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DISTRIBUTIONAL SYMMETRIES ON CCR ALGEBRAS

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Introduction

Distributional symmetries for random processes are one of the most central topics in both classical and quantum probability. Its beginning is almost parallel to the development of the foundations of modern probability and quantum mechanics.

A random sequence is, informally, a family of random variables indexed by the positive integers $X = (X_1, X_2, \ldots)$. Let Ω be its sample set, G a semigroup and α any action of G on the set of functions from the countable cartesian product Ω^∞ to itself[1]. Then, X is called G-invariant, if $X \overset{\text{distr.}}{=} X \circ g$, for all $g \in G$, where $X \circ g(\omega) = X(g(\omega))$, for all $\omega \in \Omega^\infty$.

The most popular form of G-invariances are stationarity and exchangeability, when G acts directly on the indices of the process, that is, $\alpha_g((\omega_j)_j) = (\omega_{g(j)})_j$, for all sequence of elementary events $(\omega_j)_j$, $\omega_j \in \Omega, j \in \mathbb{N}$. A random sequence is stationary, if it is invariant under such action performed by the shift operator $\theta(k) = k + 1$, for all positive integer k. It is exchangeable, if its distribution is invariant under any permutation of the indices. G-invariance "feels" independence among the variables involved, as well as the order between the class of transformations on the indices. For example, sequences of independent and identically distributed random variables (iid sequences, in short) are particular instances of G-invariant random sequences, in that G can be whatever set of index set transformations. Exchangeable sequences are clearly stationary, as the shift operator is a permutation when restricted to finite subsets of indices, but the converse does not hold.

It is arguable that the first major work in classical distributional symmetries pertains exchangeability. As [BR08] reports, the very first publication where the concept appears is [Haa28]. The most notable contribution is that of de Finetti in [Fin31] that inspired many ideas and investigations, even if with some chronological delay.

Let X be a sequence of binary random variables, that is, each X_j is Bernoulli for some success probability parameter p_j, j any positive integer. The joint law cannot be written a priori, unless we have some additional

[1] i.e. $\alpha(gh) = \alpha(g)\alpha(h), g, h \in G$, (and $\alpha(1_G) = \mathrm{id}_\Omega$ in case G is a monoid).

information. If X is a sequence of iid, then any of its finite-dimensional distribution is given by the products of its marginals. A binary sequence X is called conditionally iid, if its distribution factorizes according to a common success probability P which is random itself. If X is a binary sequence that is conditionally iid, then the success probability is given by a single random variable P a.s. $0 \leq P \leq 1$ and averaging out any finite-dimensional distribution of X results in the probability mixture

$$\mathbf{P}(X_1 = x_1, \ldots, X_n = x_n) = \int_0^1 \mathrm{d}\nu(p)\; p^r(1-p)^{n-r}\,, \tag{0.0.1}$$

for each $n \in \mathbb{N}$, $x_1, \ldots x_n \in \{0,1\}$ where $r = \sum_{i=1}^n x_i$ and ν is an undetermined probability distribution. A conditional iid structure reflects the fact that the sequence finite-dimensional distributions can be expressed as an integral of measures. If a binary sequence finite-dimensional distributions can be expressed in integral form as above, then it is called mixed iid. The conditional iid property is, clearly, stronger than exchangeability. The condition of being mixed iid is what de Finetti showed to be equivalent to exchangeability in his "fenomeni aleatori" [Fin31], for the particular case of binary exchangeable sequences. More precisely, if X is a random sequence of $\{0,1\}$-valued exchangeable random variables, then there exists a probability measure ν on $[0,1]$ such that (0.0.1). De Finetti's theorem is a consequence of general classical ergodic theory.

Suppose G is a group of transformations acting on the index set and let $\mathfrak{X}$ be the set of all random sequences divided by the equivalence relation of equality in distribution. Then, $(\mathfrak{X}, G)$ is a dynamical system. The G-invariant sequences are the elements of the fixed-point set $\mathfrak{X}^G = \left\{ X \,\middle|\, X \circ g \overset{\text{distr.}}{=} X \right\}$. For any $X, Y \in \mathfrak{X}^G$, $rX + sY \in \mathfrak{X}^G$, for all $r, s > 0$. Thus, $\mathfrak{X}^G$ is a convex cone. The ergodic decomposition theorem, indirectly, tells us that, for each X in the subset of fixed-points $\mathfrak{X}^G$, $X \overset{\text{distr.}}{=} \int Y \mathrm{d}\mu$ for a unique probability measure supported on the subset of sequence that are extreme points of $\mathfrak{X}^G$, and that these are the i.i.d. sequences. In other words, probability distributions that are invariant under the action of finite permutations are mixtures of product probability measures.

The study of distributional symmetries in classical probability extends itself beyond exchangeability and stationarity. A sequence X is called spreadable, or "contractable" [Kal05], if its joint distribution is invariant under the action on indices by monotone increasing mappings $X \circ g = (X_{g(1)}, X_{g(2)}, \ldots)$, for any g mapping of $\mathbb{N}$ to itself with $g(j) < g(j+1)$ for all positive integers j. If $\mathbb{J}_\mathbb{N}$ denotes the set of all such maps, then $\mathbb{J}_\mathbb{N}$ is a semigroup under map composition. In other words, the action on the index set of $\mathbb{J}_\mathbb{N}$ is inherently a

dissipative dynamics and present a quite different situation in comparison to the dynamics defined by finite permutations. Clearly, spreadable sequences are stationary. Likewise, any exchangeable sequence is spreadable. It is a striking fact that the converse is true and due to Ryll-Nardzewski [Ryl57]. It is sufficient to be spreadable for a process to have a directing random measure. As a result, selecting an infinite subsequence of marginals or applying a permutation (also made up by an infinite count of transpositions) does not alter the original sequence distribution.

All the distributional symmetries defined so far emerge by a condition for actions defined on the indices.

A random sequence X is said to be rotatable, if finite-dimensional distribution exhibits a spherical symmetry. An immediate example is the standard sequence i.e. an iid sequence of centered unit variance Gaussians, multiplied by a non-negative real scalar $\sigma \geq 0$. Rotatability is the strongest distributional symmetry whereas stationarity is the weakest. One is led to believe that perhaps all exchangeable sequences are rotatable, similarly to what happens for spreadability. Freedman's theorem [Fre63] tells us that the example we just made is fundamentally the only possible. A random sequence $X = (X_1, X_2, \ldots)$ is rotatable iff, for independent random variables $Y_j, \sigma, j \in \mathbb{N}$, with Y_j standard normal and $X_j = \sigma Y_j$ almost surely, for all positive integer j, where $Y = (Y_1, Y_2, \ldots)$ and σ an almost surely non-negative random variable, $X_j = \sigma Y_j$ almost surely, for all positive integer j. Freedman's statement, similarly to de Finetti's theorem, tells us that a random sequence is rotatable only if it is a mixture of iids. Nevertheless, it implies a prescribed mixturing distribution that has to be gaussian even if still undetermined in that its variance is random. In this regard, rotatability imposes more restrictive constraints on a random sequence than those of exchangeability. Freedman's theorem is equivalent[2] to Schoenberg's theorem [Sch38, Theorem 2 (2.2)].

Three years before the first paper where exchangeability appeared, Heisenberg suggested his matrix formulation of quantum mechanics. His formulation attends systems with a finite number of particles that resent of quantum effects. Supported by the contribution of Born and Jordan, Heisenberg's quantization considers positions and momenta of a particle as self-adjoint matrices $q_1, \ldots, q_n, p_1, \ldots, p_n$ subject to the so-called canonical commutation relations

$$\begin{cases} p_j p_k - p_k p_j &= 0, \\ q_j q_k - q_k q_j &= 0, \\ p_j q_k - q_k p_j &= \frac{\hbar}{i}\delta_{jk}, \end{cases} \tag{0.0.2}$$

[2] [Kal21, p.865]

for all $j, k \in \{1, \ldots, n\}$, where $\hbar$ is the Planck constant and the Kronecker symbol appears. A calculation shows that, for each particle j, at least one variable between p_j and q_j is unbounded. To give significance of this issue, it was suggested that matrices modeling observables need to be infinite-dimensional. Concurrently to Heisenberg's view, Schrödinger gave his wave formulation where the particle state is encoded in a single real square summable function of unit norm; the evolution follows his celebrated Schrödinger's equation. Subsequently, in a series of papers, von Neumann, Stone and Murray gave a sound mathematical framework for the matrix and wave interpretations. These were pointing essentially to the same theory, where essentially the same means up to unitary equivalence of the corresponding representations. Operator algebras met quantum mechanics of finite particle systems. When the number of particles is finite, it can be shown under suitable assumptions[3] that relations (0.0.2) uniquely determine the Hilbert space where observables can be represented, up to unitary equivalence. Quantum mechanics classifies particles according to their collective behaviour and the two main classes are the bosons and the fermions. Particles that are indistinguishable under any permutation given by an odd number of transpositions are called Fermi particles. They satisfy the Pauli exclusion principle that forbids them to share the same state, as, instead, bosons behave. Finite systems of fermions follow the so-called canonical anticommutation relations, which are formally the (0.0.2) with the anticommutator replacing the third relation. To the contrary, particles that are indistinguishable under permutations of positive sign follow the Bose-Einstein statistics, can condensate infinitely in the same energy level, and the relations for their position and momentum operators follow exactly (0.0.2). The system can be described as a tensor product of Hilbert spaces, each labelling a particle, and symmetry principles should give account of the particle species. It has been suggested that there are other physically meaningful relations and statistics than (0.0.2) e.g. the parastatistics [Gre53]. However, the idealization needed by thermodynamics in quantum statistical mechanics requires the number of particle to be considered as infinite. For these there is no unique quantization process and the formalism of noncommutative probability comes into play.

We ask ourselves the question: is there a quantum version of the classical theorems, specifically of Freedman's theorem?

Noncommutative probability has similar results to those of classical probability, but not a unique interpretation of what shall be called random or stochastically independent. Each of these interpretations sustains a differ-

[3] the theorem is often dubbed as von Neumann's uniqueness theorem although it results from the work of others including Rellich and Weyl

ent generalization of noncommutative independence and, consequently, an approach to distributional symmetries.

The classical independence relation features symmetry, associativity, the normalization property and universality (i.e. congruence with respect to measurable transformations). Universal notions of independence that formalize these intuitive properties are provided in [BS02] and with respect to the category of unital C*-algebras. Specifically, if a product is required to reproduce such properties and being categorical with respect to unital C*-algebras, then it needs to be either the linear tensor product $\otimes$ or the free product $*$, [BS02, Theorem 3]; it is forced to be the former, if the symmetry of the independence relation is required. However, for applicative purposes, being categorical or symmetric can be restrictive. Random matrices are widely employed in applications and their theory pertains to free independence (e.g. the distribution of a sum of two large random matrices is the free convolution of their limiting distribution [Voi91; Spe93]). Nevertheless, for applications in quantum optics [Wal73], one rather consider boolean independence [CFL17]. For this reason, there are considered weaker versions of independence i.e. quasi-universality of the related products [Mur02]. It turns out that there are five forms of independence [Mur11, p.338], which are the so-called boolean, monotone, antimonotone, free independence and tensor independence [Mur11, Theorem 5.1]. Each of these generalized stochastic independence notions corresponds to a *-algebra generated by suitable creation and annihilation operators acting in Fock spaces subject to commutation relations. In [BS94; Bożl2], for these models is introduced a unifying second quantization scheme. This is based on specifying a deformation operator required to be either Yang-Baxter-Hecke, or a contraction.

Far-reaching general results can be stated all the same for processes on infinite free products, to which free independence applies. The results proven come at the cost, though, of working with symmetries implemented by the natural coaction of the quantum orthogonal/unitary groups on the $*$-algebra of non-commutative polynomials [Wan98; BCS12]. Doing so, the Gaussian distribution must be replaced by free operator-valued circular distribution, see [KS09; Cur10].

When distributional symmetries are still discussed with respect to classical groups and semigroups but in the noncommutative probability framework, results of the same generality of classical theorems are absent.

It appears quite natural to focus on specific models, mostly suggested by applicative fields, where some kind of commutation relations are taken into account. For instance, if the deforming operator is not a contraction but given by the opposite of the identity operator, the scheme yields the C*-algebra

of the so-called Boolean Commutation Relations [CF15; Fid15]. On this algebra, proven to be the unitalization of the compact operator algebra, any discrete-time boolean quantum stochastic process lies in a segment whose extreme points are the vacuum state and the state at infinity; as a consequence, all forms of distributional symmetries are equivalent, and precisely they mean that the process is conditionally iid, [Fid15, Theorem 3.4]. A deformation given by the Pusz-Woronowicz operator defines the (non-unital) C*-algebra of monotone commutation relations [Wys05] linked to monotone independence. For this algebra the permutation group does not act as automorphisms because of the order on sequences labeling each basis vector, see e.g. [Wys07; Wys10; CFG18; CDR23]. Nevertheless, any monotone stationary discrete-time quantum process is spreadable, as they are all points of a segment as in the boolean case [CFL17, Theorem 5.12.].

The situation drastically gets more severe when the deformation is driven by a real parameter $-1 \leq q \leq 1$ inducing the so-called q-deformed commutation relations on annihilators and creators. When $q = -1$, the deformation gives the fermionic algebra and the analysis in [CF12; CF15], later extended to rotatable states in [CRZ23; CVR23], shows that the degenerate situation of the boolean case partially resurfaces, in that all distributional symmetries are the same, except that stationary states need not to be exchangeable, as the example provided in [CR22a, Proposition 4.1.]. These results are based on the existence of a grading and the parity of automorphisms involved. When $-1 < q < 1$, the emerging picture is quite "desolating". While, by [DF10, Theorem 3.3 and Corollary 3.4], stationarity reduces to spreadability, exchangeable states are proven to be stationary, as the only example of these is the Fock vacuum [CF15, Proposition 6.2]. In conclusion, with the exception of $q = -1$, distributional symmetries are trivially the same.

Looking at previous work within tensor independence, it emerges that rotatability is quite subsumed or absent. Further, the investigation of distributional symmetries for processes on canonical commutation relation algebras (CCR) has been neglected so far.

Is a noncommutative Freedman theorem still possible? Will spreadability also collapse into exchangeability for bosonic processes?

My goal is to answer to these questions, opening up the study of distributional symmetries of bosonic process transformed by the classical actions, and ultimately characterize unitary invariance. The thesis is organized as follows.

In Chapter 1, we review the classical distributional symmetry theorems in the modern language of σ-algebras. The discussion shall presents concepts later generalized to the noncommutative case.

In Chapter 2, we give essentials of C*-algebra theory. In particular, we

assess the C*-dynamical system theory in use for our results.

In Chapter 3, we review the founding concepts of quantum probability, comparing with the classical case to give a flavor of the paradigm-shift made. We make precise what we mean by a quantum stochastic process, what is the form of independence in our context. We discuss issues about the existence and, specifically, uniqueness of barycentric formulae of G-invariant states when the C*-algebra is not commutative. A general result for distributional symmetries of states on infinite tensor products is reviewed.

In Chapter 4, we illustrate the major known facts about C*-algebras for the canonical commutation relations. Specifically, we give the working definition of CCR algebra and examine its key properties. In this context, we set-up our assumptions and review its construction as a C*-algebra. In addition, motivations to consider the CCR algebra as the natural environment to study orthogonal and unitary invariance shall be highlighted by exhibiting an auxiliary subalgebra which is at the core of our subsequent arguments.

In Chapter 5, we exhibit the novel results attained within the investigation of the weak*-compact convex set of unitarily invariant states on CCR algebras. This chapter contains the work published in [Cri+24].

Chapter 1

Classical Distributional Symmetries

1.1 Basic definitions and facts

We review some well-known definitions, facts and fundamental theorems on classical distributional symmetries we are interested in, to set up some notation and to describe the picture where our original result of Chapter 5 is placed.

Let $(\Omega, \mathscr{A}, \mathbf{P})$ be a probability space, $\mathbb{Z}$ the set of integers, and $X = (X_k, k \in \mathbb{Z})$ a random sequence, $X_k : \Omega \to S$, for all $k \in \mathbb{Z}$, where S is a Polish space (i.e. separable, completely metrizable space). Let G be a set of measurable mappings from $S^{\mathbb{Z}}$ to itself, where $S^{\mathbb{Z}}$ is the Borel measurable space of functions $\mathbb{Z} \to S$. Then, for each $g \in G$, $g(X)$ is a random sequence. If G is stable under map composition, we say that G has an action on X. A random sequence X is called G-invariant, if $g(X) \stackrel{\text{distr.}}{=} X$, for all $g \in G$, where $\stackrel{\text{distr.}}{=}$ means the equality in distribution. Among all transformations for random sequences, those acting on the indices play an particularly important rôle. Let Ω^∞ denote the cartesian product of Ω by itself infinite countably many times. We say that a class of measurable transformations G acts on X, if, for all $g \in G$, $g(X) = X \circ f_g$, for some map $f_g : \mathbb{Z} \to \mathbb{Z}$, where $X \circ f$ denotes the random sequence $X_{f(k)}(\omega) = X(\omega_{f(k)})$, $k \in \mathbb{Z}$, for all $f : \mathbb{Z} \to \mathbb{Z}$, sequence of elementary events $\omega = \{\omega_k\}_{k\in\mathbb{Z}} \in \Omega^\infty$. In fact, any $G \subseteq \mathbb{Z}^{\mathbb{Z}}$ stable under composition acts yields an action on indices for any given random sequence X. When we say that a semigroup $G \subseteq \mathbb{Z}^{\mathbb{Z}}$ acts on X, we implicitly refer to the action on indices and identify f_g with g We list traditional classes for the action on the indices of a given random sequence with their notation. We denote by $\theta \in \mathbb{Z}^{\mathbb{Z}}$ the shift-operator $\theta \in \mathbb{Z}^{\mathbb{Z}}$ given by $\theta(k) = k + 1$, for all $k \in \mathbb{Z}$;

$\mathbb{J}_{\mathbb{Z}}$ denotes the set of all monotone (strictly) increasing mappings $\psi \in \mathbb{Z}^{\mathbb{Z}}$, $\psi(k) < \psi(k')$, if $k < k'$, $k, k' \in \mathbb{Z}$, that is a semigroup under map composition; finally, $\mathrm{Sym}(\mathbb{Z})$ denotes the group of all permutations of $\mathbb{Z}$.

Definition 1.1.1 (Stationary, Spreadable, Exchangeable Sequences)**.** *Let G be a semigroup $G \subseteq \mathbb{Z}^{\mathbb{Z}}$ and X a G-invariant random sequence. Then, X is called*

(i) stationary, if $G = \mathbb{S}_{\mathbb{Z}}$, where $\mathbb{S}_{\mathbb{Z}} = \langle\theta\rangle := \{\theta^k \,|\, k \in \mathbb{Z}\}$ is the semigroup generated by $\{\theta\}$ that is a group if $\mathbb{Z}$ is a group, for example the set of integers;

(ii) spreadable, if $G = \mathbb{J}_{\mathbb{Z}}$;

(iii) exchangeable, if $G = \mathrm{Sym}(\mathbb{Z})$.

Sequences of independent and identically distributed sequence of random variables are, intuitively, a very special situation, in fact, definitely the most special, as they enjoy all kinds of distributional symmetries in Definition 1.1.1. Examples of sequences that can discriminate between G-invariances of Definition 1.1.1 are a bit more elaborated. A stationary but not exchangeable sequence can be built as follows.

Let X_1 be uniformly distributed over the complex unit circle centered at the origin. Let $X_2 = \theta X_1$, for any fixed $\theta \in \mathbb{C}$ $|\theta| = 1$, $\theta \neq \pm 1$. Then, X_2 is equal in distribution to X_1, nevertheless, $\mathbb{P}((X_1, X_2) \in I \times \theta I) = \mathbb{P}(X_1 \in I)$, while $\mathbb{P}((X_2, X_1) \in I \times \theta I) = 0$ unless $\theta I = \theta^{-1} I$, that is, $\theta = \pm 1$, for any $I = \{e^{i\eta} | \eta \in [-t, t]\}$ for any fixed positive real $t < \pi$. Setting a sequence of points $\{\theta_k\}_{k\in\mathbb{N}}$,in the unit circle avoiding $\{\pm 1\}$ yields a sequence $(X_k)_{k\in\mathbb{N}}$ where $X_k = \theta_k X_1$, for all $k \in \mathbb{N}$ which is stationary, as all marginals are uniformly distributed over the unit circle. However, no pair of coordinates can be switched without altering the joint law (for θ_k being designed to avoid ± 1). Thus, stationary sequences might not be exchangeable. An infinite sequence of drawings with replacements from an urn containing an unknown quantity of black (failure) and white balls (success) is one (and perhaps the most common) example of an exchangeable sequence of classical random variables.

Our contributions concern the invariance under G the group of rotations, which we introduce.

Notation. *We adopt the standard abbreviation $[n] = \{1, \ldots, n\}$, for all positive integer n. In accordance, we denote by $\mathbb{O}[n]$ the orthogonal group of $n \times n$ real matrices, and by $\mathbb{O}_{\mathbb{Z}}$ the inductive limit of $\{\mathbb{O}[n]\}_{n\in\mathbb{N}}$. We adopt the standard abbreviation $[n] = \{1, \ldots, n\}$, for all positive integer n.*

By the Daniell-Kolmogorov theorem, the law of X is determined by the family of finite-dimensional distributions

$$\{\mathcal{L}((X_{j_1},\dots,X_{j_n}))\,|\,(j_1,\dots,j_n)\in\mathbb{Z}^n, n\in\mathbb{N}\}$$

where $\mathcal{L}(\cdot)$ denotes law of a random variable. Then, for all $O\in\mathbb{O}_{\mathbb{Z}}$, for each $n\in\mathbb{N}$, $\underline{j}=(j_1,\dots,j_n)\in\mathbb{Z}^n$, O has a restriction, say $O\restriction_{\underline{j}}\in\mathbb{O}[n]$. Thus, OX is defined as the random sequence of finite-dimensional distributions

$$\left\{\mathcal{L}(O\restriction_{\underline{j}}(X_{j_1},\dots,X_{j_n}))\,\middle|\,(j_1,\dots,j_n)\in\mathbb{Z}^n, n\in\mathbb{N}.\right\}$$

Definition 1.1.2 (Rotatable Sequences)**.** *X is said to be rotatable, if X is such that $(X_{j_1},\dots,X_{j_n})$ is $\mathbb{O}[n]$-invariant, for all $n\in\mathbb{N}$, $j_1,\dots,j_n\in\mathbb{Z}$, where $\mathbb{O}[n]$ is the n-dimensional orthogonal matrix group on $\mathbb{R}^n$.*

There are immediate relationships between the distributional symmetries of Definitions 1.1.1-1.1.2. We shall see which of these proved to be robust under the paradigm shift towards quantum probability.

For any function $f\in\mathbb{Z}^{\mathbb{Z}}$, the set $\{k\in\mathbb{Z}\,|\,f(k)\neq k\}$ is called its support. We denote by $\mathbb{P}_{\mathbb{Z}}$ the subgroup of $\mathrm{Sym}(\mathbb{Z})$ of finitary permutations, defined as all those $g\in\mathbb{P}_{\mathbb{Z}}$ such that the support of g is finite. The following are alternative equivalent conditions to those in Definition 1.1.1,

$$\begin{cases} X \text{ is stationary iff } X_k \overset{\text{distr.}}{=} X_{k+1}, \text{ for all } k\in\mathbb{Z};\\ X \text{ is spreadable iff } X \text{ is } (X_1,\dots,X_n)\overset{\text{distr.}}{=}(X_{\psi(1)},\dots X_{\psi(n)}),\\ \text{ for all } n\in\mathbb{N},\psi\in\mathbb{J}_{\mathbb{Z}};\\ X \text{ is exchangeable iff } X \text{ is } \mathbb{P}_{\mathbb{Z}}\text{-invariant};\\ X \text{ is rotatable iff } X \text{ is } (X_1,\dots,X_n)\overset{\text{distr.}}{=} O(X_1,\dots X_n),\\ \text{ for all } O\in\mathbb{O}[n], n\in\mathbb{N}. \end{cases} \tag{1.1.1}$$

We denote by $F\subseteq_{\text{fin}}\mathbb{Z}$ the fact F is a finite subset of integers. Clearly, if $\mathbb{S}_{\mathbb{Z}}=\{\theta^k\,|\,k\in\mathbb{Z}\}$ denotes the semigroup given by the compositional powers of the shift, then $\mathbb{S}_{\mathbb{Z}}\subseteq\mathbb{J}_{\mathbb{Z}}$, and all spreadable sequences are stationary. Any monotone (strictly) increasing mapping $\psi\in\mathbb{J}_{\mathbb{Z}}$ is injective and not necessarily bijective. However, for each $F\subseteq_{\text{fin}}\mathbb{Z}$, $\psi\in\mathbb{J}_{\mathbb{Z}}$ induces a finitary permutation of $\mathbb{Z}$, so that we may write $\psi\in\mathbb{P}_{\mathbb{Z}}$, with slight abuse of notation. In addition, if $F\subseteq_{\text{fin}}\mathbb{Z}$, any finitary permutation $\pi\in\mathbb{P}_{\mathbb{Z}}$ is associated to a unique finite-dimensional permutation matrix. We elaborate this fact.

