that for every $A \in \mathbf{Alg}_F$ we have $\Delta(\mathcal{G}(A))$ is abelian and for every morphism $\varphi : F[G] \to A$ of commutative algebras, there exists a homomorphism $\epsilon_A : F[\Delta(\mathcal{G}(A))] \to A$ with the property $\epsilon_A \circ \mathcal{G}(\varphi) = \varphi$ and for every morphism ϕ in $\mathbf{Alg}_F$ we have $\mathcal{G}(\phi)$ preserves elements of finite conjugacy class. Suppose $\mathcal{G}$ satisfies the connecting property for abelian subgroupschemes, then $\mathcal{G}$ satisfies a polynomial identity if and only if $\mathcal{G}$ has an abelian subgroupscheme of bounded index.*

A concrete example of a group scheme satisfying the hypotheses of Theorem 20 is the *additive group scheme* $\mathbf{G}_{add} : \mathbf{Alg}_F \to \mathbf{Grp}$ defined by $\mathbf{G}_{add}(A) = (A, +)$, where $(A, +)$ denotes the underlying additive subgroup of the algebra A. Here Problem 13 is solved by $Id(\mathbf{G}_{add}) = Id(\mathbf{G}_{add}(A)) = Id(F)$ for any $A \in \mathbf{Alg}_F$. Analogously, the *multiplicative group scheme* $\mathbf{G}_{mult} : \mathbf{Alg}_F \to \mathbf{Grp}$ defined by $\mathbf{G}_{mult}(A) = (\mathcal{U}(A), \cdot)$, where $\mathcal{U}(A)$ is the set of invertible elements of A and $\cdot$ is the product of A as an algebra; we have $Id(\mathbf{G}_{mult}) = Id(\mathbf{G}_{mult}(A)) = Id(F)$ for any $A \in \mathbf{Alg}_F$.

The *general linear* affine group scheme $\mathbf{GL}_n$ also satisfies the hypotheses above although it does not satisfy any polynomial identity. On this purpose, it could be of interest constructing affine group schemes $\mathcal{G}$ so that for any $A \in \mathbf{Alg}_F$, $\mathcal{G}(A)$ is a linear CC-group that is a natural generalization of an FC-group: the connected component containing the identity of any linear CC-group is, in fact, a finite index abelian subgroup (see [7]).

Certainly, finite group schemes do not satisfy the hypotheses of Theorem 20. Nevertheless, if $\mathcal{G}$ is finite of bounded size ($|\mathcal{G}(A)| \leq n$ for a fixed n), $\mathcal{G}$ obviously satisfies a polynomial identity (the standard polynomial of degree n) and the subgroupscheme $\mathcal{K}$ of $\mathcal{G}$ defined by $\mathcal{K}(A) = \{e_{\mathcal{G}(A)}\}$ for any $A \in \mathbf{Alg}_F$ is abelian and of bounded index.

This last example fits on a constant group scheme determined by a finite group of order, to say, n. This means finite group schemes of bounded size do satisfy Passman's result although they do not satisfy the hypotheses of Theorem 20. The following natural problem arises.

Problem 21. Find conditions similar to those of Passman's result on classes of group schemes satisfying polynomial identities.

Another natural question arises: do identities of group schemes depend on their representative algebras in a certain sense? On the purpose, we introduce the following. Let $\mathcal{G}$ and $\mathcal{H}$ be group schemes and θ any morphism from $\mathcal{G}$ to $\mathcal{H}$. If A and B are commutative algebras representing $\mathcal{G}$ and $\mathcal{H}$ respectively, we shall denote by $\theta^* : B \to A$ the morphism of algebras naturally determined by θ. We say θ is a *closed embedding* if θ^* is surjective. This automatically implies $\theta(R) : \mathcal{G}(R) \to \mathbf{GL}_n(R)$ is injective for any $R \in \mathbf{Alg}_F$ if $\mathcal{G} \to \mathbf{GL}_n$ is a closed embedding. Recalling an affine group scheme is *algebraic* if its representative algebra is finitely generated, we have the next result (see [9]).

Theorem 22. *Let $\mathcal{G}$ be an algebraic affine group scheme. Then for sufficiently large n, there exists a closed embedding from $\mathcal{G}$ to $\mathbf{GL}_n$.*

By the definition of closed embedding and by Theorem 22 and its proof, we can trigger the following: if $\mathcal{G}$ is an algebraic affine group scheme, with representative algebra A, satisfying a polynomial identity, then there exists a finite dimensional subcomodule V of A that generates A as an algebra; let us fix $n := \dim_F(V)$ being the minimal possible, then $\mathcal{G}$ is closely imbedded in $\mathbf{GL}_n$. If $\mathcal{G}$ satisfies a polynomial identity, then $Id(\mathcal{G}) = Id(M_m(F))$, where m is less than the maximum degree of any irreducible finite dimensional representation of $GL_n(A)$ for any $A \in \mathbf{Alg}_F$ because of the argument given in Remark 17.

