

Ph.D. Thesis

Variational and Geometric Analysis for Quasilinear Elliptic Equations and Systems

Natalino Borgia

Supervised by

Prof. Silvia Cingolani



UNIVERSITÀ
DEGLI STUDI DI BARI
ALDO MORO

Many people think that mathematical ideas are static. They think that the ideas originated at some time in the historical past and remain unchanged for all future times. There are good reasons for such a feeling. After all, the formula for the area of a circle was πr^2 in Euclid's day and at the present time is still πr^2 . But to one who knows mathematics from the inside, the subject has rather the feeling of a living thing. It grows daily by the accretion of new information, it changes daily by regarding itself and the world from new vantage points, it maintains a regulatory balance by consigning to the oblivion of irrelevancy a fraction of its past accomplishments.

~ Philip J. Davis ~



UNIVERSITÀ
DEGLI STUDI DI BARI
ALDO MORO

Università degli Studi di Bari Aldo Moro

Department of Mathematics

Ph.D. in Computer Science and Mathematics

XXXVIII Cycle

Mathematical Analysis MATH-03/A

Variational and Geometric Analysis for Quasilinear Elliptic Equations and Systems

Ph.D. Candidate:
Natalino Borgia

Supervisor:
Prof. Silvia Cingolani

Ph.D. Coordinator:
Prof. Giovanna Castellano

Ph.D. Thesis, 2026

Abstract

In this thesis, we employ variational and geometric analysis to investigate quasilinear elliptic equations and systems arising from nonlinear phenomena with a significant environmental impact. We focus on quasilinear elliptic systems driven by various nonlinear operators, such as the p -Laplacian, and nonlinear sources that are allowed to exhibit both subcritical and critical growth. We aim to establish the existence of solutions for perturbation of specific eigenvalue problems, by employing variational and topological methods. To establish existence results for autonomous systems of quasilinear PDEs in the spirit of the celebrated paper by Amann and Zehnder [7], we develop a local Morse theory for functional associated to quasilinear elliptic systems. Our objective is to bridge relationship between the local behaviour of the functional near its critical points and differential notions like the Morse index. To this aim, by refining topological arguments introduced by Cingolani and Degiovanni in Banach product spaces in [56], we establish the finiteness of the critical groups and we derive a Poincaré-Hopf formula in a Banach product space, in presence of both subcritical and critical nonlinear coupling. Furthermore, we address the lack of regularity in the associated Euler functional by employing a penalization method combined with uniform L^∞ -estimates. We also establish uniform boundedness results for anisotropic quasilinear systems, that are of interest within regularity theory. To show existence results for non-autonomous systems of quasilinear PDEs in the spirit of the celebrated paper [118] of Landesman and Lazer, we consider the eigenvalue problem for quasilinear elliptic systems introduced by de Thélin in [74]. Leveraging the intrinsic geometric structure of the eigenvalue problem, we prove the simplicity and isolation of the first eigenvalue λ_1 . Furthermore, we show the existence of a sequence of eigenvalues by employing a suitable deformation lemma proved by Bonnet in [29]. Subsequently, we analyze a non-autonomous perturbation of the eigenvalue problem. In particular, the established spectral properties enable us to derive new sufficient Landesman-Lazer type conditions within the framework of quasilinear elliptic systems. While the quasilinear elliptic systems under consideration typically involve the p -Laplacian for $1 < p < N$, where N denotes the dimension of the space $\mathbb{R}^N$, we also investigate the geometric role of the N -Laplacian in the N -dimensional Euclidean Onofri inequality. This inequality was established by Del Pino and Dolbeault [77] for smooth functions with compact support. Specifically, after extending this inequality to a suitable weighted Sobolev space via a density argument, we exploit its connection with the Liouville equation on $\mathbb{R}^N$ to prove an equivalence with the sharp logarithmic Moser-Trudinger inequality on the unit ball $B_1 \subset \mathbb{R}^N$, implicitly contained in the work [42] of Carleson and Chang. For the specific case $N = 2$, we further highlight a geometric link to the unit sphere $\mathbb{S}^2$ through the stereographic projection.

Abstract

In questa tesi, utilizziamo l'analisi variazionale e geometrica per studiare equazioni e sistemi quasilineari ellittici che nascono da fenomeni non lineari con un impatto significativo sull'ambiente. Ci concentriamo su sistemi quasilineari ellittici guidati da diversi operatori non lineari come il p -Laplaciano, e sorgenti non lineari che possono presentare crescita sia sottocritica che critica. Il nostro obiettivo è stabilire l'esistenza di soluzioni per una perturbazione di specifici problemi agli autovalori, utilizzando metodi sia variazionali che topologici. Per stabilire risultati di esistenza per sistemi autonomi di PDE quasilineari nello spirito del celebre lavoro di Amann e Zehnder [7], sviluppiamo una teoria di Morse locale per funzionali associati a sistemi quasilineari ellittici. Il nostro obiettivo è creare un collegamento tra il comportamento locale del funzionale vicino i suoi punti critici e nozioni differenziali come l'indice di Morse. A tal fine, affinando gli argomenti topologici introdotti da Cingolani e Degiovanni in [56] per spazi di Banach prodotto, stabiliamo la finitezza dei gruppi critici e deriviamo una formula di Poincaré-Hopf in uno spazio di Banach prodotto, in presenza di accoppiamenti non lineari sia sottocritici che critici. Inoltre, affrontiamo la mancanza di regolarità nel funzionale associato utilizzando un metodo di penalizzazione combinato con stime L^∞ -uniformi. Stabiliamo inoltre risultati di limitatezza uniforme per sistemi quasilineari anisotropi, che rivestono un certo interesse nell'ambito della teoria della regolarità. Per mostrare risultati di esistenza per sistemi non-autonomi di PDE quasilineari nello spirito del celebre lavoro [118] di Landesman e Lazer, consideriamo il problema agli autovalori per sistemi quasilineari ellittici introdotto da de Thélin in [74]. Facendo leva sulla struttura geometrica intrinseca del problema agli autovalori, proviamo la semplicità e l'isolatezza del primo autovalore λ_1 . Inoltre, mostriamo l'esistenza di una successione di autovalori utilizzando un opportuno lemma di deformazione, dimostrato da Bonnet in [29]. Successivamente, analizziamo una perturbazione non-autonoma del problema agli autovalori. In particolare, le proprietà spettrali ottenute ci permettono di derivare nuove sufficienti condizioni di tipo Landesman-Lazer all'interno del framework di sistemi quasilineari ellittici. Mentre i sistemi in considerazione tipicamente coinvolgono il p -Laplaciano per $1 < p < N$, dove N denota la dimensione dello spazio $\mathbb{R}^N$, il ruolo geometrico dell' N -Laplaciano lo investighiamo nella disuguaglianza di Onofri euclidea N -dimensionale. Questa disuguaglianza è stata ottenuta da Del Pino e Dolbeault in [77] per funzioni regolari a supporto compatto. Nello specifico, dopo aver esteso la disuguaglianza ad un opportuno spazio di Sobolev pesato tramite un argomento di densità, sfruttiamo la connessione con l'equazione di Liouville in $\mathbb{R}^N$ per dimostrare un risultato di equivalenza con la ottimale disuguaglianza logaritmica di Moser-Trudinger sulla palla unitaria $B_1 \subset \mathbb{R}^N$, contenuta implicitamente nel lavoro [42] di Carleson e Chang. Nel caso specifico di $N = 2$, sottolineiamo il collegamento con la sfera unitaria due-dimensionale $\mathbb{S}^2$ attraverso la proiezione stereografica.

Notation

- We denote by $N \in \mathbb{N}$ the dimension of the space $\mathbb{R}^N$. In particular, $N \geq 2$ unless otherwise stated.
- We denote by $\Omega \subset \mathbb{R}^N$ a bounded domain (open connected set) with smooth boundary $\partial\Omega$.
- We denote by B_1 the open unit ball of $\mathbb{R}^N$, and we write V_N for its volume and ω_{N-1} for the measure of its surface.
- Given a set A , we denote by $|A|$ its Lebesgue measure.
- Given a function $f : U \rightarrow \mathbb{R}$ with $U \subseteq \mathbb{R}^N$, for any $i = 1, \dots, N$ we write the first partial derivative as $\frac{\partial f}{\partial x_i}$. We denote by $\nabla f(x)$ the gradient of f evaluated at x .
- For $w : \Omega \rightarrow \mathbb{R}$ and $\Psi : \mathbb{R}^N \rightarrow \mathbb{R}$, we denote by $\nabla \Psi(\nabla w(x))$ the gradient of Ψ evaluated at $\nabla w(x)$.
- For any $r > 1$, we denote by $L^r(\Omega)$ the standard Lebesgue space of measurable functions $f : \Omega \rightarrow \mathbb{R}$ such that $\int_{\Omega} |f|^r dx < +\infty$. We denote by $\|\cdot\|_r$ the usual norm in $L^r(\Omega)$.
- For any $r > 1$, we denote by $W_0^{1,r}(\Omega)$ the standard Sobolev space of functions in $L^r(\Omega)$, vanishing on the boundary, whose first-order weak derivatives also belong to $L^r(\Omega)$. We denote by $\|\cdot\|_{1,r}$ the usual gradient norm in $W_0^{1,r}(\Omega)$. Moreover, we denote by $W^{-1,r'}(\Omega)$ the dual space of $W_0^{1,r}(\Omega)$.
- For any $r > 1$, we denote by $r^* := Nr/(N-r)$ the critical Sobolev exponent of r , and by $r' := r/(r-1)$ the conjugate exponent.
- Given $1 < p, q < N$, we denote by X the Banach product space $X := W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. We equip the space X with the norm

$$\|z\| = \|u\|_{1,p} + \|v\|_{1,q}, \quad z = (u, v) \in X.$$

In particular, X is not uniformly convex under the norm defined above (see [63]).

- We denote by $(\mathbb{S}^2, g_0)$ the two-dimensional unit sphere $\mathbb{S}^2$ endowed with the standard metric g_0 , which is represented by the matrix

$$g_0(x) := \begin{pmatrix} \lambda(x) & 0 \\ 0 & \lambda(x) \end{pmatrix}, \quad \text{where } \lambda : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad \lambda(x) := \frac{4}{(1 + |x|^2)^2}.$$

- We denote by $W^{1,2}(\mathbb{S}^2, g_0)$ the Sobolev space consisting of functions that, along with their distributional first derivatives, are square-integrable with respect to the Riemannian volume form $d\nu_{g_0} := \lambda(x)dx$.
- We denote by $\nabla_{g_0} f$ the gradient of f with respect to the metric g_0 .

Contents

Introduction	ii
1 A Poincaré-Hopf formula for functional associated to Quasilinear Elliptic Systems	1
1.1 Browder degree in Banach spaces	1
1.2 (p, q) -area operators	3
1.3 $(S)_+$ -condition	5
1.4 A Poincaré-Hopf formula for systems	16
Appendix A	18
2 Anisotropic Quasilinear Systems and Regularity Results	20
2.1 Assumptions and Main results	20
2.2 L^τ -Integrability for any $1 < \tau < \infty$	23
2.3 Uniformly locally L^∞ -estimate	30
Appendix B	34
3 Amann-Zehnder type results for Resonance Quasilinear Elliptic Systems	37
3.1 Amann-Zehnder type results	37
3.2 PS and CPS conditions	42
3.3 Cohomological linking	48
3.4 Critical Groups Estimates at zero	52
3.5 Geometry of J_a and Proof of the main results	56
4 On the de Thélin eigenvalue problem and Landesman-Lazer conditions	62
4.1 de Thélin eigenvalue problem	62
4.2 Simplicity of λ_1	68
4.3 Isolation of λ_1	70
4.4 A sequence of eigenvalues	75
4.5 Palais-Smale condition of J	78
4.6 Geometry of J	84
Appendix C	90
5 N-dimensional Euclidean Onofri Inequality in Weighted Sobolev Space	93
5.1 Euclidean Onofri Inequality in $C_0^\infty(\mathbb{R}^N)$	93
5.2 Euclidean Onofri Inequality in $W_{\mu_N}(\mathbb{R}^N)$	97
5.3 Logarithmic Moser-Trudinger inequality on the balls of $\mathbb{R}^N$	108
5.4 Equivalence between Euclidean Onofri Inequality and Logarithmic Moser-Trudinger Inequality	112
5.5 Insight on $N = 2$: connection with $\mathbb{S}^2$	121
Appendix D	125
Bibliography	129

Introduction

Several nonlinear partial differential equations (PDEs) and systems arise from nonlinear phenomena with significant environmental impacts. In this thesis, we investigate quasilinear elliptic systems driven by different operators, involving isotropic and anisotropic ones, which come from several nonlinear physical processes, such as non-Newtonian mechanics, nonlinear elasticity, glaciology, combustion theory, and population biology (see for instance [83, 103, 116, 124, 125, 138, 146]). While isotropic operators act uniformly in all directions, anisotropic operators exhibit a dependency on the spatial direction, introducing a bias or preference toward specific axes or orientations. The most common nonlinear isotropic operator is the p -Laplacian Δ_p , which generalizes the standard Laplacian Δ and is defined for any $1 < p < \infty$ as $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$. In particular, ∇u identifies the direction of steepest ascent, while its magnitude $|\nabla u|$ quantifies the slope's intensity. The term $|\nabla u|^{p-2}$ acts as a weight, or stiffness factor, determined by the gradient itself. The divergence operator, “div”, characterizes the flux from a point. The case $p = 2$ recovers the standard linear Laplacian Δ . For $p > 2$, the operator is degenerate, exhibiting increased stiffness in regions with large gradients and becoming ‘softer’ where the gradient is small. Conversely, for $1 < p < 2$, the operator is singular and tends to blow up as the gradient approaches zero. From a physical standpoint, the most prominent application lies in the transition from Newtonian to non-Newtonian fluid dynamics [18]. While the standard Laplacian Δ describes Newtonian flows, the p -Laplacian Δ_p characterizes non-Newtonian behavior, where viscosity is a function of the applied stress. Specifically, for $1 < p < 2$, the fluid is pseudoplastic (shear-thinning), exhibiting decreased viscosity as the shear rate increases. Conversely, for $p > 2$, the fluid is dilatant (shear-thickening), showing increased resistance in response to greater applied force. Specifically, this thesis focuses on the analytical and geometrical aspects of properties arising from the p -Laplacian, which represents the core link between the different problems addressed here. From the mathematical point of view, the study of quasilinear PDEs presents several specific difficulties and challenging theoretical problems.

A first target of this thesis is to develop a local Morse theory for functional associated to quasilinear elliptic systems in presence of subcritical and critical nonlinear coupling. We aim to establish a relationship between the local behaviour of the functional near its critical points and differential notions like the Morse index. In particular, we aim to derive a Poincaré-Hopf formula in a Banach product space. We recall that in the Hilbert space setting, the celebrated Poincaré-Hopf formula has been extended to functionals f whose gradient ∇f is a compact perturbation of the identity. This extension allows for the application of Leray-Schauder degree theory, rather than the Brouwer degree typically used for C^1 functions $f : \mathbb{R}^N \rightarrow \mathbb{R}$ (see, e.g., [22, 23, 46, 120, 129]). When Y is a Banach space (not necessarily Hilbert), the derivative f' is naturally defined from Y into its dual Y' . Consequently, this does not fit the standard framework of Leray-Schauder degree, but rather that of the Browder degree developed in [37] for demicontinuous maps of class $(S)_+$. In our context, the functional setting is defined by the Banach product space $X := W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$, where $1 < p, q < N$ and $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a smooth bounded domain. Specifically, we consider the following quasilinear elliptic system:

$$\begin{cases} -\operatorname{div} [\nabla \Psi_{a,p}(\nabla u)] = F_s(u, v) & x \in \Omega, \\ -\operatorname{div} [\nabla \Psi_{b,q}(\nabla v)] = F_t(u, v) & x \in \Omega, \\ u = v = 0 & x \in \partial\Omega, \end{cases} \quad (\text{I.1})$$

where, given $c \geq 0$ and $r > 1$, we denote by $\Psi_{c,r} : \mathbb{R}^N \rightarrow \mathbb{R}$ the function defined as

$$\Psi_{c,r}(\xi) := \frac{1}{r} \left[\left(c + |\xi|^2 \right)^{\frac{r}{2}} - c^{\frac{r}{2}} \right].$$

Hence, in (I.1) we consider $\Psi_{a,p}$ and $\Psi_{b,q}$ for $1 < p, q < N$ and $a, b \geq 0$. The nonlinearity $F \in C^1(\mathbb{R}^2)$ is allowed to exhibit both subcritical and critical growth. We show that the Euler C^1 -functional $J_{a,b} : X \rightarrow \mathbb{R}$ associated with (I.1) has a Fréchet derivative $J'_{a,b} : X \rightarrow X'$ that is locally a map of class $(S)_+$. By employing topological arguments introduced by Cingolani and Degiovanni in [56], we establish the finiteness of the critical groups of $J_{a,b}$, thereby extending results previously obtained for quasilinear elliptic equations at critical growth to the case of quasilinear systems at critical growth. Furthermore, the use of the Browder degree [37] enables us to establish a Poincaré–Hopf formula in the setting of quasilinear elliptic systems.

An other target of this thesis is to show existence results for autonomous systems of quasilinear PDEs in the spirit of the celebrated paper by Amann and Zehnder [7]. The main idea in [7] is that if the "linearized" behavior at the origin is sufficiently different from the behavior at infinity (measured by their respective Morse indices), then a non-trivial solution must exist. To extend this approach to systems within a Banach framework, we employ several innovative techniques. First, we establish a general abstract Linking Theorem, extending the result of Cingolani, Degiovanni and Vannella [58, Theorem 7.1] concerning symmetric cones to a broader setting of (p, q) -homogeneous sets. Second, the nonlocal coupling induces a lack of regularity in the functional. This phenomenon can be overcome by using a penalization method and uniform regularity estimates. A Chapter of the thesis is therefore devoted to establish uniform L^∞ -estimates results for systems involving anisotropic quasilinear elliptic equations. Consequently, we investigate the following non-autonomous system:

$$\begin{cases} -\operatorname{div} [\nabla \Psi_1(\nabla u)] = H_s(\delta, x, u, v) & \text{in } \Omega, \\ -\operatorname{div} [\nabla \Psi_2(\nabla v)] = H_t(\delta, x, u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \quad (\text{I.2})$$

where Ω and N are defined as above. The nonlinearity H is allowed to exhibit both subcritical and critical growth and may depend on a parameter $\delta \in I \subseteq \mathbb{R}$. Moreover, $\Psi_1, \Psi_2 : \mathbb{R}^N \rightarrow \mathbb{R}$ are convex functions satisfying quite general conditions for $1 < p, q < N$, as detailed in Chapter 2; these conditions allow for the recovery of the previously introduced operators $\Psi_{a,p}$ and $\Psi_{b,q}$. We establish the L^∞ -boundedness of all solutions to (I.2) and we prove that these solutions are uniformly bounded with respect to δ within any sufficiently small ball of the functional space setting X . To address the singularities arising when $1 < p < 2$ or $1 < q < 2$, we employ a suitable generalization of Stampacchia's Lemma (see [25]). Furthermore, we handle the possibly critical growth of H by proving results that hold up to the critical threshold, extending those established in [161]. We stress that in quasilinear elliptic systems of the form (I.2), an additional difficulty in proving such results is the possible coupling of the functions u and v in the nonlinearity H .

Building upon these preliminary results, we pass to detect nontrivial solutions for quasilinear systems where the nonlinearity interacts with the spectrum of an associated eigenvalue problem, in the spirit of the celebrated paper [7] of Amann and Zehnder. We analyze the following autonomous quasilinear system:

$$\begin{cases} -\operatorname{div} [\nabla \Psi_{a,p}(\nabla u)] = G_s(u, v) & x \in \Omega, \\ -\operatorname{div} [\nabla \Psi_{a,q}(\nabla v)] = G_t(u, v) & x \in \Omega, \\ u = v = 0 & x \in \partial\Omega, \end{cases} \quad (\text{I.3})$$

where Ω is as before, $2 \leq p, q < N$, $a \geq 0$, and $G \in C^1(\mathbb{R}^2, \mathbb{R})$ is a (p, q) -asymptotically linear term at infinity, with $\nabla G(0, 0) = (0, 0)$, and it is of class C^2 in a suitable neighborhood of the origin. Since $(0, 0) \in X$ is an a priori trivial solution of (I.3), we aim to find nontrivial solutions in both the asymptotic resonant and nonresonant cases, relative to the following eigenvalue problem:

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u + \frac{\lambda}{\beta+1} |u|^\alpha |v|^\beta v & x \in \Omega, \\ -\Delta_q v = \lambda |v|^{q-2} v + \frac{\lambda}{\alpha+1} |u|^\alpha |v|^\beta u, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \quad (\text{I.4})$$

where Ω, N, p, q are as before and $\alpha, \beta > 0$ are real numbers satisfying $(\alpha + 1)/p + (\beta + 1)/q = 1$.

As mentioned, since the functional is defined on the product space $X = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$, we exploit recent Morse-theoretic techniques in Banach spaces (see [59, 60, 62, 160]); moreover, we derive a Saddle Linking Theorem specifically tailored to (p, q) -homogeneous sets. To prove the non-triviality of the solution, we compute the critical groups at the origin using differential notions. However, applying Morse arguments in Banach spaces presents severe difficulties, as the classical Morse Lemma and the generalized Morse Lemma [104] are not directly applicable in this context. Moreover, the energy functional J_a associated with (I.3) is not of class C^2 ; hence the classical notion of Morse index cannot be defined. To overcome this lack of regularity, we introduce a suitable quadratic form $Q : X \rightarrow \mathbb{R}$, such that m_0 and m_0^* , defined as the supremum of the dimensions of the subspaces of X where Q is negative definite and negative semidefinite respectively, could serve as the Morse index and the large Morse index. Finally, we relate these differential notions m_0 and m_0^* to the behavior of a penalized version of J_a , which satisfies C^2 regularity only within a ball centered at the origin. It is still open the singular cases and the mixed interaction of singular and degenerate quasilinear PDEs.

In the third Chapter we pass to study spectral properties of the following eigenvalue problem for quasilinear elliptic systems, originally introduced by de Th  lin in [74]:

$$\begin{cases} -\Delta_p u = \lambda |u|^{\alpha-1} |v|^{\beta+1} u & \text{in } \Omega, \\ -\Delta_q v = \lambda |u|^{\alpha+1} |v|^{\beta-1} v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \quad (\text{I.5})$$

where Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary, $N \geq 2$, $\lambda \in \mathbb{R}$, $1 < p, q < N$ and $\alpha, \beta > 0$ are real numbers satisfying $(\alpha + 1)/p + (\beta + 1)/q = 1$. In [74, Theorem 1] de Th  lin proved that problem (I.5) admits a smallest eigenvalue $\lambda_1 > 0$. In this thesis we show that the first eigenvalue λ_1 is simple in a suitable sense, a result largely contained in [74], and subsequently we prove that λ_1 is isolated, following the approach in [5, Theorem 1] and employing Picone's identity (see [15, Lemma 24]). Furthermore, we provide a variational characterization for a sequence of eigenvalues $\{\lambda_k\}_k$ by applying a suitable deformation lemma for C^1 -submanifolds proved by Bonnet in [29]. In addition, we use spectral properties on λ_1 and λ_2 to address, in the spirit of the pioneering paper [118] of Landesman and Lazer, the following non-autonomous perturbation of problem (I.5):

$$\begin{cases} -\Delta_p u = \lambda_1 |u|^{\alpha-1} |v|^{\beta+1} u + \frac{1}{\alpha+1} [F_s(x, u, v) - h_1(x)] & x \in \Omega, \\ -\Delta_q v = \lambda_1 |u|^{\alpha+1} |v|^{\beta-1} v + \frac{1}{\beta+1} [F_t(x, u, v) - h_2(x)] & x \in \Omega, \\ u = v = 0 & x \in \partial\Omega, \end{cases} \quad (\text{I.6})$$

where Ω , N , p , q , α and β are defined as above. The nonlinearity $F : \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a C^1 -Carathéodory function, while $h_1 \in L^{p'}(\Omega)$ and $h_2 \in L^{q'}(\Omega)$. Inspired from the work [16] of Arcoya and Orsina, we prove the existence of a weak solution for the perturbed problem (I.6). This is achieved under sufficient Landesman-Lazer conditions involving p, q, F, h_1, h_2 , and the component-wise positive eigenfunction (φ_1, ψ_1) associated with λ_1 , normalized such that $\|\varphi_1\|_{1,p} + \|\psi_1\|_{1,q} = 1$. These conditions, that will be specified in detail in Chapter 4, ensure that the "nonlinear interference due to F " and the "external force" (h_1, h_2) balanced out in a way that prevents the solution from sliding off to infinity. In particular, while the Euler functional associated with (I.6) always satisfies the Palais-Smale (PS) condition, it is coercive only in certain cases. On the other hand, when coercivity is lacking, verifying that the functional exhibits the geometric structure required by the Saddle Point Theorem is not straightforward. Indeed, since the set of eigenfunctions associated with λ_1 for quasilinear systems is generally not a vector subspace of X , the standard splitting argument employed in [16] cannot be directly applied here. To overcome this difficulty, we exploit the isolation of λ_1 and the variational characterization of λ_2 to recover the saddle point geometry and thereby ensure the existence of a solution.

In the quasilinear systems discussed previously, we focused on the p -Laplacian (or its generalizations) where $p < N$. However, the N -Laplacian operator Δ_N also plays a fundamental role in quasilinear elliptic equations with an underlying geometric structure. For instance, the N -Laplacian arises in the following N -dimensional Liouville equation

$$\begin{cases} -\Delta_N U(x) = e^{U(x)} & x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} e^{U(x)} dx < +\infty. \end{cases} \quad (\text{I.7})$$

Denoting with $c_N := N^N \left(\frac{N}{N-1}\right)^{N-1}$, this problem has an explicit solution given by

$$U(x) = \ln \left(\frac{c_N}{\left(1 + |x|^{\frac{N}{N-1}}\right)^N} \right).$$

Specifically, defining $\mu_N(x) := \frac{1}{V_N \left(1 + |x|^{\frac{N}{N-1}}\right)^N}$, the function $v_N(x) := \ln \mu_N(x)$ is connected to

Liouville equation (I.7) because

$$-\Delta_N v_N(x) = c_N V_N e^{v_N(x)}.$$

The functions $\mu_N(x)$ and $v_N(x)$ arise in the N -dimensional Euclidean Onofri inequality established by Del Pino and Dolbeault in [77]. By considering the endpoint of the optimal Gagliardo–Nirenberg interpolation inequalities, first discovered in [75] and later extended in [76], they proved the N -dimensional Euclidean Onofri inequality for any $u \in C_0^\infty(\mathbb{R}^N)$:

$$\ln \left(\int_{\mathbb{R}^N} e^u d\mu_N \right) \leq \frac{1}{\omega_N} \int_{\mathbb{R}^N} H_N(u, \mu_N) dx + \int_{\mathbb{R}^N} u d\mu_N, \quad (\text{I.8})$$

where $\widetilde{\omega}_N := N^N \left(\frac{N}{N-1} \right)^{N-1} \omega_{N-1}$ and

$$H_N(u, \mu_N) := |\nabla v_N + \nabla u|^N - |\nabla v_N|^N - N|\nabla v_N|^{N-2} \nabla v_N \cdot \nabla u.$$

The function v_N is involved in the definition of the aforementioned operator; specifically, the connection with the Liouville equation (I.7) is established via the term $N|\nabla v_N|^{N-2} \nabla v_N \cdot \nabla u$ and the Gauss-Green formula. The N -dimensional Euclidean Onofri inequality was achieved with different techniques also by Agueh, Boroushaki and Ghoussoub in [3] for any $u \in W^{1,N}(\mathbb{R}^N)$. In this thesis, we are interested in determining the natural space in which such inequality remains valid. We then introduce the weighted Sobolev space

$$W_{\mu_N}(\mathbb{R}^N) := \{u \in L^1(\mathbb{R}^N, d\mu_N) : |\nabla u| \in L^N(\mathbb{R}^N, dx), |\nabla u|^2 |\nabla v_N|^{N-2} \in L^1(\mathbb{R}^N, dx)\},$$

endowed with the norm

$$\|u\|_{\mu_N} := \int_{\mathbb{R}^N} |u| d\mu_N + \|\nabla u\|_N + \left(\int_{\mathbb{R}^N} |\nabla u|^2 |\nabla v_N|^{N-2} dx \right)^{\frac{1}{2}}.$$

We extend inequality (I.8) to this setting by proving that $C_0^\infty(\mathbb{R}^N)$ is dense in $W_{\mu_N}(\mathbb{R}^N)$ with respect to the norm $\|\cdot\|_{\mu_N}$. Furthermore, we establish an equivalence with the following sharp logarithmic Moser-Trudinger inequality on the unit ball $B_1 \subset \mathbb{R}^N$, which can be derived from the work of Carleson and Chang [42]:

$$\ln \left(\frac{1}{|B_1|} \int_{B_1} e^u dx \right) < \frac{1}{\widetilde{\omega}_N} \int_{B_1} |\nabla u|^N dx + \sum_{k=1}^{N-1} \frac{1}{k} \quad \forall u \in W_0^{1,N}(B_1). \quad (\text{I.9})$$

The case $N = 2$ is of particular interest from both mathematical and physical perspectives. In fact inequality (I.8) becomes

$$\ln \left(\int_{\mathbb{R}^2} e^u d\mu_2 \right) \leq \frac{1}{16\pi} \int_{\mathbb{R}^2} |\nabla u|^2 dx + \int_{\mathbb{R}^2} u d\mu_2$$

for any $u \in W_{\mu_2}(\mathbb{R}^2) := \{u \in L^1(\mathbb{R}^2, d\mu_2) : |\nabla u| \in L^2(\mathbb{R}^2, dx)\}$. Furthermore, via stereographic projection, this inequality is equivalent to the Onofri inequality (see [134]) on the two-dimensional unit sphere $\mathbb{S}^2$ equipped with the standard metric g_0 :

$$\ln \left(\frac{1}{4\pi} \int_{\mathbb{S}^2} e^u dv_{g_0} \right) \leq \frac{1}{16\pi} \int_{\mathbb{S}^2} |\nabla_{g_0} u|_{g_0}^2 dv_{g_0} + \frac{1}{4\pi} \int_{\mathbb{S}^2} u dv_{g_0} \quad \forall u \in W^{1,2}(\mathbb{S}^2, g_0). \quad (\text{I.10})$$

Onofri inequality plays an important role in spectral analysis for the Laplace-Beltrami operator thanks to Polyakov's formula (see [135, 136, 142, 143]) and it was proved in 1982 by Onofri in [134] using conformal invariance and an earlier result by Aubin [19]. Inequality (I.10) can be obtained via several methods, including various limiting procedures, mass transportation methods in the radial case, and rigidity methods associated with a related nonlinear flow (we refer to the papers by Dolbeault, Esteban, Jankowiak, and Tarantello [88, 89], and the references therein for full proofs and further details). The Onofri inequality also has important applications; for instance, in chemotaxis it appears in the study of the Keller-Segel model (see [39, 101]).

Concerning inequality (I.9), in dimension $N = 2$ it becomes

$$\ln \left(\frac{1}{\pi} \int_{B_1} e^u dx \right) < \frac{1}{16\pi} \int_{B_1} |\nabla u|^2 dx + 1 \quad \forall u \in W_0^{1,2}(B_1). \quad (\text{I.11})$$

A generalization of previous inequality was provided in [38] (see also [47]). Specifically, if $\Omega \subseteq \mathbb{R}^2$ is a bounded domain with smooth boundary, then for any $u \in W_0^{1,2}(\Omega)$ we have

$$\ln \left(\frac{1}{|\Omega|} \int_{\Omega} e^u dx \right) \leq \frac{1}{16\pi} \int_{\Omega} |\nabla u|^2 dx + 1 + 4\pi \sup_{\Omega} \gamma(x) + \ln \frac{\pi}{|\Omega|}, \quad (\text{I.12})$$

where γ is the Robin's function of Ω . This inequality plays a crucial role in the study of the mean field equation

$$-\Delta u = \frac{\rho e^u}{\int_{\Omega} e^u dx} \quad \text{in } \Omega, \quad (\text{I.13})$$

which appears in the statistical mechanic description of vortex formations in 2d-models for turbulent flows in fluid dynamics (see e.g. [38, 111]). We also mention [20, 21, 51, 52, 123] for existence results for (I.13) and related problems.

The thesis is organized as follows. Chapter 1 is devoted to show the finiteness of the critical groups and a local Poincaré-Hopf type formula for functionals associated to quasilinear elliptic systems. In Chapter 2 we present new results concerning L^∞ -estimates for solutions to non-autonomous quasilinear systems involving operators in divergence form and nonlinearities up to critical growth. Building upon results accomplished in previous Chapters, in Chapter 3 we deal with Amann-Zehnder type results for quasilinear elliptic systems, in order to detect a nontrivial weak solution in both resonant and nonresonant case. Chapter 4 is devoted to Landesman-Lazer type conditions for quasilinear elliptic systems, after proving spectral properties on the de Thélin eigenvalue problem associated. Finally, Chapter 5 is devoted to extending the N -dimensional Euclidean Onofri inequality to the weighted Sobolev space $W_{\mu_N}(\mathbb{R}^N)$. Furthermore, we establish its equivalence with the sharp logarithmic Moser-Trudinger inequality on the unit ball $B_1 \subset \mathbb{R}^N$, providing an insight into the case $N = 2$ that highlights the underlying geometric link with the two-dimensional unit sphere $\mathbb{S}^2$.

This thesis is based on the papers [14, 30–34].

A Poincaré-Hopf formula for functional associated to Quasilinear Elliptic Systems

In this Chapter we present some new results about the finiteness of the critical groups and a local Poincaré-Hopf type formula for functionals associated to quasilinear elliptic systems, involving p -Laplacian or p -area operators. In particular, we show that the Fréchet derivative of such a functionals is locally of class $(S)_+$ when the nonlinearity exhibits critical growth, so that we extend the results proved in [56] by Cingolani and Degiovanni for quasilinear elliptic equations at critical growth.

This Chapter is based on the paper [33].

1.1 Browder degree in Banach spaces

In the context of Hilbert spaces, the celebrated Poincaré-Hopf formula has been extended for functionals f such that the map ∇f is a compact perturbation of the identity. This allows to use the Leray-Schauder degree theory, instead of the Brouwer degree for C^1 functions $f : \mathbb{R}^N \rightarrow \mathbb{R}$ (see e.g. [129, Theorem 8.5], [46, Theorem 3.2], [120, Theorem 3.2], [22, Theorem 1.3] and [23]).

Conversely if Y is a Banach (not Hilbert) space, then the derivative f' is naturally defined from Y into Y' and this is not the usual setting of Leray-Schauder degree. In [114], Kim and Wang obtained a Poincaré-Hopf type result in the special case in which the Banach space is densely embedded in a Hilbert space and the functional f induces a vector field ∇f on the Banach space through the inner product of the Hilbert space. They applied extensions of the Leray-Schauder degree for maps defined from the Banach space into itself.

In [56], Cingolani and Degiovanni established a Poincaré-Hopf formula in a general reflexive Banach space Y . Their main idea was to replace the Leray-Schauder degree by the Browder degree developed in [37] for maps that are demicontinuous (namely continuous from the strong topology of the domain to the weak topology of the codomain) and of class $(S)_+$, that is:

Definition 1.1.1. Let $(Y, \|\cdot\|)$ be a reflexive Banach space, whose dual is denoted by Y' . Let $D \subseteq Y$ and let us consider a map $F : D \rightarrow Y'$. F is said to be of class $(S)_+$ if, for every sequence $\{w_k\}_k$ in D weakly convergent to w in Y with

$$\limsup_k \langle F(w_k), w_k - w \rangle \leq 0,$$

we have $\|w_k - w\| \rightarrow 0$.

We notice that if Y is a Hilbert space and Y' is identified with Y in the standard way, then any compact perturbation of the identity is of class $(S)_+$. However, the class $(S)_+$ allows to consider further interesting cases, also in the Hilbert setting.

As noticed in [56, 78], any compact perturbation of a map of class $(S)_+$ is still of class $(S)_+$. Therefore the class includes the perturbations of the p -Laplace operator by terms with subcritical growth. We refer to [86, Theorem 10] where $-\Delta_p : W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega)$ is continuous and of class $(S)_+$. In [56] the case with critical growth has been covered, which is interesting also for $p = 2$. We also mention the paper [6] where Almi and Degiovanni showed how the degree for maps of class $(S)_+$ can be used to define, by a suitable approximation technique, a degree for quasilinear elliptic equations with natural growth conditions.

In this Chapter we prove the finiteness of the critical groups and a local Poincaré-Hopf type formula for the functional $J_{a,b} : X \rightarrow \mathbb{R}$ defined for any $z = (u, v) \in X$ as

$$J_{a,b}(z) = \frac{1}{p} \int_{\Omega} (a + |\nabla u|^2)^{\frac{p}{2}} dx + \frac{1}{q} \int_{\Omega} (b + |\nabla v|^2)^{\frac{q}{2}} dx - \int_{\Omega} F(u, v) dx, \quad (1.1.1)$$

where $1 < p, q < N$ and $a, b \geq 0$. Moreover, we assume that $F \in C^1(\mathbb{R}^2)$ and that there exists $C > 0$ such that for any $(s, t) \in \mathbb{R}^2$ we have

$$|F_s(s, t)| \leq C \left(|s|^{p^*-1} + |t|^{q^* \frac{p^*-1}{p^*}} + 1 \right), \quad (1.1.2)$$

$$|F_t(s, t)| \leq C \left(|s|^{p^* \frac{q^*-1}{q^*}} + |t|^{q^*-1} + 1 \right). \quad (1.1.3)$$

Previous assumptions and Nemytskii operator theory give that $J_{a,b}$ is of class C^1 on X .

As in [56] for the p -Laplace operator (see also [6, 58]), we show that also the case with critical growth for the functional $J_{a,b}$, associated to a quasilinear system, gives rise locally to an operator of class $(S)_+$. However, in our setting we have to face some additional difficulties in order to show that the Fréchet derivative $J'_{a,b}$ of the Euler functional associated to a quasilinear system is locally an operator of class $(S)_+$. More precisely, we generalize to a (p, q) -area type operator which returns us the (p, q) -Laplacian as a special case. About the nonlinearity F , in a quasilinear system setting new difficulties arise in computations due to the coupling of u with v , and we are able to manage them also when F is allowed to grow critically. Therefore, we partially answer to question (i) in the final comments of [26], where nonlinearity that exhibits critical growth is not treated because of a loss of compactness.

Precisely, we will prove the following.

Theorem 1.1.2. *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Let $1 < p, q < N$ and $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ a C^1 -function satisfying (1.1.2) and (1.1.3). Let $a, b \geq 0$ and $J_{a,b}$ the C^1 -functional defined in (1.1.1). Then there exists a radius $R := R(N, p, q, C)$ such that, for every $z_0 = (u_0, v_0) \in X$, the map $J'_{a,b}$ is of class $(S)_+$ on*

$$\overline{B_R(z_0)} := \{z \in X : \|z - z_0\| \leq R\}.$$

Let us now recall that for any $q \in \mathbb{N}$, the q -th critical group of a functional f at $\bar{u} \in \text{Dom} f$ is defined as

$$C_q(f, \bar{u}) = H_q(\{f \leq f(\bar{u})\}, \{f \leq f(\bar{u})\} \setminus \{\bar{u}\}),$$

where H_* denotes singular homology (see e.g. [44, 46, 129, 147, 148]).

By applying Theorem 1.1.2 and retracing the same ideas contained in [56, Theorem 1.1], we also obtain the finiteness of the critical groups and a local Poincaré-Hopf formula for the functional $J_{a,b}$, that is:

Theorem 1.1.3. *If $\bar{z}$ is an isolated critical point of the functional $J_{a,b}$ defined in (1.1.1), then $C_*(J_{a,b}, \bar{z})$ is of finite type and we have*

$$\deg(J'_{a,b}, V, 0) = \sum_{m \geq 0} (-1)^m \operatorname{rank} C_m(J_{a,b}, \bar{z})$$

for every sufficiently small neighborhood V of $\bar{z}$.

We mention [43] for the computation of the critical groups of $J_{a,b}$ when $a = b$ and $2 < p, q < N$. Furthermore, in Chapter 3 the computation of the critical groups is needed to obtain Amann-Zender type results for quasilinear systems. It is still an open problem the evaluation of the critical groups for the functional $J_{a,b}$ when p or q are less than 2, or $a \neq b$. Let us refer the reader to [2, 17, 55, 57–60] for critical groups estimates for quasilinear elliptic equations (see also [140]).

The following Chapter is organized as follows: in Section 1.2 we show that p -area and q -area operators are of class $(S)_+$, even when we consider their simultaneous action in the product space X . Section 1.3 is devoted to the proof of Theorem 1.1.2. Finally, Section 1.4 is devoted to the proof of Theorem 1.1.3.

1.2 (p, q) -area operators

We begin to recognize that the r -area operator is of class $(S)_+$ for $r > 1$. More precisely, for any $c \geq 0$ and $r > 1$, we consider the map $H_{c,r} : W_0^{1,r}(\Omega) \rightarrow W^{-1,r'}(\Omega)$ defined for any $w \in W_0^{1,r}(\Omega)$ by

$$H_{c,r}(w) := -\operatorname{div} \left(\left(c + |\nabla w|^2 \right)^{\frac{r-2}{2}} \nabla w \right),$$

where the action of the map $H_{c,r}(w) : W_0^{1,r}(\Omega) \rightarrow \mathbb{R}$ is given for any $\phi \in W_0^{1,r}(\Omega)$ by

$$\langle H_{c,r}(w), \phi \rangle := \int_{\Omega} \left(c + |\nabla w|^2 \right)^{\frac{r-2}{2}} \nabla w \cdot \nabla \phi \, dx.$$

Lemma 1.2.1. *For any $c \geq 0$ and $r > 1$, the map $H_{c,r} : W_0^{1,r}(\Omega) \rightarrow W^{-1,r'}(\Omega)$ is of class $(S)_+$.*

Proof. To get the thesis, we will apply [6, Theorem 3.5].

Let $c \geq 0$ and $r > 1$ and let us consider the map $d_{c,r} : \mathbb{R}^N \rightarrow \mathbb{R}^N$ defined as

$$d_{c,r}(\xi) = (c + |\xi|^2)^{\frac{r-2}{2}} \xi.$$

We know that the map $d_{c,r}$ is strictly monotone for any $c \geq 0$ and for any $r > 1$ (see Lemma 1.4.3 for the proof). Now, let us assume $r \geq 2$. It is immediate to see that for any $\xi \in \mathbb{R}^N$ we have

$$d_{c,r}(\xi) \cdot \xi \geq |\xi|^r.$$

On the other hand, we have

$$\lim_{|\xi| \rightarrow +\infty} \frac{|d_{c,r}(\xi)|}{|\xi|^{r-1}} = \lim_{|\xi| \rightarrow +\infty} \frac{(c + |\xi|^2)^{\frac{r-2}{2}} |\xi|}{|\xi|^{r-1}} = 1,$$

hence there exists a constant C , depending on c , such that for any $\xi \in \mathbb{R}^N$ we have

$$|d_{c,r}(\xi)| \leq C + C|\xi|^{r-1}.$$

If $1 < r < 2$, then it is immediate to see that for any $\xi \in \mathbb{R}^N$ we have

$$|d_{c,r}(\xi)| \leq |\xi|^{r-1}.$$

On the other hand, we have

$$\lim_{|\xi| \rightarrow +\infty} \frac{d_{c,r}(\xi) \cdot \xi}{|\xi|^r} = \lim_{|\xi| \rightarrow +\infty} \frac{(c + |\xi|^2)^{\frac{r-2}{2}} |\xi|^2}{|\xi|^r} = 1,$$

hence there exists a constant C , depending on c , such that for any $\xi \in \mathbb{R}^N$ we have

$$d_{c,r}(\xi) \cdot \xi \geq C|\xi|^r + C.$$

In any case, for any $c \geq 0$ and for any $r > 1$ assumptions of [6, Theorem 3.5] are satisfied, by which we get the thesis. $\square$

Now, for any $a, b \geq 0$ and $p, q > 1$, we consider the map $H_{a,p,b,q} : X \rightarrow X'$ defined for any $z = (u, v) \in X$ as

$$H_{a,p,b,q}(z) = [H_{a,p}(u), H_{b,q}(v)], \quad (1.2.4)$$

where the action of the map $H_{a,p,b,q}(z) : X \rightarrow \mathbb{R}$ is given for any $\bar{z} = (\bar{u}, \bar{v}) \in X$ by

$$\langle H_{a,p,b,q}(z), \bar{z} \rangle = \int_{\Omega} \left(a + |\nabla u|^2 \right)^{\frac{p-2}{2}} \nabla u \cdot \nabla \bar{u} \, dx + \int_{\Omega} \left(b + |\nabla v|^2 \right)^{\frac{q-2}{2}} \nabla v \cdot \nabla \bar{v} \, dx.$$

Proposition 1.2.2. *The map $H_{a,p,b,q} : X \rightarrow X'$ is of class $(S)_+$.*

Proof. Let $\{z_k\}_k = \{(u_k, v_k)\}_k \subset X$ and $z = (u, v) \in X$ such that $z_k \rightharpoonup z$ in X and

$$\limsup_{k \rightarrow \infty} \langle H_{a,p,b,q}(z_k), z_k - z \rangle \leq 0. \quad (1.2.5)$$

We need to prove $z_k \rightarrow z$ in X .

Since $v_k \rightharpoonup v$ in $W_0^{1,q}(\Omega)$ and $H_{b,q}(v) \in W_0^{-1,q'}(\Omega)$, we have

$$\int_{\Omega} \left(b + |\nabla v|^2 \right)^{\frac{q-2}{2}} \nabla v \cdot \nabla (v_k - v) \, dx \rightarrow 0. \quad (1.2.6)$$

By using the monotonicity of the map $d_{b,q} : \mathbb{R}^N \rightarrow \mathbb{R}^N$ defined in (1.4.24), we deduce

$$\begin{aligned} 0 &\leq \int_{\Omega} [d_{b,q}(\nabla v_k) - d_{b,q}(\nabla v)] \cdot \nabla (v_k - v) \, dx \\ &= \int_{\Omega} \left(b + |\nabla v_k|^2 \right)^{\frac{q-2}{2}} \nabla v_k \cdot \nabla (v_k - v) \, dx - \int_{\Omega} \left(b + |\nabla v|^2 \right)^{\frac{q-2}{2}} \nabla v \cdot \nabla (v_k - v) \, dx, \end{aligned}$$

hence

$$\int_{\Omega} \left(b + |\nabla v|^2 \right)^{\frac{q-2}{2}} \nabla v \cdot \nabla (v_k - v) \, dx \leq \int_{\Omega} \left(b + |\nabla v_k|^2 \right)^{\frac{q-2}{2}} \nabla v_k \cdot \nabla (v_k - v) \, dx$$

and so

$$\begin{aligned}
& \int_{\Omega} \left(a + |\nabla u_k|^2 \right)^{\frac{p-2}{2}} \nabla u_k \cdot \nabla (u_k - u) \, dx + \int_{\Omega} \left(b + |\nabla v|^2 \right)^{\frac{q-2}{2}} \nabla v \cdot \nabla (v_k - v) \, dx \\
& \leq \int_{\Omega} \left(a + |\nabla u_k|^2 \right)^{\frac{p-2}{2}} \nabla u_k \cdot \nabla (u_k - u) \, dx + \int_{\Omega} \left(b + |\nabla v_k|^2 \right)^{\frac{q-2}{2}} \nabla v_k \cdot \nabla (v_k - v) \, dx \quad (1.2.7) \\
& = \langle H_{a,p,b,q}(z_k), z_k - z \rangle.
\end{aligned}$$

Taking into account last inequality, (1.2.5) and (1.2.6), we deduce

$$\limsup_{k \rightarrow \infty} \langle H_{a,p}(u_k), u_k - u \rangle \leq \limsup_{k \rightarrow \infty} \langle H_{a,p,b,q}(z_k), z_k - z \rangle \leq 0,$$

hence, by Lemma 1.2.1, we deduce $u_k \rightarrow u$ in $W_0^{1,p}(\Omega)$. Similarly, we deduce $v_k \rightarrow v$ in $W_0^{1,q}(\Omega)$. Finally, we have $z_k \rightarrow z$ in X . $\square$

1.3 $(S)_+$ -condition

Let us now consider the following quasilinear system

$$\begin{cases} -\operatorname{div} \left((a + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \right) = F_s(u, v) & x \in \Omega, \\ -\operatorname{div} \left((b + |\nabla v|^2)^{\frac{q-2}{2}} \nabla v \right) = F_t(u, v) & x \in \Omega, \\ u = v = 0 & x \in \partial\Omega, \end{cases} \quad (1.3.8)$$

where $N \geq 2$, $1 < p, q < N$, and $a, b \geq 0$. We recall that the Euler functional $J_{a,b} : X \rightarrow \mathbb{R}$ associated to (1.3.8) is defined for any $z = (u, v) \in X$ as

$$J_{a,b}(z) = \frac{1}{p} \int_{\Omega} \left(a + |\nabla u|^2 \right)^{\frac{p}{2}} \, dx + \frac{1}{q} \int_{\Omega} \left(b + |\nabla v|^2 \right)^{\frac{q}{2}} \, dx - \int_{\Omega} F(u, v) \, dx.$$

Moreover, $J_{a,b}$ is of class C^1 on X and for any $z_0 = (u_0, v_0), z = (u, v) \in X$ it results

$$\begin{aligned}
\langle J'_{a,b}(z_0), z \rangle &= \int_{\Omega} (a + |\nabla u_0|^2)^{\frac{p-2}{2}} \nabla u_0 \cdot \nabla u \, dx + \int_{\Omega} (b + |\nabla v_0|^2)^{\frac{q-2}{2}} \nabla v_0 \cdot \nabla v \, dx \\
&\quad - \int_{\Omega} (F_s(u_0, v_0)u + F_t(u_0, v_0)v) \, dx.
\end{aligned}$$

In this Section we show that the map $J'_{a,b} : X \rightarrow X'$ is locally of class $(S)_+$.

Remark 1.3.1. For every $h \in \mathbb{N}$, $h \geq 1$ we define $\mathcal{T}_h, \mathcal{R}_h : \mathbb{R} \rightarrow \mathbb{R}$ as

$$\mathcal{T}_h(s) := \min\{\max\{s, -h\}, h\} \quad \text{and} \quad \mathcal{R}_h(s) := s - \mathcal{T}_h(s). \quad (1.3.9)$$

If $w \in W_0^{1,r}(\Omega)$ for some $r > 1$, then $\mathcal{T}_h(w), \mathcal{R}_h(w) \in W_0^{1,r}(\Omega)$. Moreover, we have

$$\mathcal{T}_h(w(x)) = \begin{cases} h & \text{a.e. in } \{x \in \Omega : w(x) \geq h\}, \\ w(x) & \text{a.e. in } \{x \in \Omega : -h < w(x) < h\}, \\ -h & \text{a.e. in } \{x \in \Omega : w(x) \leq -h\}; \end{cases}$$

$$\begin{aligned}\nabla \mathcal{T}_h(w(x)) &= \begin{cases} \nabla w(x) & \text{a.e. in } \{x \in \Omega : |w(x)| < h\}, \\ 0 & \text{a.e. in } \{x \in \Omega : |w(x)| \geq h\}; \end{cases} \\ \mathcal{R}_h(w(x)) &= \begin{cases} w(x) - h & \text{a.e. in } \{x \in \Omega : w(x) \geq h\}, \\ 0 & \text{a.e. in } \{x \in \Omega : -h < w(x) < h\}, \\ w(x) + h & \text{a.e. in } \{x \in \Omega : w(x) \leq -h\}; \end{cases} \\ \nabla \mathcal{R}_h(w(x)) &= \begin{cases} \nabla w(x) & \text{a.e. in } \{x \in \Omega : |w(x)| > h\}, \\ 0 & \text{a.e. in } \{x \in \Omega : |w(x)| \leq h\}; \end{cases}\end{aligned}$$

and

$$\nabla w(x) = \begin{cases} \nabla \mathcal{T}_h(w(x)) & \text{a.e. in } \{x \in \Omega : |w(x)| < h\}, \\ \nabla \mathcal{R}_h(w(x)) & \text{a.e. in } \{x \in \Omega : |w(x)| > h\}, \\ 0 & \text{a.e. in } \{x \in \Omega : |w(x)| = h\}. \end{cases} \quad (1.3.10)$$

We will use the following result.

Lemma 1.3.2. *Let Ω be a bounded domain of $\mathbb{R}^N$. Let $\{w_k\}_k \subset L^1(\Omega)$ and $w \in L^1(\Omega)$ such that $w_k(x) \rightarrow w(x)$ a.e. $x \in \Omega$ as $k \rightarrow \infty$. Let $h \in \mathbb{N}$, $h \geq 1$ and let us define, almost everywhere, the sets*

$$\Omega_{h,+}^{w_k} := \{x \in \Omega : |w_k(x)| \geq h\} \quad \text{and} \quad \Omega_{h,+}^w := \{x \in \Omega : |w(x)| \geq h\}.$$

Then

$$\limsup_{k \rightarrow \infty} |\Omega_{h,+}^{w_k}| \leq |\Omega_{h,+}^w|.$$

Proof. Since $w_k(x) \rightarrow w(x)$ a.e. $x \in \Omega$ as $k \rightarrow \infty$, w_k converges to w also in measure, that is

$$\forall \varepsilon > 0, \quad |\{x \in \Omega : |w_k(x) - w(x)| \geq \varepsilon\}| \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (1.3.11)$$

Let $\varepsilon > 0$ and let us define

$$\begin{aligned} D_{k,h} &:= \{x \in \Omega : |w_k(x) - w(x)| + |w(x)| \geq h\}; \\ A_{k,h}^\varepsilon &:= \{x \in \Omega : |w(x)| \geq h - \varepsilon, \quad |w_k(x) - w(x)| + |w(x)| \geq h\}; \\ B_{k,h}^\varepsilon &:= \{x \in \Omega : |w(x)| < h - \varepsilon, \quad |w_k(x) - w(x)| + |w(x)| \geq h\}; \\ \mathcal{A}_h^\varepsilon &:= \{x \in \Omega : |w(x)| \geq h - \varepsilon\}; \\ \mathcal{B}_{k,h}^\varepsilon &:= \{x \in \Omega : |w_k(x) - w(x)| \geq \varepsilon\}. \end{aligned}$$

Since

$$\Omega_{h,+}^{w_k} \subseteq D_{k,h} = A_{k,h}^\varepsilon \cup B_{k,h}^\varepsilon \subseteq \mathcal{A}_h^\varepsilon \cup \mathcal{B}_{k,h}^\varepsilon,$$

for any $k, h \in \mathbb{N}^*$ and for any $\varepsilon > 0$ we have

$$|\Omega_{h,+}^{w_k}| \leq |\mathcal{A}_h^\varepsilon| + |\mathcal{B}_{k,h}^\varepsilon|.$$

By (1.3.11), we deduce

$$|\mathcal{B}_{k,h}^\varepsilon| \rightarrow 0 \quad \text{as } k \rightarrow \infty \quad \forall \varepsilon > 0,$$

by which

$$\limsup_{k \rightarrow \infty} |\Omega_{k,h}^+| \leq |\mathcal{A}_h^\varepsilon|, \quad \forall \varepsilon > 0. \quad (1.3.12)$$

Since $(\mathcal{A}_h^{1/n})_{n \in \mathbb{N}}$ is a decreasing sequence of subsets with $|\mathcal{A}_h^1| < \infty$ and

$$\bigcap_{n=1}^{\infty} \mathcal{A}_h^{1/n} = \Omega_{h,+}^w,$$

by continuity of Lebesgue measure we infer

$$\lim_{n \rightarrow \infty} |\mathcal{A}_h^{1/n}| = |\Omega_{h,+}^w|,$$

by which, together with (1.3.12), we deduce

$$\limsup_{k \rightarrow \infty} |\Omega_{h,+}^{w_k}| \leq |\Omega_{h,+}^w|.$$

□

We will frequently use also the following result of functional analysis (see [36]).

Proposition 1.3.3. *Let $1 < r < \infty$ and $(\Omega, \mathcal{M}, \mu)$ a measure space with σ -finite measure μ . Let $\{f_k\}_k \subset L^r(\Omega)$ a sequence such that:*

- i) $\{f_k\}_k$ is bounded in $L^r(\Omega)$;
- ii) $f_k(x) \rightarrow f(x)$ a.e. $x \in \Omega$.

Then

$$f_k \rightharpoonup f \text{ in } L^r(\Omega).$$

Proof of Theorem 1.1.2. Let $a, b \geq 0$ and $z_0 = (u_0, v_0) \in X$. Let $R > 1$ and let us consider a sequence $\{z_k\}_k = \{(u_k, v_k)\}_k \subset \overline{B_R(z_0)}$ weakly convergent to some $z = (u, v)$ in X with

$$\limsup_{k \rightarrow \infty} \langle J'_{a,b}(z_k), z_k - z \rangle \leq 0. \quad (1.3.13)$$

By means of the map $H_{a,p,b,q}$ defined in (1.2.4), we can rewrite the preceding assumption as follows:

$$\limsup_{k \rightarrow \infty} \left\{ \langle H_{a,p,b,q}(z_k), z_k - z \rangle - \int_{\Omega} [F_s(u_k, v_k)(u_k - u) + F_t(u_k, v_k)(v_k - v)] dx \right\} \leq 0. \quad (1.3.14)$$

Let $c \geq 0$, $r > 1$ and let us consider the functional $f : W_0^{1,r}(\Omega) \rightarrow \mathbb{R}$ defined as

$$f(w) := \frac{1}{r} \int_{\Omega} (c + |\nabla w|^2)^{\frac{r}{2}} dx, \quad (1.3.15)$$

Since f is convex (see Lemma 1.4.4), for any $w, w_k \in W_0^{1,r}(\Omega)$ we have

$$f(w) \geq f(w_k) + \langle f'(w_k), w - w_k \rangle,$$

by which

$$\int_{\Omega} \left(c + |\nabla w_k|^2 \right)^{\frac{r-2}{2}} \nabla w_k \cdot \nabla (w_k - w) \, dx \geq \frac{1}{r} \int_{\Omega} \left(c + |\nabla w_k|^2 \right)^{\frac{r}{2}} \, dx - \frac{1}{r} \int_{\Omega} \left(c + |\nabla w|^2 \right)^{\frac{r}{2}} \, dx.$$

Hence, we get

$$\begin{aligned} 0 &\geq \limsup_{k \rightarrow \infty} \langle J'_{a,b}(z_k), z_k - z \rangle \\ &\geq \limsup_{k \rightarrow \infty} \left\{ \frac{1}{p} \int_{\Omega} \left(a + |\nabla u_k|^2 \right)^{\frac{p}{2}} \, dx - \frac{1}{p} \int_{\Omega} \left(a + |\nabla u|^2 \right)^{\frac{p}{2}} \, dx \right. \\ &\quad + \frac{1}{q} \int_{\Omega} \left(b + |\nabla v_k|^2 \right)^{\frac{q}{2}} \, dx - \frac{1}{q} \int_{\Omega} \left(b + |\nabla v|^2 \right)^{\frac{q}{2}} \, dx \\ &\quad \left. - \int_{\Omega} [F_s(u_k, v_k)(u_k - u) + F_t(u_k, v_k)(v_k - v)] \, dx \right\}. \end{aligned} \quad (1.3.16)$$

We know that $u_k \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and $v_k \rightharpoonup v$ in $W_0^{1,q}(\Omega)$. Moreover, we will use the fact that $\{u_k\}_k$ is bounded in $L^{p^*}(\Omega)$ and $\{v_k\}_k$ is bounded in $L^{q^*}(\Omega)$. For every $h \in \mathbb{N}$, $h \geq 1$, let us consider the functions $\mathcal{T}_h, \mathcal{R}_h : \mathbb{R} \rightarrow \mathbb{R}$ defined in (1.3.9). We have:

- i) $\mathcal{T}_h(u_k) \rightharpoonup \mathcal{T}_h(u)$ in $W_0^{1,p}(\Omega)$, and $\mathcal{T}_h(v_k) \rightharpoonup \mathcal{T}_h(v)$ in $W_0^{1,q}(\Omega)$;
- ii) $\mathcal{T}_h(u_k) \rightarrow \mathcal{T}_h(u)$ in $L^\sigma(\Omega)$ if $1 \leq \sigma < \infty$, and $\mathcal{T}_h(v_k) \rightarrow \mathcal{T}_h(v)$ in $L^\tau(\Omega)$ if $1 \leq \tau < \infty$.
- iii) $\mathcal{R}_h(u_k) \rightharpoonup \mathcal{R}_h(u)$ in $W_0^{1,p}(\Omega)$, and $\mathcal{R}_h(v_k) \rightharpoonup \mathcal{R}_h(v)$ in $W_0^{1,q}(\Omega)$;
- iv) $\mathcal{R}_h(u_k) \rightarrow \mathcal{R}_h(u)$ in $L^\sigma(\Omega)$ if $1 \leq \sigma < p^*$, and $\mathcal{R}_h(v_k) \rightarrow \mathcal{R}_h(v)$ in $L^\tau(\Omega)$ if $1 \leq \tau < q^*$.

Taking into account Remark 1.3.1, we have

$$\begin{aligned} \int_{\Omega} \left(a + |\nabla u|^2 \right)^{\frac{p}{2}} \, dx &= \int_{|u|>h} \left(a + |\nabla \mathcal{R}_h(u)|^2 \right)^{\frac{p}{2}} \, dx + \int_{|u|<h} \left(a + |\nabla \mathcal{T}_h(u)|^2 \right)^{\frac{p}{2}} \, dx + a^{\frac{p}{2}} \int_{|u|=h} \, dx \\ &= \int_{\Omega} \left(a + |\nabla \mathcal{R}_h(u)|^2 \right)^{\frac{p}{2}} \, dx + \int_{\Omega} \left(a + |\nabla \mathcal{T}_h(u)|^2 \right)^{\frac{p}{2}} \, dx - a^{\frac{p}{2}} |\Omega|. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \int_{\Omega} \left(a + |\nabla u_k|^2 \right)^{\frac{p}{2}} \, dx &= \int_{\Omega} \left(a + |\nabla \mathcal{R}_h(u_k)|^2 \right)^{\frac{p}{2}} \, dx + \int_{\Omega} \left(a + |\nabla \mathcal{T}_h(u_k)|^2 \right)^{\frac{p}{2}} \, dx - a^{\frac{p}{2}} |\Omega|, \\ \int_{\Omega} \left(b + |\nabla v_k|^2 \right)^{\frac{q}{2}} \, dx &= \int_{\Omega} \left(b + |\nabla \mathcal{R}_h(v_k)|^2 \right)^{\frac{q}{2}} \, dx + \int_{\Omega} \left(b + |\nabla \mathcal{T}_h(v_k)|^2 \right)^{\frac{q}{2}} \, dx - b^{\frac{q}{2}} |\Omega|, \\ \int_{\Omega} \left(b + |\nabla v|^2 \right)^{\frac{q}{2}} \, dx &= \int_{\Omega} \left(b + |\nabla \mathcal{R}_h(v)|^2 \right)^{\frac{q}{2}} \, dx + \int_{\Omega} \left(b + |\nabla \mathcal{T}_h(v)|^2 \right)^{\frac{q}{2}} \, dx - b^{\frac{q}{2}} |\Omega|. \end{aligned}$$

Considering these relations in (1.3.16), we get

$$0 \geq \limsup_{k \rightarrow \infty} \left\{ \frac{1}{p} \int_{\Omega} \left(a + |\nabla \mathcal{R}_h(u_k)|^2 \right)^{\frac{p}{2}} \, dx - \frac{1}{p} \int_{\Omega} \left(a + |\nabla \mathcal{R}_h(u)|^2 \right)^{\frac{p}{2}} \, dx \right. \quad (1.3.17)$$

$$\begin{aligned}
& + \frac{1}{q} \int_{\Omega} \left(b + |\nabla \mathcal{R}_h(v_k)|^2 \right)^{\frac{q}{2}} dx - \frac{1}{q} \int_{\Omega} \left(b + |\nabla \mathcal{R}_h(v)|^2 \right)^{\frac{q}{2}} dx \\
& - \int_{\Omega} [F_s(u_k, v_k)(\mathcal{R}_h(u_k) - \mathcal{R}_h(u)) + F_t(u_k, v_k)(\mathcal{R}_h(v_k) - \mathcal{R}_h(v))] dx \\
& - \int_{\Omega} [F_s(u_k, v_k)(\mathcal{T}_h(u_k) - \mathcal{T}_h(u)) + F_t(u_k, v_k)(\mathcal{T}_h(v_k) - \mathcal{T}_h(v))] dx \\
& + \frac{1}{p} \int_{\Omega} \left(a + |\nabla \mathcal{T}_h(u_k)|^2 \right)^{\frac{p}{2}} dx - \frac{1}{p} \int_{\Omega} \left(a + |\nabla \mathcal{T}_h(u)|^2 \right)^{\frac{p}{2}} dx \\
& + \frac{1}{q} \int_{\Omega} \left(b + |\nabla \mathcal{T}_h(v_k)|^2 \right)^{\frac{q}{2}} dx - \frac{1}{q} \int_{\Omega} \left(b + |\nabla \mathcal{T}_h(v)|^2 \right)^{\frac{q}{2}} dx \Big\}
\end{aligned}$$

Considering *ii*) and the boundedness of $\{u_k\}_k$ in $L^{p^*}(\Omega)$ and of $\{v_k\}_k$ in $L^{q^*}(\Omega)$, we get

$$\int_{\Omega} F_s(u_k, v_k)(\mathcal{T}_h(u_k) - \mathcal{T}_h(u)) dx \rightarrow 0, \quad \text{and} \quad \int_{\Omega} F_t(u_k, v_k)(\mathcal{T}_h(v_k) - \mathcal{T}_h(v)) dx \rightarrow 0.$$

Moreover, by *i*) and the weak lower semicontinuity of the functional f defined in (1.3.15), we get

$$\frac{1}{p} \int_{\Omega} \left(a + |\nabla \mathcal{T}_h(u)|^2 \right)^{\frac{p}{2}} dx \leq \liminf_{k \rightarrow \infty} \left[\frac{1}{p} \int_{\Omega} \left(a + |\nabla \mathcal{T}_h(u_k)|^2 \right)^{\frac{p}{2}} dx \right],$$

and

$$\frac{1}{q} \int_{\Omega} \left(b + |\nabla \mathcal{T}_h(v)|^2 \right)^{\frac{q}{2}} dx \leq \liminf_{k \rightarrow \infty} \left[\frac{1}{q} \int_{\Omega} \left(b + |\nabla \mathcal{T}_h(v_k)|^2 \right)^{\frac{q}{2}} dx \right].$$

Considering also that $\limsup_{k \rightarrow \infty} (a_k + b_k) \geq \limsup_{k \rightarrow \infty} a_k + \liminf_{k \rightarrow \infty} b_k$, by (1.3.17) we deduce

$$\begin{aligned}
0 \geq \limsup_{k \rightarrow \infty} & \left\{ \frac{1}{p} \int_{\Omega} \left(a + |\nabla \mathcal{R}_h(u_k)|^2 \right)^{\frac{p}{2}} dx - \frac{1}{p} \int_{\Omega} \left(a + |\nabla \mathcal{R}_h(u)|^2 \right)^{\frac{p}{2}} dx \right. \\
& + \frac{1}{q} \int_{\Omega} \left(b + |\nabla \mathcal{R}_h(v_k)|^2 \right)^{\frac{q}{2}} dx - \frac{1}{q} \int_{\Omega} \left(b + |\nabla \mathcal{R}_h(v)|^2 \right)^{\frac{q}{2}} dx \\
& \left. - \int_{\Omega} [F_s(u_k, v_k)(\mathcal{R}_h(u_k) - \mathcal{R}_h(u)) + F_t(u_k, v_k)(\mathcal{R}_h(v_k) - \mathcal{R}_h(v))] dx \right\}.
\end{aligned}$$

Let us now define, almost everywhere, the following sets:

$$\Omega_{h,+}^{u_k} := \{x \in \Omega : |u_k(x)| \geq h\} \quad \text{and} \quad \Omega_{h,-}^{u_k} := \{x \in \Omega : |u_k(x)| < h\}.$$

Taking into account Remark 1.3.1, we observe that

$$\begin{aligned}
\int_{\Omega} \left(a + |\nabla \mathcal{R}_h(u_k)|^2 \right)^{\frac{p}{2}} dx & = a^{\frac{p}{2}} |\Omega_{h,-}^{u_k}| + \int_{\Omega_{h,+}^{u_k}} \left(a + |\nabla \mathcal{R}_h(u_k)|^2 \right)^{\frac{p}{2}} dx \\
& \geq a^{\frac{p}{2}} |\Omega_{h,-}^{u_k}| + \int_{\Omega_{h,+}^{u_k}} |\nabla \mathcal{R}_h(u_k)|^p dx \\
& = a^{\frac{p}{2}} |\Omega| + \int_{\Omega} |\nabla \mathcal{R}_h(u_k)|^p dx - a^{\frac{p}{2}} |\Omega_{h,+}^{u_k}|.
\end{aligned}$$

By Lemma 1.3.2 we have

$$\limsup_{k \rightarrow \infty} |\Omega_{h,+}^{u_k}| \leq |\Omega_{h,+}^u|.$$

Similarly, we have

$$\int_{\Omega} \left(b + |\nabla \mathcal{R}_h(v_k)|^2 \right)^{\frac{q}{2}} dx \geq b^{\frac{q}{2}} |\Omega| + \int_{\Omega} |\nabla \mathcal{R}_h(v_k)|^q dx - b^{\frac{q}{2}} |\Omega_{h,+}^{v_k}|$$

with

$$\limsup_{k \rightarrow \infty} |\Omega_{h,+}^{v_k}| \leq |\Omega_{h,+}^v|.$$

Hence, at now we have proved

$$\begin{aligned} \limsup_{k \rightarrow \infty} & \left(\frac{1}{p} \int_{\Omega} |\nabla \mathcal{R}_h(u_k)|^p dx + \frac{1}{q} \int_{\Omega} |\nabla \mathcal{R}_h(v_k)|^q dx \right. \\ & \left. - \int_{\Omega} F_s(u_k, v_k) (\mathcal{R}_h(u_k) - \mathcal{R}_h(u)) dx - \int_{\Omega} F_t(u_k, v_k) (\mathcal{R}_h(v_k) - \mathcal{R}_h(v)) dx \right) \\ & \leq \frac{1}{p} \int_{\Omega} \left(a + |\nabla \mathcal{R}_h(u)|^2 \right)^{\frac{p}{2}} dx - \frac{1}{p} a^{\frac{p}{2}} |\Omega| + \frac{1}{p} a^{\frac{p}{2}} |\Omega_{h,+}^u| \\ & + \frac{1}{q} \int_{\Omega} \left(b + |\nabla \mathcal{R}_h(v)|^2 \right)^{\frac{q}{2}} dx - \frac{1}{q} b^{\frac{q}{2}} |\Omega| + \frac{1}{q} b^{\frac{q}{2}} |\Omega_{h,+}^v|. \end{aligned}$$

Now, on the one hand by Chebyshev inequality we have

$$|\Omega_{h,+}^u| \rightarrow 0 \quad \text{and} \quad |\Omega_{h,+}^v| \rightarrow 0 \quad \text{as } h \rightarrow \infty,$$

and on the other hand by dominated convergence theorem we have

$$\int_{\Omega} \left(\alpha + |\nabla \mathcal{R}_h(u)|^2 \right)^{\frac{p}{2}} dx \rightarrow \alpha^{\frac{p}{2}} |\Omega| \quad \text{and} \quad \int_{\Omega} \left(\beta + |\nabla \mathcal{R}_h(v)|^2 \right)^{\frac{q}{2}} dx \rightarrow \beta^{\frac{q}{2}} |\Omega| \quad \text{as } h \rightarrow \infty.$$

Therefore, denoting with $o_h(1)$ a quantity that goes to 0 as $h \rightarrow \infty$, we infer that

$$\begin{aligned} \limsup_{k \rightarrow \infty} & \left(\frac{1}{p} \int_{\Omega} |\nabla \mathcal{R}_h(u_k)|^p dx + \frac{1}{q} \int_{\Omega} |\nabla \mathcal{R}_h(v_k)|^q dx \right. \\ & \left. - \int_{\Omega} F_s(u_k, v_k) (\mathcal{R}_h(u_k) - \mathcal{R}_h(u)) dx - \int_{\Omega} F_t(u_k, v_k) (\mathcal{R}_h(v_k) - \mathcal{R}_h(v)) dx \right) \\ & \leq o_h(1). \end{aligned}$$

By continuity of F_s and the convergence a.e. $x \in \Omega$ of $u_k(x)$ to $u(x)$ and of $v_k(x)$ to $v(x)$, we deduce that $F_s(u_k(x), v_k(x)) \rightarrow F_s(u(x), v(x))$ a.e. $x \in \Omega$. By (1.1.2) and the boundedness of $\{u_k\}_k$ in $L^{p^*}(\Omega)$ and of $\{v_k\}_k$ in $L^{q^*}(\Omega)$, we also deduce that the sequence $\{F_s(u_k, v_k)\}_k$ is bounded in $L^{(p^*)'}(\Omega)$. Hence we can apply Proposition 1.3.3 and say that $F_s(u_k, v_k) \rightharpoonup F_s(u, v)$ in $L^{(p^*)'}(\Omega)$, and with similar argument also $F_t(u_k, v_k) \rightharpoonup F_t(u, v)$ in $L^{(q^*)'}(\Omega)$.