Let $f\in\mathbb{P}_{\mathbb{Z}}$ and $n\in\mathbb{N}$ the least positive integer such that the support of f is contained in I_{2n+1}, where $I_{2n+1}=[-n,n]\cap\mathbb{Z}$. Then, $f\restriction_{I_{2n+1}}\in\mathbb{P}[2n+1]$, by composing with the bijection $I_{2n+1}\to[2n+1]$, $k\mapsto 2k+1$. Fix a linear

basis for $\mathbb{R}^{2n+1}$, $\{e_1, \dots, e_{2n+1}\}$. Then, it is defined the linear map such that $O_f(e_j) = e_{f(j)}$, $j \in [2n+1]$, whose representing matrix is $(\mathbf{1}_{\{f(i)\}}(j))_{i,j\in[2n+1]}$, where $\mathbf{1}_B(j)$ denotes the characteristic function of any set B. Thus, $\mathbb{P}_\mathbb{Z} \to \cup_{n\in\mathbb{N}}\mathbb{O}[2n+1]$ given by $f \mapsto (\mathbf{1}_{\{f(i)\}}(j)_{i,j\in[2n+1]}$ is a well-defined map which extends to a map $\mathbb{P}_\mathbb{Z} \to \mathbb{O}_\mathbb{Z}$, that is an injective group-homomorphism. It follows that $\mathbb{P}_\mathbb{Z}$ has an isomorphic image contained in $\mathbb{O}_\mathbb{Z}$. We write $\mathbb{P}_\mathbb{Z} \lesssim \mathbb{O}_\mathbb{Z}$ to denote this fact.

From the observations made,

$$\begin{cases} \text{If } X \text{ is spreadable, then } X \text{ is stationary.} \\ \text{If } X \text{ is exchangeable, then } X \text{ is spreadable.} \\ \text{If } X \text{ is rotatable, then } X \text{ is exchangeable.} \end{cases} \tag{1.1.2}$$

All the mentioned symmetries, except stationarity, have been characterized in known results that are fundamental for classical distributional symmetries. As they inspired further investigations in the quantum probability set-up, we shall review them in the next sections.

1.2 De Finetti and Ryll-Nardzewski's theorem

We review the main theorems in exchangeability and spreadability when the index set of the stochastic process is countable. See [Kal05], for a wide survey, including alternative choices for the index set. The cited results are taken from [Kle14] with some adaptation. All random sequences in this chapter are intended to be on a (Kolmogorov) probability space $(\Omega, \mathscr{A}, \mathbb{P})$ and S-valued, for S a locally compact Polish space. Let $\mathcal{M}_1(S)$ be the convex cone of probability measures over S.

Definition 1.2.1 (Conditionally, Mixed Iid)**.** *A random sequence X is called conditionally iid, if there exists a random Borel probability measure*[1] *ν on $\mathcal{M}_1(S)$ such that*

$$\mathbf{P}\left(\prod_{k\in\mathbb{Z}} X_k \in \cdot \,\middle|\, \nu\right) = \otimes_{k\in\mathbb{Z}}\nu =: \nu^\infty \tag{1.2.1}$$

and ν^∞ denotes the (countably) infinite power measure, of ν, called the directing measure. If (1.2.1) holds under expectation, that is,

$$\mathbf{P}\left[\prod_{k\in\mathbb{Z}} X_k \in \cdot\right] = \mathbf{E}\nu^\infty\,, \tag{1.2.2}$$

[1] See the Appendix Section A for details.

then X is said to be a mixed iid sequence. In this case, ν is called a mixturing measure [2].

The conditional iid structure property of Definition 1.2.1(1.2.1) is, clearly, stronger than exchangeability for a random sequences. If there is a directing measure ν on S, $\mathbf{P}\left(\prod_{j=1}^{k}\xi_j|\nu\right) = \prod_{j=1}^{k}\mathbf{P}\left(\xi_j|\nu\right)$, for all $k \in \mathbb{N}$, then, for all $g \in \mathbb{P}_{\mathbb{Z}}$ with with support contained in $[k]$,

$$\begin{aligned}\mathbf{P}\left(\cap_{j=1}^{k}X_{\sigma_j} \in \cdot\right) = \mathbf{E}\left[\mathbf{P}\left(\cap_{j=1}^{k}X_{\sigma_j} \in \cdot|\nu\right)\right] = \mathbf{E}\left[\prod_{j=1}^{k}\mathbf{P}\left(X_{\sigma_j} \in \cdot|\nu\right)\right]\\ = \mathbf{E}\left[\cap_{j=1}^{k}\mathbf{P}\left(X_j \in \cdot|\nu\right)\right] = \mathbf{P}\left(\prod_{j=1}^{k}X_j \in \cdot\right),\end{aligned} \tag{1.2.3}$$

that is, X is exchangeable. Any iid sequence, is vacuously a conditionally iid sequence, the directing measure equals the unique possible for the trivial σ-algebra $\{\varnothing, \Omega\}$.

If X is a conditionally iid sequence directed by measure ν which is distributed according ot a further probability measure μ, then, (1.2.2) rewrites in integral form to

$$\mathbf{P}(X_{k_1} \in B_{k_1}, \ldots, X_{k_n} \in B_{k_n}) = \int \mathrm{d}\nu(\mu)\mathbf{P}(X_{k_1} \in B_{k_1}, \ldots, X_{k_n} \in B_{k_n}|\mu)\,, \tag{1.2.4}$$

for all $B_{k_1}, \ldots, B_{k_n}$ Borel measurable subsets of S, integers $k_1, \ldots, k_n \in \mathbb{Z}$, $n \in \mathbb{N}$, where the integral is over $\mathcal{M}_1(S)$, all the probability measures on S.

Let X be a random sequence, with coordinate variables X_k, $k \in \mathbb{Z}$. We denote by $(x_j)_{j\in F}$ a tuple of points $x_j \in S$, $j \in F$ where $F \subseteq_{\text{fin}} \mathbb{Z}$. The notations $\sigma(X_k)$, $\sigma((X_j)_{j\in F})$ and $\sigma(X)$, refer to σ-algebras generated by X_k, the random vector $(X_j)_{j\in F}$, where $F \subseteq_{\text{fin}} \mathbb{Z}$ and the whole sequence X, respectively. Further, we denote by $\sigma(S')$ and $\sigma(\mathcal{S})$ the σ-algebra generated by any subset S' and any sub-σ-algebra of Borel subsets of S, respectively. For F a finite subset of $\mathbb{Z}$, call a function $f : S^{\mathbb{Z}} \to \mathbb{R}$ F-symmetric, if

$$f((x_j)_{j\in F}) = f((x_{g(j)})_{j\in F})$$

and, simply, symmetric, if f is F-symmetric for all finite subsets F of $\mathbb{Z}$. Let $\mathrm{F}(S^{\mathbb{Z}}, \mathbb{R})$ denote the measurable functions $S^{\mathbb{Z}} \to \mathbb{R}$ and by $\mathrm{F}^{\mathbb{P}_{\mathbb{Z}}}(S^{\mathbb{Z}}, \mathbb{R})$ the subset of symmetric measurable functions.

Definition 1.2.2 (Tail and exchangeable σ-algebras)**.** *The σ-algebra*

$$\mathcal{T}(X) = \cap_{K\subseteq_{\text{fin}}\mathbb{Z}}\sigma(\cup_{k\in\mathbb{Z}\setminus K}\sigma(X_k)) \tag{1.2.5}$$

[2] The seemingly alternative term "mixing measure" is not an option, as this is near mixing time, the random variable for Markov chains.

is called the tail σ-algebra of X. Similarly, the σ-algebra

$$\mathscr{E}(X) = X^{-1}\left(\sigma\left(\mathrm{F}^{\mathbb{P}_{\mathbb{Z}}}(S^{\mathbb{Z}}, \mathbb{R})\right)\right) \tag{1.2.6}$$

is called the σ-algebra of exchangeable events for X, or briefly the exchangeable σ-algebra.

The following is [Kle14, Theorem 12.26] with slight modifications suiting our context.

Theorem 1.2.3 (de Finetti's representation theorem)**.** *Let X be a random sequence on a probability space $(\Omega, \mathscr{A}, \mathbf{P})$. Then, the following conditions are equivalent:*

1. *X is exchangeable*
2. *X is conditionally iid directed by a $\mathcal{T}(X)$-measurable random probability measure ν.*

In this case, the tail σ-algebra $\mathcal{T}(X)$ can be replaced by the exchangeable algebra $\mathscr{E}(X)$ of X.

Since mixed iid sequences are clearly, exchangeable, Theorem 1.2.3 shows a random sequence is equivalently conditionally iid, or mixed iid. De Finetti's original statement, is recovered when $S = \{0, 1\}$. In particular it tells us that, if X is exchangeable and $X_j \in \{0, 1\}$ almost surely, for all $k \in \mathbb{N}$, then the event of $l \in [n]$ successes occurs with probability

$$\mathbf{P}\left(X_1 + \ldots + X_n = l\right) = \int \mathrm{d}P \binom{n}{l} P^l (1 - P)^{n-l}.$$

where ν is an undetermined probability distribution, the so-called random directing measure, of P. What is intriguing is that in the original formulation the probability parameter is random, because in the philosophical research of de Finetti, it has to be such, according to the subjective interpretation of probability, which is attributed to him, [Pla94]. We give below some intuitive explanation behind the mixture formula above, as in [Pla94]. In a urn model, suppose there are $m \geq 2$ urns containing black and white balls, and a trial consists in drawing with replacement a ball from one among the m urns. The trial is declared successful, if the ball is white. If a_i is the chance that urn i is picked and p_i is the probability of drawing a white ball from the urn labeled i, then the probability of drawing a white ball r times on n trials is $\sum_{i=1}^{n} \binom{n}{r} p_i^r (1 - p_i)^{n-r} a_i$. In other words, the n trials is a convex combination of Bernoulli success probabilities p_i, $i \in \{1, \ldots, n\}$. Suppose ideally that the

urns shrink down to single points on the interval $[0,1]$ and that one has given a probability density $a(x)$ of choosing a point x and a chance $p(x)$ that a white ball is picked out from it. The heuristics of replacing a finite summation with an integration lead us to $\int \binom{n}{r} p(x)^r (1-p(x))^{n-r} a(x) \mathrm{d}x$ as the formula to draw a white ball. The measure with respect to which the exchangeable sequence is conditionally iid is random and still undetermined or unknown. Indeed, by Theorem 1.2.3, there exists a $\mathcal{T}$-measurable function $F\colon \Omega \times \mathbb{R} \to \mathbb{R}$, where $\mathcal{T}$ is the tail σ-algebra of X, such that, for each $n \in \mathbb{N}$, $x_1, \dots x_n \in \mathbb{R}$,

$$\mathbf{P}\left(X_1 \le x_1, \dots, X_n \le x_n | \mathcal{T}\right) = F(\omega, x_1) \cdot \dots F(\omega, x_n)$$

If $n = 1$, then $0 \le F(x_1) \le 1$ and $1 = \int F(x_1)\, \mathrm{d}\, x_1 = F(0) + F(1)$, conditionally on $\mathcal{T}$. Thus, using the notation $P = F(1)$, $F(0) = 1 - P$, if $n \in \mathbb{N}, x_1, \dots, x_n \in \mathbb{R}$, then,

$$\mathbf{P}\left(X_1 = x_1, \dots, X_n = x_n | \mathcal{T}\right) = F(x_1) \cdot \dots \cdot F(x_n) = \prod_{j : x_j = 1} F(x_j) \cdot \prod_{j' : x_{j'} = 0} F(x_{j'})$$
$$= P^{\#\{j : x_j = 1\}} \cdot (1-P)^{\#\{j : x_j = 0\}} = P^{\sum_{j=1}^n x_j} \cdot (1-P)^{n - \sum_{j=1}^n x_j}.$$

Taking expectation on both sides of the identity concludes the original de Finetti's theorem for binary exchangeable random sequences.

Theorem 1.2.3 links exchangeability to the mixturing of iid measures, or roughly speaking, to their convex combinations. However, it leaves out a comparison with the other forms of distributional invariance in Definition 1.1.1. The gap is filled by the Ryll-Nardzewski's theorem [Ryl57]. We display its synthetic and modern version, found in [Kal05, Theorem 1.1.] adapted to our setup.

Theorem 1.2.4 (Ryll-Nardzewski, 1957)**.** *Let X be a random sequence. If X is spreadable, then X is conditionally iid.*

Theorem 1.2.4 unveils that it suffices to be spreadable for a process to have a directing random measure. As a result, selecting an infinite subsequence of marginals or applying a permutation (also made up by an infinite count of transpositions) does not alter the original sequence distribution.

The argument of Ryll-Nardzewski [Ryl57] features the σ-algebra of invariant events under shift. Mirroring advancements in refining de Finetti's theorem, the literature proposes alternative proofs to Theorem 1.2.4. For instance [Kal99] establishes an argument based on the so-called asymptotic invariance property and the Birkhoff's mean ergodic theorem [Bir31]. Further notable examples of proofs that avoid any use of ergodic theory. On one hand, [Kal05] presents one based on L^2-bounds, while [Ald81] finds an

ingenious martingale argument. The possibility of having results similar to Theorem 1.2.3 and Theorem 1.2.4 has been also explored in the literature of quantum stochastic processes.

1.3 Rotatability

It is clear that the invariance under orthogonal matrix action, Definition 1.1.2, implies all the other distributional symmetries, Definition 1.1.2, i.e. the spreadable, or equivalently exchangeable.

Example 1.3.1 (Standard Gaussians are rotatable)**.** *Let $Y = (Y_k, k \in J)$ be an iid sequence of random variables with Y_1 distributed as a centered unit variance Gaussian random variable. Then, the finite-dimensional Lévy characteristic function*[3] *reads as*

$$\mathbf{E}\left[\exp\left(i \sum_{j\in F} t_j Y_j\right)\right] = \exp\left(-\frac{1}{2}\sum_{j\in F} t_j^2\right), \tag{1.3.1}$$

for each $F \subseteq_{\text{fin}} J$, $(t_k, k \in F) \in \mathbb{R}^{\#F}$.

Theorem 1.3.2 (Freedman, 1962)**.** *Let X be a rotatable sequence. Then, there exists an almost surely non-negative random variables σ with $\sigma \geq 0$ a.s. and a standard normal sequence $Y_1, Y_2, \ldots$ that are independent from each other and such that*

$$X_j \overset{\text{a.s.}}{=} \sigma^2 Y_j, \qquad j = 1, 2, \ldots \tag{1.3.2}$$

More explicitly, let X, Y, σ be independent, $\sigma^2 \geq 0$ a.s. and $X \overset{\text{distr.}}{=} \sigma^2 Y$, Then, if μ_Z is the distribution of $Z \in \{X, Y, \sigma^2\}$,

$$\mu_X = \int_0^\infty \mathrm{d}\mu_{\sigma^2}(x) \int \mathrm{d}y \frac{1}{\sqrt{2\pi}\, x} e^{-\frac{y^2}{2x^2}}$$

Indeed, for all $t \in \mathbb{R}$,

$$\mathbf{P}(X \leq t) = \int_{-\infty}^{t} \mathrm{d}\left(\mu_{\sigma^2} \otimes \mu_Y\right)$$

Independence gives

$$\mathbf{P}(X \leq t) = \int_{sy\leq t} \mathrm{d}\mu_{\sigma^2} \mathrm{d}\mu_Y$$

[3] or Fourier transform.

and

$$\int_{sy\le t} \mathrm{d}\mu_{\sigma^2}\mathrm{d}\mu_Y = \int \mathrm{d}\mu_{\sigma^2}(s) \int_{y\le t/s} \mathrm{d}\mu_Y(y)$$

Since $\int \mathrm{d}\mu_{\sigma^2}(s) = \int_0^\infty \mathrm{d}\mu_{\sigma^2}(s)$ and

$$\int_{y\le t/s} \mathrm{d}\mu_Y(y) = \int_{y\le t/s} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \mathrm{d}y = \int_{y\le t} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2s^2}} \frac{1}{s} \mathrm{d}y,$$

it follows

$$\mathbf{P}\left(X \le t\right) = \int_0^\infty \mathrm{d}\mu_{\sigma^2}(s) \int_{y\le t} \frac{1}{\sqrt{2\pi s^2}} e^{-\frac{y^2}{2s^2}} \mathrm{d}y, \qquad \forall t \in \mathbb{R}\,. \tag{1.3.3}$$

We read in (1.3.3) the convex combination (1.2.4).

The proof makes use of de Finetti's theorem, as in the Freedman [Fre63]. We refer to [Kal21] for further details on the literature of classical distributional symmetries.

For the sake of completeness, we also report an earlier known result, equivalent to Freedman's theorem, which relates it to the problem of positive-definiteness in Hilbert spaces.

Theorem 1.3.3 (Theorem 2.2., [Sch38])**.** *Let $\mathcal{H}$ be a Hilbert space. Let $f : \mathcal{H} \to \mathbb{C}$ be a function. Then, the following are equivalent:*

1. *f is positive-definite,*

$$\sum_{j,k=1}^{n} f(x_j - x_k) c_j \overline{c_k} \ge 0\,,$$

for all $x_1, \dots, x_n \in \mathcal{H}, c_1, \dots, c_n \in \mathbb{C}$, $n \in \mathbb{N}$.

2. *f has spherical symmetry i.e. there exists $f_0 : \mathbb{R}_+ \to \mathbb{C}$ such that $f(x) = f_0(\|x\|)$ for all $x \in \mathcal{H}$, and f_0 is a Laplace-Stieltjes integral, meaning that*

$$f_0(t) = \int_0^{+\infty} e^{-t\sigma^2} \mathrm{d}\alpha(\sigma^2),\ \ t \ge 0\,,$$

for some $\alpha : \mathbb{R}_+ \to \mathbb{R}$ bounded and non-decreasing real function.

The key link is Bochner's theorem [Boc33, p.406] for which positive-definiteness in Hilbert spaces is equivalent to the invariance under rotations of the projections onto finite-dimensional spaces [Sch38, p.815, par.1.2.].

Chapter 2

Basics on C*-algebras

In this chapter, we collect basic facts about C*-algebras that introduce the formalism in which processes in quantum probability are expressed. The guiding references are [Mur90; Far19], [Sak98] supplemented by additional sources as needed. All vector spaces are understood to be over the field of the complex numbers, unless otherwise specified.

A *-Banach algebra A with norm $\|\cdot\|$, satisfying the so-called C*-condition $\|a^*a\| = \|a\|^2, a \in A$, is said to be a C*-algebra. The C*-condition and the norm submultiplicativity ensure that the involution is an isometry, and then a continuous map. A C*-algebra A is called unital, if there is a neutral element for the vector multiplication; otherwise it is termed non-unital.

We shall assume our C*-algebras come always with a unital element. For this reason, we omit the adjective "unital", while we write "non-unital C*-algebra" when this is the case. We warn the reader that forthcoming considerations can be presented for non-unital C*-algebras up to adding some technicality, that is, the standard procedure of unitalisation. All the due modifications to consider for the non-unital case will be notified at the due moment. In this dissertation, we use indifferently the expressions "sub-C*-algebra" or "C*-subalgebra" referring to a C*-algebra B that is mapped by the inclusion in a larger C*-algebra A with matching unital elements $1_B \mapsto 1_A$. In fact, B can be identified with the corresponding subalgebra that is a closed subset of A (norm topology).

Example 2.0.1. *The following algebras of functions are examples of commutative C*-algebras. All multiplications are intended to be pointwise multiplications, X denote a topological space and (M, μ) a σ-finite Borel measure space with measure μ.*

1. *$\mathbb{C}$, with *-operation the complex conjugation, and the Hermitian inner*

product norm. It is the only simple commutative invertible (i.e. all non-zero elements are invertible) algebra up to isomorphism.

2. *$A = C(X)$, for X a compact Hausdorff space and $C(X)$ the continuous functions over X with involution given by pointwise complex-conjugation and sup-norm. A is unital.*

3. *$A = C_0(X)$, for X a locally compact Hausdorff space and $C_0(X)$ the continuous vanishing at infinity functions over X, with involution given by pointwise complex-conjugation and sup-norm. A is non-unital.*

4. *$A = L^\infty(M, \mu)$ the essentially bounded measurable functions over (M, μ) with involution given by pointwise complex-conjugation and essential supremum norm $\|a\|_\infty = \inf\{t > 0 \,|\, \mu(\{x \in M \,|\, |a(x)| > t\}) = 0\}$. A is unital.*

Example 2.0.2 (Not a C*-algebra)**.** *Let $\mathbb{D} = \{z \in \mathbb{C} \,|\, |z| \leq 1\}$ be the unit closed disc centered at the origin and H, the functions which are continuous on $\mathbb{D}$ and analytic in its interior. H with pointwise algebra operations and with involution operation $f^*(z) = \overline{f(\overline{z})}$ is a commutative *-algebra. By Morera's theorem the sup-norm is complete. However, H is not a C*-algebra, as, for instance, $1 = \|f\|^2 \neq \|f^* f\| = e^2$, for the function $f \in H$, $f(z) = e^{iz}$, $z \in \mathbb{D}$.*

Example 2.0.3. *The following are basic examples of noncommutative C*-algebras, [Mur90].*

1. *$A = M_n(\mathbb{C})$, the $\mathbb{C}$-vector space of $n \times n$ matrices with complex entries, that is, $A = \left\{\sum_{j,k=1}^n a_{jk} e_j \otimes e_k \,\middle|\, a_{jk} \in \mathbb{C}, j, k \in [n]\right\}$. It is an algebra with the usual matrix product. This is identified with $\mathrm{B}(\mathbb{C}^n)$, the bounded linear operators on $\mathbb{C}^n$, since $x \mapsto ax$, with the usual matrix product $ax = \sum_{l=1}^n \left(\sum_{j,k=1}^n a_{jk} e_j \otimes e_k\right) \delta_{kl} x_l e_l = \sum_{j,k=1}^n a_{jk} x_k e_k$, is a linear mapping for each $a \in M_n(\mathbb{C}^n)$. In what follows, we will define the association $a \mapsto f_a(z) = az$ as a (faithful) representation[1] of $M_n(\mathbb{C})$ as an algebra of operators on $\mathbb{C}^n$. Write $a = (a_{jk})_{j,k\in[n]}$ for the matrix $a \in M_n(\mathbb{C})$, the matrix Hermitian transpose operation $a^* = (\overline{a_{kj}})_{j,k\in[n]}$ is an involution operation. The sup-norm[2] $\sup_{z\neq 0} \frac{\|f_a(z)\|}{\|z\|}$ is both complete and C*, since $\|a^* az\|^2 = \|a^2 z\|^2$, for all $a \in M_n(\mathbb{C})$, $z \in \mathbb{C}^n$. In particular, $n \times n$ upper triangular matrices form a subalgebra of the C*-algebra $M_n(\mathbb{C})$.*

[1] This term is going to be made rigorous in the upcoming Section 2.2.1.
[2] also called operator norm.

2. *Let H be a Hilbert space.* $\mathrm{B}(H)$ *is a C*-algebra with * defined as the Hilbert operator adjoint operation and equipped with the operator norm. We refer to* $\mathrm{B}(H)$ *as a concrete C*-algebra [Dav96], or sometimes still as an operator algebra.*

Among the pedagogical cases of Example 2.0.1-2.0.3, $C(X)$ and $\mathrm{B}(H)$ are of particular interest. The well-known Gelfand-Naimark theorem and Gelfand-Naimark-Segal representation theorem tell us that, for X a compact Hausdorff space, $C(X)$ is the archetypical example of a commutative (unital) C*-algebra while, for H a Hilbert space, $\mathrm{B}(H)$ is that of a general (possibly noncommutative) one. The proofs of the mentioned theorems rely on concepts and other important theoretical facts that we recall. Let A, B be *-algebras. A homomorphism $f : A \to B$ that is *-preserving i.e. $f(a^*) = f(a)^*$, for all $a \in A$, is called a *-homomorphism. When we want to stress that domain and codomain algebras are C*, we sometimes say C* algebra-homomorphism. When a *-homomorphism range is a field i.e. $\mathbb{R}$ or $\mathbb{C}$, we rather call it a selfadjoint functional. A *-isomorphism is defined as any bijective *-homomorphism (because the inverse of a *-homomorphism is still a *-homomorphism). A *-homomorphism is called isometric, if it preserves the norm. Clearly, any isometric *-homomorphism is injective.

C*-algebras form a category where arrows are the *-homomorphisms. They have associated their dual (Banach) spaces. If A is a C*-algebra, its dual, denoted by A^*, is defined as the set of all continuous (or equivalently, bounded) linear functionals. A^* is a Banach space for the sup-norm, $\|f\| = \sup_{\|a\|\leq 1} \|f(a)\|$, for all $f \in A^*$.

In the next sections, we shall see how C*-algebras bridge effectively to the algebra of Hilbert space bounded linear operators.