It could be interesting now to relate the identities of $\mathcal{G}$ with the identities of the quotient group scheme $\mathcal{G}/\mathcal{H}$. We recall briefly what a quotient of group scheme is. Let $\mathcal{H}$ be a normal subgroupscheme of the group scheme $\mathcal{G}$, then $\mathcal{G}/\mathcal{H}$ is the image $\mathcal{K}_0$ of a morphism of group schemes (that is a group scheme too) $\theta : \mathcal{G} \to \mathcal{K}$ having $\mathcal{H}$ as kernel. In this case, if R, S_0 and S are the representative objects of $\mathcal{G}$, $\mathcal{K}_0$ and $\mathcal{K}$ respectively and $\theta^* : S \to R$ is the comorphism associated to θ, we denote by I the ideal $\ker(\theta^*)$. We set $S_0 = S/I$ and we remark it can be embedded in R. Now, for any $A \in \mathbf{Alg}_F$,

$$\mathcal{H}(A) = (\ker(\theta))(A) = \{x \in \mathcal{G}(A) | \theta(A)(x) = e_{\mathcal{K}(A)}\} = \ker(\theta(A)).$$

In the hypotheses of Theorem 20, we consider the normal subgroupscheme $\mathcal{H} : \mathbf{Alg}_F \to \mathbf{Grp}$ and the quotient group scheme $\mathcal{G}/\mathcal{H}$. It is worth mentioning right here the functor $\mathcal{F} : \mathbf{Alg}_F \to \mathbf{Grp}$ mapping A to $\mathcal{G}(A)/\mathcal{H}(A)$ usually is not representable and does not work as a model for the quotient of two group schemes; nevertheless we need to link those two conditions. Remark for every $A \in \mathbf{Alg}_F$, any morphism from R to A can be seen as a morphism from S_0 to A because of the embedding of S_0 into R. Hence, because $\mathcal{G}(A)$ can be embedded in $\mathcal{K}_0(A)$ via θ, we get

$$Id(F[\mathcal{G}(A)]) \subseteq Id(F[\mathcal{K}_0(A)])$$

for every $A \in \mathbf{Alg}_F$ that means $Id(\mathcal{G}) \subseteq Id(\mathcal{G}/\mathcal{H})$. Furthermore, if we require for every $A \in \mathbf{Alg}_F$ $Id(F[\mathcal{G}(A)/\mathcal{H}(A)]) \subseteq Id(F[\mathcal{G}(A)])$, then

$$Id(F[\mathcal{K}_0(A)]) \subseteq Id(F[\mathcal{G}(A)])$$

because $\mathcal{G}(A)/\mathcal{H}(A) \subseteq \mathcal{K}_0(A)$ for every $A \in \mathbf{Alg}_F$, that is $Id(\mathcal{G}/\mathcal{H}) \subseteq Id(\mathcal{G})$. We sum up the above argument into a corollary.

Corollary 23. *Let $\mathcal{G}$ be an affine group scheme such that for every $A \in \mathbf{Alg}_F$ we have $\Delta(\mathcal{G}(A))$ is abelian and for every morphism $\varphi : F[G] \to A$ of commutative algebras, there exists a homomorphism $\epsilon_A : F[\Delta(\mathcal{G}(A))] \to A$ with the property $\epsilon_A \circ \mathcal{G}(\varphi) = \varphi$ and for every morphism ϕ in $\mathbf{Alg}_F$ we have $\mathcal{G}(\phi)$ preserves elements of finite conjugacy class. Suppose $\mathcal{G}$ satisfies the connecting property for abelian subgroupschemes. If $\mathcal{G}$ satisfies a polynomial identity and $\mathcal{H}$ is the subgroupscheme of finite index from Theorem 20, if for every $A \in \mathbf{Alg}_F$ we have $Id(F[\mathcal{G}(A)/\mathcal{H}(A)]) \subseteq Id(F[\mathcal{G}(A)])$, then $Id(\mathcal{G}) = Id(\mathcal{G}/\mathcal{H})$.*

4. Remarks

In this final section we would like to speak a bit more about the case of group schemes over fields of any charactersitic. As already pointed out, if $\mathcal{G}$ is a group scheme over a field of characteristic 0 satisfying a polynomial identity, its identities are those of a matrix algebra $M_n(F)$; besides this, in view of the classical work by Kemer (see [14] and [15]), $Id(M_n(F))$ is finitely generated as a T-ideal. We cannot say the same in the case the ground field is of characteristic $p > 0$. First of all, we do not know if the identities of a group algebra $F[G]$ are finitely generated in this case. On this purpose, in [19] the authors arose the following question: given a group G, is there any finite dimensional algebra R such that $Id(R) \subseteq Id(F[G])$ (observe this is certainly verified over fields of characteristic 0)? This is a sort of Representability Theorem (see the work by Kemer once again) for group algebras. Instantly, we can ask the same for affine group schemes: let $\mathcal{G}$ be a group scheme over a field of characteristic $p > 0$ so that $\mathcal{G}$ satisfies a polynomial identity, then we ask if there exists a finite dimensional algebra R such that $Id(R) \subseteq Id(\mathcal{G})$? We highlight this is not that trivial in the following sense. Suppose further for any $A \in \mathbf{Alg}_F$ there exists a finite dimensional algebra R_A such that $Id(R_A) \subseteq Id(\mathcal{G}(A))$, then $\cap_{A\in \mathbf{Alg}_F} Id(R_A) \subseteq Id(\mathcal{G})$. From the classical theory of algebras with polynomial identities, if we denote by R the subdirect product of all of the R_A's, then $Id(R) \subseteq Id(\mathcal{G})$ but R rather fails to be finite dimensional. This explains why our question is far from being trivial. One more question: if A is the representative algebra of $\mathcal{G}$, are R and A related in some sense?

Declaration of competing interest

The authors declare that they have no conflict of interest.

Data availability

No data was used for the research described in the article.

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