Since $\mathcal{R}_h(u) \in L^{p^*}(\Omega)$ and $\mathcal{R}_h(v) \in L^{q^*}(\Omega)$, we infer that

$$\int_{\Omega} F_s(u_k, v_k) \mathcal{R}_h(u) dx \rightarrow \int_{\Omega} F_s(u, v) \mathcal{R}_h(u) dx, \quad \int_{\Omega} F_t(u_k, v_k) \mathcal{R}_h(v) dx \rightarrow \int_{\Omega} F_t(u, v) \mathcal{R}_h(v) dx,$$

whence

$$\begin{aligned} \limsup_{k \rightarrow \infty} & \left(\frac{1}{p} \int_{\Omega} |\nabla \mathcal{R}_h(u_k)|^p dx + \frac{1}{q} \int_{\Omega} |\nabla \mathcal{R}_h(v_k)|^q dx \right. \\ & \quad \left. - \int_{\Omega} F_s(u_k, v_k) \mathcal{R}_h(u_k) dx - \int_{\Omega} F_t(u_k, v_k) \mathcal{R}_h(v_k) dx \right) \\ & \leq - \int_{\Omega} F_s(u, v) \mathcal{R}_h(u) dx - \int_{\Omega} F_t(u, v) \mathcal{R}_h(v) dx + o_h(1). \end{aligned}$$

By *iv*) we know that

$$\int_{\Omega} |\mathcal{R}_h(u_k)| dx \rightarrow \int_{\Omega} |\mathcal{R}_h(u)| dx \quad \text{and} \quad \int_{\Omega} |\mathcal{R}_h(v_k)| dx \rightarrow \int_{\Omega} |\mathcal{R}_h(v)| dx \quad \text{as } k \rightarrow \infty.$$

Considering also (1.1.2) and (1.1.3), we get

$$\begin{aligned} \limsup_{k \rightarrow \infty} & \left(\frac{1}{p} \int_{\Omega} |\nabla \mathcal{R}_h(u_k)|^p dx + \frac{1}{q} \int_{\Omega} |\nabla \mathcal{R}_h(v_k)|^q dx \right. \\ & \quad - C \int_{\Omega} |u_k|^{p^*-1} |\mathcal{R}_h(u_k)| dx - C \int_{\Omega} |v_k|^{q^* \frac{p^*-1}{p^*}} |\mathcal{R}_h(u_k)| dx \\ & \quad - C \int_{\Omega} |u_k|^{p^* \frac{q^*-1}{q^*}} |\mathcal{R}_h(v_k)| dx - C \int_{\Omega} |v_k|^{q^*-1} |\mathcal{R}_h(v_k)| dx \Big) \\ & \leq C \int_{\Omega} |u|^{p^*-1} |\mathcal{R}_h(u)| dx + C \int_{\Omega} |v|^{q^* \frac{p^*-1}{p^*}} |\mathcal{R}_h(u)| dx \\ & \quad + C \int_{\Omega} |u|^{p^* \frac{q^*-1}{q^*}} |\mathcal{R}_h(v)| dx + C \int_{\Omega} |v|^{q^*-1} |\mathcal{R}_h(v)| dx \\ & \quad + 2C \int_{\Omega} |\mathcal{R}_h(u)| dx + 2C \int_{\Omega} |\mathcal{R}_h(v)| dx + o_h(1). \end{aligned}$$

By Gagliardo–Nirenberg–Sobolev inequality there exist $S(p, N) > 0$ and $S(q, N) > 0$ such that

$$S(p, N) \|\bar{u}\|_{p^*}^p \leq \int_{\Omega} |\nabla \bar{u}|^p dx \quad \text{and} \quad S(q, N) \|\bar{v}\|_{q^*}^q \leq \int_{\Omega} |\nabla \bar{v}|^q dx$$

for any $\bar{u} \in W_0^{1,p}(\Omega)$ and for any $\bar{v} \in W_0^{1,q}(\Omega)$, by which for any h and k we get

$$\frac{1}{p} \int_{\Omega} |\nabla \mathcal{R}_h(u_k)|^p dx \geq \frac{S(p, N)}{p} \|\mathcal{R}_h(u_k)\|_{p^*}^p \quad \text{and} \quad \frac{1}{q} \int_{\Omega} |\nabla \mathcal{R}_h(v_k)|^q dx \geq \frac{S(q, N)}{q} \|\mathcal{R}_h(v_k)\|_{q^*}^q.$$

Hence

$$\begin{aligned} \limsup_{k \rightarrow \infty} & \left(\frac{S(p, N)}{p} \|\mathcal{R}_h(u_k)\|_{p^*}^p + \frac{S(q, N)}{q} \|\mathcal{R}_h(v_k)\|_{q^*}^q \right. \\ & \quad - C \int_{\Omega} |u_k|^{p^*-1} |\mathcal{R}_h(u_k)| dx - C \int_{\Omega} |v_k|^{q^* \frac{p^*-1}{p^*}} |\mathcal{R}_h(u_k)| dx \\ & \quad - C \int_{\Omega} |u_k|^{p^* \frac{q^*-1}{q^*}} |\mathcal{R}_h(v_k)| dx - C \int_{\Omega} |v_k|^{q^*-1} |\mathcal{R}_h(v_k)| dx \Big) \\ & \leq C \int_{\Omega} |u|^{p^*-1} |\mathcal{R}_h(u)| dx + C \int_{\Omega} |v|^{q^* \frac{p^*-1}{p^*}} |\mathcal{R}_h(u)| dx \end{aligned} \tag{1.3.18}$$

$$\begin{aligned}
& + C \int_{\Omega} |u|^{p^* \frac{q^*-1}{q^*}} |\mathcal{R}_h(v)| \, dx + C \int_{\Omega} |v|^{q^*-1} |\mathcal{R}_h(v)| \, dx \\
& + 2C \int_{\Omega} |\mathcal{R}_h(u)| \, dx + 2C \int_{\Omega} |\mathcal{R}_h(v)| \, dx + o_h(1).
\end{aligned}$$

Let us now consider the term

$$\int_{\Omega} |u_k|^{p^*-1} |\mathcal{R}_h(u_k)| \, dx.$$

By Hölder inequality with conjugated exponents p^*/p and $p^*/(p^* - p)$, we have

$$\begin{aligned}
\int_{\Omega} |u_k|^{p^*-1} |\mathcal{R}_h(u_k)| \, dx &= \int_{\Omega} |u_k|^{p^*-p} |\mathcal{T}_h(u_k) + \mathcal{R}_h(u_k)|^{p-1} |\mathcal{R}_h(u_k)| \, dx \\
&\leq c(p) \left(\int_{\Omega} |u_k|^{p^*-p} |\mathcal{T}_h(u_k)|^{p-1} |\mathcal{R}_h(u_k)| \, dx + \int_{\Omega} |u_k|^{p^*-p} |\mathcal{R}_h(u_k)|^p \, dx \right) \\
&\leq c(p) \left(\int_{\Omega} |u_k|^{p^*-p} |\mathcal{T}_h(u_k)|^{p-1} |\mathcal{R}_h(u_k)| \, dx + \int_{\Omega} |u_0|^{p^*-p} |\mathcal{R}_h(u_k)|^p \, dx \right) \\
&\quad + c(p) \int_{\Omega} |u_k - u_0|^{p^*-p} |\mathcal{R}_h(u_k)|^p \, dx \\
&\leq c(p) \left(\int_{\Omega} |u_k|^{p^*-p} |\mathcal{T}_h(u_k)|^{p-1} |\mathcal{R}_h(u_k)| \, dx + \int_{\Omega} |u_0|^{p^*-p} |\mathcal{R}_h(u_k)|^p \, dx \right) \\
&\quad + c(p) \|u_k - u_0\|_{p^*}^{p^*-p} \|\mathcal{R}_h(u_k)\|_{p^*}^p.
\end{aligned}$$

By Proposition 1.3.3 we infer that $|\mathcal{R}_h(u_k)|^p \rightharpoonup |\mathcal{R}_h(u)|^p$ in $L^{\frac{p^*}{p}}(\Omega)$. Since $|u_0|^{p^*-p} \in L^{\frac{p^*}{p^*-p}}(\Omega)$, we have

$$\int_{\Omega} |u_0|^{p^*-p} |\mathcal{R}_h(u_k)|^p \, dx \rightarrow \int_{\Omega} |u_0|^{p^*-p} |\mathcal{R}_h(u)|^p \, dx.$$

On the other hand, by dominated convergence theorem we have

$$\int_{\Omega} |u_k|^{p^*-p} |\mathcal{T}_h(u_k)|^{p-1} |\mathcal{R}_h(u_k)| \, dx \rightarrow \int_{\Omega} |u|^{p^*-p} |\mathcal{T}_h(u)|^{p-1} |\mathcal{R}_h(u)| \, dx.$$

Summing up, denoting with $o_k(1)$ a quantity the goes to zero as $k \rightarrow \infty$, we have shown that

$$\begin{aligned}
\int_{\Omega} |u_k|^{p^*-1} |\mathcal{R}_h(u_k)| \, dx &\leq c(p) \left(\int_{\Omega} |u|^{p^*-p} |\mathcal{T}_h(v)|^{q-1} |\mathcal{R}_h(u)| \, dx + \int_{\Omega} |u_0|^{p^*-p} |\mathcal{R}_h(u)|^p \, dx \right) \\
&\quad + c(p) \|u_k - u_0\|_{p^*}^{p^*-p} \|\mathcal{R}_h(u_k)\|_{p^*}^p + o(1).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\int_{\Omega} |v_k|^{q^*-1} |\mathcal{R}_h(v_k)| \, dx &\leq c(q) \left(\int_{\Omega} |v|^{q^*-q} |\mathcal{T}_h(v)|^{q-1} |\mathcal{R}_h(v)| \, dx + \int_{\Omega} |v_0|^{q^*-q} |\mathcal{R}_h(v)|^q \, dx \right) \\
&\quad + c(q) \|v_k - v_0\|_{q^*}^{q^*-q} \|\mathcal{R}_h(v_k)\|_{q^*}^q + o(1).
\end{aligned}$$

Let us now consider the term

$$\int_{\Omega} |u_k|^{p^* \frac{q^*-1}{q^*}} |\mathcal{R}_h(v_k)| \, dx.$$

Denoting with

$$\gamma := p^* \frac{q^* - 1}{q^*} - \frac{p}{q'} = \frac{p^* q^* - p^* - \frac{p}{q'} q^*}{q^*},$$

we can write

$$\begin{aligned} & \int_{\Omega} |u_k|^{p^* \frac{q^* - 1}{q^*}} |\mathcal{R}_h(v_k)| \, dx \\ &= \int_{\Omega} |u_k|^{\gamma} |\mathcal{T}_h(u_k) + \mathcal{R}_h(u_k)|^{\frac{p}{q'}} |\mathcal{R}_h(v_k)| \, dx \\ &\leq c(p, q) \left(\int_{\Omega} |u_k|^{\gamma} |\mathcal{T}_h(u_k)|^{\frac{p}{q'}} |\mathcal{R}_h(v_k)| \, dx + \int_{\Omega} |u_k|^{\gamma} |\mathcal{R}_h(u_k)|^{\frac{p}{q'}} |\mathcal{R}_h(v_k)| \, dx \right) \\ &\leq c(p, q) \left(\int_{\Omega} |u_k|^{\gamma} |\mathcal{T}_h(u_k)|^{\frac{p}{q'}} |\mathcal{R}_h(v_k)| \, dx + \int_{\Omega} |u_0|^{\gamma} |\mathcal{R}_h(u_k)|^{\frac{p}{q'}} |\mathcal{R}_h(v_k)| \, dx \right. \\ &\quad \left. + \int_{\Omega} |u_k - u_0|^{\gamma} |\mathcal{R}_h(u_k)|^{\frac{p}{q'}} |\mathcal{R}_h(v_k)| \, dx \right). \end{aligned}$$

By applying the extension of Hölder inequality with conjugated exponents $\frac{p^*}{\gamma}$, $\frac{q'}{p} p^*$ and q^* , we have

$$\int_{\Omega} |u_k - u_0|^{\gamma} |\mathcal{R}_h(u_k)|^{\frac{p}{q'}} |\mathcal{R}_h(v_k)| \, dx \leq \|u_k - u_0\|_{p^*}^{\gamma} \|\mathcal{R}_h(u_k)\|_{p^*}^{\frac{p}{q'}} \|\mathcal{R}_h(v_k)\|_{q^*}.$$

Moreover, by Young's inequality with conjugated exponents q' and q , we have

$$\|\mathcal{R}_h(u_k)\|_{p^*}^{\frac{p}{q'}} \|\mathcal{R}_h(v_k)\|_{q^*} \leq \frac{1}{q'} \|\mathcal{R}_h(u_k)\|_{p^*}^p + \frac{1}{q} \|\mathcal{R}_h(v_k)\|_{q^*}^q,$$

whence

$$c(p, q) \int_{\Omega} |u_k - u_0|^{\gamma} |\mathcal{R}_h(u_k)|^{\frac{p}{q'}} |\mathcal{R}_h(v_k)| \, dx \leq c(p, q) \|u_k - u_0\|_{p^*}^{\gamma} \left(\|\mathcal{R}_h(u_k)\|_{p^*}^p + \|\mathcal{R}_h(v_k)\|_{q^*}^q \right).$$

We obtained

$$\begin{aligned} \int_{\Omega} |u_k|^{p^* \frac{q^* - 1}{q^*}} |\mathcal{R}_h(v_k)| \, dx &\leq c(p, q) \left(\int_{\Omega} |u_k|^{\gamma} |\mathcal{T}_h(u_k)|^{\frac{p}{q'}} |\mathcal{R}_h(v_k)| \, dx + \int_{\Omega} |u_0|^{\gamma} |\mathcal{R}_h(u_k)|^{\frac{p}{q'}} |\mathcal{R}_h(v_k)| \, dx \right) \\ &\quad + \|u_k - u_0\|_{p^*}^{\gamma} c(p, q) \left(\|\mathcal{R}_h(u_k)\|_{p^*}^p + \|\mathcal{R}_h(v_k)\|_{q^*}^q \right). \end{aligned}$$

Observe now that by dominated convergence theorem we have

$$\int_{\Omega} |u_k|^{\gamma} |\mathcal{T}_h(u_k)|^{\frac{p}{q'}} |\mathcal{R}_h(v_k)| \, dx \rightarrow \int_{\Omega} |u|^{\gamma} |\mathcal{T}_h(u)|^{\frac{p}{q'}} |\mathcal{R}_h(v)| \, dx.$$

By Proposition 1.3.3, we deduce that $|\mathcal{R}_h(u_k)|^{\frac{p}{q'}} |\mathcal{R}_h(v_k)| \rightharpoonup |\mathcal{R}_h(u)|^{\frac{p}{q'}} |\mathcal{R}_h(v)|$ in $L^{\left(\frac{p^*}{\gamma}\right)' }(\Omega)$.

Since $|u_0| \in L^{\frac{p^*}{\gamma}}(\Omega)$, we have

$$\int_{\Omega} |u_0|^{\gamma} |\mathcal{R}_h(u_k)|^{\frac{p}{q'}} |\mathcal{R}_h(v_k)| \, dx \rightarrow \int_{\Omega} |u_0|^{\gamma} |\mathcal{R}_h(u)|^{\frac{p}{q'}} |\mathcal{R}_h(v)| \, dx.$$

Hence,

$$\begin{aligned} \int_{\Omega} |u_k|^{p^* \frac{q^*-1}{q^*}} |\mathcal{R}_h(v_k)| \, dx &\leq c(p, q) \left(\int_{\Omega} |u|^\gamma |\mathcal{T}_h(u)|^{\frac{p}{q'}} |\mathcal{R}_h(v)| \, dx + \int_{\Omega} |u_0|^\gamma |\mathcal{R}_h(u)|^{\frac{p}{q'}} |\mathcal{R}_h(v)| \, dx \right) \\ &\quad + c(p, q) \|u_k - u_0\|_{p^*}^\gamma \left(\|\mathcal{R}_h(u_k)\|_{p^*}^p + \|\mathcal{R}_h(v_k)\|_{q^*}^q \right) + o_k(1). \end{aligned}$$

Arguing in a similar way on the term

$$\int_{\Omega} |v_k|^{q^* \frac{p^*-1}{p^*}} |\mathcal{R}_h(u_k)| \, dx,$$

and denoting with

$$\delta := q^* \frac{p^*-1}{p^*} - \frac{q}{p'} = \frac{q^* p^* - q^* - \frac{q}{p'} p^*}{p^*},$$

we get

$$\begin{aligned} \int_{\Omega} |v_k|^{q^* \frac{p^*-1}{p^*}} |\mathcal{R}_h(u_k)| \, dx &\leq c(p, q) \left(\int_{\Omega} |v|^\delta |\mathcal{T}_h(v)|^{\frac{q}{p'}} |\mathcal{R}_h(v)| \, dx + \int_{\Omega} |v_0|^\delta |\mathcal{R}_h(v)|^{\frac{q}{p'}} |\mathcal{R}_h(u)| \, dx \right) \\ &\quad + c(p, q) \|v_k - v_0\|_{q^*}^\delta \left(\|\mathcal{R}_h(v_k)\|_{q^*}^q + \|\mathcal{R}_h(u_k)\|_{p^*}^p \right) + o_k(1). \end{aligned}$$

Altogether, denoting with $K := K(p, q, C)$ the greatest positive constant in previous inequalities, by (1.3.18) we deduce

$$\begin{aligned} &\limsup_{k \rightarrow \infty} \left[\frac{S(p, N)}{p} \|\mathcal{R}_h(u_k)\|_{p^*}^p - K \|u_k - u_0\|_{p^*}^{p^*-p} \|\mathcal{R}_h(u_k)\|_{p^*}^p \right. \\ &\quad + \frac{S(q, N)}{q} \|\mathcal{R}_h(v_k)\|_{q^*}^q - K \|v_k - v_0\|_{q^*}^{q^*-q} \|\mathcal{R}_h(v_k)\|_{q^*}^q \\ &\quad - K \|u_k - u_0\|_{p^*}^\gamma \left(\|\mathcal{R}_h(u_k)\|_{p^*}^p + \|\mathcal{R}_h(v_k)\|_{q^*}^q \right) \\ &\quad \left. - K \|v_k - v_0\|_{q^*}^\delta \left(\|\mathcal{R}_h(v_k)\|_{q^*}^q + \|\mathcal{R}_h(u_k)\|_{p^*}^p \right) \right] \\ &\leq K \int_{\Omega} |u|^{p^*-1} |\mathcal{R}_h(u)| \, dx + K \int_{\Omega} |v|^{q^* \frac{p^*-1}{p^*}} |\mathcal{R}_h(u)| \, dx \\ &\quad + K \int_{\Omega} |u|^{p^* \frac{q^*-1}{q^*}} |\mathcal{R}_h(v)| \, dx + K \int_{\Omega} |v|^{q^*-1} |\mathcal{R}_h(v)| \, dx \\ &\quad + K \int_{\Omega} |\mathcal{R}_h(u)| \, dx + K \int_{\Omega} |\mathcal{R}_h(v)| \, dx + o_h(1) \\ &\quad + K \left(\int_{\Omega} |u|^{p^*-p} |\mathcal{T}_h(u)|^{p-1} |\mathcal{R}_h(u)| \, dx + \int_{\Omega} |u_0|^{p^*-p} |\mathcal{R}_h(u)|^p \, dx \right) \\ &\quad + K \left(\int_{\Omega} |v|^{q^*-q} |\mathcal{T}_h(v)|^{q-1} |\mathcal{R}_h(v)| \, dx + \int_{\Omega} |v_0|^{q^*-q} |\mathcal{R}_h(v)|^q \, dx \right) \\ &\quad + K \left(\int_{\Omega} |u|^\gamma |\mathcal{T}_h(u)|^{\frac{p}{q'}} |\mathcal{R}_h(v)| \, dx + \int_{\Omega} |u_0|^\gamma |\mathcal{R}_h(u)|^{\frac{p}{q'}} |\mathcal{R}_h(v)| \, dx \right) \\ &\quad + K \left(\int_{\Omega} |v|^\delta |\mathcal{T}_h(v)|^{\frac{q}{p'}} |\mathcal{R}_h(v)| \, dx + \int_{\Omega} |v_0|^\delta |\mathcal{R}_h(v)|^{\frac{q}{p'}} |\mathcal{R}_h(u)| \, dx \right). \end{aligned}$$

Considering the left side of previous inequality, we have

$$\begin{aligned}
& \frac{S(p, N)}{p} \|\mathcal{R}_h(u_k)\|_{p^*}^p - K \|u_k - u_0\|_{p^*}^{p^*-p} \|\mathcal{R}_h(u_k)\|_{p^*}^p \\
& + \frac{S(q, N)}{q} \|\mathcal{R}_h(v_k)\|_{q^*}^q - K \|v_k - v_0\|_{q^*}^{q^*-q} \|\mathcal{R}_h(v_k)\|_{q^*}^q \\
& - K \|u_k - u_0\|_{p^*}^\gamma \left(\|\mathcal{R}_h(u_k)\|_{p^*}^p + \|\mathcal{R}_h(v_k)\|_{q^*}^q \right) \\
& - K \|v_k - v_0\|_{q^*}^\delta \left(\|\mathcal{R}_h(v_k)\|_{q^*}^q + \|\mathcal{R}_h(u_k)\|_{p^*}^p \right) \\
& = \left[\frac{S(p, N)}{p} - K \left(\|u_k - u_0\|_{p^*}^{p^*-p} + \|u_k - u_0\|_{p^*}^\gamma + \|v_k - v_0\|_{q^*}^\delta \right) \right] \|\mathcal{R}_h(u_k)\|_{p^*}^p \\
& + \left[\frac{S(q, N)}{q} - K \left(\|v_k - v_0\|_{q^*}^{q^*-q} + \|v_k - v_0\|_{q^*}^\delta + \|u_k - u_0\|_{p^*}^\gamma \right) \right] \|\mathcal{R}_h(v_k)\|_{q^*}^q.
\end{aligned}$$

Now, let us consider

$$\overline{B_R(z_0)} := \{z \in X : \|z - z_0\| \leq r\},$$

where we recall $R > 0$ is a general radius.

Since $\{z_k\}_k = \{(u_k, v_k)\}_k \subset \overline{B_R(z_0)}$, we can choose R small enough such that for any k we have

$$\frac{S(p, N)}{p} - K \left(\|u_k - u_0\|_{p^*}^{p^*-p} + \|u_k - u_0\|_{p^*}^\gamma + \|v_k - v_0\|_{q^*}^\delta \right) \geq M$$

and

$$\frac{S(q, N)}{q} - K \left(\|v_k - v_0\|_{q^*}^{q^*-q} + \|v_k - v_0\|_{q^*}^\delta + \|u_k - u_0\|_{p^*}^\gamma \right) \geq M$$

for some $M > 0$. Therefore,

$$\begin{aligned}
& M \limsup_{k \rightarrow \infty} \left(\|\mathcal{R}_h(u_k)\|_{p^*}^p + \|\mathcal{R}_h(v_k)\|_{q^*}^q \right) \\
& \leq K \int_{\Omega} |u|^{p^*-1} |\mathcal{R}_h(u)| \, dx + K \int_{\Omega} |v|^{q^* \frac{p^*-1}{p^*}} |\mathcal{R}_h(u)| \, dx \\
& + K \int_{\Omega} |u|^{p^* \frac{q^*-1}{q^*}} |\mathcal{R}_h(v)| \, dx + K \int_{\Omega} |v|^{q^*-1} |\mathcal{R}_h(v)| \, dx \\
& + K \int_{\Omega} |\mathcal{R}_h(u)| \, dx + K \int_{\Omega} |\mathcal{R}_h(v)| \, dx + o_h(1) \\
& + K \left(\int_{\Omega} |u|^{p^*-p} |\mathcal{T}_h(u)|^{p-1} |\mathcal{R}_h(u)| \, dx + \int_{\Omega} |u_0|^{p^*-p} |\mathcal{R}_h(u)|^p \, dx \right) \\
& + K \left(\int_{\Omega} |v|^{q^*-q} |\mathcal{T}_h(v)|^{q-1} |\mathcal{R}_h(v)| \, dx + \int_{\Omega} |v_0|^{q^*-q} |\mathcal{R}_h(v)|^q \, dx \right) \\
& + K \left(\int_{\Omega} |u|^\gamma |\mathcal{T}_h(u)|^{\frac{p}{q^*}} |\mathcal{R}_h(v)| \, dx + \int_{\Omega} |u_0|^\gamma |\mathcal{R}_h(u)|^{\frac{p}{q^*}} |\mathcal{R}_h(v)| \, dx \right) \\
& + K \left(\int_{\Omega} |v|^\delta |\mathcal{T}_h(v)|^{\frac{q}{p^*}} |\mathcal{R}_h(v)| \, dx + \int_{\Omega} |v_0|^\delta |\mathcal{R}_h(v)|^{\frac{q}{p^*}} |\mathcal{R}_h(u)| \, dx \right).
\end{aligned}$$

Taking into account Remark 1.3.1, we can apply dominated convergence theorem (with respect to h) to the right side of previous inequality, obtaining that

$$\lim_{h \rightarrow \infty} \limsup_{k \rightarrow \infty} \left(\|\mathcal{R}_h(u_k)\|_{p^*}^p + \|\mathcal{R}_h(v_k)\|_{q^*}^q \right) = 0,$$

by which we deduce

$$\lim_{h \rightarrow \infty} \limsup_{k \rightarrow \infty} \|\mathcal{R}_h(u_k)\|_{p^*} = 0 \quad \text{and} \quad \lim_{h \rightarrow \infty} \limsup_{k \rightarrow \infty} \|\mathcal{R}_h(v_k)\|_{q^*} = 0.$$

Now, observing that

$$\|u_k - u\|_{p^*} \leq \|\mathcal{T}_h(u_k) - \mathcal{T}_h(u)\|_{p^*} + \|\mathcal{R}_h(u_k)\|_{p^*} + \|\mathcal{R}_h(u)\|_{p^*},$$

we have

$$\limsup_{k \rightarrow \infty} \|u_k - u\|_{p^*} \leq \limsup_{k \rightarrow \infty} \|\mathcal{R}_h(u_k)\|_{p^*} + \|\mathcal{R}_h(u)\|_{p^*}.$$

By dominated convergence theorem, we have

$$\lim_{h \rightarrow \infty} \|\mathcal{R}_h(u)\|_{p^*} = 0,$$

hence we deduce

$$\|u_k - u\|_{p^*} \rightarrow 0,$$

and similarly

$$\|v_k - v\|_{q^*} \rightarrow 0.$$

Finally, we infer that

$$\int_{\Omega} [F_s(u_k, v_k)(u_k - u) + F_t(u_k, v_k)(v_k - v)] \, dx \rightarrow 0,$$

and coming back to (1.3.14), we deduce

$$\limsup_{k \rightarrow \infty} \langle H_{a,p,b,q}(z_k), z_k - z \rangle \leq 0,$$

whence, by applying Proposition 1.2.2, we conclude that $\|z_k - z\| \rightarrow 0$.

□

1.4 A Poincaré-Hopf formula for systems

In this Section we prove Theorem 1.1.3, that is each isolated critical point of the functional $J_{a,b}$ defined in (1.1.1) and associated to system (1.3.8), has critical groups of finite type and the Poincaré-Hopf formula holds. We retrace the same ideas contained in [56], where the interested reader can find all proofs and details.

We will use the following result (see [56, Proposition 4.1]), which shows that a Poincaré-Hopf formula holds in finite dimension.

Proposition 1.4.1. *Let Y be a finite dimensional space, W an open subset of Y and $f : W \rightarrow \mathbb{R}$ a functional of class C^1 . Let $\bar{w} \in W$ and let $r, \sigma > 0$ be such that $\overline{B_{4r}(\bar{w})} \subseteq W$ and*

(a) $f'(w) \neq 0$ for every $w \in \overline{B_{4r}(\bar{w})}$ with $|f(w) - f(\bar{w})| \geq 3\sigma r/5$;

(b)

$$\inf \left\{ \|f'(w)\| : w \in \overline{B_{4r}(\bar{w})} \setminus B_r(\bar{w}) \right\} > \sigma.$$

If we set

$$\begin{aligned} \beta(w) &:= f(\bar{w}) + \sigma \left(r - (\|w - \bar{w}\| - r)^+ \right), \\ \Sigma &:= \{w \in Y : f(\bar{w}) - 2\sigma r \leq f(w) \leq \beta(w)\}, \\ A &:= \{w \in \Sigma : f(w) \leq f(\bar{w}) - \sigma r\}, \end{aligned}$$

then $H_*(\Sigma, A)$ is of finite type and we have

$$\deg(f', B_r(\bar{w}), 0) = \sum_{m \geq 0} (-1)^m \operatorname{rank} H_m(\Sigma, A).$$

Let us recall also the following result, that gives information on the critical groups.

Proposition 1.4.2. *Let Y be a normed vector space and $f : Y \rightarrow \mathbb{R}$ a C^1 function. Let $\bar{w} \in Y$ and $r, \sigma > 0$ such that $\overline{B_{4r}(\bar{w})}$ is complete and*

(i) f satisfies (PS) on $\overline{B_{4r}(\bar{w})}$;

(ii) $f'(w) \neq 0$ for every $w \in \overline{B_{4r}(\bar{w})} \setminus \{\bar{w}\}$;

(iii)

$$\inf \left\{ \|f'(w)\| : w \in \overline{B_{4r}(\bar{w})} \setminus B_r(\bar{w}) \right\} > \sigma.$$

If we set

$$\begin{aligned} \beta(w) &:= f(\bar{w}) + \sigma \left(r - (\|w - \bar{w}\| - r)^+ \right), \\ \Sigma &:= \{w \in Y : f(\bar{w}) - 2\sigma r \leq f(w) \leq \beta(w)\}, \\ A &:= \{w \in \Sigma : f(w) \leq f(\bar{w}) - \sigma r\}, \end{aligned}$$

then we have $C_m(f, \bar{w}) \approx H_m(\Sigma, A)$ for every $m \geq 0$.

Proof of Theorem 1.1.3. Let $\bar{z}$ be an isolated critical point of the functional $J_{a,b}$ defined in (1.1.1). Moreover, taking into account that $J_{a,b} : X \rightarrow \mathbb{R}$ is of class C^1 and by means of Theorem 1.1.2 its derivative is locally of class $(S)_+$, there exist $L, r > 0$ such that $J_{a,b}$ is Lipschitz continuous of constant L on $\overline{B_{4r}(\bar{z})}$, $J'_{a,b}$ is of class $(S)_+$ on $\overline{B_{4r}(\bar{z})}$ and $J'_{a,b}(z) \neq 0$ for every $z \in \overline{B_{4r}(\bar{z})} \setminus \{\bar{z}\}$. In particular (see [56, Lemma 4.1, Lemma 4.2 and Lemma 4.3]), there exist $\sigma \in [0, L]$ and a finite dimensional subspace Y of X , with $\bar{z} \in Y$, such that, setting

$$\begin{aligned} \beta(z) &:= J_{a,b}(\bar{z}) + \sigma \left(r - (\|z - \bar{z}\| - r)^+ \right), \\ \Sigma &:= \{z \in X : J_{a,b}(\bar{z}) - 2\sigma r \leq J_{a,b}(z) \leq \beta(z)\}, \\ A &:= \{z \in \Sigma : J_{a,b}(z) \leq J_{a,b}(\bar{z}) - \sigma r\}, \end{aligned}$$

the following facts hold:

- $H_*(\Sigma, A)$ is of finite type;
- for any $m \geq 0$, we have

$$H_m(\Sigma, A) \approx H_m(\Sigma \cap Y, A \cap Y) \quad \text{for any } m \geq 0. \quad (1.4.19)$$

Since assumptions of Proposition 1.4.1 and Proposition 1.4.2 are satisfied, we also deduce

$$\deg \left((J_{a,b}|_{U \cap Y})', B_r(\bar{z}) \cap Y, 0 \right) = \sum_{m \geq 0} (-1)^m \text{rank } H_m(\Sigma \cap Y, A \cap Y) \quad (1.4.20)$$

and

$$C_m(J_{a,b}, \bar{z}) \approx H_m(\Sigma, A) \quad \text{for any } m \geq 0. \quad (1.4.21)$$

Moreover, the definition and properties of degree for maps of class $(S)_+$ (see [37]) give

$$\deg \left(J'_{a,b}, B_r(\bar{z}), 0 \right) = \deg \left((J_{a,b}|_{U \cap Y})', B_r(\bar{z}) \cap Y, 0 \right). \quad (1.4.22)$$

Altogether, by (1.4.19), (1.4.20), (1.4.21) and (1.4.22), we infer that

$$\deg \left(J'_{a,b}, B_r(\bar{z}), 0 \right) = \sum_{m \geq 0} (-1)^m \text{rank } C_m(J_{a,b}, \bar{z}). \quad (1.4.23)$$

□

Appendix A

Lemma 1.4.3. *For any $c \geq 0$ and for any $r > 1$, the map $d_{c,r} : \mathbb{R}^N \rightarrow \mathbb{R}^N$ defined as*

$$d_{c,r}(\xi) = (c + |\xi|^2)^{\frac{r-2}{2}} \xi \quad (1.4.24)$$

is strictly monotone, that is

$$[d_{c,r}(\xi) - d_{c,r}(\eta)] \cdot (\xi - \eta) > 0 \quad (1.4.25)$$

for any $\xi, \eta \in \mathbb{R}^N$, with $\eta \neq \xi$.

Proof. Let $\xi, \eta \in \mathbb{R}^N$, with $\eta \neq \xi$. If $|\xi| = |\eta|$, then

$$(d_{c,r}(\xi) - d_{c,r}(\eta)) \cdot (\xi - \eta) = (c + |\eta|^2)^{\frac{r-2}{2}} |\xi - \eta|^2 > 0.$$

Let now $|\xi| \neq |\eta|$. First of all, we observe that

$$\begin{aligned} [d_{c,r}(\xi) - d_{c,r}(\eta)] \cdot (\xi - \eta) &= \left[(c + |\xi|^2)^{\frac{r-2}{2}} \xi - (c + |\eta|^2)^{\frac{r-2}{2}} \eta \right] \cdot (\xi - \eta) \\ &= (c + |\xi|^2)^{\frac{r-2}{2}} |\xi|^2 + (c + |\eta|^2)^{\frac{r-2}{2}} |\eta|^2 - \xi \cdot \eta \left[(c + |\xi|^2)^{\frac{r-2}{2}} + (c + |\eta|^2)^{\frac{r-2}{2}} \right], \end{aligned}$$

hence (1.4.25) is equivalent to

$$\xi \cdot \eta \left[(c + |\xi|^2)^{\frac{r-2}{2}} + (c + |\eta|^2)^{\frac{r-2}{2}} \right] < |\xi|^2 (c + |\xi|^2)^{\frac{r-2}{2}} + |\eta|^2 (c + |\eta|^2)^{\frac{r-2}{2}}. \quad (1.4.26)$$

Let us consider the real function $f : [0, +\infty) \rightarrow [0, +\infty)$ defined as

$$f(t) = \left(c + t^2\right)^{\frac{r-2}{2}} t.$$

If $c = 0$, then $f(t) = t^{r-1}$ is strictly increasing, hence strictly monotone.

If $c > 0$, for any $t \geq 0$ we have

$$f'(t) = \left(c + t^2\right)^{\frac{r-2}{2}} + t^2(r-2) \left(c + t^2\right)^{\frac{r-4}{2}} = \left(c + t^2\right)^{\frac{r-4}{2}} [c + (r-1)t^2] > 0,$$

hence $f(t)$ is strictly increasing, so strictly monotone.

Therefore

$$(f(s) - f(t))(s - t) > 0,$$

that is

$$ts \left[\left(c + t^2\right)^{\frac{r-2}{2}} + \left(c + s^2\right)^{\frac{r-2}{2}} \right] < t^2 \left(c + t^2\right)^{\frac{r-2}{2}} + s^2 \left(c + s^2\right)^{\frac{r-2}{2}}$$

for any $s, t \in \mathbb{R}$ with $s \neq t$.

By applying previous one with $t = |\xi|$ and $s = |\eta|$, we obtain

$$|\xi||\eta| \left(\left(c + |\xi|^2\right)^{\frac{r-2}{2}} + \left(c + |\eta|^2\right)^{\frac{r-2}{2}} \right) < |\xi|^2 \left(c + |\xi|^2\right)^{\frac{r-2}{2}} + |\eta|^2 \left(c + |\eta|^2\right)^{\frac{r-2}{2}}$$

by which, using also that $\xi \cdot \eta \leq |\xi \cdot \eta| \leq |\xi||\eta|$, we deduce (1.4.26) and so (1.4.25). □

Lemma 1.4.4. *Let $c \geq 0$ and $r > 1$. The functional $f : W_0^{1,r}(\Omega) \rightarrow \mathbb{R}$ defined as*

$$f(w) := \frac{1}{r} \int_{\Omega} \left(c + |\nabla w|^2\right)^{\frac{r}{2}} dx, \quad (1.4.27)$$

is convex.

Proof. Let us begin by proving the convexity of the function $g : [0, +\infty) \rightarrow [0, +\infty)$ defined as

$$g(s) := \left(c + s^2\right)^{\frac{r}{2}}.$$

If $c = 0$, it's clear.

Now let $c > 0$. For any $s \geq 0$ we have

$$g'(s) = r s \left(c + s^2\right)^{\frac{r}{2}-1} \geq 0 \quad \text{and} \quad g''(s) = r \left(c + s^2\right)^{\frac{r}{2}-2} \left(c + (r-1)s^2\right) \geq 0,$$

by which we deduce that g is increasing and convex.

Let us now consider the function $\psi : \mathbb{R}^N \rightarrow \mathbb{R}$ defined as

$$\psi(\xi) := g(|\xi|) = \left(c + |\xi|^2\right)^{\frac{r}{2}}.$$

Considering the triangular inequality of $|\cdot|_{\mathbb{R}^N}$ and the fact that g is increasing and convex, we deduce that ψ is convex, in fact for every $\xi_1, \xi_2 \in \mathbb{R}^N$ and for every $t \in [0, 1]$, we have

$$\psi(t\xi_1 + (1-t)\xi_2) \leq g(t|\xi_1| + (1-t)|\xi_2|) \leq t g(|\xi_1|) + (1-t) g(|\xi_2|) = t\psi(\xi_1) + (1-t)\psi(\xi_2).$$

Finally, taking into account the increasing monotonicity and the linearity of the Lebesgue integral, we deduce that the functional $f : W_0^{1,r}(\Omega) \rightarrow \mathbb{R}$ defined in (1.4.27) is convex. □

Anisotropic Quasilinear Systems and Regularity Results

In this Chapter, we present new results concerning L^∞ -estimates for solutions to non-autonomous quasilinear systems involving operators in divergence form and nonlinearities up to critical growth. These operators take the form $-\operatorname{div}[\nabla\Psi(\nabla u)]$, where the function $\Psi : \mathbb{R}^N \rightarrow \mathbb{R}$ satisfies a general assumption that involves several nonlinear operators, such as the p -Laplacian and p -area for $1 < p < N$, arising in various nonlinear physical phenomena.

To prove L^∞ -estimates for this class of systems, the classical Stampacchia's Lemma [149] could not be directly applied due to the singular cases occurring when $1 < p < 2$. We overcome this difficulty through a careful application of a generalization of Stampacchia's Lemma, established by Mammoliti in [25]. In the context of quasilinear elliptic systems of two equations, where the framework is defined by the product space $X := W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$, an additional challenge arises from the potential coupling of u and v within the nonlinearity. Furthermore, since the nonlinearity may exhibit critical growth, we must proceed rigorously by establishing preliminary results up to the critical threshold. Finally, the nonlinearity may also depend on the independent variable $x \in \Omega$ and a parameter δ in a given interval $I \subseteq \mathbb{R}$.

We establish the L^∞ -boundedness of all solutions; furthermore, we show that these solutions are bounded uniformly in δ on any sufficiently small ball of X . In particular, our results can also be applied to establish the L^∞ -boundedness of solutions to eigenvalue problems for quasilinear elliptic systems, such as the de Thélin eigenvalue problem [74], which will be addressed in Chapter 4.

This Chapter is based on the paper [34].

2.1 Assumptions and Main results

Let us consider the following non-autonomous quasilinear system

$$\begin{cases} -\operatorname{div} [\nabla\Psi_1(\nabla u)] = H_s(\delta, x, u, v) & \text{in } \Omega, \\ -\operatorname{div} [\nabla\Psi_2(\nabla v)] = H_t(\delta, x, u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.1.1)$$

Denoting with $I \subseteq \mathbb{R}$ an interval, the function $H : I \times \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is such that $H(\delta, x, \cdot, \cdot) \in C^1(\mathbb{R}^2, \mathbb{R})$ for any $\delta \in I$ and for a.e. $x \in \Omega$; $H_s(\delta, \cdot, s, t)$, $H_t(\delta, \cdot, s, t)$ are measurable for every $(\delta, s, t) \in I \times \mathbb{R}^2$, and $H(\delta, x, 0, 0) \in L^1(\Omega)$ for any $\delta \in I$. We assume:

(Ψ) $\Psi_1, \Psi_2 : \mathbb{R}^N \rightarrow \mathbb{R}$ of class C^1 with $\Psi_1(0) = 0$, $\nabla \Psi_1(0) = 0$ and $\Psi_2(0) = 0$, $\nabla \Psi_2(0) = 0$. Moreover, given $c \geq 0$, $r > 1$ and denoting with $\Psi_{c,r} : \mathbb{R}^N \rightarrow \mathbb{R}$ the function defined as

$$\Psi_{c,r}(\xi) := \frac{1}{r} \left[\left(c + |\xi|^2 \right)^{\frac{r}{2}} - c^{\frac{r}{2}} \right], \quad (2.1.2)$$

we assume that there exist $a \geq 0$, $1 < p < N$ and $\frac{1}{p} < \nu_1 \leq C_1$ such that $(\Psi_1 - \nu_1 \Psi_{a,p})$ and $(C_1 \Psi_{a,p} - \Psi_1)$ are both convex, and there exist $b \geq 0$, $1 < q < N$ and $\frac{1}{q} < \nu_2 \leq C_2$ such that $(\Psi_2 - \nu_2 \Psi_{b,q})$ and $(C_2 \Psi_{b,q} - \Psi_2)$ are both convex.

($\mathcal{H}$) there exists $C_0 > 0$ such that for a.e. $x \in \Omega$ and every $(\delta, s, t) \in I \times \mathbb{R}^2$ we have

$$|H_s(\delta, x, s, t)| \leq C_0 \left(1 + |s|^{p^*-1} + |t|^{q^* \frac{p^*-1}{p^*}} \right),$$

$$|H_t(\delta, x, s, t)| \leq C_0 \left(1 + |s|^{p^* \frac{q^*-1}{q^*}} + |t|^{q^*-1} \right).$$

Condition (Ψ) was introduced in [58], where Cingolani Degiovanni and Vannella obtained regularity results for the scalar version of system (2.1.1), considering a nonlinearity that is independent of the parameter δ and is allowed to exhibit critical growth. (See also [85, 105, 122, 157, 158] and [53, 109] for recent results). In particular, assumption (Ψ) involves various anisotropic operators. For instance, we mention the interesting papers [12, 13] in which, letting $B(t) := t^p/p$ for $t > 0$, Antonini, Cianchi, Ciruolo, Farina and Maz'ya considered

$$\Psi_1 = B \circ \mathcal{H},$$

where the norm $\mathcal{H} : \mathbb{R}^N \rightarrow \mathbb{R}$ (different from the Euclidean norm) is of class C^2 in $\mathbb{R}^N \setminus \{O\}$, and its anisotropic unit ball is uniformly convex (see for instance the paper [68] of Cozzi, Farina and Valdinoci). Notice that if $\mathcal{H}$ is the Euclidean norm $\mathcal{H}(\xi) = |\xi|$, then we are in the isotropic case, and choosing $B(t) := t^p/p$ for $t > 0$, we get $-\operatorname{div} [\nabla \Psi_1 (\nabla u)] = -\Delta_p u$, hence we recover the p -Laplacian. From a mathematical point of view, the study of anisotropic operators is interesting since the lack of isotropy produces a richer geometric structure. From a physical point of view, anisotropic operators and setting arise in different contexts, such as surface energy (see e.g. [102]) crystallography (see e.g. [154, 155, 164]), thermodynamics (see e.g. [107]), noise-removal procedures in digital image processing (see e.g. [95]).

Quasilinear elliptic systems were studied by Marino and Winkert in [127], where L^∞ -estimates results are obtained under general conditions on the data (see assumption ($\tilde{H}$) and Theorem 3.4. in [127]) that allowed the application of Moser's iteration technique (see also [41, 73, 106, 126] for existence results and [28, 54, 71, 130, 131] for regularity results).

Here we aim to establish L^∞ -estimates for systems of the form (2.1.1) under the broad assumptions (Ψ) and ($\mathcal{H}$). It is immediate to see that, for any $1 < p < N$, $a \geq 0$ and $1 < q < N$, $b \geq 0$, the functions $\Psi_1 = \Psi_{a,p}$ and $\Psi_2 = \Psi_{b,q}$ satisfy assumption (Ψ) (it is enough to take $\nu_1 = C_1 = 1$ and $\nu_2 = C_2 = 1$ respectively). Therefore, taking into account that

$$\nabla \Psi_{c,r}(\xi) := \begin{cases} \left(c + |\xi|^2 \right)^{\frac{r-2}{2}} \xi & \text{if } \xi \neq 0, \\ 0 & \text{if } \xi = 0, \end{cases} \quad (2.1.3)$$

system (2.1.1) generalizes models involving (p, q) -Laplacian and (p, q) -area type operators.

In fact, if we take $\Psi_1 = \Psi_{0,p}$ and $\Psi_2 = \Psi_{0,q}$, and moreover we take a function H that does not depend on the variable δ , system (2.1.1) becomes

$$\begin{cases} -\Delta_p u = H_s(x, u, v) & \text{in } \Omega, \\ -\Delta_q v = H_t(x, u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.1.4)$$

In the case $p = q = 2$, a priori bounds for semilinear elliptic systems similar to (2.1.4) have been established in [64, 65, 70, 144] by employing different approaches.

Now, if we consider $\Psi_1 = \Psi_{a,p}$ and $\Psi_2 = \Psi_{b,q}$ with $a, b \geq 0$, by (2.1.3) problem (2.1.1) becomes

$$\begin{cases} -\operatorname{div} \left((a + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \right) = H_s(\delta, x, u, v) & \text{in } \Omega; \\ -\operatorname{div} \left((b + |\nabla v|^2)^{\frac{q-2}{2}} \nabla v \right) = H_t(\delta, x, u, v) & \text{in } \Omega; \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.1.5)$$

Obtaining L^∞ -estimates for weak solutions of (2.1.5) is challenging due to the coupling between u and v in $(\mathcal{H})$. In [43], Carmona, Cingolani, Martínez-Aparicio and Vannella proved such estimates assuming $a = b$, $2 \leq p, q < N$ and a subcritical nonlinearity $H : \mathbb{R}^2 \rightarrow \mathbb{R}$.

Under assumptions (Ψ) and $(\mathcal{H})$, we extend the results established for system (2.1.5) in several directions. Specifically, we consider the range $1 < p, q < N$ and allow the constants a and b to differ. Furthermore, the nonlinearity H may exhibit critical growth and can depend explicitly on both the spatial variable x and a parameter δ . Finally, we treat more general operators involving functions Ψ_1 and Ψ_2 satisfying condition (Ψ) .

To include the singular case (i.e. when $1 < p < 2$ or $1 < q < 2$), the classical Stampacchia's Lemma [149, Lemma 4.1] is no longer applicable. Instead, we employ [25, Lemma 2.1] (stated in Section 2.3), the proof of which relies on the following generalization of Stampacchia's Lemma established in [25, Lemma A.1]:

Lemma 2.1.1. *Let $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a non increasing function such that*

$$\varphi(h) \leq \frac{c_0}{(h-k)^\rho} k^{\theta\rho} [\varphi(k)]^{1+\lambda} \quad \forall h > k \geq k_0,$$

where $c_0 > 0$, $k_0 \geq 0$, $\rho > 0$, $0 \leq \theta < 1$ and $\lambda > 0$. Then there exists $k^* > 0$ such that $\varphi(k^*) = 0$.

Lastly, given that H may grow critically, we generalize the technical lemma from [161, Lemma 2.1] (see Section 2.2 for the statement), provided the radius R is chosen to be sufficiently small:

Lemma 2.1.2. *Let $r \in (1, N)$ and denote by r^* the conjugate Sobolev exponent of r . For any $\varepsilon > 0$ and $w_0 \in W_0^{1,r}(\Omega)$, there exist $\sigma > 0$ and $R > 0$ such that*

$$\int_{\{|w(x)| \geq \sigma\}} |w(x)|^{r^*} dx < \varepsilon$$

for any $w \in \overline{B_R(w_0)} = \{w \in W_0^{1,r}(\Omega) : \|w - w_0\|_{1,r} \leq R\}$.

Weak solutions of (2.1.1) correspond to the critical points of the Euler functional $I_{\delta, \Psi_1, \Psi_2} : X \rightarrow \mathbb{R}$ defined for any $z = (u, v) \in X$ as

$$I_{\delta, \Psi_1, \Psi_2}(z) = \int_{\Omega} \Psi_1(\nabla u) dx + \int_{\Omega} \Psi_2(\nabla v) dx - \int_{\Omega} H(\delta, x, u, v) dx.$$

We can now state the main result of this Chapter:

Theorem 2.1.3. *If (u, v) is a solution of (2.1.1) and $(\mathcal{H})$ and (Ψ) hold, then $(u, v) \in (L^\infty(\Omega))^2$. Moreover, for any fixed $z_0 = (u_0, v_0) \in X$ there exists a suitably small $R > 0$ such that for any $\delta \in I$ and denoting by*

$$D_{R,\delta}(z_0) := \{z \in X : \|z - z_0\| \leq R, I'_{\delta, \Psi_1, \Psi_2}(z) = 0\},$$

there exists $C > 0$ such that

$$\|u\|_\infty, \|v\|_\infty \leq C \quad \forall z = (u, v) \in D_{R,\delta}(z_0),$$

where both R and C depend on $z_0, C_0, |\Omega|, p, q, N, a, b, \nu_1$ and ν_2 , but not on δ .

So in the present Chapter we show carefully that, for any arbitrary $z_0 \in X$ there exists $R > 0$ such that the $(L^\infty(\Omega))^2$ -norm of any weak solution to (2.1.1) belonging to $B_R(z_0)$ is bounded by a constant C that is independent of $\delta \in I$.

As a consequence of Theorem 2.1.3, we obtain L^∞ -boundedness of solutions also for system (2.1.4). System of the form (2.1.4) include also different types of eigenvalue problem, such as the eigenvalue problem introduced by de Th  lin in [74]:

$$\begin{cases} -\Delta_p u = \lambda |u|^{\alpha-1} |v|^{\beta+1} u & \text{in } \Omega, \\ -\Delta_q v = \lambda |u|^{\alpha+1} |v|^{\beta-1} v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.1.6)$$

where $1 < p, q < N$ and $\alpha, \beta \geq 0$ satisfy $(\alpha + 1)/p + (\beta + 1)/q = 1$.

By our main result we deduce that if (u, v) is an eigenfunction associated to an eigenvalue $\bar{\lambda}$ of (2.1.6), then both $u \in L^\infty(\Omega)$ and $v \in L^\infty(\Omega)$, and their norms in $L^\infty(\Omega)$ are bounded by a constant that depends on $\bar{\lambda}$.

2.2 L^τ -Integrability for any $1 < \tau < \infty$

Let Ψ_1 be a function as in assumption (Ψ) . Then Ψ_1 is strictly convex and satisfies

$$\nu_1 \Psi_{a,p}(\xi) \leq \Psi_1(\xi) \leq C_1 \Psi_{a,p}(\xi) \quad \forall \xi \in \mathbb{R}^N, \quad (2.2.7)$$

$$\nu_1 (a + |\xi|^2)^{\frac{p-2}{2}} |\xi|^2 \leq \nabla \Psi_1(\xi) \cdot \xi \leq C_1 (a + |\xi|^2)^{\frac{p-2}{2}} |\xi|^2 \quad \forall \xi \in \mathbb{R}^N, \quad (2.2.8)$$

and the number p for which Ψ_1 satisfies assumption (Ψ) is unique.

In fact, by convexity of $(\Psi_1 - \nu_1 \Psi_{a,p})$ and $(C_1 \Psi_{a,p} - \Psi_1)$, and by using the fact that $\Psi_1(0) = 0$, $\Psi_{a,p}(0) = 0$, $\nabla \Psi_1(0) = 0$ and $\nabla \Psi_{a,p}(0) = 0$, we easily get (2.2.7). Observe now that by the strict convexity of $\Psi_{a,p}$ (see the Proof of Lemma 1.4.4), we get

$$\begin{aligned} 0 < \nu_1 (\nabla \Psi_{a,p}(\xi) - \nabla \Psi_{a,p}(\eta)) \cdot (\xi - \eta) &\leq (\nabla \Psi_1(\xi) - \nabla \Psi_1(\eta)) \cdot (\xi - \eta) \\ &\leq C_1 (\nabla \Psi_{a,p}(\xi) - \nabla \Psi_{a,p}(\eta)) \cdot (\xi - \eta) \end{aligned}$$

for any $\xi, \eta \in \mathbb{R}^N$ with $\xi \neq \eta$, and so we deduce that Ψ_1 is strictly convex and, evaluating in $\eta = 0$, we also infer (2.2.8). Moreover, by (2.2.7) we also get the uniqueness of p .

Similarly, we deduce that Ψ_2 considered in assumption (Ψ) is strictly convex and satisfies

$$\nu_2 \Psi_{b,q}(\xi) \leq \Psi_2(\xi) \leq C_2 \Psi_{b,q}(\xi) \quad \forall \xi \in \mathbb{R}^N,$$

$$\nu_2(b + |\xi|^2)^{\frac{p-2}{2}} |\xi|^2 \leq \nabla \Psi_2(\xi) \cdot \xi \leq C_2(b + |\xi|^2)^{\frac{p-2}{2}} |\xi|^2 \quad \forall \xi \in \mathbb{R}^N,$$

and the number q for which Ψ_2 satisfies assumption (Ψ) is unique.

Let us now consider the Euler functional $I_{\delta, \Psi_1, \Psi_2} : X \rightarrow \mathbb{R}$ associated to system (2.1.1), defined for any $z = (u, v) \in X$ as

$$I_{\delta, \Psi_1, \Psi_2}(z) = \int_{\Omega} \Psi_1(\nabla u) dx + \int_{\Omega} \Psi_2(\nabla v) dx - \int_{\Omega} H(\delta, x, u, v) dx. \quad (2.2.9)$$

By assumption $(\mathcal{H})$ and Nemytskii operator theory it follows that the part of (2.2.9) involving the nonlinearity H is of class C^1 on X . By Lemma 2.3.4 and Proposition 2.3.5 also the part of (2.2.9) involving the anisotropic-type operators is of class C^1 on X . Hence the functional $I_{\delta, \Psi_1, \Psi_2}$ is of class C^1 on X and for any $z_0 = (u_0, v_0)$, $z = (u, v) \in X$ we have

$$\begin{aligned} \langle I'_{\delta, \Psi_1, \Psi_2}(z_0), z \rangle &= \int_{\Omega} \nabla \Psi_1(\nabla u_0) \cdot \nabla u dx + \int_{\Omega} \nabla \Psi_2(\nabla v_0) \cdot \nabla v dx \\ &\quad - \int_{\Omega} (H_s(\delta, x, u_0, v_0)u + H_t(\delta, x, u_0, v_0)v) dx. \end{aligned}$$

Since in $(\mathcal{H})$ we are assuming that the function H could grow critically, in order to prove Theorem 2.1.3 we need to proceed carefully. Let us recall the following result (see [161, Lemma 2.1]):

Lemma 2.2.1. *Let $r \in (1, N)$ and denote by r^* the conjugate Sobolev exponent of r , namely $r^* = rN/(N - r)$. If $R, \varepsilon > 0$, $w_0 \in W_0^{1,r}(\Omega)$ and $\tilde{r} \in [1, r^*)$, there is $\sigma > 0$ such that*

$$\int_{\{|w(x)| \geq \sigma\}} |w(x)|^{\tilde{r}} dx < \varepsilon$$

for any $w \in \overline{B_R(w_0)} := \{w \in W_0^{1,r}(\Omega) : \|w - w_0\|_{1,r} \leq R\}$.

We need to improve previous lemma up to critical growth, and we can do this provided we take a suitably small radius R .

Lemma 2.2.2. *Let $r \in (1, N)$ and denote by r^* the conjugate Sobolev exponent of r . For any $\varepsilon > 0$ and $w_0 \in W_0^{1,r}(\Omega)$, there exist $\sigma > 0$ and $R > 0$ such that*

$$\int_{\{|w(x)| \geq \sigma\}} |w(x)|^{r^*} dx < \varepsilon$$

for any $w \in \overline{B_R(w_0)}$.

Proof. By contradiction, assume that there are $\varepsilon > 0$ and $w_0 \in W_0^{1,r}(\Omega)$ such that, for any $\sigma > 0$ and $R > 0$ there exists $w_{\sigma,R} \in \overline{B_R(w_0)}$ such that

$$\int_{\{|w_{\sigma,R}(x)| \geq \sigma\}} |w_{\sigma,R}(x)|^{r^*} dx \geq \varepsilon.$$

Hence, we deduce that for any $R > 0$ there exists a sequence $\{w_k\}_k \subset \overline{B_R(w_0)}$ such that

$$\int_{\{|w_k(x)| \geq k\}} |w_k(x)|^{r^*} dx \geq \varepsilon \quad \forall k \in \mathbb{N}. \quad (2.2.10)$$

Up to subsequence, there exists $\bar{w} \in W_0^{1,r}(\Omega)$ such that w_k converges to $\bar{w}$ weakly in $W_0^{1,r}(\Omega)$ and strongly in $L^t(\Omega)$ for any $1 \leq t < r^*$.

Denoting with $E_k := \{x \in \Omega : |w_k(x)| \geq k\}$, by Chebyshev inequality we know that

$$|E_k| \leq \frac{1}{k} \int_{\Omega} |w_k| dx,$$

by which we deduce that $|E_k| \rightarrow 0$ as $k \rightarrow +\infty$.

Observe now that

$$\begin{aligned} \int_{\{|w_k(x)| \geq k\}} |w_k(x)|^{r^*} dx &= \int_{E_k} |w_k(x) - w_0(x) + w_0(x)|^{r^*} dx \\ &\leq 2^{r^*} \int_{E_k} |w_k(x) - w_0(x)|^{r^*} dx + 2^{r^*} \int_{E_k} |w_0(x)|^{r^*} dx \\ &\leq 2^{r^*} \int_{\Omega} |w_k(x) - w_0(x)|^{r^*} dx + 2^{r^*} \int_{E_k} |w_0(x)|^{r^*} dx. \end{aligned} \quad (2.2.11)$$

Since $|E_k| \rightarrow 0$ as $k \rightarrow +\infty$ and $w_0 \in L^{r^*}(\Omega)$, by absolute continuity of the Lebesgue integral we deduce

$$\int_{E_k} |w_0(x)|^{r^*} dx \rightarrow 0 \quad \text{as } k \rightarrow +\infty.$$

Taking into account Sobolev-Gagliardo-Nirenberg inequality, there exists a constant $C := C(r, N)$ such that

$$\int_{\Omega} |w_k(x) - w_0(x)|^{r^*} dx \leq C \|w_k - w_0\|_{1,r}^{r^*} \leq CR^{r^*} \quad \forall k \in \mathbb{N}.$$

By choosing $R = \frac{1}{2C^{\frac{1}{r^*}}} \bar{\varepsilon}^{\frac{1}{r^*}}$ with $0 < \bar{\varepsilon} < \varepsilon$ and passing to $\limsup_{k \rightarrow \infty}$ in (2.2.11), we obtain

$$\limsup_{k \rightarrow \infty} \int_{\{|w_k(x)| \geq n\}} |w_k(x)|^{r^*} dx \leq \bar{\varepsilon},$$

that is a contradiction with (2.2.10). □

To prove Theorem 2.1.3, inspired by [43], [105] and [161], we firstly prove the following result.

Proposition 2.2.3. *For any fixed $z_0 = (u_0, v_0) \in X$ there exists a suitably small $R > 0$ such that for any $\delta \in I$, denoting by*

$$D_{R,\delta}(z_0) = \{z \in X : \|z - z_0\| \leq R, I'_{\delta, \Psi_1, \Psi_2}(z) = 0\},$$

and for any $\gamma > 1$ there exists $C > 0$ such that

$$\int_{\Omega} |\bar{u}|^{\gamma p^*} dx \leq C \quad \text{and} \quad \int_{\Omega} |\bar{v}|^{\gamma q^*} dx \leq C \quad \forall \bar{z} = (\bar{u}, \bar{v}) \in D_{R,\delta}(z_0),$$

where both R and C depend on $z_0, C_0, |\Omega|, p, q, N, a, b, \nu_1, \nu_2$ and γ but not on δ .

Proof. For simplicity, we will omit dx . For every $\gamma, t, k > 1$ we define

$$h_{k,\gamma}(s) := \begin{cases} s|s|^{\gamma-1} & \text{if } |s| \leq k, \\ \gamma k^{\gamma-1}s + \text{sign}(s)(1-\gamma)k^\gamma & \text{if } |s| > k, \end{cases}$$

$$\Phi_{k,t,\gamma}(s) := \int_0^s \left| h'_{k,\gamma}(r) \right|^{\frac{t}{\gamma}} dr.$$

Observe that $h_{k,\gamma}$ and $\Phi_{k,t,\gamma}$ are C^1 -functions with bounded derivative, depending on γ, t and k . In particular, if $w \in W_0^{1,r}(\Omega)$ then $h_{k,\gamma}(w)$ and $\Phi_{k,t,\gamma}(w)$ are in $W_0^{1,r}(\Omega)$.

We see that

$$|h'_{k,\gamma}(s)| \leq \gamma |h_{k,\gamma}(s)|^{\frac{\gamma-1}{\gamma}}, \quad (2.2.12)$$

and for every $t \geq \gamma$ there exists a positive constant C , depending on γ and t but independent of k , such that

$$|s|^{\frac{t}{\gamma}-1} |\Phi_{k,t,\gamma}(s)| \leq C |h_{k,\gamma}(s)|^{\frac{t}{\gamma}} \quad (2.2.13)$$

$$|\Phi_{k,t,\gamma}(s)| \leq C |h_{k,\gamma}(s)|^{\frac{1}{\gamma}(1+t\frac{\gamma-1}{\gamma})} \quad (2.2.14)$$

and

$$\left| h_{k,\gamma} \left(|s|^{\frac{q^*}{p^*}} \right) \right|^{p^*} \leq C \left| h_{k^{\frac{p^*}{q^*},\gamma}(s)} \right|^{q^*}. \quad (2.2.15)$$

Moreover, for any $u \in W_0^{1,p}(\Omega)$ we have

$$\nabla h_{k,\gamma}(u) = h'_{k,\gamma}(u) \nabla u, \quad (2.2.16)$$

and

$$\nabla \Phi_{k,\gamma p,\gamma}(u) = \Phi'_{k,\gamma p,\gamma}(u) \nabla \bar{u} = \left| h'_{k,\gamma}(u) \right|^p \nabla u. \quad (2.2.17)$$

For any $p > 1$, $a \geq 0$ and $s \geq 0$, we have

$$s^p \leq \left(a + s^2 \right)^{\frac{p-2}{2}} s^2 + a^{\frac{p}{2}}. \quad (2.2.18)$$

The inequality is obvious if $p \geq 2$ or $s = 0$.

Otherwise, if $p \in (1, 2)$ and $s \neq 0$, we have

$$s^p \leq \left(a + s^2 \right)^{\frac{p}{2}} = \left(a + s^2 \right) \left(a + s^2 \right)^{\frac{p-2}{2}} \leq \left(a + s^2 \right)^{\frac{p-2}{2}} s^2 + a^{\frac{p}{2}}.$$

Let us fix $R > 0$. For any $\delta \in I$, let us consider $\bar{z} = (\bar{u}, \bar{v}) \in D_{R,\delta}(z_0)$.