2.1 Spectra

The concept of spectrum is a well-known fundamental tool for a first analysis of C*-algebras. Let A be (a unital) algebra. For each $a \in A$, the set

$$\sigma(a) = \{\lambda \in \mathbb{C} \,|\, \lambda 1 - a \text{ not invertible in } A\} \subseteq \mathbb{C} \tag{2.1.1}$$

is called the spectrum of the element a, while $r(a) = \sup_{\lambda \in \sigma(a)} |\lambda|$ is called its spectral radius. We underline that the spectrum (2.1.1) is defined regardless of any unital element existence which can be added up by "unitization" procedure. The spectrum (2.1.1) is needed to establish various technical facts that build up proofs for the fundamental statements of the basic theory of C*-algebras. We mention a few of them.

Proposition 2.1.1 (Gelfand). *For all Banach algebras A, for all $a \in A$, $\sigma(a)$ is a non-empty compact subset of the complex plane.*

An element $a \in A$ is called selfadjoint, if $a^* = a$; it is called normal, if $a^*a = aa^*$. Any normal element $a \in A$ satisfies $r(a) = \|a\|$. This fact follows from the so-called Beurling formula,

$$r(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \inf_{n\geq 1} \|x^n\|^{1/n}$$

which is valid even for unital Banach algebras, that is, regardless of the C*-condition. As a consequence, $\|a\| = \sqrt{r(a^*a)}$, which shows that any *-algebra has at most one complete norm that satisfies the C*-condition.

By an ideal $I \subseteq A$, we mean a ring-theoretic ideal that is also a linear subspace of A; by a closed ideal we mean that it is a closed subset in the norm-topology. Any C*-algebra A is called simple, if for any two-sided closed ideal $I \subseteq A$, either $I = \{0\}$ of $I = A$. A *-ideal is an ideal that is stable for the *-operation (or selfadjoint, in short). On the note of the ubiquitous interplay between (linear) algebra and topology in C*-algebra theory, this definition can be given just in terms of closedness, since any closed subspace of a C*-algebra algebra is an ideal if and only if it is a *-ideal; the explanation of this (as well as other known facts in the sequel) is omitted because it involves slightly further theory i.e. approximate units. See [Far19, Lemma 2.1.1.].

Example 2.1.2 (Compact operators). *The set of compact operators $\mathsf{K}(H)$ is a non-trivial closed ideal of $\mathrm{B}(H)$.*

In light of Proposition 2.1.1 one can prove immediately the

Proposition 2.1.3 (Gelfand-Mazur). *Let A be a Banach algebra. If A is a unital and all nonzero elements are invertibles, then $A \simeq \mathbb{C}$.*

Indeed, if A is as in the hypothesis, then, by Proposition 2.1.1, $\sigma(a) \neq \varnothing$, for all $a \in A$. Therefore, there exists $\lambda \in \mathbb{C}$ such that $a - \lambda 1$ is not invertible, but $a - \lambda 1 \in A$, so that $a = \lambda 1$, and $a \in \mathbb{C}$.

Write with $\sigma(A)$ the set of linear functionals $\varphi : A \to \mathbb{C}$ that are multiplicative[3], $\varphi(ab) = \varphi(a)\varphi(b)$. *-homomorphisms between C*-algebras are norm-decreasing [Mur90, Theorem 2.1.7]. On one hand, this implies that any C*-algebra multiplicative linear functional is continuous, on the other, that C*-algebra injective *-homomorphisms are, in fact, isometric mappings.

[3]Some authors (e.g. [Far19]) call them characters, but we rather use this word when the context is limited to that of group representations and by definition the range is the torus, denoted $\mathbb{T}$.

If A is a commutative C*-algebra, then the spectrum of the algebra $\sigma(A)$ is in bijection with the maximal ideal space of A. Indeed, the kernel of any multiplicative linear functional $f : A \to \mathbb{C}$ is a maximal ideal of A. Conversely, if I is a maximal ideal of A, then it has codimension one, so that the projection functional $\varphi(x) = \lambda$ if $x = x_0 + \lambda 1$, $x_0 \in I$, is a multiplicative linear functional. Further, any multiplicative linear functional kernel as an ideal is a closed subset, because of its maximality. As a result, the algebra spectrum is related to the element spectrum as follows

$$\sigma(a) = \{\psi(a) \,|\, \psi \in \sigma(A)\} \,. \tag{2.1.2}$$

The spectrum inherits the weak* topology of the Banach dual space A^*. Each $\varphi \in \sigma(A)$ is continuous, that is, $\sigma(A) \subseteq A^*$. In fact, since any $\psi \in \sigma(A)$ is unital and norm-decreasing, $\sigma(A)$ is contained in the weak* unit ball. In addition, $\sigma(A)$ is a weak*-closed, therefore, by Banach-Alaoglu theorem, it is also weak*-compact and Hausdorff. Multiplicative linear functionals are selfadjoint, because any multiplicative linear functional is real on selfadjoint elements.

2.2 Gelfand Transform

Let A be a commutative C*-algebra. The evaluation map $\hat{\cdot} : A \to C(\sigma(A))$ that to each $a \in A$ assigns the function $\hat{a} : \sigma(A) \to \mathbb{C}$ defined on every $\psi \in \sigma(A)$ by

$$\hat{a}(\psi) = \psi(a) \tag{2.2.1}$$

is called the Gelfand transform. The Gelfand transform is a unital homomorphism, a group isomorphism when restricted to the open set given by the group of multiplicative invertible elements and a *-preserving map. It is isometric, since

$$\|\hat{a}\| = \sup\{|\hat{a}(\psi)| \,|\, \psi \in \sigma(A)\} = \sup\{|\lambda| \,|\, \lambda \in \sigma(a)\} = r(a) = \|a\|$$

where the last identity holds because A is commutative and then every element is normal. Thus, the Gelfand transform is injective and continuous. Further, $\hat{A}$ is a C*-subalgebra, and in particular an algebra of continuous functions over a compact that include the identity function and separate points in $\sigma(A)$. Therefore, the Gelfand transform $a \mapsto \hat{a}$ is surjective, by Stone-Weierstrass theorem.

These considerations sketch the proof of the fundamental

Theorem 2.2.1 (Gelfand-Naimark theorem, unital case)**.** *Let A be commutative and unital. Then, $A \simeq C(\sigma(A))$. Precisely, the Gelfand transform $\hat{\cdot} : A \to C(\sigma(A))$; defined by $a \mapsto \hat{a}$ is an isometric *-isomorphism.*

We emphasize that the Gelfand-Naimark theorem also holds for unital Banach algebras provided they are semisimple i.e. the kernel of the Gelfand transform, or radical of the algebra, is trivial. For non-unital Banach algebras the same conclusion holds except that the algebra spectrum is not necessarily compact but locally compact. Theorem 2.2.1 tells us that the Gelfand transform generalizes the Fourier transform.

Let A be a C*-algebra and $S \subseteq A$. We denote by $C^*(S)$ the smallest subalgebra of A containing S. This is definable as the intersection of all unital C*-subalgebras containing S, since C*-algebras are stable under arbitrary intersections.

A first consequence of Theorem 2.2.1 is that selfadjoint elements have real spectra. Let $a \in A$ be selfadjoint. By Theorem 2.2.1, the Gelfand transform is an isometric *-isomorphism from $C^*(\{1, a\})$, the commutative C*-algebra generated by a, into the C*-algebra of continuous functions over the compact $\sigma(C^*(\{1, a\}))$, that sends a to $\hat{a}$, where $\hat{a}(\psi) = \psi(a)$ for all $\psi \in \sigma(C^*(\{1, a\}))$. As $a = a^*$, it follows, $\hat{a} = \hat{a^*} = (\hat{a})^* = \bar{\hat{a}}$, that is, $\psi(a) = \hat{a}(\psi) \in \mathbb{R}$. Finally, by (2.1.2), $\sigma(a) \subseteq \mathbb{R}$.

A second consequence involves normal elements. If $a \in A$ is a normal element, then the subalgebra $C^*(\{1, a, a^*\})$ is commutative. Let $a \in A$ be normal and $B = C^*(\{1, a, a^*\})$. Then, $\sigma(a)$ is in bijection with $\sigma(B)$, by (2.1.2), because spectra in C*-algebras do not depend on the ambient algebra of invertibility, see e.g. [Tak79, Proposition I.4.8]. Thus, Gelfand-Naimark theorem, promotes this bijection to a homeomorphism. It is known that if A is commutative, then metrizability of $\sigma(A)$ implies the separability of A. If A is commutative and separable then, by Theorem 2.2.1, A is isometrically isomorphic to $C(\sigma(A))$. By Banach-Alaoglu theorem, the separable $C(\sigma(A))$ has weak*-compact unit ball, so that $\sigma(A)$ is metrizable.

Definition 2.2.2. *A C*-algebra representation is a pair (H, π) such that $\pi : A \to \mathrm{B}(H)$ and A is a C*-algebra, H is a Hilbert space and π is a *-homomorphism. The set $\pi(A)\xi = \{\pi(a)\xi \,|\, a \in A\}$ is called orbit of a vector ξ. Any subspace $F \subseteq H$, such that $\pi(A)F \subseteq F$ is called an invariant subspace. A representation (H, π) is called*

i) faithful, if $\ker \pi = \{0\}$, or equivalently, if π is injective.

ii) cyclic, if $\pi(A)\xi$ is norm-dense in H, for some $\xi \in H$ and, in this case, ξ is said to be cyclic for π;

iii) irreducible, if, either $F = \{0\}$ or $F = H$ holds, for each invariant subspace $F \subseteq H$.

Theorem 2.2.1 represents commutative C*-algebras with a faithful representation and allows to think commutative C*-algebras as continuous function algebras.

Example 2.2.3 (Continuous functions as multiplication operators)**.** *Let A be commutative and unital C*-algebra. Let $X = \sigma(A)$ be the compact Hausdorff spectrum of A. Then, by Theorem 2.1.1, $A \simeq C(X)$ for the isometric *-isomorphism provided by the Gelfand transform (2.2.1). Let $H = L^2(X)$ the Hilbert space of square-summable complex-valued functions over X. Then, for each $a \in A$, $\hat{a} \in C(X)$ and it is mapped to a bounded linear operator on H that is the multiplication by $\hat{a}$, $m_a : h \mapsto \hat{a}h$, where $h \in H$. The mapping $\hat{a} \mapsto m_a$ is clearly a continuous unital *-monomorphism, that when composed with the Gelfand transform yields a faithful representation $A \to \mathrm{B}(\mathcal{H})$.*

The other mentioned basic fundamental theorem, the Gelfand-Naimark-Segal representation theorem, treats the case of noncommutative C*-algebras. This essentially says that any C*-algebra can be represented as an algebra of operators as discussed in the upcoming Section 2.6.

Another consequence of Theorem 2.2.1 is the so-called continuous functional calculus, an extension of the polynomial and holomorphic calculi. Mentioning this tool allows us to introduce positive elements, and consequently define states of a C*-algebra.

2.3 Continuous functional calculus

Let $a \in A$ be a normal element and $B = C^*(\{1, a, a^*\})$ commutative C*-subalgebra of A it generates.

If X, Y are compact Hausdorff spaces and $\phi : X \to Y$ continuous, then the map $\phi^{\mathsf{t}} : C(Y) \to C(X)$ given by $\phi^{\mathsf{t}}(f) = f \circ \phi$ is a *-homomorphism, and an isomorphism, if ϕ is a homeomorphism. ϕ^{t} is called the transpose of ϕ [Mur90, p.42]. The Gelfand transform (2.2.1) is a *-isometric isomorphism. For merely typographical reasons and limited to this section, we momentarily denote it with $\mathcal{G} : B \to C(\sigma(B))$ and with $\mathcal{G}^{-1} : C(\sigma(B)) \to B$ its inverse. For the fixed element a, $\mathcal{G}(a) : \sigma(B) \to \sigma(a)$ is a homeomorphism, therefore, the transpose $(\mathcal{G}(a))^{\mathsf{t}}$ is a *-isomorphism $C(\sigma(a)) \to C(\sigma(B))$ and the composition

$$\Phi_a = \mathcal{G}^{-1} \circ (\mathcal{G}(a))^{\mathsf{t}} \tag{2.3.1}$$

is a *-isomorphism $\Phi_a : C(\sigma(a)) \to B$ such that for $\mathrm{id}_{\sigma(a)}$, the identity function on $\sigma(a)$, $\mathrm{id}_{\sigma(a)}(z) = z$, $\Phi_a(\mathrm{id}_{\sigma(a)}) = \mathcal{G}^{-1}(\mathcal{G}(a))^{\mathsf{t}}(\mathrm{id}_{\sigma(a)}) = a$, and $\Phi_a(1_{C(\sigma(a))}) = 1_B$, where $1_{C(\sigma(a))}$ denotes the constant function 1.

Since $\Phi_a \circ p$ is defined on each bivariate polynomial $p(z, \overline{z})$ as the unique map that sends the inclusion function z to 1_B and preserves the constant function 1, Stone-Weierstrass theorem allows to give the following

Definition 2.3.1 (Continuous functional calculus, Definition 2.5. [Con97])**.** *Let A be a C*-algebra, $a \in A$ a normal element, and Φ_a the mapping (2.3.1). The element $f(a) \in A$ where $f(a) = \Phi_a(f)$ is called the continuous functional calculus of a.*

The continuous functional calculus has a number of consequences worth mentioning.

Injective homomorphisms of C*-algebras need to be isometric; in particular, faithful representations are isometric. The continuous functional calculus has the so-called spectral mapping property with respect to continuous functions and C*-algebras i.e. if A is a C*-algebra and $a \in A$, then $f(\sigma(a)) = \sigma(f(a))$, for all continuous $f : \sigma(a) \to \mathbb{C}$. An interesting fact about the spectra of normal elements is that, if $a \in A$ is normal and $\lambda \notin \sigma(a)$, then $\|(a-\lambda)^{-1}\|^{-1} = \operatorname{dist}(\lambda, \sigma(a))$, for the functional $\operatorname{dist}(\lambda, S) = \inf_{s \in S} \|\lambda - s\|$, defined for all convex subset S. In particular, if $a, b \in A$ are normal, then the Hausdorff distance[4] of their spectra is bounded $\operatorname{dist}(\sigma(a), \sigma(b)) \leq \|a - b\|$, [Far19, Lemma 1.4.7].

Subsets of basic importance in any C*-algebra are the group of unitary elements, i.e. each $u \in A$ that is an isometry ($u^*u = 1$) and a co-isometry ($uu^* = 1$); the subset of projections i.e. $p \in A$ such that $p = p^* = p^2$. While the unitaries are normal elements, the projections form a subset of A_{sa}.

By the continuous functional calculus, for all normal $a \in A$, a is a unitary iff $\sigma(a) \subseteq \partial\mathbb{D}$, the boundary of the unit disk. In more detail, if $f : \sigma(a) \to \mathbb{C}$, $f(z) = |z|$, then $f(z) = z^*z$, and $f(a) = a^*a = 1$ But Φ_a is isometric *-isomorphism. Therefore, $f \equiv 1$, the constant function on $\sigma(a)$, which implies $\sigma(a) \subseteq \partial\mathbb{D}$. By a similar argument, any normal a is a projection iff $\sigma(a) \subseteq \{0, 1\}$. The importance of projections is underlined by the fact that, for any C*-algebra A, A is projectionless iff $\sigma(A)$ is connected.

The continuous functional calculus makes it possible to take the square-root of any element $a \in A$, provided it is selfadjoint and $\sigma(a) \subseteq \mathbb{R}_+$. In this spirit, an element $a \in A$ is said to be positive, if $\sigma(a) \geq 0$ with elementwise meaning (i.e. $\lambda \geq 0$, for all $\lambda \in \sigma(a)$), and $a = a^*$. Let A^+ be their set. As a consequence of the spectral mapping property of the continuous functional calculus, for all $a \in A$ and $t \geq 0$, $ta \in A_+$ and no non-zero element a is such that both $a \in A_+$ and $-a \in A_+$. Moreover, for all $t \geq 0$ and $a \in A_{\text{sa}}$, with $\|a\| \leq t$, then $a - t1$ is selfadjoint and $\|a - t1\| = r(a - t1) = \sup_{\lambda \in \sigma(a)} |\lambda - t| = \sup_{\lambda \in \sigma(a)} \lambda - t$,

[4]The distance given by $\operatorname{dist}(x, y) := \sup_{x \in K_1} \sup_{y \in K_2} \|x - y\|$, for all $x, y \in \mathbb{C}$.

since $\|a\| \leq t$. Thus, $a \in A_+$ iff $\|a - t1\| \leq t$, for all selfadjoint with norm bounded by a positive real number t. From this fact, one has that $a + b \in A_+$, for all $a, b \in A_+$, and consequently, that A^+ is a convex cone.

Being a cone, the set of positive elements A^+ partially orders A. Further, A^+ generates linearly A. More precisely, every element $a \in A$ is a linear combination of at most four elements in A_+.

On one hand, for any $a \in A$, one can write $a = a_1 + i\, a_2$, where a_1, a_2 are uniquely determined by $a_1 = 2^{-1}(a + a^*)$ and $a_2 = (2i)^{-1}(a - a^*)$ and called the real and imaginary part of a. On the other, any self-adjoint element $a \in A$ has unique positives $a_+, a_- \in A^+$ such that $a = a_+ - a_-$ and $a_+ a_- = 0$, which is also a consequence of the continuous functional calculus, since the functions $f(t) = t \vee 0$ and $f(t) = t \wedge 0$, for all $t \in \sigma(a)$, are real continuous on $\sigma(a)$ with $t = f(t) + g(t)$ and $f(t)g(t) = 0$, for all $t \in \sigma(a)$. Thus, for all $a \in A$, $a = (a_1)_+ - (a_1)_- + i\,((a_2)_+ - (a_2)_-)$, where a_1, a_2 are the real and imaginary part of a, respectively. By this decomposition, a is normal, whenever its real and imaginary part commute. In addition, it follows that any $a \in A$ can be written a sum of at most four unitary elements. To see this, let $a \in A$, and $a = x_1 + ix_2$ for the selfadjoint $x_1, x_2 \in A$, given by $x_j = \|a\|^{-1} a_j$, $j \in [2]$, where a_1 and a_2 are the real and imaginary parts of a, respectively. The element $1 - x_1^2$ is positive, since $\sigma(x_1) \subseteq (0, 1)$, by the spectral mapping property and $\|x_1\| \leq 1$. Therefore,

$$x_{1,k} = \frac{x_1 + (-1)^k i\sqrt{1 - x_1^2}}{2}, \;\; k \in \{1, 2\},$$

are unitaries such that $x_1 = x_{1,1} + x_{1,2}$. Similar considerations apply to $1 - x_2^2$, and consequently $a = \|a\| \left(\sum_{k=1}^2 x_{1,k} + i \sum_{k=1}^2 x_{2,k}\right)$. In the sequel, for any given C*-algebra A, similarly to what happens in the case $A = \mathbb{C}$, we denote $\mathrm{Re}(a)$ and with $\mathrm{Im}(a)$ the real and imaginary part of an element $a \in A$, respectively.

As a result of the observations made, if $a \in A$, then a is positive iff $a = b^*b$, for some $b \in A$.

Example 2.3.2 (Positive elements in concrete C*-algebras)**.** *In the concrete algebra* $\mathrm{B}(H)$*, one can consider the usual condition of positivity that declares* $S \in \mathrm{B}(H)$ *positive, iff* $\langle Sh, h\rangle \geq 0$*, for all* $h \in H$*.*

Indeed, suppose S *is selfadjoint and the spectrum is contained in the nonnegative real line. Then,* $S = T^*T$ *for some* $T \in \mathrm{B}(H)$ *and* $\langle Sh, h\rangle = \langle T^*Th, h\rangle = \|Th\|^2 \geq 0$*, for all* $h \in H$*. For the converse, let* $S \in \mathrm{B}(H)$ *be such that* $\langle Sh, h\rangle \geq 0$*, for all* $h \in H$*. Then, if* $h \in H$*,* $\langle S^*h, h\rangle = \overline{\langle h, S^*h\rangle} = \overline{\langle Sh, h\rangle} = \langle Sh, h\rangle$*, which, by polarization, yields* $S = S^*$*. Thus,* S *is selfadjont and* $\sigma(T) \subseteq \mathbb{R}$*. As is known, the spectrum of a bounded linear operator*

is the union of its approximate spectrum[5] $\sigma_{\mathrm{ap}}(T)$ *and its compression spectrum*[6] $\sigma_{\mathrm{cp}}(T)$*. Based on this fact, we prove that* $\sigma(T)$ *is contained in the nonnegative real line.*

Let $t \in \sigma(T)$ *be such that* $t < 0$*. We note that* $|\langle t1-Th, h\rangle| = |t\,\|h\| + \langle Th, h\rangle| \geq t\,\|h\|^2$*. Then,* $t1 - T$ *is bounded from below. Thus,* $t \notin \sigma_{\mathrm{ap}}(T)$*. However,* $\sigma_{\mathrm{cp}}(T) = \varnothing$*, if* T *is selfadjoint. Here is why. If* $t \in \sigma_{\mathrm{cp}}(T)$ *then* $\mathrm{Ran}(t1 - T)$ *is not dense in* H*. Thus, the orthogonal complement of the closure, which is* $\mathrm{Ker}(t1 - T^*)$*, is not trivial; therefore,* $\mathrm{Ker}(t1 - T)$ *is not trivial, meaning* $t1 - T$ *is not injective which contradicts its being bounded from below. In conclusion,* $t \geq 0$*.*

Positive elements play a key rôle in determining basic properties of states, that is, noncommutative probability distributions, as well as C*-algebra representations.

2.4 States

Definition 2.4.1 (State)**.** *Let* A *be a unital *-Banach algebra. A linear functional* $\varphi \in A^*$ *is called positive, if* $\varphi(a^*a) \geq 0$*, for all* $a \in A$*; unital, if* $\varphi(1) = 1$*. Let* A *be a C*-algebra, a linear functional* $\varphi \in A^*$ *is called a state, if* φ *is positive and unital. We denote by* A^*_+ *the set of positive linear functionals and with* $\mathcal{S}(A)$ *its subset of states,* $\mathcal{S}(A) \subseteq A^*_+$*. A state* $\varphi \in A^*$ *is called tracial, if* $\varphi(ab) = \varphi(ba)$*, for all* $a, b \in A$*.*

The set of positive linear functionals A^*_+ is a convex cone and a subset of the Banach dual of A, the space A^* of continuous linear functionals on A, which is a locally convex space. The state space $\mathcal{S}(A) \subseteq A^*_+$ is also convex. However, if $\varphi \in A^*_+$ is a positive linear functional, then φ is bounded and can be normalized. In other words, states can be defined as normalized positive linear functionals. Since, similarly any linear functional φ with $\|\varphi\| = \varphi(1)$ is a positive functional [Mur90, Corollary 3.3.4], states can also be defined, in fact, as linear functionals with $\varphi(1) = \|\varphi\|$ and $\varphi(1) = 1$. States are selfadjoint, as a consequence of the decomposition in the positive ad negative part of an element; for each $\varphi \in \mathcal{S}(A)$, $(a, b) \mapsto \varphi(a^*b)$ defines a sesquilinear form that satisfies the Cauchy-Schwarz inequality $|\varphi(a^*b)|^2 \leq \varphi(a^*a)\varphi(b^*b)$, for all $a, b \in A$. It follows that the norm of a state can be computed over the positives. For each C*-algebra A and C*-subalgebra $B \subseteq A$, each $\varphi_0 \in \mathcal{S}(B)$ can be extended to a state $\varphi \in \mathcal{S}(A)$ such that $\varphi\restriction_B = \varphi_0$. Indeed, for all

[5] The set of all and only $\lambda \in \mathbb{C}$ such that there exists sequence $(x_n)_n$ of normalized vectors $\|x_n\| = 1$, $\forall n \in \mathbb{N}$ for which $\|(\lambda 1 - T)x_n\| \to 0$ as $n \to \infty$.

[6] The set of all and only $\lambda \in \mathbb{C}$ such that $\mathrm{Ran}(\lambda 1 - T)$ is not dense in H.

$\varphi \in \mathcal{S}(B)$, by Hahn-Banach theorem, $\varphi : B \to \mathbb{C}$ is extended to a bounded linear functional $\varphi : A \to \mathbb{C}$ such that $\|\varphi_0\| = \|\varphi\|$. Since φ_0 is positive, then $\|\varphi_0\| = \varphi_0(1) = 1$ and $\varphi \in \mathcal{S}(A)$.

Example 2.4.2 (States of $\mathrm{B}(H)$, vector states)**.** *Let H be a Hilbert space. Let $h \in H$ be a norm-one vector. Then, $T \mapsto \langle Th, h\rangle$ defines a linear functional $\mathrm{B}(H) \to \mathbb{C}$, clearly positive and bounded because of Cauchy-Schwarz inequality, $|\langle Th, h\rangle| \leq \|T\| \, \|h\|^2$. States of this kind are called vector states.*

2.5 Extreme states

We introduce some convex analysis theory around C*-algebra states. The notions we refer to are defined in more detail and abstraction in Section A.1 of the Appendix.

Let V be a locally convex topological vector space.