In particular,

$$\left\langle I'_{\delta,\Psi_1,\Psi_2}(\bar{z}), (\Phi_{k,\gamma p,\gamma}(\bar{u}), 0) \right\rangle = 0 \quad \text{for any } k, \gamma > 1.$$

By Sobolev-Gagliardo-Nirenberg inequality, there exists $c := c(p, N) > 0$ such that

$$\left(\int_{\Omega} |u|^{p^*} \right)^{\frac{p}{p^*}} \leq c \int_{\Omega} |\nabla u|^p \quad \text{for any } u \in W_0^{1,p}(\Omega).$$

Furthermore, taking into account (2.2.8), (2.2.16), (2.2.17) and (2.2.18), we have

$$\begin{aligned} \left(\int_{\Omega} |h_{k,\gamma}(\bar{u})|^{p^*} \right)^{\frac{p}{p^*}} &\leq c \int_{\Omega} |\nabla \bar{u}|^p |h'_{k,\gamma}(\bar{u})|^p \\ &\leq c \int_{\Omega} (a + |\nabla \bar{u}|^2)^{\frac{p-2}{2}} |\nabla \bar{u}|^2 |h'_{k,\gamma}(\bar{u})|^p + c a^{\frac{p}{2}} \int_{\Omega} |h'_{k,\gamma}(\bar{u})|^p \\ &\leq \frac{c}{\nu_1} \int_{\Omega} \nabla \Psi_1(\nabla \bar{u}) \cdot \nabla \bar{u} |h'_{k,\gamma}(\bar{u})|^p + c a^{\frac{p}{2}} \int_{\Omega} |h'_{k,\gamma}(\bar{u})|^p \\ &= \frac{c}{\nu_1} \int_{\Omega} \nabla \Psi_1(\nabla \bar{u}) \cdot \nabla \Phi_{k,\gamma p,\gamma}(\bar{u}) + c a^{\frac{p}{2}} \int_{\Omega} |h'_{k,\gamma}(\bar{u})|^p \\ &= \frac{c}{\nu_1} \int_{\Omega} H_s(\delta, x, \bar{u}, \bar{v}) \Phi_{k,\gamma p,\gamma}(\bar{u}) + c a^{\frac{p}{2}} \int_{\Omega} |h'_{k,\gamma}(\bar{u})|^p \\ &\leq \tilde{c} \int_{\Omega} |H_s(\delta, x, \bar{u}, \bar{v})| |\Phi_{k,\gamma p,\gamma}(\bar{u})| + \tilde{c} \int_{\Omega} |h'_{k,\gamma}(\bar{u})|^p. \end{aligned}$$

Considering also $(\mathcal{H})$, we get

$$\left(\int_{\Omega} |h_{k,\gamma}(\bar{u})|^{p^*} \right)^{\frac{p}{p^*}} \leq \hat{c} \left(\int_{\Omega} (|\bar{u}|^{p^*-1} + 1) |\Phi_{k,\gamma p,\gamma}(\bar{u})| + \int_{\Omega} |\bar{v}|^{q^* \frac{p^*-1}{p^*}} |\Phi_{k,\gamma p,\gamma}(\bar{u})| \right) + \hat{c} \int_{\Omega} |h'_{k,\gamma}(\bar{u})|^p. \quad (2.2.19)$$

By (2.2.12) and Hölder's inequality with $\frac{p^*\gamma}{p(\gamma-1)}$ and $\left(\frac{p^*\gamma}{p(\gamma-1)}\right)'$, we get

$$\hat{c} \int_{\Omega} |h'_{k,\gamma}(\bar{u})|^p \leq \hat{c} \gamma^p \int_{\Omega} |h_{k,\gamma}(\bar{u})|^{p \frac{\gamma-1}{\gamma}} \leq c_0 \left(\int_{\Omega} |h_{k,\gamma}(\bar{u})|^{p^*} \right)^{\frac{p}{p^*} \frac{\gamma-1}{\gamma}}, \quad (2.2.20)$$

where $c_0 > 0$ depends on $C_0, |\Omega|, p, N, a, \nu_1$ and γ but not on k and δ .

For any $\sigma > 1$, $r > 1$ and w in $W_0^{1,r}(\Omega)$, we denote by

$$\Omega_{\sigma,w} := \{x \in \Omega : |w(x)| > \sigma\}.$$

From now on, when necessary, we redefine the positive constant c_1 , depending on $C_0, |\Omega|, p, q, N, a, \nu_1$ and γ but not on k and δ . Therefore, using (2.2.13), (2.2.14) and Hölder inequality with conjugated exponents $\frac{p^*\gamma}{p\gamma+1-p}$ and $\left(\frac{p^*\gamma}{p\gamma+1-p}\right)'$, and with conjugated exponents $\frac{p^*}{p^*-p}$ and $\frac{p^*}{p}$ respectively, we have

$$\begin{aligned} \int_{\Omega} (|\bar{u}|^{p^*-1} + 1) |\Phi_{k,\gamma p,\gamma}(\bar{u})| &= \int_{\Omega} |\Phi_{k,\gamma p,\gamma}(\bar{u})| + \int_{\Omega \setminus \Omega_{\sigma,\bar{u}}} |\bar{u}|^{p^*-1} |\Phi_{k,\gamma p,\gamma}(\bar{u})| + \int_{\Omega_{\sigma,\bar{u}}} |\bar{u}|^{p^*-1} |\Phi_{k,\gamma p,\gamma}(\bar{u})| \\ &\leq (\sigma^{p^*-1} + 1) \int_{\Omega} |\Phi_{k,\gamma p,\gamma}(\bar{u})| + \int_{\Omega_{\sigma,\bar{u}}} |\bar{u}|^{p^*-p} |\bar{u}|^{p-1} |\Phi_{k,\gamma p,\gamma}(\bar{u})| \\ &\leq 2\sigma^{p^*-1} \int_{\Omega} |\Phi_{k,\gamma p,\gamma}(\bar{u})| + c_1 \int_{\Omega_{\sigma,\bar{u}}} |\bar{u}|^{p^*-p} |h_{k,\gamma}(\bar{u})|^p \\ &\leq c_1 \sigma^{p^*-1} \int_{\Omega} |h_{k,\gamma}(\bar{u})|^{\frac{p\gamma+1-p}{\gamma}} + c_1 \int_{\Omega_{\sigma,\bar{u}}} |\bar{u}|^{p^*-p} |h_{k,\gamma}(\bar{u})|^p. \end{aligned} \quad (2.2.21)$$

$$\leq c_1 \sigma^{p^*-1} \left(\int_{\Omega} |h_{k,\gamma}(\bar{u})|^{p^*} \right)^{\frac{p}{p^*} \frac{\gamma p + 1 - p}{\gamma p}} + c_1 \|\bar{u}\|_{L^{p^*}(\Omega_{\sigma, \bar{u}})}^{p^*-p} \left(\int_{\Omega} |h_{k,\gamma}(\bar{u})|^{p^*} \right)^{\frac{p}{p^*}}.$$

Now, we deal with the integral $\int_{\Omega} |\bar{v}|^{q^* \frac{p^*-1}{p^*}} |\Phi_{k,\gamma p,\gamma}(\bar{u})|$ in (2.2.19). By using (2.2.13), (2.2.14), (2.2.15), the fact that $\Phi_{k,\gamma p,\gamma}(s)$ is nondecreasing for $s \geq 0$ and Hölder inequality as before, we obtain

$$\begin{aligned} \int_{\Omega} |\bar{v}|^{q^* \frac{p^*-1}{p^*}} |\Phi_{k,\gamma p,\gamma}(\bar{u})| &= \int_{\Omega \setminus \Omega_{\sigma, \bar{v}}} |\bar{v}|^{q^* \frac{p^*-1}{p^*}} |\Phi_{k,\gamma p,\gamma}(\bar{u})| + \int_{\Omega_{\sigma, \bar{v}}} |\bar{v}|^{\frac{q^*}{p^*}(p^*-p)} |\bar{v}|^{\frac{q^*}{p^*}(p-1)} |\Phi_{k,\gamma p,\gamma}(\bar{u})| \\ &\leq \sigma^{q^* \frac{p^*-1}{p^*}} \int_{\Omega} |\Phi_{k,\gamma p,\gamma}(\bar{u})| + \int_{\Omega_{\sigma, \bar{v}} \cap \{|\bar{v}|^{\frac{q^*}{p^*}} \leq |\bar{u}|\}} |\bar{v}|^{\frac{q^*}{p^*}(p^*-p)} |\bar{u}|^{p-1} |\Phi_{k,\gamma p,\gamma}(\bar{u})| \\ &\quad + \int_{\Omega_{\sigma, \bar{v}} \cap \{|\bar{v}|^{\frac{q^*}{p^*}} \geq |\bar{u}|\}} |\bar{v}|^{\frac{q^*}{p^*}(p^*-p)} (|\bar{v}|^{\frac{q^*}{p^*}})^{p-1} |\Phi_{k,\gamma p,\gamma}(\bar{v})| \\ &\leq c_1 \sigma^{q^* \frac{p^*-1}{p^*}} \int_{\Omega} |h_{k,\gamma}(\bar{u})|^{\frac{p\gamma+1-p}{\gamma}} + c_1 \int_{\Omega_{\sigma, \bar{v}} \cap \{|\bar{v}|^{\frac{q^*}{p^*}} \leq |\bar{u}|\}} |\bar{v}|^{\frac{q^*}{p^*}(p^*-p)} |h_{k,\gamma}(\bar{u})|^p \\ &\quad + c_1 \int_{\Omega_{\sigma, \bar{v}} \cap \{|\bar{v}|^{\frac{q^*}{p^*}} \geq |\bar{u}|\}} |\bar{v}|^{\frac{q^*}{p^*}(p^*-p)} |h_{k,\gamma}(\bar{v})|^p \\ &\leq c_1 \sigma^{q^* \frac{p^*-1}{p^*}} \left(\int_{\Omega} |h_{k,\gamma}(\bar{u})|^{p^*} \right)^{\frac{p}{p^*} \frac{\gamma p + 1 - p}{\gamma p}} \\ &\quad + c_1 \|\bar{v}\|_{L^{q^*}(\Omega_{\sigma, \bar{v}})}^{\frac{q^*}{p^*}(p^*-p)} \left(\left(\int_{\Omega} |h_{k,\gamma}(\bar{u})|^{p^*} \right)^{\frac{p}{p^*}} + \left(\int_{\Omega} \left| h_{k,\frac{p^*}{q^*},\gamma}(\bar{v}) \right|^{q^*} \right)^{\frac{p}{p^*}} \right). \end{aligned}$$

Combining with (2.2.19), (2.2.20) and (2.2.21), we get

$$\begin{aligned} \left(\int_{\Omega} |h_{k,\gamma}(\bar{u})|^{p^*} \right)^{\frac{p}{p^*}} &\leq c_1 \sigma^{p^*-1} \left(\int_{\Omega} |h_{k,\gamma}(\bar{u})|^{p^*} \right)^{\frac{p}{p^*} \frac{\gamma p + 1 - p}{\gamma p}} + c_1 \|\bar{u}\|_{L^{p^*}(\Omega_{\sigma, \bar{u}})}^{p^*-p} \left(\int_{\Omega} |h_{k,\gamma}(\bar{u})|^{p^*} \right)^{\frac{p}{p^*}} \\ &\quad + c_1 \sigma^{\frac{q^*}{p^*}(p^*-1)} \left(\int_{\Omega} |h_{k,\gamma}(\bar{u})|^{p^*} \right)^{\frac{p}{p^*} \frac{\gamma p + 1 - p}{\gamma p}} \\ &\quad + c_1 \|\bar{v}\|_{L^{q^*}(\Omega_{\sigma, \bar{v}})}^{\frac{q^*}{p^*}(p^*-p)} \left(\left(\int_{\Omega} |h_{k,\gamma}(\bar{u})|^{p^*} \right)^{\frac{p}{p^*}} + \left(\int_{\Omega} \left| h_{k,\frac{p^*}{q^*},\gamma}(\bar{v}) \right|^{q^*} \right)^{\frac{p}{p^*}} \right) \\ &\quad + c_1 \left(\int_{\Omega} |h_{k,\gamma}(\bar{u})|^{p^*} \right)^{\frac{p}{p^*} \frac{\gamma-1}{\gamma}}, \end{aligned}$$

where the constant c_1 depends on $C_0, |\Omega|, p, q, N, a, \nu_1$ and γ but not on k and δ .

By using Lemma 2.2.2 with $\varepsilon = (4c_1)^{\frac{p^*}{p^*-p^*}}$ and $u_0 \in W_0^{1,p}(\Omega)$, and with $\varepsilon = (4c_1)^{\frac{p^*}{p^*-p^*}}$ and $v_0 \in W_0^{1,q}(\Omega)$ respectively, we infer that there exist $\sigma_1 > 1$ and $R_1 > 0$ such that, for any $\sigma \geq \sigma_1$ and for any $(\bar{u}, \bar{v}) \in D_{R_1, \delta}(z_0)$,

$$c_1 \|\bar{u}\|_{L^{p^*}(\Omega_{\sigma, \bar{u}})}^{p^*-p} + c_1 \|\bar{v}\|_{L^{q^*}(\Omega_{\sigma, \bar{v}})}^{\frac{q^*}{p^*}(p^*-p)} \leq \frac{1}{2}.$$

Hence, we infer that

$$\frac{1}{2} \left(\int_{\Omega} |h_{k,\gamma}(\bar{u})|^{p^*} \right)^{\frac{p}{p^*}} \leq c_1 \left(\sigma^{p^*-1} + \sigma^{\frac{q^*}{p^*}(p^*-1)} \right) \left(\int_{\Omega} |h_{k,\gamma}(\bar{u})|^{p^*} \right)^{\frac{p}{p^*} \frac{\gamma p + 1 - p}{\gamma p}} \quad (2.2.22)$$

$$\begin{aligned}
& + c_1 \|\bar{v}\|_{L^{q^*}(\Omega_{\sigma, \bar{v}})}^{\frac{q^*}{p^*}(p^*-p)} \left(\int_{\Omega} \left| h_{\frac{p^*}{k^{q^*}}, \gamma}(\bar{v}) \right|^{q^*} \right)^{\frac{p}{p^*}} \\
& + c_1 \left(\int_{\Omega} |h_{k, \gamma}(\bar{u})|^{p^*} \right)^{\frac{p}{p^*} \frac{\gamma-1}{\gamma}}.
\end{aligned}$$

If $\eta \in (0, 1)$, by using Young inequality with conjugated exponents $\frac{1}{\eta}$ and $\frac{1}{1-\eta}$, we obtain that

$$ax^\eta \leq \left(\frac{x}{8}\right)^\eta 8a \leq \frac{x}{8} + (8a)^{1/(1-\eta)} \quad \forall a, x \geq 0. \quad (2.2.23)$$

Therefore, since $\frac{\gamma p+1-p}{\gamma p} < 1$ and $\frac{\gamma-1}{\gamma} < 1$, we infer

$$c_1 \left(\sigma^{p^*-1} + \sigma^{\frac{q^*}{p^*}(p^*-1)} \right) \left(\int_{\Omega} |h_{k, \gamma}(\bar{u})|^{p^*} \right)^{\frac{p}{p^*} \frac{\gamma p+1-p}{\gamma p}} \leq \frac{1}{8} \left(\int_{\Omega} |h_{k, \gamma}(\bar{u})|^{p^*} \right)^{\frac{p}{p^*}} + c_1 \left(\sigma^{p^*-1} + \sigma^{\frac{q^*}{p^*}(p^*-1)} \right)^{\frac{\gamma p}{p-1}}$$

and

$$c_1 \left(\int_{\Omega} |h_{k, \gamma}(\bar{u})|^{p^*} \right)^{\frac{p}{p^*} \frac{\gamma-1}{\gamma}} \leq \frac{1}{8} \left(\int_{\Omega} |h_{k, \gamma}(\bar{u})|^{p^*} \right)^{\frac{p}{p^*}} + c_1,$$

so that (2.2.22) becomes

$$\frac{1}{4} \left(\int_{\Omega} |h_{k, \gamma}(\bar{u})|^{p^*} \right)^{\frac{p}{p^*}} \leq c_1 \left[\left(\sigma^{p^*-1} + \sigma^{\frac{q^*}{p^*}(p^*-1)} \right)^{\frac{\gamma p}{p-1}} + 1 \right] + c_1 \|\bar{v}\|_{L^{q^*}(\Omega_{\sigma, \bar{v}})}^{\frac{q^*}{p^*}(p^*-p)} \left(\int_{\Omega} \left| h_{\frac{p^*}{k^{q^*}}, \gamma}(\bar{v}) \right|^{q^*} \right)^{\frac{p}{p^*}}.$$

Thus, we have shown that for any $k, \gamma > 1$ there exists a positive constant c_1 depending on $C_0, |\Omega|, p, q, N, a, \nu_1$ and γ but not on k and δ , and there are $\sigma_1 > 1$ and a suitably small $R_1 > 0$, depending on c_1 and (u_0, v_0) , such that for any $(\bar{u}, \bar{v}) \in D_{R_1, \delta}(z_0)$ and any $\sigma \geq \sigma_1$ we have

$$\int_{\Omega} |h_{k, \gamma}(\bar{u})|^{p^*} \leq c_1 \left[\left(\sigma^{p^*-1} + \sigma^{\frac{q^*}{p^*}(p^*-1)} \right)^{\frac{\gamma p^*}{p-1}} + 1 \right] + c_1 \|\bar{v}\|_{L^{q^*}(\Omega_{\sigma, \bar{v}})}^{\frac{q^*}{p^*}(p^*-p)} \int_{\Omega} \left| h_{\frac{p^*}{k^{q^*}}, \gamma}(\bar{v}) \right|^{q^*}. \quad (2.2.24)$$

Reasoning in a similar way, exploiting that

$$\left\langle I'_{\delta, \Psi_1, \Psi_2}(\bar{z}), \left(0, \Phi_{\tilde{k}, \gamma q, \gamma}(\bar{v}) \right) \right\rangle = 0 \quad \text{for any } \tilde{k}, \gamma > 1,$$

we infer that for any $\tilde{k}, \gamma > 1$ there exists a positive constant c_2 depending on $C_0, |\Omega|, p, q, N, b, \nu_2$ and γ but not on $\tilde{k}$ and δ , and there are $\sigma_2 > 1$ and a suitably small $R_2 > 0$, depending on c_2 and (u_0, v_0) , such that for any $(\bar{u}, \bar{v}) \in D_{R_2, \delta}(z_0)$ and any $\sigma \geq \sigma_2$ we have

$$\int_{\Omega} |h_{\tilde{k}, \gamma}(\bar{v})|^{q^*} \leq c_2 \left[\left(\sigma^{q^*-1} + \sigma^{\frac{p^*}{q^*}(q^*-1)} \right)^{\frac{\gamma q^*}{q-1}} + 1 \right] + c_2 \|\bar{u}\|_{L^{p^*}(\Omega_{\sigma, \bar{u}})}^{\frac{p^*}{q^*}(q^*-q)} \int_{\Omega} \left| h_{\frac{q^*}{\tilde{k}^{p^*}}, \gamma}(\bar{u}) \right|^{p^*}. \quad (2.2.25)$$

Let us consider a suitable $\bar{\sigma} > 1$ and a suitably small $\bar{R} > 0$ such that both (2.2.24) and (2.2.25) hold for any $\sigma \geq \bar{\sigma}$ and for any $0 < R \leq \bar{R}$.

Setting $\tilde{k} = k^{\frac{p^*}{q^*}}$ in (2.2.25) and substituting in (2.2.24) we obtain

$$\begin{aligned}
\int_{\Omega} |h_{k,\gamma}(\bar{u})|^{p^*} &\leq c_1 \left[\left(\sigma^{p^*-1} + \sigma^{\frac{q^*}{p^*}(p^*-1)} \right)^{\frac{\gamma p^*}{p^*-1}} + 1 \right] \\
&\quad + c_1 c_2 \|\bar{v}\|_{L^{q^*}(\Omega_{\sigma,\bar{v}})}^{\frac{q^*}{p^*}(p^*-p)} \left[\left(\sigma^{q^*-1} + \sigma^{\frac{p^*}{q^*}(q^*-1)} \right)^{\frac{\gamma q^*}{q^*-1}} + 1 \right] \\
&\quad + c_1 c_2 \|\bar{v}\|_{L^{q^*}(\Omega_{\sigma,\bar{v}})}^{\frac{q^*}{p^*}(p^*-p)} \|\bar{u}\|_{L^{p^*}(\Omega_{\sigma,\bar{u}})}^{\frac{p^*}{q^*}(q^*-q)} \int_{\Omega} |h_{k,\gamma}(\bar{u})|^{p^*}.
\end{aligned}$$

By using again Lemma 2.2.2 with $\varepsilon = 2^{\frac{q}{q-q^*}}$ and $u_0 \in W_0^{1,p}(\Omega)$, and with $\varepsilon = (c_1 c_2)^{\frac{p}{p-p^*}}$ and $v_0 \in W_0^{1,q}(\Omega)$ respectively, we infer that there exist $\tilde{\sigma} \geq \bar{\sigma}$ and $\tilde{R} \in (0, \bar{R}]$ such that, for any $\sigma \geq \tilde{\sigma}$ and for any $(\bar{u}, \bar{v}) \in D_{\tilde{R},\delta}(z_0)$,

$$c_1 c_2 \|\bar{v}\|_{L^{q^*}(\Omega_{\sigma,\bar{v}})}^{\frac{q^*}{p^*}(p^*-p)} \leq 1 \quad \text{and} \quad \|\bar{u}\|_{L^{p^*}(\Omega_{\sigma,\bar{u}})}^{\frac{p^*}{q^*}(q^*-q)} \leq \frac{1}{2}.$$

Hence, we finally get that for any $(\bar{u}, \bar{v}) \in D_{R,\delta}(z_0)$

$$\int_{\Omega} |h_{k,\gamma}(\bar{u})|^{p^*} \leq c_3 \quad \text{for any } k, \gamma > 1,$$

and similarly

$$\int_{\Omega} |h_{k,\gamma}(\bar{v})|^{q^*} \leq c_3 \quad \text{for any } k, \gamma > 1,$$

where the positive constant c_3 depends on $C_0, |\Omega|, p, q, N, a, \nu_1, b, \nu_2$ and γ but not on k and δ , and in addition $R > 0$ is a suitable radius.

Thus we can apply Fatou Lemma, and passing to limit for $k \rightarrow \infty$ we get

$$\int_{\Omega} |\bar{u}|^{\gamma p^*} \leq C \quad \text{and} \quad \int_{\Omega} |\bar{v}|^{\gamma q^*} \leq C \quad \forall (\bar{u}, \bar{v}) \in D_{R,\delta}(z_0),$$

where the positive constant C still depends on $C_0, |\Omega|, p, q, N, a, \nu_1, b, \nu_2$ and γ but not on δ . $\square$

2.3 Uniformly locally L^∞ -estimate

In this Section we prove Theorem 2.1.3, and we will use a result proved in [25, Appendix]. In particular, in [25, Lemma A.1] it was shown the following generalization of Stampacchia Lemma:

Lemma 2.3.1. *Let $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a nonincreasing function such that*

$$\varphi(h) \leq \frac{c_0}{(h-k)^\rho} k^{\theta\rho} [\varphi(k)]^{1+\lambda} \quad \forall h > k \geq k_0,$$

where $c_0 > 0$, $k_0 \geq 0$, $\rho > 0$, $0 \leq \theta < 1$ and $\lambda > 0$. Then there exists $k^* > 0$ such that $\varphi(k^*) = 0$.

Thanks to previous result, in [25, Lemma 2.1] it was proved the following Theorem:

Theorem 2.3.2. *Let $r > 1$ and let w be a function in $W_0^{1,r}(\Omega)$ such that, for k greater then some k_0 ,*

$$\int_{A_k} |\nabla w|^r dx \leq c k^{\theta r} |A_k|^{\frac{r}{r^*} + \varepsilon}, \quad (2.3.26)$$

where $\varepsilon > 0$, $0 \leq \theta < 1$, $r^* = \frac{Nr}{N-r}$ and

$$A_k = \{x \in \Omega : |w(x)| > k\}.$$

Then, the norm of w in $L^\infty(\Omega)$ is bounded by a constant that depends on $c, \theta, r, N, \varepsilon, k_0$ and $|\Omega|$.

Proof of Theorem 2.1.3. From now on, we will omit dx and we will redefine the positive constant C when necessary. Fixing $m > N$, by Proposition 2.2.3 and assumption $(\mathcal{H})$ we infer that there are $C, R > 0$ depending on $C_0, |\Omega|, p, q, N, a, \nu_1, b, \nu_2$ and m but not on δ , such that for any $(\bar{u}, \bar{v}) \in D_{R,\delta}(u_0, v_0)$ we have $H_s(\delta, x, \bar{u}, \bar{v}) \in L^m(\Omega)$, $H_t(\delta, x, \bar{u}, \bar{v}) \in L^m(\Omega)$ and

$$\|H_s(\delta, x, \bar{u}, \bar{v})\|_m^{p'}, \quad \|H_t(\delta, x, \bar{u}, \bar{v})\|_m^{q'} \leq C. \quad (2.3.27)$$

Fixed $(\bar{u}, \bar{v}) \in D_{R,\delta}(z_0)$, for any $k \geq 1$ we denote by

$$A_k^{\bar{u}} = \{x \in \Omega : |\bar{u}(x)| > k\},$$

and we consider the map $\mathcal{R}_k : \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$\mathcal{R}_k(r) := \begin{cases} 0 & \text{if } |r| \leq k, \\ \text{sign}(r) \left[|r| - k \right] & \text{if } |r| > k. \end{cases}$$

Since $\mathcal{R}_k(\bar{u}) \in W_0^{1,p}(\Omega)$ and $\langle I'_{\delta, \Psi_1, \Psi_2}(\bar{u}, \bar{v}), (\mathcal{R}_k(\bar{u}), 0) \rangle = 0$, by using (2.2.8), (2.2.18), and denoting by $\bar{f}_s := H_s(\delta, x, \bar{u}, \bar{v})$, we have

$$\begin{aligned} \int_{A_k^{\bar{u}}} |\nabla \bar{u}|^p &= \int_{A_k^{\bar{u}}} |\nabla \mathcal{R}_k(\bar{u})|^p = \int_{A_k^{\bar{u}}} |\nabla \bar{u}|^p \mathcal{R}'_k(\bar{u})^p = \int_{A_k^{\bar{u}}} |\nabla \bar{u}|^p \mathcal{R}'_k(\bar{u}) \\ &\leq \int_{A_k^{\bar{u}}} \left(a + |\nabla \bar{u}|^2 \right)^{\frac{p-2}{2}} |\nabla \bar{u}|^2 \mathcal{R}'_k(\bar{u}) + a^{\frac{p}{2}} \int_{A_k^{\bar{u}}} \mathcal{R}'_k(\bar{u}) \\ &\leq \int_{\Omega} \left(a + |\nabla \bar{u}|^2 \right)^{\frac{p-2}{2}} |\nabla \bar{u}|^2 \mathcal{R}'_k(\bar{u}) + a^{\frac{p}{2}} |A_k^{\bar{u}}| \\ &\leq \frac{1}{\nu_1} \int_{\Omega} \nabla \Psi_1(\nabla \bar{u}) \cdot \nabla \bar{u} \mathcal{R}'_k(\bar{u}) + a^{\frac{p}{2}} |A_k^{\bar{u}}| \\ &= \frac{1}{\nu_1} \int_{\Omega} \nabla \Psi_1(\nabla \bar{u}) \cdot \nabla \mathcal{R}_k(\bar{u}) + a^{\frac{p}{2}} |A_k^{\bar{u}}| \\ &= \frac{1}{\nu_1} \int_{\Omega} \bar{f}_s \mathcal{R}_k(\bar{u}) + a^{\frac{p}{2}} |A_k^{\bar{u}}| \\ &= \frac{1}{\nu_1} \int_{A_k^{\bar{u}}} \bar{f}_s \mathcal{R}_k(\bar{u}) + a^{\frac{p}{2}} |A_k^{\bar{u}}| \\ &\leq \frac{1}{\nu_1} \int_{A_k^{\bar{u}}} |\bar{f}_s| |\mathcal{R}_k(\bar{u})| + a^{\frac{p}{2}} |A_k^{\bar{u}}|. \end{aligned} \quad (2.3.28)$$

By Sobolev-Gagliardo-Nirenberg inequality, Hölder inequality with conjugated exponents $(p^*)'$ and p^* , and Young inequality with conjugated exponents p' and p , we have

$$\int_{A_k^{\bar{u}}} |\bar{f}_s| |\mathcal{R}_k(\bar{u})| \leq \left(\int_{A_k^{\bar{u}}} |\bar{f}_s|^{p^{**}} \right)^{\frac{1}{p^{**'}}} \left(\int_{A_k^{\bar{u}}} |\mathcal{R}_k(\bar{u})|^{p^*} \right)^{\frac{1}{p^*}}$$

$$\begin{aligned}
&= \left(\int_{A_k^{\bar{u}}} |\bar{f}_s|^{p^{*'}} \right)^{\frac{1}{p^{*'}}} \left(\int_{\Omega} |\mathcal{R}_k(\bar{u})|^{p^*} \right)^{\frac{1}{p^*}} \\
&\leq C \left(\int_{A_k^{\bar{u}}} |\bar{f}_s|^{p^{*'}} \right)^{\frac{1}{p^{*'}}} \left(\int_{\Omega} |\nabla \mathcal{R}_k(\bar{u})|^p \right)^{\frac{1}{p}} \\
&= C \left(\int_{A_k^{\bar{u}}} |\bar{f}_s|^{p^{*'}} \right)^{\frac{1}{p^{*'}}} \left(\int_{A_k^{\bar{u}}} |\nabla \bar{u}|^p \right)^{\frac{1}{p}} \\
&\leq \frac{1}{p'} C^{p'} \left(\int_{A_k^{\bar{u}}} |\bar{f}_s|^{p^{*'}} \right)^{\frac{p'}{p^{*'}}} + \frac{1}{p} \int_{A_k^{\bar{u}}} |\nabla \bar{u}|^p.
\end{aligned}$$

Therefore, we have

$$\frac{1}{\nu_1} \int_{A_k^{\bar{u}}} |\bar{f}_s| |\mathcal{R}_k(\bar{u})| \leq \frac{1}{\nu_1 p'} C^{p'} \left(\int_{A_k^{\bar{u}}} |\bar{f}_s|^{p^{*'}} \right)^{\frac{p'}{p^{*'}}} + \frac{1}{\nu_1 p} \int_{A_k^{\bar{u}}} |\nabla \bar{u}|^p. \quad (2.3.29)$$

Taking into account (2.3.28) and (2.3.29), we have proved

$$\left(1 - \frac{1}{\nu_1 p} \right) \int_{A_k^{\bar{u}}} |\nabla \bar{u}|^p \leq \frac{C^{p'}}{\nu_1 p'} \left(\int_{\Omega} |\bar{f}_s|^{p^{*'}} \right)^{\frac{p'}{p^{*'}}} + a^{\frac{p}{2}} |A_k^{\bar{u}}|, \quad (2.3.30)$$

by which, considering that $\nu_1 > \frac{1}{p}$, we get

$$\int_{A_k^{\bar{u}}} |\nabla \bar{u}|^p \leq \frac{p}{p'(\nu_1 p - 1)} C^{p'} \left(\int_{\Omega} |\bar{f}_s|^{p^{*'}} \right)^{\frac{p'}{p^{*'}}} + \frac{\nu_1 p}{\nu_1 p - 1} a^{\frac{p}{2}} |A_k^{\bar{u}}|, \quad (2.3.31)$$

Since $m > N$, we deduce that

$$m > p^{*'} = \frac{Np}{Np - N + p} \quad \text{and} \quad p' \left(\frac{1}{p^{*'}} - \frac{1}{m} \right) > 1.$$

Now, by (2.3.27) and Hölder inequality with conjugated exponents $\frac{m}{p^{*'}}$ and $\frac{m}{m - p^{*'}}$, we have

$$\begin{aligned}
\left(\int_{A_k^{\bar{u}}} |\bar{f}_s|^{p^{*'}} \right)^{\frac{p'}{p^{*'}}} &\leq \left(\int_{A_k^{\bar{u}}} |\bar{f}_s|^m \right)^{\frac{p'}{m}} |A_k^{\bar{u}}|^{p' \left(\frac{1}{p^{*'}} - \frac{1}{m} \right)} \\
&= \|H_s(\delta, x, \bar{u}, \bar{v})\|_m^{p'} |A_k^{\bar{u}}|^{p' \left(\frac{1}{p^{*'}} - \frac{1}{m} \right)} \\
&\leq C |A_k^{\bar{u}}|^{p' \left(\frac{1}{p^{*'}} - \frac{1}{m} \right)}.
\end{aligned} \quad (2.3.32)$$

Considering (2.3.31) and (2.3.32), we have proved that there exists $C > 0$ such that

$$\int_{A_k^{\bar{u}}} |\nabla \bar{u}|^p \leq C \left(|A_k^{\bar{u}}|^{p' \left(\frac{1}{p^{*'}} - \frac{1}{m} \right)} + |A_k^{\bar{u}}| \right). \quad (2.3.33)$$

Now, by using Lemma 2.2.1, there exists $k_0 \geq 1$ such that for any $(\bar{u}, \bar{v}) \in D_{R,\delta}(z_0)$ and for any k greater then k_0 , we get

$$|A_k^{\bar{u}}| < 1.$$

In fact, by using Lemma 2.2.1 with $u_0 \in W_0^{1,p}(\Omega)$, $r = 1$ and $\varepsilon = 1$, we deduce that there exists $\sigma_1 > 0$ such that

$$\int_{\{|u(x)| \geq \sigma_1\}} |u(x)| dx < 1$$

for any $u \in \overline{B_R(u_0)} = \{u \in W_0^{1,p}(\Omega) : \|u - u_0\|_{1,p} \leq R\}$.

Let us denote with $k_1 := \max\{1, \sigma_1\}$.

Observe that

$$1 > \int_{\{|u(x)| \geq \sigma_1\}} |u(x)| dx \geq \int_{A_{k_1}^u} |u(x)| dx > \int_{A_{k_1}^u} k_1 dx = k_1 |A_{k_1}^u|,$$

by which

$$|A_k^u| \leq |A_{k_1}^u| < \frac{1}{k_1} \leq 1$$

for any $u \in \overline{B_R(u_0)}$ and for any $k \geq k_1$.

Similarly, we deduce that there exists $k_2 \geq 1$ such that

$$|A_k^v| < 1$$

for any $v \in \overline{B_R(v_0)} = \{v \in W_0^{1,q}(\Omega) : \|v - v_0\|_{1,q} \leq R\}$ and for any $k \geq k_2$.

Denoting with $k_0 := \max\{k_1, k_2\}$, for any $(\bar{u}, \bar{v}) \in D_{R,\delta}(u_0, v_0)$ and for any $k \geq k_0$, we get

$$|A_k^{\bar{u}}| < 1.$$

Considering also $p' \left(\frac{1}{p^{*'}} - \frac{1}{m} \right) > 1$, by (2.3.33) we deduce that for any $k \geq k_0$ we have

$$\int_{A_k^{\bar{u}}} |\nabla \bar{u}|^p \leq 2\bar{C} |A_k^{\bar{u}}|. \quad (2.3.34)$$

Therefore, by applying Theorem 2.3.2 with $\varepsilon = 1 - \frac{p}{p^*}$ and $\theta = 0$, we deduce that $\bar{u} \in L^\infty(\Omega)$. Similarly, we deduce that $\bar{v} \in L^\infty(\Omega)$.

Finally, we have proved that there exist a suitable small $R > 0$ and a constant $C > 0$ depending on $C_0, |\Omega|, p, q, N, a, \nu_1, b$ and ν_2 but not on δ , such that for any $(\bar{u}, \bar{v}) \in D_{R,\delta}(z_0)$, $\bar{u}$ and $\bar{v}$ are in $L^\infty(\Omega)$ and

$$\|\bar{u}\|_\infty, \|\bar{v}\|_\infty \leq C.$$

□

Remark 2.3.3. We point out that if $p \geq 2$ the constant C does not depend on a . In fact, in the proof of Theorem 2.1.3 we can use instead of (2.2.18) the inequality

$$s^p \leq \left(a + s^2 \right)^{\frac{p-2}{2}} s^2,$$

that holds for any $p \geq 2$ and for any $s \geq 0$.

Similarly, if $q \geq 2$ the constant C does not depend on b .

Appendix B

Lemma 2.3.4. *Let $\Psi : \mathbb{R}^N \rightarrow \mathbb{R}$ be a convex C^1 -function. Assume that there exist $c > 0, d > 0$ and $r > 1$ such that*

$$|\Psi(\xi)| \leq c |\xi|^r + d \quad \forall \xi \in \mathbb{R}^N. \quad (2.3.35)$$

Then there exists a constant $\tilde{c}$ such that

$$|\nabla \Psi(\xi)| \leq \tilde{c} \left(|\xi|^{r-1} + 1 \right) \quad \forall \xi \in \mathbb{R}^N. \quad (2.3.36)$$

Proof. By convexity of Ψ , we know that

$$\Psi(\xi + h) - \Psi(\xi) \geq \nabla \Psi(\xi) \cdot h \quad \forall \xi, h \in \mathbb{R}^N.$$

Let $\xi \in \mathbb{R}^N$ with $\xi \neq 0$. Taking into account previous relation and (2.3.35), we get

$$\begin{aligned} |\nabla \Psi(\xi)| &= \sup_{|\eta| \leq 1} \nabla \Psi(\xi) \cdot \eta = \sup_{|\eta| \leq 1} \frac{\nabla \Psi(\xi) \cdot |\xi| \eta}{|\xi|} \\ &\leq \sup_{|\eta| \leq 1} \frac{\Psi(\xi + |\xi| \eta) - \Psi(\xi)}{|\xi|} \leq \sup_{|\eta| \leq 1} \frac{|\Psi(\xi + |\xi| \eta) - \Psi(\xi)|}{|\xi|} \leq \bar{c} |\xi|^{r-1} + \frac{\tilde{d}}{|\xi|}, \end{aligned}$$

where $\bar{c}$ and $\tilde{d}$ are suitable constants.

Observe now that

$$\lim_{|\xi| \rightarrow +\infty} \frac{\bar{c} |\xi|^{r-1} + \frac{\tilde{d}}{|\xi|}}{\bar{c} |\xi|^{r-1}} = 1,$$

hence there exists a constant $R > 0$ such that

$$\bar{c} |\xi|^{r-1} + \frac{\tilde{d}}{|\xi|} \leq 2\bar{c} |\xi|^{r-1} \quad \text{if } |\xi| > R,$$

by which

$$|\nabla \Psi(\xi)| \leq 2\bar{c} |\xi|^{r-1} \quad \text{if } |\xi| > R.$$

Now, since Ψ is a function of class C^1 in $\mathbb{R}^N$, we deduce that there exists $M > 0$ such that

$$|\nabla \Psi(\xi)| \leq M \quad \text{if } |\xi| \leq R.$$

Finally, denoting with $\tilde{c} := \max\{2\bar{c}, M\}$, we get the thesis. $\square$

Proposition 2.3.5. *Let $\Psi : \mathbb{R}^N \rightarrow \mathbb{R}$ be a convex C^1 -function. Let $r > 1$ and let us assume that there exist $c > 0$ and $d > 0$ such that*

$$|\Psi(\xi)| \leq c |\xi|^r + d \quad \forall \xi \in \mathbb{R}^N. \quad (2.3.37)$$

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let us define $f : W_0^{1,r}(\Omega) \rightarrow \mathbb{R}$ as

$$f(w) := \int_{\Omega} \Psi(\nabla w(x)) \, dx. \quad (2.3.38)$$

Then f is a C^1 -functional on $W_0^{1,r}(\Omega)$ and for any $w_0, w \in W_0^{1,r}(\Omega)$ we have

$$\langle f'(w_0), w \rangle = \int_{\Omega} \nabla \Psi(\nabla w_0(x)) \cdot \nabla w(x) dx. \quad (2.3.39)$$

Proof. The functional f is well defined, in fact for any $w \in W_0^{1,r}(\Omega)$ inequality (2.3.37) implies

$$|f(w)| \leq \int_{\Omega} |\Psi(\nabla w(x))| dx \leq c \|\nabla w\|_r^r + d |\Omega| < +\infty.$$

Let us now prove that f is Gateaux differentiable in any $w_0 \in W_0^{1,r}(\Omega)$, where the Gateaux differential in w_0 is the map $f'(w_0)$ whose action is defined in (2.3.39).

Hence, we need to show that for any $w_0, w \in W_0^{1,r}(\Omega)$ we have

$$\lim_{t \rightarrow 0} \frac{f(w_0 + tw) - f(w_0)}{t} = \int_{\Omega} \nabla \Psi(\nabla w_0(x)) \cdot \nabla w(x) dx. \quad (2.3.40)$$

Observe that

$$\frac{f(w_0 + tw) - f(w_0)}{t} = \int_{\Omega} \frac{\Psi(\nabla w_0(x) + t \nabla w(x)) - \Psi(\nabla w_0(x))}{t} dx.$$

Let $\xi, \eta \in \mathbb{R}^N$ and let us define the function $h : [-1, 1] \rightarrow \mathbb{R}$ as $h(t) := \Psi(\xi + t \eta)$.

The function h is of class C^1 with $h'(t) = \nabla \Psi(\xi + t \eta) \cdot \eta$ and $h'(0) = \nabla \Psi(\xi) \cdot \eta$, whence

$$\lim_{t \rightarrow 0} \frac{\Psi(\nabla w_0(x) + t \nabla w(x)) - \Psi(\nabla w_0(x))}{t} = \nabla \Psi(\nabla w_0(x)) \cdot \nabla w(x) \quad \text{a.e. in } \Omega.$$

By Lemma 2.3.4, we know that

$$|\nabla \Psi(\xi)| \leq \tilde{c} (|\xi|^{r-1} + 1) \quad \forall \xi \in \mathbb{R}^N. \quad (2.3.41)$$

By Lagrange Theorem applied to the function h , there exists $\theta \in \mathbb{R}$ with $|\theta| \leq |t| \leq 1$ such that

$$\begin{aligned} \left| \frac{\Psi(\nabla w_0(x) + t \nabla w(x)) - \Psi(\nabla w_0(x))}{t} \right| &= |\nabla \Psi(\nabla w_0(x) + \theta \nabla w(x)) \cdot \nabla w(x)| \\ &\leq |\nabla \Psi(\nabla w_0(x) + \theta \nabla w(x))| |\nabla w(x)|. \end{aligned}$$

By (2.3.41), we get

$$\begin{aligned} |\nabla \Psi(\nabla w_0(x) + \theta \nabla w(x))| |\nabla w(x)| &\leq \tilde{c} (|\nabla w_0(x) + \theta \nabla w(x)|^{r-1} + 1) |\nabla w(x)| \\ &\leq \tilde{c} (|\nabla w_0(x)| + |\nabla w(x)|)^{r-1} + 1) |\nabla w(x)| \\ &\leq C |\nabla w_0(x)|^{r-1} |\nabla w(x)| + C |\nabla w(x)|^r + \tilde{c} |\nabla w(x)|, \end{aligned}$$

with

$$C |\nabla w_0(x)|^{r-1} |\nabla w(x)| + C |\nabla w(x)|^r + \tilde{c} |\nabla w(x)| \in L^1(\Omega).$$

Hence, by applying dominated convergence theorem we get (2.3.40).

Let us prove that the Gateaux derivative $f' : W_0^{1,r}(\Omega) \rightarrow W^{-1,r'}(\Omega)$ is continuous.

Let $\{w_k\}_k \subset W_0^{1,r}(\Omega)$ be a sequence such that $w_k \rightarrow w$ in $W_0^{1,r}(\Omega)$. Up to subsequence, we have:

- i) $\nabla w_k \rightarrow \nabla w$ in $(L^r(\Omega))^N$ as $k \rightarrow \infty$;
- ii) $\nabla w_k(x) \rightarrow \nabla w(x)$ a.e. $x \in \Omega$ as $k \rightarrow \infty$;
- iii) there exists $g \in L^1(\Omega)$ such that $|\nabla w_k(x)|^r \leq g(x)$ a.e. $x \in \Omega$, and for all $k \in \mathbb{N}$.

We have

$$\langle f'(w) - f'(w_k), \bar{w} \rangle = \int_{\Omega} (\nabla \Psi(\nabla w(x)) - \nabla \Psi(\nabla w_k(x))) \cdot \nabla \bar{w}(x) dx$$

and

$$\begin{aligned} |\langle f'(w) - f'(w_k), \bar{w} \rangle| &= \left| \int_{\Omega} (\nabla \Psi(\nabla w(x)) - \nabla \Psi(\nabla w_k(x))) \cdot \nabla \bar{w}(x) dx \right| \\ &\leq \int_{\Omega} |(\nabla \Psi(\nabla w(x)) - \nabla \Psi(\nabla w_k(x))) \cdot \nabla \bar{w}(x)| dx \\ &\leq \int_{\Omega} |\nabla \Psi(\nabla w(x)) - \nabla \Psi(\nabla w_k(x))| |\nabla \bar{w}(x)| dx \\ &\leq \left(\int_{\Omega} |\nabla \Psi(\nabla w(x)) - \nabla \Psi(\nabla w_k(x))|^{\frac{r}{r-1}} dx \right)^{\frac{r-1}{r}} \|\nabla \bar{w}\|_r \end{aligned}$$

Therefore, we deduce

$$\|f'(w) - f'(w_k)\|_{W_0^{-1,r}(\Omega)} \leq \left(\int_{\Omega} |\nabla \Psi(\nabla w(x)) - \nabla \Psi(\nabla w_k(x))|^{\frac{r}{r-1}} dx \right)^{\frac{r-1}{r}}.$$

Taking into account ii) and the fact that Ψ is a C^1 function, we easily deduce that

$$\nabla \Psi(\nabla w_k(x)) \rightarrow \nabla \Psi(\nabla w(x)) \quad \text{a.e. in } \Omega.$$

Considering (2.3.41) and iii), we have

$$\begin{aligned} |\nabla \Psi(\nabla w(x)) - \nabla \Psi(\nabla w_k(x))|^{\frac{r}{r-1}} &\leq (|\nabla \Psi(\nabla w(x))| + |\nabla \Psi(\nabla w_k(x))|)^{\frac{r}{r-1}} \\ &\leq C \left(|\nabla \Psi(\nabla w(x))|^{\frac{r}{r-1}} + |\nabla \Psi(\nabla w_k(x))|^{\frac{r}{r-1}} \right) \\ &\leq \tilde{C} \left((|\nabla w(x)|^{r-1} + 1)^{\frac{r}{r-1}} + (|\nabla w_k(x)|^{r-1} + 1)^{\frac{r}{r-1}} \right) \\ &\leq \tilde{\tilde{C}} (|\nabla w(x)|^r + |\nabla w_k(x)|^r + 1) \\ &\leq 2\tilde{\tilde{C}}g(x) + \tilde{\tilde{C}} \end{aligned}$$

a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$, where $2\tilde{\tilde{C}}g(x) + \tilde{\tilde{C}} \in L^1(\Omega)$.

Therefore, we can apply dominated convergence theorem to say that

$$\left(\int_{\Omega} |\nabla \Psi(\nabla w(x)) - \nabla \Psi(\nabla w_k(x))|^{\frac{r}{r-1}} dx \right)^{\frac{r-1}{r}} \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

whence

$$\|f'(w) - f'(w_k)\| \rightarrow 0 \text{ as } k \rightarrow \infty.$$

This holds for a subsequence of the original sequence $\{w_k\}_k$. By using the Lemma of sub-subsequence we obtain that f' is continuous, and so f is of class C^1 . □

Amann-Zehnder type results for Resonance Quasilinear Elliptic Systems

The interaction between a nonlinearity and the spectrum of $-\Delta$ at 0 and infinity has been employed in the celebrated paper [7] of Amann and Zehnder, under the assumption of nonresonance at infinity. The quasilinear counterpart of the Amann-Zehnder existence result has been obtained by Cingolani, Degiovanni and Vannella in [58] (see also [55, 92]) for a class of quasilinear elliptic equations, introducing a Saddle Theorem where linear subspaces are substituted by symmetric cones (see Theorem 7.1 in [58]).

In this Chapter, we present Amann-Zehnder type results for a class of quasilinear elliptic systems whose principal parts consist of (p, q) -Laplace or (p, q) -area type operators. The nonlinearities are assumed to exhibit (p, q) -linear growth at infinity, covering both the asymptotic resonant and nonresonant cases. The space setting is given by the Banach product space $X := W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$, hence we exploit recent Morse-theoretic techniques in Banach spaces (see [59, 60, 62, 160]). First, we establish a Saddle Point Theorem specifically tailored for (p, q) -homogeneous sets, since we lose the structure of symmetric cones. Second, we establish a relationship between the local behaviour of the functional near its critical points and differential notions like the Morse index. Moreover, since the nonlocal coupling induces a lack of regularity in the functional, we overcome this difficulty by using a penalization method and uniform regularity estimates shown in Chapter 2.

The present Chapter is based on the paper [32].

3.1 Amann-Zehnder type results

In 1980, Amann and Zehnder [7] studied the asymptotically linear elliptic problem

$$\begin{cases} -\Delta u = g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.1.1)$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $g : \mathbb{R} \rightarrow \mathbb{R}$ is a C^1 -function such that $g(0) = 0$ and there exists $\bar{\lambda} \in \mathbb{R}$ such that

$$\lim_{|s| \rightarrow \infty} g'(s) = \bar{\lambda}.$$

They proved that problem (3.1.1) admits a nontrivial solution u , supposing that $\bar{\lambda}$ is not an eigenvalue of $-\Delta$, the so-called nonresonance condition at infinity, and that there exists some eigenvalue of $-\Delta$ between $\bar{\lambda}$ and $g'(0)$. Using Morse theory for manifolds with boundary, the same result has been found by Chang in [45]. Successively, in [119] Lazer and Solimini recognized that such nontrivial solution can be detected combining mini-max characterization of the critical point and Morse index estimates (see also [128]). More precisely, the basic idea in [119] is to recognize that the energy functional associated with the asymptotically linear problem (3.1.1) has a saddle geometry, which implies that a suitable Poincaré polynomial is not trivial, and also to show that a certain critical group at zero is trivial, to ensure the existence of a solution $u \neq 0$ of (3.1.1).

Recently the quasilinear counterpart of the Amann-Zehnder existence result has been obtained in [58] (see also [55, 92]) for a class of quasilinear elliptic equations, involving either the p -Laplace operator or the operator related to the p -area functional, and a nonlinearity with p -linear growth at infinity.

More precisely, in [58] Cingolani, Degiovanni and Vannella studied the following problem

$$\begin{cases} -\operatorname{div} \left[(k^2 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \right] = g(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (3.1.2)$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 1$, with $\partial\Omega$ of class $C^{1,\alpha}$ for some $\alpha \in]0, 1]$, while $k \geq 0$, $p > 1$ are real numbers and $g : \mathbb{R} \rightarrow \mathbb{R}$ is a C^1 -function such that $g(0) = 0$ and there exists $\bar{\lambda} \in \mathbb{R}$ such that

$$\lim_{|s| \rightarrow \infty} \frac{g(s)}{|s|^{p-2}s} = \bar{\lambda}.$$

Problem (3.1.2) gives rise to new difficulties when $p \neq 2$. In particular, as regards the eigenvalue problem $-\Delta_p u = \lambda |u|^{p-2}u$, the spectral properties are not yet well understood. Denoted $\sigma(-\Delta_p)$ the set of such eigenvalues λ , one can define in at least three different ways a diverging sequence (λ_m) of eigenvalues of $-\Delta_p$ (see [55, 92]), but it is not known if they agree for $m \geq 3$ and if the whole set $\sigma(-\Delta_p)$ is covered. Hence it is more difficult to recognize a saddle structure with a related information on a suitable Poincaré polynomial. On the other hand, for functionals defined on Banach spaces serious difficulties arise in extending Morse theory, and so in providing an estimate of the critical groups at zero by some Hessian type notion. Therefore problem (3.1.2) was addressed in [58] exploiting new techniques of Morse theory in Banach spaces (see [59, 60, 62, 160]) and introducing a Saddle Theorem where linear subspaces are substituted by symmetric cones (see Theorem 7.1 in [58]).

In this Chapter we present Amann-Zehnder type results for a class of quasilinear elliptic systems whose principal parts consist of (p, q) -Laplace or (p, q) -area type operators. Precisely we will seek for nontrivial solutions for the following autonomous quasilinear system

$$\begin{cases} -\operatorname{div} \left((a + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \right) = G_s(u, v) & x \in \Omega, \\ -\operatorname{div} \left((a + |\nabla v|^2)^{\frac{q-2}{2}} \nabla v \right) = G_t(u, v) & x \in \Omega, \\ u = v = 0 & x \in \partial\Omega, \end{cases} \quad (3.1.3)$$

where $2 \leq p < N$, $2 \leq q < N$, $a \geq 0$, and $G \in C^1(\mathbb{R}^2, \mathbb{R})$ satisfies the following conditions:

(a₁) $\nabla G(0, 0) = (0, 0)$ and there exists $\bar{\lambda} \in \mathbb{R}$ such that

$$\lim_{|(s,t)| \rightarrow \infty} \frac{G_s(s, t) - \bar{\lambda} F_s(s, t)}{|s|^{p-1} + |t|^{\frac{p-1}{p}}} = 0 \quad \text{and} \quad \lim_{|(s,t)| \rightarrow \infty} \frac{G_t(s, t) - \bar{\lambda} F_t(s, t)}{|s|^{\frac{q-1}{q}} + |t|^{q-1}} = 0,$$

where $F \in C^1(\mathbb{R}^2, \mathbb{R})$ is defined by

$$F(s, t) = \frac{1}{p}|s|^p + \frac{1}{q}|t|^q + \frac{1}{(\alpha+1)(\beta+1)}|s|^\alpha|t|^\beta st$$

with $\alpha, \beta > 0$ such that $(\alpha+1)/p + (\beta+1)/q = 1$;

(a₂) there is a suitable neighborhood U of $(0, 0)$ such that $G \in C^2(U, \mathbb{R})$.

Weak solutions of problem (3.1.3) correspond to critical points of the Euler functional $J_a : X \rightarrow \mathbb{R}$ defined for any $z = (u, v) \in X$ by setting

$$J_a(z) = \frac{1}{p} \int_{\Omega} (a + |\nabla u|^2)^{\frac{p}{2}} dx + \frac{1}{q} \int_{\Omega} (a + |\nabla v|^2)^{\frac{q}{2}} dx - \int_{\Omega} G(u, v) dx. \quad (3.1.4)$$

Observe now that for any $(s, t) \in \mathbb{R} \times \mathbb{R}$ we have

$$F_s(s, t) = |s|^{p-2}s + \frac{1}{\beta+1}|s|^\alpha|t|^\beta t \quad \text{and} \quad F_t(s, t) = |t|^{q-2}t + \frac{1}{\alpha+1}|s|^\alpha|t|^\beta s.$$

By $(\alpha+1)/p + (\beta+1)/q = 1$, we deduce $\frac{\alpha}{p-1} + \frac{p(\beta+1)}{q(p-1)} = 1$ and $\frac{\beta}{q-1} + \frac{q(\alpha+1)}{p(q-1)} = 1$, and so by Young's inequality we infer that there exists a constant $\bar{C}$ such that for any $(s, t) \in \mathbb{R} \times \mathbb{R}$ we have

$$|F_s(s, t)| \leq \bar{C} \left(|s|^{p-1} + |t|^{q \frac{p-1}{p}} \right) \quad \text{and} \quad |F_t(s, t)| \leq \bar{C} \left(|t|^{q-1} + |s|^{p \frac{q-1}{q}} \right).$$

Considering now the limits in assumption (a₁), we infer that there is $C > 0$ such that for any $(s, t) \in \mathbb{R} \times \mathbb{R}$ we have

$$|G_s(s, t)| \leq C \left(1 + |s|^{p-1} + |t|^{q \frac{p-1}{p}} \right) \quad \text{and} \quad |G_t(s, t)| \leq C \left(1 + |t|^{q-1} + |s|^{p \frac{q-1}{q}} \right). \quad (3.1.5)$$

Therefore the functional J_a is of class C^1 on X and for any $z_0 = (u_0, v_0), z = (u, v) \in X$ it results

$$\begin{aligned} \langle J'_a(z_0), z \rangle &= \int_{\Omega} (a + |\nabla u_0|^2)^{\frac{p-2}{2}} \nabla u_0 \cdot \nabla u dx + \int_{\Omega} (a + |\nabla v_0|^2)^{\frac{q-2}{2}} \nabla v_0 \cdot \nabla v dx \\ &\quad - \int_{\Omega} (G_s(u_0, v_0)u + G_t(u_0, v_0)v) dx. \end{aligned}$$

Existence, nonexistence and regularity results for quasilinear elliptic systems are obtained by various authors, see for instance [26, 35, 40, 72, 74, 87, 94, 163]. In this Chapter we are interested to find nontrivial solution of (3.1.3) in both the asymptotic resonant and nonresonant cases, that is we consider the following nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u + \frac{\lambda}{\beta+1} |u|^\alpha |v|^\beta v & x \in \Omega, \\ -\Delta_q v = \lambda |v|^{q-2}v + \frac{\lambda}{\alpha+1} |u|^\alpha |v|^\beta u, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \quad (3.1.6)$$

where $2 \leq p < N$, $2 \leq q < N$, $\alpha > 0$ and $\beta > 0$ are real numbers satisfying $(\alpha+1)/p + (\beta+1)/q = 1$. The real number λ is called an eigenvalue of (3.1.6) if there exists a nontrivial solution (u, v) of (3.1.6).

In [137], the existence of an unbounded sequence of minimax eigenvalues was proved by using $\mathbb{Z}_2$ -cohomological index of Fadell and Rabinowitz [97, 98].

Precisely, let $\Phi : X \rightarrow \mathbb{R}$, $\Psi : X \rightarrow \mathbb{R}$ be the following functionals

$$\Phi(u, v) = \frac{1}{p} \int_{\Omega} |\nabla u(x)|^p dx + \frac{1}{q} \int_{\Omega} |\nabla v(x)|^q dx,$$

$$\Psi(u, v) = \int_{\Omega} F(u, v) dx.$$

By Young's inequality we infer that

$$\frac{1}{p} \frac{\beta}{\beta + 1} |s|^p + \frac{1}{q} \frac{\alpha}{\alpha + 1} |t|^q \leq F(s, t) \leq \frac{1}{p} \frac{\beta + 2}{\beta + 1} |s|^p + \frac{1}{q} \frac{\alpha + 2}{\alpha + 1} |t|^q. \quad (3.1.7)$$

Note that Φ and Ψ are (p, q) -homogeneous, i.e.

$$\Phi(t^{\frac{1}{p}} u, t^{\frac{1}{q}} v) = t \Phi(u, v), \quad \Psi(t^{\frac{1}{p}} u, t^{\frac{1}{q}} v) = t \Psi(u, v) \quad (3.1.8)$$

for all $t > 0$ and $(u, v) \in X$.

Moreover λ is an eigenvalue if and only if there is $(u, v) \in X \setminus \{(0, 0)\}$ such that

$$\Phi'(u, v) = \lambda \Psi'(u, v).$$

In [137] it is proved that (3.1.6) has a nondecreasing and unbounded sequence of eigenvalues with the variational characterization

$$\lambda_k = \inf_{A \in \Sigma_k} \sup_{(u, v) \in A} \Phi(u, v), \quad (3.1.9)$$

where Σ is the C^1 manifold

$$\Sigma := \{(u, v) \in X \mid \Psi(u, v) = 1\}$$

and

$$\Sigma_k := \{A \subset \Sigma \mid A \text{ is symmetric, compact and } i(A) \geq k\}.$$

Here $i(A)$ denotes the $\mathbb{Z}_2$ -cohomological index of A , introduced by Fadell-Rabinowitz [97, 98]. For a matter of convenience, we also put $\lambda_0 = -\infty$.

In [150] it is shown that the first eigenvalue λ_1 is simple and isolated with a first strictly positive eigenfunction $\psi = (\psi_1, \psi_2)$, i.e. $\psi_1 > 0, \psi_2 > 0$ in Ω (see also [26, 73, 74, 93]). However it is not clear if the set of the eigenvalues described by (3.1.9) contains all the eigenvalues of problem (3.1.6).

Furthermore we remark that in the context of quasilinear systems, with $p \neq q$, the level set $E_{\lambda_k} = \{(u, v) \in X \mid \Phi(u, v) = \lambda_k \Psi(u, v)\}$ is not a linear space or a cone, but by (3.1.8) we infer that E_{λ_k} is a (p, q) -homogeneous set, i.e. $(t^{\frac{1}{p}} u, t^{\frac{1}{q}} v) \in E_{\lambda_k}$ for any $t \geq 0$ and $(u, v) \in E_{\lambda_k}$. This makes delicate to recognize a saddle type geometry for the energy functional associated to (3.1.3).

In Section 3.3 we derive a quite general abstract Linking Theorem (see Theorem 3.3.3), that extends Theorem 7.1 in [58] which deals with symmetric cones. This abstract result can be successfully applied in the framework of systems and allows to detect the existence of a minimax critical point for J_a with a suitable nontrivial critical group. Finally, in order to prove that the solution we found is not trivial, we need to compute the critical groups at the origin in terms of

differential notions. However the applicability of Morse arguments in Banach spaces present severe difficulties since classical Morse Lemma and generalized Morse lemma [104] are so far known, also due to the lack of Fredholm properties of the second derivative of the functionals. Moreover the energy functional J_a is not C^2 , so that we cannot define the classical notion of Morse index. In order to overcome this lack of regularity, the first idea is to consider the quadratic form $Q : X \rightarrow \mathbb{R}$ defined as

$$\begin{aligned} Q(z) = Q(u, v) &= a^{\frac{p-2}{2}} \left(\int_{\Omega} |\nabla u|^2 dx \right) + a^{\frac{q-2}{2}} \left(\int_{\Omega} |\nabla v|^2 dx \right) \\ &\quad - \int_{\Omega} (G_{ss}(0, 0)u^2 + G_{tt}(0, 0)v^2 + 2G_{st}(0, 0)uv) dx \\ &= a^{\frac{p-2}{2}} \|u\|_{1,2}^2 + a^{\frac{q-2}{2}} \|v\|_{1,2}^2 - \int_{\Omega} H_G(0, 0)[z]^2 dx, \end{aligned} \quad (3.1.10)$$

where $H_G(0, 0)$ is the Hessian matrix of G at $(0, 0)$. Then we consider m_0 and m_0^* defined as the supremum of the dimensions of the subspaces of X where Q is negative definite and negative semidefinite respectively. These objects will play the role of Morse index and large Morse index.

The second problem becomes to relate such differential notions m_0 and m_0^* to the behaviour of the functional J_a near the origin. This delicate issue will be obtained by means of a penalized functional of J_a , which is C^2 just in a ball centered at the origin. We stress that even if $p \geq 2$ and $q \geq 2$, the functional F is just C^1 on X , differently from the case of a single equation. Roughly speaking, we say that the eigenvalue problem (3.1.6) is intrinsically less regular than the eigenvalue problem for the single p -Laplace equation, due to the presence of the right-hand side coupled terms.

The main goals of this Chapter are the following:

Theorem 3.1.1. *Suppose that $2 \leq p < N, 2 \leq q < N, a \geq 0$ and $(a_1) - (a_2)$ hold. Assume that $\bar{\lambda}$ is not an eigenvalue for system (3.1.6) and denote by m_{∞} the integer such that*

$$\lambda_{m_{\infty}} < \bar{\lambda} < \lambda_{m_{\infty}+1}.$$

If

$$m_{\infty} \notin [m_0, m_0^*],$$

then there exists a nontrivial weak solution $z = (u, v)$ of (3.1.3).

We remark that in Theorem 3.1.1 the parameter $\bar{\lambda}$ is not an eigenvalue of (3.1.6) and $\lambda_m < \bar{\lambda} < \lambda_{m+1}$ for some $m \in \mathbb{N}$. This condition seems to be possible, for instance, if $m = 1$ taking into account that λ_1 is isolated and it is the only eigenvalue of (3.1.6) to which corresponds a componentwise positive eigenfunction [150]. However such nonresonant restriction would be quite severe and it becomes relevant to face with systems at resonance.

In the following theorems the value $\bar{\lambda}$ is allowed to be an eigenvalue of (3.1.6), under an additional condition of G at infinity.

Theorem 3.1.2. *Suppose that $2 \leq p < N, 2 \leq q < N, a \geq 0$ and $(a_1) - (a_2)$ hold. Denote by m_{∞} the integer such that*

$$\lambda_{m_{\infty}} < \bar{\lambda} \leq \lambda_{m_{\infty}+1}.$$

If

$$m_{\infty} \notin [m_0, m_0^*]$$

and

$$(b_-) \quad \lim_{|(s,t)| \rightarrow \infty} \left[G(s, t) - \frac{1}{p} G_s(s, t)s - \frac{1}{q} G_t(s, t)t \right] = -\infty,$$

then there exists a nontrivial weak solution $z = (u, v)$ of (3.1.3).

Theorem 3.1.3. *Suppose that $2 \leq p < N, 2 \leq q < N$, $a = 0$ and $(a_1) - (a_2)$ hold. Denote by m_∞ the integer such that*

$$\lambda_{m_\infty} \leq \bar{\lambda} < \lambda_{m_\infty+1}.$$

If

$$m_\infty \notin [m_0, m_0^*]$$

and

$$(b_+) \quad \lim_{|(s,t)| \rightarrow \infty} \left[G(s,t) - \frac{1}{p} G_s(s,t)s - \frac{1}{q} G_t(s,t)t \right] = +\infty,$$

then there exists a nontrivial weak solution $z = (u, v)$ of (3.1.3).

We remark that the nontrivial solutions of (3.1.3), including those found through Theorems 3.1.1, 3.1.2 and 3.1.3, cannot be semitrivial if we assume that there exists $\varepsilon > 0$ such that $G_t(s, 0) \neq 0$ for any $s \in (-\varepsilon, \varepsilon) \setminus \{0\}$ and $G_s(0, t) \neq 0$ for any $t \in (-\varepsilon, \varepsilon) \setminus \{0\}$ (see Proposition 3.5.2 and also [48, Lemma 3.1]).

Finally, we notice that if $\alpha = \beta = 0$ then $p = q = 2$. In this case, minimax arguments and Morse theory are applied in [100] for deriving existence of nontrivial solutions of a semilinear elliptic system with a C^2 nonlinear function G .

This Chapter is organized as follows: in Section 3.2, we show that the functional J_a associated with system (3.1.3) satisfies the PS condition or the CPS condition, depending on whether resonance conditions occur. In Section 3.3, we prove a Saddle Theorem based on cohomological linking and provide related results concerning (p, q) -homogeneous sets. Consequently, in Section 3.5 we show that suitable (p, q) -homogeneous sets satisfy the geometric assumptions of the Saddle Theorem, thereby obtaining a solution. Finally, in Section 3.4 we present estimates of critical groups at zero, which allow us to ensure that the obtained solution is nontrivial.

3.2 PS and CPS conditions

Proposition 3.2.1. *If $\{z_n\}_n$ is a bounded sequence in X such that $\|J'_\alpha(z_n)\| \rightarrow 0$, then $\{z_n\}_n$ has a convergent subsequence in X .*

Proof. Let $\{z_n\}_n = \{(u_n, v_n)\}_n \subset X$ be a bounded sequence such that $\|J'_a(z_n)\| \rightarrow 0$. Since $\{z_n\}_n$ is bounded in X , there exists a subsequence, still denoted by $\{z_n\}_n$, that converges to some $z = (u, v)$ weakly in X and strongly in $L^p(\Omega) \times L^q(\Omega)$.

In particular, by $\|J'_a(z_n)\| \rightarrow 0$ we get

$$\langle J'_a(z_n), (u_n - u, 0) \rangle = \int_{\Omega} (a + |\nabla u_n|^2)^{\frac{p-2}{2}} \nabla u_n \cdot \nabla (u_n - u) dx + \int_{\Omega} G_s(u_n, v_n)(u_n - u) dx \rightarrow 0.$$

Moreover, since $u_n \rightarrow u$ in $L^p(\Omega)$, by first inequality in (3.1.5) we deduce

$$\int_{\Omega} G_s(u_n, v_n)(u_n - u) dx \rightarrow 0.$$

Therefore we obtain

$$\langle H_{a,p}(u_n), u_n - u \rangle = \int_{\Omega} (a + |\nabla u_n|^2)^{\frac{p-2}{2}} \nabla u_n \cdot \nabla (u_n - u) dx \rightarrow 0.$$

By Lemma 1.2.1 we know that $H_{a,p}$ is of class $(S)_+$, so that $u_n \rightarrow u$ strongly in $W_0^{1,p}(\Omega)$. In the same way, we also get that $v_n \rightarrow v$ strongly in $W_0^{1,q}(\Omega)$.

□

Proposition 3.2.2. *If $\bar{\lambda}$ is not an eigenvalue of (3.1.6), then J_a satisfies the (PS) condition at any level $c \in \mathbb{R}$, namely if $\{z_n\}_n = \{(u_n, v_n)\}_n$ is a sequence in X such that $J_a(z_n) \rightarrow c$ and $\|J'_a(z_n)\| \rightarrow 0$, then $\{z_n\}_n$ has a convergent subsequence in X .*

Proof. Let $\{z_n\}_n = \{(u_n, v_n)\}_n$ be a sequence in X such that $J_a(z_n)$ is bounded and $\|J'_a(z_n)\| \rightarrow 0$. Due to the previous proposition, it suffices to prove that the sequence $\{z_n\}_n$ is bounded in X . By contradiction, there exists a subsequence, still denoted with $\{z_n\}_n$, such that $\|(u_n, v_n)\| \rightarrow \infty$. In particular

$$r_n := \|u_n\|_{1,p}^p + \|v_n\|_{1,q}^q \rightarrow +\infty.$$

Let us set

$$\bar{u}_n = \frac{u_n}{r_n^{1/p}} \quad \text{and} \quad \bar{v}_n = \frac{v_n}{r_n^{1/q}},$$

and observe that $\bar{z}_n := (\bar{u}_n, \bar{v}_n)$ satisfies

$$\|\bar{u}_n\|_{1,p}^p + \|\bar{v}_n\|_{1,q}^q = \frac{\|u_n\|_{1,p}^p + \|v_n\|_{1,q}^q}{r_n} = 1 \quad \text{for any } n \in \mathbb{N}.$$

Hence $\|\bar{z}_n\| \leq 2$ for any $n \in \mathbb{N}$, so there exists a subsequence, still denoted by $\{\bar{z}_n\}_n$, that converges to some $\bar{z} = (\bar{u}, \bar{v})$ weakly in X and strongly in $L^p(\Omega) \times L^q(\Omega)$.

In particular

$$\frac{1}{r_n^{\frac{p-1}{p}}} \langle J'_a(z_n), (\bar{u}_n - \bar{u}, 0) \rangle \rightarrow 0 \quad (3.2.11)$$

and

$$\frac{1}{r_n^{\frac{q-1}{q}}} \langle J'_a(z_n), (0, \bar{v}_n - \bar{v}) \rangle \rightarrow 0. \quad (3.2.12)$$

Considering first inequality in (3.1.5), we deduce

$$\frac{1}{r_n^{\frac{p-1}{p}}} \left| \int_{\Omega} G_s(z_n)(\bar{u}_n - \bar{u}) dx \right| \leq C \left(\int_{\Omega} |\bar{u}_n - \bar{u}| dx + \int_{\Omega} |\bar{u}_n|^{p-1} |\bar{u}_n - \bar{u}| dx + \int_{\Omega} |\bar{v}_n|^{\frac{q(p-1)}{p}} |\bar{u}_n - \bar{u}| dx \right).$$

Hence, since $\bar{u}_n \rightarrow \bar{u}$ in $L^p(\Omega)$, by Hölder inequality we obtain

$$\lim_{n \rightarrow \infty} \frac{1}{r_n^{\frac{p-1}{p}}} \int_{\Omega} G_s(z_n)(\bar{u}_n - \bar{u}) dx = 0$$

which, combined with (4.5.32), gives

$$\int_{\Omega} \left(\frac{a}{r_n^{2/p}} + |\nabla \bar{u}_n|^2 \right)^{\frac{p-2}{2}} \nabla \bar{u}_n \cdot \nabla (\bar{u}_n - \bar{u}) dx \rightarrow 0.$$

Using now the convexity (see Lemma 1.4.4) of the functional $f : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined as

$$f(u) := \int_{\Omega} \left(\frac{a}{r_n^{2/p}} + |\nabla u|^2 \right)^{\frac{p}{2}} dx,$$

we get

$$\begin{aligned} \|\bar{u}\|_{1,p}^p &\leq \liminf_{n \rightarrow \infty} \|\bar{u}_n\|_{1,p}^p \leq \limsup_{n \rightarrow \infty} \|\bar{u}_n\|_{1,p}^p \leq \limsup_{n \rightarrow \infty} \int_{\Omega} \left(\frac{a}{r_n^{2/p}} + |\nabla \bar{u}_n|^2 \right)^{\frac{p}{2}} dx \\ &\leq \limsup_{n \rightarrow \infty} \left[\int_{\Omega} \left(\frac{a}{r_n^{2/p}} + |\nabla \bar{u}|^2 \right)^{\frac{p}{2}} + p \int_{\Omega} \left(\frac{a}{r_n^{2/p}} + |\nabla \bar{u}_n|^2 \right)^{\frac{p-2}{2}} \nabla \bar{u}_n \cdot \nabla (\bar{u}_n - \bar{u}) dx \right] \leq \|\bar{u}\|_{1,p}^p, \end{aligned}$$

i.e. $\|\bar{u}_n\|_{1,p}^p \rightarrow \|\bar{u}\|_{1,p}^p$, and by uniform convexity of $W_0^{1,p}(\Omega)$, we deduce $\bar{u}_n \rightarrow \bar{u}$ in $W_0^{1,p}(\Omega)$. Analogously, by (3.2.12), we get that $\bar{v}_n \rightarrow \bar{v}$ strongly in $W_0^{1,q}(\Omega)$.

In particular

$$\|\bar{u}\|_{1,p}^p + \|\bar{v}\|_{1,q}^q = \lim_{n \rightarrow \infty} \left\{ \|\bar{u}_n\|_{1,p}^p + \|\bar{v}_n\|_{1,q}^q \right\} = 1,$$

so that $(\bar{u}, \bar{v}) \neq (0, 0)$.

Since $\|J'_a(z_n)\| \rightarrow 0$, for any $\tilde{z} = (\tilde{u}, \tilde{v}) \in X$ we have

$$\left\langle J'_a(z_n), \left(\frac{\tilde{u}}{r_n^{\frac{p-1}{p}}}, \frac{\tilde{v}}{r_n^{\frac{q-1}{q}}} \right) \right\rangle \rightarrow 0, \quad (3.2.13)$$

where

$$\begin{aligned} \left\langle J'_a(z_n), \left(\frac{\tilde{u}}{r_n^{\frac{p-1}{p}}}, \frac{\tilde{v}}{r_n^{\frac{q-1}{q}}} \right) \right\rangle &= \int_{\Omega} \left(\frac{a}{r_n^{2/p}} + |\nabla \bar{u}_n|^2 \right)^{\frac{p-2}{2}} \nabla \bar{u}_n \cdot \nabla \tilde{u} dx \\ &\quad + \int_{\Omega} \left(\frac{a}{r_n^{2/q}} + |\nabla \bar{v}_n|^2 \right)^{\frac{q-2}{2}} \nabla \bar{v}_n \cdot \nabla \tilde{v} dx \\ &\quad - \bar{\lambda} \left[\frac{1}{r_n^{\frac{p-1}{p}}} \int_{\Omega} F_s(z_n) \tilde{u} dx + \frac{1}{r_n^{\frac{q-1}{q}}} \int_{\Omega} F_t(z_n) \tilde{v} dx \right] \\ &\quad - \bar{\lambda} \left[\frac{1}{r_n^{\frac{p-1}{p}}} \int_{\Omega} r_1(z_n) \tilde{u} dx + \frac{1}{r_n^{\frac{q-1}{q}}} \int_{\Omega} r_2(z_n) \tilde{v} dx \right]. \end{aligned}$$

By assumption (a_1) , we get

$$G_s(s, t) = \bar{\lambda} F_s(s, t) + r_1(s, t) \quad \text{and} \quad G_t(s, t) = \bar{\lambda} F_t(s, t) + r_2(s, t),$$

where $r_1 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $r_2 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions such that

$$\lim_{|(s,t)| \rightarrow \infty} \frac{r_1(s, t)}{|s|^{p-1} + |t|^{q \frac{p-1}{p}}} = 0 \quad \text{and} \quad \lim_{|(s,t)| \rightarrow \infty} \frac{r_2(s, t)}{|t|^{q-1} + |s|^{p \frac{q-1}{q}}} = 0. \quad (3.2.14)$$

Hence there is $c > 0$ such that

$$|r_1(s, t)| \leq c \left(|s|^{p-1} + |t|^{q \frac{p-1}{p}} + 1 \right) \quad \text{and} \quad |r_2(s, t)| \leq c \left(|t|^{q-1} + |s|^{p \frac{q-1}{q}} + 1 \right) \quad (3.2.15)$$

for any $(s, t) \in \mathbb{R}^2$. We want to prove that

$$\lim_{n \rightarrow \infty} \frac{1}{r_n^{\frac{p-1}{p}}} \int_{\Omega} |r_1(z_n(x))| |\tilde{u}(x)| dx = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{1}{r_n^{\frac{q-1}{q}}} \int_{\Omega} |r_2(z_n(x))| |\tilde{v}(x)| dx = 0.$$

Recalling that $\|\bar{u}_n\|_{1,p}, \|\bar{v}_n\|_{1,q} \leq 1$, let $\bar{c} > 0$ be such that

$$\|\bar{u}_n\|_p^{p-1} + \|\bar{v}_n\|_q^{q \frac{p-1}{p}} \leq \bar{c} \quad \text{for any } n \in \mathbb{N}. \quad (3.2.16)$$

Let us fix $\varepsilon > 0$. By (3.2.14), there is $\delta_\varepsilon > 0$ such that

$$|(s, t)| > \delta_\varepsilon \Rightarrow |r_1(s, t)| < \frac{\varepsilon}{2\bar{c}} \left(|s|^{p-1} + |t|^{q \frac{p-1}{p}} \right). \quad (3.2.17)$$

Denoting by $\Omega_{n,\varepsilon}^- = \{x \in \Omega : |z_n(x)| \leq \delta_\varepsilon\}$ and $\Omega_{n,\varepsilon}^+ = \Omega \setminus \Omega_{n,\varepsilon}^-$,

$$x \in \Omega_{n,\varepsilon}^- \Rightarrow |u_n(x)|, |v_n(x)| \leq \delta_\varepsilon$$

so, by (3.2.15),

$$\frac{1}{r_n^{\frac{p-1}{p}}} \int_{\Omega_{n,\varepsilon}^-} |r_1(z_n(x))| |\tilde{u}(x)| dx \leq \frac{c}{r_n^{\frac{p-1}{p}}} \left(|\delta_\varepsilon|^{p-1} + |\delta_\varepsilon|^{q \frac{p-1}{p}} + 1 \right) \|\tilde{u}\|_1$$

and, choosing n big enough,

$$\frac{1}{r_n^{\frac{p-1}{p}}} \int_{\Omega_{n,\varepsilon}^-} |r_1(z_n(x))| |\tilde{u}(x)| dx < \frac{\varepsilon}{2} \|\tilde{u}\|_p. \quad (3.2.18)$$

On the other hand, by (3.2.17), Hölder inequality and (3.2.16),

$$\begin{aligned} & \frac{1}{r_n^{\frac{p-1}{p}}} \int_{\Omega_{n,\varepsilon}^+} |r_1(z_n(x))| |\tilde{u}(x)| dx \\ & \leq \frac{\varepsilon}{2\bar{c}} \frac{1}{r_n^{\frac{p-1}{p}}} \int_{\Omega_{n,\varepsilon}^+} \left(|u_n(x)|^{p-1} + |v_n(x)|^{q \frac{p-1}{p}} \right) |\tilde{u}(x)| dx \\ & \leq \frac{\varepsilon}{2\bar{c}} \int_{\Omega} \left(|\bar{u}_n(x)|^{p-1} + |\bar{v}_n(x)|^{q \frac{p-1}{p}} \right) |\tilde{u}(x)| dx \\ & \leq \frac{\varepsilon}{2} \|\tilde{u}\|_p \end{aligned}$$

which, together with (3.2.18), gives

$$\lim_{n \rightarrow \infty} \frac{1}{r_n^{\frac{p-1}{p}}} \int_{\Omega} |r_1(z_n(x))| |\tilde{u}(x)| dx = 0.$$

Similarly, we also get

$$\lim_{n \rightarrow \infty} \frac{1}{r_n^{\frac{q-1}{q}}} \int_{\Omega} |r_2(z_n(x))| |\tilde{v}(x)| dx = 0.$$

Since $\bar{u}_n \rightarrow \bar{u}$ in $W_0^{1,p}(\Omega)$ and $\bar{v}_n \rightarrow \bar{v}$ in $W_0^{1,q}(\Omega)$, we get

$$\lim_{n \rightarrow \infty} \int_{\Omega} \left(\frac{a}{r_n^{2/p}} + |\nabla \bar{u}_n|^2 \right)^{\frac{p-2}{2}} \nabla \bar{u}_n \cdot \nabla \tilde{u} dx = \int_{\Omega} |\nabla \bar{u}|^{p-2} \nabla \bar{u} \cdot \nabla \tilde{u} dx,$$

$$\lim_{n \rightarrow \infty} \int_{\Omega} \left(\frac{a}{r_n^{2/q}} + |\nabla \bar{v}_n|^2 \right)^{\frac{q-2}{2}} \nabla \bar{v}_n \cdot \nabla \bar{v} \, dx = \int_{\Omega} |\nabla \bar{v}|^{q-2} \nabla \bar{v} \cdot \nabla \bar{v} \, dx,$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{r_n^{\frac{p-1}{p}}} \int_{\Omega} F_s(u_n(x), v_n(x)) \tilde{u}(x) \, dx &= \int_{\Omega} F_s(\bar{u}(x), \bar{v}(x)) \tilde{u}(x) \, dx, \\ \lim_{n \rightarrow \infty} \frac{1}{r_n^{\frac{q-1}{q}}} \int_{\Omega} F_t(u_n(x), v_n(x)) \tilde{v}(x) \, dx &= \int_{\Omega} F_t(\bar{u}(x), \bar{v}(x)) \tilde{v}(x) \, dx. \end{aligned}$$

Considering these convergences in (3.2.13), we have proved

$$\begin{aligned} \int_{\Omega} |\nabla \bar{u}|^{p-2} \nabla \bar{u} \cdot \nabla \tilde{u} \, dx &= \bar{\lambda} \int_{\Omega} F_s(\bar{u}, \bar{v}) \tilde{u} \, dx \\ \int_{\Omega} |\nabla \bar{v}|^{q-2} \nabla \bar{v} \cdot \nabla \tilde{v} \, dx &= \bar{\lambda} \int_{\Omega} F_t(\bar{u}, \bar{v}) \tilde{v} \, dx \end{aligned}$$

for any $\tilde{z} = (\tilde{u}, \tilde{v}) \in X$. In other words, $(\bar{u}, \bar{v}) \neq (0, 0)$ solves (3.1.6) with $\lambda = \bar{\lambda}$, so the contradiction proves that $\{z_n\}_n$ is bounded in X . $\square$

Proposition 3.2.3. *If $\bar{\lambda}$ is an eigenvalue of (3.1.6) and (b_-) holds, so that*

$$\lim_{|(s,t)| \rightarrow \infty} \left[G(s, t) - \frac{1}{p} G_s(s, t) s - \frac{1}{q} G_t(s, t) t \right] = -\infty,$$

then J_a satisfies the (CPS) condition at any level $c \in \mathbb{R}$, namely any sequence $\{z_n\}_n = \{(u_n, v_n)\}_n$ in X such that $J_a(z_n) \rightarrow c$ and $\|J'_a(z_n)\|(1 + \|z_n\|) \rightarrow 0$ has a convergent subsequence in X .

Moreover, if $\bar{\lambda}$ is an eigenvalue of (3.1.6) and (b_+) holds, so that

$$\lim_{|(s,t)| \rightarrow \infty} \left[G(s, t) - \frac{1}{p} G_s(s, t) s - \frac{1}{q} G_t(s, t) t \right] = +\infty,$$

then J_0 satisfies the (CPS) condition at any level $c \in \mathbb{R}$.

Proof. Let $\{z_n\}_n = \{(u_n, v_n)\}_n$ be a sequence in X such that $J_a(z_n)$ is bounded and

$$\|J'_a(z_n)\|(1 + \|z_n\|) \rightarrow 0.$$

Firstly we notice that previous condition implies $\|J'_a(z_n)\| \rightarrow 0$, hence by Proposition 3.2.1 it suffices to prove that the sequence $\{z_n\}_n$ is bounded in X .

By contradiction, there exists a subsequence, still denoted with $\{z_n\}_n$, such that $\|(u_n, v_n)\| \rightarrow \infty$. As in the proof of Proposition 3.2.2, we set

$$r_n := \|u_n\|_{1,p}^p + \|v_n\|_{1,q}^q \rightarrow +\infty,$$

and

$$\bar{u}_n = \frac{u_n}{r_n^{1/p}} \quad \text{and} \quad \bar{v}_n = \frac{v_n}{r_n^{1/q}},$$

so that $\{\bar{z}_n\}_n = \{(\bar{u}_n, \bar{v}_n)\}_n$ is bounded in X and there exists a subsequence, still denoted by $\{\bar{z}_n\}_n$, that converges to some $\bar{z} = (\bar{u}, \bar{v})$ weakly in X and strongly in $L^p(\Omega) \times L^q(\Omega)$. Arguing as in the proof of Proposition 3.2.2, we deduce that $\{\bar{z}_n\}_n$ converges to $\bar{z} = (\bar{u}, \bar{v})$ strongly in X , with $\bar{z} \neq (0, 0)$.

Let us show that $\{z_n\}_n$ is bounded in X when (b_-) holds and $a \geq 0$.

Denoting by

$$K(s, t) = G(s, t) - \frac{1}{p}G_s(s, t)s - \frac{1}{q}G_t(s, t)t,$$

by (CPS) condition there exists a constant $C \in \mathbb{R}$ such that

$$\begin{aligned} C &\geq J_a(z_n) - \left\langle J'_a(z_n), \left(\frac{u_n}{p}, \frac{v_n}{q}\right) \right\rangle \\ &= \frac{1}{p} \int_{\Omega} \left(a + |\nabla u_n|^2\right)^{\frac{p}{2}} dx - \frac{1}{p} \int_{\Omega} \left(a + |\nabla u_n|^2\right)^{\frac{p-2}{2}} |\nabla u_n|^2 dx \\ &\quad + \frac{1}{q} \int_{\Omega} \left(a + |\nabla v_n|^2\right)^{\frac{q}{2}} dx - \frac{1}{q} \int_{\Omega} \left(a + |\nabla v_n|^2\right)^{\frac{q-2}{2}} |\nabla v_n|^2 dx \\ &\quad - \left[\int_{\Omega} \left(G(u_n(x), v_n(x)) - \frac{1}{p}G_s(u_n(x), v_n(x))u_n(x) - \frac{1}{q}G_t(u_n(x), v_n(x))v_n(x) \right) dx \right] \\ &= \frac{a}{p} \int_{\Omega} \left(a + |\nabla u_n|^2\right)^{\frac{p-2}{2}} dx + \frac{a}{q} \int_{\Omega} \left(a + |\nabla v_n|^2\right)^{\frac{q-2}{2}} dx - \int_{\Omega} K(z_n(x)) dx \\ &\geq - \int_{\Omega} K(z_n(x)) dx. \end{aligned} \tag{3.2.19}$$

From (b_-) , there is $c_1 \in \mathbb{R}$ such that

$$K(s, t) \leq c_1 \quad \forall (s, t) \in \mathbb{R}^2,$$

hence, by reverse Fatou's lemma we get

$$\limsup_{n \rightarrow \infty} \int_{\Omega} K(z_n(x)) dx \leq \int_{\Omega} \limsup_{n \rightarrow \infty} K(z_n(x)) dx$$

and taking into account inequality (3.2.19) we deduce

$$-C \leq \int_{\Omega} \limsup_{n \rightarrow \infty} K(z_n(x)) dx. \tag{3.2.20}$$

As we are assuming $r_n \rightarrow +\infty$, by (b_-) we get that, for almost every $x \in \Omega$

$$\bar{z}(x) \neq (0, 0) \Rightarrow |z_n(x)| = \left| (r_n^{1/p} \bar{u}_n(x), r_n^{1/q} \bar{v}_n(x)) \right| \rightarrow \infty \Rightarrow K(z_n(x)) \rightarrow -\infty.$$

So, from (3.2.20), we infer that $\bar{z}(x) = (0, 0)$ almost everywhere in Ω , which contradicts $\bar{z} \neq (0, 0)$. Consequently $\{z_n\}_n$ is bounded and, by Proposition 3.2.1, has a convergent subsequence in X .

If instead (b_+) holds, $J_0(z_n) \rightarrow c \in \mathbb{R}$ and $\|J'_0(z_n)\|(1 + \|z_n\|) \rightarrow 0$, then

$$\lim_{n \rightarrow \infty} \int_{\Omega} K(z_n(x)) dx = -c.$$

Then, reasoning as before, we infer again that $\{z_n\}_n$ has a convergent subsequence in X . $\square$

3.3 Cohomological linking

Throughout this section, Y denotes a Banach space and $f : Y \rightarrow \mathbb{R}$ a C^1 function. We also denote by H^* the Alexander-Spanier cohomology [148] with coefficients in $\mathbb{Z}_2 = \{-1, 1\}$.

Let us recall the following definitions (see [43, 46, 79, 80]).

Definition 3.3.1. Let D, S, A be three subsets of Y , m a nonnegative integer.

We say that (D, S) *links A cohomologically in dimension m* (over $\mathbb{Z}_2$) if $S \subseteq D$, $S \cap A = \emptyset$ and the restriction homomorphism $H^m(Y, Y \setminus A) \rightarrow H^m(D, S)$ is not identically zero.

If (D, S) links A cohomologically in some dimension, then (D, S) links A (c.f. [55, Definition 5.1]). Moreover if (D, S) links A , then $D \cap A \neq \emptyset$.

Definition 3.3.2. Let G be an abelian group. Let the m -th critical group of f at $z \in Y$ with coefficient in G be defined by

$$C_m(f; z) = H^m(f^c, f^c \setminus \{z\}),$$

where $c = f(z)$, $f^c = \{u \in Y \mid f(u) \leq c\}$.

In general it can happen that $C_m(f, z)$ is not finitely generated for some m and that $C_m(f, z) \neq 0$ for infinitely many m 's.

Now we assume that f is Gauteax differentiable and we denote by U an open subset of X . If z is an isolated critical point of f and $f' : U \rightarrow X'$ is a demicontinuous function (namely it is continuous from the strong topology to the weak topology) and of class $(S)_+$ in a neighborhood of u , then $C_*(f, z)$ is of finite type (see [56, Theorem 1.1] and [6, Theorem 3.4]).

Let us introduce a general result concerning a C^1 -functional in a Banach space which extends Theorem 3.2 in [55] to the case in which f satisfies only the (CPS) condition.

Theorem 3.3.3. Let Y be a Banach space, $f \in C^1(Y, \mathbb{R})$, D, S, A be three subsets of Y , $m \in \mathbb{N}$.

Assume that

(D, S) links A cohomologically in dimension m over $\mathbb{Z}_2$,

$$\sup_S f < \inf_A f = a, \quad b = \sup_D f < +\infty,$$

f verifies the (CPS) condition at any level $c \in [a, b]$ and $f^{-1}([a, b])$ contains a finite number of critical points.

Then there exists a critical point z of f with $a \leq f(z) \leq b$ and $C_m(f; z) \neq \{0\}$.

Proof. Since $D \cap A \neq \emptyset$, then $a \leq b$. We aim to apply [79, Theorem 5.2]. If (E, d) is a metric space, $f : E \rightarrow \mathbb{R}$ is a continuous function and $z \in E$, let us consider the notion of *weak slope* $|df|(z)$ as defined in [81, Definition 2.1] (see also [112, Definition 5.1]). Consequently we say that

- z is a critical point if $|df|(z) = 0$;
- f satisfies the (PS) condition at a level $c \in \mathbb{R}$ if any sequence z_n in E such that $f(z_n) \rightarrow c$ and $|df|(z_n) \rightarrow 0$ has a convergent subsequence in E .

In particular, if E is also a Banach space and f is C^1 , we have

$$|df|(z) = \|f'(z)\| \quad \forall z \in E.$$

Due to [66, Theorem 4.1, Remark 4.4], there exists a metric $\tilde{d}$ on Y , topologically equivalent to the metric induced by $\|\cdot\|_Y$, such that $(Y, \tilde{d})$ is complete and, denoting by $|\tilde{d}f|$ the weak slope of f with respect to the metric $\tilde{d}$,

$$|\tilde{d}f|(z) = \|f'(z)\|_{Y'}(1 + \|z\|_Y) \quad \forall z \in Y.$$

Therefore, as f satisfies the (CPS) condition at any level $c \in [a, b]$, then f satisfies the (PS) condition at any level $c \in [a, b]$, with respect to $\tilde{d}$.

By [79, Theorem 7.5] the assumptions of [79, Theorem 5.2] are satisfied. Then the assertion follows, taking into account [79, Proposition 7.3 and Remark 5.3]. $\square$

Denoting by $\mathcal{A}$ the class of symmetric subsets of Y , Fadell and Rabinowitz (see [97, 98]) constructed an index theory $i : \mathcal{A} \rightarrow \mathbb{N} \cup \{\infty\}$ with the following properties:

- (i) Definiteness: $i(A) \geq 0$, $i(A) = 0 \Leftrightarrow A = \emptyset$;
- (ii) Monotonicity: If there is an odd continuous map $A \rightarrow A'$, then

$$i(A) \leq i(A'),$$

in particular, equality holds if A and A' are homeomorphic;

- (iii) Subadditivity:

$$i(A \cup A') \leq i(A) + i(A');$$

- (iv) Continuity: If A is closed, there is a neighborhood $U \in \mathcal{A}$ of A such that

$$i(U) = i(A);$$

- (v) Neighborhood of zero: If U is a bounded symmetric neighborhood of 0 in Y ,

$$i(\partial U) = \dim Y;$$

- (vi) Stability: If A is closed and $A * \mathbb{Z}_2$ is the join of A with $\mathbb{Z}_2$, realized in $Y \oplus \mathbb{R}$,

$$i(A * \mathbb{Z}_2) = i(A) + 1$$

where $A * \mathbb{Z}_2$ is the union of all line segments in $Y \oplus \mathbb{R}$, joining $\{1\}$ and $\{-1\}$ to points of A ;

- (vii) Piercing property: Assume that A, A_0, A_1 are closed and

$$\varphi : A \times [0, 1] \rightarrow A_0 \cup A_1$$

is an odd continuous map such that $\varphi(A \times [0, 1])$ is closed, and

$$\varphi(A \times \{0\}) \subset A_0, \quad \varphi(A \times \{1\}) \subset A_1.$$

Then

$$i(\varphi(A \times [0, 1]) \cap A_0 \cap A_1) \geq i(A).$$

We can state the next theorem whose proof follows from [55, Theorem 3.6] and [80, Theorem 2.7]. For reader's convenience, we give a sketch of the proof.

Theorem 3.3.4. *Let S, A be two symmetric subsets of Y with $S \cap A = \emptyset$. Assume that $0 \in A$ and $i(S) = i(Y \setminus A) = m < \infty$.*

Then (Y, S) links A cohomologically in dimension m over $\mathbb{Z}_2$.

Proof. We want to prove that $H^m(Y, Y \setminus A) \rightarrow H^m(Y, S)$ is not identically zero. So, considering the exact sequence

$$H^m(Y, Y \setminus A) \rightarrow H^m(Y, S) \rightarrow H^m(Y \setminus A, S)$$

it suffices to show that

$$H^m(Y, S) \rightarrow H^m(Y \setminus A, S) \quad \text{is not injective.} \quad (3.3.21)$$

If $m = 0$, then $S = \emptyset$ and $A = Y$, so $H^0(Y, S) \neq \{0\}$ while $H^0(Y \setminus A, S) = \{0\}$, whence (3.3.21) is proved.

Let us denote by $j : H^{m-1}(Y \setminus A) \rightarrow H^{m-1}(S)$ and $i : H^{m-1}(Y) \rightarrow H^{m-1}(S)$ the homomorphisms induced by inclusions. We recall that $H^q(Y) = \{0\}$ if $q \neq 0$, while the dimension of $H^0(Y)$ is 1.

By Lemma 3.4 in [55], we infer that

(i) if $m = 1$, the dimension of $im(j)$ is at least 2, while the dimension of $im(i)$ is at most 1.

(ii) if $m \geq 2$, the dimension of $im(j)$ is at least 1, while $im(i) = \{0\}$.

Hence $im(i)$ is a proper subset of $im(j)$, for any $m \geq 1$.

Consider now the commutative diagram

$$\begin{array}{ccccc} H^{m-1}(Y) & \xrightarrow{i} & H^{m-1}(S) & \xrightarrow{\delta} & H^m(Y, S) \\ \downarrow & & \text{Id} \downarrow & & \downarrow \\ H^{m-1}(Y \setminus A) & \xrightarrow{j} & H^{m-1}(S) & \xrightarrow{\bar{\delta}} & H^m(Y \setminus A, S) \end{array}$$

where the horizontal rows are exact sequences and the vertical rows are induced by inclusions.

Let $\omega \in im(j) \setminus im(i)$, then $\delta\omega \neq 0$ in $H^m(Y, S)$, while $\bar{\delta}\omega = (\delta\omega)|_{(Y \setminus A, S)} = 0$.

Hence (3.3.21) is proved also for any $m \geq 1$, as $H^m(Y, S) \rightarrow H^m(Y \setminus A, S)$ is not injective. $\square$

Now we consider the Banach space $X = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. We recall that the sequence (λ_m) is defined by (3.1.9), i.e.

$$\lambda_m = \inf_{A \in \Sigma_m} \sup_{(u,v) \in A} \Phi(u, v)$$

where $\Sigma_m = \{A \subset \Sigma \mid A \text{ is symmetric, compact and } i(A) \geq m\}$. Let us recall the following Lemma, proved in [80].

Lemma 3.3.5. *For every symmetric and open subset A of Σ*

$$i(A) = \sup\{i(K) \mid K \text{ is compact and symmetric with } K \subseteq A\}.$$

Remark 3.3.6. Denoting by $\mathcal{B}_m = \{A \subset \Sigma \mid A \text{ is symmetric and } i(A) \geq m\}$, the previous Lemma assures that

$$\lambda_m = \inf_{A \in \mathcal{B}_m} \sup_{(u,v) \in A} \Phi(u, v).$$

Proposition 3.3.7. *Let $\alpha \in \mathbb{R}$ and $r > 0$ and set*

$$X^\alpha = \{z \in X \mid \Phi(z) \leq \alpha\Psi(z)\}, \quad \Sigma^\alpha = \Sigma \cap X^\alpha,$$

$$S_r^\alpha = \{z \in X^\alpha \mid \Phi(z) = r\},$$

$$\overset{\circ}{X}^\alpha = \{z \in X \mid \Phi(z) < \alpha\Psi(z)\}, \quad \overset{\circ}{\Sigma}^\alpha = \Sigma \cap \overset{\circ}{X}^\alpha.$$

Then

- (a) $i(\Sigma^\alpha) \geq m$, for any $\alpha \geq \lambda_m$;
- (b) $\lambda_m < \lambda_{m+1} \Rightarrow i(\Sigma^\alpha) = m$, for any $\alpha \in [\lambda_m, \lambda_{m+1})$;
- (c) $\lambda_m < \lambda_{m+1} \Rightarrow i(\overset{\circ}{\Sigma}^{\lambda_{m+1}}) = m$;
- (d) Σ^α is an odd strong deformation retract of $X^\alpha \setminus \{0\}$;
- (e) $\overset{\circ}{\Sigma}^\alpha$ is an odd strong deformation retract of $\overset{\circ}{X}^\alpha \setminus \{0\}$;
- (f) S_r^α is an odd strong deformation retract of $X^\alpha \setminus \{0\}$.

Proof. Let us consider $\alpha > \lambda_1$, otherwise the proof is even simpler.

By contradiction, assume that $i(\Sigma^{\lambda_m}) \leq m - 1$. Continuity assures that there is a closed neighborhood N of Σ^{λ_m} such that $i(N) = i(\Sigma^{\lambda_m})$. As N is also a neighborhood of the critical set $\{z \mid \Phi|_\Sigma(z) = \lambda_m\}$, by the equivariant deformation theorem, there exist $\varepsilon > 0$ and an odd continuous map $\eta : \Sigma^{\lambda_m + \varepsilon} \rightarrow \Sigma^{\lambda_m - \varepsilon} \cup N = N$. So, due to monotonicity, $i(\Sigma^{\lambda_m + \varepsilon}) \leq i(N) = i(\Sigma^{\lambda_m}) \leq m - 1$. By definition of λ_m , there is $\bar{A} \in \Sigma_m$ such that $\sup \Phi(\bar{A}) < \lambda_m + \varepsilon$. Hence $\bar{A} \subset \Sigma^{\lambda_m + \varepsilon}$ and we have the contradiction $m \leq i(\bar{A}) \leq i(\Sigma^{\lambda_m + \varepsilon}) \leq m - 1$, which proves (a).

Now, if $\lambda_m < \lambda_{m+1}$ and $\alpha \in [\lambda_m, \lambda_{m+1})$, by (a) we know that $i(\Sigma^\alpha) \geq m$. By contradiction, assume that $i(\Sigma^\alpha) \geq m + 1$. Recalling Remark 3.3.6, $\Sigma^\alpha \in \mathcal{B}_{m+1}$ and

$$\lambda_{m+1} = \inf_{A \in \mathcal{B}_{m+1}} \sup_A \Phi \leq \sup_{\Sigma^\alpha} \Phi = \alpha < \lambda_{m+1}$$

so we conclude that $i(\Sigma^\alpha) = m$.

Moreover, still due to (a), $i(\overset{\circ}{\Sigma}^{\lambda_{m+1}}) \geq m$. Let K be a symmetric and compact subset of $\overset{\circ}{\Sigma}^{\lambda_{m+1}}$, so that $\max \Phi(K) \leq \alpha$ for a suitable $\alpha \in [\lambda_m, \lambda_{m+1})$. (b) gives that $i(K) \leq i(\Sigma^\alpha) = m$, hence (c) comes from Lemma 3.3.5.

For any $s \in [0, 1]$ and $z = (u, v) \in X \setminus \{0\}$, we define $\gamma_{s,z} = 1 - s + \frac{s}{\Psi(z)}$ and $\eta(s, (u, v)) = (\gamma_{s,z}^{1/p} u, \gamma_{s,z}^{1/q} v)$.

η is clearly continuous and odd. In addition, due to the (p, q) -homogeneity of Φ and Ψ , showed in (3.1.8),

$$\Phi(\eta(s, z)) = \gamma_{s,z} \Phi(z) \quad \text{and} \quad \Psi(\eta(s, z)) = \gamma_{s,z} \Psi(z).$$

Hence (d) and (e) are proved as

$$z \in X^\alpha \setminus \{0\} \Rightarrow \eta(0, z) = z, \quad \eta(1, z) \in \Sigma^\alpha \quad \text{and} \quad \eta(s, z) \in X^\alpha \setminus \{0\} \quad \forall s \in [0, 1]$$

$$z \in \Sigma \Rightarrow \eta(s, z) = z \quad \text{for any } s \in [0, 1].$$

Analogously we obtain (f) setting

$$\delta_{s,z} = 1 - s + s \frac{r}{\Phi(z)} \quad \text{and} \quad \bar{\eta}(s, (u, v)) = (\delta_{s,z}^{1/p} u, \delta_{s,z}^{1/q} v). \quad \square$$

Let $m \in \mathbb{N}$ be such that $\lambda_m < \lambda_{m+1}$.

If $m \geq 1$ we set

$$\begin{aligned} X_-^m &:= \{z = (u, v) \in X \mid \Phi(u, v) \leq \lambda_m \Psi(u, v)\} \\ X_+^m &:= \{z = (u, v) \in X \mid \Phi(u, v) \geq \lambda_{m+1} \Psi(u, v)\} \end{aligned}$$

and if $m = 0$

$$X_-^0 := \{0\} \quad X_+^0 := X.$$

We underline that the sets X_-^m and X_+^m are symmetric and (p, q) -homogeneous, i.e.,

$$(u, v) \in X_\pm^m \Rightarrow (\delta^{\frac{1}{p}} u, \delta^{\frac{1}{q}} v) \in X_\pm^m \quad \text{for any } \delta > 0.$$

In particular, if $p \neq q$ and $m \geq 1$, the sets X_+^m and X_-^m are not cones.

Theorem 3.3.8. *Setting*

$$D_r = \{z \in X_-^m \mid \Phi(z) \leq r\} \quad S_r = \{z \in X_-^m \mid \Phi(z) = r\},$$

(D_r, S_r) links X_+^m cohomologically in dimension m over $\mathbb{Z}_2$, for any $r > 0$.

Proof. From Proposition 3.3.7 we obtain that

$$i(S_r) = i(X \setminus X_+^m) = m.$$

As S_r and X_+^m are symmetric, $0 \in X_+^m$ and $S_r \cap X_+^m = \emptyset$, Theorem 3.3.4 gives that

$$(X, S_r) \text{ links } X_+^m \text{ cohomologically in dimension } m \text{ over } \mathbb{Z}_2. \quad (3.3.22)$$

Consider the commutative diagram

$$\begin{array}{ccccccccc} H^{m-1}(X) & \rightarrow & H^{m-1}(X_-^m \setminus \{0\}) & \rightarrow & H^m(X, X_-^m \setminus \{0\}) & \rightarrow & H^m(X) & \rightarrow & H^m(X_-^m \setminus \{0\}) \\ \downarrow \gamma_1 & & \downarrow \gamma_2 & & \downarrow \gamma_3 & & \downarrow \gamma_4 & & \downarrow \gamma_5 \\ H^{m-1}(D_r) & \longrightarrow & H^{m-1}(S_r) & \longrightarrow & H^m(D_r, S_r) & \longrightarrow & H^m(D_r) & \longrightarrow & H^m(S_r) \end{array}$$

where the horizontal rows are exact sequences and the vertical rows are induced by inclusions.

In particular, γ_1 , γ_2 , γ_4 and γ_5 are isomorphisms, as D_r is homotopic to X and S_r is homotopic to $X_-^m \setminus \{0\}$. Hence also γ_3 is an isomorphism, which, combined with (3.3.22), completes the proof. $\square$

3.4 Critical Groups Estimates at zero

From now on we assume that $p, q \in [2, N)$, $a \geq 0$, $H \in C^1(\mathbb{R}^2, \mathbb{R})$ and there exist $C > 0$, $p' \in (p, p^*)$, $q' \in (q, q^*)$ such that

$$\begin{aligned} |H_s(s, t)| &\leq C \left(|s|^{p'-1} + |t|^{q' \frac{p'-1}{p'}} + 1 \right), \\ |H_t(s, t)| &\leq C \left(|s|^{p' \frac{q'-1}{q'}} + |t|^{q'-1} + 1 \right). \end{aligned} \quad (3.4.23)$$

We denote by $I_{a,H}$ the C^1 -functional defined for any $z = (u, v) \in X$ by

$$I_{a,H}(z) = \frac{1}{p} \int_{\Omega} \left(a + |\nabla u|^2 \right)^{\frac{p}{2}} dx + \frac{1}{q} \int_{\Omega} \left(a + |\nabla v|^2 \right)^{\frac{q}{2}} dx - \int_{\Omega} H(u, v) dx. \quad (3.4.24)$$

Theorem 3.4.1. *Let $p, q \in [2, N)$, $a \geq 0$. Assume that $H \in C^1(\mathbb{R}^2, \mathbb{R})$ satisfies (3.4.23) and $\nabla H(0, 0) = (0, 0)$.*

If $\mathbf{0} = (0, 0)$ is an isolated critical point for $I_{a,H}$, there are $\overline{H} \in C^1(\mathbb{R}^2, \mathbb{R})$ and $\bar{\eta} > 0$ such that:

- $\overline{H}(s, t) = H(s, t)$ when $|s|, |t| \leq \bar{\eta}$;
- $\mathbf{0}$ is an isolated critical point for $I_{a,\overline{H}}$ and $C_m(I_{a,H}, \mathbf{0}) = C_m(I_{a,\overline{H}}, \mathbf{0})$, for any $m \in \mathbb{N}$;
- $\overline{H}(\mathbb{R}^2) = \overline{H}([-2\bar{\eta}, 2\bar{\eta}]^2) \subset H([-2\bar{\eta}, 2\bar{\eta}]^2)$.

In particular, if $(s, t) \notin [-2\bar{\eta}, 2\bar{\eta}]^2$, then

$$\overline{H}(s, t) = \begin{cases} H(0, 0) & \text{if } |s| \geq 2\bar{\eta}, |t| \geq 2\bar{\eta}, \\ \overline{H}(0, t) & \text{if } |s| \geq 2\bar{\eta}, |t| \leq 2\bar{\eta}, \\ \overline{H}(s, 0) & \text{if } |s| \leq 2\bar{\eta}, |t| \geq 2\bar{\eta}. \end{cases}$$

In addition, if H is C^2 at least in an open set containing $[-2\bar{\eta}, 2\bar{\eta}]^2$, then $\overline{H} \in C^2(\mathbb{R}^2, \mathbb{R})$.

Proof. Consider a C^∞ -function $\theta : \mathbb{R} \rightarrow [0, 1]$ such that $\theta(\tau) = 1$ for $|\tau| \leq 1$ and $\theta(\tau) = 0$ for $|\tau| \geq 2$. For every $\delta \in [0, 1]$, we define

$$H^\delta(s, t) = H(\theta(\delta s)s, \theta(\delta t)t),$$

so that

$$H_s^\delta(s, t) = H_s(\theta(\delta s)s, \theta(\delta t)t)(\theta'(\delta s)\delta s + \theta(\delta s)) \quad (3.4.25)$$

$$H_t^\delta(s, t) = H_t(\theta(\delta s)s, \theta(\delta t)t)(\theta'(\delta t)\delta t + \theta(\delta t)).$$

As the function $\tau \mapsto |\theta'(\tau)\tau + \theta(\tau)|$ is bounded in $\mathbb{R}$, by (3.4.23) and (3.4.25) there is a constant $C_0 > 0$, independent from δ , such that

$$\begin{aligned} |H_s^\delta(s, t)| &\leq C_0 \left(|s|^{p'-1} + |t|^{q' \frac{p'-1}{p'}} + 1 \right), \\ |H_t^\delta(s, t)| &\leq C_0 \left(|s|^{p' \frac{q'-1}{q}} + |t|^{q'-1} + 1 \right). \end{aligned} \quad (3.4.26)$$

Hence the functional $I_{a, H^\delta} : X \rightarrow \mathbb{R}$ is C^1 for any $a \geq 0$ and $\delta \in [0, 1]$, and for any $z_0 = (u_0, v_0) \in X$, $z = (u, v) \in X$ it results

$$\begin{aligned} \langle I'_{a, H^\delta}(z_0), z \rangle &= \int_{\Omega} (a + |\nabla u_0|^2)^{\frac{p-2}{2}} \nabla u_0 \nabla u \, dx + \int_{\Omega} (a + |\nabla v_0|^2)^{\frac{q-2}{2}} \nabla v_0 \nabla v \, dx \\ &\quad - \int_{\Omega} (H_s^\delta(u_0, v_0)u + H_t^\delta(u_0, v_0)v) dx. \end{aligned}$$

Let $r > 0$ be such that $\mathbf{0}$ is the unique critical point of $I_{a, H^0} = I_{a, H}$ in

$$D_r = \{z = (u, v) \in X : \|\nabla u\|_p + \|\nabla v\|_q \leq r\}.$$

The map $\delta \mapsto I_{a, H^\delta}$ is continuous from $[0, 1]$ to $C_b^1(D_r, \mathbb{R})$. Moreover, reasoning as in the proof of Proposition 3.2.1, we see that I_{a, H^δ} satisfies the Palais-Smale condition in D_r , for any $a \geq 0$ and $\delta \in [0, 1]$.

We claim that there exists $\bar{\delta} \in (0, 1]$ such that $\mathbf{0}$ is the unique critical point of I_{a, H^δ} in D_r , for any $\delta \in [0, \bar{\delta}]$.

Assume, by contradiction, that $\delta_j \rightarrow 0$ and $z_j = (u_j, v_j) \in D_r \setminus \{\mathbf{0}\}$ is a critical point of $I_{a, H^{\delta_j}}$. Taking into account (3.4.26), by Theorem 3.1.1 in Chapter 2 (see also [161, Theorem 1.1]) we infer that every u_j and v_j are in $L^\infty(\Omega)$ and there is $C_1 > 0$, depending on Ω , p , q , p' , q' and r but independent from j , such that

$$\|u_j\|_\infty, \|v_j\|_\infty \leq C_1,$$

thus $|\delta_j u_j(x)| < 1$ and $|\delta_j v_j(x)| < 1$ in Ω , when j is big enough.

Therefore, by (3.4.25), $z_j = (u_j, v_j)$ is a critical point of $I_{a, H}$ and a contradiction follows.

Setting $\overline{H} = H_{\bar{\delta}}$, from [67, Theorem 5.2] we deduce that $C_m(I_{a,H}, \mathbf{0}) = C_m(I_{a,\overline{H}}, \mathbf{0})$ (for related results, see also [46, Theorem I.5.6]) and the assertion follows. $\square$

Assume that $H \in C^2(\mathbb{R}^2, \mathbb{R})$ and there exist $C > 0$, $p' \in (p, p^*)$, $q' \in (q, q^*)$ such that

$$\begin{aligned} |H_{ss}(s, t)| &\leq C \left(|s|^{p'-2} + |t|^{q' \frac{p'-2}{p'}} + 1 \right), \\ |H_{st}(s, t)|, |H_{ts}(s, t)| &\leq C \left(|s|^{p'-1-\frac{p'}{q'}} + |t|^{q'-1-\frac{q'}{p'}} + 1 \right), \\ |H_{tt}(s, t)| &\leq C \left(|s|^{p' \frac{q'-2}{q'}} + |t|^{q'-2} + 1 \right). \end{aligned} \quad (3.4.27)$$

The functional $I_{a,H}$ defined in (3.4.24) is of class C^2 and for any $z_0 = (u_0, v_0)$, $z_1 = (u_1, v_1)$, $z_2 = (u_2, v_2) \in X$, we have

$$\begin{aligned} \langle I''_{a,H}(z_0)z_1, z_2 \rangle &= \int_{\Omega} \left((a + |\nabla u_0|^2)^{\frac{p-2}{2}} (\nabla u_1 | \nabla u_2) + (p-2)(a + |\nabla u_0|^2)^{\frac{p-4}{2}} (\nabla u_0 | \nabla u_1) (\nabla u_0 | \nabla u_2) \right) dx \\ &\quad + \int_{\Omega} \left((a + |\nabla v_0|^2)^{\frac{q-2}{2}} (\nabla v_1 | \nabla v_2) + (q-2)(a + |\nabla v_0|^2)^{\frac{q-4}{2}} (\nabla v_0 | \nabla v_1) (\nabla v_0 | \nabla v_2) \right) dx \\ &\quad - \int_{\Omega} (H_{ss}(u_0, v_0)u_1u_2 + H_{tt}(u_0, v_0)v_1v_2 + H_{st}(u_0, v_0)u_1v_2 + H_{ts}(u_0, v_0)u_2v_1) dx. \end{aligned}$$

We recall the following notion.

Definition 3.4.2. If Y is a Banach space, $I \in C^2(Y, \mathbb{R})$ and z_0 a critical point of I , the Morse index $m(I, z_0)$ of I at z_0 is the supremum of the dimensions of the subspaces of Y where $I''(z_0)$ is negative definite. The large Morse index $m^*(I, z_0)$ is the sum of $m(I, z_0)$ and the dimension of the kernel of $I''(z_0)$.

Arguing as in [43, Theorem 1.4] we can establish the following critical groups estimates for the C^2 -functionals I_a associated to systems of (p, q) -area equations. Precisely, we have

Theorem 3.4.3. Let $a > 0$ and $H \in C^2(\mathbb{R}^2, \mathbb{R})$ satisfying assumptions (3.4.27). If z_0 is a critical point of the functional $I_{a,H}$, then $m(I_{a,H}, z_0)$ and $m^*(I_{a,H}, z_0)$ are finite and

$$C_m(I_{a,H}, z_0) = \{0\}$$

whenever $m < m(I_{a,H}, z_0)$ or $m > m^*(I_{a,H}, z_0)$.

In the following theorem we refer to m_0 and m_0^* which have been defined through (3.1.10).

Theorem 3.4.4. Let $a \geq 0$ and $p, q \in [2, N)$. Let $G \in C^1(\mathbb{R}^2, \mathbb{R})$ a function that satisfies the conditions (a_1) and (a_2) . If $\mathbf{0}$ is an isolated critical point of the functional J_a defined in (3.1.4), then

$$C_m(J_a, \mathbf{0}) = \{0\} \quad \text{whenever } m < m_0 \text{ or } m > m_0^*.$$

Proof. From (3.4.24), $J_a = I_{a,G}$. Due to assumption (a_1) and Theorem 3.4.1, there is $\overline{G} \in C^1(\mathbb{R}^2, \mathbb{R})$ and $\bar{\eta} > 0$ such that

$$|s|, |t| \leq \bar{\eta} \quad \Rightarrow \quad \overline{G}(s, t) = G(s, t); \quad (3.4.28)$$

$$|s|, |t| \geq 2\bar{\eta} \quad \Rightarrow \quad \overline{G}(s, t) = G(0, 0); \quad (3.4.29)$$

$$C_m(J_a, \mathbf{0}) = C_m(I_{a,\overline{G}}, \mathbf{0}), \quad \text{for any } m \in \mathbb{N}. \quad (3.4.30)$$

Taking into account assumption (a_2) , where U is an open set containing $[-2\bar{\eta}, 2\bar{\eta}] \times [-2\bar{\eta}, 2\bar{\eta}]$, Theorem 3.4.1 assures that $\overline{G} \in C^2(\mathbb{R}^2, \mathbb{R})$. Furthermore by (3.4.29) $\overline{G}$ satisfies (3.4.27), so that $I_{a,\overline{G}} \in C^2(X, \mathbb{R})$. Moreover (3.4.28) and (3.1.10) imply $\langle I''_{a,\overline{G}}(\mathbf{0})z, z \rangle = Q(z)$ for any z in X , thus $m(I_{a,\overline{G}}, \mathbf{0}) = m_0$ and $m^*(I_{a,\overline{G}}, \mathbf{0}) = m_0^*$.

If $a > 0$, then Theorem 3.4.3 holds, thus $C_m(I_{a,\overline{G}}, \mathbf{0}) = 0$ whenever $m < m(I_{a,\overline{G}}, \mathbf{0})$ or $m > m^*(I_{a,\overline{G}}, \mathbf{0})$, so by (3.4.30) the assertion follows.

If instead $a = 0$, we have to make a further distinction.

If $H_G(0, 0)$ is null or negative semidefinite we have $m_0 = 0$, $m_0^* = +\infty$ and the assertion is obvious.

If $H_G(0, 0)$ is negative definite, we have $m_0 = m_0^* = 0$ and there is $\mu > 0$ such that

$$H_G(0, 0) [\xi]^2 \leq -\mu |\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^2. \quad (3.4.31)$$

Moreover, due to assumption (a_1) , there are $p' > p$, $q' > q$, $C > 0$ such that $W_0^{1,p}(\Omega) \subset L^{p'}(\Omega)$, $W_0^{1,q}(\Omega) \subset L^{q'}(\Omega)$ and

$$|G(s, t)| \leq C(|s|^{p'} + |t|^{q'} + 1).$$

Combining with (3.4.31) and redefining $C > 0$, we get

$$-G(s, t) \geq -G(0, 0) - C(|s|^{p'} + |t|^{q'}) \quad \text{for any } (s, t) \in \mathbb{R}^2,$$

thus

$$\begin{aligned} J_0(u, v) - J_0(\mathbf{0}) &\geq \frac{1}{p} \|u\|_p^p + \frac{1}{q} \|v\|_q^q - C(|u|_{p'}^{p'} + |v|_{q'}^{q'}) \\ &\geq \|u\|_p^p \left(\frac{1}{p} - C\|u\|_p^{p'-p} \right) + \|v\|_q^q \left(\frac{1}{q} - C\|v\|_q^{q'-q} \right) \geq 0 \end{aligned}$$

if $\|(u, v)\|$ is small enough.

Hence $\mathbf{0}$ is a local minimum for J_0 and, by the excision property, we have

$$C_m(J_0, \mathbf{0}) \approx H^m(\{0\}, \emptyset),$$

so the assertion follows.

If $H_G(0, 0)$ is positive semidefinite or indefinite, we have $m_0 = m_0^* = +\infty$.

As $I_{a,\overline{G}} \in C^2(X, \mathbb{R})$, from [117, Theorem 3.1] we infer that $C_m(I_{a,\overline{G}}, \mathbf{0}) = \{0\}$ for any m , so by (3.4.30) the assertion follows. $\square$

3.5 Geometry of J_a and Proof of the main results

Let us recall that $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ is the C^1 -function defined by

$$F(s, t) = \frac{1}{p}|s|^p + \frac{1}{q}|t|^q + \frac{1}{(\alpha+1)(\beta+1)}|s|^\alpha|t|^\beta st.$$

Assuming that $G(0, 0) = 0$, from assumption (a_1) we see that

$$G(s, t) = \bar{\lambda}F(s, t) + R(s, t),$$

where $R \in C^1(\mathbb{R}^2, \mathbb{R})$, $R(0, 0) = 0$ and

$$\lim_{|(s,t)| \rightarrow \infty} \frac{R_s(s, t)}{|s|^{p-1} + |t|^{q \frac{p-1}{p}}} = 0, \quad \lim_{|(s,t)| \rightarrow \infty} \frac{R_t(s, t)}{|s|^{p \frac{q-1}{q}} + |t|^{q-1}} = 0. \quad (3.5.32)$$

Theorem 3.5.1. *Following the previous notations, we have*

$$\lim_{|(s,t)| \rightarrow \infty} \frac{R(s, t)}{F(s, t)} = 0.$$

Proof. From Young's inequality and (3.1.7), there exists $c > 0$ be such that

$$\frac{1}{p}|s|^p + |s||t|^{q \frac{p-1}{p}} + \frac{1}{q}|t|^q \leq cF(s, t) \quad \forall (s, t) \in \mathbb{R}^2.$$

By (3.5.32), for every $\varepsilon > 0$ there are $\eta_\varepsilon, c_\varepsilon, \delta_\varepsilon > 0$ such that

$$\begin{aligned} |(s, t)| > \eta_\varepsilon &\Rightarrow |R_s(s, t)| \leq \frac{\varepsilon}{2c} \left(|s|^{p-1} + |t|^{q \frac{p-1}{p}} \right), \\ |R_t(s, t)| &\leq \frac{\varepsilon}{2c} \left(|s|^{p \frac{q-1}{q}} + |t|^{q-1} \right); \end{aligned}$$

$$|(s, t)| \leq \eta_\varepsilon \Rightarrow |R_s(s, t)| \leq c_\varepsilon, \quad |R_t(s, t)| \leq c_\varepsilon;$$

$$|(s, t)| > \delta_\varepsilon \Rightarrow F(s, t) > \frac{4c_\varepsilon \eta_\varepsilon}{\varepsilon}.$$

For any $r \in \mathbb{R}$ let us denote by $r^+ = \max\{0, r\}$ and $r^- = \min\{0, r\}$. As

$$R(s, t) = \int_0^s R_s(\sigma, t) d\sigma + \int_0^t R_t(0, \tau) d\tau,$$

we infer that

$$\begin{aligned} |R(s, t)| &\leq \int_{s^-}^{s^+} |R_s(\sigma, t)| d\sigma + \int_{t^-}^{t^+} |R_t(0, \tau)| d\tau \\ &= \int_{\{\sigma \in [s^-, s^+] : |(\sigma, t)| \leq \eta_\varepsilon\}} |R_s(\sigma, t)| d\sigma + \int_{\{\sigma \in [s^-, s^+] : |(\sigma, t)| > \eta_\varepsilon\}} |R_s(\sigma, t)| d\sigma \\ &\quad + \int_{\{\tau \in [t^-, t^+] : |\tau| \leq \eta_\varepsilon\}} |R_t(0, \tau)| d\tau + \int_{\{\tau \in [t^-, t^+] : |\tau| > \eta_\varepsilon\}} |R_t(0, \tau)| d\tau \\ &\leq 2c_\varepsilon \eta_\varepsilon + \frac{\varepsilon}{2} F(s, t) \end{aligned}$$

therefore

$$|(s, t)| > \delta_\varepsilon \Rightarrow \frac{|R(s, t)|}{F(s, t)} < \varepsilon.$$

□

Proposition 3.5.2. *If (a_1) holds and (u, v) is a weak solution to problem (3.1.3), then*

(i) $(u, v) \in \left(C^{1,\eta}(\overline{\Omega})\right)^2$ for some $\eta \in (0, 1)$;

(ii) for every bounded set $A \subset X$, there exists $\eta \in (0, 1)$ and $M > 0$ such that $\|u\|_{C^{1,\eta}(\overline{\Omega})} \leq M$ and $\|v\|_{C^{1,\eta}(\overline{\Omega})} \leq M$ for every $(u, v) \in A$ solving (3.1.3);

(iii) if there is $\varepsilon > 0$ such that

$$\begin{aligned} G_s(0, t) &\neq 0 & \forall t \in (-\varepsilon, \varepsilon) \setminus \{0\} \\ (*) \\ G_t(s, 0) &\neq 0 & \forall s \in (-\varepsilon, \varepsilon) \setminus \{0\}, \end{aligned}$$

then every nontrivial solution (u, v) of (3.1.3) is also not semitrivial, i.e. $u \neq 0$ and $v \neq 0$.

Proof. If (a_1) holds and A is a bounded subset of X , by Theorem 1.1 in [161] all the weak solutions $(u, v) \in A$ are in $(L^\infty(\Omega))^2$ and there is $M_0 > 0$ such that $\|u\|_\infty \leq M_0$ and $\|v\|_\infty \leq M_0$. Hence, due to [122], we infer (i) and (ii).

Let us prove (iii) arguing by contradiction. Suppose that (a_1) and $(*)$ hold and that there is an $(u, 0) \neq (0, 0)$ solving (3.1.3). By (i) we get that $u \in C(\overline{\Omega}) \setminus \{0\}$, $M = \max_{\overline{\Omega}} |u| > 0$ and there is $\bar{x} \in \Omega$ such that $0 < |u(\bar{x})| < \min\{\varepsilon, M\}$. Hence, by $(*)$, $G_t(u(\bar{x}), 0) \neq 0$, in contradiction with (3.1.3). $\square$

Lemma 3.5.3. *If $r \geq 2$, $a \geq 0$ and $\varepsilon > 0$, there is $c(r, a, \varepsilon) > 0$ such that*

$$\frac{1}{r}(a + t^2)^{\frac{r}{2}} \leq \frac{1}{r}(1 + \varepsilon)|t|^r + c(r, a, \varepsilon)$$

for any $t \in \mathbb{R}$.

Proof. The assertion is trivial if $r = 2$, so let us consider the case $r > 2$.

Setting $f_a(t) = \frac{1}{r}(a + t^2)^{\frac{r}{2}} - \frac{1}{r}(1 + \varepsilon)|t|^r$, we see that $\lim_{|t| \rightarrow +\infty} f_a(t) = -\infty$.

Moreover $f'_a(t) = t|t|^{r-2} \left(\left(\frac{a}{t^2} + 1 \right)^{\frac{r-2}{2}} - 1 - \varepsilon \right)$ and, denoting by $t_a = \left(\frac{\frac{a}{2}}{(1+\varepsilon)^{\frac{2}{r-2}} - 1} \right)^{1/2}$, we get

$$\max_{t \in \mathbb{R}} f_a(t) = f_a(\pm t_a) = \frac{a^{r/2}}{r} \frac{1 + \varepsilon}{\left((1 + \varepsilon)^{\frac{2}{r-2}} - 1 \right)^{\frac{r-2}{2}}}.$$

$\square$

Setting

$$L(s, t) = \frac{1}{p}|s|^p + \frac{1}{q}|t|^q \quad L_a(s, t) = \frac{1}{p}(a + s^2)^{\frac{p}{2}} + \frac{1}{q}(a + t^2)^{\frac{q}{2}}$$

and recalling Theorem 3.5.1 we obtain the following result.

Corollary 3.5.4. *For any $\varepsilon > 0$ there is $c_\varepsilon > 0$ such that*

$$L(s, t) \leq L_a(s, t) \leq (1 + \varepsilon)L(s, t) + c_\varepsilon \quad (3.5.33)$$

$$-(\bar{\lambda} + \varepsilon)F(s, t) - c_\varepsilon \leq -G(s, t) \leq -(\bar{\lambda} - \varepsilon)F(s, t) + c_\varepsilon \quad (3.5.34)$$

for any $(s, t) \in \mathbb{R}^2$.

Lemma 3.5.5. *Let $\Gamma : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a C^1 function such that*

$$\lim_{|(s,t)| \rightarrow \infty} \frac{\Gamma(s,t)}{F(s,t)} = 0 \quad (3.5.35)$$

and

$$\lim_{|(s,t)| \rightarrow \infty} \Gamma(s,t) - \frac{1}{p}\Gamma_s(s,t)s - \frac{1}{q}\Gamma_t(s,t)t = -\infty. \quad (3.5.36)$$

Then

$$\lim_{|(s,t)| \rightarrow \infty} \Gamma(s,t) = -\infty. \quad (3.5.37)$$

In the same way, if Γ satisfies condition (3.5.35) and

$$\lim_{|(s,t)| \rightarrow \infty} \Gamma(s,t) - \frac{1}{p}\Gamma_s(s,t)s - \frac{1}{q}\Gamma_t(s,t)t = +\infty, \quad (3.5.38)$$

then

$$\lim_{|(s,t)| \rightarrow \infty} \Gamma(s,t) = +\infty. \quad (3.5.39)$$

Proof. Let observe $F(x^{1/p}s, x^{1/q}t) = xF(s,t)$, for any $x \geq 0$ and $(s,t) \in \mathbb{R}^2$.

Hence if $(s,t) \in \mathbb{R}^2 \setminus \{(0,0)\}$ then, by (3.5.35),

$$\lim_{x \rightarrow +\infty} \frac{\Gamma(x^{1/p}s, x^{1/q}t)}{x} = 0. \quad (3.5.40)$$

Let us fix $M > 0$. By (3.5.36) there exists $\delta > 0$ such that

$$\Gamma(s,t) - \frac{1}{p}\Gamma_s(s,t)s - \frac{1}{q}\Gamma_t(s,t)t < -M \quad \text{when } |(s,t)| > \delta.$$

If $(s',t') \in \mathbb{R}^2$ is such that $|(s',t')| > \delta$, then

$$\begin{aligned} & \frac{d}{dx} \frac{\Gamma(x^{\frac{1}{p}}s', x^{\frac{1}{q}}t') + M}{x} \\ &= \frac{\frac{1}{p}\Gamma_s(x^{\frac{1}{p}}s', x^{\frac{1}{q}}t')x^{\frac{1}{p}}s' + \frac{1}{q}\Gamma_t(x^{\frac{1}{p}}s', x^{\frac{1}{q}}t')x^{\frac{1}{q}}t' - \Gamma(x^{\frac{1}{p}}s', x^{\frac{1}{q}}t') - M}{x^2} > 0 \quad \text{if } x \geq 1 \end{aligned}$$

so that

$$\frac{\Gamma(x^{\frac{1}{p}}s', x^{\frac{1}{q}}t') + M}{x} \leq \frac{\Gamma(\bar{x}^{\frac{1}{p}}s', \bar{x}^{\frac{1}{q}}t') + M}{\bar{x}} \quad \text{whenever } 1 \leq x \leq \bar{x}.$$

By (3.5.40), passing to the limit for $\bar{x} \rightarrow +\infty$, we get

$$\frac{\Gamma(x^{\frac{1}{p}}s', x^{\frac{1}{q}}t') + M}{x} \leq 0 \quad \forall x \geq 1$$

hence $\Gamma(s',t') \leq -M$, which proves (3.5.37).

Finally (3.5.39) can be proved applying (3.5.37) to the function $\bar{\Gamma} = -\Gamma$. □

Now, we can prove Theorem 3.1.1, Theorem 3.1.2 and Theorem 3.1.3.

Proof of Theorem 3.1.1.

As $\bar{\lambda}$ is not an eigenvalue for system (3.1.6), let $m_\infty \in \mathbb{N}$ be such that $\lambda_{m_\infty} < \bar{\lambda} < \lambda_{m_\infty+1}$ and assume $m_\infty \notin [m_0, m_0^*]$.

Setting

$$\begin{aligned} X_- &= \{z \in X \mid \Phi(z) \leq \lambda_{m_\infty} \Psi(z)\} \\ X_+ &= \{z \in X \mid \Phi(z) \geq \lambda_{m_\infty+1} \Psi(z)\} \\ D_r &= \{z \in X_- \mid \Phi(z) \leq r\} \quad S_r = \{z \in X_- \mid \Phi(z) = r\}, \end{aligned}$$

Theorem 3.3.8 assures that (D_r, S_r) links X_+ cohomologically in dimension m_∞ over $\mathbb{Z}_2$, for any $r > 0$.

We take into consideration only the case $m_\infty \geq 1$, as when $m_\infty = 0$ the proof is similiar and simpler.

Let $\alpha', \alpha'', \beta'$ be such that

$$\lambda_{m_\infty} < \alpha' < \alpha'' < \bar{\lambda} < \beta' < \lambda_{m_\infty+1}.$$

By Corollary 3.5.4, there is $C > 0$ such that

$$\begin{aligned} -\beta' F(s, t) - C &\leq -G(s, t) \leq -\alpha'' F(s, t) + C \\ L(s, t) &\leq L_a(s, t) \leq \frac{\alpha'}{\lambda_{m_\infty}} L(s, t) + C. \end{aligned}$$

Therefore, for any $(u, v) \in X_+$,

$$\begin{aligned} J_a(u, v) &= \int_\Omega L_a(|\nabla u(x)|, |\nabla v(x)|) dx - \int_\Omega G(u(x), v(x)) dx \\ &\geq \int_\Omega L(|\nabla u(x)|, |\nabla v(x)|) dx - \int_\Omega (\beta' F(u(x), v(x)) + C) dx \\ &\geq (\lambda_{m_\infty+1} - \beta') \int_\Omega F(u(x), v(x)) dx - C|\Omega| \\ &\geq -C|\Omega| \end{aligned}$$

so that

$$\inf_{z \in X_+} J_a(z) = \bar{a} > -\infty.$$

On the other hand, for any $(u, v) \in X_-$,

$$\begin{aligned} J_a(u, v) &= \int_\Omega L_a(|\nabla u(x)|, |\nabla v(x)|) dx - \int_\Omega G(u(x), v(x)) dx \\ &\leq \frac{\alpha'}{\lambda_{m_\infty}} \int_\Omega L(|\nabla u(x)|, |\nabla v(x)|) dx + C|\Omega| - \alpha'' \int_\Omega F(u(x), v(x)) dx + C|\Omega| \\ &\leq 2C|\Omega| + \left(\frac{\alpha' - \alpha''}{\lambda_{m_\infty}} \right) \int_\Omega L(|\nabla u(x)|, |\nabla v(x)|) dx \end{aligned}$$

so that

$$\lim_{\substack{\|z\| \rightarrow \infty \\ z \in X_-}} J_a(z) = -\infty \quad \Rightarrow \quad \exists r > 0 : \sup_{S_r} J_a < \bar{a} = \inf_{X_+} J_a.$$

Moreover, as D_r is compact, $\bar{b} = \max_{D_r} J_a < +\infty$.

If $J_a^{-1}([\bar{a}, \bar{b}])$ contains an infinite number of critical points, the theorem is obviously proved. Otherwise, applying Theorem 3.3.3 combined with Proposition 3.2.2, there exists a critical point z_0 of J_a with $\bar{a} \leq J_a(z_0) \leq \bar{b}$ and

$$C_{m_\infty}(J_a, z_0) \neq \{0\}.$$

As $m_\infty \notin [m_0, m_0^*]$, Theorem 3.4.4 gives that

$$C_{m_\infty}(J_a, \mathbf{0}) = \{0\},$$

hence $z_0 \neq \mathbf{0}$. □

Proof of Theorem 3.1.2. As

$$\lim_{|(s,t)| \rightarrow \infty} \left[G(s,t) - \frac{1}{p} G_s(s,t)s - \frac{1}{q} G_t(s,t)t \right] = -\infty$$

and $F(s,t) - \frac{1}{p} F_s(s,t)s - \frac{1}{q} F_t(s,t)t = 0$ for any $(s,t) \in \mathbb{R}^2$, by Theorem 3.5.1 and Lemma 3.5.5 we get

$$\lim_{|(s,t)| \rightarrow \infty} G(s,t) - \bar{\lambda} F(s,t) = -\infty,$$

hence $G(s,t) - \bar{\lambda} F(s,t)$ is bounded from above.

Therefore, recalling Corollary 3.5.4 and letting α', α'' be such that

$$\lambda_{m_\infty} < \alpha' < \alpha'' < \bar{\lambda} \leq \lambda_{m_\infty+1},$$

there is $C > 0$ such that

$$\begin{aligned} -\bar{\lambda} F(s,t) - C &\leq -G(s,t) \leq -\alpha'' F(s,t) + C \\ L(s,t) &\leq L_a(s,t) \leq \frac{\alpha'}{\lambda_{m_\infty}} L(s,t) + C. \end{aligned}$$

So we can conclude as in the proof of Theorem 3.1.1, defining X_- and X_+ in the same way. □

Proof of Theorem 3.1.3.

Firstly, reasoning as in the previous proof, by assumption (b_+) we get

$$\lim_{|(s,t)| \rightarrow \infty} G(s,t) - \bar{\lambda} F(s,t) = +\infty. \quad (3.5.41)$$

Let $\beta' \in (\bar{\lambda}, \lambda_{m_\infty+1})$. By Corollary 3.5.4 there is $C > 0$ such that

$$-\beta' F(s,t) - C \leq -G(s,t).$$

We define X_- and X_+ as in the proof of Theorem 3.1.1 and, reasoning in the same way, we see that

$$\inf_{z \in X_+} J_0(z) = \bar{a} > -\infty.$$

So it suffices to prove that, for any $M > 0$, we can choose a suitably big $r > 0$ such that

$$\sup_{z \in \bar{S}_r} J_0(z) < -M.$$

By contradiction, there is $M > 0$ and a sequence $\{z_n\}_n = \{(u_n, v_n)\}_n$ in X_- such that $\Phi(z_n) = n$ and

$$J_0(z_n) \geq -M. \quad (3.5.42)$$

As $z_n \in X_-$ we see that

$$J_0(z_n) = \Phi(z_n) - \int_{\Omega} G(z_n(x)) dx \leq \int_{\Omega} \bar{\lambda} F(z_n(x)) - G(z_n(x)) dx. \quad (3.5.43)$$

Let $\delta_n = 1/\Phi(z_n)$, $\bar{u}_n = \delta_n^{1/p} u_n$, $\bar{v}_n = \delta_n^{1/q} v_n$, $\bar{z}_n = (\bar{u}_n, \bar{v}_n)$.