Definition 2.5.1 (Face)**.** *Any subset F of a convex subset $K \subseteq V$ is called a face, if it contains convex combinations of its points and, for all $x, y \in K, t \in (0,1)$, if $tx + (1-t)y \in F$, then $x, y \in F$.*

Definition 2.5.2 (Extreme point)**.** *Any singleton $\{x\} \subseteq K$ that is a face of any convex subset $K \subseteq V$ is called an extreme point of K. We denote by $\mathcal{E}K$ the set of extreme points of K.*

Lemma 2.5.3 (Lemma 4.118, [BS23])**.** *Let A be unital commutative C*-algebra. Then, $\sigma(A) = \mathcal{E}\mathcal{S}(A)$.*

Remark 2.5.4. Since $\varphi(1) = \|\varphi\| = 1$, then $\mathcal{S}(A) \subseteq A_1^*$ the dual unit ball which is weak*-compact, and it is a closed subset, similarly to what said for the algebra spectrum. Thus, $\mathcal{S}(A)$ is a weak*-compact convex set. The Krein-Milman theorem (Theorem A.1.4) ensures that its set of extreme points $\mathcal{E}\mathcal{S}(A)$ is non-empty and that its weak*-closure convex hull is the whole state space,

$$\mathcal{S}(A) = \overline{\mathrm{conv}\, \mathcal{E}\mathcal{S}(A)} \tag{2.5.1}$$

where $\mathrm{conv}(S)$ is the convex hull of a subset $S \subseteq \mathcal{S}(A)$.

The works of [BL59; CM63; Phe66] and (2.5.1) establish that states on C*-algebras are, in fact, represented by an integral formula. However, it is not guaranteed the barycentric measure uniqueness. See the Appendix Section A.1 for a more extensive exposition on this point.

By [Mur90, Theorem 5.1.8.], an extreme state can be characterized as pure, i.e. a state $\varphi \in \mathcal{S}(A)$ such that, for all positive linear functional $\omega \in A^+$ with $\omega \leq \varphi$ there exists a constant $t \in (0,1)$ such that $\omega = t\varphi$. For this reason and as done in [Far19], we make no distinction between pure and extreme states and, further, call "mixed" every non-pure state. This latter terminology is clarified in Section 3.2 of Chapter 3.

Remark 2.5.4 assures that there exist extreme states in any given C*-algebra. This "abundance of states" can also be proven by Segal's extension theorem for pure, or extreme, states (Theorem 2.5.6 below).

Definition 2.5.5 (Commutant)**.** *Let H be a Hilbert space. For each $A \subseteq \mathrm{B}(H)$, the set $A' = \{T \in \mathrm{B}(H) \,|\, TS = ST, S \in A\}$ is called the commutant of A. The relative commutant of any subalgebra B of A is defined as the set $\{a \in A \,|\, ab = ba, b \in B\}$ denoted by $A' \cap B$.*

Theorem 2.5.6 (Segal, [Tak79], Lemma 4.11)**.** *Let A be a C*-algebra and $A_0 \subseteq A$ any of its subalgebras. If $\omega \in \mathcal{S}(A)$ such that the restriction $\omega_0 = \omega\restriction_{A_0}$, is $\omega_0 \in \mathcal{ES}(A_0)$, then $\omega(ab) = \omega(a)\omega(b)$, for all $a \in A$ and $b \in A' \cap A_0$ where $A' \cap A_0$ is the relative commutant.*

Definition 2.5.7. *By a kernel, we mean a map $\kappa : S \times S \to \mathrm{B}(H)$ where S is a set and H a Hilbert space. It is termed positive-definite, if*

$$\sum_{j,k=1}^{n} \langle \kappa(x_j, x_k) h_k, h_j \rangle \geq 0$$

for all $n \in \mathbb{N}$, $h_1, \ldots, h_n \in H$, for all $x_1, \ldots, x_n \in X$.

Definition 2.5.8. *Let S be a semigroup with an involution J i.e. J is an idempotent unary operation on S with $J(xy) = J(y)J(x)$, $x, y \in S$. Then, a operator-valued map $T : S \to \mathrm{B}(H)$ is called completely positive, if the kernel $x, y \mapsto T(J(x)y)$ is a positive-definite.*

An example of semigroup with involution is the algebraic dual of a *-algebra A, the set of linear functionals over A; it is an abelian semigroup for the componentwise addition and involution given by $\varphi^*(a) = \overline{\varphi(a^*)}$, $a \in A$, for all linear functionals φ. Another example includes the positive-definite functions mentioned in Theorem 1.3.3, when the domain Hilbert space is regarded as its additive semigroup with involution $x \mapsto -x$. States are particular completely positive maps[7], as any positive linear map between C*-algebras $\varphi : A \to B$ is completely positive, if either A or B is commutative. See [EL77, Section 4].

[7] after generalizing Definition 2.5.8 by allowing the maps range to be any abstract C*-algebra

Definition 2.5.9 (Conditional expectation in C*-algebras, Definition 3.3.3, [Far19]). *Let A be a C*-algebra and $B \subseteq A$ a C*-subalgebra. A completely positive contraction $\Theta : A \to B$ such that $\Theta(bac) = b\Theta(a)c$, for all $a \in A$, $b, c \in B$, is called a conditional expectation (on B).*

Based on a well-known result of Tomiyama [Tom57], the above definition is equivalent to the condition that $\Theta : A \to A$ is a projection of norm one i.e. $\Theta^2 = \Theta$ (map composition), and $1 = \|\Theta\| = \sup_{a \in A^+} \|\Theta(a)\|$, for all $a \in A$.

2.6 The GNS construction and representation

Below, we give a quick recap of basic C*-representation theory, useful to present the fundamental GNS construction and GNS representation theorems. These are known to be generalizations of abstract harmonic analysis for unitary representations and positive-type functions of locally compact groups.

Any representations $(\pi, H_1), (\rho, H_2)$ are said to be intertwined, if, for some $T \in \mathrm{B}(H_2, H_1)$, $T\rho(a) = \pi(a)T$, for all $a \in A$ and T is called an intertwiner for π and ρ. The relation of being intertwined is an equivalence relation. If representations $(\pi, H_1), (\rho, H_2)$ are intertwined by a unitary operator, they are called unitarily equivalent; if $H_1 \subseteq H_2$ and ρ and π are intertwined by the projection $P_{H_1} \in \mathrm{B}(H_2)$, then (ρ, H_1) is called a subrepresentation of (π, H_2). For a family of representations $\{(H_j, \pi_j)\}_j$ of a given C*-algebra A, the mapping $\oplus_j \pi_j$ is the *-homomorphism $A \to \oplus_j \mathrm{B}(H_j)$ defined by $\oplus_j \pi_j(a)(\oplus_j h_j) = \oplus_j(\pi_j(a)(h_j))$. The representation $(\oplus_j H_j, \oplus_j \pi_j)$ is called the direct sum of the family.

Any concrete C*-algebra has vector states (Example 2.4.2). Also any "abstract" C*-algebra state is a vector state up to representing it, which is always possible.

Theorem 2.6.1 (Gelfand-Naimark-Segal construction). *Let A be a unital C*-algebra A. For each state $\varphi \in \mathcal{S}(A)$, there exists a cyclic representation $(H_\varphi, \pi_\varphi, \xi_\varphi)$ such that, for a unique cyclic vector ξ_φ,*

$$\varphi(a) = \langle \pi_\varphi(a)\xi_\varphi, \xi_\varphi \rangle, \tag{2.6.1}$$

for all $a \in A$. Furthermore, $(H_\varphi, \pi_\varphi, \xi_\varphi)$ is unique up to unitary equivalence, that is, if (F, ρ, η) is another cyclic representation with $\varphi(a) = \langle \rho(a)\eta, \eta \rangle$, for all $a \in A$, then, there exists a unitary intertwiner between π and ρ, such that $U\xi_\varphi = \eta$.

For each C*-algebra A, the triples $(\pi_\varphi, H_\varphi, \xi_\varphi), \varphi \in \mathcal{S}(A)$ are referred to as the GNS representations of A.

Lemma 2.6.2 ([Mur90], Theorem 3.3.6.)**.** *Let A be a C*-algebra. If $a \in A$ is normal, then there exists $\varphi \in \mathcal{S}(A)$ such that $|\varphi(a)| = \|a\|$. We call such state a norming state for a.*

As an immediate consequence of the existence of norming states for normal elements, $\mathcal{S}(A)$ separates points in A, that is, for all nonzero $a \in A$, there exists $\varphi \in \mathcal{S}(A)$ such that $\varphi(a) \neq 0$. Indeed, each $a \in A$ can be written as $a = a_1 + ia_2$, for selfadjoint elements $a_1, a_2 \in A$. Since a_1 is normal, it has a norming state φ, and $|\varphi(a)| = |\varphi(a_1) + i\varphi(a_2)| \geq |\varphi(a_1)| = \|a_1\| \neq 0$, that implies $\varphi(a) \neq 0$. If an element is positive then the norming state can be taken to be pure, [Mur90, Theorem 5.1.11.]. Thus pure states separate elements as well.

In Theorem 2.6.1, the existence part is based on envisioning a Hilbert space structure in the C*-algebra suitably divided by the closed left-ideal $N = \overline{\text{Span}}\{a \in A \,|\, \varphi(a^*a) = 0\}$; essentially, the quotient space A/N is a pre-Hilbert space with inner product

$$\langle\langle [a], [b] \rangle\rangle = \varphi(b^*a), \qquad a, b \in A$$

where $[x] = x + N$, $x \in A$. For the uniqueness part of Theorem 2.6.1, for all cyclic representations (H, π, ξ) and (F, ρ, η) with (2.6.1),

$$\|\rho(a)\eta\|^2 = \langle \rho(a)^*\rho(a)\eta, \eta \rangle = \langle \pi(a)^*\pi(a)\xi, \xi \rangle = \|\pi(a)\xi\|^2$$

Therefore, the linear extension of the map that sends $\pi(a)\xi$ to $\rho(a)\eta$ is an isometry; by cyclicity, it is surjective and a unitary map i.e. It sends unitaries to unitaries.

Besides Theorem 2.6.1, the following facts are fundamental in the praxis.

1. Any representation[8] is the direct sum of cyclic representations;

2. Cyclicity of representations is preserved by unitary equivalence, since $\overline{\pi(A)\xi} = H$, then $\overline{U^*\rho(A)U\xi} = H$ and $\overline{\rho(A)\xi} = UHU^* = H$ (closure in norm), for (π, H, ξ) a cyclic representation of a C*-algebra A, unitarily equivalent to the representation (ρ, F) by the intertwiner U.

3. Any cyclic representation is unitarily equivalent to a GNS-representation.

Therefore,

$$\oplus_{\varphi \in \mathcal{S}(A)} (H_\varphi, \pi_\varphi, \xi_\varphi) \tag{2.6.2}$$

[8]We are assuming all the mentioned C*-algebras are unital; if non-unital, this fact refers to all non-degenerate representations i.e. if $\xi \in \mathcal{H}$ is such that $\forall T \in \pi(\mathfrak{A})\ T\xi = 0$, then $\xi = 0$.

contains any other representation as a subrepresentation, up to unitary equivalence. This is the reason why (2.6.2) is called the universal representation of A. Moreover, it is faithful as follows by Lemma 2.6.2, because $\oplus_{\varphi\in\mathcal{S}(A)}\pi_\varphi(a) = 0$ implies , $\pi_\varphi(a) = 0$, for all $\varphi \in \mathcal{S}(A)$, that is, $\pi_\varphi(a^*a) = 0$ and then $0 = \varphi(a^*a) = \|a^*a\|$, whence $\|a\| = 0$. The universal representation witnesses the situation portrayed in the celebrated

Theorem 2.6.3 (Gelfand-Naimark-Segal representation theorem, 1943)**.** *Any unital C*-algebra A is isometrically *isomorphic to a concrete C*-algebra of operators. In other words, any C*-algebra is faithfully represented on a Hilbert space by an isometric representation.*

A consequence of Theorem 2.6.3 is that, for any C*-algebra A, the norm of elements can be expressed as a supremum of the modulus of their images under states,

$$\|a\| = \sup_{\varphi\in\mathcal{S}(A)} \sqrt{\varphi(a^*a)}\,, \qquad a \in A\,. \tag{2.6.3}$$

Another consequence is that, if A is a separable C*-algebra, then the GNS -representation $(\mathcal{H}_\varphi, \pi_\varphi, \xi_\varphi)$ for any $\varphi \in \mathcal{S}(A)$, provides a separable Hilbert space $\mathcal{H}_\varphi$ on which it is represented as a concrete algebra. The direct sum (2.6.2) may represent a given C*-algebra on a very broad Hilbert space that, likely, lacks nice properties (e.g. the Hilbert space might not be separable). We illustrate some more useful known alternatives. In first place, there is the direct sum

$$\oplus_{\varphi\in\mathcal{ES}(A)}(H_\varphi, \pi_\varphi, \xi_\varphi) \tag{2.6.4}$$

which is called the atomic representation of A. We can diminish further its complexity by considering its subrepresentation

$$\oplus_{\varphi\in\mathcal{P}}(H_\varphi, \pi_\varphi, \xi_\varphi) \tag{2.6.5}$$

where $\mathcal{P}$ is any maximal subset of nonequivalent pure states. Representation (2.6.5) is called the reduced atomic representation of A. Representations (2.6.4) and (2.6.5) are still faithful and distinguish themselves with respect to (2.6.2) in that summands are irreducibles. The next theorem ([Seg47, Theorem 1]) explains why this is the case.

Theorem 2.6.4. *Let A be a unital C*-algebra and $\varphi \in \mathcal{S}(A)$. Then, $\varphi \in \mathcal{ES}(A)$ if and only if $(H_\varphi, \pi_\varphi, \xi_\varphi)$ is an irreducible representation.*

Together with the abundance of states observed in Remark 2.5.4, the theorem tells us that there are plenty of irreducible representations for any given C*-algebra. The proof of Theorem 2.6.4 is based on the following facts

that we report without proof. In first place, if A is a C*-algebra and $\varphi \in \mathcal{S}(A)$, then for all positive linear functional $\omega \in A^*_+$, there exists a unique operator in the commutant of the GNS representation by φ, $v \in (\pi_\varphi(A))'$ such that, for all $a \in A$, $\omega(a) = \langle \pi_\varphi(a) v \xi_\varphi, \xi_\varphi \rangle$ and $0 \leq v \leq 1$, in the sense of operator positivity [Mur90, Theorem 5.1.2.]. Second, for any C*-algebra A and representation π on Hilbert space $\mathcal{H}$, π is irreducible iff every vector $h \in \mathcal{H}$ is cyclic for π [Mur90, Theorem 5.1.5]. The last ingredient is known as the generalization of the Schur's lemma.

Lemma 2.6.5 (Schur's lemma)**.** *Suppose π is a representation of A on some Hilbert space H. Then, π irreducible if and only if $\pi(A)' = \mathbb{C}1$*

A consequence of Schur's lemma is that in commutative unital C*-algebras A, characters are extreme states, $\sigma(A) \subseteq \mathcal{ES}(A)$. In short, if $\varphi \in \sigma(A)$, then $\varphi(a^*a) \geq 0$, for all $a \in A^+$, and $\varphi(1) = 1$, by multiplicativity, so $\varphi \in \mathcal{S}(A)$. Since its GNS-representation is one-dimensional, the multiplicative linear functional φ is extreme. More precisely, its GNS-representation is one-dimensional for what follows. Let $\varphi \in \sigma(A)$ and (H, π, ξ) its GNS. Then, for all $h \in H$, by cyclicity, $0 \neq \|h\| = \|\pi(a)\xi\| = \varphi(a^*a)$ for some $a \in A$. Similarly $0 \neq \varphi(b^*b) = \|\pi(b)\xi\|^2$, for all $b \in A$. However, the element $x = \varphi(b)a - \varphi(a)b \in A$, is such that $\varphi(x) = 0$, that is, $\|\pi(x)\xi\|^2 = \varphi(x^*x) = 0$. Thus, $\|\varphi(b)\pi(a)\xi - \varphi(a)\pi(b)\xi\|^2 = 0$, that is, $h = \pi(a)\xi$ is in the linear span of $\pi(b)\xi$.

2.7 Tensor products

We collect essential facts about the theory of tensor product for Banach spaces, Hilbert spaces, and (C)*-algebras. For details we refer to [Tak79]. Let V, W be vector spaces. The linear functional that sends any $f \in V^*$ to $f(v)w \in W$ is called the simple tensor of $v \in V$ and $w \in W$ and denoted by $v \otimes w$. Let $V \odot W$ denote the algebraic tensor product of V and W, defined as, for instance[9], the vector space spanned by the formal linear combinations $\sum_{j=1}^n \lambda_j v_j \otimes w_j$ on the simple tensors $v_j \otimes w_j$, for $\{v_j\}_j$ and $\{w_j\}_j$ linear basis of V and W. If X, Y are Banach spaces a norm β on the $X \odot Y$ is said to be a cross-norm, if

$$\beta(x \otimes y) = \beta(x)\beta(y), \tag{2.7.1}$$

[9]There are definitions that are basis-free, but all of them are equivalent to these "basis-constrained" definitions.

for all $x \in X$, $y \in Y$. The completion under β of $X \odot Y$ yields a Banach space $X \otimes_\beta Y$. Let $\|\cdot\|_X$ and $\|\cdot\|_Y$ be the norm of X and Y respectively. The norm

$$\beta_p(x) = \inf\left\{\sum_{j=1}^{k} \|x_j\|\|y_j\| : x = \sum_{i=1}^{k} x_j \otimes y_j\right\} \tag{2.7.2}$$

is the greatest cross-norm, i.e. $\beta_p(x) \geq \beta(x) \geq \beta_i(x)$, for any other cross-norm β on $X \odot Y$. The functional

$$\beta_i(x) = \sup\left\{\left|\sum_{j=1}^{n} f(x_j))g(y_j)\right| : f \in X^*, \|f\|_X \leq 1; g \in Y^*, \|g\|_Y \leq 1\right\} \tag{2.7.3}$$

is a cross-norm for $X \odot Y$ [Tak79, Proposition 2.1.] as well.

Let X, Y be Banach spaces. The norm (2.7.2) is called the projective cross-norm; this defines the projective tensor products $X \otimes_p Y$, by completion of $X \odot Y$. The injective tensor product, $X \otimes_i Y$, of *-Banach spaces is defined similarly. It can be shown using the the adjunct norm [Tak79, (5), p.189] that tensor products of Banach spaces are Banach for a unique cross-norm, if its injective and projective cross-norm coincide. While this fact also affects tensor products of *-Banach algebras, tensor products are known to behave nicely on Hilbert spaces. Let H be a Hilbert space and H^*, its Banach dual. Let H_1, H_2 be Hilbert spaces with inner products $\langle\cdot,\cdot\rangle_1, \langle\cdot,\cdot\rangle_2$, respectively. On algebraic tensor product $H_1 \odot H_2$, the form given by

$$\langle\sum_{j=1}^{n} v_{1,j} \otimes v_{2,j}, \sum_{j=1}^{n} w_{1,j} \otimes w_{2,j}\rangle = \sum_{j=1}^{n}\langle v_{1,j}, w_{1,j}\rangle\langle v_{2,j}, w_{2,j}\rangle\,, \tag{2.7.4}$$

for all $v_{1,j}, v_{2,j} \in H_1, w_{1,j}, w_{2,j} \in H_2, j \in [n], n \in \mathbb{N}$, is sesquilinear and positive-definite. It makes $H_1 \odot H_2$ a pre-Hilbert space.

Definition 2.7.1. *$H_1 \otimes H_2$ is defined as the Hilbert space completion of the $H_1 \odot H_2$ equipped with inner product (2.7.4).*

We denote by $H^{\otimes n}$ the n-fold tensor product of H by itself. In particular, $\mathcal{H}^{\otimes 0}$ is a one-dimensional space, for which $\mathcal{H}^{\otimes 0} = \mathbb{C}\Omega$, where we have fixed the notation of a linearly generating vector Ω, called conventionally, the vacuum vector. We write $x^{\otimes n} \in H^{\otimes n}$, for all $x \in H$, that is defined as $x^{\otimes 0} = 1 \in \mathbb{C}$, $x^{\otimes 1} = x$ and $x^{\otimes n} = x^{\otimes(n-1)} \otimes x$, for $n \geq 2$. The direct sum of a family of Hilbert spaces $\{H_j\}_j$, with unspecified index set, is the Hilbert space that is the linear span of elements denoted by $\oplus_j h_j$, identified with the sequence $\{h_j\}_j$ of elements $h_j \in H_j$ that has finitely many non-zero terms, and completed with respect to the norm induced by the inner product $\langle\oplus_j h_j, \oplus_j k_j\rangle_{\oplus_j H_j} = \sum_j \langle h_k, k_j\rangle$. The

direct sum of Hilbert spaces defines the tensor product of a pair of Hilbert spaces [KR83, Remark 2.6.8]. For all H_1, H_2 Hilbert spaces, $T_1 \in \mathrm{B}(H_1)$ and $T_2 \in \mathrm{B}(H_2)$, the tensor product $T_1 \otimes T_2$ is the unique bounded linear operator of $H_1 \otimes H_2$ such that $h_1 \otimes h_2 \to T_1 h_1 \otimes T_2 h_2$, for all $h_1 \in H_1, h_2 \in H_2$, [KR83, Proposition 2.6.12.]. If H_1, H_2 are Hilbert spaces and $T_1 \in \mathrm{B}(H_1), T_2 \in \mathrm{B}(H_2)$ the operator $T_1 \otimes T_2$ is called their spatial tensor product. In fact, even if T_1, T_2 are unbounded, their algebraic tensor product, denoted $T_1 \odot T_2$, is defined. Let A_1, A_2 be *-Banach algebras. Then, $A_1 \odot A_2$ is naturally a *-algebra with multiplication given by componentwise multiplication and involution $(a_1 \otimes a_2)^* = (a_1^* \otimes a_2^*)$, $a_1 \in A_1, a_2 \in A_2$. Similarly, the algebraic tensor products of *-algebras is a *-algebra with the involution acting componentwise on simple tensors. We note that any *-algebra has at most one complete C*-norm, whereas algebraic tensor product may have many C*-norm that complete them to C*-tensor products.

Definition 2.7.2 (C*-norm on algebraic tensor products)**.** *Let A_1, A_2 C*-algebras. Then, a norm $N : A_1 \odot A_2 \to [0, +\infty)$ is said to be a C*-norm on $A_1 \odot A_2$, if it has the C*-condition.*

$A_1 \odot A_2$ can be turned into many C*-algebras, one for each possibly available C*-norm.

Let A_1, A_2 be C*-algebras. Whenever a C*-norm is defined on their algebraic product, any representation $\pi : A_1 \otimes A_2 \to \mathrm{B}(H)$ has unique *-homomorphisms (called its restrictions) $\pi_j : A_j \to \mathrm{B}(H)$, $j \in [2]$, such that $\pi(a_1 \otimes a_2) = \pi_1(a_1)\pi_2(a_2)$, for all $a_1 \in A_1, a_2 \in A_2$.

Definition 2.7.3. *The functional on $A_1 \odot A_2$ given by*

$$\|x\|_{\max} = \sup \|\pi(x)\|, \qquad x \in A_1 \odot A_2 \tag{2.7.5}$$

where the supremum runs over all representations π of $A_1 \odot A_2$, is called the projective C-norm on $A_1 \odot A_2$. The norm*

$$\|x\|_{\min} = \sup \|(\pi_1 \otimes \pi_2)x)\|, \qquad x \in A_1 \odot A_2 \tag{2.7.6}$$

where the supremum runs over all representations π_j of A_j, $j \in [2]$, is called the injective C-norm.*

The projective C*-norm defines the so-called projective C*-tensor product, with notation $A_1 \otimes_{\max} A_2$, as the the completion of $A_1 \odot A_2$ with respect to (2.7.5). Similarly, it is defined the injective C*-tensor product $A_1 \otimes_{\min} A_2$ with respect to (2.7.6). If (2.7.5) and (2.7.6) coincide, then $A_1 \odot A_2$ is C* for a unique C*-norm, say the projective, and we shall write simply $A_1 \otimes A_2$.

Because of Theorem 2.6.3, $\|\cdot\|_{\max}$ is the largest C*-norm on $A_1 \odot A_2$. Immediately, $\|x\|_{\min} \leq \|x\|_{\max}$, $x \in A_1 \odot A_2$, by Definition 2.7.3. Moreover, $\|\pi_1(a_1) \otimes \pi_2(a_2)\| = \|\pi_1(a_1)\| \, \|\pi_2(a_2)\|$, for all $\pi_j : A_j \to \mathrm{B}(H_j)$ representations, $j \in [2]$, where on the left-hand side is applied the spatial tensor product of operators. It follows that $\|\cdot\|_{\min}$ is a cross-norm. For the projective C*-norm, $\|x\|_{\max} \leq \beta_p(x)$, for all $x \in A_1 \otimes A_2$, since any representation of the algebraic tensor product is extended to the projective tensor product of A_1 and A_2 as Banach spaces. Since

$$\|a_1\| \, \|a_2\| = \|a_1 \otimes a_2\|_{\min} \leq \|a_1 \otimes a_2\|_{\max} \leq \beta_p(a_1 \otimes a_2) = \|a_1\| \, \|a_2\| \, ,$$

for all $a_1 \otimes a_2 \in A_1 \otimes A_2$, then the projective C*-norm is also a cross-norm.