As $\Phi(\bar{z}_n) = 1$, up to subsequence $\{\bar{z}_n\}_n$ converges to some $\bar{z} = (\bar{u}, \bar{v})$ weakly in X and strongly in $L^p(\Omega) \times L^q(\Omega)$.

It is easy to see that every $\bar{z}_n \in X_-$, as well as $\bar{z} \in X_-$.

Moreover

$$\Psi(\bar{z}) = \lim_{n \rightarrow \infty} \Psi(\bar{z}_n) \geq \lim_{n \rightarrow \infty} \frac{1}{\lambda_{m_\infty}} \Phi(\bar{z}_n) = \frac{1}{\lambda_{m_\infty}} > 0,$$

hence

$$\bar{z} \neq 0. \quad (3.5.44)$$

Since

$$|u_n(x)|^p + |v_n(x)|^q = \Phi(z_n) \left(|\delta_n^{1/p} u_n(x)|^p + |\delta_n^{1/q} v_n(x)|^q \right) \geq n (|\bar{u}_n(x)|^p + |\bar{v}_n(x)|^q)$$

and

$$|\bar{u}_n(x)|^p + |\bar{v}_n(x)|^q \longrightarrow |\bar{u}(x)|^p + |\bar{v}(x)|^q \quad \text{a.e. in } \Omega,$$

taking into account (3.5.41), (3.5.43) and (3.5.44), up to subsequence we get

$$\lim_{n \rightarrow \infty} J_0(z_n) = -\infty,$$

which contradicts (3.5.42).

□

On the de Thélin eigenvalue problem and Landesman-Lazer conditions for quasilinear systems

In this Chapter, we investigate the eigenvalue problem for quasilinear elliptic systems originally introduced by de Thélin in [74]. We prove that the first eigenvalue λ_1 is not only simple (in a suitable sense), but also isolated. Moreover, we provide a variational characterization for a sequence of eigenvalues $\{\lambda_k\}_k$, employing a deformation lemma for C^1 -submanifolds established by Bonnet in [29]. Furthermore, in the spirit of the work of Landesman and Lazer [118], we prove the existence of a weak solution for a quasilinear elliptic system at resonance with λ_1 . We introduce new sufficient Landesman-Lazer type conditions, extending the results obtained for quasilinear elliptic equations by Arcoya and Orsina in [16]. These conditions ensure that the Euler functional satisfies the PS condition and, under specific assumptions, is coercive. In the other cases, since the set of eigenfunctions associated with λ_1 for quasilinear systems is not necessarily a vector subspace of $X := W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$, the standard splitting argument used in [16] is not applicable. To overcome this difficulty, we exploit the isolation of λ_1 and the variational characterization of λ_2 to obtain a solution.

This Chapter is based on the paper [14].

4.1 de Thélin eigenvalue problem

In this Chapter, we study the following eigenvalue problem for quasilinear elliptic systems, originally introduced by de Thélin [74]:

$$\begin{cases} -\Delta_p u = \lambda |u|^{\alpha-1} |v|^{\beta+1} u & \text{in } \Omega, \\ -\Delta_q v = \lambda |u|^{\alpha+1} |v|^{\beta-1} v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.1.1)$$

where $\lambda \in \mathbb{R}$ and p, q, α, β are real numbers satisfying:

$$1 < p < N, \quad 1 < q < N, \quad \alpha > 0, \quad \beta > 0, \quad \text{and} \quad \frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1. \quad (4.1.2)$$

A real number λ is said to be an eigenvalue of (4.1.1) if there exists $(\varphi, \psi) \in X$, $(\varphi, \psi) \neq (0, 0)$ such that for any $(u, v) \in X$ we have

$$\begin{aligned}
& (\alpha + 1) \int_{\Omega} |\nabla \varphi|^{p-2} \nabla \varphi \cdot \nabla u \, dx + (\beta + 1) \int_{\Omega} |\nabla \psi|^{q-2} \nabla \psi \cdot \nabla v \, dx \\
& = (\alpha + 1) \int_{\Omega} |\varphi|^{\alpha-1} |\psi|^{\beta+1} \varphi u \, dx + (\beta + 1) \int_{\Omega} |\varphi|^{\alpha+1} |\psi|^{\beta-1} \psi v \, dx.
\end{aligned} \tag{4.1.3}$$

The nontrivial weak solution (φ, ψ) is then called eigenfunction associated with the eigenvalue λ . Notice that if (φ, ψ) is a nontrivial (i.e. $(\varphi, \psi) \neq (0, 0)$) eigenfunction of (4.1.1) associated with an eigenvalue $\lambda \in \mathbb{R}$, then it cannot be semitrivial (i.e. $\varphi \neq 0 \neq \psi$). In fact, if for example $\psi \equiv 0$ and we test (4.1.3) in $(u, v) = (\varphi, 0)$, we get

$$\int_{\Omega} |\nabla \varphi|^p \, dx = 0,$$

by which $\varphi \equiv 0$, that is a contradiction with the nontriviality of (φ, ψ) .

Let us introduce the functionals $\Phi : X \rightarrow \mathbb{R}$ and $\Psi : X \rightarrow \mathbb{R}$ defined as

$$\Phi(u, v) = \frac{\alpha + 1}{p} \int_{\Omega} |\nabla u|^p \, dx + \frac{\beta + 1}{q} \int_{\Omega} |\nabla v|^q \, dx \tag{4.1.4}$$

and

$$\Psi(u, v) = \int_{\Omega} |u|^{\alpha+1} |v|^{\beta+1} \, dx \tag{4.1.5}$$

for any $(u, v) \in X$, and let us denote with Σ the C^1 manifold

$$\Sigma := \left\{ z = (u, v) \in X : \Psi(z) = 1 \right\}.$$

In [74, Theorem 1] de Th  lin proved that problem (4.1.1) admits a smallest eigenvalue $\lambda_1 > 0$ defined as

$$\lambda_1 := \inf_{z \in \Sigma} \Phi(z). \tag{4.1.6}$$

In [74, Theorem 2] he also proved, by means of Tolksdorf regularity results [157] and Vazquez's maximum principle [162], that there exists a nontrivial eigenfunction (φ_0, ψ_0) associated with λ_1 such that $\varphi_0 \in C^{1,\eta}(\bar{\Omega})$, $\psi_0 \in C^{1,\xi}(\bar{\Omega})$ and $\varphi_0 > 0$, $\psi_0 > 0$ in Ω . In addition, by using results of Diaz and Saa [84] and Anane [9], in [74, Theorem 3] it was proved that if (φ_0, ψ_0) is an eigenfunction associated with λ_1 satisfying $\varphi_0 > 0$, $\psi_0 > 0$ in Ω and $(\varphi_0, \psi_0) \in \Sigma$, then is unique.

In this Chapter we prove that the eigenvalue λ_1 is isolated, and it is simple in the sense stated below.

Firstly, if λ is an eigenvalue with a nontrivial eigenfunction (φ, ψ) , by using the relation $(\alpha + 1)/p + (\beta + 1)/q = 1$ it turns out that

$$E := \left\{ (|\theta|^{\frac{1}{p}} \varphi, |\theta|^{\frac{1}{q}} \psi) \operatorname{sgn}(\theta) : \theta \in \mathbb{R} \right\} \cup \left\{ (-|\theta|^{\frac{1}{p}} \varphi, |\theta|^{\frac{1}{q}} \psi) \operatorname{sgn}(\theta) : \theta \in \mathbb{R} \right\}$$

is a set of eigenfunctions associated with λ . However, if $(\bar{\varphi}, \bar{\psi})$ is another eigenfunction associated with the same eigenvalue λ , it could be the case that $(\bar{\varphi}, \bar{\psi}) \notin E$. [74, Theorem 3] proves that this possibility is excluded for the eigenvalue λ_1 (simplicity of λ_1).

Theorem 4.1.1. *The set of all eigenfunctions associated with λ_1 is given by*

$$E_1 := \left\{ (|\theta|^{\frac{1}{p}} \varphi_0, |\theta|^{\frac{1}{q}} \psi_0) \operatorname{sgn}(\theta) : \theta \in \mathbb{R} \right\} \cup \left\{ (-|\theta|^{\frac{1}{p}} \varphi_0, |\theta|^{\frac{1}{q}} \psi_0) \operatorname{sgn}(\theta) : \theta \in \mathbb{R} \right\}$$

where (φ_0, ψ_0) is the unique nontrivial eigenfunction satisfying $(\varphi_0, \psi_0) \in \Sigma$, $\varphi_0 > 0$ and $\psi_0 > 0$ in Ω .

Previous result and Picone's identity (see [5, Theorem 1] and [15, Lemma 24]), allow us to prove that if (φ, ψ) is an eigenfunction of (4.1.1) associated with the eigenvalue λ with $\lambda > \lambda_1$, then both φ and ψ change sign (see Proposition 4.3.2).

By means of this result, we are able to prove that:

Theorem 4.1.2. *The eigenvalue λ_1 is isolated, that is there exists $a > \lambda_1$ such that λ_1 is the only eigenvalue of (4.1.1) in $[0, a]$.*

Moreover, inspired by [92], we also construct a sequence $\{\lambda_k\}_k$ of eigenvalues of (4.1.1) with a suitable variational characterization involving the unit sphere S^{k-1} of $\mathbb{R}^k$ for any $k \geq 1$. Such a sequence will be constructed on the C^1 manifold

$$\mathcal{M} := \left\{ z = (u, v) \in X : \Phi(z) = 1 \right\}.$$

In particular, setting

$$\mathcal{M}_k := \{ A = \alpha(S^{k-1}) \subset \mathcal{M} \mid \alpha : S^{k-1} \rightarrow \mathcal{M} \text{ is odd and continuous} \},$$

and

$$c_k := \sup_{A \in \mathcal{M}_k} \min_{(u,v) \in A} \Psi(u, v),$$

we prove in Proposition 4.4.3 that the sequence $\{\lambda_k\}_k \subset \mathbb{R}$ where

$$\lambda_k := \frac{1}{c_k}, \tag{4.1.7}$$

is a sequence of eigenvalues of (4.1.1), and λ_1 is exactly the first eigenvalue defined in (4.1.6) (see Remark 4.4.4).

In order to construct a sequence of eigenvalues of (4.1.1), we notice that the manifold $\mathcal{M}$ is not of class $C^{1,1}$ when $1 < p < 2$ or $1 < q < 2$. We will apply a deformation lemma for C^1 submanifolds of a Banach space (see Theorem 4.4.2) based in [29, Theorem 2.5]. This theorem requires a coupled Palais-Smale condition.

It is unknown if the set of the eigenvalues described by (4.1.7) contains all the eigenvalues of problem (4.1.1). Actually, the eigenvalue structure in the context of quasilinear elliptic systems is a challenging open problem.

Really, also for the single scalar eigenvalue problem $-\Delta_p u = \eta |u|^{p-2} u$, the spectral properties of $-\Delta_p$ are not yet well understood. We mention [9] and [11, 92, 139] for results concerning the first eigenvalue and the second one, (see also [55] and [61]). Moreover it is also possible to define, in at least three different ways, a diverging sequence $\{\eta_k\}_k$ of eigenvalues (see [69, 92, 139, 153]), but it is unknown whether these definitions are equivalent for $k \geq 3$ or whether the whole set of eigenvalues is covered.

In the context of quasilinear elliptic systems, we also recall that there are several ways to define an eigenvalue problem (see for instance [150] for an other type of eigenvalue problem different from

(4.1.1), and also [26] for a more general class of eigenvalue problems involving (p, q) -homogeneous functions).

Then we consider the eigenvalue problem (4.1.1), by proving that λ_1 is isolated and simple, and admits the sequence $\{\lambda_k\}_k$ of eigenvalues defined in (4.1.7), with the idea of proving the same results for other eigenvalue problems in a forthcoming paper.

Here we also concern with sufficient Landesman-Lazer type conditions for a quasilinear elliptic systems in resonance around λ_1 . When talking about Landesman-Lazer conditions, one generally refers to the pioneering work [118], where Landesman and Lazer considered the boundary value problem

$$\begin{cases} -\Delta u = \lambda u + f(u) - h(x) & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \quad (4.1.8)$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$, h is a real function in $L^2(\Omega)$, λ is a positive number and f is a real valued function which is bounded and continuous on $\mathbb{R}$. Moreover, there exist the limits

$$f_{-\infty} := \lim_{s \rightarrow -\infty} f(s) \quad \text{and} \quad f^{+\infty} := \lim_{s \rightarrow +\infty} f(s).$$

They studied the existence of solutions of (4.1.8) in the resonance case, i.e. they took into account the linear eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases}$$

and considered the case in which λ is a simple eigenvalue, that is the eigenspace is one-dimensional and spanned by the vector $w \neq 0$.

Denoting with $\Omega^+ := \{x \in \Omega : w(x) > 0\}$, $\Omega^- := \{x \in \Omega : w(x) < 0\}$ and assuming that $f_{-\infty} < f(s) < f^{+\infty}$ for any $s \in \mathbb{R}$, Landesman and Lazer proved that the inequalities

$$f_{-\infty} \int_{\Omega^+} |w| dx - f^{+\infty} \int_{\Omega^-} |w| dx < \int_{\Omega} h w dx < f_{+\infty} \int_{\Omega^+} |w| dx - f_{-\infty} \int_{\Omega^-} |w| dx$$

are sufficient for the existence of a solution of the boundary value problem (4.1.8) (see also [108] for an improvement of the proof in which λ is also isolated, and [8] for the existence of multiple solutions).

An extension of previous result was obtained in [4], where Lazer et al. proved existence of solutions of (4.1.8) without assuming $f_{-\infty} < f(s) < f^{+\infty}$ for any $s \in \mathbb{R}$, and showing that also the reverse inequalities

$$f_{+\infty} \int_{\Omega^+} |w| dx - f_{-\infty} \int_{\Omega^-} |w| dx < \int_{\Omega} h w dx < f_{-\infty} \int_{\Omega^+} |w| dx - f^{+\infty} \int_{\Omega^-} |w| dx$$

are sufficient for the existence of a solution of (4.1.8).

The quasilinear counterpart, that is quasilinear elliptic equations involving p -Laplacian, was studied by several authors in different contexts (see for instance [10, 16, 27]).

Here we also concern with the existence of a weak solution for a quasilinear elliptic systems in resonance around λ_1 , under new sufficient Landesman-Lazer type conditions, extending the results by Arcoya and Orsina [16].

We then consider the following system:

$$\begin{cases} -\Delta_p u = \lambda_1 |u|^{\alpha-1} |v|^{\beta+1} u + \frac{1}{\alpha+1} [F_s(x, u, v) - h_1(x)] & x \in \Omega, \\ -\Delta_q v = \lambda_1 |u|^{\alpha+1} |v|^{\beta-1} v + \frac{1}{\beta+1} [F_t(x, u, v) - h_2(x)] & x \in \Omega, \\ u = v = 0 & x \in \partial\Omega. \end{cases} \quad (4.1.9)$$

Here the nonlinearity $F : \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a C^1 -Carathéodory function (that is, $F(x, s, t)$ is measurable with respect to $x \in \Omega$ for every $(s, t) \in \mathbb{R}^2$, and of class C^1 with respect to $(s, t) \in \mathbb{R}^2$ for a.e. $x \in \Omega$), while $h_1 \in L^{p'}(\Omega)$ and $h_2 \in L^{q'}(\Omega)$, where $p' := p/(p-1)$ and $q' := q/(q-1)$.

We assume that F satisfies also the following conditions:

(F1) $F(x, 0, 0) \in L^1(\Omega)$ and there exists $M > 0$ such that

$$|F_s(x, s, t)| \leq M \quad \text{and} \quad |F_t(x, s, t)| \leq M \quad \text{for all } (x, s, t) \in \Omega \times \mathbb{R}^2.$$

(F2) For almost every $x \in \Omega$, there exist

$$\begin{aligned} F_s^{++}(x) &= \lim_{\substack{s \rightarrow +\infty \\ t \rightarrow +\infty}} F_t(x, s, t), & F_t^{++}(x) &= \lim_{\substack{s \rightarrow +\infty \\ t \rightarrow +\infty}} F_s(x, s, t), \\ F_s^{+-}(x) &= \lim_{\substack{s \rightarrow +\infty \\ t \rightarrow -\infty}} F_t(x, s, t), & F_t^{+-}(x) &= \lim_{\substack{s \rightarrow +\infty \\ t \rightarrow -\infty}} F_s(x, s, t), \\ F_s^{-+}(x) &= \lim_{\substack{s \rightarrow -\infty \\ t \rightarrow +\infty}} F_t(x, s, t), & F_t^{-+}(x) &= \lim_{\substack{s \rightarrow -\infty \\ t \rightarrow +\infty}} F_s(x, s, t), \\ F_s^{--}(x) &= \lim_{\substack{s \rightarrow -\infty \\ t \rightarrow -\infty}} F_t(x, s, t), & F_t^{--}(x) &= \lim_{\substack{s \rightarrow -\infty \\ t \rightarrow -\infty}} F_s(x, s, t). \end{aligned}$$

Moreover, setting

$$(\varphi_1, \psi_1) := \left(\frac{\varphi_0}{\left(\|\varphi_0\|_{1,p}^p + \|\psi_0\|_{1,q}^q \right)^{\frac{1}{p}}}, \frac{\psi_0}{\left(\|\varphi_0\|_{1,p}^p + \|\psi_0\|_{1,q}^q \right)^{\frac{1}{q}}} \right),$$

we infer that (φ_1, ψ_1) is an eigenfunction associated with λ_1 with

$$\|\varphi_1\|_{1,p}^p + \|\psi_1\|_{1,q}^q = 1.$$

In particular, by (p, q) -homogeneity we can write E_1 given by Theorem 4.1.1 also as

$$E_1 = \left\{ (|\theta|^{\frac{1}{p}} \varphi_1, |\theta|^{\frac{1}{q}} \psi_1) \operatorname{sgn}(\theta) : \theta \in \mathbb{R} \right\} \cup \left\{ (-|\theta|^{\frac{1}{p}} \varphi_1, |\theta|^{\frac{1}{q}} \psi_1) \operatorname{sgn}(\theta) : \theta \in \mathbb{R} \right\}. \quad (4.1.10)$$

We then introduce the following sufficient Landesman-Lazer type conditions, involving p, q, F, h_1, h_2 and the eigenfunction (φ_1, ψ_1) (we omit dx for clarity of reading):

if $p < q$, either

$$\left\{ \begin{array}{l} \int_{\Omega} F_s^{++} \varphi_1 < \int_{\Omega} h_1 \varphi_1 < \int_{\Omega} F_s^{--} \varphi_1 \\ \text{and} \\ \int_{\Omega} F_s^{+-} \varphi_1 < \int_{\Omega} h_1 \varphi_1 < \int_{\Omega} F_s^{-+} \varphi_1, \end{array} \right. \quad (4.1.11)$$

or

$$\left\{ \begin{array}{l} \int_{\Omega} F_s^{--} \varphi_1 < \int_{\Omega} h_1 \varphi_1 < \int_{\Omega} F_s^{++} \varphi_1 \\ \text{and} \\ \int_{\Omega} F_s^{-+} \varphi_1 < \int_{\Omega} h_1 \varphi_1 < \int_{\Omega} F_s^{+-} \varphi_1; \end{array} \right. \quad (4.1.12)$$

if $p = q$, either

$$\left\{ \begin{array}{l} \int_{\Omega} F_s^{++} \varphi_1 + F_t^{++} \psi_1 < \int_{\Omega} h_1 \varphi_1 + h_2 \psi_1 < \int_{\Omega} F_s^{--} \varphi_1 + F_t^{--} \psi_1 \\ \text{and} \\ \int_{\Omega} F_s^{+-} \varphi_1 - F_t^{+-} \psi_1 < \int_{\Omega} h_1 \varphi_1 - h_2 \psi_1 < \int_{\Omega} F_s^{-+} \varphi_1 - F_t^{-+} \psi_1, \end{array} \right. \quad (4.1.13)$$

or

$$\left\{ \begin{array}{l} \int_{\Omega} F_s^{--} \varphi_1 + F_t^{--} \psi_1 < \int_{\Omega} h_1 \varphi_1 + h_2 \psi_1 < \int_{\Omega} F_s^{++} \varphi_1 + F_t^{++} \psi_1 \\ \text{and} \\ \int_{\Omega} F_s^{-+} \varphi_1 - F_t^{-+} \psi_1 < \int_{\Omega} h_1 \varphi_1 - h_2 \psi_1 < \int_{\Omega} F_s^{+-} \varphi_1 - F_t^{+-} \psi_1; \end{array} \right. \quad (4.1.14)$$

if $p > q$, either

$$\left\{ \begin{array}{l} \int_{\Omega} F_t^{++} \psi_1 < \int_{\Omega} h_2 \psi_1 < \int_{\Omega} F_t^{--} \psi_1 \\ \text{and} \\ \int_{\Omega} F_t^{-+} \psi_1 < \int_{\Omega} h_2 \psi_1 < \int_{\Omega} F_t^{+-} \psi_1, \end{array} \right. \quad (4.1.15)$$

or

$$\left\{ \begin{array}{l} \int_{\Omega} F_t^{--} \psi_1 < \int_{\Omega} h_2 \psi_1 < \int_{\Omega} F_t^{++} \psi_1 \\ \text{and} \\ \int_{\Omega} F_t^{+-} \psi_1 < \int_{\Omega} h_2 \psi_1 < \int_{\Omega} F_t^{-+} \psi_1. \end{array} \right. \quad (4.1.16)$$

Our main result is the following.

Theorem 4.1.3. *Let $F : \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be a C^1 -Carathéodory function satisfying (F1) and (F2). Consider $h_1 \in L^{p'}(\Omega)$ and $h_2 \in L^{q'}(\Omega)$. If we assume that*

- *in the case $p < q$, either condition (4.1.11) or condition (4.1.12) holds true;*
- *in the case $p = q$, either condition (4.1.13) or condition (4.1.14) holds true;*
- *in the case $p > q$, either condition (4.1.15) or condition (4.1.16) holds true;*

then there exists at least a weak solution $(u, v) \in X$ for problem (4.1.9).

In the proof of Theorem 4.1.3, we show that in any case the Euler functional J associated with (4.1.9) satisfies the compactness condition of Palais-Smale.

In addition, under the assumptions (4.1.11) if $p < q$, (4.1.13) if $p = q$ and (4.1.15) if $p > q$, the functional J is coercive on X , and so (4.1.9) has a solution.

On the other hand, that is under the assumptions (4.1.12) if $p < q$, (4.1.14) if $p = q$ and (4.1.16) if $p > q$, the extension of the results contained in [16] to problem (4.1.9) is not straightforward, since the set E_1 given by (4.1.10) is not a vector subspace of X .

Indeed, in [16] it was possible to apply the saddle point theorem of Rabinowitz (see [145]), exploiting the fact that whenever $W_0^{1,p}(\Omega)$ is splitted as the direct sum of $\langle \varphi \rangle$ and V , where φ is a nontrivial eigenfunction associated with the first eigenvalue μ_1 of $-\Delta_p$ on Ω with respect to Dirichlet boundary condition, then from standard argument it follows that there exists $\bar{\lambda} > \mu_1$ such that $\int_{\Omega} |\nabla u|^p dx \geq \bar{\lambda} \int_{\Omega} |u|^p dx$ for any $u \in V$. In particular, this argument requires just the fact that the eigenspace $\langle \varphi \rangle$ associated with μ_1 is a finite-dimensional vector subspace of $W_0^{1,p}(\Omega)$, without using any other information on the spectrum of the p -Laplacian, even the fact that μ_1 is isolated.

Here we cannot apply the same argument, since the set E_1 given by (4.1.10) is never a vector subspace of X , even in the case $p = q$, in which case we have the union of two vector subspaces. We overcome this difficulty by using the fact that λ_1 is isolated and the variational characterization of λ_2 , therefore we are able to prove that two suitable sets satisfy the geometry of saddle point and are linked in a way that allows the application of standard minimax theorems (see [151, Theorem 3.4]).

This Chapter is organized as follows: in Section 4.2 we prove the simplicity of λ_1 in the sense stated in Theorem 4.1.1; Section 4.3 is devoted to the proof of Theorem 4.1.2, that is the isolation of λ_1 ; in Section 4.4 we show that the sequence $\{\lambda_k\}_k$ defined in (4.1.7) is a sequence of eigenvalues of (4.1.1). Finally, in Section 4.5 we show that the Euler functional J associated with problem (4.1.9) satisfies the compactness condition of Palais-Smale in under any of the previously introduced Landesman-Lazer sufficient conditions, while Section 4.6 is devoted to the geometry of J , so that Theorem 4.1.3 is proved.

4.2 Simplicity of λ_1

The functionals $\Phi : X \rightarrow \mathbb{R}$ and $\Psi : X \rightarrow \mathbb{R}$ defined in (4.1.4) and (4.1.5) are of class C^1 and for any $(u_0, v_0), (u, v) \in X$ we have

$$\langle \Phi'(u_0, v_0), (u, v) \rangle = (\alpha + 1) \int_{\Omega} |\nabla u_0|^{p-2} \nabla u_0 \cdot \nabla u dx + (\beta + 1) \int_{\Omega} |\nabla v_0|^{q-2} \nabla v_0 \cdot \nabla v dx$$

and

$$\langle \Psi'(u_0, v_0), (u, v) \rangle = (\alpha + 1) \int_{\Omega} |u_0|^{\alpha-1} |v_0|^{\beta+1} u_0 u \, dx + (\beta + 1) \int_{\Omega} |u_0|^{\alpha+1} |v_0|^{\beta-1} v_0 v \, dx.$$

By (4.1.2) we get that Φ and Ψ are (p, q) -homogeneous functionals, i.e.

$$\Phi(\theta^{\frac{1}{p}} u, \theta^{\frac{1}{q}} v) = \theta \Phi(u, v), \quad \Psi(\theta^{\frac{1}{p}} u, \theta^{\frac{1}{q}} v) = \theta \Psi(u, v) \quad \forall \theta > 0, (u, v) \in X.$$

Notice that $\Psi(u, v) = 0$ whenever $u(x)v(x) = 0$ a.e. $x \in \Omega$. Hence, denoting with $\tilde{X}$ the set of $(u, v) \in X$ such that the product uv is different from the zero function on Ω , i.e.

$$\tilde{X} := \{(u, v) \in X : uv \neq 0\},$$

we can characterize $\lambda_1 := \inf_{z \in \Sigma} \Phi(z)$ also as

$$\lambda_1 = \inf_{z \in \tilde{X}} \frac{\Phi(z)}{\Psi(z)}.$$

In fact, it suffices to show that $A = B$ where

$$A := \left\{ \frac{\Phi(u, v)}{\Psi(u, v)}; (u, v) \in \tilde{X} \right\} \quad \text{and} \quad B := \left\{ \Phi(u, v); (u, v) \in \Sigma \right\}.$$

We have $B \subseteq A$ (in fact $\Sigma \subset \tilde{X}$ and $\Psi(u, v) = 1$ when $(u, v) \in \Sigma$).

Let now $z \in A$, that is $z = \frac{\Phi(u, v)}{\Psi(u, v)}$ for some $(u, v) \in \tilde{X}$.

Since $\Psi(u, v) \neq 0$, we can define

$$(\bar{u}, \bar{v}) := \left(\frac{u}{\Psi(u, v)^{\frac{1}{p}}}, \frac{v}{\Psi(u, v)^{\frac{1}{q}}} \right).$$

In particular, we have $(\bar{u}, \bar{v}) \in \Sigma$, by which $\Phi(\bar{u}, \bar{v}) \in B$. Moreover we get

$$\Phi(\bar{u}, \bar{v}) = \frac{\Phi(u, v)}{\Psi(u, v)} = z,$$

and then $z \in B$. We have proved $A \subseteq B$ and we conclude that $A = B$.

In [74, Theorem 2] de Thélin showed through Tolksdorf regularity results [157] that every nontrivial eigenfunction (φ, ψ) satisfies $\varphi \in C^{1,\eta}(\bar{\Omega})$, $\psi \in C^{1,\xi}(\bar{\Omega})$. In addition, through Vazquez's maximum principle [162], for every nontrivial eigenfunction (φ_0, ψ_0) associated with λ_1 with $\varphi_0 \geq 0$ and $\psi_0 \geq 0$, we have necessarily $\varphi_0 > 0$, $\psi_0 > 0$ in Ω and $\frac{\partial \varphi_0}{\partial \nu}, \frac{\partial \psi_0}{\partial \nu} < 0$ on $\partial\Omega$, where ν denotes the outward normal to $\partial\Omega$. This result is easily extended to any eigenvalue λ of (4.1.1).

Lemma 4.2.1. *If (φ, ψ) is a nontrivial eigenfunction associated with an eigenvalue $\lambda \geq \lambda_1$ of (4.1.1), then $\varphi \in C^{1,\eta}(\bar{\Omega})$ and $\psi \in C^{1,\xi}(\bar{\Omega})$ for some $\eta, \xi \in (0, 1)$.*

If $\varphi \geq 0$ and $\psi \geq 0$, then $\varphi > 0, \psi > 0$ in Ω and $\frac{\partial \varphi}{\partial \nu}, \frac{\partial \psi}{\partial \nu} < 0$ on $\partial\Omega$.

Proof. Let (φ, ψ) be a nontrivial eigenfunction associated with an eigenvalue $\lambda \geq \lambda_1$. By [32, Theorem 1.3] we know that both $\varphi \in L^\infty(\Omega)$ and $\psi \in L^\infty(\Omega)$.

Since assumptions of [122, Theorem 1] are satisfied, we infer that $\varphi \in C^{1,\eta}(\bar{\Omega})$ and $\psi \in C^{1,\xi}(\bar{\Omega})$ for some $\eta, \xi \in (0, 1)$.

Now, let us assume $\varphi \geq 0$ and $\psi \geq 0$ and let us consider the first equation of (4.1.1):

$$\Delta_p \varphi = -\lambda |\varphi|^{\alpha-1} |\psi|^{\beta+1} \psi.$$

Since assumptions of [162, Theorem 5] are satisfied, we get $\varphi > 0$ in Ω and $\frac{\partial \varphi}{\partial \nu} < 0$ on $\partial\Omega$. Similarly we obtain $\psi > 0$ in Ω and $\frac{\partial \psi}{\partial \nu} < 0$ on $\partial\Omega$. $\square$

Proof of Theorem 4.1.1. It is contained in [74], but for the reader's convenience, we provide a sketch here. First of all, we observe that any $(\varphi, \psi) \in E_1$ is an eigenfunction of (4.1.1) associated with the eigenvalue λ_1 . Conversely, let (φ, ψ) be an eigenfunction associated with λ_1 . Since

$$\frac{\Phi(|\varphi|, |\psi|)}{\Psi(|\varphi|, |\psi|)} = \frac{\Phi(\varphi, \psi)}{\Psi(\varphi, \psi)} = \lambda_1,$$

we deduce that $(|\varphi|, |\psi|)$ is an eigenfunction associated with λ_1 .

We know that $|\varphi| \in C^{1,\eta}(\overline{\Omega})$, $|\psi| \in C^{1,\xi}(\overline{\Omega})$ and $|\varphi| > 0$, $|\psi| > 0$ in Ω . Therefore both φ and ψ cannot change sign. In fact, if by contradiction they change sign, since Ω is a connected set we would have $x_0 \in \Omega$ such that $\varphi(x_0) = 0$, and similarly for ψ , but this is not possible.

Let us assume for example $\varphi > 0$ and $\psi > 0$. Denoting with

$$\theta := \int_{\Omega} |\varphi|^{\alpha+1} |\psi|^{\beta+1} dx,$$

we have that

$$(\bar{\varphi}, \bar{\psi}) := \left(\frac{\varphi}{\theta^{\frac{1}{p}}}, \frac{\psi}{\theta^{\frac{1}{q}}} \right)$$

is an eigenfunction associated with λ_1 such that $\bar{\varphi} > 0$, $\bar{\psi} > 0$ and $(\bar{\varphi}, \bar{\psi}) \in \Sigma$. By [74, Theorem 3] there is a unique eigenfunction (φ_0, ψ_0) associated with λ_1 satisfying $(\varphi_0, \psi_0) \in \Sigma$, $\varphi_0 > 0$ and $\psi_0 > 0$ in Ω and we deduce $(\bar{\varphi}, \bar{\psi}) = (\varphi_0, \psi_0)$, hence $(\varphi, \psi) \in E_1$. We have proved the result assuming $\varphi > 0$ and $\psi > 0$.

The same result holds if we argue as before on the vector $(\varphi, -\psi)$ if $\varphi > 0$ and $\psi < 0$, $(-\varphi, \psi)$ if $\varphi < 0$ and $\psi > 0$, and $(-\varphi, -\psi)$ if $\varphi < 0$ and $\psi < 0$. $\square$

4.3 Isolation of λ_1

To prove that λ_1 is isolated, we follow the ideas of [5, Theorem 1] by using the Picone's identity (see [15, Lemma 24]).

Theorem 4.3.1. *Let $u \geq 0$ and $v > 0$ be differentiable functions, and let $r > 1$. Denote*

$$\begin{aligned} L_r(u, v) &= |\nabla u|^r + (r-1) \left(\frac{u}{v} \right)^r |\nabla v|^r - r \left(\frac{u}{v} \right)^{r-1} |\nabla v|^{r-2} \nabla v \cdot \nabla u, \\ R_r(u, v) &= |\nabla u|^r - |\nabla v|^{r-2} \nabla v \cdot \nabla \left(\frac{u^r}{v^{r-1}} \right). \end{aligned}$$

Then

$$L_r(u, v) = R_r(u, v) \geq 0.$$

Moreover, $L_r(u, v) = 0$ a.e. in Ω if and only if $u = kv$ for some constant k .

Proposition 4.3.2. *Let (φ, ψ) be an eigenfunction of (4.1.1) associated with the eigenvalue λ with $\lambda > \lambda_1$. Then both φ and ψ change sign.*

Proof. Let (φ, ψ) be an eigenfunction of (4.1.1) associated with the eigenvalue λ with $\lambda > \lambda_1$. By contradiction, assume that both φ and ψ have constant sign, for example $\varphi \geq 0$ and $\psi \geq 0$. We know that $\varphi \in C^{1,\eta}(\bar{\Omega})$ and $\psi \in C^{1,\xi}(\bar{\Omega})$ for some $\eta, \xi \in (0, 1)$, and that $\varphi > 0, \psi > 0$ in Ω .

Using that (φ_0, ψ_0) is an eigenfunction associated with λ_1 and (φ, ψ) is an eigenfunction associated with λ , for any $(u, v) \in X$ we have

$$\begin{aligned} & \int_{\Omega} |\nabla \varphi_0|^{p-2} \nabla \varphi_0 \cdot \nabla u \, dx + \int_{\Omega} |\nabla \psi_0|^{q-2} \nabla \psi_0 \cdot \nabla v \, dx \\ &= \lambda_1 \left[\int_{\Omega} |\varphi_0|^{\alpha-1} |\psi_0|^{\beta+1} \varphi_0 u \, dx + \int_{\Omega} |\varphi_0|^{\alpha+1} |\psi_0|^{\beta-1} \psi_0 v \, dx \right], \end{aligned} \quad (4.3.17)$$

and

$$\int_{\Omega} |\nabla \varphi|^{p-2} \nabla \varphi \cdot \nabla u \, dx + \int_{\Omega} |\nabla \psi|^{q-2} \nabla \psi \cdot \nabla v \, dx = \lambda \left[\int_{\Omega} |\varphi|^{\alpha-1} |\psi|^{\beta+1} \varphi u \, dx + \int_{\Omega} |\varphi|^{\alpha+1} |\psi|^{\beta-1} \psi v \, dx \right].$$

By testing (4.3.17) in $(u, v) = \left(\frac{\alpha+1}{p} \varphi_0, \frac{\beta+1}{q} \psi_0 \right)$, we get

$$\frac{\alpha+1}{p} \int_{\Omega} |\nabla \varphi_0|^p \, dx + \frac{\beta+1}{q} \int_{\Omega} |\nabla \psi_0|^q \, dx = \lambda_1 \int_{\Omega} \varphi_0^{\alpha+1} \psi_0^{\beta+1} \, dx.$$

By testing second equation in $(u, v) = \left(\frac{\alpha+1}{p} \frac{\varphi_0^p}{\varphi^{p-1}}, \frac{\beta+1}{q} \frac{\psi_0^q}{\psi^{q-1}} \right)$, we get

$$\begin{aligned} & \frac{\alpha+1}{p} \int_{\Omega} |\nabla \varphi|^{p-2} \nabla \varphi \cdot \nabla \left(\frac{\varphi_0^p}{\varphi^{p-1}} \right) \, dx + \frac{\beta+1}{q} \int_{\Omega} |\nabla \psi|^{q-2} \nabla \psi \cdot \nabla \left(\frac{\psi_0^q}{\psi^{q-1}} \right) \, dx \\ &= \lambda \left[\frac{\alpha+1}{p} \int_{\Omega} \varphi^{\alpha+1-p} \psi^{\beta+1} \varphi_0^p \, dx + \frac{\beta+1}{q} \int_{\Omega} \varphi^{\alpha+1} \psi^{\beta+1-q} \psi_0^q \, dx \right]. \end{aligned}$$

By Theorem 4.3.1, and recalling that $\lambda_1 < \lambda$, we have

$$\begin{aligned} 0 &\leq \frac{\alpha+1}{p} \int_{\Omega} L_p(\varphi_0, \varphi) \, dx + \frac{\beta+1}{q} \int_{\Omega} L_q(\psi_0, \psi) \, dx \\ &= \frac{\alpha+1}{p} \int_{\Omega} R_p(\varphi_0, \varphi) \, dx + \frac{\beta+1}{q} \int_{\Omega} R_q(\psi_0, \psi) \, dx \\ &= \frac{\alpha+1}{p} \int_{\Omega} |\nabla \varphi_0|^p \, dx - \frac{\alpha+1}{p} \int_{\Omega} |\nabla \varphi|^{p-2} \nabla \varphi \cdot \nabla \left(\frac{\varphi_0^p}{\varphi^{p-1}} \right) \, dx \\ &\quad + \frac{\beta+1}{q} \int_{\Omega} |\nabla \psi_0|^q \, dx - \frac{\beta+1}{q} \int_{\Omega} |\nabla \psi|^{q-2} \nabla \psi \cdot \nabla \left(\frac{\psi_0^q}{\psi^{q-1}} \right) \, dx \\ &= \lambda_1 \int_{\Omega} \varphi_0^{\alpha+1} \psi_0^{\beta+1} \, dx \\ &\quad - \lambda \left[\frac{\alpha+1}{p} \int_{\Omega} \varphi^{\alpha+1-p} \psi^{\beta+1} \varphi_0^p \, dx + \frac{\beta+1}{q} \int_{\Omega} \varphi^{\alpha+1} \psi^{\beta+1-q} \psi_0^q \, dx \right] \\ &\leq \lambda \left[\int_{\Omega} (\varphi_0^{\alpha+1} \psi_0^{\beta+1} - \frac{\alpha+1}{p} \varphi^{\alpha+1-p} \psi^{\beta+1} \varphi_0^p - \frac{\beta+1}{q} \varphi^{\alpha+1} \psi^{\beta+1-q} \psi_0^q) \, dx \right]. \end{aligned} \quad (4.3.18)$$

However, by Young inequality with conjugated exponents $p/(\alpha + 1)$ and $q/(\beta + 1)$, we also have

$$\begin{aligned} \varphi_0^{\alpha+1} \psi_0^{\beta+1} &= \left(\varphi_0^{\alpha+1} \frac{\psi^{\frac{(\alpha+1)(\beta+1)}{p}}}{\varphi^{\frac{(\alpha+1)(\beta+1)}{q}}} \right) \left(\psi_0^{\beta+1} \frac{\varphi^{\frac{(\alpha+1)(\beta+1)}{q}}}{\psi^{\frac{(\alpha+1)(\beta+1)}{p}}} \right) \\ &\leq \frac{\alpha+1}{p} \varphi_0^p \psi^{\beta+1} \varphi^{\alpha+1-p} + \frac{\beta+1}{q} \psi_0^q \varphi^{\alpha+1} \psi^{\beta+1-q}, \end{aligned}$$

whence, considering (4.3.18), we get

$$L_p(\varphi_0, \varphi) = 0 \quad \text{and} \quad L_q(\psi_0, \psi) = 0.$$

Applying again Theorem 4.3.1, we deduce $\varphi_0 = c_1 \varphi$ and $\psi_0 = c_2 \psi$ for some positive constant c_1 and c_2 . Observe now that by (4.3.17) we infer

$$\int_{\Omega} |\nabla \varphi_0|^p dx = \lambda_1 \int_{\Omega} \varphi_0^{\alpha+1} \psi_0^{\beta+1} dx = \int_{\Omega} |\nabla \psi_0|^q dx.$$

Substituting $\varphi_0 = c_1 \varphi$ and $\psi_0 = c_2 \psi$, we get $c_1^p = c_2^q$. Denoting by $\theta := c_1^p$, we have shown that $(\varphi_0, \psi_0) = (\theta^{\frac{1}{p}} \varphi, \theta^{\frac{1}{q}} \psi)$ by which (φ_0, ψ_0) is an eigenfunction associated with λ , hence a contradiction.

The proof is similar in the other sign cases: either $\psi \leq 0 \leq \varphi$, or $\psi, \varphi \leq 0$ or $\varphi \leq 0 \leq \psi$. $\square$

Proof of Theorem 4.1.2. By the characterization of λ_1 , for every eigenvalue λ of (4.1.1) we have $\lambda \geq \lambda_1$. We prove the theorem by contradiction, assuming that $\{\mu_n\}_n$ is a sequence of eigenvalues of (4.1.1) such that $\mu_n > \lambda_1$ for any $n \geq 2$ and $\mu_n \rightarrow \lambda_1$ as $n \rightarrow +\infty$. For every $n \geq 2$ consider an eigenfunction $z_n = (\varphi_n, \psi_n) \in X$ associated with μ_n with

$$\|\varphi_n\|_{1,p}^p + \|\psi_n\|_{1,q}^q = 1.$$

Hence the sequence $\{z_n\}_n$ is bounded and there exists a subsequence, still denoted by $\{z_n\}_n$, that converges to some $z = (\varphi, \psi)$ weakly in X and strongly in $L^p(\Omega) \times L^q(\Omega)$. Since φ_n converges to φ strongly in $L^p(\Omega)$ and ψ_n converges to ψ strongly in $L^q(\Omega)$, we have $\varphi_n(x) \rightarrow \varphi(x)$ and $\psi_n(x) \rightarrow \psi(x)$ a.e. in Ω as $n \rightarrow +\infty$. Moreover, passing to a subsequence, we can assume that there exists $\bar{\varphi} \in L^p(\Omega)$ and $\bar{\psi} \in L^q(\Omega)$ such that $|\varphi_n(x)| \leq \bar{\varphi}(x)$ and $|\psi_n(x)| \leq \bar{\psi}(x)$ a.e. in Ω and for all $n \geq 1$.

Since z_n is an eigenfunction associated with μ_n , for any $(u, v) \in X$ it satisfies

$$\begin{aligned} &\int_{\Omega} |\nabla \varphi_n|^{p-2} \nabla \varphi_n \cdot \nabla u dx + \int_{\Omega} |\nabla \psi_n|^{q-2} \nabla \psi_n \cdot \nabla v dx \\ &= \mu_n \left[\int_{\Omega} |\varphi_n|^{\alpha-1} |\psi_n|^{\beta+1} \varphi_n u dx + \int_{\Omega} |\varphi_n|^{\alpha+1} |\psi_n|^{\beta-1} \psi_n v dx \right]. \end{aligned} \quad (4.3.19)$$

By testing previous equation in $(u, v) = (\varphi_n - \varphi, 0)$, we get

$$\int_{\Omega} |\nabla \varphi_n|^{p-2} \nabla \varphi_n \cdot \nabla (\varphi_n - \varphi) dx = \mu_n \int_{\Omega} |\varphi_n|^{\alpha-1} |\psi_n|^{\beta+1} \varphi_n (\varphi_n - \varphi) dx. \quad (4.3.20)$$

By Young inequality with conjugated exponents $p/(\alpha + 1)$ and $q/(\beta + 1)$, we get

$$\left| |\varphi_n(x)|^{\alpha-1} |\psi_n(x)|^{\beta+1} \varphi_n(x) (\varphi_n(x) - \varphi(x)) \right|$$

$$\begin{aligned}
&\leq |\varphi_n(x)|^\alpha |\psi_n(x)|^{\beta+1} (|\varphi_n(x)| + |\varphi(x)|) \\
&\leq 2\bar{\varphi}(x)^{\alpha+1} \bar{\psi}(x)^{\beta+1} \\
&\leq 2 \left(\frac{\alpha+1}{p} \bar{\varphi}(x)^p + \frac{\beta+1}{q} \bar{\psi}(x)^q \right) \in L^1(\Omega).
\end{aligned}$$

Hence, by dominated convergence theorem we have

$$\lim_{n \rightarrow \infty} \int_{\Omega} |\varphi_n|^{\alpha-1} |\psi_n|^{\beta+1} \varphi_n (\varphi_n - \varphi) dx = 0.$$

Considering also that $\mu_n \rightarrow \lambda_1 > 0$ and passing to limit as $n \rightarrow +\infty$ in (4.3.20), we get

$$\lim_{n \rightarrow \infty} \int_{\Omega} |\nabla \varphi_n|^{p-2} \nabla \varphi_n \cdot \nabla (\varphi_n - \varphi) dx = 0.$$

Now, the weak lower semicontinuity and the convexity of $\|\cdot\|_{1,p}^p$ imply

$$\begin{aligned}
\|\varphi\|_{1,p}^p &\leq \liminf_{n \rightarrow \infty} \|\varphi_n\|_{1,p}^p \leq \limsup_{n \rightarrow \infty} \|\varphi_n\|_{1,p}^p = \limsup_{n \rightarrow \infty} \int_{\Omega} |\nabla \varphi_n|^p dx \\
&\leq \limsup_{n \rightarrow \infty} \left[\int_{\Omega} |\nabla \varphi|^p dx + p \int_{\Omega} |\nabla \varphi_n|^{p-2} \nabla \varphi_n \cdot \nabla (\varphi_n - \varphi) dx \right] = \|\varphi\|_{1,p}^p,
\end{aligned}$$

i.e. $\|\varphi_n\|_{1,p} \rightarrow \|\varphi\|_{1,p}$, and by uniform convexity of $W_0^{1,p}(\Omega)$ we deduce $\varphi_n \rightarrow \varphi$ strongly in $W_0^{1,p}(\Omega)$. Analogously, we get that $\psi_n \rightarrow \psi$ strongly in $W_0^{1,q}(\Omega)$.

In particular,

$$\|\varphi\|_{1,p}^p + \|\psi\|_{1,q}^q = \lim_{n \rightarrow \infty} (\|\varphi_n\|_{1,p}^p + \|\psi_n\|_{1,q}^q) = 1$$

and $z = (\varphi, \psi) \neq 0$. By testing (4.3.19) in $(u, v) = \left(\frac{\alpha+1}{p} \varphi_n, \frac{\beta+1}{q} \psi_n\right)$, we get

$$\frac{\alpha+1}{p} \int_{\Omega} |\nabla \varphi_n|^p dx + \frac{\beta+1}{q} \int_{\Omega} |\nabla \psi_n|^q dx = \mu_n \int_{\Omega} |\varphi_n|^{\alpha+1} |\psi_n|^{\beta+1} dx,$$

and passing to limit as $n \rightarrow +\infty$ we deduce

$$\frac{\alpha+1}{p} \int_{\Omega} |\nabla \varphi|^p dx + \frac{\beta+1}{q} \int_{\Omega} |\nabla \psi|^q dx = \lambda_1 \int_{\Omega} |\varphi|^{\alpha+1} |\psi|^{\beta+1} dx.$$

By last identity and $z = (\varphi, \psi) \neq (0, 0)$, we get $z \in \tilde{X}$ and

$$\lambda_1 = \frac{\frac{\alpha+1}{p} \int_{\Omega} |\nabla \varphi|^p dx + \frac{\beta+1}{q} \int_{\Omega} |\nabla \psi|^q dx}{\int_{\Omega} |\varphi|^{\alpha+1} |\psi|^{\beta+1} dx} = \frac{\Phi(z)}{\Psi(z)},$$

i.e. $z = (\varphi, \psi)$ is an eigenfunction associated with λ_1 .

By Theorem 4.1.1 and (4.1.10) we conclude that only one of the following possibilities occurs:

$$(\varphi, \psi) = (\varphi_1, \psi_1), \quad (\varphi, \psi) = (-\varphi_1, -\psi_1), \quad (\varphi, \psi) = (-\varphi_1, \psi_1), \quad (\varphi, \psi) = (\varphi_1, -\psi_1).$$

Let us assume $(\varphi, \psi) = (\varphi_1, \psi_1)$.

By Proposition 4.3.2 we know that both φ_n and ψ_n change sign for any $n \geq 2$, hence both

$\varphi_n^- := -\min\{\varphi_n, 0\}$ and $\psi_n^- := -\min\{\psi_n, 0\}$ are different from zero.

By testing (4.3.19) in $(u, v) = \left(\frac{\alpha+1}{p}\varphi_n^-, \frac{\beta+1}{q}\psi_n^-\right)$, we get

$$\begin{aligned} \frac{\alpha+1}{p} \int_{\Omega} |\nabla \varphi_n^-|^p dx + \frac{\beta+1}{q} \int_{\Omega} |\nabla \psi_n^-|^q dx &= \mu_n \int_{\Omega} |\varphi_n^-|^{\alpha+1} |\psi_n^-|^{\beta+1} dx, \\ &= \mu_n \int_{\Omega_n^-} |\varphi_n^-|^{\alpha+1} |\psi_n^-|^{\beta+1} dx \end{aligned}$$

where $\Omega_n^- := \{x \in \Omega : \varphi_n(x) < 0 \text{ or } \psi_n(x) < 0\}$.

Applying Young and Hölder inequalities we obtain

$$\begin{aligned} \int_{\Omega_n^-} |\varphi_n^-|^{\alpha+1} |\psi_n^-|^{\beta+1} dx &\leq \left[\frac{\alpha+1}{p} \int_{\Omega_n^-} |\varphi_n^-|^p dx + \frac{\beta+1}{q} \int_{\Omega_n^-} |\psi_n^-|^q dx \right] \\ &\leq \left[\frac{\alpha+1}{p} |\Omega_n^-|^{\frac{p}{N}} \|\varphi_n^-\|_{p^*}^p + \frac{\beta+1}{q} |\Omega_n^-|^{\frac{q}{N}} \|\psi_n^-\|_{q^*}^q \right]. \end{aligned}$$

For $p^* = Np/(N-p)$ and $q^* = Nq/(N-q)$ we use now the Sobolev embeddings $\|u\|_{p^*} \leq \mathcal{S}(p, N)\|\nabla u\|_p$ and $\|v\|_{q^*} \leq \mathcal{S}(q, N)\|\nabla v\|_q$ to get from the above identity that

$$\begin{aligned} \frac{\alpha+1}{p} \int_{\Omega} |\nabla \varphi_n^-|^p dx + \frac{\beta+1}{q} \int_{\Omega} |\nabla \psi_n^-|^q dx \\ \leq \mu_n C \max \left\{ |\Omega_n^-|^{\frac{p}{N}}, |\Omega_n^-|^{\frac{q}{N}} \right\} \left[\frac{\alpha+1}{p} \int_{\Omega} |\nabla \varphi_n^-|^p dx + \frac{\beta+1}{q} \int_{\Omega} |\nabla \psi_n^-|^q dx \right], \end{aligned}$$

where $C := \max \{\mathcal{S}(p, N)^p, \mathcal{S}(q, N)^q\}$. This implies that

$$\frac{1}{\mu_n C} \leq \max \left\{ |\Omega_n^-|^{\frac{p}{N}}, |\Omega_n^-|^{\frac{q}{N}} \right\},$$

whence

$$0 < \frac{1}{C\lambda_1} \leq \liminf_{n \rightarrow \infty} \max \left\{ |\Omega_n^-|^{\frac{p}{N}}, |\Omega_n^-|^{\frac{q}{N}} \right\}.$$

Since, up to subsequence, $\varphi_n(x) \rightarrow \varphi_1(x)$, $\nabla \varphi_n(x) \rightarrow \nabla \varphi_1(x)$, $\psi_n(x) \rightarrow \psi_1(x)$, and $\nabla \psi_n(x) \rightarrow \nabla \psi_1(x)$ a.e. in Ω , we can apply Severini-Egorov theorem to say that in the complement of a set of arbitrarily small measure we have φ_n converges uniformly to φ_1 , $\nabla \varphi_n$ converges uniformly to $\nabla \varphi_1$, ψ_n converges uniformly to ψ_1 and $\nabla \psi_n$ converges uniformly to $\nabla \psi_1$. Since $\varphi_1 > 0$, $\psi_1 > 0$ in Ω and $\frac{\partial \varphi_1}{\partial \nu}, \frac{\partial \psi_1}{\partial \nu} < 0$ on $\partial\Omega$, we conclude that there exists a subset of Ω of arbitrarily small measure such that in its complement φ_n and ψ_n are positive for n sufficiently large, that is a contradiction with the last inequality.

Let us recall that in the previous argument we have assumed $(\varphi, \psi) = (\varphi_1, \psi_1)$.

In the rest of cases we can argue similarly. For instance, if $(\varphi, \psi) = (\varphi_1, -\psi_1)$, then we arrive to a contradiction by considering $\varphi_n^- := -\min\{\varphi_n, 0\}$, $\psi_n^+ := \max\{\psi_n, 0\}$, and

$$\Omega_n^{-,+} := \{x \in \Omega : \varphi_n(x) < 0 \text{ or } \psi_n(x) > 0\}.$$

□

4.4 A sequence of eigenvalues

Let us introduce the C^1 functional $Q : X \setminus \{(0, 0)\} \rightarrow \mathbb{R}$ defined by

$$Q(z) := \frac{\Psi(z)}{\Phi(z)}, \quad (4.4.21)$$

where Φ and Ψ are defined in (4.1.4) and (4.1.5) respectively. Observe that

$$Q'(z) = \frac{1}{\Phi(z)} [\Psi'(z) - Q(z) \Phi'(z)] \quad \forall z = (u, v) \in X \setminus \{(0, 0)\}. \quad (4.4.22)$$

We note that a real number $\lambda \neq 0$ is an eigenvalue of (4.1.1) if and only if there exists $\tilde{z} \in \tilde{X}$ such that

$$\Phi'(\tilde{z}) = \lambda \Psi'(\tilde{z}),$$

or equivalently, if $1/\lambda$ is a critical value for Q . Let us denote with $\mathcal{M}$ the set

$$\mathcal{M} := \left\{ z = (u, v) \in X : \Phi(z) = 1 \right\}. \quad (4.4.23)$$

$\mathcal{M}$ is a C^1 manifold of codimension 1 in X (see for instance [82, Example 27.2]) and for any $z \in \mathcal{M}$ the tangent space $T_z \mathcal{M}$ of $\mathcal{M}$ at z can be identified with

$$\text{Ker} (\Phi'(z)) = \{h \in X : \langle \Phi'(z), h \rangle = 0\}.$$

Observe that by definition of Q and (4.4.22) the restriction $Q_{\mathcal{M}} := Q|_{\mathcal{M}}$ of Q to $\mathcal{M}$ satisfies

$$Q_{\mathcal{M}}(z) = \Psi(z) \quad \text{and} \quad Q'(z) = \Psi'(z) - \Psi(z) \Phi'(z) \quad \forall z = (u, v) \in \mathcal{M}.$$

Moreover, (see [82, Example 27.3]), the derivative $Q'_{\mathcal{M}}$ of $Q_{\mathcal{M}}$ is defined on the tangent bundle $T\mathcal{M}$, and in particular for any $z \in \mathcal{M}$ the map $Q'_{\mathcal{M}}(z)$, defined on $T_z \mathcal{M}$, can be identified as

$$Q'_{\mathcal{M}}(z) = [Q'(z) - \langle Q'(z), e(z) \rangle \Phi'(z)]|_{T_z \mathcal{M}}, \quad \forall z = (u, v) \in \mathcal{M}, \quad (4.4.24)$$

where $e : \mathcal{M} \rightarrow X$ is given by

$$e(z) = \left(\frac{u}{p}, \frac{v}{q} \right) \quad \forall z = (u, v) \in \mathcal{M}.$$

Notice that

$$\langle \Psi'(z), e(z) \rangle = \Psi(z) \quad \text{and} \quad \langle \Phi'(z), e(z) \rangle = \Phi(z) = 1 \quad \forall z = (u, v) \in \mathcal{M},$$

hence by (4.4.24) we deduce

$$Q'_{\mathcal{M}}(z) = [\Psi'(z) - \Psi(z) \Phi'(z)]|_{T_z \mathcal{M}} = Q'(z)|_{T_z \mathcal{M}} \quad \forall z = (u, v) \in \mathcal{M}. \quad (4.4.25)$$

We recall that a real number c is a critical value of $Q_{\mathcal{M}}$ if there exists $z \in \mathcal{M}$ such that $Q_{\mathcal{M}}(z) = c$ and $Q'_{\mathcal{M}}(z) \equiv 0$. It follows from standard arguments that critical values of $Q_{\mathcal{M}}$

correspond to critical values of Q . In particular, we deduce that $\lambda \neq 0$ is an eigenvalue of (4.1.1) if and only if $1/\lambda$ is a critical value for $Q_{\mathcal{M}}$.

Given $z \in \mathcal{M}$, we have $Q'_{\mathcal{M}}(z) \in T_z^* \mathcal{M}$, where $T_z^* \mathcal{M}$ denotes the dual space of $T_z \mathcal{M}$ endowed with the norm $\|\cdot\|_z^*$. Denoting with $\|\cdot\|^*$ the norm of X^* , by [140, Proposition 3.54] it follows that

$$\|Q'_{\mathcal{M}}(z)\|_z^* = \min_{\mu \in \mathbb{R}} \|Q'(z) - \mu \Phi'(z)\|^* = \min_{\mu \in \mathbb{R}} \|\Psi'(z) - (\Psi(z) + \mu) \Phi'(z)\|^*. \quad (4.4.26)$$

As it has been mentioned, the construction of a sequence of eigenvalues of (4.1.1) uses a deformation lemma for C^1 submanifolds of a Banach space (see Theorem 4.4.2) based in [29, Theorem 2.5] that requires a coupled Palais-Smale condition.

Firstly we observe that 0 is the only critical value of Φ , therefore for any $\varepsilon > 0$ sufficiently small, the manifold

$$\mathcal{M}_\varepsilon := \left\{ z = (u, v) \in X : \Phi(z) = 1 + \varepsilon \right\}$$

is well defined, hence definitions and results proved for Q on $\mathcal{M}$ still hold on $\mathcal{M}_\varepsilon$.

Lemma 4.4.1. *For every $c > 0$ the functionals Q and Φ satisfy the following coupled Palais-Smale condition at level c on $\mathcal{M}$ (denoted by $(PS)_{\mathcal{M},c}$), that is any sequence $\{z_n\}_n = \{(u_n, v_n)\}_n \subset X$ such that*

- i) $z_n \in \mathcal{M}_{\varepsilon_n}$, where $\varepsilon_n > 0$ and $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$;
- ii) $Q(z_n) \rightarrow c$ as $n \rightarrow \infty$;
- iii) $\|Q'_{\mathcal{M}_{\varepsilon_n}}(z_n)\|_{z_n}^* \rightarrow 0$ or $\|\Phi'(z_n)\| \rightarrow 0$ as $n \rightarrow \infty$;

has a converging subsequence in X .

Proof. Let $\{z_n\}_n = \{(u_n, v_n)\}_n \subset X$ a sequence satisfying i) – iii).

By condition i) we deduce that

$$\|u_n\|_{1,p}^p \leq \frac{p}{\alpha+1} \Phi(z_n) = \frac{p(1+\varepsilon_n)}{\alpha+1} \quad \text{and} \quad \|v_n\|_{1,q}^q \leq \frac{q}{\beta+1} \Phi(z_n) = \frac{q(1+\varepsilon_n)}{\beta+1},$$

that is $\{z_n\}_n$ is bounded in X , and so there exists a subsequence, still denoted by $\{z_n\}_n$, that converges to some $z = (u, v)$ weakly in X and strongly in $L^p(\Omega) \times L^q(\Omega)$. By condition iii) we have two possibilities.

Firstly we assume that $\|\Phi'(z_n)\| \rightarrow 0$ as $n \rightarrow \infty$ holds true. In particular, we have

$$\langle \Phi'(z_n), u_n - u \rangle \rightarrow 0,$$

that is

$$\int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla (u_n - u) dx \rightarrow 0.$$

Now, the weak lower semicontinuity and the convexity of $\|\cdot\|_{1,p}^p$ imply

$$\|u\|_{1,p}^p \leq \liminf_{n \rightarrow \infty} \|u_n\|_{1,p}^p \leq \limsup_{n \rightarrow \infty} \|u_n\|_{1,p}^p = \limsup_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n|^p dx$$

$$\leq \limsup_{n \rightarrow \infty} \left[\int_{\Omega} |\nabla u|^p dx + p \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla (u_n - u) dx \right] = \|u\|_{1,p}^p,$$

i.e. $\|u_n\|_{1,p}^p \rightarrow \|u\|_{1,p}^p$, and by uniform convexity of $W_0^{1,p}(\Omega)$, we deduce $u_n \rightarrow u$ in $W_0^{1,p}(\Omega)$. Analogously, we get that $v_n \rightarrow v$ strongly in $W_0^{1,q}(\Omega)$.

Now, let us assume the other possibility, i.e. $\|Q'_{\mathcal{M}_{\varepsilon_n}}(z_n)\|_{z_n}^* \rightarrow 0$ as $n \rightarrow \infty$. Strong convergences of u_n to u in $L^p(\Omega)$ and of v_n to v in $L^q(\Omega)$ give $\Psi(z_n) \rightarrow \Psi(z)$. Moreover, by *i*) we infer that $\Phi(z_n) \rightarrow 1$. By using also *ii*), we get

$$\Psi(z_n) \rightarrow \Psi(z) = c.$$

By (4.4.26) applied to $z_n \in \mathcal{M}_{\varepsilon_n}$ and the convergence to zero of $\|Q'_{\mathcal{M}_{\varepsilon_n}}(z_n)\|_{z_n}^*$, there exists a sequence $\{\mu_n\}_n \subset \mathbb{R}$ such that

$$\Psi'(z_n) - (\Psi(z_n) + \mu_n)\Phi'(z_n) \rightarrow 0 \quad \text{in } X^* \quad \text{as } n \rightarrow \infty.$$

Testing on the vector $(u_n/p, v_n/q)$, we get

$$\Psi(z_n) - (\Psi(z_n) + \mu_n)\Phi(z_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and using that $\Phi(z_n) \rightarrow 1$ and $\Psi(z_n) \rightarrow c$, we deduce $\mu_n \rightarrow 0$ as $n \rightarrow \infty$. Testing now on the vector $(u_n - u, 0)$, we get

$$\int_{\Omega} |u_n|^{\alpha-1} |v_n|^{\beta+1} u_n (u_n - u) dx - (\Psi(z_n) + \mu_n) \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla (u_n - u) dx \rightarrow 0,$$

and considering the strong convergences of u_n to u in $L^p(\Omega)$ and v_n to v in $L^q(\Omega)$ we deduce

$$(\Psi(z_n) + \mu_n) \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla (u_n - u) dx \rightarrow 0.$$

Since $\Psi(z_n) + \mu_n \rightarrow c$ with $c \neq 0$, we get

$$\int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla (u_n - u) dx \rightarrow 0.$$

Arguing as before we conclude that $u_n \rightarrow u$ strongly in $W_0^{1,p}(\Omega)$. Similarly we deduce that $v_n \rightarrow v$ strongly in $W_0^{1,q}(\Omega)$. □

We will apply the following deformation result for C^1 submanifolds of a Banach space, which is a consequence of [29, Theorem 2.5].

Theorem 4.4.2. *Let X be a Banach space and $\Phi \in C^1(X, \mathbb{R})$. Assume that 1 is a regular value of Φ . If Q is a C^1 functional on a neighborhood of $\mathcal{M} = \{x \in X : \Phi(x) = 1\}$ and Q and Φ satisfy $(PS)_{\mathcal{M},c}$ for a regular value $c \in \mathbb{R}$ of $Q_{\mathcal{M}}$, then there exists $\hat{\varepsilon} > 0$ such that for all $0 < \varepsilon < \hat{\varepsilon}$ there is an homeomorphism η of $\mathcal{M}$ onto $\mathcal{M}$ such that:*

1. $\eta(z) = z$ if $Q(z) \notin [c - \hat{\varepsilon}, c + \hat{\varepsilon}]$;
2. $Q(\eta(z)) \geq Q(z)$ for all $z \in \mathcal{M}$;
3. $Q(\eta(z)) \geq c + \varepsilon$ for all $z \in \mathcal{M}$ such that $Q(z) \geq c - \varepsilon$;

4. If $\mathcal{M}$ is symmetric ($\mathcal{M} = -\mathcal{M}$) and if Q is even, then η is odd.

Now, we will construct a sequence of eigenvalues for (4.1.1).

Proposition 4.4.3. *Let $k \in \mathbb{N}$ and let S^{k-1} be the unit sphere in $\mathbb{R}^k$. Set*

$$\mathcal{M}_k := \{A = \alpha(S^{k-1}) \subset \mathcal{M} \mid \alpha : S^{k-1} \rightarrow \mathcal{M} \text{ is odd and continuous}\}.$$

If

$$c_k := \sup_{A \in \mathcal{M}_k} \min_{(u,v) \in A} Q_{\mathcal{M}}(u, v), \quad (4.4.27)$$

then

$$\lambda_k := \frac{1}{c_k} \quad (4.4.28)$$

is an eigenvalue of problem (4.1.1).

Proof. As it has been mentioned a real number λ is an eigenvalue of (4.1.1) if and only if $1/\lambda$ is a critical value of $Q_{\mathcal{M}}$. By contradiction assume that c_k is a regular value of $Q_{\mathcal{M}}$. By Theorem 4.4.2 there exists $\varepsilon > 0$ an homeomorphism η of $\mathcal{M}$ onto $\mathcal{M}$ satisfying properties (1) – (4). By definition of c_k , we deduce that there exists an odd and continuous map $\alpha : S^{k-1} \rightarrow \mathcal{M}$ such that $\min_{(u,v) \in A} Q_{\mathcal{M}}(u, v) > c_k - \varepsilon$ for $A = \alpha(S^{k-1})$. Hence, taking into account property (4), we deduce that $\eta(A) \in \mathcal{M}_k$. However, by (3) of the cited Theorem we infer that $\min_{(u,v) \in \eta(A)} Q_{\mathcal{M}}(u, v) \geq c_k + \varepsilon$, which contradicts the definition of c_k . Therefore c_k is a critical value of $Q_{\mathcal{M}}$ and λ_k is an eigenvalue of (4.1.1). $\square$

Remark 4.4.4. We point out that the first eigenvalue λ_1 defined in (4.4.28) coincides with the eigenvalue λ_1 defined in (4.1.2). In fact, by definition we know that

$$\mathcal{M}_1 := \{A = \{z, -z\} \mid z \in \mathcal{M}\},$$

and since $\Psi(-z) = \Psi(z)$ we also have

$$c_1 = \sup_{A \in \mathcal{M}_1} \min_{\bar{z} \in A} \Psi(\bar{z}) = \sup_{z \in \mathcal{M}} \Psi(z) = \sup_{z \in X \setminus \{0,0\}} \frac{\Psi(z)}{\Phi(z)} = \frac{1}{\inf_{z \in \tilde{X}} \frac{\Phi(z)}{\Psi(z)}} = \frac{1}{\lambda_1}.$$

4.5 Palais-Smale condition of J

In order to study problem (4.1.9), let us consider the functional $J : X \rightarrow \mathbb{R}$ defined for any $z = (u, v) \in X$ as

$$\begin{aligned} J(z) &:= \frac{\alpha+1}{p} \int_{\Omega} |\nabla u|^p \, dx + \frac{\beta+1}{q} \int_{\Omega} |\nabla v|^q \, dx - \lambda_1 \int_{\Omega} |u|^{\alpha+1} |v|^{\beta+1} \, dx \\ &\quad - \int_{\Omega} F(x, u, v) \, dx + \int_{\Omega} h_1 u \, dx + \int_{\Omega} h_2 v \, dx \\ &= \Phi(u, v) - \lambda_1 \Psi(u, v) - \int_{\Omega} F(x, u, v) \, dx + \int_{\Omega} h_1 u \, dx + \int_{\Omega} h_2 v \, dx. \end{aligned} \quad (4.5.29)$$

It turns out that the functional J is of class C^1 on X with

$$\begin{aligned} \langle J'(z_0), z \rangle = & (\alpha + 1) \int_{\Omega} |\nabla u_0|^{p-2} \nabla u_0 \cdot \nabla u \, dx + (\beta + 1) \int_{\Omega} |\nabla v_0|^{q-2} \nabla v_0 \cdot \nabla v \, dx \\ & - \lambda_1 \left[(\alpha + 1) \int_{\Omega} |u_0|^{\alpha-1} |v_0|^{\beta+1} u_0 u \, dx + (\beta + 1) \int_{\Omega} |u_0|^{\alpha+1} |v_0|^{\beta-1} v_0 v \, dx \right] \\ & - \int_{\Omega} [F_s(x, u_0, v_0) u + F_t(x, u_0, v_0) v] \, dx + \int_{\Omega} h_1 u \, dx + \int_{\Omega} h_2 v \, dx \end{aligned}$$

for any $z_0 = (u_0, v_0), z = (u, v) \in X$.

Lemma 4.5.1. *Let $F : \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be a C^1 -Carathéodory function satisfying **(F1)** and **(F2)**. Consider $h_1 \in L^{p'}(\Omega)$ and $h_2 \in L^{q'}(\Omega)$. If we assume that*

- *in the case $p < q$, either condition (4.1.11) or condition (4.1.12) holds true;*
- *in the case $p = q$, either condition (4.1.13) or condition (4.1.14) holds true;*
- *in the case $p > q$, either condition (4.1.15) or condition (4.1.16) holds true;*

then J satisfies the (PS) condition, namely if $\{z_n\}_n = \{(u_n, v_n)\}_n$ is a sequence in X such that there exists a positive constant c such that

$$|J(z_n)| \leq c \quad \text{for any } n \in \mathbb{N} \quad (4.5.30)$$

and

$$\|J'(z_n)\| \rightarrow 0, \quad (4.5.31)$$

then $\{z_n\}_n$ has a convergent subsequence in X .

Proof. Let $\{z_n\}_n = \{(u_n, v_n)\}_n \subset X$ satisfying (4.5.30) and (4.5.31). Let us begin by proving that the sequence $\{z_n\}_n = \{(u_n, v_n)\}_n$ is bounded in X . By contradiction, there exists a subsequence, still denoted with $\{z_n\}_n$, such that $\|(u_n, v_n)\| \rightarrow \infty$. In particular, as $n \rightarrow +\infty$ we have

$$r_n := \|u_n\|_{1,p}^p + \|v_n\|_{1,q}^q \rightarrow +\infty.$$

Let us set

$$\bar{u}_n = \frac{u_n}{r_n^{\frac{1}{p}}} \quad \text{and} \quad \bar{v}_n = \frac{v_n}{r_n^{\frac{1}{q}}},$$

and observe that $\bar{z}_n := (\bar{u}_n, \bar{v}_n)$ satisfies

$$\|\bar{u}_n\|_{1,p}^p + \|\bar{v}_n\|_{1,q}^q = \frac{\|u_n\|_{1,p}^p + \|v_n\|_{1,q}^q}{r_n} = 1 \quad \text{for any } n \in \mathbb{N}.$$

Hence $\|\bar{z}_n\| \leq 2$ for any $n \in \mathbb{N}$ and there exists a subsequence, still denoted by $\{\bar{z}_n\}_n$, that converges to some $\bar{z} = (\bar{u}, \bar{v})$ weakly in X and strongly in $L^p(\Omega) \times L^q(\Omega)$.

In particular, as $n \rightarrow \infty$ we have

$$\begin{aligned}
\frac{1}{r_n^{\frac{p-1}{p}}} \langle J'(z_n), (\bar{u}_n - \bar{u}, 0) \rangle &= (\alpha + 1) \int_{\Omega} |\nabla \bar{u}_n|^{p-2} \nabla \bar{u}_n \cdot \nabla (\bar{u}_n - \bar{u}) \, dx \\
&\quad - \lambda_1 (\alpha + 1) \int_{\Omega} |\bar{u}_n|^{\alpha-1} |\bar{v}_n|^{\beta+1} \bar{u}_n (\bar{u}_n - \bar{u}) \, dx \\
&\quad - \frac{1}{r_n^{\frac{p-1}{p}}} \int_{\Omega} F_s(x, u_n, v_n) (\bar{u}_n - \bar{u}) \, dx \\
&\quad + \frac{1}{r_n^{\frac{p-1}{p}}} \int_{\Omega} h_1 (\bar{u}_n - \bar{u}) \, dx \rightarrow 0,
\end{aligned} \tag{4.5.32}$$

and

$$\frac{1}{r_n^{\frac{q-1}{q}}} \langle J'(z_n), (0, \bar{v}_n - \bar{v}) \rangle \rightarrow 0. \tag{4.5.33}$$

Since $\bar{u}_n \rightarrow \bar{u}$ in $L^p(\Omega)$, $h \in L^{p'}(\Omega)$ and F_s is bounded we deduce as $n \rightarrow +\infty$ that

$$\frac{1}{r_n^{\frac{p-1}{p}}} \int_{\Omega} h_1 (\bar{u}_n - \bar{u}) \, dx \rightarrow 0 \quad \text{and} \quad \frac{1}{r_n^{\frac{p-1}{p}}} \int_{\Omega} F_s(x, u_n, v_n) (\bar{u}_n - \bar{u}) \, dx \rightarrow 0.$$

Moreover, considering $(\alpha + 1)/p + (\beta + 1)/q = 1$ and the strong convergences $\bar{u}_n \rightarrow \bar{u}$ in $L^p(\Omega)$ and $\bar{v}_n \rightarrow \bar{v}$ in $L^q(\Omega)$, by dominated convergence theorem we get

$$\lim_{n \rightarrow \infty} \int_{\Omega} |\bar{u}_n|^{\alpha-1} |\bar{v}_n|^{\beta+1} \bar{u}_n (\bar{u}_n - \bar{u}) \, dx = 0.$$

Therefore, taking into account these convergences in (4.5.32), we have proved

$$\lim_{n \rightarrow \infty} \int_{\Omega} |\nabla \bar{u}_n|^{p-2} \nabla \bar{u}_n \cdot \nabla (\bar{u}_n - \bar{u}) \, dx = 0.$$

How done in the proof of Theorem 4.1.2, by using also (4.5.33) we finally obtain that $\{\bar{u}_n, \bar{v}_n\}_n$ converges to $\bar{z} = (\bar{u}, \bar{v})$ strongly in X , where $\bar{z} \neq (0, 0)$ satisfies

$$\|\bar{u}\|_{1,p}^p + \|\bar{v}\|_{1,q}^q = 1.$$

Now, to prove that $\bar{z}$ is an eigenfunction associated with the eigenvalue λ_1 , it suffices to show that $\bar{u} \neq 0$, $\bar{v} \neq 0$ and

$$\frac{\Phi(\bar{z})}{\Psi(\bar{z})} = \lambda_1.$$

In the following, we assume both $\|u_n\|_{1,p} \rightarrow \infty$ and $\|v_n\|_{1,q} \rightarrow \infty$. In the case that one of both sequences is bounded, computations are simpler and left to reader. Considering just the boundedness form above coming from (4.5.30), we have

$$\Phi(u_n, v_n) - \lambda_1 \Psi(u_n, v_n) - \int_{\Omega} F(x, u_n, v_n) \, dx + \int_{\Omega} h_1 u_n \, dx + \int_{\Omega} h_2 v_n \, dx \leq c.$$

Dividing by r_n and taking into account the definition of $\bar{u}_n$ and $\bar{v}_n$, we have

$$\Phi(\bar{u}_n, \bar{v}_n) - \lambda_1 \Psi(\bar{u}_n, \bar{v}_n) - \int_{\Omega} \frac{F(x, u_n, v_n)}{r_n} \, dx + \int_{\Omega} \frac{h_1 u_n}{r_n} \, dx + \int_{\Omega} \frac{h_2 v_n}{r_n} \, dx \leq \frac{c}{r_n},$$

whence

$$\limsup_{n \rightarrow \infty} \left[\Phi(\bar{u}_n, \bar{v}_n) - \lambda_1 \Psi(\bar{u}_n, \bar{v}_n) - \int_{\Omega} \frac{F(x, u_n, v_n)}{r_n} dx + \int_{\Omega} \frac{h_1 u_n}{r_n} dx + \int_{\Omega} \frac{h_2 v_n}{r_n} dx \right] \leq 0. \quad (4.5.34)$$

Since $h_1 \in L^{p'}(\Omega)$ and $h_2 \in L^{q'}(\Omega)$, by applying Hölder and Poincaré inequality and the convergence to infinity of the sequence $\{r_n\}_n$, we have

$$\lim_{n \rightarrow \infty} \int_{\Omega} \frac{h_1 u_n}{r_n} dx = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_{\Omega} \frac{h_2 v_n}{r_n} dx = 0.$$

For almost every $(x, s, t) \in \Omega \times \mathbb{R}^2$ mean value theorem gives

$$F(x, s, t) = F(x, 0, 0) + \int_0^1 F_s(x, \tau s, \tau t) s d\tau + \int_0^1 F_t(x, \tau s, \tau t) t d\tau,$$

hence by assumption **(F1)** we deduce

$$|F(x, s, t)| \leq |F(x, 0, 0)| + M|s| + M|t|.$$

Therefore,

$$\left| \int_{\Omega} \frac{F(x, u_n, v_n)}{r_n} dx \right| \leq \frac{\|F(x, 0, 0)\|_1}{r_n} + K_1 \|u_n\|_{1,p}^{1-p} + K_2 \|v_n\|_{1,q}^{1-q},$$

whence

$$\lim_{n \rightarrow \infty} \int_{\Omega} \frac{F(x, u_n, v_n)}{r_n} dx = 0.$$

Considering these convergences to zero into (4.5.34), we obtain

$$\limsup_{n \rightarrow \infty} \left[\Phi(\bar{u}_n, \bar{v}_n) - \lambda_1 \Psi(\bar{u}_n, \bar{v}_n) \right] \leq 0.$$

Now, since $\bar{u}_n \rightarrow \bar{u}$ in $L^p(\Omega)$ and $\bar{v}_n \rightarrow \bar{v}$ in $L^q(\Omega)$, by Lebesgue theorem we deduce

$$\Psi(\bar{u}_n, \bar{v}_n) \rightarrow \Psi(\bar{u}, \bar{v}) \quad \text{as } n \rightarrow \infty$$

and by the strong convergences $\bar{u}_n \rightarrow \bar{u}$ in $W_0^{1,p}(\Omega)$ and $\bar{v}_n \rightarrow \bar{v}$ in $W_0^{1,q}(\Omega)$ we also get

$$\Phi(\bar{u}_n, \bar{v}_n) \rightarrow \Phi(\bar{u}, \bar{v}) \quad \text{as } n \rightarrow \infty.$$

Therefore, we have proved $\Phi(\bar{u}, \bar{v}) \leq \lambda_1 \Psi(\bar{u}, \bar{v})$, i.e.

$$\frac{\alpha+1}{p} \int_{\Omega} |\nabla \bar{u}|^p dx + \frac{\beta+1}{q} \int_{\Omega} |\nabla \bar{v}|^q dx \leq \lambda_1 \int_{\Omega} |\bar{u}|^{\alpha+1} |\bar{v}|^{\beta+1} dx.$$

Since $\bar{z} \neq (0, 0)$, the left side of previous inequality is strictly greater than zero, by which $\bar{u} \neq 0$, $\bar{v} \neq 0$ and

$$\frac{\Phi(\bar{z})}{\Psi(\bar{z})} = \lambda_1,$$

that is $\bar{z} = (\bar{u}, \bar{v})$ is an eigenfunction associated with the eigenvalue λ_1 .

Since also $\bar{z}$ satisfies $\|\bar{u}\|_{1,p}^p + \|\bar{v}\|_{1,q}^q = 1$, we conclude by Theorem 4.1.1 and (4.1.10) that only one of the following possibilities occurs:

$$(\bar{u}, \bar{v}) = (\varphi_1, \psi_1), \quad (\bar{u}, \bar{v}) = (-\varphi_1, -\psi_1), \quad (\bar{u}, \bar{v}) = (-\varphi_1, \psi_1), \quad (\bar{u}, \bar{v}) = (\varphi_1, -\psi_1).$$

By (4.5.31), we deduce that, up to subsequence, there exists a decreasing sequence $\{\varepsilon_n\}_n$, converging to zero, such that

$$\left| \left\langle J'(z_n), \left(\frac{u_n}{p}, \frac{v_n}{q} \right) \right\rangle \right| \leq \varepsilon_n \|z_n\| \quad \forall n \in \mathbb{N}, \quad \forall z = (u, v) \in X.$$

Hence, taking into account also (4.5.30), we have

$$(1 + \varepsilon_n \|z_n\|) C \geq \left| J(z_n) - \left\langle J'(z_n), \left(\frac{u_n}{p}, \frac{v_n}{q} \right) \right\rangle + \int_{\Omega} F(x, 0, 0) dx \right|,$$

which by the mean value theorem and the definition of $\bar{u}_n$ and $\bar{v}_n$ implies

$$\begin{aligned} (1 + \varepsilon_n \|z_n\|) C \geq & \left| r_n^{\frac{1}{p}} \left\{ \left(1 - \frac{1}{p} \right) \int_{\Omega} h_1 \bar{u}_n dx - \int_{\Omega} \int_0^1 F_s(x, \tau u_n, \tau v_n) \bar{u}_n d\tau dx \right. \right. \\ & \left. \left. + \frac{1}{p} \int_{\Omega} F_s(x, u_n, v_n) \bar{u}_n dx \right\} \right. \\ & \left. + r_n^{\frac{1}{q}} \left\{ \left(1 - \frac{1}{q} \right) \int_{\Omega} h_2 \bar{v}_n dx - \int_{\Omega} \int_0^1 F_t(x, \tau u_n, \tau v_n) \bar{v}_n d\tau dx \right. \right. \\ & \left. \left. + \frac{1}{q} \int_{\Omega} F_t(x, u_n, v_n) \bar{v}_n dx \right\} \right|. \end{aligned} \quad (4.5.35)$$

Let us assume $(\bar{u}, \bar{v}) = (\varphi_1, \psi_1)$.

Since $\bar{u}_n \rightarrow \varphi_1$ in $L^p(\Omega)$ and $\bar{v}_n \rightarrow \psi_1$ in $L^q(\Omega)$, we have

$$\lim_{n \rightarrow \infty} \int_{\Omega} h_1 \bar{u}_n dx = \int_{\Omega} h_1 \varphi_1 dx \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_{\Omega} h_2 \bar{v}_n dx = \int_{\Omega} h_2 \psi_1 dx.$$

Now, let us observe that $u_n(x) = \bar{u}_n(x) r_n^{\frac{1}{p}}$, where $r_n^{\frac{1}{p}} \rightarrow +\infty$ and $\bar{u}_n(x) \rightarrow \varphi_1(x)$ a.e. $x \in \Omega$. Since $\varphi_1 > 0$ in Ω , we deduce $u_n(x) \rightarrow +\infty$ a.e. $x \in \Omega$, and similarly we have $v_n(x) \rightarrow +\infty$ a.e. $x \in \Omega$. Hence, by assumption **(F2)** we get

$$F_s(x, u_n(x), v_n(x)) \rightarrow F_s^{++}(x) \quad \text{a.e. } x \in \Omega,$$

and

$$F_t(x, u_n(x), v_n(x)) \rightarrow F_t^{++}(x) \quad \text{a.e. } x \in \Omega,$$

so by Lebesgue theorem we deduce

$$\lim_{n \rightarrow \infty} \int_{\Omega} F_s(x, u_n, v_n) \bar{u}_n dx = \int_{\Omega} F_s^{++} \varphi_1 dx$$

and

$$\lim_{n \rightarrow \infty} \int_{\Omega} F_t(x, u_n, v_n) \bar{v}_n dx = \int_{\Omega} F_t^{++} \psi_1 dx.$$

In a similar way, by using also Fubini-Tonelli's Theorem we derive

$$\lim_{n \rightarrow \infty} \int_{\Omega} \int_0^1 F_s(x, \tau u_n, \tau v_n) \bar{u}_n d\tau dx = \int_{\Omega} F_s^{++} \varphi_1 dx$$

and

$$\lim_{n \rightarrow \infty} \int_{\Omega} \int_0^1 F_t(x, \tau u_n, \tau v_n) \bar{v}_n d\tau dx = \int_{\Omega} F_t^{++} \psi_1 dx.$$

Considering these convergences in (4.5.35), and denoting with $o(1)$ any quantity that goes to zero as n goes to infinity, we have proved

$$(1 + \varepsilon_n \|z_n\|) C \geq \left| r_n^{\frac{1}{p}} \left(1 - \frac{1}{p}\right) \left[\int_{\Omega} h_1 \varphi_1 dx - \int_{\Omega} F_s^{++} \varphi_1 dx + o(1) \right] \right. \\ \left. + r_n^{\frac{1}{q}} \left(1 - \frac{1}{q}\right) \left[\int_{\Omega} h_2 \psi_1 dx - \int_{\Omega} F_t^{++} \psi_1 dx + o(1) \right] \right|. \quad (4.5.36)$$

If $p < q$, dividing by $r_n^{\frac{1}{p}} \left(1 - \frac{1}{p}\right)$ and since $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$, passing to limit we get

$$0 \geq \left| \int_{\Omega} h_1 \varphi_1 dx - \int_{\Omega} F_s^{++} \varphi_1 dx \right|,$$

that is a contradiction with both (4.1.11) and (4.1.12).

If $p = q$, in a similar way we get

$$0 \geq \left| \int_{\Omega} h_1 \varphi_1 + h_2 \psi_1 dx - \int_{\Omega} F_s^{++} \varphi_1 + F_t^{++} \psi_1 dx \right|,$$

that is a contradiction with both (4.1.13) and (4.1.14).

If $p > q$, dividing by $r_n^{\frac{1}{q}} \left(1 - \frac{1}{q}\right)$ and passing to limit we obtain

$$0 \geq \left| \int_{\Omega} h_2 \psi_1 dx - \int_{\Omega} F_t^{++} \psi_1 dx \right|,$$

that is a contradiction with both (4.1.15) and (4.1.16).

Let us recall that we assumed $(\bar{u}, \bar{v}) = (\varphi_1, \psi_1)$.

If it would be $(\bar{u}, \bar{v}) = (-\varphi_1, -\psi_1)$, we deduce inequality (4.5.36) in which F_s^{++} and F_t^{++} are replaced by F_s^{--} and F_t^{--} respectively, and reasoning as before we get a contradiction under the same assumptions.

If now we consider $(\bar{u}, \bar{v}) = (\varphi_1, -\psi_1)$, convergences in (4.5.35) give

$$(1 + \varepsilon_n \|z_n\|) C \geq \left| r_n^{\frac{1}{p}} \left(1 - \frac{1}{p}\right) \left[\int_{\Omega} h_1 \varphi_1 dx - \int_{\Omega} F_s^{+-} \varphi_1 dx + o(1) \right] \right. \\ \left. + r_n^{\frac{1}{q}} \left(1 - \frac{1}{q}\right) \left[\int_{\Omega} F_t^{+-} \psi_1 dx - \int_{\Omega} h_2 \psi_1 dx + o(1) \right] \right|,$$

by which we infer a contradiction with (4.1.11) and (4.1.12) if $p < q$, with (4.1.13) and (4.1.14) if $p = q$, and with (4.1.15) and (4.1.16) if $p > q$.

If it would be $(\bar{u}, \bar{v}) = (-\varphi_1, \psi_1)$, we deduce previous inequality in which F_s^{+-} and F_t^{+-} are replaced by F_s^{-+} and F_t^{-+} respectively, and we reach a contradiction under the same hypothesis.

We have so proved that the sequence $\{z_n\}_n = \{(u_n, v_n)\}_n$ satisfying (4.5.30) and (4.5.31) is bounded in X . This implies that there exists $z = (u, v) \in X$ such that, up to subsequence, $\{z_n\}_n$ converges to z weakly in X and strongly in $L^p(\Omega) \times L^q(\Omega)$.

In particular, as $n \rightarrow \infty$ we have

$$\langle J'(z_n), (u_n - u, 0) \rangle \rightarrow 0$$

and

$$\langle J'(z_n), (0, v_n - v) \rangle \rightarrow 0.$$

Arguing as previously done to show the strong convergence of $\bar{u}_n$ to $\bar{u}$ in $W_0^{1,p}(\Omega)$ and the strong convergence of $\bar{v}_n$ to $\bar{v}$ in $W_0^{1,q}(\Omega)$, we conclude that up to subsequence $z_n \rightarrow z$ strongly in X . $\square$

4.6 Geometry of J

Proposition 4.6.1. *Let $F : \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be a C^1 -Carathéodory function satisfying (F1) and (F2). Consider $h_1 \in L^{p'}(\Omega)$ and $h_2 \in L^{q'}(\Omega)$. If we assume that*

- *in the case $p < q$, condition (4.1.11) holds true;*
- *in the case $p = q$, condition (4.1.13) holds true;*
- *in the case $p > q$, condition (4.1.15) holds true;*

then J is coercive, i.e.,

$$\lim_{\|(u,v)\| \rightarrow +\infty} J(u, v) = +\infty.$$

Proof. Arguing by contradiction, suppose that there exists a constant c and a sequence $\{(u_n, v_n)\}_n$ in X such that

$$J(u_n, v_n) \leq c \quad \text{for any } n \in \mathbb{N}, \tag{4.6.37}$$

and

$$\|(u_n, v_n)\| \rightarrow +\infty \quad \text{as } n \rightarrow +\infty.$$

Let us set

$$r_n := \|u_n\|_{1,p}^p + \|v_n\|_{1,q}^q, \quad \bar{u}_n := \frac{u_n}{r_n^{\frac{1}{p}}} \quad \text{and} \quad \bar{v}_n := \frac{v_n}{r_n^{\frac{1}{q}}},$$

and observe that $\bar{z}_n := (\bar{u}_n, \bar{v}_n)$ satisfies

$$\|\bar{u}_n\|_{1,p}^p + \|\bar{v}_n\|_{1,q}^q = 1 \quad \text{for any } n \in \mathbb{N}.$$

Hence $\|\bar{z}_n\| \leq 2$ for any $n \in \mathbb{N}$ and there exists a subsequence, still denoted by $\{\bar{z}_n\}_n$, that converges to some $\bar{z} = (\bar{u}, \bar{v})$ weakly in X and strongly in $L^p(\Omega) \times L^q(\Omega)$.

Dividing (4.6.37) by r_n , arguing as in the proof of (PS) condition and considering also the characterization of λ_1 and the weak lower semicontinuity of both $\|\cdot\|_{1,p}^p$ and $\|\cdot\|_{1,q}^q$, we get

$$\Phi(\bar{u}, \bar{v}) \leq \liminf_{n \rightarrow \infty} \Phi(\bar{u}_n, \bar{v}_n) \leq \limsup_{n \rightarrow \infty} \Phi(\bar{u}_n, \bar{v}_n) \leq \lambda_1 \Psi(\bar{u}, \bar{v}) \leq \Phi(\bar{u}, \bar{v}).$$

In particular, since $\|\bar{u}_n\|_{1,p}^p + \|\bar{v}_n\|_{1,q}^q = 1$ for any $n \in \mathbb{N}$, we deduce that $(\bar{u}, \bar{v})$ is a nontrivial eigenfunction associated with λ_1 , hence there exists $\theta > 0$ such that only one of the following possibilities occurs:

$$(\bar{u}, \bar{v}) = (\theta^{\frac{1}{p}} \varphi_1, \theta^{\frac{1}{q}} \psi_1), \quad (\bar{u}, \bar{v}) = (-\theta^{\frac{1}{p}} \varphi_1, -\theta^{\frac{1}{q}} \psi_1),$$

$$(\bar{u}, \bar{v}) = (\theta^{\frac{1}{p}} \varphi_1, -\theta^{\frac{1}{q}} \psi_1), \quad (\bar{u}, \bar{v}) = (-\theta^{\frac{1}{p}} \varphi_1, \theta^{\frac{1}{q}} \psi_1).$$

Let us assume $(\bar{u}, \bar{v}) = (\theta^{\frac{1}{p}} \varphi_1, \theta^{\frac{1}{q}} \psi_1)$.

By the mean value theorem, the characterization of λ_1 and the definition of the normalized sequence $\bar{z}_n = (\bar{u}_n, \bar{v}_n)$, we infer that

$$\begin{aligned} c &\geq J(u_n, v_n) \\ &\geq - \int_{\Omega} F(x, u_n, v_n) dx + \int_{\Omega} h_1 u_n dx + \int_{\Omega} h_2 v_n dx \\ &= r_n^{\frac{1}{p}} \left[\int_{\Omega} h_1 \bar{u}_n dx - \int_{\Omega} \int_0^1 F_s(x, \tau u_n, \tau v_n) \bar{u}_n d\tau dx \right] \\ &\quad + r_n^{\frac{1}{q}} \left[\int_{\Omega} h_2 \bar{v}_n dx - \int_{\Omega} \int_0^1 F_t(x, \tau u_n, \tau v_n) \bar{v}_n d\tau dx \right] - \int_{\Omega} F(x, 0, 0) dx. \end{aligned}$$

Since $(\bar{u}_n, \bar{v}_n) \rightarrow (\theta^{\frac{1}{p}} \varphi_1, \theta^{\frac{1}{q}} \psi_1)$ strongly in $L^p(\Omega) \times L^q(\Omega)$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\Omega} h_1 \bar{u}_n dx &= \theta^{\frac{1}{p}} \int_{\Omega} h_1 \varphi_1 dx, \\ \lim_{n \rightarrow \infty} \int_0^1 \int_{\Omega} F_s(x, \tau u_n, \tau v_n) \bar{u}_n d\tau dx &= \theta^{\frac{1}{p}} \int_{\Omega} F_s^{++} \varphi_1 dx, \end{aligned}$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\Omega} h_2 \bar{v}_n dx &= \theta^{\frac{1}{q}} \int_{\Omega} h_2 \psi_1 dx, \\ \lim_{n \rightarrow \infty} \int_0^1 \int_{\Omega} F_t(x, \tau u_n, \tau v_n) \bar{v}_n d\tau dx &= \theta^{\frac{1}{q}} \int_{\Omega} F_t^{++} \psi_1 dx. \end{aligned}$$

Therefore, considering these limits in the previous inequality we deduce that

$$\begin{aligned} c &\geq \theta^{\frac{1}{p}} r_n^{\frac{1}{p}} \left[\int_{\Omega} h_1 \varphi_1 dx - \int_{\Omega} F_s^{++} \varphi_1 dx + o(1) \right] \\ &\quad + \theta^{\frac{1}{q}} r_n^{\frac{1}{q}} \left[\int_{\Omega} h_2 \psi_1 dx - \int_{\Omega} F_t^{++} \psi_1 dx + o(1) \right] - \int_{\Omega} F(x, 0, 0) dx. \end{aligned}$$

In the case $p < q$, dividing by $\theta^{\frac{1}{p}} r_n^{\frac{1}{p}}$ and passing to limit as $n \rightarrow \infty$, we have

$$0 \geq \int_{\Omega} h_1 \varphi_1 dx - \int_{\Omega} F_s^{++} \varphi_1 dx,$$

which contradicts (4.1.11).

When $p = q$, similarly we obtain

$$0 \geq \int_{\Omega} h_1 \varphi_1 dx + h_2 \psi_1 dx - \int_{\Omega} F_s^{++} \varphi_1 + F_t^{++} \psi_1 dx,$$

contradicting (4.1.13).

If $p > q$, dividing by $\theta^{\frac{1}{q}} r_n^{\frac{1}{q}}$ and passing to limit as $n \rightarrow \infty$ we get

$$0 \geq \int_{\Omega} h_2 \psi_1 dx - \int_{\Omega} F_t^{++} \psi_1 dx,$$

contradicting (4.1.15).

These contradiction show that $(\bar{u}, \bar{v})$ cannot be equal to $(\theta^{\frac{1}{p}} \varphi_1, \theta^{\frac{1}{q}} \psi_1)$.

In a similar way if it would be $(\bar{u}, \bar{v}) = (-\theta^{\frac{1}{p}} \varphi_1, -\theta^{\frac{1}{q}} \psi_1)$, one obtains a contradiction with the hypotheses (4.1.11), (4.1.13) and (4.1.15), involving F_s^{--} and F_t^{--} .

If now we assume $(\bar{u}, \bar{v}) = (\theta^{\frac{1}{p}} \varphi_1, -\theta^{\frac{1}{q}} \psi_1)$, we obtain

$$\begin{aligned} c \geq & \theta^{\frac{1}{p}} r_n^{\frac{1}{p}} \left[\int_{\Omega} h_1 \varphi_1 dx - \int_{\Omega} F_s^{+-} \varphi_1 dx + o(1) \right] \\ & + \theta^{\frac{1}{q}} r_n^{\frac{1}{q}} \left[\int_{\Omega} F_t^{+-} \psi_1 dx - \int_{\Omega} h_2 \psi_1 dx + o(1) \right] - \int_{\Omega} F(x, 0, 0) dx, \end{aligned}$$

and arguing as before we get a contradiction with (4.1.11), (4.1.13) and (4.1.15).

Similarly, if it would be $(\bar{u}, \bar{v}) = (-\theta^{\frac{1}{p}} \varphi_1, \theta^{\frac{1}{q}} \psi_1)$, one obtains a contradiction with the hypotheses (4.1.11), (4.1.13) and (4.1.15), involving F_s^{-+} and F_t^{-+} .

□

Under the assumption (4.1.12) if $p < q$, (4.1.14) if $p = q$ and (4.1.16) if $p > q$, we will apply the following deformation result in order to prove that the functional J has critical points of saddle point type (see [151, Theorem 3.4]).

Lemma 4.6.2. *Let f be a functional of class C^1 on a Banach space V that satisfies (PS) condition. Let $c \in \mathbb{R}$ be a regular value of f and $\bar{\varepsilon} > 0$. Then there is $\varepsilon \in (0, \bar{\varepsilon})$ and a continuous one parameter family of homeomorphisms $\phi : V \times [0, 1] \rightarrow V$ with the following properties:*

- (i) $\phi(z, \tau) = z$ if $\tau = 0$ or if $|f(z) - c| \geq \bar{\varepsilon}$;
- (ii) $f(\phi(z, \tau))$ is non-increasing in τ for any $z \in V$;
- (iii) If $f(z) \leq c + \varepsilon$, then $f(\phi(z, 1)) \leq c - \varepsilon$.

Proof of Theorem 4.1.3. Let us consider assumption (4.1.11) if $p < q$, (4.1.13) if $p = q$ and (4.1.15) if $p > q$. In these cases we have proved that the functional J is coercive.

Now, by weak lower semicontinuity of both $\|\cdot\|_{1,p}$ and $\|\cdot\|_{1,q}$, and by Lebesgue theorem, we deduce that J is sequentially weakly lower semicontinuous, in fact for any sequence $\{z_n\}_n = \{(u_n, v_n)\}_n$ in X weakly convergent to some $z = (u, v) \in X$, we have

$$\begin{aligned}
 J(z) &= \Phi(u, v) - \lambda_1 \Psi(u, v) - \int_{\Omega} F(x, u, v) dx + \int_{\Omega} h_1 u dx + \int_{\Omega} h_2 v dx \\
 &\leq \liminf_{n \rightarrow \infty} \frac{\alpha + 1}{p} \int_{\Omega} |\nabla u_n|^p dx + \liminf_{n \rightarrow \infty} \frac{\beta + 1}{q} \int_{\Omega} |\nabla v_n|^q dx \\
 &\quad - \lambda_1 \Psi(u, v) - \int_{\Omega} F(x, u, v) dx + \int_{\Omega} h_1 u dx + \int_{\Omega} h_2 v dx \\
 &\leq \liminf_{n \rightarrow \infty} \Phi(u_n, v_n) + \lim_{n \rightarrow \infty} \left[-\lambda_1 \Psi(u_n, v_n) - \int_{\Omega} F(x, u_n, v_n) dx + \int_{\Omega} h_1 u_n dx + \int_{\Omega} h_2 v_n dx \right] \\
 &= \liminf_{n \rightarrow \infty} J(z_n).
 \end{aligned}$$

This and the coercivity of J imply by Weierstrass Theorem that J has a global minimum point, i.e. there exists a weak solution for problem (4.1.9).

Let us now consider assumption (4.1.12) if $p < q$, (4.1.14) if $p = q$ and (4.1.16) if $p > q$.

We will show that

$$\begin{aligned}
 \lim_{\theta \rightarrow +\infty} J(\theta^{\frac{1}{p}} \varphi_1, \theta^{\frac{1}{q}} \psi_1) &= -\infty, & \lim_{\theta \rightarrow +\infty} J(-\theta^{\frac{1}{p}} \varphi_1, -\theta^{\frac{1}{q}} \psi_1) &= -\infty, \\
 \lim_{\theta \rightarrow +\infty} J(\theta^{\frac{1}{p}} \varphi_1, -\theta^{\frac{1}{q}} \psi_1) &= -\infty, & \lim_{\theta \rightarrow +\infty} J(-\theta^{\frac{1}{p}} \varphi_1, \theta^{\frac{1}{q}} \psi_1) &= -\infty.
 \end{aligned} \tag{4.6.38}$$

By characterization of λ_1 and the mean value formula, for any $\theta > 0$ we infer that

$$\begin{aligned}
 J(\theta^{\frac{1}{p}} \varphi_1, \theta^{\frac{1}{q}} \psi_1) &= \theta^{\frac{1}{p}} \int_{\Omega} h_1 \varphi_1 dx + \theta^{\frac{1}{q}} \int_{\Omega} h_2 \psi_1 dx - \int_{\Omega} F(x, \theta^{\frac{1}{p}} \varphi_1, \theta^{\frac{1}{q}} \psi_1) dx \\
 &= \theta^{\frac{1}{p}} \left[\int_{\Omega} h_1 \varphi_1 dx - \int_{\Omega} \int_0^1 F_s(x, \tau \theta^{\frac{1}{p}} \varphi_1, \tau \theta^{\frac{1}{q}} \psi_1) \varphi_1 d\tau dx \right] \\
 &\quad + \theta^{\frac{1}{q}} \left[\int_{\Omega} h_2 \psi_1 dx - \int_{\Omega} \int_0^1 F_t(x, \tau \theta^{\frac{1}{p}} \varphi_1, \tau \theta^{\frac{1}{q}} \psi_1) \psi_1 d\tau dx \right] \\
 &\quad - \int_{\Omega} F(x, 0, 0) dx.
 \end{aligned}$$

By **(F2)**, Fubini-Tonelli and Lebesgue theorems, it follows

$$\begin{aligned}
 \lim_{\theta \rightarrow +\infty} \int_{\Omega} \int_0^1 F_s(x, \tau \theta^{\frac{1}{p}} \varphi_1, \tau \theta^{\frac{1}{q}} \psi_1) \varphi_1 d\tau dx &= \int_{\Omega} F_s^{++} \varphi_1 dx; \\
 \lim_{\theta \rightarrow +\infty} \int_{\Omega} \int_0^1 F_t(x, \tau \theta^{\frac{1}{p}} \varphi_1, \tau \theta^{\frac{1}{q}} \psi_1) \psi_1 d\tau dx &= \int_{\Omega} F_t^{++} \psi_1 dx.
 \end{aligned}$$

Therefore,

$$J(\theta^{\frac{1}{p}} \varphi_1, \theta^{\frac{1}{q}} \psi_1) = \theta^{\frac{1}{p}} \left[\int_{\Omega} h_1 \varphi_1 dx - \int_{\Omega} F_s^{++} \varphi_1 dx + o(1) \right]$$

$$\begin{aligned}
& + \theta^{\frac{1}{q}} \left[\int_{\Omega} h_2 \psi_1 dx - \int_{\Omega} F_t^{++} \psi_1 dx + o(1) \right] \\
& - \int_{\Omega} F(x, 0, 0) dx,
\end{aligned}$$

by which

$$\lim_{\theta \rightarrow +\infty} J(\theta^{\frac{1}{p}} \varphi_1, \theta^{\frac{1}{q}} \psi_1) = -\infty.$$

Arguing in a similar way, considering the other inequalities in assumptions (4.1.12), (4.1.14) and (4.1.16) involving F_s^{--} and F_t^{--} , F_s^{+-} and F_t^{+-} , and F_s^{-+} and F_t^{-+} respectively, we get the other limits in (4.6.38).

Let λ_2 be the eigenvalue of (4.1.1) defined in (4.4.28) and let us set

$$\Lambda_2 := \{(u, v) \in X : \Phi(u, v) \geq \lambda_2 \Psi(u, v)\}.$$

By **(F1)** and the mean value formula, we have

$$|F(x, s, t)| \leq M|s| + M|t| + |F(x, 0, 0)|.$$

Denoting by $\lambda_{1,p}$ and $\lambda_{1,q}$ the first eigenvalue on Ω under homogeneous Dirichlet boundary condition of Δ_p and Δ_q respectively, by Hölder and Poincaré inequality and for any $(u, v) \in X$ we have

$$\begin{aligned}
& \left| - \int_{\Omega} F(x, u, v) dx + \int_{\Omega} h_1 u dx + \int_{\Omega} h_2 v dx \right| \\
& \leq \int_{\Omega} |F(x, 0, 0)| dx + M \int_{\Omega} |u| dx + M \int_{\Omega} |v| dx + \int_{\Omega} |h_1| |u| dx + \int_{\Omega} |h_2| |v| dx \\
& \leq \int_{\Omega} |F(x, 0, 0)| dx + \left(M |\Omega|^{\frac{1}{p'}} + \|h_1\|_{p'} \right) \|u\|_p + \left(M |\Omega|^{\frac{1}{q'}} + \|h_2\|_{q'} \right) \|v\|_q \\
& \leq \int_{\Omega} |F(x, 0, 0)| dx + \frac{\left(M |\Omega|^{\frac{1}{p'}} + \|h_1\|_{p'} \right)}{\lambda_{1,p}^{\frac{1}{p}}} \|u\|_{1,p} + \frac{\left(M |\Omega|^{\frac{1}{q'}} + \|h_2\|_{q'} \right)}{\lambda_{1,q}^{\frac{1}{q}}} \|v\|_{1,q},
\end{aligned}$$

On the other hand, for any $(u, v) \in \Lambda_2$ we get

$$\begin{aligned}
\Phi(u, v) - \lambda_1 \Psi(u, v) & \geq \left(1 - \frac{\lambda_1}{\lambda_2} \right) \Phi(u, v) \\
& = \frac{\alpha + 1}{p} \left(1 - \frac{\lambda_1}{\lambda_2} \right) \|u\|_{1,p}^p + \frac{\beta + 1}{q} \left(1 - \frac{\lambda_1}{\lambda_2} \right) \|v\|_{1,q}^q.
\end{aligned}$$

Altogether, for any $(u, v) \in \Lambda_2$ we have

$$\begin{aligned}
J(u, v) & \geq \frac{\alpha + 1}{p} \left(1 - \frac{\lambda_1}{\lambda_2} \right) \|u\|_{1,p}^p - \frac{\left(M |\Omega|^{\frac{1}{p'}} + \|h_1\|_{p'} \right)}{\lambda_{1,p}^{\frac{1}{p}}} \|u\|_{1,p} \\
& + \frac{\beta + 1}{q} \left(1 - \frac{\lambda_1}{\lambda_2} \right) \|v\|_{1,q}^q - \frac{\left(M |\Omega|^{\frac{1}{q'}} + \|h_2\|_{q'} \right)}{\lambda_{1,q}^{\frac{1}{q}}} \|v\|_{1,q} \\
& - \int_{\Omega} |F(x, 0, 0)| dx.
\end{aligned}$$

Since λ_1 is isolated, we have $\lambda_2 > \lambda_1$ and so we deduce that J is coercive on Λ_2 , and in particular it is bounded from below. Considering also (4.6.38), we have shown that there exists $\Theta > 0$ such that, setting

$$E_1^\Theta = \left\{ (|\theta|^{\frac{1}{p}} \varphi_1, |\theta|^{\frac{1}{q}} \psi_1) \operatorname{sgn}(\theta) : |\theta| \geq \Theta \right\} \cup \left\{ (-|\theta|^{\frac{1}{p}} \varphi_1, |\theta|^{\frac{1}{q}} \psi_1) \operatorname{sgn}(\theta) : |\theta| \geq \Theta \right\},$$

we have

$$\gamma := \sup_{(u,v) \in E_1^\Theta} J(u,v) < \inf_{(u,v) \in \Lambda_2} J(u,v) := \delta.$$

Consider the family of mappings

$$\Gamma := \{h \in C([-1, 1], X) : h(1) = -h(-1) \in E_1^\Theta\}.$$

Notice that $\tilde{h}(\tau) := (\tau \Theta^{\frac{1}{p}} \varphi_1, \tau \Theta^{\frac{1}{q}} \psi_1)$, $\tau \in [-1, 1]$, belongs to Γ and thus $\Gamma \neq \emptyset$.

Let us show that $h(B_1) \cap \Lambda_2 \neq \emptyset$ for every $h \in \Gamma$, where $B_1 := [-1, 1]$.

Indeed, this is immediate if $(0, 0) \in h(B_1)$. When $(0, 0) \notin h(B_1)$, we consider the odd map $\pi : X \setminus (0, 0) \rightarrow \mathcal{M}$ given by

$$\pi(u, v) := \left(\frac{u}{\Phi(u, v)^{\frac{1}{p}}}, \frac{v}{\Phi(u, v)^{\frac{1}{q}}} \right), \quad (u, v) \in X \setminus (0, 0),$$

and the map $\alpha_0 : S^1 \rightarrow \mathcal{M}$ defined for every $(\xi_1, \xi_2) \in S^1$ by

$$\alpha_0(\xi_1, \xi_2) := \begin{cases} \pi(h(\xi_1)) & \text{if } \xi_2 \geq 0, \\ -\pi(h(-\xi_1)) & \text{if } \xi_2 < 0. \end{cases}$$

Observe that by the definition of α_0 , we have $\alpha_0(-\xi_1, -\xi_2) = -\alpha_0(\xi_1, \xi_2)$ for every $(\xi_1, \xi_2) \in S^1$ with $\xi_2 \neq 0$. On the other hand, since $h(1) = -h(-1)$, the oddness of π implies that $\alpha_0(-1, 0) = -\alpha_0(1, 0)$ and, consequently, α_0 is odd in all S^1 . In addition, using again that $h(1) = -h(-1)$, the map α_0 is also continuous. In particular, $\alpha_0(S^1) \in \mathcal{M}_2$, where $\mathcal{M}_2$ is defined in Proposition 4.4.3.

Recalling that

$$\lambda_2 := \frac{1}{c_2} \quad \text{where} \quad c_2 := \sup_{A \in \mathcal{M}_2} \min_{(u,v) \in A} Q_{\mathcal{M}}(u, v) = \sup_{A \in \mathcal{M}_2} \min_{(u,v) \in A} \frac{\Psi(u, v)}{\Phi(u, v)},$$

we have

$$\min_{(u,v) \in \alpha_0(S^1)} \frac{\Psi(u, v)}{\Phi(u, v)} \leq c_2.$$

Let $(u_0, v_0) \in \alpha_0(S^1)$ be such that $\frac{\Psi(u_0, v_0)}{\Phi(u_0, v_0)} \leq c_2$, i.e., $\Phi(u_0, v_0) \geq \lambda_2 \Psi(u_0, v_0)$ and so $(u_0, v_0) \in \alpha_0(S^1) \cap \Lambda_2$. By definition of α_0 , this implies that there exists $\tau \in [-1, 1]$ such that $\pi(h(\tau)) \in \Lambda_2$. Now we observe that the (p, q) -homogeneity of the functionals Φ and Ψ implies that $h(\tau) \in \Lambda_2$ and thus $h(B_1) \cap \Lambda_2 \neq \emptyset$.

In particular,

$$c := \inf_{h \in \Gamma} \max_{\tau \in B_1} J(h(\tau)) \geq \delta > \gamma.$$

Now, we show that c is a critical value of J .

Indeed, let us assume by contradiction that c is a regular value of J . Fixing $0 < \bar{\varepsilon} < c - \gamma$, by Lemma 4.6.2 there exists $\varepsilon \in (0, \bar{\varepsilon})$ and a family of homeomorphisms $\phi : X \times [0, 1] \rightarrow X$ satisfying the following properties:

- (i) $\phi(z, \tau) = z$ if $\tau = 0$ or if $|J(z) - c| \geq \bar{\varepsilon}$;
- (ii) $J(\phi(z, \tau))$ is non-increasing in τ for any $z \in X$;
- (iii) If $J(z) \leq c + \varepsilon$, then $J(\phi(z, 1)) \leq c - \varepsilon$.

For any $z \in E_1^\Theta$, we have $J(z) \leq \gamma < c - \bar{\varepsilon}$ and so by property (i) we infer that ϕ leaves the set E_1^Θ fixed. By definition of c there exists $h \in \Gamma$ such that $\max_{\tau \in [-1, 1]} J(h(\tau)) < c + \varepsilon$. Let us define $\hat{h}(\cdot) := \phi(h(\cdot), 1)$. Since $h(\pm 1) \in E_1^\Theta$, we get $\hat{h}(\pm 1) = \phi(h(\pm 1), 1) = h(\pm 1)$, from which we deduce that $\hat{h}(-1) = -\hat{h}(1)$ and hence $\hat{h} \in \Gamma$. Now, property (iii) gives $\max_{\tau \in [-1, 1]} J(\hat{h}(\tau)) \leq c - \varepsilon$, which contradicts the definition of c . □

Appendix C

In this Section we analyze the definitions contained in the paper [29] of Bonnet. Let E be a Banach space and let us denote with S the C^1 manifold defined by

$$S := \{x \in E : g(x) = 1\},$$

where $g \in C^1(E, \mathbb{R})$ and 1 is a regular value for g . For any $x \in S$, we denote with $T_x S$ the tangent space of S at x . Let f be a C^1 functional defined on a neighborhood of S . The restriction $f_S := f|_S$ of f to S is a C^1 functional on S and the derivative f'_S of f_S is a map defined on the tangent bundle TS and is such that

$$f'_S(x) = f'(x)|_{T_x S} \quad \forall x \in S.$$

In particular, for any $x \in S$ we have $f'_S(x) \in T_x^* S$, where $T_x^* S$ denotes the dual space of $T_x S$, endowed with the norm $\|\cdot\|_x^*$. Denoting with $\|\cdot\|^*$ the norm of E^* , by [140, Proposition 3.54] it follows that

$$\|f'_S(x)\|_x^* = \min_{\mu \in \mathbb{R}} \|f'(x) - \mu g'(x)\|^*. \quad (4.6.39)$$

Definition 4.6.3. We say that f_S satisfies the Palais-Smale condition at the level c (henceforth denoted by $(PS)_c$) if any sequence $\{x_n\}_n \subset S$ such that

$$f_S(x_n) \rightarrow c \quad \text{as } n \rightarrow +\infty,$$

$$f'_S(x_n) \rightarrow 0 \quad \text{as } n \rightarrow +\infty,$$

possesses a convergent subsequence.

Lemma 4.6.4. *Suppose that c is a regular value of f_S and that f_S satisfies $(PS)_c$. Then there exist $\delta, \varepsilon > 0$ such that*

$$\forall x \in S, \quad |f(x) - c| \leq \varepsilon \implies \|f'_S(x)\|_x^* > \delta. \quad (4.6.40)$$

Proof. By contradiction, $\forall \delta > 0$ and $\forall \varepsilon > 0$ there exists $x := x(\delta, \varepsilon) \in S$ such that

$$|f(x) - c| \leq \varepsilon \quad \text{and} \quad \|f'_S(x)\|_x^* \leq \delta.$$

In particular, for any $n \in \mathbb{N}$ there exists $x_n \in S$ such that

$$|f(x_n) - c| \leq \frac{1}{n} \quad \text{and} \quad \|f'_S(x_n)\|_{x_n}^* \leq \frac{1}{n}.$$

Therefore, $\{x_n\}_n$ is a Palais-Smale sequence at level c . Since f_S satisfies $(PS)_c$, there exists $\bar{x} \in S$ such that $f_S(\bar{x}) = c$ and $f'_S(\bar{x}) = 0$, that is a contradiction with the fact that c is a regular value. $\square$

Now, let $a > 0$ and let us consider

$$F_a := \{S_b : b \in [0, a]\}$$

and

$$F_a^* := \{S_b : b \in [-a, 0]\},$$

where

$$S_b = \{x \in E : g(x) = 1 + b\}.$$

Definition 4.6.5. Suppose that $c \in \mathbb{R}$ is a regular value of f_S , and let $a > 0$. A family F_a is said to be admissible for f at c if f is defined in S_b for any $b \in [0, a]$ and if there exist $\mu, \delta, \varepsilon > 0$ such that

$$\forall b \in [0, a], \forall x \in S_b, \quad |f(x) - c| \leq \varepsilon \implies (\|f'_{S_b}(x)\|_x^* > \delta \text{ and } \|g'(x)\| > \mu). \quad (4.6.41)$$

Similar definitions hold if we consider F_a^* .

Definition 4.6.6. Let $c \in \mathbb{R}$. We say that f and g satisfy a coupled Palais-Smale condition at level c on S (henceforth denoted by $\widehat{(PS)}_{S,c}$) if any sequence $\{x_n\}_n$ such that

- $x_n \in S_{\varepsilon_n}$ where $\varepsilon_n > 0$ and $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$;
- $f(x_n) \rightarrow c$ as $n \rightarrow \infty$;
- $\|f'_{S_{\varepsilon_n}}(x_n)\|_{x_n}^* \rightarrow 0$ or $\|g'(x_n)\| \rightarrow 0$ as $n \rightarrow \infty$;

possesses a convergent subsequence.

Lemma 4.6.7. *Suppose that c is a regular value of f_S and that f and g satisfy $\widehat{(PS)}_{S,c}$. Then there exists $a > 0$ such that F_a is admissible for f at c .*

Proof. By contradiction $\forall a > 0$ and $\forall \mu, \delta, \varepsilon > 0$ there exist $b \in [0, a]$ and $\bar{x} \in S_b$ such that

$$|f(\bar{x}) - c| \leq \varepsilon \quad \text{and} \quad (\|f'_{S_b}(\bar{x})\|_{\bar{x}}^* \leq \varepsilon \quad \text{or} \quad \|g'(\bar{x})\| \leq \mu).$$

In particular, for any $n \in \mathbb{N}$ there exist $b_n \in [0, \frac{1}{n}]$ and $x_n \in S_{b_n}$ such that

$$|f(x_n) - c| \leq \frac{1}{n} \quad \text{and} \quad \left(\|f'_{S_{b_n}}(x_n)\|_{x_n}^* \leq \frac{1}{n} \quad \text{or} \quad \|g'(x_n)\| \leq \frac{1}{n} \right).$$

Notice that $\beta_n \rightarrow 0$, $f(x_n) \rightarrow a$ and that

$$\|f'_{S_{b_n}}(x_n)\|_{x_n}^* \rightarrow 0 \quad \text{or} \quad \|g'(x_n)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Since $\{x_n\}_n$ is sequence of the type of previous definition and f and g satisfy $(\widehat{PS})_{S,c}$, we infer that $\{x_n\}_n$ has a converging subsequence. Hence there exists $\bar{x} \in S$ such that $f(\bar{x}) = c$ and

$$f'_S(\bar{x}) = 0 \quad \text{or} \quad g'(\bar{x}) = 0.$$

If we consider $g'(\bar{x}) = 0$, we get a contradiction with the fact that 1 is a regular value of g .

If we consider $f'_S(\bar{x}) = 0$, we get a contradiction with the fact that c is a regular value of f_S .

In any case, we have a contradiction. □

Analogous definitions and results hold if we consider the family F_a^* .

We can now recall [29, Theorem 2.5].

Theorem 4.6.8. *Let $S := \{x \in E : g(x) = 1\}$ be a C^1 submanifold of a Banach space E , where $g \in C^1(E, \mathbb{R})$ and 1 is a regular value of g . Suppose that f is a C^1 functional on a neighborhood of S , $c \in \mathbb{R}$ is a regular value of f_S and that there exists $a > 0$ such that F_a or F_a^* is admissible for f at the value c . Then there exists $\hat{\varepsilon} > 0$ such that for all $0 < \varepsilon < \hat{\varepsilon}$ there exists an homeomorphism η of S onto S such that:*

1. $\eta(x) = x$ if $f(x) \notin [c - \hat{\varepsilon}, c + \hat{\varepsilon}]$;
2. $f(\eta(x)) \geq f(x)$ for all $x \in S$;
3. $f(\eta(x)) \geq c + \varepsilon$ for all x such that $f(x) \geq c - \varepsilon$;
4. If S is symmetric ($S = -S$) and if f is even, then η is odd.

As a consequence of Lemma 4.6.7 and previous theorem, we get the following.

Theorem 4.6.9. *Let $S := \{x \in E : g(x) = 1\}$ be a C^1 submanifold of a Banach space E , where $g \in C^1(E, \mathbb{R})$ and 1 is a regular value of g . Suppose that f is a C^1 functional on a neighborhood of S , $c \in \mathbb{R}$ is a regular value of f_S and f and g satisfy $(\widehat{PS})_{S,c}$. Then there exists $\hat{\varepsilon} > 0$ such that for all $0 < \varepsilon < \hat{\varepsilon}$ there exists an homeomorphism η of S onto S satisfying properties (1) – (4) of Theorem 4.6.8.*

N -dimensional Euclidean Onofri Inequality in Weighted Sobolev Space

In this Chapter, we focus on the N -dimensional Euclidean Onofri inequality established by Del Pino and Dolbeault in [77] for smooth functions with compact support. We prove that the N -Laplacian operator, Δ_N , is intrinsically linked to this inequality through its connection with the Liouville equation in $\mathbb{R}^N$. Furthermore, we extend the Euclidean Onofri inequality to a suitable weighted Sobolev space. We also establish an equivalence result with the logarithmic Moser-Trudinger inequality on the balls of $\mathbb{R}^N$; in the specific case $N = 2$, we highlight the connection between the 2-dimensional Euclidean Onofri inequality and the two-dimensional unit sphere $\mathbb{S}^2$, by using the stereographic projection from the North pole.

This Chapter is based on the works [30, 31].

5.1 Euclidean Onofri Inequality in $C_0^\infty(\mathbb{R}^N)$

If Ω is a bounded domain of $\mathbb{R}^N$ with $N \geq 2$, it is well known that the Sobolev space $W_0^{1,N}(\Omega)$ is continuously embedded into $L^q(\Omega)$ for any $1 \leq q < \infty$. In particular, there exists a constant $c := c(q, N, \Omega)$ such that

$$\|u\|_q^q \leq c \|\nabla u\|_N^q \quad \text{for any } u \in W_0^{1,N}(\Omega).$$

Denoting by $\mathcal{H} := \{u \in W_0^{1,N}(\Omega) \mid \|\nabla u\|_N \leq 1\}$, it is equivalent to state that

$$\sup_{u \in \mathcal{H}} \int_{\Omega} |u|^q dx \leq c(q, N, \Omega) \quad \text{for any } 1 \leq q < \infty.$$

Even if $W_0^{1,N}(\Omega) \hookrightarrow L^q(\Omega)$ for any $1 \leq q < \infty$, we have $W_0^{1,N}(\Omega) \not\hookrightarrow L^\infty(\Omega)$. In fact if Ω is for instance the unit ball, then a counterexample is given by the function $u(x) = \ln(1 - \ln|x|)$.

However, in [159] Trudinger (see also Yudovich [165] and Pohozaev [141]) improved previous power type estimate obtaining an exponential type one. He proved that there exist $\alpha > 0$ and a constant $c := c(N) > 1$ such that

$$\sup_{u \in \mathcal{H}} \frac{1}{|\Omega|} \int_{\Omega} e^{\alpha|u|^{\frac{N}{N-1}}} dx \leq c.$$

Trudinger's proof [159] makes use of power series expansion of the exponential function and some Sobolev estimates for the individual terms carefully observing the dependence on the exponent of the expansion. An improvement on Trudinger's result was achieved by Moser in [132], who found the best number α_N such that the above statement holds for any $\alpha \leq \alpha_N$ and it is false for $\alpha > \alpha_N$. More precisely, denoting by $\alpha_N := N \omega_{N-1}^{\frac{1}{N-1}}$ where ω_{N-1} denotes the measure of the surface of the unit sphere of $\mathbb{R}^N$, Moser proved in [132, Theorem 1] that there exists a constant $C_N > 1$ such that

$$\sup_{u \in \mathcal{H}} \frac{1}{|\Omega|} \int_{\Omega} e^{\alpha|u|^{\frac{1}{N-1}}} dx \leq C_N \quad \forall \alpha \leq \alpha_N. \quad (5.1.1)$$

The integral on the left is actually finite for any positive α and for any $u \in \mathcal{H}$, but if $\alpha > \alpha_N$ it can be made arbitrarily large by an appropriate choice of u , hence the supremum on $\mathcal{H}$ is equal to $+\infty$.

Moser's proof of the above result relies on symmetrization techniques and a change of variable that reduces the N -dimensional problem to a *one*-dimensional one. He obtained with the same idea a corresponding inequality of (5.1.1) also on the two-sphere $\mathbb{S}^2$ endowed with the standard metric g_0 . In particular, Moser proved in [132, Theorem 2] that there exists a constant $S > 1$ such that

$$\sup_{u \in \bar{\mathcal{H}}} \frac{1}{4\pi} \int_{\mathbb{S}^2} e^{4\pi u^2} d\nu_{g_0} \leq S, \quad (5.1.2)$$

where $\bar{\mathcal{H}} := \{u \in W^{1,2}(\mathbb{S}^2, g_0) \mid \bar{u} = 0, \|\nabla_{g_0} u\|_2 \leq 1\}$ and $\bar{u} := \frac{1}{4\pi} \int_{\mathbb{S}^2} u d\nu_{g_0}$.

Although the proof of the two previous results is based on the same techniques, as pointed out by Moser in his paper it seems impossible to deduce one theorem from the other, therefore the equivalence between [132, Theorem 1] and [132, Theorem 2] is still an open problem.

It is also important to mention the work of Adams [1] for functions of higher order derivatives, and the work of Fontana [99] for compact smooth Riemannian manifolds.

Starting from (5.1.1) and (5.1.2), and applying Young's inequality, one derives the so called logarithmic Moser-Trudinger inequalities. Specifically, (5.1.1) yields the following *logarithmic Moser-Trudinger inequality on bounded domains of $\mathbb{R}^N$* :

$$\ln \left(\frac{1}{|\Omega|} \int_{\Omega} e^u dx \right) \leq \frac{1}{\widetilde{\omega_N}} \int_{\Omega} |\nabla u|^N dx + \ln C_N, \quad (5.1.3)$$

for any $u \in W_0^{1,N}(\Omega)$, where $\widetilde{\omega_N} := N^N \left(\frac{N}{N-1} \right)^{N-1} \omega_{N-1}$ and $C_N > 1$ is the constant in (5.1.1). Based on the work of Carleson and Chang [42], the following improvement of the previous inequality can be established for the unit ball $B_1 \subset \mathbb{R}^N$:

$$\ln \left(\frac{1}{|B_1|} \int_{B_1} e^u dx \right) < \frac{1}{\widetilde{\omega_N}} \int_{B_1} |\nabla u|^N dx + \sum_{k=1}^{N-1} \frac{1}{k}, \quad (5.1.4)$$

for any $u \in W_0^{1,N}(B_1)$.

Moreover, inequality (5.1.4) is sharp in the sense that the functional $J : W_0^{1,N}(B_1) \rightarrow \mathbb{R}$ defined as

$$J(u) := \frac{1}{\widetilde{\omega_N}} \int_{B_1} |\nabla u|^N dx - \ln \left(\frac{1}{|B_1|} \int_{B_1} e^u dx \right), \quad (5.1.5)$$

satisfies

$$\inf_{W_0^{1,N}(B_1)} J = - \sum_{k=1}^{N-1} \frac{1}{k}.$$

Regarding the *logarithmic Moser–Trudinger inequality on $\mathbb{S}^2$* , it reads as follows:

$$\ln \left(\frac{1}{4\pi} \int_{\mathbb{S}^2} e^u d\nu_{g_0} \right) \leq \frac{1}{16\pi} \int_{\mathbb{S}^2} |\nabla_{g_0} u|_{g_0}^2 d\nu_{g_0} + \frac{1}{4\pi} \int_{\mathbb{S}^2} u d\nu_{g_0} + c \quad (5.1.6)$$

for any $u \in W^{1,2}(\mathbb{S}^2, g_0)$, where $c := \ln S$ and $S > 1$ is the constant appearing in (5.1.2).

In [134], Onofri proved that the best constant c for which the previous inequality holds actually vanishes. This leads to the so-called *logarithmic Onofri–Moser–Trudinger inequality*, or simply the *Onofri inequality*:

$$\ln \left(\frac{1}{4\pi} \int_{\mathbb{S}^2} e^u d\nu_{g_0} \right) \leq \frac{1}{16\pi} \int_{\mathbb{S}^2} |\nabla_{g_0} u|_{g_0}^2 d\nu_{g_0} + \frac{1}{4\pi} \int_{\mathbb{S}^2} u d\nu_{g_0} \quad (5.1.7)$$

for any $u \in W^{1,2}(\mathbb{S}^2, g_0)$.

By means of the stereographic projection from the North Pole of $\mathbb{S}^2$, the Onofri inequality (5.1.7) can be equivalently reformulated as the *two-dimensional Euclidean Onofri inequality*:

$$\ln \left(\int_{\mathbb{R}^2} e^u d\mu_2 \right) \leq \frac{1}{16\pi} \int_{\mathbb{R}^2} |\nabla u|^2 dx + \int_{\mathbb{R}^2} u d\mu_2 \quad (5.1.8)$$

for any $u \in W_{\mu_2}(\mathbb{R}^2) := \{u \in L^1(\mathbb{R}^2, d\mu_2) : |\nabla u| \in L^2(\mathbb{R}^2, dx)\}$, where

$$\mu_2(x) := \frac{1}{\pi(1+|x|^2)^2} \quad \text{and} \quad d\mu_2 := \mu_2(x) dx.$$

Extension of (5.1.8) to any dimension $N \geq 2$, was obtained by del Pino and Dolbeault in [77]. Denoting by V_N the volume of the unit ball in $\mathbb{R}^N$, they obtained the following *N -dimensional Euclidean Onofri inequality* for any $u \in C_0^\infty(\mathbb{R}^N)$:

$$\ln \left(\int_{\mathbb{R}^N} e^u d\mu_N \right) \leq \frac{1}{\widetilde{\omega}_N} \int_{\mathbb{R}^N} H_N(u, \mu_N) dx + \int_{\mathbb{R}^N} u d\mu_N, \quad (5.1.9)$$

where

$$\begin{aligned} \widetilde{\omega}_N &:= N^N \left(\frac{N}{N-1} \right)^{N-1} \omega_{N-1}, \\ \mu_N(x) &:= \frac{1}{V_N \left(1 + |x|^{\frac{N}{N-1}} \right)^N}, \quad d\mu_N(x) := \mu_N(x) dx, \end{aligned} \quad (5.1.10)$$

and

$$H_N(u, \mu_N) := |\nabla v_N + \nabla u|^N - |\nabla v_N|^N - N|\nabla v_N|^{N-2} \nabla v_N \cdot \nabla u,$$

with

$$v_N(x) := \ln \mu_N(x). \quad (5.1.11)$$

Inequality (5.1.9) was obtained in [77] by considering the endpoint of optimal Gagliardo–Nirenberg interpolation inequalities, discovered in [75] and extended in [76]. We point out that the Euclidean Onofri inequality (5.1.9) was achieved with different techniques also by Agueh, Boroushaki, and

Ghoussoub in [3], for any $u \in W^{1,N}(\mathbb{R}^N)$. In this Chapter we extend inequality (5.1.9) to a suitable weighted Sobolev space. Precisely, we introduce the following weighted Sobolev space

$$W_{\mu_N}(\mathbb{R}^N) := \{u \in L^1(\mathbb{R}^N, d\mu_N) : |\nabla u| \in L^N(\mathbb{R}^N, dx), |\nabla u|^2 |\nabla v_N|^{N-2} \in L^1(\mathbb{R}^N, dx)\},$$

endowed with the norm

$$\|u\|_{\mu_N} := \int_{\mathbb{R}^N} |u| d\mu_N + \|\nabla u\|_N + \left(\int_{\mathbb{R}^N} |\nabla u|^2 |\nabla v_N|^{N-2} dx \right)^{\frac{1}{2}}. \quad (5.1.12)$$

Our main result is the extension of inequality (5.1.9) to $W_{\mu_N}(\mathbb{R}^N)$. To this aim, we firstly show that smooth compactly supported functions of $\mathbb{R}^N$ are dense in $W_{\mu_N}(\mathbb{R}^N)$ with respect to the norm stated above.

Theorem 5.1.1. *Assume $N \geq 2$. Then*

$$W_{\mu_N}(\mathbb{R}^N) = \overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|_{\mu_N}}.$$

Therefore, by density we are able to prove the following.

Theorem 5.1.2. *Assume $N \geq 2$. For any $u \in W_{\mu_N}(\mathbb{R}^N)$ we have*

$$\ln \left(\int_{\mathbb{R}^N} e^u d\mu_N \right) \leq \frac{1}{\overline{\omega_N}} \int_{\mathbb{R}^N} H_N(u, \mu_N) dx + \int_{\mathbb{R}^N} u d\mu_N. \quad (5.1.13)$$

Let us now consider the functional $I : W_{\mu_N}(\mathbb{R}^N) \rightarrow \mathbb{R}$, associated to (5.1.2), defined as

$$I(u) := \frac{1}{\overline{\omega_N}} \int_{\mathbb{R}^N} H_N(u, \mu_N) dx + \int_{\mathbb{R}^N} u d\mu_N - \ln \left(\int_{\mathbb{R}^N} e^u d\mu_N \right). \quad (5.1.14)$$

Without using (5.1.4) and (5.1.13), we also prove the following identity between the infimum.

Theorem 5.1.3. *For any $N \geq 2$, we have*

$$\inf_{W_{\mu_N}(\mathbb{R}^N)} I = \inf_{W_0^{1,N}(B_1)} J + \sum_{k=1}^{N-1} \frac{1}{k}.$$

Theorem 5.1.3 shows that the sharp inequalities (5.1.4) and (5.1.13) are equivalent. In particular, we can give a simple alternative proof of (5.1.13). On the one hand, since $I(0) = 0$, Theorem 5.1.3 yields

$$0 = \inf_{W_{\mu_N}(\mathbb{R}^N)} I = \min_{W_{\mu_N}(\mathbb{R}^N)} I.$$

On the other hand it follows from (5.1.4) that for any $u \in W_0^{1,N}(B_1)$ we have

$$J(u) > - \sum_{k=1}^{N-1} \frac{1}{k} = \inf_{W_0^{1,N}(B_1)} J.$$

Indeed, we provide a refinement of inequality (5.1.4); specifically, in the proof of Theorem 5.1.3 we establish that for any $u \in W_0^{1,N}(B_1)$ we have

$$\ln \left(\frac{1}{V_N} \int_{B_1} e^u dx + N - 1 \right) \leq \frac{1}{\overline{\omega_N}} \int_{B_1} |\nabla u|^N dx + \sum_{k=1}^{N-1} \frac{1}{k}. \quad (5.1.15)$$

Theorem 5.1.3 becomes more interesting in dimension $N = 2$, considering also the equivalence between (5.1.7) and (5.1.8). Indeed, as announced in [30] (see also [110]), it shows that the sharp two-dimensional logarithmic Moser-Trudinger inequality (5.1.4) by Carleson and Chang is equivalent to the Onofri's inequality (5.1.7) on $\mathbb{S}^2$.

As discussed, the inequalities with non-optimal constants (5.1.3) and (5.1.6) are direct corollaries of the Moser-Trudinger inequality for bounded domains (5.1.1) and its analogue for the sphere (5.1.2). In [132], Moser proved these theorems independently, although their proofs are based on similar arguments relying on symmetrization techniques. He explicitly states that it seems impossible to deduce one Theorem from the other. Although the equivalence between Moser's theorems is still an open problem, we have proved the equivalence between the corresponding logarithmic inequalities with sharp constants, i.e. (5.1.4) for $N = 2$ and (5.1.7).

The Chapter is structured as follows: Section 5.2 focuses on the weighted Sobolev space $W_{\mu_N}(\mathbb{R}^N)$ and the proofs of Theorems 5.1.1 and 5.1.2. The logarithmic Moser-Trudinger inequality on bounded domains is discussed in Section 5.3. Section 5.4 details the proof of Theorem 5.1.3 concerning the equivalence between (5.1.4) and (5.1.13), while Section 5.5 examines the case $N = 2$ and its relation to the unit sphere $\mathbb{S}^2$.

5.2 Euclidean Onofri Inequality in $W_{\mu_N}(\mathbb{R}^N)$

We notice that μ_N defined in (5.1.10) is a probability density on $\mathbb{R}^N$, namely

$$\int_{\mathbb{R}^N} d\mu_N = 1.$$

In fact

$$\int_{\mathbb{R}^N} \mu_N(x) dx = N \int_0^{+\infty} \frac{1}{\left(1 + \rho^{\frac{N}{N-1}}\right)^N} \rho^{N-1} d\rho = (N-1) \int_0^{+\infty} \frac{1}{(1+t)^2} \left(\frac{t}{1+t}\right)^{N-2} dt = 1.$$

Moreover, we have

$$\nabla v_N(x) = -\frac{N^2}{N-1} \frac{|x|^{\frac{1}{N-1}}}{1 + |x|^{\frac{N}{N-1}}} \frac{x}{|x|}.$$

By computations (see Lemma 5.5.4), it follows that

$$-\Delta_N v_N(x) = N^N \left(\frac{N}{N-1}\right)^{N-1} V_N \mu_N(x) = c_N V_N e^{v_N(x)}, \quad (5.2.16)$$

where $c_N := N^N \left(\frac{N}{N-1}\right)^{N-1}$ and $\Delta_N := \operatorname{div}(|\nabla u|^{N-2} \nabla u)$ denotes the N -Laplacian operator.