The next known fact informs us about the factorization of extreme states in the C*-tensor products we are after.

Theorem 2.7.4 ([Tak79], Theorem 4.14.)**.** *Let A, B be C*-algebras and denote by $\otimes_{\min}$ the completion of their algebraic tensor product with respect to the injective cross-norm (2.7.3). Then, the following are equivalent*

1. *If $\varphi \in \mathcal{ES}(A \otimes_{\min} B)$, then $\varphi = \varphi_1 \otimes \varphi_2$ for $\varphi_1 \in \mathcal{ES}(A)$ and $\varphi_2 \in \mathcal{ES}(B)$*
2. *either A or B is commutative*
3. *The norm in (2.7.3) agrees with the norm in (2.7.6).*

Therefore, a unique C*-norm is also determined if cross-norms (2.7.2) and (2.7.3) coincide, although they are not C*.

In addition to Theorem 2.7.4, it is known that, if either A or B is abelian, then any C*-norm on $A \odot B$ is the injective C*-norm [Tak79, Lemma 4.18] and that this is the smallest C*-norm [Tak79, Theorem 4.19]. Consequently, all C*-norms are also cross-norms (2.7.1).

The C*-norm is unique whenever the C*-algebras A, B are finite-dimensional, basically because they are direct sum of finite-size matrix algebras.

Example 2.7.5 (Direct sum of matrix algebras)**.** *By [Mur90, Theorem 6.3.9.], all finite-dimensional C*-algebras are direct sum of finite-size complex matrix algebras. Let $A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$, for $n_1, \ldots, n_k \in \mathbb{N}$. If B is a C*-algebra, then $A \odot B$ is isomorphic to the *-algebra $\oplus_{j=1}^{n}(M_{n_j}(\mathbb{C}) \odot B)$ via the *-isomorphism uniquely determined by $(a_1, \ldots, a_n) \otimes b \mapsto (a_1 \otimes b, \ldots, a_n \otimes b)$. The latter is a *-algebra whose spatial norm is a complete C*-norm.*

More in general, the uniqueness of the C*-norm is guaranteed, if either A_1 or A_2 in expression $A_1 \odot A_2$ is nuclear.

Definition 2.7.6 (Nuclear algebra)**.** *A C*-algebra A is called nuclear, if for any C*-algebra B, the injective and projective cross-norms are equal, $\beta_i = \beta_p$.*

Example 2.7.7 (Compact operators)**.** *Let H be a Hilbert space. The C*-algebra of compact operators $\mathsf{K}(H)$ is nuclear.*

If $\{e_j\}_j$ is an orthonormal basis of H, the projection on the diagonal simple tensors $P = \sum_{j=1}^{n} e_j \otimes e_j$ gives the space $P\mathsf{K}(H)P$ which is finite-dimensional because isomorphic to $M_n(\mathbb{C})$. A family of subspace, where each is spanned by a finite subset of the fixed orthonormal basis vectors is an inductive system. It follows that the family of projections $\{P_F\}_F$ where $P_F = \sum_{j\in F} e_j \otimes e_j$, for $F \subseteq_{\text{fin}} \mathbb{Z}$ is a inductive system for their are (partially) ordered as C-algebra elements. As a result, $\{P_F\mathsf{K}(H)P_F\}_F$ is an inductive system of nuclear subalgebras, because each $P_F\mathsf{K}(H)P_F$, $F \subseteq_{\text{fin}}$ finite-dimensional. By the "abundance" of nuclear algebras, [Mur90, Theorem 6.3.10.], their inductive limit is nuclear. Since $\lim_F P_F h = h$, for all $h \in H$, so that also $\lim_F P_F T = T$ on every finite-rank operator T, the inductive limit of system of subalgebras $\{P_F\mathsf{K}(H)P_F\}_F$ is $\mathsf{K}(H)$. Thus, $\mathsf{K}(H)$ is nuclear.*

The following is proven by Takesaki in [Tak64, Theorem 1], where it is also established that all type I C*-algebras are nuclear and nuclearity is stable under inductive limit formations.

Theorem 2.7.8. *If A is commutative, then A is nuclear.*

We are ready to introduce noncommutative probability spaces in the next chapter.

Chapter 3

Noncommutative Probability

In this chapter, we introduce and review essential concepts in quantum probability. More precisely, we see how the C*-formalism developed generalizes classical probability. First of all, states generalize classical probability measures.

Let X be a compact Hausdorff space. The Riesz-Markov theorem tells us that $C(X)^*$ is a Banach space isometrically isomorphic to the Banach space of Radon measures on X. Precisely, for all $\varphi \in C(X)^*$, there exists a unique regular Borel finite measure $\mu_\varphi \in \mathcal{M}(X)$ such that

$$\varphi(f) = \int f(x)\mathrm{d}\mu_\varphi(x) \tag{3.0.1}$$

with $\|\varphi\|_{C(X)^*} = \|\mu_\varphi\|_{\mathrm{TV}}$, and, since $\|\mu_\varphi\|_{\mathrm{TV}} = \mu_\varphi(X)$, μ_φ is probability iff φ is normalized.

Let $\mathfrak{A}$ be a commutative unital C*-algebra. Then, $\sigma(\mathfrak{A})$ is weak*-compact and Hausdorff. By Gelfand-Naimark theorem, $\mathfrak{A}$ is isometrically *-isomorphic to $C(\sigma(\mathfrak{A}))$. As a consequence, the Banach dual $\mathfrak{A}^*$ is isometrically *-isomorphic to $C(\sigma(\mathfrak{A}))^*$. In turn, these are the Radon measures on $\sigma(\mathfrak{A})$ and each state $\varphi \in \mathcal{S}(\mathfrak{A})$ has the integral representation (3.0.1). Thus, for commutative C*-algebras, states are probability measures and arise as mixtures under the same conditions determined by the classical theory. Lastly, as observed in Section 2.6, the extreme measures correspond to multiplicative states $\mathcal{S}(\mathfrak{A}) \cap \sigma(\mathfrak{A})$ that is, Dirac point mass measures.

We investigate existence and, specifically, uniqueness of barycentric formulae (3.0.1) of G-invariant states when $\mathfrak{A}$ is not commutative and G is any among $\mathbb{Z}$, $\mathbb{J}_\mathbb{Z}$, $\mathbb{P}_\mathbb{Z}$ or $\mathbb{O}_\mathbb{Z}$.

In the next sections, we make precise what we mean by a quantum stochastic process, what is the independence in our context, and make, en passant, comparisons with existing work in the literature.

3.1 Quantum stochastic processes

Definition 3.1.1 (Quantum Probability space, Random variable)**.** *A quantum probability space is a pair* $(\mathfrak{A}, \varphi)$ *where* $\mathfrak{A}$ *is a C*-algebra and* $\varphi \in \mathcal{S}(\mathfrak{A})$*. Let* $(\mathfrak{A}, \varphi)$ *be a quantum probability space and* $\mathfrak{B}$ *another C*-algebra. A* $\mathfrak{A}$*-valued quantum random variable* j *is a unital *-homomorphism* $j : \mathfrak{B} \to \mathfrak{A}$*.*

Example 3.1.2. *A operator-valued quantum random variable is a representation* $\pi : \mathfrak{A} \to \mathrm{B}(\mathcal{H})$ *for some C*-algebra* $\mathfrak{A}$*. If* π *is cyclic for some* $\xi \in \mathcal{H}$*, its distribution is the vector state it induces* $a \mapsto \langle \pi(a)\xi, \xi \rangle$*.*

Definition 3.1.3 (Quantum Stochastic Process)**.** *A quantum stochastic process of index set* J *is a quadruple* $(\mathfrak{A}, \mathcal{H}, \{\pi_k\}_{k\in J}, \xi)$*, where* $\mathfrak{A}$ *is a C*-algebra, called the sample algebra,* $\mathcal{H}$ *is a Hilbert space on which* $\mathfrak{A}$*, for all* $k \in J$*,* $\pi_k : \mathfrak{A} \to \mathrm{B}(\mathcal{H})$ *is a representation of* $\mathfrak{A}$ *on* $\mathcal{H}$ *and* $\xi \in \mathcal{H}$ *is a unit cyclic vector for the von Neumann algebra* $\bigvee_k \pi_k(\mathfrak{A})$*.*

Continuous time index set J are considered with regard to distributional symmetries in [AL93], to which the reader is referred for a more general definition. As we deal with discrete-time processes, we assume J to be countable and fix $J = \mathbb{Z}$ for the rest of thesis; whenever the index set is missing from the notation, it is understood to be $\mathbb{Z}$, the additive discrete group of integers. In the sequel, we also identify $\mathbb{S}_{\mathbb{Z}} = \mathbb{Z}$.

In [CFL16, Theorem 2.3], Definition 3.1.3 is shown to have an equivalent formulation, that is more convenient to deal with (semi)group actions realizing distributional invariance notions. This is based on the infinite free product of a single C*-algebra.

We recall what is a C*-algebra universally generated by a pair $(\mathcal{G}|\mathcal{R})$ where $\mathcal{G}$ is a set of generators and $\mathcal{R}$ is a set of relations, since, as authors [Far19; Bla06] pointed out, there is no common, standard, rigorous definition of what the relations need to be for a good praxis.

Consider the language for normed *-algebras. Let $\mathcal{G} = \{x_j \,|\, j \in J\}$ be a set of variables and $\mathcal{R}$ a set of relations on it such that, for each $j \in J$, it contains the formula $\|x_j\| \leq \mathsf{c}_j$, for some real scalar constant symbol c_j, and, formulae $\mathsf{K}_l \mathsf{f}_l(\|t_1\|, \ldots, \|t_n\|)$, where K_l is a 1-ary relation symbol, f_l is a function symbol and $t^l_1, \ldots, t^l_n$ *-noncommutative polynomials[1] in variables belonging to $\mathcal{G}$, for each l in a set L.

The pair $(\mathcal{G}|\mathcal{R})$ is said to be *-represented, if there exists a set $\{T_j \,|\, j \in J\}$ of bounded linear operators over some Hilbert space $\mathcal{H}$, non-negative real

[1] In fact, polynomial terms can defined recursively as follows. Every single constant symbol or variable is a polynomial term, if t_1, t_2 are terms then $t_1 + t_2$ is a term, $t_1 t_2$ is a term, and λt_1 is a term for each constant scalar symbol.

numbers c_j, continuous functions $f_l : K \to [0, +\infty)$ with $K_l \subseteq \mathbb{C}$, for all $l \in L$, and an interpretation map that sends $\mathsf{c}_j \mapsto c_j$, $x_j \mapsto T_j$, $\mathsf{f}_l \mapsto f_l$, $\mathsf{K}_l \mapsto K_l$, for each $l \in L$, so that, for all $j \in J$, $\|T_j\| \leq c_j$ and, for all $l \in L$, $f_l(\|t_1^l(T_1, \ldots, T_n)\|, \ldots, \|t_n^l(T_1, \ldots, T_n)\|) \in K_l$, where $\{\mathsf{f}_l\}_{l\in L}$ $\{\mathsf{K}_l\}_{l\in L}$ are set of functions and relation symbols respectively, belonging to the language of $\mathcal{R}$.

For a pair of generators and relations $(\mathcal{G}|\mathcal{R})$, any representation spans a selfadjoint *-algebra of operators on $\mathcal{H}$ satisfying $\mathcal{R}$ and containing $\{T_j \,|\, j \in J\}$. The functional

$$p(x) = \sup \|\pi(x)\|$$

where the supremum runs[2] over all *-representations of $(\mathcal{G}|\mathcal{R})$ is finite, because we required that $\mathcal{R}$ contains an upper bound for each generator. Then, it is a seminorm and defines a C*-norm after quotienting by its annihilator set $\{T \in \mathrm{B}(\mathcal{H}) \,|\, p(T) = 0\}$ which completes the *-representation to a C*-representation.

Any C*-algebra can be seen as universally generated. It suffices to include in $\mathcal{G}$ all the elements and determining an upper bound for the norm of all *-noncommutative polynomial on all elements. Further, any universal C*-algebra is unique up to isomorphism.

An example of universal C*-algebra is the free product of a family of unital C*-algebras, for $\mathcal{R}$ containing only relations imposing bounds on operator uniform norms. For any family of unital C*-algebras $\{\mathfrak{A}_k\}_k$, their unital free product is defined to be the C*-algebra, denoted $*_k\mathfrak{A}_k$, that has unique *-monomorphisms $i_k : \mathfrak{A}_k \hookrightarrow *_k\mathfrak{A}_k$, called the free product canonical (or coordinate) mappings, such that for any other C*-algebra $\mathfrak{B}$ and family of *-homomorphisms $\Phi_k : \mathfrak{A}_k \to \mathfrak{B}$, $k \in \mathbb{Z}$, there exists a unique *-homomorphism Φ on $*_k\mathfrak{A}_k$ such that $\Phi \circ i_k = \Phi_k$, for all $k \in \mathbb{Z}$. See [Avi82; VDN92]. For any unital C*-algebra, $\mathfrak{A}$, we denote

$$\mathfrak{P} = *_k\mathfrak{A}\,. \tag{3.1.1}$$

Any $\varphi \in \mathcal{S}(\mathfrak{P})$ might be called the distribution of a quantum stochastic process defined on $\mathfrak{P}$. For the sake of clarity, let $\varphi \in \mathcal{S}(\mathfrak{P})$ and denote by $\{i_k\}_k$ the family of coordinate mappings and with $(\mathcal{H}, \pi, \xi)$ the GNS representation of $(\mathfrak{P}, \varphi)$. Then, $\pi : \mathfrak{P} \to \mathrm{B}(\mathcal{H})$, and $\varphi(a) = \langle \pi(a)\xi, \xi\rangle$, for all $a \in \mathfrak{P}$. The map $\pi_k(a) = \pi \circ i_k\,(a)$, for all $k \in \mathbb{Z}$ and $a \in \mathfrak{A}$, defines a family of representations of $\mathfrak{A}$ on $\mathcal{H}$ and the vector $\xi \in \mathcal{H}$, being cyclic for π, it is cyclic for the von Neumann algebra $\bigvee_k \pi_k(A)$. Viceversa, let $(\mathfrak{A}, \mathcal{H}, \{\pi_k\}_{k\in\mathbb{Z}}, \xi)$ be any quadruple as in Definition 3.1.3. For each $k \in \mathbb{Z}$, $\pi_k : \mathfrak{A} \to \mathrm{B}(\mathcal{H})$ is a representation and a homomorphism of $\mathfrak{A}$. Then, by universality, there exists

[2] similarly for the projective C*-norm (2.7.5).

a homomorphism $\pi : \mathfrak{P} \to \mathrm{B}(\mathcal{H})$ such that $\pi_k = \pi \circ i_k$, for all $k \in \mathbb{Z}$. Next, observe that the C*-algebra $\mathfrak{P}$ is generated by the norm closure of the linear span of elements $i_{s_1}(a_{s_1}) \dots i_{s_n}(a_{s_1})$, for all $a_{s_1}, \dots, a_{s_n} \in \mathfrak{A}$, each finite subsequence of consecutively distinct indices $s_1 \neq \dots \neq s_n \in \mathbb{Z}$, and any positive integer $n \in \mathbb{N}$. Thus, a state $\varphi \in \mathcal{S}(\mathfrak{P})$ is uniquely determined by

$$\varphi\left(\prod_{k=1}^{n} i_{s_k}(a)\right) = \left\langle \prod_{k=1}^{n} \pi_{s_k}(a)\xi, \xi \right\rangle, \tag{3.1.2}$$

for all $a \in \mathfrak{P}$, $s_k \in \mathbb{Z}$, $k \leq n, k, n \in \mathbb{N}$. Note that this is the analogue of the Daniell-Kolmogorov theorem, when the family of correlation kernels

$$\{\varphi(a_1^* \cdot \dots \cdot a_n^* \cdot b_n \cdot \dots \cdot b_1) \,|\, a_j, b_k \in \mathfrak{A}, 1 \leq j, k \leq n, n \in \mathbb{N}\} \tag{3.1.3}$$

is interpreted as the finite-dimensional distributions, for any C*-algebra $\mathfrak{A}$. In conclusion, states on infinite free products (3.1.1) can be identified as distribution of quantum stochastic processes.

If a quantum stochastic process is given as a state on the infinite free product (3.1.1), then the classical case is recovered as soon as the coordinate maps commute.

Agreeing with classical intuition, exchangeable states on $\mathfrak{P}$ are, in fact, stationary where exchangeable and stationary are given by the invariance conditions on finite-dimensional quantum joint distributions (3.1.2), [CF16].

Typically, one is interested in studying states on the C*-algebra in which some sort of commutation relations holds. Let $[a, b] = ab - ba$, for all $a, b \in \mathfrak{A}$, denote the commutator in a C*-algebra $\mathfrak{A}$, and

$$I = \overline{\mathrm{Span}\,\{a\,[i_k(a_1), i_l(a_2)]\,b \,|\, j, l \in \mathbb{Z}, k \neq l, a_1, a_2 \in \mathfrak{A}, a, b \in \mathfrak{P}\}} \tag{3.1.4}$$

be the two-sided closed *-ideal, containing all commutators in $\mathfrak{P}$. The literature focuses in studying states of the quotient C*-algebras

$$\mathfrak{P}_I = \mathfrak{P}/I \tag{3.1.5}$$

modulo two-sided closed *-ideals I alternative to (3.1.4) and reproducing commutation relations of quantum mechanics and their deformed versions [BS94; Bożl2]. See [BB24] for an overview. When I is as in (3.1.4), $\mathfrak{P}_I$ turns out to be the infinite projective tensor product (3.3.2), which is the universal object in the category of unital C*-algebras with commuting range morphisms.

We are interested in the case when covariant systems arise on $\mathfrak{P}_I$ for (semi)groups of transformations drawn from $\mathbb{Z}, \mathbb{J}_\mathbb{Z}, \mathbb{P}_\mathbb{Z}$, and $\mathbb{O}_\mathbb{Z}$. In the literature, these cases are successfully analyzed by means of the known C*-dynamical system theory that studies such systems, as we shall illustrate in the next section.

3.2 C*-dynamical systems

This section clarifies how to formulate G-invariance for states, to rephrase de Finetti and Freedman's theorems.

Let $\mathfrak{A}$ be a C*-algebra. We denote by $\mathrm{End}(\mathfrak{A})$ (resp. $\mathrm{Aut}(\mathfrak{A})$) the semigroup of *-endomorphisms (resp. *-automorphisms) of $\mathfrak{A}$, respectively. Let G be a semigroup, a monoid or a group. If a semigroup G acts on $\mathfrak{A}$ and α denotes this action, then, for all $g \in G$, $\alpha(g) \in \mathrm{End}(\mathfrak{A})$, whereas $\alpha(g) \in \mathrm{Aut}(\mathfrak{A})$, whenever G is a group.

Definition 3.2.1 (C*-dynamics)**.** *A triple $(\mathfrak{A}, G, \alpha)$ is called a C*-dynamical (or covariant) system, if $\mathfrak{A}$ is a C*-algebra and G a (semi)group acting on $\mathfrak{A}$ by the homomorphism $\alpha : G \to \mathrm{End}(\mathfrak{A})$. We call $(\mathfrak{A}, G, \alpha)$ reversible (or conservative), when G is a group acting on $\mathfrak{A}$, and irreversible (or dissipative) otherwise.*

As observed in [CF16], dissipative dynamics can be studied by requiring that the action is through completely positive maps. In our definition, we do not assume for G any property (e.g. local compactness), because, in fact, we have in mind precisely the instances of Chapter 1. Note that any covariant system $(\mathfrak{A}, \mathbb{J}_{\mathbb{Z}}, \alpha)$ is irreversible. Specifically, each α_g's has only a left inverse. Clearly, if $(\mathfrak{A}, G, \alpha)$ is a covariant system, then $(\mathfrak{A}, S, \alpha{\restriction}_S)$ is a covariant system, for any semigroup $S \subseteq G$. If $(\mathfrak{A}, G, \alpha)$ is a reversible system, then $(\mathfrak{A}, H, \alpha{\restriction}_H)$ is reversible, for each $H \leq G$, where $\leq$ stands for the subgroup relation. Thus, if $\mathbb{O}_{\mathbb{Z}}$ acts on $\mathfrak{A}$ by automorphism, then $(\mathfrak{A}, G, \alpha{\restriction}_G)$ is reversible for $G = \mathbb{P}_{\mathbb{Z}}$.

Definition 3.2.2 (G-invariant states)**.** *Let $(\mathfrak{A}, G, \alpha)$ be a C*-dynamical system. Then, any state $\varphi \in \mathcal{S}(\mathfrak{A})$ is said to be G-invariant, if*

$$\varphi(\alpha_g(a)) = \varphi(a)\,, \tag{3.2.1}$$

for all $a \in \mathfrak{A}$, $g \in G$, where α_g is the image of g under α. For any C-dynamical system $(\mathfrak{A}, G, \alpha)$, denote by $\mathcal{S}^G(\mathfrak{A}) \subseteq \mathcal{S}(\mathfrak{A})$ the subset of G-invariant states on $\mathfrak{A}$. As usual, we refer to the elements of $\mathcal{ES}^G(\mathfrak{A})$ as the ergodic states.*

Let V be a locally convex topological vector space (in short TVS) and V^* its topological dual.

Definition 3.2.3. *Let $X \subseteq V$ be a compact convex subset. Let $\mu \in \mathcal{M}_1(X)$ be a Borel probability measure on X. Then, μ is called supported on a measurable $S \subseteq X$, if $\mu(X \setminus S) = 0$; it is called pseudosupported on a measurable $S \subseteq X$, if $\mu(B) = 0$, for all Borel regular subset $B \subseteq X$ such that $B \cap S = \varnothing$. For convenience, we denote by $\mathrm{supp}(\mu)$ the support of a measure μ.*

Definition 3.2.4 (Barycentric decomposition)**.** *Let $X \subseteq V$ be any compact convex subset and $x \in X$. If $\mu \in \mathcal{M}_1(X)$*

$$f(x) = \int_X f(y)\mathrm{d}\mu(y)\,, \tag{3.2.2}$$

for all $f \in V^*$, then, μ is said to be a barycentric decomposition[3] of x.

Definition 3.2.5 (Choquet simplex, Bauer simplex)**.** *A simplex X such that each point has a unique barycentric decomposition that is a probability supported on $\mathcal{E}X$ is said to be a Choquet simplex. A Bauer simplex is a Choquet simplex whose extreme points form a closed subset.*

We investigate the weak*-compact convex set of G-invariant states, searching for a barycentric decomposition that is uniquely determined for each given G-invariant state. Extensive work has been done [DS58; BL59; CM63; Phe66] in determining the existence and uniqueness of a Borel probability measure such that (3.2.2) and under general hypothesis. Nevertheless, these criteria are not always applicable to states. We report a consequent reading for the weak*compact convex of C*-algebra states. Let $\mathfrak{A}$ be a separable C*-algebra. Then, $\mathcal{S}(\mathfrak{A})$ is metrizable and Choquet's theorems A.1.6 and A.1.7 entail that, in any weak* closed convex subset $S \subseteq \mathcal{S}(\mathfrak{A})$ every $\varphi \in S$ admits a barycentric decomposition μ that is a Radon probability on $\mathcal{S}(\mathfrak{A})$ with support $\mathcal{E}S$. In other, words, for all $\varphi \in S$, there exists a Radon probability measure μ on $\mathcal{S}(\mathfrak{A})$ supported on $\mathcal{E}S$ such that

$$\varphi = \int_{\mathcal{E}S} \omega\, \mu(d\omega). \tag{3.2.3}$$

If $\mathfrak{A}$ is nonseparable, it is known that for any G-invariant state φ in the weak* compact convex $\mathcal{S}^G(\mathfrak{A})$, the integral formula (3.2.3) remains valid for maximal Radon probability measures pseudosupported on $\mathcal{E}S$, by the Bishop-de Leeuw theorem (see Theorem A.1.5); but no uniqueness is guaranteed. We present a known property of C*-dynamics that ensures uniqueness of the measure such that (3.2.3), as it turns out to be useful in many situations, including our work.

Before that, we would like to give reading of de Finetti's theorem in the C*-formalism so far introduced.

Let $\varphi \in \mathcal{S}(\mathfrak{P})$ and assume it stands for a classical process, that is, a random sequence of commutative random variables. Then, φ corresponds to a quantum stochastic process $(\mathfrak{A}, \mathcal{H}, \{\pi_k\}_k, \xi)$ and the ranges of the marginals

[3] In the literature, expressions equivalent to "μ is the barycentric decomposition of point" are "x is the resultant of μ", "x is the center of gravity of μ".

$\pi_k, k \in \mathbb{Z}$ shall commute with each other, $[\pi_j(a_j), \pi_k(a_k)] = 0$, for all $a_j, a_k \in \mathfrak{A}$, $j, k \in \mathbb{Z}$. However, by a classical process, we intend that the distributions of the marginals $\pi_k, k \in \mathbb{Z}$ are commutative as well. These are the vector states which are classical measures, if the sample algebra is commutative. Thus, for φ to be a classical process it must be that $\varphi \in \mathcal{S}(*_{\mathbb{Z}}\mathfrak{A})$ and $\mathfrak{A}$ is commutative. On the other hand, by [CF16, Proposition 2.6], $\varphi = \omega \circ P$, for P the pull-back on $*_k\mathfrak{A}$, for some $\omega \in \mathcal{S}(\mathfrak{P}_I)$ and I as in (3.1.4). Since $\mathfrak{A}$ is commutative, $\mathfrak{P}_I \simeq C(\prod_{\mathbb{Z}} \sigma(\mathfrak{A}))$, where $\prod_{\mathbb{Z}}$ is the Tychonoff product of the compact Hausdorff $\sigma(\mathfrak{A})$. In addition, product measures, $\otimes_{\mathbb{Z}}\nu$, for ν a probability measure on $\sigma(\mathfrak{A})$, are exactly the extreme points of exchangeable probability measures and these are the iid processes, according to classical independence. If the classical process φ is exchangeable, de Finetti's theorem applies and $\varphi = \int \psi \mathrm{d}\mu_\varphi(\psi)$, for a unique undetermined probability measure μ_φ supported on extreme exchangeable process, $\nu \in \mathcal{M}_1(\sigma(\mathfrak{A}))$. In other words, the compact convex of exchangeable (classical) states is a Choquet simplex. Similar considerations can be made when φ is rotatable, applying Freedman's theorem.

When $\mathfrak{A}$ is noncommutative, the extreme invariant states may have the analogous meaning of iid processes. If they factorize, according to suitable products of states, then they are viewed as iid processes, for their associated concept of independence, alternatively generalizing classical independence.

Notation. *For a Hilbert space $\mathcal{H}$, $\mathcal{U}(\mathcal{H})$ denotes the group of unitary (bounded linear) operators on $\mathcal{H}$.*

Let $(\mathfrak{A}, G, \alpha)$ be a reversible C*-dynamical system. For all $\omega \in \mathcal{S}^G(\mathfrak{A})$, there is a strongly continuous one-parameter unitary group $\{U_g^\omega : g \in G\}$ where each $U_g^\omega \in \mathcal{U}(\mathcal{H}_\varphi)$, for all $g \in G$, is defined as

$$U_g^\omega \pi_\omega(a) = \pi_\omega(\alpha_g(a))\xi_\omega\,, \tag{3.2.4}$$

for all $a \in \mathfrak{A}$. It turns out that $U_g^\omega \xi_\omega = \xi_\omega$, for all $g \in G$. Furthermore, the representation of G is isometric

$$\begin{aligned}&\left\|U_g^\omega \pi_\omega(a)\right\|^2 = \left\|\pi_\omega(\alpha_g(a))\xi_\omega\right\|^2 = \langle \pi_\omega(\alpha_g(a))^* \pi_\omega(\alpha_g(a))\xi_\omega, \xi_\omega\rangle \\ &= \langle \pi_\omega(\alpha_g(a^* a))\xi_\omega, \xi_\omega\rangle = \omega(\alpha_g(a^* a)) = \|a\|^2\,.\end{aligned}$$

Moreover,

$$U_g^\omega \pi_\omega(a)(U_g^\omega)^* = \pi_\omega(\alpha_g(a))\,, \tag{3.2.5}$$

for all $a \in \mathfrak{A}$. In other words, a C*-dynamics can be seen as the adjoint map action of the group of unitaries on the GNS Hilbert space. It is known

that the unitaries in (3.2.4) allow to compute the reduced cross product C*-algebra. See [Far19, Section 2.4.3].

The fixed-point set (or invariant vector set)

$$\mathcal{H}_\omega^G = \bigcap_{g\in G}\{\zeta \in \mathcal{H}_\omega : U_g^\omega \zeta = \zeta\} \tag{3.2.6}$$

is a closed non-empty subspace, because $\mathbb{C}\xi_\omega \subseteq \mathcal{H}_\omega^G \subseteq \mathcal{H}_\omega$, where $\xi_\omega \in \mathcal{H}_\omega$ denotes the GNS vector. As $\xi \in \mathcal{H}_\omega^G$, this subspace is not trivial. Denote by E_ω the orthogonal projection onto $\mathcal{H}_\omega^G$.

Definition 3.2.6 (G-abelian reversible C*-dynamical system)**.** *A reversible C*-dynamical system $(\mathfrak{A}, G, \alpha)$ is said to be G-abelian if, for any $\omega \in \mathcal{S}^G(\mathfrak{A})$, the family of operators $\{E_\omega \pi_\omega(a) E_\omega : a \in \mathfrak{A}\}$ is commutative.*

For the closed space of fixed-points (3.2.6), if $\mathcal{H}_\omega^G = \mathbb{C}\xi_\omega$, then ω is ergodic. The converse implication is known to be valid, if $(\mathfrak{A}, G, \alpha)$ is G-abelian. The following is also established regarding G-abelianness.

Theorem 3.2.7 ([Sak98], Theorem 3.1.14.)**.** *If $(\mathfrak{A}, G, \alpha)$ is G-abelian, then $\mathcal{S}^G(\mathfrak{A})$ is a Choquet simplex.*

Since [Dop+67], G-abelianness of a covariant system is often checked by asymptotically abelianness.

Definition 3.2.8. *A C*-dynamical system $(\mathfrak{A}, G, \alpha)$ is called asymptotically abelian, if there exists a sequence $\{g_n\}_{n\in\mathbb{N}} \subset G$ such that, for every $a, b \in \mathfrak{A}$,*

$$\lim_{n\to\infty} \|\alpha_{g_n}(a)b - b\,\alpha_{g_n}(a)\| = 0\,.$$

Indeed, if $(\mathfrak{A}, G, \alpha)$ is reversible and asymptotically abelian, then it is G-abelian, [Sak98, Proposition 3.1.16]. Therefore, one has given a general strategy to establish barycentric decompositions of G-invariant states in covariant systems. Namely, one can check if there exists a sequence $\{g_n\}_{n\in\mathbb{N}}$ in G that verifies the asymptotic abelianness of $(\mathfrak{A}, G, \alpha)$, deduce that $(\mathfrak{A}, G, \alpha)$ is G-abelian, and conclude by Theorem 3.2.7 that the simplex $\mathcal{S}^G(\mathfrak{A})$ is Choquet. We give a criterion retrieved in [CVR23, Theorem 3.5.], written in more general terms.

Lemma 3.2.9 (Face criterion)**.** *Let V be a locally compact TVS. Let C_1, C_2 be Choquet simplices such that $C_1 \subseteq C_2$. Then, C_1 is a face of C_2.*

Proof. Let $x \in C_1$ and suppose $y, z \in C_2$ and $t \in [0,1]$, such that $x = ty + (1-t)z$. By Definition 3.2.5, $x = \int w\mathrm{d}\mu(w)$ for a unique probability measure

supported on $\mathcal{E}C_1$, the extreme points of C_1. Similarly, $y = \int w \mathrm{d}\mu_y(w)$ and $z = \int w \mathrm{d}\mu_z(w)$ for probability measures $\mu_y, \mu_z \in \mathcal{M}_1(V)$ supported on $\mathcal{E}C_2$ uniquely determined by and for each y, z. It follows that $\mu = t\mu_y + (1-t)\mu_z$, where $t \in [0,1]$. Then, for each measurable $B \subseteq V$, either $\mu_y(B) = 0$ or $\mu(B) > 0$. Likewise, for μ_z, meaning that $\mathrm{supp}(\mu_y) \cup \mathrm{supp}(\mu_z) \subseteq \mathrm{supp}(\mu) \subseteq \mathcal{E}C_1$ 3.2.3. Therefore, μ_y, μ_z are supported on $\mathcal{E}C_1$, which implies $y, z \in C_1$. In conclusion, for all $x \in C_1$, for each $y, z \in C_2 \setminus C_1$ and $t \in [0,1]$ such that $x = ty + (1-t)z$, then either $t = 0$ or $t = 1$. □

In Section 3.1, we motivated the importance of focusing on quotients of the infinite free product. Before any quotient is considered, distributional symmetries refer to the infinite free product C*-algebra. For these processes, free independence applies and the natural framework of study is that of Free Probability. We are interested in quotients that give rise to tensor products and tensor independence.

3.3 Størmer's theorem

Let $\mathfrak{A}$ be a C*-algebra and I_0 the two-sided closed ideal of the infinite free product $\mathfrak{P} = *_k\mathfrak{A}$ generated by the set of elements

$$\begin{aligned} &i_{s_1}(a_1)\dots i_{s_{k-1}}(a_{k-1})i_{s_k}(a_k)i_{s_{k+1}}(a_{k+1})\dots i_{s_n}(a_n) \\ &+ i_{s_1}(a_1)\dots i_{s_{k-1}}(a_{k-1})i_{s_k}(b_k)i_{s_{k+1}}(a_{k+1})\dots i_{s_n}(a_n) \\ &- i_{s_1}(a_1)\dots i_{s_{k-1}}(a_{k-1})i_{s_k}(a_k+b_k)i_{s_{k+1}}(a_{k+1})\dots i_{s_n}(a_n)\,, \end{aligned} \tag{3.3.1}$$

for all $s_k \in \mathbb{Z}$, $a_k, b_k \in \mathfrak{A}$, $k \in \{1,\dots,n\}, n \in \mathbb{N}$.

Denoting $\mathfrak{P}_0 = \mathfrak{P}_{I_0}$, in this case

$$\mathfrak{P}_0 = \otimes_k\mathfrak{A} \tag{3.3.2}$$

is the infinite tensor product C*-algebra on index set $\mathbb{Z}$ and normed by the projective C*-norm. Let $\mathfrak{P}_0 = \otimes_{\mathbb{Z}}\mathfrak{A}_j$, for a family of (unital) C*-algebras $\{\mathfrak{A}_j\}_j$, with $\{i_j\}_j$ the family of its coordinate mappings.

Definition 3.3.1 (Free product state)**.** *Let $\{\mathfrak{A}_k\}_k$ a family of C*-algebras indexed by $\mathbb{Z}$, and $\psi_k \in \mathcal{S}(\mathfrak{A}_k)$, for all $k \in \mathbb{Z}$. Write $*_k\mathfrak{A}_k$ for the C*-algebra free product of the family $\{\mathfrak{A}_k\}_k$. Then, the free product state of $\{\psi_k\}_k$ is the unique state on $*_k\mathfrak{A}_k$ defined by*

$$i_{s_1}(a_{s_1})\dots i_{s_n}(a_{s_n}) \mapsto \psi_{s_1}(a_{s_1})\dots\psi_{s_n}(a_{s_n})\,, \tag{3.3.3}$$

for all $a_{s_1} \in \mathfrak{A}_{s_1}, \dots, a_{s_n} \in \mathfrak{A}_{s_n}$, $s_1, \dots, s_n \in \mathbb{Z}$, $n \in \mathbb{N}$. We denote this state with $*_k\psi_k$.

The group $\mathbb{P}_{\mathbb{Z}}$ acts on infinite tensor products $\mathfrak{P}_0$ and the action is that of a large group of automorphisms [Stø67]. For infinite tensor products exchangeable states are characterized by the invariance with respect to the subgroup of finitary permutations, similarly to the classical case. If $\psi_k = \psi \in \mathcal{S}(\mathfrak{A})$, for all $k \in \mathbb{Z}$ in Definition 3.3.1, relations (3.3.1) imply that any free product state $\varphi = *_k \psi_k$, defined in (3.3.3), becomes an infinite (tensor) product state $\varphi = \otimes_k \psi$.

The tensor independence, in a quantum probability space $(\mathfrak{A}, \varphi)$, reads as

$$\varphi(a_{s_1} \dots a_{s_n}) = \varphi(a_{s_1}) \dots \varphi(a_{s_n}),$$

for all $n \in \mathbb{N}, s_1, \dots, s_n \in \mathbb{Z}$, $a_{s_1}, \dots, a_{s_n} \in \mathfrak{A}$. In addition, tensor product states $\otimes_k \varphi_0$ are joint distributions of iid sequence of quantum random variables.

By Theorem 2.7.4, we see that the exchangeability assumption is fundamental in absence of commutativity.

Proposition 3.3.2 ([Stø69], Theorem 2.7.)**.** *Let $\mathfrak{A}$ be a unital C*-algebra and $\varphi \in \mathcal{S}^{\mathbb{P}_J}(\mathfrak{P}_0)$ be an exchangeable state on the infinite tensor product (3.3.2). Then, the following conditions are equivalent,*

1. *$\varphi = \otimes_k \varphi_0$, for some $\varphi_0 \in \mathcal{S}(\mathfrak{A})$*
2. *$\varphi \in \mathcal{ES}^{\mathbb{P}_J}(\mathfrak{A})$*
3. *φ is such that $\lim_{n\to\infty} \varphi(\alpha(g_n)ab) = \varphi(a)\varphi(b)$.*

Definition 3.3.3 (Strongly clustering)**.** *A state $\varphi \in \mathcal{S}^{\mathbb{P}_{\mathbb{Z}}}(\mathfrak{A})$ is called strongly clustering, if it satisfies the third of the conditions in Proposition 3.3.2.*

Proposition 3.3.4 ([Stø69], Theorem 2.8)**.** *The mapping*

$$\begin{aligned} \mathcal{S}(\mathfrak{A}) &\to \mathcal{ES}^{\mathbb{P}_J}(\mathfrak{P}_0) \\ \varphi_0 &\mapsto \otimes_k \varphi_0 \end{aligned} \tag{3.3.4}$$

is a homeomorphism. In particular, there is a positive unital linear mapping $\Phi : \mathfrak{P}_0 \to C(\mathcal{S}(\mathfrak{A}))$ such that $\Phi(\mathfrak{P}_0)$ is norm dense and the adjoint map

$$\begin{aligned} \Phi^* : &\mathcal{M}_1(\mathcal{S}(\mathfrak{A})) \to \mathcal{S}(\mathfrak{P}_0) \\ &\Phi^*(\mu) = \int \Phi(b)(x) \mathrm{d}\mu(x), \end{aligned} \tag{3.3.5}$$

for all $\mu \in \mathcal{M}_1(\mathcal{S}(\mathfrak{A}))$, is an affine isomorphism.

Proposition 3.3.4 tells us that any identically distributed tensor independent ([NS06, Definition 5.1.]) quantum distribution $\otimes_k \varphi$ is extreme, in the convex of all such exchangeable distributions.

Theorem 3.3.5 ([Stø69], Theorem 3.1.)**.** $\mathcal{S}^{\mathbb{P}_J}(\mathfrak{P}_0)$ *is a Choquet simplex.*

In other words, de Finetti's theorem holds for quantum stochastic processes arising in infinite tensor product C*-algebras, where the iid sequences are seen as the (tensor-)independent and identically distributed quantum random variables.

We emphasize what is the counterpart of the tail σ-algebra of classical probability the property of being conditionally iid for quantum stochastic processes, used in the literature of distributional symmetries. For any free product $\mathfrak{P} = *_k \mathfrak{A}$, the tail algebra of any state $\varphi \in \mathcal{S}(\mathfrak{P})$ is defined as the subalgebra

$$\mathfrak{Z}_\varphi^\perp := \bigcap_{\substack{I\subset\mathbb{Z},\\ I \text{ finite}}} \left(\bigcup_{\substack{K\cap I=\emptyset,\\ K \text{ finite}}} \left(\bigvee_{k\in K} \pi_k(\mathfrak{A}) \right) \right)'' \tag{3.3.6}$$

where $\pi_k, k \in \mathbb{Z}$ are the coordinate maps of any[4] quantum stochastic process distributed with φ. A state $\varphi \in \mathcal{S}(\mathfrak{P})$ is conditionally iid with respect to (3.3.6), if, by definition, there exists a conditional expectation $E_\varphi : \bigvee_{j\in\mathbb{Z}} \pi_j(\mathfrak{A}) \to \mathfrak{Z}_\varphi^\perp$ such that

1. $E_\varphi(xy) = E_\varphi(x)E_\varphi(y)$, for all $x, y \in \bigvee_{j\in\mathbb{Z}} \pi_j(\mathfrak{A})$;
2. $E_\varphi(\pi_j(a)) = E_\varphi(\pi_k(a))$, for all integers $j, k \in \mathbb{Z}$, $a \in \mathfrak{A}$.

See [CF15, Section 4].

[4] up to equality in distribution, see Section 3.1.

3.4 Bose Fock space

We introduce the Bose Fock space. Let $\mathcal{H}$ be a separable Hilbert space and fix an orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$. Let $\{P_+^{(n)}\}_{n\in\mathbb{N}}$ a family of projections on $\mathcal{H}^{\otimes n}$ defined recursively by the linear extensions of $P_+^{(0)} = \mathrm{id}_{\mathrm{B}(\mathcal{H})}$, if $n = 0$, and that of

$$P_+^{(n)}(e_1 \otimes \ldots \otimes e_n) = \frac{1}{n!} \sum_{\pi\in\mathbb{P}[n]} e_{\pi_1} \otimes \ldots \otimes e_{\pi_n}\,, \tag{3.4.1}$$

for all subset of n indices $s_j \in \mathbb{N}$, $j \in [n]$ and $n \geq 1$. The Bose creation and annihilation operators have expressions

$$a_+(e_k) = P_+ a(e_k) P_+ \text{ and } a_+^\dagger(e_k) = P_+ a^\dagger(e_k) P_+ \tag{3.4.2}$$

where $a(e_k)$ and $a^\dagger(e_k)$ are the full Fock space annihilation and creation operators respectively. The Fock direct sum of the images under projections (3.4.1) is called the Bose Fock space,

$$\mathcal{F}_+(\mathcal{H}) = \oplus_{n=0}^\infty P_+^{(n)}(\mathcal{H}^{\otimes n}). \tag{3.4.3}$$

The *-algebra generated by (3.4.2) acts on $\mathcal{F}_+(\mathcal{H})$. By their definition, annihilation and creations are mutually (essentially) adjoint, $a_+^\dagger(e_k) = a_+^*(e_k)$, $k \in \mathbb{Z}$, and that they obey the so-called canonical commutation relations,

$$a_k^+ a_l^{+\dagger} - a_l^{+\dagger} a_k^+ = \delta_{kl} 1\,, \tag{3.4.4}$$

for all integers k, l, where we have denoted $a_k^+ = a_+(e_k)$, and δ_{kl} is the Kronecker symbol. From (3.4.4), annihilation by any $f \in \mathcal{H}$ is an unbounded operator, because, on each n^{th} particle space,

$$\left\| a(f) P_+^{(n)}(g) \right\| = \sqrt{n} \left\| P_+^{(n)}(g) \right\| \|f\|\,, \tag{3.4.5}$$

for all $g \in \mathcal{H}$. In other words, the number operator defined by[5]

$$Nf = \oplus_{n\geq 0}\, n P_+^{(n)}(f)\,, \tag{3.4.6}$$

for each $f \in \mathcal{F}_+(\mathcal{H})$ such that $\sum_{n=0}^\infty \left\| n P_+^{(n)}(f) \right\|_{\mathcal{H}^{\otimes n}}^2 < \infty$, is unbounded.

As it happens for the full Fock space functor, the Bose Fock space construction leaves defined a functor $\mathcal{F}$ in the category of Hilbert spaces with morphisms given by contraction operators. In fact, let $\mathcal{F}(T)$ be the second-quantization of any contraction $T : \mathcal{H} \to \mathcal{K}$, that is, $\mathcal{F}(T) = \oplus_{n=0}^\infty T^{\otimes n}$, where

[5] It can be introduced, equivalently, via second-quantization of the identity operator.

$T^{\otimes n}$ has the meaning given in Section 2.7 of Chapter 2. By [EL77, Lemma 6.1], one has that $\mathcal{F}(TS) = \mathcal{F}(T)\mathcal{F}(S)$, $\mathcal{F}(1) = 1$, where 1 is the identity map, and $\mathcal{F}(T)^* = \mathcal{F}(T^*)$, , for all contractions between Hilbert spaces, $T : \mathcal{H}_2 \to \mathcal{H}_3$ and $S : \mathcal{H}_1 \to \mathcal{H}_2$. Likewise, there is a functor available, known as CCR, that replaces Bose Fock functor $\mathcal{F}_+$ meeting our needs to work with bounded linear operators of a C*-algebra.

The Bose particle creations and annihilations are also formulated as differential operators. A representation on the space of square-summable real functions $L^2(\mathbb{R})$ is provided in [Pet90]; for a single particle, its position Q and momentum P can be seen as operators

$$Q(f) = xf(x)\,, \tag{3.4.7}$$

for all $f \in L^2(\mathbb{R})$, and

$$Pf = -i\mathrm{D}f(x)\,, \tag{3.4.8}$$

for all $f \in L^2(\mathbb{R}), \mathrm{D}f(x) \in L^2(\mathbb{R})$, where D, is the ordinary derivative. Denoting F and F^{-1} the Fourier transform and its inverse on $L^2(\mathbb{R})$, one has $Pf = \mathrm{F}^{-1}Q\mathrm{F}f$ for each differentiable function f in the Schwarz space. See also [Par92, Example 12.10].

Bose creation and annihilation operators can be thought of as infinite matrices with respect to this L^2 representation. Fix the orthonormal basis $\{f_n\}_{n\in\mathbb{N}}$ in $L^2(\mathbb{R})$ given by

$$f_n(x) = (2^n\, n!\, \sqrt{\pi})^{-\frac{1}{2}} \exp(-\tfrac{x^2}{2} - x^2)\mathrm{D}^n \exp(-x^2)\,, \tag{3.4.9}$$

for all $n \in \mathbb{N}$, where $(-1)^n h_n(x) = \exp(x^2)\mathrm{D}^n \exp(-x^2)$, for all $n \in \mathbb{N}$, are the Hermite polynomials. Then, position and momentum of bosons are the matrices $(Q_{jk})_{j,k}, (P_{j,k})_{j,k}$ with entries

$$\begin{aligned} Q_{jk} &= (\delta_{j+1,k} + \delta_{j,k+1})\sqrt{\min\{j,k\}}\,, \\ P_{jk} &= -i2^{-1}\sqrt{2}(\delta_{j+1,k} - \delta_{j,k+1})\sqrt{\min\{j,k\}}\,, \end{aligned} \tag{3.4.10}$$

for all integers $j, k \geq 1$. However, (3.4.10) do not act as bounded operators.

According to all the representations we recalled, bosons act as unbounded linear operators. By the Fock representation of the C*-algebra of canonical commutation relations is understood something different. In the next chapter, we narrow the discussion of noncommutative probability to the case of bosonic algebras. In particular, we shall review in detail the representation that allows us to work within C*-algebra theory of Chapter 2.

Chapter 4

The CCR algebra

It is known, since [GW54], that there are uncountably infinitely many unitarily inequivalent irreducible representations of the canonical commutation relations. A pertinent C*-algebra with representations continuous on rays was proposed by Segal [Seg59], while [Rue99] contains a study of their use in statistical mechanics within the study of bosons. Among these, the Weyl and the Fock representation are undoubtedly significant for quantum statistical mechanics and are central to our discussion. On one hand, an original result in the last chapter refers to the Fock representation. On the other, the Weyl representation gives the formulation in which we define our working definition of C*-algebra of canonical commutation relations. These reasons motivate a brief overview of the basic properties of the Weyl and Fock representation for the canonical commutation relations. As a result, their relationship with Bose creation and annihilation operators are highlighted.

4.1 Weyl operators

Let G be an abelian group. A map $b : G \times G \to \mathbb{T}$ such that $b(0, g) = b(g, 0) = 1$, for all $g \in G$, and $b(x, y)b(x + y, z) = b(x, y + z)b(y, z)$, for all $x, y, z \in G$, is called a multiplier (or bicharacter). A projective representation is defined as a mapping $u : G \to \mathcal{U}(\mathcal{H})$ such that $u(x)u(y) = u(x + y)b(x, y)$, for some bicharacter b, for all $x, y \in G$.

In fact, projective representations we are interested in are those of abelian groups that are additive group of Hilbert spaces.

Definition 4.1.1 (Representation of the canonical commutation relations, Definition 7.3, [EL77])**.** *Let $\mathcal{H}$ be a Hilbert space and $b : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ the bicharacter given by*

$$b(x, y) = e^{\frac{1}{2} i \operatorname{Im}\langle x, y\rangle}, \tag{4.1.1}$$

for all $x, y \in \mathcal{H}$. Then, any projective representation with respect to b is called a representation of the canonical commutation relations (3.4.4).

The term multiplier for bicharacters is due to the fact that positive-definiteness of kernels affects properties of CCR representations when multiplied by bicharacters. We make this assertion precise while giving known facts on CCR representations. Let $\mathcal{H}$ be a Hilbert space. In finite-dimensional real spaces, translations and rotations for $\mathcal{H}$ are given by affine transformations

$$(v, O) : \mathcal{H} \to \mathcal{H} \qquad h \mapsto v + Oh\,, \tag{4.1.2}$$

for some $v \in \mathcal{H}$ and $O \in \mathbb{O}[n]$, where n is the dimension of $\mathcal{H}$. If $\mathcal{H}$ is infinite-dimensional and complex, each (v, U) is a homeomorphism, still acting as (4.1.2) on $\mathcal{H}$.