By (5.2.16) we can observe that the function $v_N(x)$ is connected to the following Liouville equation

$$\begin{cases} -\Delta_N U(x) = e^{U(x)} & x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} e^{U(x)} dx < +\infty. \end{cases} \quad (5.2.17)$$

This problem has an explicit solution given by

$$U(x) = \ln \left(\frac{c_N}{\left(1 + |x|^{\frac{N}{N-1}}\right)^N} \right) \quad x \in \mathbb{R}^N.$$

Observe now that we have

$$-\Delta_N v_N(x) = -\Delta_N U(x) = e^{U(x)} = \frac{c_N}{\left(1 + |x|^{\frac{N}{N-1}}\right)^N} = c_N V_N \mu_N(x) = c_N V_N e^{v_N(x)}.$$

We also recall that thanks to scaling and invariance translation, a family $U_{\lambda, \bar{x}}$ of solutions of (5.2.17) is given by

$$U_{\lambda, \bar{x}}(x) = \ln \left(\frac{c_N \lambda^N}{\left(1 + \lambda^{\frac{N}{N-1}} |x - \bar{x}|^{\frac{N}{N-1}}\right)^N} \right)$$

for all $\lambda > 0$ and for any $\bar{x} \in \mathbb{R}^N$.

Moreover, it follows that

$$\int_{\mathbb{R}^N} e^{U_{\lambda, \bar{x}}(x)} dx = \int_{\mathbb{R}^N} e^{U(x)} dx = c_N V_N.$$

In fact

$$\int_{\mathbb{R}^N} e^{U(x)} dx = c_N \int_{\mathbb{R}^N} \frac{1}{\left(1 + |x|^{\frac{N}{N-1}}\right)^N} dx = c_N \omega_{N-1} \int_0^{+\infty} \frac{1}{\left(1 + \rho^{\frac{N}{N-1}}\right)^N} \rho^{N-1} d\rho = c_N V_N.$$

In particular, in [96, Theorem 1.1.] it was shown that if U is a solution of (5.2.17), then it is of the form $U_{\lambda, \bar{x}}$ for some $\lambda > 0$ and $\bar{x} \in \mathbb{R}^N$.

Now, we show how inequality (5.1.9) involves somehow the N -Laplacian operator Δ_N , by means of $H_N(u, \mu_N)$, that we recall is given by

$$H_N(u(x), \mu_N(x)) = |\nabla v_N(x) + \nabla u(x)|^N - |\nabla v_N(x)|^N - N |\nabla v_N(x)|^{N-2} \nabla v_N(x) \cdot \nabla u(x).$$

If $u \in C_0^\infty(\mathbb{R}^N)$, then there exists a suitable $r > 0$ such that the open ball B_r , centered in the origin e with radius r , contains the support of u . Since u is zero on the boundary of the ball B_r , Gauss-Green formula gives

$$-N \int_{B_r} |\nabla v_N(x)|^{N-2} \nabla v_N(x) \cdot \nabla u(x) dx = \int_{B_r} \Delta_N v_N(x) u(x) dx,$$

We have thus shown that the term $-|\nabla v_N|^{N-2} \nabla v_N \cdot \nabla u$ in $H_N(u, \mu_N)$ contains Δ_N .

Now, let us denote by

$$R_N(X, Y) := |X + Y|^N - |X|^N - N |X|^{N-2} X \cdot Y \quad (X, Y) \in \mathbb{R}^N \times \mathbb{R}^N. \quad (5.2.18)$$

Considering the definition of $H_N(u(x), \mu_N(x))$, we get

$$H_N(u(x), \mu_N(x)) := R_N(\nabla v_N(x), \nabla u(x)).$$

Let us now prove the following estimate on R_N .

Lemma 5.2.1. *For any $N \in \mathbb{N}$, $N \geq 2$, there exists a constant c_N such that*

$$0 \leq R_N(X, Y) \leq c_N(|Y|^N + |Y|^2|X|^{N-2}) \quad \forall X, Y \in \mathbb{R}^N.$$

Proof. If $N = 2$, $R_2(X, Y) = |Y|^2$ does not depend on X and the thesis is trivial. Let $N \geq 3$ and let $X, Y \in \mathbb{R}^N$. We consider the C^2 function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(t) = |tY + X|^N.$$

By a direct computation, we have

$$f'(t) = N |tY + X|^{N-2} (tY + X) \cdot Y,$$

and

$$f''(t) = N(N-2) |tY + X|^{N-4} ((tY + X) \cdot Y)^2 + N |tY + X|^{N-2} |Y|^2$$

if $tY + X \neq 0$, and $f''(t) = 0$ otherwise.

Now, we observe that if $t \in [0, 1]$ we have

$$0 \leq f''(t) \leq 2c_N (|Y|^N + |Y|^2|X|^{N-2}), \quad (5.2.19)$$

where $c_N = N(N-1)2^{N-4}$. By Taylor's theorem with Lagrange remainder, there exists $\bar{t} \in]0, 1[$ such that

$$\frac{1}{2} f''(\bar{t}) = f(1) - f(0) - f'(0) = |Y + X|^N - |X|^N - N |X|^{N-2} X \cdot Y.$$

Taking into account (5.2.19), we deduce that

$$0 \leq R_N(X, Y) \leq c_N (|Y|^N + |Y|^2|X|^{N-2}).$$

□

Now, recalling that

$$H_N(u(x), \mu_N(x)) := R_N(\nabla v_N(x), \nabla u(x)),$$

an immediate consequence of Lemma 5.2.1 is the following integrability condition for $H_N(u, \mu_N)$.

Corollary 5.2.2. *Let $u \in W_{\mu_N}(\mathbb{R}^N)$. Then*

$$\int_{\mathbb{R}^N} H_N(u, \mu_N) dx < +\infty.$$

We stress the fact that we have shown how $|\nabla u| \in L^N(\mathbb{R}^N, dx)$ and $|\nabla u|^2 |\nabla v_N|^{N-2} \in L^1(\mathbb{R}^N, dx)$ in the definition of $W_{\mu_N}(\mathbb{R}^N)$ are sufficient conditions to obtain the integrability of $H_N(u, \mu_N)$. Therefore one can wonder what happens if we do not consider $|\nabla u|^2 |\nabla v_N|^{N-2} \in L^1(\mathbb{R}^N, dx)$, i.e. it is interesting to compare $W_{\mu_N}(\mathbb{R}^N)$ with $W^{1,N}(\mathbb{R}^N)$.

Lemma 5.2.3. *If $u \in W^{1,N}(\mathbb{R}^N)$ is such that $|\nabla u|^2 |\nabla v_N|^{N-2} \in L^1(\mathbb{R}^N, dx)$, then $u \in W_{\mu_N}(\mathbb{R}^N)$. Moreover, we have $W_{\mu_N}(\mathbb{R}^N) \setminus W^{1,N}(\mathbb{R}^N) \neq \emptyset$.*

Proof. Let $u \in W^{1,N}(\mathbb{R}^N)$ such that $|\nabla u|^2 |\nabla v_N|^{N-2} \in L^1(\mathbb{R}^N, dx)$.

$$\int_{\mathbb{R}^N} |u(x)| d\mu_N = \int_{\mathbb{R}^N} |u(x)| \frac{1}{V_N \left(1 + |x|^{\frac{N}{N-1}}\right)^N} dx.$$

By assumption $\|u\|_N < +\infty$, while

$$\gamma_N := \int_{\mathbb{R}^N} \frac{1}{\left(1 + |x|^{\frac{N}{N-1}}\right)^{\frac{N^2}{N-1}}} dx < +\infty.$$

By Hölder inequality we have

$$\int_{\mathbb{R}^N} |u(x)| d\mu_N \leq \frac{\gamma_N^{\frac{N-1}{N}}}{V_N} \|u\|_N < +\infty.$$

Therefore $u \in W_{\mu_N}(\mathbb{R}^N)$.

Finally, the constant $u \equiv 1$ is such that $u \in W_{\mu_N}(\mathbb{R}^N) \setminus W^{1,N}(\mathbb{R}^N)$. □

The next lemma shows that the condition $|\nabla u|^2 |\nabla v_N|^{N-2} \in L^1(\mathbb{R}^N, dx)$ is also necessary for the integrability of $H(u, \mu_N)$, at least in even dimension.

Lemma 5.2.4. *If $N \geq 4$ is even, then*

$$H(u, \mu_N) \geq \frac{N}{2} |\nabla u|^2 |\nabla v_N|^{N-2}.$$

Proof. The proof is based on the following inequalities, which hold for $a, b \in \mathbb{R}$ and $k \in \mathbb{N}$, with $a \geq 0$ and $a + b \geq 0$:

1. if $k \geq 2$, then $(a + b)^k \geq a^k + ka^{k-1}b + b^k$.
2. if $k \geq 3$, then $(a + b)^k \geq a^k + ka^{k-1}b + kab^{k-1} + b^k$.

Both (1) and (2) can be easily proved by induction on k . Let $X, Y \in \mathbb{R}^N$. If $\frac{N}{2}$ is even, using inequality (1) with $k = \frac{N}{2}$, $a = |X|^2$ and $b = 2X \cdot Y + |Y|^2$ we get that

$$\begin{aligned} R_N(X, Y) &= \left(|X|^2 + 2X \cdot Y + |Y|^2\right)^{\frac{N}{2}} - |X|^N - N|X|^{N-2}X \cdot Y \\ &\geq \frac{N}{2}|X|^{N-2}|Y|^2 + \left(2X \cdot Y + |Y|^2\right)^{\frac{N}{2}}. \end{aligned}$$

Since $\frac{N}{2}$ is even, we get

$$R_N(X, Y) \geq \frac{N}{2}|X|^{N-2}|Y|^2. \quad (5.2.20)$$

Similarly, if $\frac{N}{2}$ is odd (hence $N \geq 6$), we can apply inequality (2) (with k, a and b as above) to get:

$$\begin{aligned}
R_N(X, Y) &\geq \frac{N}{2}|X|^{N-2}|Y|^2 + \frac{N}{2}|X|^2 \left(2X \cdot Y + |Y|^2\right)^{\frac{N}{2}-1} + \left(2X \cdot Y + |Y|^2\right)^{\frac{N}{2}} \\
&= \frac{N}{2}|X|^{N-2}|Y|^2 + \left(2X \cdot Y + |Y|^2\right)^{\frac{N}{2}-1} \left(\frac{N}{2}|X|^2 + 2X \cdot Y + |Y|^2\right) \\
&\geq \frac{N}{2}|X|^{N-2}|Y|^2 + \left(2X \cdot Y + |Y|^2\right)^{\frac{N}{2}-1} |X + Y|^2.
\end{aligned}$$

Since $\frac{N}{2} - 1$ is even, we get again (5.2.20). Hence (5.2.20) holds whenever $\frac{N}{2}$ is integer and the conclusion follows by taking $X = \nabla v_N$ and $Y = \nabla u$. $\square$

Remark 5.2.5. If $N \geq 3$ and $u \in W^{1,N}(\mathbb{R}^N)$, in general the integrability condition $|\nabla u|^2 |\nabla v_N|^{N-2} \in L^1(\mathbb{R}^N, dx)$ of Lemma 5.2.3 is not satisfied. Indeed, let

$$u(x) := \sum_{k=2}^{\infty} \frac{1}{k\sqrt{\ln k}} w(2(|x| - k)),$$

where

$$w(r) := \begin{cases} 1 + r & \text{if } r \in [-1, 0[, \\ 1 - r & \text{if } r \in [0, 1], \\ 0 & \text{otherwise.} \end{cases}$$

A direct computation shows that the function u satisfies

$$u \in W^{1,N}(\mathbb{R}^N) \quad \text{and} \quad |\nabla u|^2 |\nabla v_N|^{N-2} \notin L^1(\mathbb{R}^N, dx). \quad (5.2.21)$$

In particular, if N is even, Lemma 5.2.4 yields

$$\int_{\mathbb{R}^N} H(u, \mu_N) dx = +\infty.$$

To prove (5.2.21) we observe that the functions $x \rightarrow w(2(|x| - k))$ have disjoint supports as k varies in $\mathbb{N}$ with $k \geq 2$. Then

$$|u(x)|^N = \sum_{k=2}^{\infty} \frac{1}{k^N (\ln k)^{\frac{N}{2}}} (w(2(|x| - k)))^N.$$

Integrating in polar coordinates and using that $0 \leq w \leq 1$, we find

$$\begin{aligned}
\int_{\mathbb{R}^N} |u|^N dx &\leq \omega_{N-1} \sum_{k=2}^{\infty} \frac{1}{k^N (\ln k)^{\frac{N}{2}}} \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} \rho^{N-1} d\rho \\
&= \frac{\omega_{N-1}}{N} \sum_{k=2}^{\infty} \frac{1}{k^N (\ln k)^{\frac{N}{2}}} \left[\left(k + \frac{1}{2}\right)^N - \left(k - \frac{1}{2}\right)^N \right] \\
&\leq c_1(N) \sum_{k=2}^{\infty} \frac{1}{k (\ln k)^{\frac{N}{2}}} < +\infty
\end{aligned}$$

since $\frac{N}{2} > 1$. Similarly, we have

$$|\nabla u(x)| = 2 \sum_{k=2}^{\infty} \frac{1}{k\sqrt{\ln k}} |w'(2(|x| - k))|,$$

and

$$\int_{\mathbb{R}^N} |\nabla u(x)|^N dx = c_2(N) \sum_{k=2}^{\infty} \frac{1}{k^N (\ln k)^{\frac{N}{2}}} \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} \rho^{N-1} d\rho < +\infty.$$

Now, we observe that $|\nabla v_N(x)| \sim \frac{N^2}{N-1} \frac{1}{|x|}$ if $|x| \gg 1$, hence, taking k_0 sufficiently large, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u|^2 |\nabla v_N|^{N-2} dx &\geq c_3(N) + c_4(N) \sum_{k=k_0}^{\infty} \frac{1}{k^2 \ln k} \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} \frac{1}{\rho^{N-2}} \rho^{N-1} d\rho \\ &\geq c_5(N) \left(1 + \sum_{k=k_0}^{\infty} \frac{1}{k \ln k} \right) = +\infty. \end{aligned}$$

To extend (5.1.9) to $W_{\mu_N}(\mathbb{R}^N)$, we show that smooth compactly supported functions of $\mathbb{R}^N$ are dense in $W_{\mu_N}(\mathbb{R}^N)$ with respect to the norm (5.1.12).

Let us define

$$\mathcal{X} := \overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|_{\mu_N}}.$$

Clearly, $C_0^\infty(\mathbb{R}^N) \subset W_{\mu_N}(\mathbb{R}^N)$, so that $\mathcal{X} \subseteq W_{\mu_N}(\mathbb{R}^N)$. Our aim is to prove that the two spaces coincide. We will prove this in two steps. Firstly, by using a similar idea of Tintarev and Fieseler in [156, Remark 2.2], we show that bounded functions in $W_{\mu_N}(\mathbb{R}^N)$ belong to $\mathcal{X}$.

Lemma 5.2.6. *Let $u \in W_{\mu_N}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then $u \in \mathcal{X}$.*

Proof. For any $k \in \mathbb{N}$, $k \geq 2$, let us denote by

$$k^* := \frac{1}{\left(1 - \frac{1}{\sqrt[k]{k}}\right)^k}.$$

We define

$$\eta_k(x) := \begin{cases} \frac{1}{\sqrt[k]{k}} & \text{if } |x| \leq 1, \\ \frac{1}{\sqrt[k]{|x|}} + \frac{1}{\sqrt[k]{k}} - 1 & \text{if } 1 < |x| \leq k^*, \\ 0 & \text{if } k^* < |x|. \end{cases}$$

Let $u \in W_{\mu_N}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. We define $u_k := u \eta_k$. Observe that

$$\nabla u_k(x) := \begin{cases} \frac{1}{\sqrt[k]{k}} \nabla u(x) & \text{if } |x| < 1, \\ \left(\frac{1}{\sqrt[k]{k}} + \frac{1}{\sqrt[k]{|x|}} - 1 \right) \nabla u(x) - \frac{1}{k} \frac{u(x)}{|x|^{\frac{1}{k}+1}} \frac{x}{|x|} & \text{if } 1 < |x| < k^*, \\ 0 & \text{if } k^* < |x|. \end{cases}$$

First of all, since $u \in L^\infty(\mathbb{R}^N)$ and $\eta_k(x) \rightarrow 1$ for any $x \in \mathbb{R}^N$ as $k \rightarrow +\infty$, we can apply dominated convergence theorem to say that

$$\int_{\mathbb{R}^N} |u - u_k(x)| d\mu_N \rightarrow 0 \quad \text{as } k \rightarrow +\infty.$$

Now, we prove

$$\|\nabla(u - u_k)\|_N \rightarrow 0 \quad \text{as } k \rightarrow +\infty.$$

We will denote by $o(1)$ any quantity that goes to 0 as $k \rightarrow +\infty$.

$$\int_{\mathbb{R}^N} |\nabla u - \nabla u_k|^N dx = \int_{|x|<1} |\nabla u - \nabla u_k|^N dx + \int_{1<|x|<k^*} |\nabla u - \nabla u_k|^N dx + \int_{|x|>k^*} |\nabla u|^N dx.$$

We have

$$\int_{|x|<1} |\nabla u - \nabla u_k|^N dx = \left(1 - \frac{1}{\sqrt[k]{k}}\right)^N \int_{|x|<1} |\nabla u|^N dx = o(1).$$

Since $\|\nabla u\|_N < +\infty$ and $k^* \rightarrow +\infty$ as $k \rightarrow +\infty$, we easily obtain

$$\int_{|x|>k^*} |\nabla u - \nabla u_k|^N dx = \int_{|x|>k^*} |\nabla u|^N dx = o(1).$$

Since $u \in L^\infty(\mathbb{R}^N)$, there exists a constant M_u such that $|u(x)| \leq M_u$ for almost every $x \in \mathbb{R}^N$.

$$\begin{aligned} \int_{1<|x|<k^*} |\nabla u - \nabla u_k|^N dx &= \int_{1<|x|<k^*} \left| \left(2 - \frac{1}{\sqrt[k]{k}} - \frac{1}{\sqrt[k]{|x|}}\right) \nabla u(x) + \frac{1}{k} \frac{u(x)}{|x|^{\frac{1}{k}+1}} \frac{x}{|x|} \right|^N dx \\ &\leq C \int_{1<|x|<k^*} \left(2 - \frac{1}{\sqrt[k]{k}} - \frac{1}{\sqrt[k]{|x|}}\right)^N |\nabla u(x)|^N dx \\ &\quad + C \frac{M_u^N}{k^N} \int_{1<|x|<k^*} \frac{1}{|x|^{\frac{N}{k}+N}} dx. \end{aligned}$$

Now, we observe that

$$\frac{M_u^N}{k^N} \int_{1<|x|<k^*} \frac{1}{|x|^{\frac{N}{k}+N}} dx < \frac{c_1}{k^N} \int_1^{+\infty} \frac{1}{\rho^{\frac{N}{k}+1}} d\rho = \frac{c_2}{k^{N-1}} = o(1).$$

Moreover, we can apply dominated convergence theorem to say that

$$\int_{1<|x|<k^*} \left(2 - \frac{1}{\sqrt[k]{k}} - \frac{1}{\sqrt[k]{|x|}}\right)^N |\nabla u(x)|^N dx \rightarrow 0 \quad \text{as } k \rightarrow +\infty.$$

Hence

$$\|\nabla(u - u_k)\|_N \rightarrow 0 \quad \text{as } k \rightarrow +\infty.$$

It remains to prove, in the case $N \geq 3$, that

$$\left(\int_{\mathbb{R}^N} |\nabla u - \nabla u_k|^2 |\nabla v_N|^{N-2} dx \right)^{\frac{1}{2}} \rightarrow 0 \quad \text{as } k \rightarrow +\infty.$$

We observe that $|\nabla v_N(x)| \sim \frac{1}{|x|}$ as $|x| \rightarrow +\infty$, by which there exists a constant $r_N > 1$ such that $|\nabla v_N(x)| \leq \frac{2}{|x|}$ if $|x| \geq r_N$. Since $k \rightarrow +\infty$ implies $k^* \rightarrow +\infty$, we can assume from the beginning that k is such that $k^* > r_N$. Moreover, we recall that $|\nabla v_N|$ is bounded from above.

$$\begin{aligned}
& \int_{\mathbb{R}^N} |\nabla u - \nabla u_k|^2 |\nabla v_N|^{N-2} dx \\
&= \int_{|x|<1} |\nabla u - \nabla u_k|^2 |\nabla v_N|^{N-2} dx + \int_{1<|x|<r_N} |\nabla u - \nabla u_k|^2 |\nabla v_N|^{N-2} dx \\
&+ \int_{r_N<|x|<k^*} |\nabla u - \nabla u_k|^2 |\nabla v_N|^{N-2} dx + \int_{k^*<|x|} |\nabla u|^2 |\nabla v_N|^{N-2} dx.
\end{aligned}$$

Reasoning in a similar way to the previous case, we obtain

$$\begin{aligned}
& \int_{|x|<1} |\nabla u - \nabla u_k|^2 |\nabla v_N|^{N-2} dx = o(1), \\
& \int_{1<|x|<r_N} |\nabla u - \nabla u_k|^2 |\nabla v_N|^{N-2} dx = o(1),
\end{aligned}$$

and

$$\int_{k^*<|x|} |\nabla u|^2 |\nabla v_N|^{N-2} dx = o(1).$$

Now we observe that

$$\begin{aligned}
& \int_{r_N<|x|<k^*} |\nabla u - \nabla u_k|^2 |\nabla v_N|^{N-2} dx \\
&= \int_{r_N<|x|<k^*} \left| \left(2 - \frac{1}{\sqrt[k]{k}} - \frac{1}{\sqrt[k]{|x|}} \right) \nabla u(x) + \frac{1}{k} \frac{u(x)}{|x|^{\frac{1}{k}+1}} \frac{x}{|x|} \right|^2 |\nabla v_N|^{N-2} dx \\
&\leq C \int_{r_N<|x|<k^*} \left(2 - \frac{1}{\sqrt[k]{k}} - \frac{1}{\sqrt[k]{|x|}} \right)^2 |\nabla u(x)|^2 |\nabla v_N|^{N-2} dx \\
&+ C \frac{1}{k^2} \int_{r_N<|x|} \frac{1}{|x|^{\frac{2}{k}+N}} dx.
\end{aligned}$$

Clearly,

$$\frac{1}{k^2} \int_{r_N<|x|} \frac{1}{|x|^{\frac{2}{k}+N}} dx = o(1).$$

Moreover, taking into account that $|\nabla u|^2 |\nabla v_N|^{N-2} \in L^1(\mathbb{R}^N, dx)$, we can apply dominated convergence theorem to say that

$$\int_{r_N<|x|<k^*} \left(2 - \frac{1}{\sqrt[k]{k}} - \frac{1}{\sqrt[k]{|x|}} \right)^2 |\nabla u(x)|^2 |\nabla v_N|^{N-2} dx = o(1).$$

Finally, we have

$$\left(\int_{\mathbb{R}^N} |\nabla u - \nabla u_k|^2 |\nabla v_N|^{N-2} dx \right)^{\frac{1}{2}} \rightarrow 0 \quad \text{as } k \rightarrow +\infty.$$

We have proved that $\{u_k\}_k$ is a sequence of compactly supported functions such that $\|u - u_k\|_{\mu_N} \rightarrow 0$ as $k \rightarrow +\infty$.

We want to take a sequence $\{\tilde{u}_k\}_k \subset C_0^\infty(\mathbb{R}^N)$ such that $\|u - \tilde{u}_k\|_{\mu_N} \rightarrow 0$ as $k \rightarrow +\infty$.

Let us denote by B_{k^*} the open ball $B_{k^*}(O)$.

Since $u_k|_{B_{k^*}} \in W_0^{1,N}(B_{k^*})$, there exists $\tilde{u}_k \in C_0^\infty(B_{k^*})$ such that

$$\|u_k|_{B_{k^*}} - \tilde{u}_k\|_{L^N(B_{k^*})} + \|\nabla(u_k|_{B_{k^*}} - \tilde{u}_k)\|_{L^N(B_{k^*})} < \frac{1}{k}.$$

We can extend to 0 in $\mathbb{R}^N \setminus B_{k^*}$ and consider $\tilde{u}_k \in C_0^\infty(\mathbb{R}^N)$. Hence

$$\|u_k - \tilde{u}_k\|_N + \|\nabla(u_k - \tilde{u}_k)\|_N < \frac{1}{k}.$$

Since

$$\gamma_N := \int_{\mathbb{R}^N} \frac{1}{\left(1 + |x|^{\frac{N}{N-1}}\right)^{\frac{N^2}{N-1}}} dx < +\infty,$$

we can apply Hölder inequality and say that

$$\int_{\mathbb{R}^N} |u_k(x) - \tilde{u}_k(x)| d\mu_N < \gamma_N^{\frac{N-1}{N}} \|u_k - \tilde{u}_k\|_N < \frac{\gamma_N^{\frac{N-1}{N}}}{k} = o(1).$$

Now, let k be such that $k^* > r_N$.

Since

$$\int_{0 < |x| \leq k^*} |\nabla v_N|^N dx \leq C (\ln k^* + 1),$$

we have

$$\int_{\mathbb{R}^N} |\nabla(u_k - \tilde{u}_k)|^2 |\nabla v_N|^{N-2} dx < C \frac{(\ln k^* + 1)^{\frac{N-2}{N}}}{k^2}.$$

Now, observe that

$$\lim_{k \rightarrow +\infty} \ln k^* = \lim_{k \rightarrow +\infty} -k \ln \left(1 - \frac{1}{\sqrt[k]{k}}\right) = +\infty,$$

and

$$\ln \left(1 - \frac{1}{\sqrt[k]{k}}\right) = \ln \left(1 - e^{-\frac{\ln k}{k}}\right) = \ln \left(\frac{\ln k}{k} + o\left(\frac{\ln k}{k}\right)\right) = -(1 + o(1)) \ln k.$$

Therefore, we have

$$\lim_{k \rightarrow +\infty} \frac{(\ln k^* + 1)^{\frac{N-2}{N}}}{k^2} = \lim_{k \rightarrow +\infty} \frac{(\ln k^*)^{\frac{N-2}{N}}}{k^2} = \lim_{k \rightarrow +\infty} \frac{[\ln k]^{1 - \frac{2}{N}}}{k^{1 + \frac{2}{N}}} = 0.$$

Hence

$$\int_{\mathbb{R}^N} |\nabla(u_k - \tilde{u}_k)|^2 |\nabla v_N|^{N-2} dx \rightarrow 0.$$

Finally, we have

$$\|u - \tilde{u}_k\|_{\mu_N} \leq \|u - u_k\|_{\mu_N} + \|u_k - \tilde{u}_k\|_{\mu_N} \rightarrow 0 \quad \text{as } k \rightarrow +\infty.$$

□

Thanks to previous lemma, it remains to prove that any function $u \in W_{\mu_N}(\mathbb{R}^N)$ can be approximated by a sequence of bounded functions in $W_{\mu_N}(\mathbb{R}^N)$.

Lemma 5.2.7.

Let $u \in W_{\mu_N}(\mathbb{R}^N)$. There exists a sequence $\{u_k\}_{k \in \mathbb{N}} \subset W_{\mu_N}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that $\|u - u_k\|_{\mu_N} \rightarrow 0$ as $k \rightarrow +\infty$.

Proof. Let $u \in W_{\mu_N}(\mathbb{R}^N)$. Taking into account [115, Lemma A.4, Chapter 2], we have $|u| \in W_{\mu_N}(\mathbb{R}^N)$ with

$$\nabla|u| = \begin{cases} \nabla u & \text{a.e. in } \{x \in \mathbb{R}^N \mid u(x) > 0\}, \\ 0 & \text{a.e. in } \{x \in \mathbb{R}^N \mid u(x) = 0\}, \\ -\nabla u & \text{a.e. in } \{x \in \mathbb{R}^N \mid u(x) < 0\}. \end{cases}$$

For any $\lambda > 0$, we can define the *truncated function of u of parameter λ* as $u_\lambda := \max\{-\lambda, \min\{u, \lambda\}\}$, that is

$$u_\lambda(x) = \begin{cases} \lambda & \text{a.e. in } \{x \in \mathbb{R}^N \mid u(x) \geq \lambda\}, \\ u & \text{a.e. in } \{x \in \mathbb{R}^N \mid -\lambda < u(x) < \lambda\}, \\ -\lambda & \text{a.e. in } \{x \in \mathbb{R}^N \mid u(x) \leq -\lambda\}. \end{cases}$$

Let $k \in \mathbb{N}$, $k \geq 1$, and let $\{u_k\}_{k \in \mathbb{N}}$ be a sequence of truncated function of u of parameter k .

Clearly, $\{u_k\}_{k \in \mathbb{N}} \subset W_{\mu_N}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

Let us denote by $E_k := \{x \in \mathbb{R}^N \mid |u(x)| \geq k\}$.

Hence

$$\int_{\mathbb{R}^N} |u - u_k| d\mu_N = \int_{E_k} |u - u_k| d\mu_N \leq \int_{E_k} |u| d\mu_N.$$

Since $u \in L^1(\mathbb{R}^N, d\mu_N)$, by Chebyshev's inequality we know that

$$|E_k|_{\mu_N} \rightarrow 0 \quad \text{as } k \rightarrow +\infty.$$

Moreover, we can apply absolute continuity of the integral and say that

$$\int_{E_k} |u| d\mu_N \rightarrow 0 \quad \text{as } k \rightarrow +\infty.$$

Therefore, we deduce $u_k \rightarrow u$ in $L^1(\mathbb{R}^N, d\mu_N)$.

Let now $\varepsilon > 0$ and let us consider the set

$$\left\{ x \in \mathbb{R}^N : \left| \frac{\partial u_k}{\partial x_i}(x) - \frac{\partial u}{\partial x_i}(x) \right| \geq \varepsilon \right\}.$$

Since

$$\left\{ x \in \mathbb{R}^N : \left| \frac{\partial u_k}{\partial x_i}(x) - \frac{\partial u}{\partial x_i}(x) \right| \geq \varepsilon \right\} \subset E_k,$$

we deduce that

$$\frac{\partial u_k}{\partial x_i} \rightarrow \frac{\partial u}{\partial x_i} \quad \text{in measure with respect to } \mu_N.$$

Hence there exists a subsequence $\{u_{k_j}\}_{j \in \mathbb{N}}$ such that $\nabla u_{k_j}(x) \rightarrow \nabla u(x)$ almost everywhere in $\mathbb{R}^N$ with respect to the measure μ_N . Since the Lebesgue measure dx is absolutely continuous with respect to the measure $d\mu_N$, we deduce that $\nabla u_{k_j} \rightarrow \nabla u$ almost everywhere in $\mathbb{R}^N$ also with respect to the Lebesgue measure.

Finally, since

$$|\nabla u(x) - \nabla u_{k_j}(x)|^N \leq |\nabla u(x)|^N \quad \text{a.e. } x \in \mathbb{R}^N,$$

and

$$|\nabla u(x) - \nabla u_{k_j}(x)|^2 |\nabla v_N(x)|^{N-2} \leq |\nabla u(x)|^2 |\nabla v_N(x)|^{N-2} \quad \text{a.e. } x \in \mathbb{R}^N,$$

we can apply dominated convergence theorem to say that

$$\|\nabla u - \nabla u_{k_j}\|_N \rightarrow 0 \quad \text{as } j \rightarrow +\infty,$$

and

$$\left(\int_{\mathbb{R}^N} |\nabla u - \nabla u_{k_j}|^2 |\nabla v_N|^{N-2} dx \right)^{\frac{1}{2}} \rightarrow 0 \quad \text{as } j \rightarrow +\infty.$$

□

Taking into account Lemma 5.2.6 and Lemma 5.2.7, we have proved Theorem 5.1.1, that is

$$W_{\mu_N}(\mathbb{R}^N) = \mathcal{X} = \overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|_{\mu_N}}.$$

Now, we can extend the N -dimensional Euclidean Onofri inequality to the weighted Sobolev space $W_{\mu_N}(\mathbb{R}^N)$.

Proof of Theorem 5.1.2. Let $u \in W_{\mu_N}(\mathbb{R}^N)$. Since $W_{\mu_N}(\mathbb{R}^N) = \overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|_{\mu_N}}$, there exists a sequence $\{u_m\}_{m \in \mathbb{N}} \subset C_0^\infty(\mathbb{R}^N)$ such that $\|u - u_m\|_{\mu_N} \rightarrow 0$. By (5.1.9), for any $m \in \mathbb{N}$ we have

$$\ln \left(\int_{\mathbb{R}^N} e^{u_m} d\mu_N \right) \leq \frac{1}{\omega_N} \int_{\mathbb{R}^N} H_N(u_m, \mu_N) dx + \int_{\mathbb{R}^N} u_m d\mu_N.$$

The condition $\|u - u_m\|_{\mu_N} \rightarrow 0$ implies that there exists a subsequence, that we recall $\{u_m\}_{m \in \mathbb{N}}$, and a nonnegative function $w \in L^N(\mathbb{R}^N, dx)$ with $w^2 |\nabla v_N|^{N-2} \in L^1(\mathbb{R}^N, dx)$, such that:

- $u_m(x) \rightarrow u(x)$ a.e. in $\mathbb{R}^N$ as $m \rightarrow +\infty$;
- $\nabla u_m(x) \rightarrow \nabla u(x)$ a.e. in $\mathbb{R}^N$ as $m \rightarrow +\infty$;
- for any m , $|\nabla u_m(x)| \leq w(x)$ a.e. in $\mathbb{R}^N$.

We easily obtain that

$$\int_{\mathbb{R}^N} u_m d\mu_N \rightarrow \int_{\mathbb{R}^N} u d\mu_N \quad \text{as } m \rightarrow +\infty.$$

It is immediate to observe that

$$H_N(u_m(x), \mu_N(x)) \rightarrow H_N(u(x), \mu_N(x)) \quad \text{a.e. in } \mathbb{R}^N \quad \text{as } m \rightarrow +\infty.$$

Taking into account Lemma 5.2.1, we have

$$\begin{aligned} 0 \leq H_N(u_m(x), \mu_N(x)) &\leq \frac{c_N}{2} \left(|\nabla u_m(x)|^N + |\nabla u_m(x)|^2 |\nabla v_N(x)|^{N-2} \right) \\ &\leq \frac{c_N}{2} \left(w(x)^N + w(x)^2 |\nabla v_N(x)|^{N-2} \right), \end{aligned}$$

almost everywhere in $\mathbb{R}^N$ and for any m .

Since $w^N + w^2 |\nabla v_N|^{N-2} \in L^1(\mathbb{R}^N, dx)$, we can apply dominated convergence theorem to say that

$$\int_{\mathbb{R}^N} H_N(u_m, \mu_N) dx \rightarrow \int_{\mathbb{R}^N} H_N(u, \mu_N) dx.$$

Finally, by Fatou's Lemma we have

$$\int_{\mathbb{R}^N} e^u d\mu_N \leq \liminf_{m \rightarrow +\infty} \left(\int_{\mathbb{R}^N} e^{u_m} d\mu_N \right) \leq e^{\frac{1}{\omega_N} \int_{\mathbb{R}^N} H_N(u, \mu_N) dx + \int_{\mathbb{R}^N} u d\mu_N},$$

and passing to logarithm we can conclude that

$$\ln \left(\int_{\mathbb{R}^N} e^u d\mu_N \right) \leq \frac{1}{\omega_N} \int_{\mathbb{R}^N} H_N(u, \mu_N) dx + \int_{\mathbb{R}^N} u d\mu_N.$$

□

Notice that the functional $I : W_{\mu_N}(\mathbb{R}^N) \rightarrow \mathbb{R}$, associated to (5.1.2) and defined as

$$I(u) := \frac{1}{\omega_N} \int_{\mathbb{R}^N} H_N(u, \mu_N) dx + \int_{\mathbb{R}^N} u d\mu_N - \ln \left(\int_{\mathbb{R}^N} e^u d\mu_N \right)$$

satisfies

$$\inf_{W_{\mu_N}(\mathbb{R}^N)} I = 0,$$

where the infimum is realized for any constant in $W_{\mu_N}(\mathbb{R}^N)$.

5.3 Logarithmic Moser-Trudinger inequality on the balls of $\mathbb{R}^N$

As previously noted, Moser [132] investigated the borderline case of Sobolev embeddings for $W_0^{1,N}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$, refining the results of Trudinger [159] (see also Yudovich [165] and Pohozaev [141]). More precisely, Moser established the following theorem:

Theorem 5.3.1 (Moser-Trudinger inequality on bounded domain of $\mathbb{R}^N$).

Let Ω be a bounded domain in $\mathbb{R}^N$ with $N \geq 2$. Let $\alpha_N := N \omega_{N-1}^{\frac{1}{N-1}}$, where ω_{N-1} denotes the measure of the surface of the unit sphere of $\mathbb{R}^N$. Denoting by $\mathcal{H} := \{u \in W_0^{1,N}(\Omega) \mid \|\nabla u\|_N \leq 1\}$, there exists a constant $C_N > 1$, such that

$$\sup_{u \in \mathcal{H}} \frac{1}{|\Omega|} \int_{\Omega} e^{\alpha|u|^{\frac{N}{N-1}}} dx \leq C_N \quad \forall \alpha \leq \alpha_N. \quad (5.3.22)$$

The integral on the left is actually finite for any positive α and for any $u \in \mathcal{H}$, but if $\alpha > \alpha_N$ it can be made arbitrarily large by an appropriate choice of u , hence the supremum on $\mathcal{H}$ is equal to $+\infty$.

As a straightforward consequence of previous result and Young's inequality, one can prove the following weaker inequality:

Proposition 5.3.2 (Logarithmic M-T inequality on bounded domains of $\mathbb{R}^N$).

Let Ω be a bounded domain of $\mathbb{R}^N$ with $N \geq 2$, and let C_N be the constant appearing in (5.3.22).

For any $u \in W_0^{1,N}(\Omega)$, we have

$$\ln \left(\frac{1}{|\Omega|} \int_{\Omega} e^u dx \right) \leq \frac{1}{\widetilde{\omega}_N} \int_{\Omega} |\nabla u|^N dx + \ln C_N, \quad (5.3.23)$$

where $\widetilde{\omega}_N := N^N \left(\frac{N}{N-1} \right)^{N-1} \omega_{N-1}$.

Proof. Let $u \in W_0^{1,N}(\Omega)$, $u \neq 0$, and let $b^{\frac{N}{N-1}} := \frac{N}{N-1} \frac{\alpha_N}{\|\nabla u\|_N^{\frac{N}{N-1}}}$.

By Young's inequality with conjugated exponents N and $\frac{N}{N-1}$, we infer that

$$u \leq \frac{1}{b} (b|u|) \leq \frac{1}{Nb^N} + \frac{N-1}{N} b^{\frac{N}{N-1}} |u|^{\frac{N}{N-1}} = \frac{\|\nabla u\|_N^N}{\widetilde{\omega}_N} + \alpha_N \left(\frac{|u|}{\|\nabla u\|_N} \right)^{\frac{N}{N-1}}.$$

Considering (5.3.22), we get

$$\frac{1}{|\Omega|} \int_{\Omega} e^u dx \leq e^{\frac{\|\nabla u\|_N^N}{\widetilde{\omega}_N}} \frac{1}{|\Omega|} \int_{\Omega} e^{\alpha_N \left(\frac{|u|}{\|\nabla u\|_N} \right)^{\frac{N}{N-1}}} dx \leq e^{\frac{\|\nabla u\|_N^N}{\widetilde{\omega}_N}} C_N.$$

Passing to logarithm, we obtain

$$\ln \left(\frac{1}{|\Omega|} \int_{\Omega} e^u dx \right) \leq \frac{1}{\widetilde{\omega}_N} \int_{\Omega} |\nabla u|^N dx + \ln C_N.$$

If $u \equiv 0$, inequality (5.3.23) still holds. □

A refinement of inequality (5.3.23) for the case where Ω is the unit ball B_1 of $\mathbb{R}^N$ follows from the work of Carleson and Chang [42]. More precisely, they proved the following result [42, Lemma 1]:

Lemma 5.3.3. Let $\delta > 0$ and let

$$\mathcal{K}_{\delta} := \left\{ C^1 - \text{function } w \text{ defined in } 0 \leq t < \infty, w(0) = 0, w'(t) \geq 0, \text{ and } \int_0^{+\infty} w'(t)^N dt \leq \delta \right\}.$$

For any $c > 0$, we have

$$\sup_{w \in \mathcal{K}_{\delta}} \int_0^{+\infty} e^{cw(t)-t} dt < e^{\left(\frac{N-1}{N}\right)^{N-1} \left(\frac{c^N \delta}{N}\right)} e^{1+\frac{1}{2}+\dots+\frac{1}{N-1}}. \quad (5.3.24)$$

Moreover, when $c^N \delta \rightarrow +\infty$, the inequality in (5.3.24) tends asymptotically to an equality.

Proposition 5.3.4 (Logarithmic M-T inequality on the unit ball B_1 of $\mathbb{R}^N$).

For any $u \in W_0^{1,N}(B_1)$, we have

$$\ln \left(\frac{1}{V_N} \int_{B_1} e^u dx \right) < \frac{1}{\widetilde{\omega}_N} \int_{B_1} |\nabla u|^N dx + \sum_{k=1}^{N-1} \frac{1}{k}. \quad (5.3.25)$$

Proof. Let $u \in W_0^{1,N}(B_1)$. We may assume that $u \geq 0$ since we can replace u by $|u|$ without increasing the integral of the gradient. Also, by exploiting the density of the space $C^1(B_1)$ in $W_0^{1,N}(B_1)$, we may assume that u has compact support and is in $C^1(B_1)$.

We recall that the symmetric decreasing rearrangement of u is the unique right-continuous radially symmetric and decreasing function $u^* : B_1 \rightarrow [0, +\infty)$ such that

$$|\{x \in B_1 : u^*(x) > t\}| = |\{x \in B_1 : u(x) > t\}| \quad \text{for every } t \geq 0.$$

Among the properties of u^* , we recall (see [121] for details) that

$$\int_{B_1} |\nabla u^*|^N dx \leq \int_{B_1} |\nabla u|^N dx$$

and

$$\int_{B_1} e^{u^*} dx = \int_{B_1} e^u dx.$$

So if the thesis holds with u^* , then it will hold also for u .

Therefore let us assume $u \in C^1(B_1)$ be radially symmetric and decreasing.

Since $u(x)$ is a function that depends just on $|x|$, it is of the form

$$u(x) = g(|x|) \quad \text{for some } g : [0, 1] \rightarrow \mathbb{R} \text{ of class } C^1.$$

Let us now introduce the change of variable $t = -N \ln |x|$, by which $e^{-t} = |x|^N$.

Now, we denote by $\tilde{c} := N^{\frac{N-1}{N}} \omega_{N-1}^{\frac{1}{N}}$ and we set $w(t) := \tilde{c}u(x)$, by which

$$u(x) = \frac{w(t)}{\tilde{c}} = \frac{w(-N \ln |x|)}{\tilde{c}} \quad \text{for any } x \in B_1 \setminus \{O\}.$$

Observe that

$$w(+\infty) := \lim_{t \rightarrow +\infty} w(t) = \tilde{c}u(O) \in \mathbb{R}^+.$$

The function w is monotone increasing in t . Moreover,

$$\frac{1}{V_N} \int_{B_1} e^{u(x)} dx = \frac{N}{\omega_{N-1}} \int_{B_1} e^{\frac{w(-N \ln |x|)}{\tilde{c}}} dx = N \int_0^1 e^{\frac{w(-N \ln \rho)}{\tilde{c}}} \rho^{N-1} d\rho = \int_0^{+\infty} e^{\frac{w(t)}{\tilde{c}} - t} dt$$

and

$$\int_{B_1} |\nabla u(x)|^N dx = N \int_0^1 w'(-N \ln \rho)^N \frac{1}{\rho^N} \rho^{N-1} d\rho = \int_0^{+\infty} w'(t)^N dt.$$

By applying Lemma 5.3.3 with $c := 1/\tilde{c}$ and $\delta := \int_0^{+\infty} w'(t)^N dt = \int_{B_1} |\nabla u(x)|^N dx$, we get

$$\frac{1}{V_N} \int_{B_1} e^{u(x)} dx = \int_0^{+\infty} e^{c w(t) - t} dt < e^{\frac{1}{\omega_N} \int_{B_1} |\nabla u|^N dx} e^{1 + \frac{1}{2} + \dots + \frac{1}{N-1}},$$

and passing to logarithm we conclude that

$$\ln \left(\frac{1}{V_N} \int_{B_1} e^u dx \right) < \frac{1}{\overline{\omega}_N} \int_{B_1} |\nabla u|^N dx + \sum_{k=1}^{N-1} \frac{1}{k}.$$

□

Notice that by Lemma 5.3.3 we also deduce that inequality (5.3.25) is sharp, that is the functional $J : W_0^{1,N}(B_1) \rightarrow \mathbb{R}$ defined as

$$J(u) := \frac{1}{\overline{\omega}_N} \int_{B_1} |\nabla u|^N dx - \ln \left(\frac{1}{V_N} \int_{B_1} e^u dx \right),$$

satisfies

$$\inf_{W_0^{1,N}(B_1)} J = - \sum_{k=1}^{N-1} \frac{1}{k}.$$

Furthermore, as both the L^N -norm of the gradient and the integral average of the exponential are invariant under translation and dilation, inequality (5.3.25) holds for any ball in $\mathbb{R}^N$. While translation invariance is immediate, the invariance under dilation is verified below:

Lemma 5.3.5. *Let $r > 0$ and let us denote by $B_r := B_r(O) \subset \mathbb{R}^N$, where $N \geq 2$.*

If $u \in W_0^{1,N}(B_1)$ then $u_r \in W_0^{1,N}(B_r)$, where $u_r(x) := u\left(\frac{x}{r}\right)$. Moreover,

$$\frac{1}{|B_r|} \int_{B_r} e^{u_r(x)} dx = \frac{1}{|B_1|} \int_{B_1} e^{u(y)} dy \quad \text{and} \quad \int_{B_r} |\nabla u_r(x)|^N dx = \int_{B_1} |\nabla u(y)|^N dy.$$

Proof. Let us consider the following diffeomorphism:

$$\begin{aligned} B_r &\xrightarrow{\varphi} B_1 \\ x &\mapsto y = \varphi(x) := \frac{x}{r}. \end{aligned}$$

Note that

$$J_\varphi(x) = \begin{pmatrix} \frac{1}{r} & 0 & \cdots & 0 \\ 0 & \frac{1}{r} & 0 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & & & \frac{1}{r} \end{pmatrix} \quad |\det J_\varphi(x)| = \frac{1}{r^N}, \quad dy = \frac{1}{r^N} dx.$$

We have

$$\int_{B_r} |u_r(x)|^N dx = \int_{B_r} \left| u\left(\frac{x}{r}\right) \right|^N dx = r^N \int_{B_1} |u(y)|^N dy < +\infty,$$

and

$$\frac{1}{|B_r|} \int_{B_r} e^{u_r(x)} dx = \frac{1}{V_N r^N} \int_{B_r} e^{u\left(\frac{x}{r}\right)} dx = \frac{1}{|B_1|} \int_{B_1} e^{u(y)} dy.$$

Observe now that $\nabla u_r(x) = \nabla(u \circ \varphi)(x) = \nabla u(\varphi(x)) J_\varphi(x) = \frac{1}{r} \nabla u\left(\frac{x}{r}\right)$.

Hence

$$\int_{B_r} |\nabla u_r(x)|^N dx = \frac{1}{r^N} \int_{B_r} \left| \nabla u \left(\frac{x}{r} \right) \right|^N dx = \int_{B_1} |\nabla u(y)|^N dy$$

□

A generalization of inequality (5.3.25) for $N = 2$ was provided in [38] (see also [47]). Specifically, if $\Omega \subseteq \mathbb{R}^2$ is a bounded domain with smooth boundary, then for any $u \in H_0^1(\Omega)$ we have

$$\ln \left(\frac{1}{|\Omega|} \int_{\Omega} e^u dx \right) \leq \frac{1}{16\pi} \int_{\Omega} |\nabla u|^2 dx + 1 + 4\pi \sup_{\Omega} \gamma(x) + \ln \frac{\pi}{|\Omega|}, \quad (5.3.26)$$

where γ is Robin's function of Ω , which is defined for $x \in \Omega$ as $\gamma(x) = H_x(x)$, where H_x is the unique solution of

$$\begin{cases} \Delta H_x = 0 & \text{in } \Omega, \\ H_x(y) = \frac{1}{2\pi} \ln |x - y| & \text{on } \partial\Omega. \end{cases}$$

Differently from (5.3.25), (5.3.26) is not always sharp. Indeed, denoting by

$$F(\Omega) := \inf_{u \in H_0^1(\Omega)} \left(\frac{1}{16\pi} \int_{\Omega} |\nabla u|^2 dx - \ln \left(\frac{1}{|\Omega|} \int_{\Omega} e^u dx \right) \right),$$

there are domains Ω for which

$$F(\Omega) > -1 - 4\pi \sup_{\Omega} \gamma(x) - \ln \frac{\pi}{|\Omega|}. \quad (5.3.27)$$

The domains for which (5.3.26) is sharp are known in the literature as domains of first kind, while the ones for which (5.3.27) holds are known as domains of second kind. Inequality (5.3.26) plays a crucial role in the study of the mean field equation

$$-\Delta u = \frac{\rho e^u}{\int_{\Omega} e^u dx} \quad \text{in } \Omega, \quad (5.3.28)$$

which appears in the statistical mechanic description of vortex formations in 2d-models for turbulent flows in fluid dynamics (see e.g. [38, 111]). We refer to [20, 21, 51, 52, 123] about existence results for (5.3.28) and related problems.

5.4 Equivalence between Euclidean Onofri Inequality and Logarithmic Moser-Trudinger Inequality

This Section is devoted to the proof of Theorem 5.1.3. Let us recall that $J : W_0^{1,N}(B_1) \rightarrow \mathbb{R}$ and $I : W_{\mu_N}(\mathbb{R}^N) \rightarrow \mathbb{R}$ are the functionals defined in (5.1.5) and (5.1.14), as

$$J(u) := \frac{1}{\omega_N} \int_{B_1} |\nabla u|^N dx - \ln \left(\frac{1}{V_N} \int_{B_1} e^u dx \right),$$

and

$$I(u) := \frac{1}{\omega_N} \int_{\mathbb{R}^N} H_N(u, \mu_N) dx + \int_{\mathbb{R}^N} u d\mu_N - \ln \left(\int_{\mathbb{R}^N} e^u d\mu_N \right).$$

We will prove that

$$\inf_{W_{\mu_N}(\mathbb{R}^N)} I = \inf_{W_0^{1,N}(B_1)} J + \sum_{k=1}^{N-1} \frac{1}{k}.$$

Proof of Theorem 5.1.3. We begin with the proof of “ $\geq$ ”.

Let $u \in W_{\mu_N}(\mathbb{R}^N)$. Since $W_{\mu_N}(\mathbb{R}^N) = \overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|_{\mu_N}}$, we can assume $u \in C_0^\infty(\mathbb{R}^N)$ and the thesis will follow by density.

Let us consider the following cutoff function $\Psi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \Psi \leq 1$, $\Psi \equiv 1$ if $|x| \leq \frac{1}{2}$, and $\Psi \equiv 0$ if $|x| \geq 1$.

Let $r > 0$ and let us define $\Psi_r(x) := \Psi\left(\frac{x}{r}\right)$, so that $\Psi_r \equiv 1$ if $|x| \leq \frac{r}{2}$, and $\Psi_r \equiv 0$ if $|x| \geq r$.

Obviously, $\Psi_r(x) \rightarrow 1$ as $r \rightarrow +\infty$, for any $x \in \mathbb{R}^N$.

For simplicity of notation, let's say $B_r := B_r(O)$; moreover, we will denote by $o(1)$ any function that goes to zero as $r \rightarrow +\infty$.

Using the characteristic function χ , we can write

$$\int_{B_r} e^{u(x)\Psi_r(x)+v_N(x)} dx = \int_{\mathbb{R}^N} e^{u(x)\Psi_r(x)+v_N(x)} \chi_{B_r}(x) dx.$$

Now we observe that $|u|$ is bounded by a constant M_u , so for any $x \in \mathbb{R}^N$ and for any $r > 0$,

$$\left| e^{u(x)\Psi_r(x)+v_N(x)} \chi_{B_r}(x) \right| \leq e^{M_u+v_N(x)} = e^{M_u} \mu_N(x) \in L^1(\mathbb{R}^N).$$

Moreover, for any $x \in \mathbb{R}^N$,

$$e^{u(x)\Psi_r(x)+v_N(x)} \chi_{B_r}(x) \rightarrow e^{u(x)+v_N(x)} \quad \text{as } r \rightarrow +\infty.$$

Therefore we can apply dominated convergence theorem and say that, as $r \rightarrow +\infty$,

$$\int_{B_r} e^{u(x)\Psi_r(x)+v_N(x)} dx \rightarrow \int_{\mathbb{R}^N} e^{u(x)+v_N(x)} dx = \int_{\mathbb{R}^N} e^u d\mu_N.$$

We can write

$$\ln \left(\int_{\mathbb{R}^N} e^u d\mu_N \right) = \lim_{r \rightarrow +\infty} \ln \left(\int_{B_r} e^{u(x)\Psi_r(x)+v_N(x)} dx \right),$$

therefore we can estimate

$$\ln \left(\int_{B_r} e^{u(x)\Psi_r(x)+v_N(x)} dx \right).$$

In general the function $u\Psi_r + v_N \notin W_0^{1,N}(B_r)$.

If we define $w := u\Psi_r + v_N - v_N(r)$, (where with $v_N(r)$ we make a slight abuse of notation to denote a constant function on B_r whose value is $v_N(x)$ as $|x| = r$), we have $w \in W_0^{1,N}(B_r)$.

$$\begin{aligned} \ln \left(\int_{B_r} e^{u(x)\Psi_r(x)+v_N(x)} dx \right) &= \ln \left(\int_{B_r} e^{w(x)+v_N(r)} dx \right) \\ &= \ln \left(\mu_N(r) |B_r| \frac{1}{|B_r|} \int_{B_r} e^{w(x)} dx \right) \end{aligned}$$

$$\begin{aligned}
&= \ln \left(\frac{1}{V_N \left(1 + r^{\frac{N}{N-1}}\right)^N} V_N r^N \right) + \ln \left(\frac{1}{|B_r|} \int_{B_r} e^{w(x)} dx \right) \\
&= N \ln \left(\frac{r}{1 + r^{\frac{N}{N-1}}} \right) + \ln \left(\frac{1}{|B_r|} \int_{B_r} e^{w(x)} dx \right) \\
&= -\frac{N}{N-1} \ln r - N \ln \left(1 + \frac{1}{r^{\frac{N}{N-1}}} \right) + \ln \left(\frac{1}{|B_r|} \int_{B_r} e^{w(x)} dx \right) \\
&= -\frac{N}{N-1} \ln r + \ln \left(\frac{1}{|B_r|} \int_{B_r} e^{w(x)} dx \right) + o(1).
\end{aligned}$$

Since $w \in W_0^{1,N}(B_r)$, we have

$$\inf_{W_0^{1,N}(B_1)} J < \frac{1}{\widetilde{\omega_N}} \int_{B_r} |\nabla w|^N dx - \ln \left(\frac{1}{|B_r|} \int_{B_r} e^{w(x)} dx \right).$$

Therefore

$$\ln \left(\int_{B_r} e^{u\Psi_r} d\mu_N \right) < \frac{1}{\widetilde{\omega_N}} \int_{B_r} |\nabla w|^N dx - \frac{N}{N-1} \ln r - \inf_{W_0^{1,N}(B_1)} J + o(1).$$

We need to compute $\frac{1}{\widetilde{\omega_N}} \int_{B_r} |\nabla w|^N dx$.

Since

$$\nabla w = \nabla(u\Psi_r) + \nabla v_N$$

and

$$H_N(u\Psi_r, \mu_N) = |\nabla w|^N - |\nabla v_N|^N - N|\nabla v_N|^{N-2} \nabla v_N \cdot \nabla(u\Psi_r),$$

we have

$$|\nabla w|^N = H_N(u\Psi_r, \mu_N) + |\nabla v_N|^N + N|\nabla v_N|^{N-2} \nabla v_N \cdot \nabla(u\Psi_r),$$

by which

$$\begin{aligned}
\int_{B_r} |\nabla w|^N dx &= \int_{B_r} H_N(u\Psi_r, \mu_N) dx \\
&\quad + \int_{B_r} |\nabla v_N|^N dx + \int_{B_r} |\nabla v_N|^{N-2} \nabla v_N \cdot \nabla(u\Psi_r) dx.
\end{aligned}$$

By Green's formula and taking into account (5.2.16), we get

$$\begin{aligned}
\frac{N}{\widetilde{\omega_N}} \int_{B_r} |\nabla v_N|^{N-2} \nabla v_N \cdot \nabla(u\Psi_r) dx &= \frac{N}{\widetilde{\omega_N}} \left(\int_{\partial B_r} u\Psi_r |\nabla v_N|^{N-2} \nabla v_N \cdot \nu d\sigma - \int_{B_r} u\Psi_r (\Delta_N v_N) dx \right) \\
&= \frac{N}{\widetilde{\omega_N}} \int_{B_r} u\Psi_r (-\Delta_N v_N) dx
\end{aligned}$$

$$\begin{aligned}
 &= \frac{N^{N+1} \left(\frac{N}{N-1}\right)^{N-1} V_N}{\widetilde{\omega_N}} \int_{B_r} u \Psi_r \mu_N dx \\
 &= \int_{B_r} u \Psi_r \mu_N dx.
 \end{aligned}$$

Now, we need to compute $\frac{1}{\widetilde{\omega_N}} \int_{B_r} |\nabla v_N|^N dx$.

$$\begin{aligned}
 \frac{1}{\widetilde{\omega_N}} \int_{B_r} |\nabla v_N|^N dx &= \frac{N^{2N}}{(N-1)^N \widetilde{\omega_N}} \int_{B_r} \frac{|x|^{\frac{N}{N-1}}}{\left(1 + |x|^{\frac{N}{N-1}}\right)^N} dx \\
 &= \frac{N^{2N} \omega_{N-1}}{(N-1)^N \widetilde{\omega_N}} \int_0^r \frac{\rho^{\frac{N}{N-1}}}{\left(1 + \rho^{\frac{N}{N-1}}\right)^N} \rho^{N-1} d\rho \\
 &= \frac{N}{N-1} \int_0^r \frac{\rho^N}{\left(1 + \rho^{\frac{N}{N-1}}\right)^N} \rho^{\frac{1}{N-1}} d\rho \\
 &= \int_0^{r^{\frac{N}{N-1}}} \frac{t^{N-1}}{(1+t)^N} dt \\
 &= \int_0^{r^{\frac{N}{N-1}}} \frac{1}{1+t} dt - \sum_{k=0}^{N-2} \binom{N-1}{k} \int_0^{r^{\frac{N}{N-1}}} \frac{t^k}{(1+t)^N} dt \\
 &= \frac{N}{N-1} \ln r - \sum_{k=0}^{N-2} \binom{N-1}{k} \int_0^{r^{\frac{N}{N-1}}} \frac{t^k}{(1+t)^N} dt + o(1).
 \end{aligned} \tag{5.4.29}$$

By induction, we prove that

$$\binom{N-1}{k} \int_0^{r^{\frac{N}{N-1}}} \frac{t^k}{(1+t)^N} dt = \frac{1}{N-k-1} + o(1) \tag{5.4.30}$$

for any $k = 0, \dots, N-2$.

If $k = 0$, we have

$$\binom{N-1}{0} \int_0^{r^{\frac{N}{N-1}}} \frac{t^0}{(1+t)^N} dt = \int_0^{r^{\frac{N}{N-1}}} \frac{1}{(1+t)^N} dt = \frac{1}{1-N} \left[\frac{1}{(1+t)^{N-1}} \right]_0^{r^{\frac{N}{N-1}}} = \frac{1}{N-1} + o(1).$$

Now, let $N \geq 3$, $k \geq 1$, and let assume (5.4.30) to hold for $k-1$.

$$\begin{aligned}
 &\binom{N-1}{k} \int_0^{r^{\frac{N}{N-1}}} \frac{t^k}{(1+t)^N} dt \\
 &= \binom{N-1}{k} \left[\int_0^{r^{\frac{N}{N-1}}} t^k (1+t)^{-N} dt \right] \\
 &= \binom{N-1}{k} \left[\left[t^k \frac{(1+t)^{1-N}}{1-N} \right]_0^{r^{\frac{N}{N-1}}} - \frac{k}{1-N} \int_0^{r^{\frac{N}{N-1}}} t^{k-1} (1+t)^{1-N} dt \right]
 \end{aligned}$$

$$\begin{aligned}
&= \binom{N-1}{k} \left[o(1) + \frac{k}{N-1} \int_0^{r^{\frac{N}{N-1}}} \frac{t^{k-1}(1+t)}{(1+t)^N} dt \right] \\
&= \binom{N-1}{k} \left[o(1) + \frac{k}{\binom{N-1}{k-1}} \binom{N-1}{k-1} \int_0^{r^{\frac{N}{N-1}}} \frac{t^{k-1}}{(1+t)^N} dt + \frac{k}{N-1} \int_0^{r^{\frac{N}{N-1}}} \frac{t^k}{(1+t)^N} dt \right] \\
&= \binom{N-1}{k} \left[o(1) + \frac{k}{\binom{N-1}{k-1}} \left(\frac{1}{N-k} + o(1) \right) + \frac{k}{N-1} \int_0^{r^{\frac{N}{N-1}}} \frac{t^k}{(1+t)^N} dt \right] \\
&= \frac{\binom{N-1}{k}}{\binom{N-1}{k-1}} \frac{k}{(N-1)(N-k)} + \frac{k}{N-1} \binom{N-1}{k} \int_0^{r^{\frac{N}{N-1}}} \frac{t^k}{(1+t)^N} dt + o(1) \\
&= \frac{1}{N-1} + \frac{k}{N-1} \binom{N-1}{k} \int_0^{r^{\frac{N}{N-1}}} \frac{t^k}{(1+t)^N} dt + o(1).
\end{aligned}$$

Therefore

$$\frac{N-k-1}{N-1} \binom{N-1}{k} \int_0^{r^{\frac{N}{N-1}}} \frac{t^k}{(1+t)^N} dt = \frac{1}{N-1} + o(1),$$

by which

$$\binom{N-1}{k} \int_0^{r^{\frac{N}{N-1}}} \frac{t^k}{(1+t)^N} dt = \frac{1}{N-k-1} + o(1).$$

By (5.4.30) we deduce that

$$\sum_{k=0}^{N-2} \binom{N-1}{k} \int_0^{r^{\frac{N}{N-1}}} \frac{t^k}{(1+t)^N} dt = \sum_{k=0}^{N-2} \frac{1}{N-k-1} + o(1) = \sum_{k=1}^{N-1} \frac{1}{k} + o(1). \quad (5.4.31)$$

Finally,

$$\frac{1}{\widetilde{w_N}} \int_{B_r} |\nabla v_N|^N dx = \frac{N}{N-1} \ln r - \sum_{k=1}^{N-1} \frac{1}{k} + o(1). \quad (5.4.32)$$

Summing up, we have shown that

$$\inf_{W_0^{1,N}(B_1)} J + \sum_{k=1}^{N-1} \frac{1}{k} < I(u\Psi_r) + o(1).$$

Now, we pass to the limit as $r \rightarrow +\infty$.

We observe that $|u|$ is bounded by a constant M_u , so for any $x \in \mathbb{R}^N$ and for any $r > 0$,

$$|u(x)\Psi_r(x)\mu_N(x)| \leq M_u \mu_N(x) \in L^1(\mathbb{R}^N).$$

Moreover, for any $x \in \mathbb{R}^N$,

$$u(x)\Psi_r(x)\mu_N(x) \rightarrow u(x)\mu_N(x) \quad \text{as } r \rightarrow +\infty.$$

We can apply dominated convergence theorem to say that

$$\int_{\mathbb{R}^N} u \Psi_r d\mu_N \rightarrow \int_{\mathbb{R}^N} u d\mu_N.$$

Now we observe that, for any $x \in \mathbb{R}^N$,

$$H_N(u(x)\Psi_r(x), \mu_N(x)) \rightarrow H_N(u(x), \mu_N(x)) \quad \text{as } r \rightarrow +\infty.$$

Moreover, considering Lemma 5.2.1 and $u \in C_0^\infty(\mathbb{R}^N)$, there exists a function $v \in L^1(\mathbb{R}^N, dx)$ such that, for any $x \in \mathbb{R}^N$ and for any $r > 1$, we have

$$0 \leq H_N(u(x)\Psi_r(x), \mu_N(x)) \leq v(x).$$

In fact

$$\begin{aligned} 0 &\leq H_N(u(x)\Psi_r(x), \mu_N(x)) \\ &\leq \frac{c_N}{2} \left(|\nabla(u(x)\Psi_r(x))|^N + |\nabla(u(x)\Psi_r(x))|^2 |\nabla v_N(x)|^{N-2} \right) \\ &\leq C \left((|u(x)| + |\nabla u(x)|)^N + (|u(x)| + |\nabla u(x)|)^2 |\nabla v_N(x)|^{N-2} \right). \end{aligned}$$

Since $u \in C_0^\infty(\mathbb{R}^N)$, we have that

$$v(x) := C \left((|u(x)| + |\nabla u(x)|)^N + (|u(x)| + |\nabla u(x)|)^2 |\nabla v_N(x)|^{N-2} \right) \in L^1(\mathbb{R}^N, dx).$$

Therefore we can apply dominated convergence theorem and say that

$$\frac{1}{\omega_N} \int_{\mathbb{R}^N} H_N(u\Psi_r, \mu_N) dx \rightarrow \frac{1}{\omega_N} \int_{\mathbb{R}^N} H_N(u, \mu_N) dx.$$

Finally,

$$\inf_{W_0^{1,N}(B_1)} J + \sum_{k=1}^{N-1} \frac{1}{k} \leq I(u),$$

by which

$$\inf_{W_0^{1,N}(B_1)} J + \sum_{k=1}^{N-1} \frac{1}{k} \leq \inf_{C_0^\infty(\mathbb{R}^N)} I.$$

By density, that is by the proof of Theorem 5.1.2, we obtain

$$\inf_{W_0^{1,N}(B_1)} J + \sum_{k=1}^{N-1} \frac{1}{k} \leq \inf_{W_{\mu_N}(\mathbb{R}^N)} I.$$

Now, we prove “ $\leq$ ”.

Let $u \in W_0^{1,N}(B_1)$ and let $r > 0$. We define the function $u_r : \mathbb{R}^N \rightarrow \mathbb{R}$ as

$$u_r(x) = \begin{cases} u\left(\frac{x}{r}\right) - v_N(x) + v_N(r) & \text{if } |x| \leq r, \\ 0 & \text{if } |x| > r. \end{cases}$$

By Lemma 5.2.3, we have $u_r \in W_{\mu_N}(\mathbb{R}^N)$, by which

$$\inf_{W_{\mu_N}(\mathbb{R}^N)} I \leq I(u_r),$$

where

$$I(u_r) = \frac{1}{\omega_N} \int_{\mathbb{R}^N} H_N(u_r, \mu_N) dx + \int_{\mathbb{R}^N} u_r d\mu_N - \ln \left(\int_{\mathbb{R}^N} e^{u_r} d\mu_N \right). \quad (5.4.33)$$

We need to compute the terms in (5.4.33). Firstly, we observe that

$$\ln \left(\int_{\mathbb{R}^N} e^{u_r} d\mu_N \right) = \ln \left(\mu_N(r) \int_{B_r} e^{u(\frac{x}{r})} dx + \int_{\mathbb{R}^N \setminus B_r} \mu_N(x) dx \right).$$

In particular,

$$\begin{aligned} \mu_N(r) \int_{B_r} e^{u(\frac{x}{r})} dx &= \frac{1}{V_N \left(1 + r^{\frac{N}{N-1}}\right)^N} \int_{B_r} e^{u(\frac{x}{r})} dx \\ &= \frac{r^N}{\left(1 + r^{\frac{N}{N-1}}\right)^N} \frac{1}{V_N} \int_{B_r} e^{u(\frac{x}{r})} \frac{1}{r^N} dx \\ &= \left(\frac{r}{1 + r^{\frac{N}{N-1}}} \right)^N \frac{1}{V_N} \int_{B_1} e^{u(y)} dy; \end{aligned}$$

and

$$\int_{\mathbb{R}^N \setminus B_r} \mu_N(x) dx = \int_{\mathbb{R}^N \setminus B_r} \frac{1}{V_N \left(1 + |x|^{\frac{N}{N-1}}\right)^N} dx = \left[\left(\frac{t}{1+t} \right)^{N-1} \right]_{r^{\frac{N}{N-1}}}^{+\infty} = 1 - \frac{r^N}{\left(1 + r^{\frac{N}{N-1}}\right)^{N-1}}.$$

Therefore

$$\begin{aligned} \ln \left(\int_{\mathbb{R}^N} e^{u_r} d\mu_N \right) &= \ln \left(\int_{B_r} e^{u(\frac{x}{r})} \mu_N(r) dx + \int_{\mathbb{R}^N \setminus B_r} \mu_N(x) dx \right) \\ &= \ln \left(\left(\frac{r}{1 + r^{\frac{N}{N-1}}} \right)^N \frac{1}{V_N} \int_{B_1} e^u dx + 1 - \frac{r^N}{\left(1 + r^{\frac{N}{N-1}}\right)^{N-1}} \right) \\ &= \ln \left(\left(\frac{r}{1 + r^{\frac{N}{N-1}}} \right)^N \left[\frac{1}{V_N} \int_{B_1} e^u dx + \left(\frac{1 + r^{\frac{N}{N-1}}}{r} \right)^N - \left(1 + r^{\frac{N}{N-1}}\right) \right] \right) \\ &= N \ln \left(\frac{r}{1 + r^{\frac{N}{N-1}}} \right) + \ln \left(\frac{1}{V_N} \int_{B_1} e^u dx + \frac{(N-1)r^N + \sum_{k=0}^{N-2} \binom{N}{k} r^{\frac{Nk}{N-1}}}{r^N} \right) \end{aligned}$$

$$= -\frac{N}{N-1} \ln r + \ln \left(\frac{1}{V_N} \int_{B_1} e^u dx + N - 1 + o(1) \right) + o(1).$$

Now, we compute $\frac{1}{\widetilde{\omega_N}} \int_{\mathbb{R}^N} H_N(u_r, \mu_N) dx$.