The set of all such mappings is the known Euclidean group, $\mathbb{E}(\mathcal{H}) = \mathcal{H} \times \mathcal{U}(\mathcal{H})$ with operation $(v_1, U_1)(v_2, U_2) = (v_1 + U_1 v_2, U_1 U_2)$, $v_1, v_2 \in \mathcal{H}$, $U_1, U_2 \in \mathcal{U}(\mathcal{H})$, identity element $(0, \mathrm{id})$ and inverse elements obtained by $(v, U)^{-1} = (U^{-1}(-v), U^{-1})$, for all $v \in \mathcal{H}, U \in \mathcal{U}(\mathcal{H})$. See also [Fol16, Section 6.7., p.202]. $\mathbb{E}(\mathcal{H})$ is a topological group with product topology between norm of $\mathcal{H}$ and the strong operator topology of $\mathcal{U}(\mathcal{H})$. If $\mathcal{H}$ is separable, then $\mathbb{E}(\mathcal{H})$ is Polish. The unitary group $\mathcal{U}(\mathcal{H})$ is isomorphic to the quotient $\mathbb{E}(\mathcal{H})/\mathcal{H}$, being $\mathcal{H}$ a normal subgroup. By (4.1.2), the group $\mathbb{E}(\mathcal{H})$ acts on the Bose Fock space and the resulting representation is the so-called Weyl representation of $\mathbb{E}(\mathcal{H})$. The Bose Fock space vectors

$$e(x) = \oplus_{n=0}^{\infty} \tfrac{1}{\sqrt{n!}}\, x^{\otimes n} \tag{4.1.3}$$

where $x^{\otimes n} = x^{\otimes(n-1)} \otimes x$, if $n \geq 1, n \in \mathbb{N}$, and $x^{\otimes 0} = \Omega$ for the fixed vacuum vector Ω, are called exponential vectors [Par92, p. 126]. Since the linear span of exponential vectors is dense in the Bose Fock space, each pair $(x, U) \in \mathbb{E}(\mathcal{H})$ defines an operator $W(x, U)$ acting on $\mathcal{H}$ by

$$W(x, U)e(y) = e(-\tfrac{1}{2}\|x\|^2 - \langle x, Ux\rangle)e(Uy + x)\,, \tag{4.1.4}$$

for all $y \in \mathcal{H}$. Operators in (4.1.4) are called the Weyl operators. The mapping $(x, U) \mapsto W(x, U)$ is a strongly continuous unitary representation of $\mathbb{E}(\mathcal{H})$ in the Bose Fock space $\mathcal{F}_+(\mathcal{H})$, called the projective unitary representation. For this representation of $\mathbb{E}(\mathcal{H})$,

$$W(x_1, U_1)W(x_2, U_2) = e^{-\frac{1}{2}\|x\|^2 - \langle Ux, x\rangle} W((x, U_1)(x_2, U_2))\,,$$

for all $x_1, x_2 \in \mathcal{H}$, $U_1, U_2 \in \mathcal{U}(\mathcal{H})$.

The mapping $U \mapsto \Gamma(U) = W(0,U)$, is a representation of $\mathcal{U}(\mathcal{H})$ acting on the Bose Fock space $\mathcal{F}_+(\mathcal{H})$, known as the second-quantization of the unitary operator $U \in \mathcal{U}(\mathcal{H})$. Its connection with Weyl operators is given by $\Gamma(U)^* W(x)\Gamma(U) = W(Ux)$, for all $x \in \mathcal{H}$, $U \in \mathcal{U}(\mathcal{H})$. The map $U \mapsto \Gamma(U)$ defines a group homomorphism $\mathcal{U}(\mathcal{H}) \to \mathcal{U}(\mathcal{F}_+(\mathcal{H}))$ that sends one-parameter subgroups of $\mathcal{U}(\mathcal{H})$ to one-parameter subgroups of $\mathcal{U}(\mathcal{F}_+(\mathcal{H}))$, preserving strong continuity.

We specify what is meant by Fock representation of the canonical commutation relations.

The exponential map $\mathsf{Exp} : \mathcal{H} \to \mathcal{F}_+(\mathcal{H})$ that sends a Hilbert space vector $x \in \mathcal{H}$ to the exponential vector (4.1.3) decomposes the inner product [EL77, Theorem 6.3.]. Since exponential vectors (4.1.3) form a total subset, the vectors

$$e(x)e^{-\frac{1}{2}\|x\|^2}, \qquad x \in \mathcal{H}\,, \tag{4.1.5}$$

are also a total set which decomposes the positive-definite kernel on $\mathcal{H} \times \mathcal{H}$ given by

$$e^{-\frac{1}{4}\|x-y\|^2 + \frac{1}{2}i\mathrm{Im}\langle x,y\rangle}, \qquad x,y \in \mathcal{H}\,. \tag{4.1.6}$$

Therefore, $\mathcal{F}_+(\mathcal{H})$ is a reproducing kernel for (4.1.6). The operators $\pi_{\mathcal{F}}(x) : \mathcal{F}_+(\mathcal{H}) \to \mathcal{F}_+(\mathcal{H})$ defined by

$$\pi_{\mathcal{F}}(x)e(x) = e(x+y)b(x,y), \qquad x,y \in \mathcal{H}\,, \tag{4.1.7}$$

where b is the bicharacter (4.1.1), are unitaries. They satisfy

$$\pi_{\mathcal{F}}(x)\pi_{\mathcal{F}}(y) = \pi_{\mathcal{F}}(x+y)b(x,y)\,, \qquad x,y \in \mathcal{H}\,,$$

for all $x, y \in \mathcal{H}$.

Definition 4.1.2 (Fock representation of the Canonical Commutation Relations)**.** *The map that sends each $x \in \mathcal{H}$ to the unitary operators $\pi_{\mathcal{F}}(x)$ where $\pi_{\mathcal{F}}(x)$ is defined as (4.1.7) is called the Fock representation of the canonical commutation relations.*

We note that it is an action on the Bose Fock space (3.4.3) as an algebra of multiplication operators, $\pi_{\mathcal{F}}(x) \in \mathrm{B}(\mathcal{F}_+(\mathcal{H}))$, for all $x \in V$. The Fock representation is non-singular i.e. the map $\mathbb{R} \to \pi_{\mathcal{F}}(\mathcal{H})$, $t \mapsto \pi_{\mathcal{F}}(tx)$ is weakly continuous. This implies that $\pi_{\mathcal{F}}(\mathcal{H})$ is a von Neumann algebra. As is known, since $\mathcal{H}$ is separable, Stone theorem is valid for the weakly continuous one-parameter unitary group $\{\pi_{\mathcal{F}}(tx)\}_{t\in\mathbb{R}}$. Therefore, we may write

$$\pi_{\mathcal{F}}(x) = e^{iB(x)} \tag{4.1.8}$$

for a set $\{B(x)\,|\,x\in\mathcal{H}\}$ of selfadjoint linear operators. The mapping from $\mathcal{H}$ into this set is known as the Bose field, and the elements as field operators. See the Appendix Section A.2.4 for further details on this point.

Remark 4.1.3. It may promote comprehension to think about field operators as noncommutative random variables associated to the CCR. This reading is going to be further clarified in Section 4.5.

Although not directly defined by suitably deformed creation and annihilation operators, the CCR Fock representation is linked to the unbounded Bose creation and annihilation operators $a^\dagger, a$. The connection is given by the formula

$$a(x) = \tfrac{1}{2}(B(x) + iB(ix)), \qquad x\in\mathcal{H}\,, \tag{4.1.9}$$

and the creation defined by taking the adjoint. The essential self-adjointness of each Bose field operator ensures the well-definedness and recovers the canonical commutation relations (3.4.4). Additionally, the unitary operators given by the Fock representation can be expressed in terms of creation and annihilation operators by (4.1.8)-(4.1.9), that is,

$$W(x) = e^{i\frac{1}{\sqrt{2}}a^\dagger(x)}e^{i\frac{1}{\sqrt{2}}a(x)}e^{-\frac{1}{4}\|x\|^2}. \tag{4.1.10}$$

Definition 4.1.4. *Let V be a real vector space. A pair (V,σ) is said to be a symplectic space, if V is a real vector space and σ is a symplectic form on V i.e. σ is 2-form, or an antisymmetric bilinear form. A symplectic form σ is called nondegenerate, if $\sigma(x,y)=0$, for all $y\in V$, implies $x=0$; otherwise, σ is called degenerate. Accordingly, (V,σ) is called (non)degenerate, if σ is (non)degenerate. We write $(V_1,\sigma_1)\preceq(V_2,\sigma_2)$ whenever $V_1\subseteq V_2$, and $\sigma_1=\sigma_2{\restriction}_{V_1}$, that is an order relation for symplectic spaces.*

Generally, if (V,σ) is a finite-dimensional symplectic space, either the dimension is even or σ is degenerate. We explain this fact.

Let σ be nondegenerate symplectic form over a real vector space V. Let L be a subspace and $L^\sigma=\{u\in V\,|\,\sigma(u,v)=0\;\forall v\in L\}$ its so-called orthocomplement. In finite dimension, for each subspace L, $\dim(V)=\dim(L)+\dim(L^\sigma)$, by bilinearity of σ. Since σ is antisymmetric, if L has dimension one, $L\subseteq L^\sigma$. In addition, for all $e\in V$, the orthocomplement $(\mathrm{Span}(\{e\})^\sigma$ has dimension $n-1$, and $L_1^\sigma\cap L_2^\sigma=(L_1+L_2)^\sigma$ for any pair of subspaces L_1,L_2. The family of all subspaces included in their orthocomplement is a poset with respect to the subspace order relation, and it has maximal elements, because V is finite-dimensional. Actually, maximal elements attain the equality: if L is a subspace with $L\subseteq L^\sigma$, then $L=L^\sigma$. Indeed, if L is maximal and $e\in L^\sigma\setminus L$, then, for all $x,y\in L$, $\alpha,\beta\in\mathbb{R}$, $\sigma(x+\alpha e,y+\beta e)=$

$\sigma(x,y) + \sigma(x,\beta e) + \sigma(\alpha e, y) + \sigma(\alpha e, \beta e) = 0$, where $\sigma(x,y) = 0$, because $L \subseteq L^\sigma$. Therefore, $L + e \subseteq L^\sigma$. But this is against maximality of L, since $L + e \subseteq L^\sigma = L^\sigma \cap (\mathrm{Span}(\{e\}))^\sigma$ and $L^\sigma \cap e^\sigma = (L + \mathrm{Span}(\{e\}))^\sigma$. Thus, $L = L^\sigma$, whenever $L \subseteq L^\sigma$, and $\dim(V) = 2\dim(L)$.

Definition 4.1.5. *Let (V,σ) be a symplectic space. The CCR algebra over (V,σ) denoted by $\mathrm{CCR}(V,\sigma)$, is defined as the universal C*-algebra with generators $\{W(x)\,|\,x \in V\}$ subject to relations*

$$\begin{aligned} W(x)^* &= W(-x)\,, & x &\in V\,, \\ W(x)W(y) &= e^{i\sigma(x,y)}W(x+y)\,, & x,y &\in V. \end{aligned} \tag{4.1.11}$$

The operators in (4.1.11) are related to the Weyl represented elements of the group $\mathbb{E}(\mathcal{H})$ by $W(x) = W(x, 1_{\mathcal{U}(\mathcal{H})})$, for all $x \in \mathcal{H}$, and they are called the Weyl form for the canonical commutation relations (3.4.4). We stress that $\mathrm{CCR}(V,\sigma)$ has the universal property with respect to the relations (4.1.11) in that, for any C*-algebra $\mathfrak{C}$ on generators satisfying them there exists an epimorphism $\psi : \mathrm{CCR}(V,\sigma) \to \mathfrak{C}$. In particular, the map $t \mapsto W(tx)$ is weakly-continuous, by the observed non-singularity of the Fock representation. The $\mathrm{CCR}(\mathcal{H},\sigma)$ is a unital C^*-algebra with unit $W(0)$, and $W(x)$ is a unitary element for every $x \in \mathcal{H}$. If $x,y \in V$ are "symplectic orthogonal" i.e. $\sigma(x,y) = 0$, then $W(x)$ and $W(y)$ commute. As a consequence, subalgebras generated by mutually symplectic orthogonal vectors $x_1,\ldots,x_n \in V$, $\sigma(x_j,x_k) = 0$, for all $j,k \in [n]$, give commutative subalgebras of CCR.

In the next sections, we present constructions to substantiate Definition 4.1.5.

The case of nondegenerate symplectic spaces is due to [Sla72] while [Man+73] extends Slawny's construction to the degenerate case. Both existence proofs of C*-algebras satisfying (4.1.11) are quite straightforward constructions. The uniqueness part in the nondegenerate case can be elicited as done in [Pet90] and relies on some abstract harmonic analysis facts. Any C*-algebra which is universally generated is by definition unique. We display these to highlight the main properties of CCR algebras. The construction in the degenerate case elicits a necessary and sufficient condition on simpleness of CCR algebras. Although it is easy to exhibit a representation of the CCR algebra, for example the one given in the proof of Slawny's theorem, it is then a demanding task to make sure that representation is faithful. When, the symplectic space is assumed to be nondegenerate, CCR algebra is simple and all representations are faithful. Since CCR is infinite-dimensional, it follows that in this case, the $\mathrm{CCR}(\mathcal{H})$ is not a type I C*-algebra in the sense of [Sak98, Definition 4.6.1. p.221]. When (V,σ) is assumed to be nondegenerate, the construction

of [Sla72] makes clear that there exists a unique finite tracial state and that $\mathrm{CCR}(V,\sigma)$ is nonseparable. As said, there are many irreducible inequivalent representations for the CCR. However, for physical applications in quantum physics the choice is nontrivial as to decide whether any given representation of the CCR is irreducible [FS00].

4.2 Constructions and properties

The constructions of [Sla72; Man+73] stand on facts from abstract harmonic analysis. Let G be a locally compact abelian group. Denote by $\hat{G}$, the (Pontrjagin) dual group of G, defined as the group of all continuous homomorphisms $G \to \mathbb{T}$, called characters, where $\mathbb{T}$ denotes the torus. Let $\mathcal{H}$ be a Hilbert space. Recall, as by Definition 2.5.7, that a complex-valued kernel on G, $b : G \times G \to \mathbb{C}$ is positive-definite iff, for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in G$, $c_1, \ldots, c_n \in \mathbb{C}$,

$$\sum_{j,k=1}^{n} \kappa(x_j, x_k)\overline{c_j}c_k \geq 0.$$

Definition 4.2.1. *Let f a complex-valued function on G. Then, f is called positive-type, if the complex-valued kernel κ on G given by*

$$\kappa(x,y) = f(x-y)\,,$$

for all $x, y \in G$, is a positive-definite kernel.

There is an analogue of Theorem 2.6.1 relating unitary representations of groups and positive-type functions. If $f : G \to \mathbb{C}$ is of positive-type, then it is represented as a vector state for a cyclic vector ξ in a suitable Hilbert space $\mathcal{H}_\xi$ induced by the positive semi-definite hermitian form

$$\langle g_1, g_2\rangle_f = \int\int g_1(x)\overline{g_2(y)}f(x-y)\mathrm{d}x\mathrm{d}y\,.$$

Since $f(x) = \langle \pi(x)\xi, \xi\rangle$, for all $x \in G$, then f is continuous locally a.e., by [Fol16, Theorem 3.20].

Theorem 4.2.2 (Bochner, [Fol16], Theorem 4.18)**.** *Each continuous positive-type function f on G, admits a unique $\mu \in \mathcal{M}(\hat{G})$ such that, for all $x \in G$, $f(x) = \int \chi(x)\mathrm{d}\mu(\chi)$.*

4.3 Degenerate case

Let V be an abelian group (with additive notation) and $\sigma : V \times V \to \mathbb{C}$ an antisymmetric and second-argument additive form, i.e. $\sigma(x,y) = -\sigma(y,x)$ and $\sigma(z,x+y) = \sigma(z,x) + \sigma(z,y)$, for all $x,y,z \in V$.

In first place, $\sigma(x,-y) = \sigma(-x,y)$, for all $x,y \in V$. Denote by $\Delta(V,\sigma)$ the linear span of the collection of point-mass complex-valued functions $\delta_x : V \to \mathbb{C}$ $\delta_x(y) = \mathbf{1}_{\{y\}}(x)$. It is a *-algebra with pointwise defined algebra operations and the involution given by $\delta_x^* = \delta_{-x}$, $x \in V$. In particular, the algebra multiplication has unital element, δ_0, and

$$\delta_x \delta_y = e^{-i\sigma(x,y)} \delta_{x+y}\,, \tag{4.3.1}$$

holds for all $x,y \in V$. It follows that $\delta_x^* = \delta_{-x} = \delta_x^{-1}$, for each $x \in V$, that is, δ_x's are unitaries and clearly satisfies (4.1.11). The 1-norm is not a C*-norm for the *-algebra $\Delta(V,\sigma)$. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{W} \subseteq \mathcal{U}(\mathcal{H})$ a collection of unitary operators satisfying (4.1.11) with linear span over the complex numbers $\mathrm{Span}(\mathcal{W})$. Then, the linear map $\pi : \Delta(V,\sigma) \to \mathrm{Span}(\mathcal{W})$ defined by $\pi(\delta_x) = U(x)$, for all $x \in V$, extends to a representation of $\overline{\Delta(V,\sigma)}$, where the completion is with respect to the norm

$$\|a\| = \sup_{\varphi} \sqrt{\varphi(a^*a)}\,, \quad a \in \Delta(V,\sigma)\,, \tag{4.3.2}$$

where φ runs over the set of unital positive linear functionals on the unital *-Banach algebra $\overline{\Delta(V,\sigma)}^0$ (completion taken with respect to the 1-norm). In [Man+73, Corollary 3.8], it is proven that (4.3.2) is also maximal. Further, for any other C*-norm on $\Delta(V,\sigma)$, there exists a non-trivial ideal I such that $\overline{\Delta(V,\sigma)}^0 = \overline{\Delta(V,\sigma)}/I$, [Dix77, p. 1.9.13]. Therefore, for general symplectic space (V,σ), the C*-algebra $\overline{\Delta(V,\sigma)}$ (norm-completion with respect to (4.3.2)) should be regarded as the $\mathrm{CCR}(V,\sigma)$ of Definition 4.1.5.

□

For each $\chi \in \hat{V}$, the linear operator τ_χ on $\overline{\Delta(V,\sigma)}$ uniquely determined by

$$\tau_\chi(\delta_x) = \chi(x)\delta_x, \qquad x \in V, \tag{4.3.3}$$

is clearly bijective, and

$$\left\| \tau\Big(\sum_j \lambda_j \delta_{x_j}\Big) \right\| = \left\| \sum_j \lambda_j \chi(x_j) \delta_{x_j} \right\| = \sum_j |\lambda_j \chi(x_j)| = \sum_j |\lambda_j| = \left\| \sum_j \lambda_j \delta_{x_j} \right\|.$$

Therefore, (4.3.3) is an isometric *-automorphism of $\overline{\Delta(V,\sigma)}$. In addition, it defines a group homomorphism

$$\tau : \hat{V} \to \mathrm{Aut}(\overline{\Delta(V,\sigma)})\,, \qquad \chi \mapsto \tau_\chi\,. \tag{4.3.4}$$

On the other hand, the mappings $\kappa_x : V \to \mathbb{C}^*$ given by

$$k_x(y) = \exp(2i\sigma(x,y)), \qquad y \in V\,,$$

for all $x \in V$, are characters that form a subgroup of $\hat{V}$. Let K be this subgroup. K is a subgroup of inner *-automorphisms because

$$\tau_{\kappa_x}(\delta_y) = e^{2i\sigma(x,y)}\delta_y = e^{i\sigma(x,y)}\delta_{-x}\delta_{y+x} = \delta_{-x}\delta_y\delta_x\,,$$

for all $x, y \in V$. Then, $\overline{K}$, the topological closure of K as a subspace in the discrete topology of V, is isomorphic to a Pontrjagin dual. Precisely, $\overline{K} \simeq \widehat{V/V_0}$. where

$$V_0 = \{x \in V \,|\, \exp\big(\,2i\sigma(x,y)\,\big) = 1,\ y \in V\} = \Big\{x \in V \,\Big|\, (\kappa_x(y))^2 = 1,\, y \in V\Big\}\,. \tag{4.3.5}$$

The relevance of V_0 is that it allows to understand the ideal structure of $\overline{\Delta(V,\sigma)}$. Precisely,

Lemma 4.3.1 (Corollary (4.24) [Man+73]). *$\overline{\Delta(V,\sigma)}$ is simple iff $V_0 = \{0\}$*

As a consequence, if σ is nondegenerate, then V_0 in (4.3.5) is trivial and $\overline{\Delta(V,\sigma)}$ is simple. If (V,σ) is degenerate, then (4.3.5) is not trivial and $\mathrm{CCR}(V,\sigma)$ is not necessarily simple.

An interesting case for our forthcoming discussion in Chapter 5 is that of the one-dimensional real-vector space $V = \mathbb{R}$ that relates to the C*-algebra of almost periodic functions.

Example 4.3.2 (Null case). *Let $V = \mathbb{R}$. Then, if e_1 is a spanning vector, the antilinearity of σ, $\sigma(e_1,e_1) = -\sigma(e_1,e_1)$, forces $\sigma \equiv 0$. Let $\mathbf{0}$ be the null symplectic form on $\mathbb{R}$. The C*-algebra $\mathfrak{C} = \mathrm{CCR}(\mathbb{R},\mathbf{0})$ is a commutative and unital. It is known that $\mathrm{CCR}(\mathbb{R},\mathbf{0})$ is isomorphic to $\mathrm{AP}(\mathbb{R})$, the almost-periodic functions on the real line, with isomorphism given by, for all $x \in \mathbb{R}$,*

$$W(x) \mapsto f_x, \ \textit{where } f_x : \mathbb{R} \to \mathbb{R} \textit{ is given by } f_x(t) = e^{itx}, \ \textit{for all } t \in \mathbb{R}.$$

The dual group

$$\beta\mathbb{R} = \widehat{(X)_d} \tag{4.3.6}$$

of $(X)_d$ for $X = \widehat{\mathbb{R}}$ equipped with the discrete topology is known as the Bohr compactification of $\mathbb{R}$. See also [Dix77, Chapter 16]. By Pontrjagin's duality theorem, $\beta\mathbb{R}$ is a compact group.

4.4 Nondegenerate case

The second of the mentioned construction is in terms of multiplication operators. We follow [Pet90].

Let (V,σ) be a nondegenerate symplectic space. We regard V as a discrete abelian group with the vector space addition. Let $\ell^2(V)$ be the square-summable complex sequences with index set V. For each $x \in V$, the right-multiplication operator $R(x)$ acts on $\ell^2(V)$, as

$$(R(x)F)(y) = \exp(i\sigma(y,x))F(x+y), \quad y \in V\,, \tag{4.4.1}$$

for each square-summable $F : V \to \mathbb{C}$. It is a unitary bounded linear operator; the set $\{R(x)\,|\,x \in V\}$ satisfies (4.1.11), since a direct computation valid for all $x, v \in V$ gives

$$R(x)R(y) = R(x+y).$$

Therefore, its norm closure is a CCR algebra over (V,σ) which acts on the Hilbert space $\ell^2(V)$.

Sometimes we refer to the functional $\mathcal{H} \to \mathbb{C}$ given by $x \mapsto \exp(-C\,\|x\|^2)$ as the Gaussian functional of parameter C. Moreover, as this is the Lévy characteristic function of a real centered Gaussian random variable with variance $\frac{1}{2C}$, we also refer to C as the variance parameter.