Since

$$H_N(u, \mu_N) = |\nabla u + \nabla v_N|^N - |\nabla v_N|^N - N|\nabla v_N|^{N-2} \nabla v_N \cdot \nabla u$$

and

$$\nabla u_r(x) = \begin{cases} \frac{1}{r} \nabla_y u \left(\frac{x}{r} \right) - \nabla v_N(x) & \text{if } |x| \leq r, \\ 0 & \text{if } |x| > r, \end{cases}$$

we have

$$H_N(u_r, \mu_N) = \begin{cases} \frac{1}{r^N} \left| \nabla_y u \left(\frac{x}{r} \right) \right|^N + (N-1) |\nabla v_N|^N - \frac{N}{r} |\nabla v_N|^{N-2} \nabla v_N \cdot \nabla_y u \left(\frac{x}{r} \right) & \text{if } |x| \leq r, \\ 0 & \text{if } |x| > r. \end{cases}$$

Therefore

$$\begin{aligned} & \frac{1}{\widetilde{\omega_N}} \int_{\mathbb{R}^N} H_N(u_r, \mu_N) dx \\ &= \frac{1}{\widetilde{\omega_N}} \int_{B_r} \left| \nabla_y u \left(\frac{x}{r} \right) \right|^N \frac{1}{r^N} dx + \frac{N-1}{\widetilde{\omega_N}} \int_{B_r} |\nabla v_N|^N dx - \frac{N}{\widetilde{\omega_N} r} \int_{B_r} |\nabla v_N|^{N-2} \nabla v_N \cdot \nabla_y u \left(\frac{x}{r} \right) dx \\ &= \frac{1}{\widetilde{\omega_N}} \int_{B_1} |\nabla u|^N dy + \frac{N-1}{\widetilde{\omega_N}} \int_{B_r} |\nabla v_N|^N dx - \frac{N}{\widetilde{\omega_N}} \int_{B_r} |\nabla v_N|^{N-2} \nabla v_N \cdot \nabla_x u \left(\frac{x}{r} \right) dx. \end{aligned}$$

Considering (5.4.32), we have

$$\frac{N-1}{\widetilde{\omega_N}} \int_{B_r} |\nabla v_N|^N dx = N \ln r - (N-1) \sum_{k=1}^{N-1} \frac{1}{k} + o(1).$$

Moreover, by Green's formula and taking into account (5.2.16), we obtain

$$\begin{aligned} & -\frac{N}{\widetilde{\omega_N}} \int_{B_r} |\nabla v_N|^{N-2} \nabla v_N \cdot \nabla_x u \left(\frac{x}{r} \right) dx \\ &= -\frac{N}{\widetilde{\omega_N}} \left(\int_{\partial B_r} u \left(\frac{x}{r} \right) |\nabla v_N|^{N-2} \nabla v_N \cdot \nu d\sigma - \int_{B_r} u \left(\frac{x}{r} \right) (\Delta_N v_N) dx \right) \\ &= \frac{N}{\widetilde{\omega_N}} \int_{B_r} u \left(\frac{x}{r} \right) (\Delta_N v_N) dx \\ &= -\frac{N^{N+1} \left(\frac{N}{N-1} \right)^{N-1} V_N}{\widetilde{\omega_N}} \int_{B_r} u \left(\frac{x}{r} \right) \mu_N dx \end{aligned}$$

$$= - \int_{B_r} u \left(\frac{x}{r} \right) d\mu_N.$$

Therefore

$$\begin{aligned} & \frac{1}{\omega_N} \int_{\mathbb{R}^N} H_N(u_r, \mu_N) dx \\ &= \frac{1}{\omega_N} \int_{B_1} |\nabla u|^N dx - \int_{B_r} u \left(\frac{x}{r} \right) d\mu_N + N \ln r - (N-1) \sum_{k=1}^{N-1} \frac{1}{k} + o(1). \end{aligned}$$

It remains to compute the term $\int_{\mathbb{R}^N} u_r d\mu_N$ in (5.4.33).

$$\int_{\mathbb{R}^N} u_r d\mu_N = \int_{B_r} u \left(\frac{x}{r} \right) d\mu_N - \int_{B_r} v_N(x) d\mu_N + v_N(r) \int_{B_r} d\mu_N.$$

In particular,

$$\begin{aligned} v_N(r) \int_{B_r} d\mu_N &= \ln \left(\frac{1}{V_N \left(1 + r^{\frac{N}{N-1}} \right)^N} \right) \left[\left(\frac{t}{1+t} \right)^{N-1} \right]_0^{r^{\frac{N}{N-1}}} \\ &= - \frac{r^N \ln V_N}{\left(1 + r^{\frac{N}{N-1}} \right)^{N-1}} - N \frac{r^N \ln \left(1 + r^{\frac{N}{N-1}} \right)}{\left(1 + r^{\frac{N}{N-1}} \right)^{N-1}} \\ &= - \ln V_N - \frac{N^2}{N-1} \ln r + o(1), \end{aligned}$$

and

$$\begin{aligned} - \int_{B_r} v_N(x) d\mu_N &= - \int_{B_r} \frac{1}{V_N \left(1 + |x|^{\frac{N}{N-1}} \right)^N} \ln \left(\frac{1}{V_N \left(1 + |x|^{\frac{N}{N-1}} \right)^N} \right) dx \\ &= \frac{\omega_{N-1}}{V_N} \int_0^r \frac{\ln \left(V_N \left(1 + \rho^{\frac{N}{N-1}} \right)^N \right)}{\left(1 + \rho^{\frac{N}{N-1}} \right)^N} \rho^{N-1} d\rho \\ &= N \ln V_N \int_0^r \frac{\rho^{N-1}}{\left(1 + \rho^{\frac{N}{N-1}} \right)^N} d\rho + N^2 \int_0^r \frac{\rho^{N-1} \ln \left(1 + \rho^{\frac{N}{N-1}} \right)}{\left(1 + \rho^{\frac{N}{N-1}} \right)^N} d\rho \\ &= \ln V_N + N^2 \int_0^r \frac{\rho^{N-1} \ln \left(1 + \rho^{\frac{N}{N-1}} \right)}{\left(1 + \rho^{\frac{N}{N-1}} \right)^N} d\rho + o(1). \end{aligned}$$

Considering (5.4.29) and (5.4.31), we have

$$N^2 \int_0^r \frac{\rho^{N-1} \ln \left(1 + \rho^{\frac{N}{N-1}} \right)}{\left(1 + \rho^{\frac{N}{N-1}} \right)^N} d\rho$$

$$\begin{aligned}
&= N(N-1) \int_0^{r^{\frac{N}{N-1}}} \frac{t^{N-2} \ln(1+t)}{(1+t)^N} dt \\
&= N(N-1) \int_0^{r^{\frac{N}{N-1}}} \frac{1}{(1+t)^2} \left(\frac{t}{1+t} \right)^{N-2} \ln(1+t) dt \\
&= N \left(\left[\left(\frac{t}{1+t} \right)^{N-1} \ln(1+t) \right]_0^{r^{\frac{N}{N-1}}} - \int_0^{r^{\frac{N}{N-1}}} \frac{t^{N-1}}{(1+t)^N} dt \right) \\
&= N \left(\frac{r^N - \left(1 + r^{\frac{N}{N-1}}\right)^{N-1}}{\left(1 + r^{\frac{N}{N-1}}\right)^{N-1}} \ln\left(1 + r^{\frac{N}{N-1}}\right) + \sum_{k=1}^{N-1} \frac{1}{k} + o(1) \right) \\
&= N \sum_{k=1}^{N-1} \frac{1}{k} + o(1).
\end{aligned}$$

Therefore

$$\int_{\mathbb{R}^N} u_r d\mu_N = \int_{B_r} u \left(\frac{x}{r} \right) d\mu_N - \frac{N^2}{N-1} \ln r + N \sum_{k=1}^{N-1} \frac{1}{k} + o(1).$$

Summing up, we have shown that

$$\inf_{W_{\mu_N}(\mathbb{R}^N)} I \leq \frac{1}{\widetilde{\omega_N}} \int_{B_1} |\nabla u|^N dx + \sum_{k=1}^{N-1} \frac{1}{k} - \ln \left(\frac{1}{V_N} \int_{B_1} e^u dx + N - 1 + o(1) \right) + o(1).$$

Passing to limit as $r \rightarrow +\infty$, we obtain

$$\inf_{W_{\mu_N}(\mathbb{R}^N)} I \leq \frac{1}{\widetilde{\omega_N}} \int_{B_1} |\nabla u|^N dx + \sum_{k=1}^{N-1} \frac{1}{k} - \ln \left(\frac{1}{V_N} \int_{B_1} e^u dx + N - 1 \right),$$

by which

$$\inf_{W_{\mu_N}(\mathbb{R}^N)} I < J(u) + \sum_{k=1}^{N-1} \frac{1}{k}.$$

Finally,

$$\inf_{W_{\mu_N}(\mathbb{R}^N)} I \leq \inf_{W_0^{1,N}(B_1)} J + \sum_{k=1}^{N-1} \frac{1}{k}.$$

□

5.5 Insight on $N = 2$: connection with $\mathbb{S}^2$.

Among the reasons that led Moser to improve Trudinger's result, it is important to mention Nirenberg's problem on $\mathbb{S}^2$, which consists in describing the set of smooth functions that can be obtained as Gaussian curvature of a Riemannian metric conformally equivalent to the standard spherical metric.

Let Σ be a smooth surface, we recall that two Riemannian metrics g_0 and g on Σ are said to be pointwise conformally equivalent if there exists $u \in C^\infty(\Sigma)$ such that $g = e^u g_0$ on Σ . In this setting, the following identity holds:

$$\Delta_{g_0} u + 2K_g e^u = 2K_{g_0},$$

where Δ_{g_0} denotes the Laplace-Beltrami operator on Σ , and K_g and K_{g_0} are the Gaussian curvatures of Σ corresponding respectively to g and g_0 . When $\Sigma = \mathbb{S}^2$ and g_0 is the standard spherical metric, then a function $K \in C^\infty(\mathbb{S}^2)$ is the Gaussian curvature of a metric which is pointwise conformally equivalent to g_0 if and only if there exists $u \in C^\infty(\mathbb{S}^2)$ satisfying

$$\Delta_{g_0} u + 2K e^u = 2 \quad \text{on} \quad (\mathbb{S}^2, g_0). \quad (5.5.34)$$

It is possible to give a variational formulation of (5.5.34) by considering the functional

$$\Phi(u) = \frac{1}{2} \int_{\mathbb{S}^2} |\nabla_{g_0} u|_{g_0}^2 dv_{g_0} + 2 \int_{\mathbb{S}^2} u dv_{g_0} - 8\pi \ln \left(\frac{1}{4\pi} \int_{\mathbb{S}^2} K e^u dv_{g_0} \right),$$

which is well defined in

$$\mathcal{A} =: \left\{ u \in W^{1,2}(\mathbb{S}^2, g_0) \mid \int_{\mathbb{S}^2} K e^u dv_{g_0} > 0 \right\},$$

provided $\max_{\mathbb{S}^2} K > 0$. A function $u \in W^{1,2}(\mathbb{S}^2, g_0)$ is a weak solution to (5.5.34) if and only if it is a critical point of Φ and

$$\int_{\mathbb{S}^2} K e^u dv_{g_0} = 4\pi.$$

As an analog of Theorem 5.3.1 for the Sobolev space $W^{1,2}(\mathbb{S}^2, g_0)$, Moser in [132] proved the following

Theorem 5.5.1 (Moser-Trudinger inequality on $(\mathbb{S}^2, g_0)$).

Let $(\mathbb{S}^2, g_0)$ be the two-dimensional unit sphere endowed with standard metric g_0 .

Let us denote with $\bar{u} := \frac{1}{4\pi} \int_{\mathbb{S}^2} u dv_{g_0}$ and with $\bar{\mathcal{H}} := \{u \in W^{1,2}(\mathbb{S}^2, g_0) \mid \bar{u} = 0, \|\nabla_{g_0} u\|_2 \leq 1\}$.

Then there exists a constant $S > 1$ such that

$$\sup_{u \in \bar{\mathcal{H}}} \frac{1}{4\pi} \int_{\mathbb{S}^2} e^{4\pi u^2} dv_{g_0} \leq S. \quad (5.5.35)$$

As a straightforward consequence of previous result and Young's inequality, one can prove the following weaker inequality:

Proposition 5.5.2 (Logarithmic M-T inequality on $(\mathbb{S}^2, g_0)$).

Let S be the constant appearing in (5.5.35). For any $u \in W^{1,2}(\mathbb{S}^2, g_0)$, we have

$$\ln \left(\frac{1}{4\pi} \int_{\mathbb{S}^2} e^u dv_{g_0} \right) \leq \frac{1}{16\pi} \int_{\mathbb{S}^2} |\nabla_{g_0} u|_{g_0}^2 dv_{g_0} + \frac{1}{4\pi} \int_{\mathbb{S}^2} u dv_{g_0} + \ln S. \quad (5.5.36)$$

Proof. Let $u \in W^{1,2}(\mathbb{S}^2, g_0)$, $\|\nabla_{g_0} u\|_2 \neq 0$, and let $b^2 = \frac{8\pi}{\|\nabla_{g_0} u\|_2^2}$.

By Young's inequality, we infer that

$$u \leq |u| \leq \frac{1}{2b^2} + \frac{1}{2}b^2u^2 = \frac{1}{16\pi}\|\nabla_{g_0}u\|_2^2 + 4\pi\frac{u^2}{\|\nabla_{g_0}u\|_2^2},$$

by which

$$\frac{1}{4\pi}\int_{\mathbb{S}^2} e^u d\nu_{g_0} \leq e^{\frac{1}{16\pi}\|\nabla_{g_0}u\|_2^2} \frac{1}{4\pi}\int_{\mathbb{S}^2} e^{4\pi\left(\frac{u}{\|\nabla_{g_0}u\|_2}\right)^2} d\nu_{g_0}.$$

Applying previous inequality to $u - \bar{u}$ and considering also (5.5.35), we obtain

$$\frac{1}{4\pi}\int_{\mathbb{S}^2} e^{u-\bar{u}} d\nu_{g_0} \leq e^{\frac{1}{16\pi}\|\nabla_{g_0}u\|_2^2} \frac{1}{4\pi}\int_{\mathbb{S}^2} e^{4\pi\left(\frac{u-\bar{u}}{\|\nabla_{g_0}u\|_2}\right)^2} d\nu_{g_0} \leq e^{\frac{1}{16\pi}\|\nabla_{g_0}u\|_2^2} S.$$

Passing to logarithm, we obtain

$$\ln\left(\frac{1}{4\pi}\int_{\mathbb{S}^2} e^u d\nu_{g_0}\right) \leq \frac{1}{16\pi}\int_{\mathbb{S}^2} |\nabla_{g_0}u|_{g_0}^2 d\nu_{g_0} + \frac{1}{4\pi}\int_{\mathbb{S}^2} u d\nu_{g_0} + \ln S.$$

If $u \equiv c$ for any $c \in \mathbb{R}$, inequality (5.5.36) still holds. □

In particular, (5.5.36) shows that Φ is bounded from below in $\mathcal{A}$. While this is not sufficient to apply standard minimization techniques, the existence of critical points for Φ can be obtained via improved versions of (5.5.36) under suitable assumptions on K . For example, in [133] Moser proves that if $u \in W^{1,2}(\mathbb{S}^2, g_0)$ is even, then (5.5.36) can be replaced by the stronger inequality:

$$\ln\left(\frac{1}{4\pi}\int_{\mathbb{S}^2} e^u d\nu_{g_0}\right) \leq \frac{1}{32\pi}\int_{\mathbb{S}^2} |\nabla_{g_0}u|_{g_0}^2 d\nu_{g_0} + \frac{1}{4\pi}\int_{\mathbb{S}^2} u d\nu_{g_0} + C, \quad (5.5.37)$$

for some $C \in \mathbb{R}$. In particular, if $K \in C^\infty(\mathbb{S}^2)$ is even, the restriction of Φ to the space of even functions in $W^{1,2}(\mathbb{S}^2, g_0)$ with zero average is coercive, and one can find a critical point of Φ by means of Palais symmetric criticality principle. This leads to one of first general existence results for the Nirenberg problem: if $K \in C^\infty(\mathbb{S}^2)$ is an even function and $\max_{\mathbb{S}^2} K > 0$, then there exists a solution of (5.5.34). We refer to [24, 49, 50, 113, 152] and references therein for further results concerning Nirenberg's problem on $\mathbb{S}^2$.

Another interesting aspect of (5.5.36) is that while the coefficient $\frac{1}{16\pi}$ is sharp, the constant $\ln S$ is not. Indeed, Onofri in [134] proved that

$$\inf_{W^{1,2}(\mathbb{S}^2, g_0)} \Phi = 0$$

and he classified all minimum points of Φ . Namely, we have the so called *logarithmic Onofri-Moser-Trudinger inequality* or *Onofri inequality*

$$\ln\left(\frac{1}{4\pi}\int_{\mathbb{S}^2} e^u d\nu_{g_0}\right) \leq \frac{1}{16\pi}\int_{\mathbb{S}^2} |\nabla_{g_0}u|_{g_0}^2 d\nu_{g_0} + \frac{1}{4\pi}\int_{\mathbb{S}^2} u d\nu_{g_0}, \quad (5.5.38)$$

for any $u \in W^{1,2}(\mathbb{S}^2, g_0)$, and equality holds if and only if $u = \ln |J_\psi| + C$, where $\psi : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ is a conformal diffeomorphism of $(\mathbb{S}^2, g_0)$, J_ψ denotes the Jacobian of ψ , and $C \in \mathbb{R}$. Inequality (5.5.38) was proved in 1982 by Onofri in [134] using conformal invariance and an earlier result by Aubin [19]. Indeed, inequality (5.5.38) can be obtained via several methods, including various limiting procedures, mass transportation methods in the radial case, and rigidity methods associated with a related nonlinear flow (we refer to [88, 89] and references therein for complete proof and details).

Onofri's inequality (5.5.38) plays a role in spectral analysis for the Laplace-Beltrami operator. Indeed, thanks to Polyakov's formula (see [135, 136, 142, 143]), we have

$$\frac{\det(-\Delta_g)}{\det(-\Delta_{g_0})} = e^{-\frac{1}{24}\Phi(u)},$$

for any metric $g = e^u g_0$ with $u \in C^\infty(\mathbb{S}^2)$. In particular, Onofri's inequality (5.5.38) implies that, in the conformal class of g_0 , the determinant of the Laplace-Beltrami operator is maximized by g_0 . We mention also important applications; for instance, in chemotaxis, it appears in the study of the Keller-Segel model (see [39, 90, 101]).

Now, let us recall that the two-dimensional Euclidean Onofri inequality states that

$$\ln \left(\int_{\mathbb{R}^2} e^u d\mu_2 \right) \leq \frac{1}{16\pi} \int_{\mathbb{R}^2} |\nabla u|^2 dx + \int_{\mathbb{R}^2} u d\mu_2 \quad (5.5.39)$$

for any $u \in W_{\mu_2}(\mathbb{R}^2) := \{u \in L^1(\mathbb{R}^2, d\mu_2) : |\nabla u| \in L^2(\mathbb{R}^2, dx)\}$, where

$$\mu_2(x) := \frac{1}{\pi(1+|x|^2)^2} \quad \text{and} \quad d\mu_2 := \mu_2(x) dx.$$

In particular, we have the following:

Proposition 5.5.3. *Inequalities (5.5.38) and (5.5.39) are equivalent.*

Proof. Let $\pi_{\mathcal{N}} : \mathbb{S}^2 \setminus \{\mathcal{N}\} \rightarrow \mathbb{R}^2$ be the stereographic projection from the north pole $\mathcal{N}$, and let us denote by $\pi_{\mathcal{N}}^{-1} : \mathbb{R}^2 \rightarrow \mathbb{S}^2 \setminus \{\mathcal{N}\}$ its inverse.

The standard metric g_0 on $\mathbb{S}^2$ is represented by the matrix

$$g_0(x) = \begin{pmatrix} \lambda(x) & 0 \\ 0 & \lambda(x) \end{pmatrix}, \quad \text{where} \quad \lambda(x) = \frac{4}{(1+|x|^2)^2}.$$

If $v : \mathbb{S}^2 \rightarrow \mathbb{R}$ is a real valued function on $\mathbb{S}^2$, denoting with $\tilde{v} := v \circ \pi_{\mathcal{N}}^{-1} : \mathbb{R}^2 \rightarrow \mathbb{R}$, by definition we have

$$\int_{\mathbb{S}^2} v d\nu_{g_0} := \int_{\mathbb{R}^2} \tilde{v}(x) \sqrt{\det g_0(x)} dx = \int_{\mathbb{R}^2} \tilde{v}(x) \lambda(x) dx.$$

Therefore we have

$$\frac{1}{4\pi} \int_{\mathbb{S}^2} e^u d\nu_{g_0} = \frac{1}{4\pi} \int_{\mathbb{R}^2} e^{\tilde{u}(x)} \lambda(x) dx = \int_{\mathbb{R}^2} e^{\tilde{u}(x)} d\mu_2$$

and

$$\frac{1}{4\pi} \int_{\mathbb{S}^2} u d\nu_{g_0} = \int_{\mathbb{R}^2} \tilde{u}(x) \lambda(x) dx = \int_{\mathbb{R}^2} \tilde{u} d\mu_2.$$

Now we recall that if $p \in \mathbb{S}^2 \setminus \{\mathcal{N}\}$, the gradient of v in p with respect to the metric g_0 is given by

$$\nabla_{g_0} v(p) = \frac{1}{\lambda(x)} \nabla \tilde{v}(x) \Big|_{x=\pi_{\mathcal{N}}(p)}.$$

In particular, we get

$$|\nabla_{g_0} u(p)|_{g_0}^2 = \nabla_{g_0} u(p) \begin{pmatrix} \lambda(x) & 0 \\ 0 & \lambda(x) \end{pmatrix} (\nabla_{g_0} u(p))^T = \frac{1}{\lambda(x)} |\nabla \tilde{v}(x)|^2 \Big|_{x=\pi_{\mathcal{N}}(p)},$$

whence

$$\int_{\mathbb{S}^2} |\nabla_{g_0} v|_{g_0}^2 d\nu_{g_0} = \int_{\mathbb{R}^2} |\nabla \tilde{v}|^2 dx.$$

Therefore

$$\frac{1}{16\pi} \int_{\mathbb{S}^2} |\nabla_{g_0} u|_{g_0}^2 d\nu_{g_0} = \frac{1}{16\pi} \int_{\mathbb{R}^2} |\nabla \tilde{u}|^2 dx.$$

It remains to show that the space

$$W^{1,2}(\mathbb{S}^2, g_0) := \left\{ u : \mathbb{S}^2 \rightarrow \mathbb{R} : \int_{\mathbb{S}^2} |u|^2 d\nu_{g_0} < +\infty \quad \text{and} \quad \int_{\mathbb{S}^2} |\nabla_{g_0} u|_{g_0}^2 d\nu_{g_0} < +\infty \right\}$$

coincides with the space

$$W := \left\{ u : \mathbb{S}^2 \rightarrow \mathbb{R} : \int_{\mathbb{S}^2} |u|^2 d\nu_{g_0} < +\infty \quad \text{and} \quad \int_{\mathbb{S}^2} |\nabla_{g_0} u|_{g_0}^2 d\nu_{g_0} < +\infty \right\}.$$

It is clear that $W^{1,2}(\mathbb{S}^2, g_0) \subseteq W$. Conversely, if $u \in W$ then $u \in W^{1,1}(\mathbb{S}^2, g_0)$, and so by Sobolev embeddings on compact manifolds we get $u \in L^2(\mathbb{S}^2, g_0)$, i.e. $u \in W^{1,2}(\mathbb{S}^2, g_0)$. $\square$

Appendix D

Lemma 5.5.4. *Let $N \in \mathbb{N}$, $N \geq 2$. Let $V_N := \omega_{N-1}/N$, where $\omega_{N-1} := |\mathbb{S}^{N-1}|$, and let us define*

$$\mu_N(x) := \frac{1}{V_N \left(1 + |x|^{\frac{N}{N-1}}\right)^N} \quad \text{and} \quad v_N(x) := \ln \mu_N(x).$$

Then

$$\Delta_N v_N(x) = -N^N \left(\frac{N}{N-1} \right)^{N-1} V_N \mu_N(x).$$

Proof. We recall that the N -Laplacian operator is defined as

$$\Delta_N u := \operatorname{div} \left(|\nabla u|^{N-2} \nabla u \right).$$

Observe that

$$\nabla v_N(x) = -\frac{N^2}{N-1} \frac{|x|^{\frac{1}{N-1}}}{1 + |x|^{\frac{N}{N-1}}} \frac{x}{|x|}, \quad \text{and} \quad |\nabla v_N(x)| = \frac{N^2}{N-1} \frac{|x|^{\frac{1}{N-1}}}{1 + |x|^{\frac{N}{N-1}}}.$$

Hence

$$|\nabla v_N(x)|^{N-2} \nabla v_N(x) = - \left(\frac{N^2}{N-1} \right)^{N-1} \frac{x}{\left(1 + |x|^{\frac{N}{N-1}}\right)^{N-1}}.$$

Now, observe that

$$\begin{aligned} \frac{\partial}{\partial x_i} \left(\frac{x_i}{\left(1 + |x|^{\frac{N}{N-1}}\right)^{N-1}} \right) &= \frac{\left(1 + |x|^{\frac{N}{N-1}}\right)^{N-1} - N \left(1 + |x|^{\frac{N}{N-1}}\right)^{N-2} |x|^{\frac{1}{N-1}} \frac{x_i^2}{|x|}}{\left(1 + |x|^{\frac{N}{N-1}}\right)^{2(N-1)}} \\ &= \frac{1}{\left(1 + |x|^{\frac{N}{N-1}}\right)^{N-1}} - N \frac{|x|^{\frac{1}{N-1}} \frac{x_i^2}{|x|}}{\left(1 + |x|^{\frac{N}{N-1}}\right)^N}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \Delta_N v_N(x) &= - \left(\frac{N^2}{N-1} \right)^{N-1} \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\frac{x_i}{\left(1 + |x|^{\frac{N}{N-1}}\right)^{N-1}} \right) \\ &= - \left(\frac{N^2}{N-1} \right)^{N-1} N \left[\frac{1}{\left(1 + |x|^{\frac{N}{N-1}}\right)^{N-1}} - \frac{|x|^{\frac{N}{N-1}}}{\left(1 + |x|^{\frac{N}{N-1}}\right)^N} \right] \\ &= -N^N \left(\frac{N}{N-1} \right)^{N-1} V_N \mu_N(x). \end{aligned}$$

□

Acknowledgments

“Tu ne quaesieris, scire nefas...”

“Do not ask, it is not permitted to know...”

“Non chiedere, non è lecito sapere...”

With these famous words begins the celebrated ‘Carpe Diem’ ode by the Latin poet Quintus Horatius Flaccus. I chose these words because I believe that experiences of such human intensity should remain private. For this reason, an Acknowledgments section is absent from both my Bachelor’s and Master’s theses. Nevertheless, I feel it is right for a sign of that intensity to be visible, even if it only shows one percent of what lies beneath.

First and foremost, I must thank my supervisor, Professor Silvia Cingolani, for helping me grow. Under her guidance, my PhD in Mathematics was far more than just a PhD in Mathematics. She introduced me to a new, diverse, and complex branch of Mathematics, supporting me every step of the way. She guided me not as a vessel for internal knowledge, but as one that must know how to reflect, through its external form, the essence of what it contains. Even during my most challenging moments, her wide-ranging experience acted as a guiding light, providing a steady path for me to follow. I must thank again my supervisor, Professor Silvia Cingolani, because She taught me not just the substance of what to think, but the wisdom of how and when to do so.

Secondly, I would like to express my sincere gratitude to Professor David Arcoya Álvarez for his warm hospitality during my two visits to the University of Granada. He provided me with the opportunity not only to broaden my research scope, but also to present the results achieved. He encouraged me not only to understand the deep mathematical and historical roots of today’s research, but also to improve how I communicate and present my results. Together with my supervisor, we initiated a collaboration that has culminated in a paper for which we are confidently awaiting the peer review results. The insightful discussions regarding the understanding of the topics of the paper have been of fundamental importance in my development as a researcher.

I would like to express my gratitude to Professor Gabriele Mancini. Our collaboration, which began together with my supervisor immediately after I completed a doctoral exam held by him, has been fundamental not only from a scientific perspective but also on a personal level.

Special thanks go to Professor Giusi Vannella. The collaboration we started together with my supervisor, provided invaluable scientific results and human support.

I would also like to thank my internal PhD commission, composed of Professor Giusi Vaira and Professor Felice Iavernaro, who followed my progress throughout these years, always providing wise advice and appreciation for the results achieved year after year.

I would also like to extend my deepest thanks to the referees of this thesis, Professor Alberto Farina and Professor José Carmona Tapia, for the time and effort they dedicated to evaluating my doctoral work.

I would like to thank both the former coordinator, Professor Francesca Mazzia, for her guidance, availability, and especially her appreciation throughout the three years of my PhD, and the current coordinator, Professor Giovanna Castellano, for her guidance and support during the final steps of this thesis.

I must also express my indebtedness to Professor Roberto Bellotti, the PI of the PNRR grant that funded this research. Without him, the words you are now reading would simply not exist.

The list of people I should thank is—if I may be allowed to use an expression that is both mathematically and philosophically incorrect— ‘countably infinite’. I cannot but continue this list by thanking Professor Antonio Azzollini, my supervisor for both my Bachelor’s and Master’s degrees, who continued to take an interest in my progress and granted me the opportunity to hold a seminar at the University of Basilicata in Potenza. I must also thank Professor Benedetta Pellacci, who gave me the opportunity to hold a seminar at the Department of Mathematics of the University of Campania ‘Luigi Vanvitelli’. I shall not forget the taste of a pre-seminar pizza for lunch, but I certainly will never forget the beauty of their mathematical library. Special thanks are also due to Professor Berardino Sciunzi for the opportunity to present a seminar at a school held at the University of Cosenza. I would like to thank also Professor Jaroslaw Mederski, Professor Bartosz Bieganski and Dr. Jacopo Schino for granting me the opportunity to present my work on several occasions, both in Warsaw and in Będlewo.

I would also like to extend my gratitude to everyone in the academic community I met during my time abroad in Spain. I owe special thanks to Rubén Fiñana Aránega for our conversations, which spanned from mathematics and linguistics to diverse subjects of a different nature.

I am also indebted to Dr. Marco Gallo for his support in various situations. I especially appreciate him giving a seminar here in Bari, where he even featured me as a character in one of his slides!

I would like to thank all my PhD colleagues at the Department of Mathematics. In particular, my heartfelt gratitude goes to Antonio Lagioia and Alessandro Cannone. Antonio welcomed me with his genuine kindness when I was still a stranger to everyone—more specifically, when I had not yet even figured out where the Department entrance was. Alessandro, on the other hand, has always been a source of inspiration to me. I have always admired his incredible balance; indeed, he even managed to convince me to drink (water) during lunch, besides teaching me many other important things on a human level. I must also thank Grazia Gargano (though I am not permitted here to use the name I have always called her), who had a significant impact on me at an institutional level. To Antonio, Alessandro, and Grazia, I have successfully taught the true meaning of Martial’s expression: ‘*Parcere personis, dicere de vitiis*’.

I would also like to thank Giuseppe Rago; between one smile and the next, we enjoyed discussions ranging from the Greek god Hermès to D’Annunzio’s aestheticism.

I am profoundly grateful to Professor Sandra Lucente for perceiving and describing me, since our first meeting, as a man of the eighteenth and nineteenth centuries.

If I were to name each of you individually, I would have to write a second thesis just for that purpose. Therefore, I ‘limit’ myself to saying that I truly want to thank the entire Department of Mathematics of the University of Bari Aldo Moro. I hope I managed to bring a smile to your face during our π -Day celebrations.

Finally, I wish to thank Professor Antonio Masiello. The words He told me at the beginning of my PhD— ‘*Nella Matematica ci vuole fantasia*’ translated in ‘*Mathematics requires imagination*’ — opened a fascinating door in my mind.

As for my friends and my family, I am sorry “*Tu ne quaesieris, scire nefas...*”. However, I wish to leave only one statement here. Even today, especially when I think of my sister Lucia, I have serious doubts about whether my cradle was swapped with someone else’s at birth.

Bibliography

- [1] D. R. Adams, *A sharp inequality of J. Moser for higher order derivatives*, Ann. of Math. **128** (1988), no. 2, 385–398. [94](#)
- [2] A. Aftalion and F. Pacella, *Morse index and uniqueness for positive solutions of radial p -Laplace equations*, Trans. Amer. Math. Soc. **356** (2004), no. 11, 4255–4272. [3](#)
- [3] M. Agueh, S. Boroushaki, and N. Ghoussoub, *A dual Moser–Onofri inequality and its extensions to higher dimensional spheres*, Ann. Fac. Sci. Toulouse Math. (6) **26** (2017), no. 2, 217–233. [vi](#), [96](#)
- [4] S. Ahmad, A. C. Lazer, and J. L. Paul, *Elementary critical point theory and perturbations of elliptic boundary value problems at resonance*, Indiana Univ. Math. J. **25** (1976), no. 10, 933–944. [65](#)
- [5] W. Allegretto and H. Y. Xi, *A Picone’s identity for the p -Laplacian and applications*, Nonlinear Anal. **32** (1998), no. 7, 819–830. [iv](#), [64](#), [70](#)
- [6] S. Almi and M. Degiovanni, *On degree theory for quasilinear elliptic equations with natural growth conditions*, Recent Trends in Nonlinear Partial Differential Equations II: Stationary Problems, Contemp. Math. **595** (2013), 1–20. [2](#), [3](#), [4](#), [48](#)
- [7] H. Amann and E. Zehnder, *Nontrivial solutions for a class of nonresonance problems and applications to nonlinear differential equations*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) **7** (1980), no. 4, 539–603. [IV](#), [V](#), [iii](#), [37](#)
- [8] A. Ambrosetti and G. Mancini, *Existence and multiplicity results for nonlinear elliptic problems with linear part at resonance: the case of the simple eigenvalue*, J. Differential Equations **28** (1978), no. 2, 220–245. [65](#)
- [9] A. Anane, *Simplicité et isolation de la première valeur propre du p -laplacien avec poids*, C. R. Acad. Sci. Paris Sér. I Math. **305** (1987), no. 16, 725–728. [63](#), [64](#)
- [10] A. Anane and J.-P. Gossez, *Strongly nonlinear elliptic problems near resonance: a variational approach*, Comm. Partial Differential Equations **15** (1990), no. 8, 1141–1159. [65](#)
- [11] A. Anane and N. Tsouli, *On the second eigenvalue of the p -Laplacian*, in: Nonlinear partial differential equations (Fès, 1994), Pitman Res. Notes Math. Ser., 343, Longman, Harlow, 1996, 1–9. [64](#)
- [12] C. A. Antonini, A. Cianchi, G. Ciraolo, A. Farina, and V. Maz’ya, *Global second-order estimates in anisotropic elliptic problems*, Proc. Lond. Math. Soc. (3) **130** (2025), no. 1, e70034. [21](#)

- [13] C. A. Antonini, G. Ciraolo, and A. Farina, *Interior regularity results for inhomogeneous anisotropic quasilinear equations*, Math. Ann. **387** (2023), no. 3, 1745–1776. [21](#)
- [14] D. Arcoya, N. Borgia, and S. Cingolani, *On the de Thélin eigenvalue problem and Landesman-Lazer conditions for quasilinear systems*, preprint. <https://doi.org/10.48550/arXiv.2601.16846> [vii](#), [62](#)
- [15] D. Arcoya and J. L. Gámez, *Bifurcation theory and related problems: anti-maximum principle and resonance*, Comm. Partial Differential Equations **26** (2001), no. 9-10, 1879–1911. [iv](#), [64](#), [70](#)
- [16] D. Arcoya and L. Orsina, *Landesman-Lazer conditions and quasilinear elliptic equations*, Nonlinear Anal. **28** (1997), no. 10, 1623–1632. [v](#), [62](#), [65](#), [68](#)
- [17] L. Asselle, S. Cingolani, and M. Starostka, *Morse homology for a class of elliptic partial differential equations*, Commun. Contemp. Math. (2025). <https://doi.org/10.1142/S0219199726500082> [3](#)
- [18] G. Astarita and G. Marucci, *Principles of Non-Newtonian Fluid Hydromechanics*, McGraw Hill, New York, 1974. [ii](#)
- [19] T. Aubin, *Meilleures constantes dans le théorème d’inclusion de Sobolev et un théorème de Fredholm non linéaire pour la transformation conforme de la courbure scalaire*, J. Funct. Anal. **32** (1979), no. 2, 148–174. [vi](#), [123](#)
- [20] D. Bartolucci, *Non-degeneracy, Mean Field Equations and the Onsager Theory of 2D Turbulence*, Arch. Ration. Mech. Anal. **230** (2018), 397–426. [vii](#), [112](#)
- [21] D. Bartolucci and A. Malchiodi, *Mean field equations and domains of first kind*, Rev. Mat. Iberoam. **38** (2022), no. 4, 1067–1086. [vii](#), [112](#)
- [22] T. Bartsch and N. Dancer, *Poincaré-Hopf type formulas on convex sets of Banach spaces*, Topol. Methods Nonlinear Anal. **34** (2009), 213–229. [ii](#), [1](#)
- [23] V. Benci, *A new approach to the Morse-Conley theory and some applications*, Ann. Mat. Pura Appl. (4) **158** (1991), 231–305. [ii](#), [1](#)
- [24] M. S. Berger, *Riemannian structures of prescribed Gaussian curvature for compact 2-manifolds*, J. Differential Geom. **5** (1971), 325–332. [123](#)
- [25] L. Boccardo, A. Dall’Aglia, and L. Orsina, *Existence and regularity results for some elliptic equations with degenerate coercivity*, Atti Sem. Mat. Fis. Univ. Modena **46** (1998), 51–82. [iii](#), [20](#), [22](#), [30](#)
- [26] L. Boccardo and D. G. de Figueiredo, *Some remarks on a system of quasilinear elliptic equations*, NoDEA Nonlinear Differential Equations Appl. **9** (2002), no. 3, 309–323. [2](#), [39](#), [40](#), [65](#)
- [27] L. Boccardo, P. Drábek, and M. Kučera, *Landesman–Lazer conditions for strongly nonlinear boundary value problems*, Comment. Math. Univ. Carolin. **30** (1989), no. 3, 411–427. [65](#)
- [28] V. Bögelein, F. Duzaar, P. Marcellini, and C. Scheven, *Boundary regularity for elliptic systems with p , q -growth*, J. Math. Pures Appl. (9) **159** (2022), 250–293. [21](#)

- [29] A. Bonnet, *A deformation lemma on a C^1 manifold*, Manuscripta Math. **81** (1993), no. 1, 339–359. [IV](#), [V](#), [iv](#), [62](#), [64](#), [76](#), [77](#), [90](#), [92](#)
- [30] N. Borgia, S. Cingolani, and G. Mancini, *New Analytical and Geometrical Aspects on Trudinger-Moser Type Inequality in 2D*, in: Singularities, Asymptotics, and Limiting Models, Springer INdAM Ser., vol. 64, Springer, Singapore, 2025, 235–254. https://doi.org/10.1007/978-981-96-3584-9_9 [vii](#), [93](#), [97](#)
- [31] N. Borgia, S. Cingolani, and G. Mancini, *On the equivalence between an Onofri-type inequality by Del Pino–Dolbeault and the sharp logarithmic Moser–Trudinger inequality*, Calc. Var. Partial Differential Equations **64** (2025), no. 3, Art. 89, 22 pp. [vii](#), [93](#)
- [32] N. Borgia, S. Cingolani, and G. Vannella, *Nontrivial solutions for resonance quasilinear elliptic systems*, Adv. Nonlinear Anal. **13** (2024), no. 1, Art. 20230155. [vii](#), [37](#), [69](#)
- [33] N. Borgia, S. Cingolani, and G. Vannella, *A Poincaré–Hopf formula for functional associated to quasilinear elliptic system*, Nonlinear Anal. Real World Appl. **87** (2026), Art. 104443, 12 pp. [vii](#), [1](#)
- [34] N. Borgia, S. Cingolani, and G. Vannella, *Uniform L^∞ -boundedness for solutions of anisotropic quasilinear systems*, Adv. Nonlinear Stud. (2025), accepted. <https://doi.org/10.1515/ans-2023-0203> (arxiv link: <https://doi.org/10.48550/arXiv.2601.16673>) [vii](#), [20](#)
- [35] Y. Bozhkov and E. Mitidieri, *Existence of multiple solutions for quasilinear systems via fibering method*, J. Differential Equations **190** (2003), no. 1, 239–267. [39](#)
- [36] Brezis, H. (2011). Functional Analysis, Sobolev Spaces and Partial Differential Equations. Springer New York. [7](#)
- [37] F. E. Browder, *Fixed point theory and nonlinear problems*, Bull. Amer. Math. Soc. (N.S.) **9** (1983), no. 1, 1–39. [ii](#), [iii](#), [1](#), [18](#)
- [38] E. Caglioti, P.-L. Lions, C. Marchioro, and M. Pulvirenti, *A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics description. Part II*, Comm. Math. Phys. **174** (1995), no. 2, 229–260. [vii](#), [112](#)
- [39] V. Calvez and L. Corrias, *The parabolic-parabolic Keller–Segel model in $\mathbb{R}^2$* , Commun. Math. Sci. **6** (2008), no. 2, 417–447. [vi](#), [124](#)
- [40] A. M. Candela, E. Medeiros, G. Palmieri, and K. Perera, *Weak solutions of quasilinear elliptic systems via the cohomological index*, Topol. Methods Nonlinear Anal. **36** (2010), no. 1, 1–18. [39](#)
- [41] A. Candela, A. Salvatore, and C. Sportelli, *Existence and multiplicity results for a class of coupled quasilinear elliptic systems of gradient type*, Adv. Nonlinear Stud. **21** (2021), no. 2, 461–488. [21](#)
- [42] L. Carleson and A. Chang, *On the existence of an extremal function for an inequality of J. Moser*, Bull. Sci. Math. (2) **110** (1986), no. 2, 113–127. [IV](#), [V](#), [vi](#), [94](#), [109](#)
- [43] J. Carmona, S. Cingolani, P. Martinez-Aparicio, and G. Vannella, *Regularity and Morse index of the solutions to critical quasilinear elliptic systems*, Comm. Partial Differential Equations **38** (2013), no. 10, 1675–1711. [3](#), [22](#), [25](#), [48](#), [54](#)

- [44] K.-C. Chang, *Variational methods for nondifferentiable functionals and their applications to partial differential equations*, J. Math. Anal. Appl. **80** (1981), no. 1, 102–129. [2](#)
- [45] K.-C. Chang, *Solutions of asymptotically linear operator equations via Morse theory*, Comm. Pure Appl. Math. **34** (1981), no. 5, 693–712. [38](#)
- [46] K.-C. Chang, *Infinite-dimensional Morse theory and multiple solution problems*, Progr. Nonlinear Differential Equations Appl., vol. 6, Birkhäuser Boston, Inc., Boston, MA, 1993. [ii](#), [1](#), [2](#), [48](#), [54](#)
- [47] S.-Y. A. Chang, C.-C. Chen, and C.-S. Lin, *Extremal functions for a mean field equation in two dimension*, in: Lectures on Partial Differential Equations in Honor of Louis Nirenberg’s 75th Birthday, Ser. Anal., vol. 2, Int. Press, Somerville, MA, 2003, 33–93. [vii](#), [112](#)
- [48] K.-C. Chang and Z.-Q. Wang, *Multiple non semi-trivial solutions for elliptic systems*, Adv. Nonlinear Stud. **12** (2012), no. 2, 363–381. [42](#)
- [49] S.-Y. A. Chang and P. C. Yang, *Prescribing Gaussian curvature on S^2* , Acta Math. **159** (1987), no. 1-2, 215–259. [123](#)
- [50] S.-Y. A. Chang and P. C. Yang, *Conformal deformation of metrics on S^2* , J. Differential Geom. **27** (2008), no. 2, 259–296. [123](#)
- [51] C.-C. Chen and C.-S. Lin, *Sharp Estimates for Solutions of Multi-Bubbles in Compact Riemann Surfaces*, Comm. Pure Appl. Math. **55** (2002), no. 6, 728–771. [vii](#), [112](#)
- [52] C.-C. Chen and C.-S. Lin, *Topological Degree for a Mean Field Equation on Riemann Surfaces*, Comm. Pure Appl. Math. **56** (2003), no. 12, 1667–1727. [vii](#), [112](#)
- [53] L. Chen and Y. Yang, *Talenti comparison results and rigidity results for anisotropic p -Laplacian operator with Robin boundary conditions*, Adv. Nonlinear Stud. (2025). doi:10.1515/ans-2023-0194. [21](#)
- [54] A. Cianchi and V. G. Maz’ya, *Optimal second-order regularity for the p -Laplace system*, J. Math. Pures Appl. (9) **132** (2019), 41–78. [21](#)
- [55] S. Cingolani and M. Degiovanni, *Nontrivial solutions for p -Laplace equations with right hand side having p -linear growth at infinity*, Comm. Partial Differential Equations **30** (2005), no. 7-9, 1191–1203. [3](#), [37](#), [38](#), [48](#), [49](#), [50](#), [64](#)
- [56] S. Cingolani and M. Degiovanni, *On the Poincaré-Hopf Theorem for functionals defined on Banach spaces*, Adv. Nonlinear Stud. **9** (2009), no. 4, 679–699. [IV](#), [V](#), [iii](#), [1](#), [2](#), [16](#), [17](#), [48](#)
- [57] S. Cingolani, M. Degiovanni, and B. Sciunzi, *Weighted Sobolev spaces and Morse estimates for quasilinear elliptic equations*, J. Funct. Anal. **286** (2024), no. 8, Art. 110346, 45 pp. [3](#)
- [58] S. Cingolani, M. Degiovanni, and G. Vannella, *Amann-Zehnder type results for p -Laplace problems*, Ann. Mat. Pura Appl. (4) **197** (2018), no. 2, 605–640. [iii](#), [2](#), [3](#), [21](#), [37](#), [38](#), [40](#)
- [59] S. Cingolani and G. Vannella, *Critical groups computations on a class of Sobolev Banach spaces via Morse index*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **20** (2003), no. 2, 271–292. [iv](#), [3](#), [37](#), [38](#)

- [60] S. Cingolani and G. Vannella, *Morse index and critical groups for p -Laplace equations with critical exponents*, Mediterr. J. Math. **3** (2006), no. 3-4, 495–512. [iv](#), [3](#), [37](#), [38](#)
- [61] S. Cingolani and G. Vannella, *Marino-Prodi perturbation type results and Morse indices of minimax critical points for a class of functionals in Banach spaces*, Ann. Mat. Pura Appl. (4) **186** (2007), no. 1, 157–185. [64](#)
- [62] S. Cingolani and G. Vannella, *Multiple positive solutions for a critical quasilinear equation via Morse theory*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **26** (2009), no. 2, 397–413. [iv](#), [37](#), [38](#)
- [63] J. A. Clarkson, *Uniformly convex spaces*, Trans. Amer. Math. Soc. **40** (1936), no. 3, 396–414. [VI](#)
- [64] P. Clément, D. G. de Figueiredo, and E. Mitidieri, *A priori estimates for positive solutions of semilinear elliptic systems via Hardy-Sobolev inequalities*, in: Differential Equations, Asymptotic Analysis, and Mathematical Physics (Potsdam, 1996), Pitman Res. Notes Math. Ser., 366, Longman, Harlow, 1997, 73–91. [22](#)
- [65] P. Clément, D. G. de Figueiredo, and E. Mitidieri, *Positive solutions of semilinear elliptic systems*, in: Djairo G. de Figueiredo - Selected Papers (ed. D. Cassani et al.), Springer, Cham, 2014, 369–386. [22](#)
- [66] J. N. Corvellec, *Quantitative deformation theorems and critical point theory*, Pacific J. Math. **187** (1999), no. 2, 263–279. [48](#)
- [67] J. N. Corvellec and A. Hantoute, *Homotopical stability of isolated critical points of continuous functionals*, Set-Valued Anal. **10** (2002), no. 2-3, 143–164. [54](#)
- [68] M. Cozzi, A. Farina, and E. Valdinoci, *Monotonicity formulae and classification results for singular, degenerate, anisotropic PDEs*, Adv. Math. **293** (2016), 343–381. [21](#)
- [69] M. Cuesta, *Eigenvalue problems for the p -Laplacian with indefinite weights*, Electron. J. Differential Equations **2001** (2001), no. 33, 1–9. [64](#)
- [70] D. G. de Figueiredo and J. Yang, *A priori bounds for positive solutions of a non-variational elliptic system*, in: Djairo G. de Figueiredo - Selected Papers (ed. D. Cassani et al.), Springer, Cham, 2014, 483–499. [22](#)
- [71] C. De Filippis and M. Piccinini, *Borderline global regularity for nonuniformly elliptic systems*, Int. Math. Res. Not. IMRN **2023** (2023), no. 20, 17324–17376. [21](#)
- [72] D. C. de Moraes Filho and M. A. S. Souto, *Systems of p -Laplacian equations involving homogeneous nonlinearities with critical Sobolev exponent degrees*, Comm. Partial Differential Equations **24** (1999), no. 7-8, 1537–1553. [39](#)
- [73] P. De Napoli and M. C. Mariani, *Quasilinear elliptic systems of resonant type and nonlinear eigenvalue problems*, Abstr. Appl. Anal. **7** (2002), no. 3, 155–167. [21](#), [40](#)
- [74] F. de Thélin, *First eigenvalue of a nonlinear elliptic system*, C. R. Acad. Sci. Paris Sér. I Math. **311** (1990), no. 10, 603–606. [IV](#), [V](#), [iv](#), [20](#), [23](#), [39](#), [40](#), [62](#), [63](#), [69](#), [70](#)
- [75] M. del Pino and J. Dolbeault, *Best constants for Gagliardo–Nirenberg inequalities and applications to nonlinear diffusions*, J. Math. Pures Appl. (9) **81** (2002), no. 9, 847–875. [v](#), [95](#)

- [76] M. del Pino and J. Dolbeault, *Nonlinear diffusions and optimal constants in Sobolev type inequalities: asymptotic behaviour of equations involving the p -Laplacian*, C. R. Math. Acad. Sci. Paris **334** (2002), no. 5, 365–370. [v](#), [95](#)
- [77] M. del Pino and J. Dolbeault, *The Euclidean Onofri inequality in higher dimensions*, Int. Math. Res. Not. IMRN **2013** (2013), no. 15, 3600–3611. [IV](#), [V](#), [v](#), [93](#), [95](#)
- [78] M. Degiovanni, *Critical groups of finite type for functionals defined on Banach spaces*, Commun. Appl. Anal. **13** (2009), no. 3, 395–410. [2](#)
- [79] M. Degiovanni, *On topological Morse theory*, J. Fixed Point Theory Appl. **10** (2011), no. 2, 197–218. [48](#), [49](#)
- [80] M. Degiovanni and S. Lancelotti, *Linking over cones and nontrivial solutions for p -Laplace equations with p -superlinear nonlinearity*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **24** (2007), no. 6, 907–919. [48](#), [49](#), [50](#)
- [81] M. Degiovanni and M. Marzocchi, *A critical point theory for nonsmooth functionals*, Ann. Mat. Pura Appl. (4) **167** (1994), 73–100. [48](#)
- [82] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin-Heidelberg, 1985. [75](#)
- [83] J. I. Díaz and F. de Thélin, *On a nonlinear parabolic problem arising in some models related to turbulent flows*, SIAM J. Math. Anal. **25** (1994), no. 4, 1085–1111. [ii](#)
- [84] J. I. Díaz and J. E. Saá, *Existence et unicité de solutions positives pour certaines équations elliptiques quasilinéaires*, C. R. Acad. Sci. Paris Sér. I Math. **305** (1987), no. 12, 521–524. [63](#)
- [85] E. DiBenedetto, *$C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal. **7** (1983), no. 8, 827–850. [21](#)
- [86] G. Dinca, P. Jebelean, and J. Mawhin, *Variational and topological methods for Dirichlet problems with p -Laplacian*, Port. Math. (N.S.) **58** (2001), no. 3, 339–378. [2](#)
- [87] L. Ding and S. W. Xiao, *Multiple positive solutions for a critical quasilinear elliptic system*, Nonlinear Anal. **72** (2010), no. 5, 2592–2607. [39](#)
- [88] J. Dolbeault, M. J. Esteban, and G. Jankowiak, *The Moser-Trudinger-Onofri inequality*, Chin. Ann. Math. Ser. B **36** (2015), no. 5, 777–802. [vi](#), [123](#)
- [89] J. Dolbeault, M. J. Esteban, and G. Tarantello, *The role of Onofri type inequalities in the symmetry properties of extremals for Caffarelli-Kohn-Nirenberg inequalities, in two space dimensions*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **7** (2008), no. 2, 313–341. [vi](#), [123](#)
- [90] J. Dolbeault and B. Perthame, *Optimal critical mass in the two dimensional Keller-Segel model in $\mathbb{R}^2$* , C. R. Math. Acad. Sci. Paris **339** (2004), no. 9, 611–616. [124](#)
- [91] M. Dong, N. Lam, and G. Lu, *Sharp weighted Trudinger–Moser and Caffarelli–Kohn–Nirenberg inequalities and their extremal functions*, Nonlinear Anal. **173** (2018), 75–98.
- [92] P. Drábek and S. B. Robinson, *Resonance problems for the p -Laplacian*, J. Funct. Anal. **169** (1999), no. 1, 189–200. [37](#), [38](#), [64](#)

- [93] P. Drábek, N. M. Stavrakakis, and N. B. Zographopoulos, *Multiple nonsemitrivial solutions for quasilinear elliptic systems*, Differential Integral Equations **16** (2003), no. 12, 1519–1531. [40](#)
- [94] S. El Manouni and K. Perera, *Existence and nonexistence results for a class of quasilinear elliptic systems*, Bound. Value Probl. **2007** (2007), Art. ID 85621, 5 pp. [39](#)
- [95] S. Esedoglu and S. J. Osher, *Decomposition of images by the anisotropic Rudin-Osher-Fatemi model*, Comm. Pure Appl. Math. **57** (2004), no. 12, 1609–1626. [21](#)
- [96] P. Esposito, *A classification result for the quasi-linear Liouville equation*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **35** (2018), no. 3, 781–801. [98](#)
- [97] E. R. Fadell and P. H. Rabinowitz, *Bifurcations for odd potential operators and an alternative topological index*, J. Funct. Anal. **26** (1977), no. 1, 48–67. [40](#), [49](#)
- [98] E. R. Fadell and P. H. Rabinowitz, *Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems*, Invent. Math. **45** (1978), no. 2, 139–174. [40](#), [49](#)
- [99] L. Fontana, *Sharp borderline Sobolev inequalities on compact Riemannian manifolds*, Comment. Math. Helv. **68** (1993), no. 3, 415–454. [94](#)
- [100] M. Furtado and F. O. V. de Paiva, *Multiplicity of solutions for resonant elliptic systems*, J. Math. Anal. Appl. **319** (2006), no. 2, 435–449. [42](#)
- [101] H. Gajewski and K. Zacharias, *Global behaviour of a reaction-diffusion system modelling chemotaxis*, Math. Nachr. **195** (1998), 77–114. [vi](#), [124](#)
- [102] Y. Giga, *Surface Evolution Equations. A Level Set Approach*, Monographs in Mathematics, vol. 99, Birkhäuser Verlag, Basel, 2006. [21](#)
- [103] R. Glowinski and J. Rappaz, *Approximation of a nonlinear elliptic problem arising in a non-Newtonian fluid flow model in glaciology*, ESAIM Math. Model. Numer. Anal. **37** (2003), no. 1, 175–186. [ii](#)
- [104] D. Gromoll and W. Meyer, *On differentiable functions with isolated critical points*, Topology **8** (1969), 361–369. [iv](#), [41](#)
- [105] M. Guedda and L. Véron, *Quasilinear elliptic equations involving critical Sobolev exponents*, Nonlinear Anal. **13** (1989), no. 8, 879–902. [21](#), [25](#)
- [106] Z. Guo, K. Perera, and W. Zou, *On critical p -Laplacian systems*, Adv. Nonlinear Stud. **17** (2017), no. 4, 641–659. [21](#)
- [107] M. E. Gurtin, *Thermomechanics of Evolving Phase Boundaries in the Plane*, Oxford Mathematical Monographs, Oxford University Press, New York, 1993. [21](#)
- [108] P. Hess, *On a theorem by Landesman and Lazer*, Indiana Univ. Math. J. **23** (1973/74), no. 9, 827–829. [65](#)
- [109] K. Ho, Y. H. Kim, P. Winkert, and C. Zhang, *The boundedness and Hölder continuity of weak solutions to elliptic equations involving variable exponents and critical growth*, J. Differential Equations **313** (2022), 503–532. [21](#)

- [110] S. Iula and G. Mancini, *Extremal functions for singular Moser–Trudinger embeddings*, Nonlinear Anal. **156** (2017), 215–248. [97](#)
- [111] G. Joyce and D. Montgomery, *Negative temperature states for the two-dimensional guiding-center plasma*, J. Plasma Phys. **10** (1973), 107–121. [vii](#), [112](#)
- [112] G. Katriel, *Mountain pass theorems and global homeomorphism theorems*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **11** (1994), no. 2, 189–209. [48](#)
- [113] J. L. Kazdan and F. W. Warner, *Curvature functions for compact 2-manifolds*, Ann. of Math. (2) **99** (1974), no. 1, 14–47. [123](#)
- [114] S. K. Kim and T. X. Wang, *The generalized Morse lemma and the Euler characteristic on Banach manifolds*, Topology Appl. **32** (1989), no. 1, 13–23. [1](#)
- [115] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Classics in Applied Mathematics, vol. 31, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000. [106](#)
- [116] S. S. Lai, G. T. Will, and S. Otani, *Model for inelastic biaxial bending of concrete members*, J. Structural Engineering **110** (1984), no. 11, 2563–2584. [ii](#)
- [117] S. Lancelotti, *Morse index estimates for continuous functionals associated with quasi-linear elliptic equations*, Adv. Differential Equations **7** (2002), no. 1, 99–128. [55](#)
- [118] E. M. Landesman and A. C. Lazer, *Nonlinear perturbations of linear elliptic boundary value problems at resonance*, J. Math. Mech. **19** (1970), no. 7, 609–623. [IV](#), [V](#), [iv](#), [62](#), [65](#)
- [119] A. C. Lazer and S. Solimini, *Nontrivial solutions of operator equations and Morse indices of critical points of min-max type*, Nonlinear Anal. **12** (1988), no. 8, 761–775. [38](#)
- [120] C. Li, S. Li, and J. Liu, *Splitting theorem, Poincaré–Hopf theorem and jumping nonlinear problems*, J. Funct. Anal. **221** (2005), no. 2, 439–455. [ii](#), [1](#)
- [121] E. H. Lieb and M. Loss, *Analysis*, Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, second ed., 2001. [110](#)
- [122] G. M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. **12** (1988), no. 11, 1203–1219. [21](#), [57](#), [69](#)
- [123] A. Malchiodi, *Topological methods for an elliptic equation with exponential nonlinearities*, Discrete Contin. Dyn. Syst. **21** (2008), no. 1, 277–294. [vii](#), [112](#)
- [124] R. Manásevich and J. Mawhin, *Periodic solutions for nonlinear systems with p -Laplacian-like operators*, J. Differential Equations **145** (1998), no. 2, 367–393. [ii](#)
- [125] P. Marcellini, *On the definition and the lower semicontinuity of certain quasiconvex integrals*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **3** (1986), no. 5, 391–409. [ii](#)
- [126] A. J. Marinho and K. Perera, *Local and nonlocal critical growth anisotropic quasilinear elliptic systems*, Calc. Var. Partial Differential Equations **64** (2025), no. 4, Art. 102, 26 pp. [21](#)

- [127] G. Marino and P. Winkert, *Global a priori bounds for weak solutions of quasilinear elliptic systems with nonlinear boundary condition*, J. Math. Anal. Appl. **482** (2020), no. 2, Art. 123555, 19 pp. [21](#)
- [128] A. Masiello and L. Pisani, *Asymptotically linear elliptic problems at resonance*, Ann. Mat. Pura Appl. (4) **171** (1996), 1–13. [38](#)
- [129] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, Applied Mathematical Sciences, vol. 74, Springer-Verlag, New York, 1989. [ii](#), [1](#), [2](#)
- [130] G. Mingione, *Bounds for the singular set of solutions to non linear elliptic systems*, Calc. Var. Partial Differential Equations **18** (2003), no. 4, 373–400. [21](#)
- [131] L. Montoro, L. Muglia, B. Sciunzi, and D. Vuono, *Regularity and symmetry results for the vectorial p -Laplacian*, Nonlinear Anal. **251** (2025), Art. 113700, 20 pp. [21](#)
- [132] J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1970/71), no. 11, 1077–1092. [94](#), [97](#), [108](#), [122](#)
- [133] J. Moser, *On a nonlinear problem in differential geometry*, in: Dynamical Systems (M. M. Peixoto, ed.), Academic Press, New York, 1973, 273–280. [123](#)
- [134] E. Onofri, *On the positivity of the effective action in a theory of random surfaces*, Comm. Math. Phys. **86** (1982), no. 3, 321–326. [vi](#), [95](#), [123](#)
- [135] B. Osgood, R. Phillips, and P. Sarnak, *Extremals of determinants of Laplacians*, J. Funct. Anal. **80** (1988), no. 1, 148–211. [vi](#), [124](#)
- [136] B. Osgood, R. Phillips, and P. Sarnak, *Compact isospectral sets of surfaces*, J. Funct. Anal. **80** (1988), no. 1, 212–234. [vi](#), [124](#)
- [137] Z. Ou and C. Tang, *Existence and multiplicity of nontrivial solutions for quasilinear elliptic systems*, J. Math. Anal. Appl. **383** (2011), no. 2, 423–438. [40](#)
- [138] M. C. Pelissier and L. Reynaud, *Étude d'un modèle mathématique d'écoulement de glacier*, C. R. Acad. Sci. Paris Sér. A **279** (1974), 531–534. [ii](#)
- [139] K. Perera, *Nontrivial critical groups in p -Laplacian problems via the Yang index*, Topol. Methods Nonlinear Anal. **21** (2003), no. 2, 301–309. [64](#)
- [140] K. Perera, R. P. Agarwal, and D. O'Regan, *Morse Theoretic Aspects of p -Laplacian Type Operators*, Mathematical Surveys and Monographs, vol. 161, American Mathematical Society, Providence, RI, 2010. [3](#), [76](#), [90](#)
- [141] S. I. Pohozaev, *On the eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$* , Dokl. Akad. Nauk SSSR **165** (1965), 36–39; English transl., Soviet Math. Dokl. **6** (1965), 1408–1411. [93](#), [108](#)
- [142] A. M. Polyakov, *Quantum geometry of bosonic strings*, Phys. Lett. B **103** (1981), no. 3, 207–210. [vi](#), [124](#)
- [143] A. M. Polyakov, *Quantum geometry of fermionic strings*, Phys. Lett. B **103** (1981), no. 3, 211–213. [vi](#), [124](#)

- [144] P. Quittner and P. Souplet, *A priori estimates and existence for elliptic systems via bootstrap in weighted Lebesgue spaces*, Arch. Ration. Mech. Anal. **174** (2004), no. 1, 49–81. [22](#)
- [145] P. H. Rabinowitz, *Some minimax theorems and applications to nonlinear partial differential equations*, Nonlinear Anal. **2** (1978), no. 2, 161–177. [68](#)
- [146] V. D. Rădulescu and B. Zhang, *Morse theory and local linking for a nonlinear degenerate problem arising in the theory of electrorheological fluids*, Nonlinear Anal. Real World Appl. **17** (2014), 311–321. [ii](#)
- [147] K. P. Rybakowski, *The Homotopy Index and Partial Differential Equations*, Universitext, Springer-Verlag, Berlin, 1987. [2](#)
- [148] E. H. Spanier, *Algebraic Topology*, McGraw-Hill Book Co., New York-Toronto, Ont.-London, 1966. [2](#), [48](#)
- [149] G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Ann. Inst. Fourier (Grenoble) **15** (1965), no. 1, 189–257. [20](#), [22](#)
- [150] N. M. Stavrakakis and N. B. Zographopoulos, *Bifurcation results for quasilinear elliptic systems*, Adv. Differential Equations **8** (2003), no. 3, 315–336. [40](#), [41](#), [64](#)
- [151] M. Struwe, *Variational Methods*, third ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 34, Springer-Verlag, Berlin, 2000. [68](#), [86](#)
- [152] M. Struwe, *A flow approach to Nirenberg’s problem*, Duke Math. J. **128** (2005), no. 1, 19–64. [123](#)
- [153] A. Szulkin, *Ljusternik-Schnirelmann theory on C^1 -manifolds*, Ann. Inst. H. Poincaré Anal. Non Linéaire **5** (1988), no. 2, 119–139. [64](#)
- [154] J. E. Taylor, *Crystalline variational problems*, Bull. Amer. Math. Soc. **84** (1978), no. 4, 568–588. [21](#)
- [155] J. E. Taylor, J. W. Cahn, and C. A. Handwerker, *Geometric models of crystal growth*, Acta Metall. Mater. **40** (1992), no. 7, 1443–1474. [21](#)
- [156] K. Tintarev and K. H. Fieseler, *Concentration Compactness: Functional-Analytic Grounds and Applications*, Imperial College Press, London, 2007. [102](#)
- [157] P. Tolksdorf, *On the Dirichlet problem for quasilinear equations in domains with conical boundary points*, Comm. Partial Differential Equations **8** (1983), no. 7, 773–817. [21](#), [63](#), [69](#)
- [158] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations **51** (1984), no. 1, 126–150. [21](#)
- [159] N. S. Trudinger, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), no. 5, 473–483. [93](#), [94](#), [108](#)
- [160] K. Uhlenbeck, *Morse theory on Banach manifolds*, J. Funct. Anal. **10** (1972), no. 4, 430–445. [iv](#), [37](#), [38](#)

- [161] G. Vannella, *Uniform L^∞ -estimates for quasilinear elliptic systems*, J. Math. **2023** (2023), Art. ID 2891340, 11 pp. [iii](#), [22](#), [24](#), [25](#), [53](#), [57](#)
- [162] J. L. Vázquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. **12** (1984), no. 3, 191–202. [63](#), [69](#), [70](#)
- [163] J. Velin and F. de Thélin, *Existence and nonexistence of nontrivial solutions for some nonlinear elliptic systems*, Rev. Mat. Univ. Complut. Madrid **6** (1993), no. 1, 153–194. [39](#)
- [164] G. Wulff, *Zur Frage der Geschwindigkeit des Wachstums und der Auflösung der Kristallflächen*, Z. Krist. **34** (1901), 449–530. [21](#)
- [165] V. I. Yudovich, *Some estimates connected with integral operators and with solutions of elliptic equations*, Dokl. Akad. Nauk SSSR **138** (1961), no. 4, 805–808; English transl., Soviet Math. Dokl. **2** (1961), 746–749. [93](#), [108](#)

Mathematics Subject Classification:

26D10 35B06 35B34 35B45 35B65 35J50 35J60 35J62
 35J70 35J92 46E30 46E35 47H11 47J30 58E05 58E35

Key words:

Asymptotically (p, q) linear	Banach spaces	Critical groups
Critical growth	Degree theory	Eigenvalue problem
Fadell–Rabinowitz index	Generalized Stampacchia lemma	Morse index
Poincaré–Hopf formula	Quasilinear elliptic systems	Resonant problems
Trudinger–Moser inequality	Two dimension	Uniform L^∞ -boundedness

Contacts: natalino.borgia@uniba.it, Dipartimento di Matematica, UNIBA (Bari).