We shall see that constructions $\overline{\Delta(V,\sigma)}$ and $R(V,\sigma)$ are isomorphic, by showing the uniqueness part of Slawny's theorem for the nondegenerate case [Pet90, Theorem 2.2.]. Put $H_1 = \ell^2(V)$, and let $\mathfrak{R}_1$ denote the smallest C*-algebra containing multiplication operators by $\{R(x)\,|\,x \in V\}$ acting on H_1 and satisfying (4.1.11). Suppose there exists another C*-algebra $\mathfrak{R}_2$ generated by operators $\{W(x)\,|\,x \in V\}$. Up to a unitarily equivalent representation to the GNS-representation, we can suppose this is a concrete C*-algebra, $\mathfrak{R}_2 \subseteq \mathrm{B}(H_2)$, for some Hilbert space H_2. The map $R(x) \mapsto W(x)$ is an algebra homomorphisms that is *-isometric and surjective, if

$$\left\|\sum_{j=1}^{n} \lambda_j W(x_j)\right\|_{\mathfrak{R}_2} = \left\|\sum_{j=1}^{n} \lambda_j R(x_j)\right\|_{\mathfrak{R}_1}\,, \tag{4.4.2}$$

for all $\lambda_1,\ldots,\lambda_n \in \mathbb{C}, x_1,\ldots,x_n \in V$, $n \in \mathbb{N}$. Therefore, uniqueness is proven upon showing (4.4.2). To this end, we pass to a representation that is unitarily equivalent to R. For each $x \in V$, the mapping $\pi(x) : H_1 \otimes V \to H_1 \otimes V$ defined by

$$\pi(x)f \otimes y = (y-x) \otimes W(x)f\,, \tag{4.4.3}$$

for all $y \in V$, $f \in H_1$. Each $\pi(y), y \in V$, is linear and, clearly, bounded. Moreover,

$$\begin{aligned}\pi(x)\pi(y)(f\otimes z) &= \pi(x)(z-y)\otimes W(y)f = \pi(x)W(y)f\otimes(z-y)\\ &= (z-y-x)\otimes W(x)W(y)f = (z-(x+y))\otimes e^{-i\sigma(x,y)}W(x+y)f\\ &= e^{-is(x,y)}\pi(x+y)(f\otimes z)\,,\end{aligned}$$

for all $x, y, z \in V$, $f \in H_1$. Thus, $\pi : V \to \mathrm{B}(H_1 \otimes V)$ is a representation with the image algebra $\mathfrak{R}_3 = \pi(V)$ generated by operators satisfying (4.1.11) in $\mathrm{B}(H_1 \otimes V)$. Note that, if $H_3 = \ell^2(V, V)$, then, $H_3 \simeq H_1 \otimes V$. Therefore, π represents the canonical commutation relations (4.1.11) on H_3. Representations π and R are intertwined. Indeed, the operator U acting on H_3 by $U[f\otimes z] = e^{i\sigma(u,z)}(f\otimes z)$, for all $f \in \mathcal{H}_1$ and $z \in V$, fulfills $U\pi(x) = R(x)U$, for all $x \in V$, and it is unitary. As a consequence, $\mathfrak{R}_3 \simeq \mathfrak{R}_1$. Therefore, proving

$$\left\|\sum_{j=1}^{n}\lambda_j W(x_j)\right\|_{\mathfrak{R}_2} = \left\|\sum_{j=1}^{n}\lambda_j \pi(x_j)\right\|_{\mathfrak{R}_3} \tag{4.4.4}$$

yields that $\mathfrak{R}_1 \simeq \mathfrak{R}_2$ and the uniqueness of the CCR construction for nondegenerate symplectic spaces. We show (4.4.4) below.

For π in (4.4.3), consider the representation $\hat{\pi} : \hat{V} \to \mathcal{U}(\mathcal{H})$ given by

$$\hat{\pi}(x)\hat{a}(\chi) = \chi(x)W(x)\hat{a}(\chi), \quad \chi \in \hat{V},\ \hat{a} \in H_4,\ x \in V \tag{4.4.5}$$

where $\hat{V}$ is the dual of the discrete group V, and $H_4 = \ell^2(\hat{V}, H_2)$. Therefore, $\hat{\pi}(x)$ acts on H_4 as a multiplication by the character $\chi(x)$, for some $x \in V$. It follows that, for each finite $n \in \mathbb{N}$, $x_1, \ldots, x_n \in V$,

$$\left\|\sum_j \lambda_j \hat{\pi}(x_j)\right\|_{\mathfrak{R}_4} = \sup_{\chi\in\hat{V}}\left\|\sum_j \chi(x_j)W(x_j)\right\|_{\mathfrak{R}_2}. \tag{4.4.6}$$

The mapping $\chi \mapsto \chi(x_j)W(x_j)$, is a continuous function $\hat{V} \to \mathbb{C}$, for each positive integer $j \le n$ and vector $x_j \in V$. It follows that the supremum can be computed on a dense subset of $\hat{V}$.

In any proper closed subgroup of $\hat{G}$, there exists $0 \neq h \in V$ such that $k(h) = 1$ for every $k \in K$, [HR63, Corollary 23.26]. As a consequence, there cannot exist a nonzero $y \in V$ such that $\chi(y) = 1$, for all $\chi \in \hat{V}$. Indeed, if $h \in V$ is such that $\exp(2i\sigma(x, y)) = 1$ for every $x \in V$, then for every $t \in \mathbb{R}$ there exists an integer $l \in \mathbb{Z}$ such that $t\sigma(x, y) = l\pi$. Thus, y is such that $\sigma(x, y) = 0$, for all $x \in V$, which contradicts the nondegeneracy assumption. Consequently, the subset

$$G_0 = \left\{e^{2i\sigma(x,\cdot)}\,\middle|\, x \in \hat{V}\right\} \tag{4.4.7}$$

is dense in $\hat{V}$. The claim (4.4.4) follows by computing the supremum limited to (4.4.7), in light of the formula for the adjoint action of the Weyl operators,

$$W(x)W(y)W(-x) = e^{i\sigma(y,x)}W(x)W(y-x) = e^{2i\sigma(y,x)}W(y)\,. \tag{4.4.8}$$

□

We shall make the assumption that the symplectic space is nondegenerate. Specifically, we focus on CCR algebras $\mathrm{CCR}(V,\sigma)$ of Definition 4.1.5 for symplectic spaces $(\mathcal{H},\sigma_{\mathcal{H}})$ where $\mathcal{H}$ is a Hilbert space and the symplectic form is given by

$$\sigma_{\mathcal{H}}(x,y) = \mathrm{Im}\langle x,y\rangle, \qquad x,y\in\mathcal{H}\,. \tag{4.4.9}$$

Furthermore, we assume $\mathcal{H}$ to be separable and, precisely,

$$\mathcal{H} = \ell^2(\mathbb{Z}) \tag{4.4.10}$$

where $\ell^2(\mathbb{Z})$ is the set of square-summable sequences indexed by the integers $\mathbb{Z}$ with inner product given by the Hermitian inner product $\langle x,y\rangle = \sum_{j\in\mathbb{Z}} x_j\overline{y_j}$ which is anti-linear in the second argument.

Notation. *For any Hilbert space, we call the symplectic form (4.4.9) standard and ease the notation by setting* $\mathrm{CCR}(\mathcal{H}) := \mathrm{CCR}(\mathcal{H},\sigma_{\mathcal{H}})$*, whenever* $\mathcal{H}$ *is a Hilbert space.*

By the nondegeneracy assumption any representation is faithful. Among them, the CCR Fock representation is also irreducible.

The functional

$$\tau\left(\sum_{x\in\mathcal{H}}\lambda(x)W(x)\right) = \lambda(0)\,, \tag{4.4.11}$$

for all $\lambda:\mathcal{H}\to\mathbb{C}$ with finite support, extends to a state on $\mathrm{CCR}(\mathcal{H})$. It is clearly tracial by direct verification of Definition 2.4.1. Moreover, (4.4.11) allows to directly compute

$$\tau((W(x)-W(y))^*(W(x)-W(y))) = 2\,, \tag{4.4.12}$$

for any distinct $x,y\in\mathcal{H}$. Thus, generators stand apart for at least $\sqrt{2}$ in norm distance, then the one-parameter unitary group $\{W(tx)\}_{t\geq 0}$ is not normcontinuous and finally $\mathrm{CCR}(\mathcal{H})$ is a nonseparable C*-algebra.

Write $\mathrm{Symp}(\mathcal{H})$ for the group of linear operators $S:\mathcal{H}\to\mathcal{H}$ such that $\sigma(Sx,Sy) = \sigma(x,y)$. We dub these operators simply by "symplectic transformations"[1], recognizing them as the so-called Bogoliubov transformations,

[1] dropping linear from the terminology of [HLV15].

[BR81, p.22]. Each symplectic transformation defines a further unitary representation of the Weyl operators, by $W(x) \mapsto W(Sx)$. By universality, each $S \in \mathrm{Symp}(\mathcal{H})$ admits a *-monomorphism $\mathrm{CCR}(\mathcal{H}) \to \mathrm{CCR}(\mathcal{H})$ uniquely determined by

$$W(x) \mapsto W(Sx)\,. \tag{4.4.13}$$

Therefore, $\mathrm{Symp}(\mathcal{H})$ acts on $\mathrm{CCR}(\mathcal{H})$.

From the discussion carried so far, it emerges that even if $\mathcal{H}$ is a separable Hilbert space, working with $\mathrm{CCR}(\mathcal{H})$ presents challenges. Choquet's theorem A.1.6 cannot be applied straightforwardly, as discussed in Section 3.2. We attend this issue, by investigating G-abelianness of the C*-dynamical systems, where G is any classical group of Chapter 1.

4.5 An infinite tensor product CCR algebra

The discussion of Section 4.1 gave first heuristics on why $\mathrm{CCR}(\mathcal{H})$ turns out to be a natural model where rotatable states can be investigated. We give more detail in this direction. Specifically, for a counterpart to Freedman's theorem 1.3.2 we can rely on the existence of a nice-behaved subalgebra that we introduce in a moment.

$\mathrm{CCR}(\mathcal{H})$ is nuclear, see [EL77, Theorem 10.10]. It follows that, for nondegenerate symplectic spaces, the spatial tensor product [KR86, p.846, Section 11.3] coincides with both the projective and injective C*-tensor products, reviewed in Section 2.7.

Let $\mathfrak{C}$ any C*-algebra and $M : \mathcal{H} \to \mathfrak{C}$ a map. Then, by [EL77, Theorem 10.1], there exists a completely positive map $F : \mathrm{CCR}(\mathcal{H}) \to \mathfrak{C}$ with $F(W(x)) = M(x)$, for all $x \in \mathcal{H}$, iff the kernel induced by the Gaussian multiplier (4.1.1) on M is positive definite. CCR is known to be a functor from the category of Hilbert spaces with arrows that are contractions to the category of the C*-algebras satisfying the canonical commutation relations, whose arrows are the completely positive maps. See [EL77, Theorem 10.3]. Moreover, it satisfies

$$\mathrm{CCR}(\oplus_{j=1}^{n}\mathcal{K}_j) \simeq \bigotimes_{j=1}^{n} \mathrm{CCR}(\mathcal{K}_j)\,, \tag{4.5.1}$$

for all finitely many Hilbert spaces $\mathcal{K}_1, \ldots, \mathcal{K}_n$, $n \in \mathbb{N}$. The tensor product on the right-hand side of (4.5.1) is unambiguous because of the mentioned nuclearity. It is well-behaved on countable tensor products.

Let $\mathcal{H}_n \subset \mathcal{H}$ denote the finite-dimensional subspace generated by the orthonormal basis finite subset $\{e_k : |k| \leq n\}$. Let σ_n be the standard symplectic

form of $\mathcal{H}_n$ which is, in fact, the restriction of (4.4.9) to $\mathcal{H}_n$. Put

$$\mathfrak{B}_n = \mathrm{CCR}(\mathcal{H}_n, \sigma_n)\,. \tag{4.5.2}$$

The family of Hilbert spaces $\{\mathcal{H}_n\}_{n\in\mathbb{N}}$ is increasing and an inductive system.

Denote its inductive limit by $\mathcal{H}_0 = \cup_{n\in\mathbb{N}}\mathcal{H}_n$. Then, the space $\mathcal{H}_0$ is dense in $\mathcal{H}$, and a nondegenerate symplectic space with form (4.4) also obtained as the inductive limit of the $\sigma_n, n \in \mathbb{N}$. We set

$$\mathfrak{B}_0 = \mathrm{CCR}(\mathcal{H}_0, \sigma_0) \tag{4.5.3}$$

where σ_0 is the inductive limit of $\{\sigma_n\}_n$ that is directed by the order specified in Definition 4.1.4. The functor CCR on the objects of the Hilbert space category respects their order. Indeed, by [BR81, Proposition 5.2.9.], if $(V_1, \sigma_1) \preceq (V_2, \sigma_2)$ and $\mathrm{CCR}(\mathcal{V}_1, \sigma_1) \simeq \mathrm{CCR}(\mathcal{V}_2, \sigma_2)$ then $\mathcal{V}_1 = \mathcal{V}_2$. In particular, each (C*-)subalgebra $\mathfrak{C} \subseteq \mathrm{CCR}(\mathcal{H})$ is such that $\mathfrak{C} = \mathrm{CCR}(\mathcal{K}, \sigma_1)$ where $(\mathcal{K}, \sigma_1) \preceq (\mathcal{H}, \sigma_\mathcal{H})$. Therefore, since $\mathcal{H}_0$ is a proper subset of $\mathcal{H}$, $\mathfrak{B}_0 \subset \mathrm{CCR}(\mathcal{H})$ is a proper subalgebra.

In the construction of $\mathfrak{B}_0$, we see another instance of a CCR, beside that provided in Example 4.3.2 and fundamental for our analysis. The construction, indeed, is based on $\mathrm{CCR}(\mathbb{C})$, the CCR algebra over the complex numbers. $\mathfrak{B}_0$ is isomorphic to $\otimes_\mathbb{Z}\mathrm{CCR}(\mathbb{C})$, for the C*-algebra isomorphism $\Psi : \mathfrak{B}_0 \to \otimes_\mathbb{Z}\mathrm{CCR}(\mathbb{C})$ uniquely determined by

$$\Psi\left(W\left(\sum_{|k|\leq n} z_k e_k\right)\right) = \prod_{|k|\leq n} i_k(W(z_k))\,, \tag{4.5.4}$$

for all $n \in \mathbb{N}$ and $z_{-n}, \ldots, z_n \in \mathbb{C}$.

Notation. *In the sequel, we shorten* $\mathfrak{A} = \mathrm{CCR}(\mathbb{C})$ *and* $\mathfrak{B} = \mathrm{CCR}(\mathcal{H})$. *Further, we identify* $\mathfrak{B}_0$ *with its isomorphic copy* $\mathfrak{B}_0 \cong \otimes_\mathbb{Z}\mathfrak{A}$.

Reprising the observation made in Remark 4.1.3, each element in $\mathfrak{B}_0$ corresponds to a sequence of pairs of non-commuting selfadjoint operators $\{(Q_i, P_i) : i \in \mathbb{Z}\}$, each of which should be conceived as a complex variable.

4.6 Quasi-free states

We investigate the states of the C*-algebra CCR, aiming to solve similar questions to those studied in the literature. By contrast with these works that do not address the algebra of canonical commutation relations, we study CCR states by means of a generalized version of the Lévy characteristic function. When studying states on the CCR algebra, it is convenient to look at the generating functionals.

Definition 4.6.1. *Let $\varphi \in \mathcal{S}(\mathrm{CCR}(\mathcal{H}))$. Then, the functional $G : \mathcal{H} \to \mathbb{C}$ defined by $G(x) = \varphi(W(x))$, for all $x \in \mathcal{H}$, is called the generating (or characteristic) functional of φ.*

The generating functional reduces to the Lévy characteristic function (or Fourier transform) of the probability measure $\varphi\restriction_{\mathfrak{C}}$, whenever $\mathfrak{C} \subseteq \mathfrak{B}$ is a commutative subalgebra of $\mathfrak{B}$. We observe that if $G : \mathcal{H} \to \mathbb{C}$ is the characteristic functional of some C*-algebra state $\varphi \in \mathcal{S}(\mathfrak{A})$, then

$$|G(x)| \leq 1 = |G(0)| = \sup_{y \in \mathcal{H}} |G(y)| = \sup_{\|y\| \leq 1} |G(y)| = \|G\|_\infty \ ,$$

for all $x \in \mathcal{H}$, where $|\cdot|$ is the complex modulus.

We recall an important class of characteristic functionals.

Definition 4.6.2 (Quasi-free states)**.** *A state $\varphi \in \mathcal{S}(\mathfrak{B})$ is called gaussian or quasi-free, if its characteristic functional is a Gaussian functional. Specifically, we denote by φ_C, for $C > 0$, the quasi-free state of characteristic function, $G_{\varphi_C}(x) = \exp(-2^{-1}C\|x\|^2)$, $x \in \mathcal{H}$.*

The CCR Fock representation is obtained as a GNS-representation of the quasi-free state with characteristic functional $\exp(-\|x\|^2)$. It is also obtained as the GNS representation of the vacuum vector state for the Bose Fock space. The GNS representation of any quasi-free state is irreducible. Indeed, it is the representation of generating functional resulting from a variance dilation of the quasi-free functional inducing the Fock representation which is $e^{-\|x\|}, x \in \mathcal{H}$. See [EL77, Theorem 7.8 and Theorem 7.9.].

Not all maps $F : \mathcal{H} \to \mathbb{C}$ are characteristic functions of states on $\mathfrak{B}$. In [MV68, Proposition 1] a class of functions, called quasi-free, is provided such that if F is quasi-free then F is of positive-type iff the functional $\omega_F(W(x)) = F(x)$, $x \in \mathcal{H}$, extends to a state on the whole $\mathfrak{B}$. The quasi-free functions have known physical interpretations as the Wightman one-point and two-point correlation functions. See [MV68] for further details. We stress that quasi-free states have also been studied in the Fermi algebra. For the CAR algebra, there is a seminal definition of [BV68] that declare quasi-free states to be states uniquely determined by the 2-point correlation functions.

Proposition 4.6.3 ([Pet90], Proposition 3.13.)**.** *Let $F : \mathcal{H} \to \mathbb{C}$. Then, there exists $\phi \in \mathcal{S}(\mathfrak{B})$ such that $F = G_\phi$ if and only if $F(0) = 1$ and the kernel*

$$\mathcal{H}^2 \to \mathbb{C}\,, \quad (f,g) \mapsto F(f-g)\exp\{-i\sigma(f,g)\}$$

is positive-definite.

Remark 4.6.4. The restriction $\tau_0 = \tau\restriction_{\mathfrak{B}_0}$, where τ is the tracial state (4.4.11), is defined on each $a = \sum_{x\in\mathcal{H}_0}\lambda(x)W(x)$ by $\tau_0(a) = \lambda(0)$. Consequently, $\tau_0(W(x)) = \mathbf{1}_{\{0\}}(x)$, for all $x \in \mathcal{H}$, and $\tau_0 = \otimes_{\mathbb{Z}}\varphi$, for $\varphi \in \mathcal{S}(\mathfrak{A})$ defined by $G_\varphi(z) = \mathbf{1}_{\{0\}}(z)$.

Further, we see the indicator function at the origin as a gaussian functional with infinite variance parameter. Explicitly, we write with $\varphi_\infty \in \mathcal{S}(\mathfrak{A})$ the unique state it defines, whereas $\omega_\infty = \otimes_{\mathbb{Z}}\varphi_\infty$ shall denote the infinite tensor product.

Definition 4.6.5 (Regular states, analytic states)**.** *A state $\omega \in \mathcal{S}(\mathfrak{B})$ is called regular, if $t \mapsto G_\omega(tx + y)$ is continuous function $\mathbb{R} \to \mathbb{C}$. Any state $\varphi \in \mathcal{S}(\mathfrak{B})$ is said to be analytic, if $G_\varphi : \mathbb{C} \to \mathbb{C}$ is an analytic function.*

Quasi-free states, Definition 4.6.2 are regular, being even analytic.

Chapter 5

CCR Invariant States

In this chapter, we present our original results. The main theorem is the characterization of states on $\mathfrak{B} = \mathrm{CCR}(\mathcal{H})$, the CCR algebra over a Hilbert space $\mathcal{H}$, which are invariant under the action of the unitaries of $\mathcal{H}$.

We first ensure that actions of the groups $\mathbb{P}_{\mathbb{Z}}$,$\mathbb{O}_{\mathbb{Z}}$, $\mathbb{U}_{\mathbb{Z}}$ and $\mathcal{U}(\mathcal{H})$ are defined. Next, we pass to analyze the dynamics of infinite orthogonal matrices on the auxiliary subalgebra $\mathfrak{B}_0 = \mathrm{CCR}(\mathcal{H}_0)$ defined in (4.5.3). We apply the strategy depicted in the C*-dynamical section of Chapter 3, that entails the uniqueness of barycentric measures for rotatable states; we check that extreme states are product states and note that the arguments can be applied to states on $\mathfrak{B}_0$ that are invariant under infinite unitary matrices. We conclude the dynamical system part of our study by highlighting the order of inclusion between the simplices found. The next step is the detailed examination of conditions under which states on $\mathfrak{B}_0$ are rotatable. We deduce a functional equation that we solve, finding a consequent particular functional form for extreme rotatable state characteristic functions. This part contains the very generalization of the Freedman's theorem. The same functional equation with an extra condition characterizes unitarily invariant states; but in this case the variance turns out to be lower bounded. This fact has a nice physical interpretation as the Heisenberg's principle. In the concluding part of this chapter, we describe the topology of the extreme unitarily invariant states and prove the uniqueness of the barycentric decomposition formula for unitarily invariant states. The formula is comparable to the classical Freedman's theorem mixture, in that the integral involved is against a measure that distributes over admissible variance parameters for Gaussian states. We show that the found barycentric decomposition can be extended to states defined over the whole CCR algebra $\mathfrak{B}$ provided that they are not only unitarily invariant but invariant under the action of all unitary operators. As a consequence, we deduce the normality in the CCR Fock representation of

regular $\mathcal{U}(\mathcal{H})$-invariant states.

5.1 Invariant States

We shall see that the classical groups act on $\mathrm{CCR}(\mathcal{H})$. We are interested in isometries that, similarly to the classical case, act directly on the index set $\mathbb{Z}$. Fix an orthonormal basis $\{e_j\}_j$.

For any injective map $\Psi : \mathbb{Z} \to \mathbb{Z}$, the mapping defined by

$$S_\Psi e_j = e_{\psi(j)}, \qquad j \in \mathbb{Z}, \tag{5.1.1}$$

extends to a unique bounded linear operator on $\mathcal{H}$. In particular, S_Ψ is a symplectic transformation, sending $W(e_j)$ to $W(e_{\psi(j)})$.

The class of isometries defined in (5.1.1) includes all mappings related to the classical group action, since the shift operator on $\mathbb{Z}$, the monoid of monotone increasing mappings, and each finitary permutation of $\mathbb{Z}$ are all injective mappings and induce an action with (5.1.1). For each $U \in \mathcal{U}(\mathcal{H})$, $\sigma(Ux, Uy) = \sigma(x, y)$, for all $x, y \in \mathcal{H}$. Then, $\mathcal{U}(\mathcal{H})$ is a subgroup of $\mathrm{Symp}(\mathcal{H})$, and thus it also acts on $\mathfrak{B}$ through automorphisms with (4.4.13). Precisely, $\alpha_U(W(e_j)) = W(Ue_j)$, for each $U \in \mathcal{U}(\mathcal{H})$ and each basis element e_j of $\mathcal{H}$, uniquely determines an automorphism of $\mathfrak{B}$ and an action of $\mathcal{U}(\mathcal{H})$,

$$\alpha : \mathcal{U}(\mathcal{H}) \to \mathrm{Aut}(\mathfrak{B}), \qquad \alpha : U \mapsto \alpha_U \,. \tag{5.1.2}$$

Therefore, each semigroup $\mathbb{Z}, \mathbb{J}_\mathbb{Z}, \mathbb{P}_\mathbb{Z}, \mathcal{U}(\mathcal{H})$, acts by endomorphisms upon $\mathfrak{B}$. In fact, $\mathbb{Z}$ and $\mathbb{P}_\mathbb{Z}$ act through automorphisms on $\mathfrak{B}$, for their elements are bijective mappings and then, their images, invertible endomorphisms.

We assess $\mathcal{U}(\mathcal{H})$-invariance on the CCR algebra over a nondegenerate symplectic space. Our results are obtained through a strategy for a smaller C*-dynamical system than $(\mathfrak{B}, \mathcal{U}(\mathcal{H}), \alpha)$, which is deduced by examining the subalgebra $\mathfrak{B}_0$ of Section 4.5, for specific auxiliary subgroups. The family of unitary matrix groups $\{\mathbb{U}[n]\}_{n\in\mathbb{N}}$ is a directed system. We denote by $\mathbb{U}_\mathbb{Z}$ its inductive limit group and refer to it as the infinite unitary matrix group. For each $U \in \mathbb{U}_\mathbb{Z}$, $\langle Ue_j, e_k\rangle$ is the entry of U in position (j, k) and there exists the smallest positive integer n such that $U \in \mathbb{U}[n]$. Consequently, the infinite orthogonal group $\mathbb{O}_\mathbb{Z}$, introduced in Chapter 1, can be written as

$$\mathbb{O}_\mathbb{Z} = \{U \in \mathbb{U}_\mathbb{Z} \,|\, \langle Ue_j, e_k\rangle \in \mathbb{R},\ j, k \in \mathbb{Z}\} \tag{5.1.3}$$

and it is seen to be a subgroup of $\mathbb{U}_\mathbb{Z}$, $\mathbb{O}_\mathbb{Z} \leq \mathbb{U}_\mathbb{Z}$. On the other hand, each element $U \in \mathbb{U}_\mathbb{Z}$ extends linearly to a unitary operator on $\mathcal{H}$, by the mapping that sends U to the operator $\widetilde{U}$ such that $\widetilde{U}x = Ux$, for all $x \in \mathcal{H}_0$ and

$\widetilde{U}x = x$, for all $x \in \mathcal{H} \setminus \mathcal{H}_0$. This mapping is, clearly, a group monomorphism and we write $\mathbb{U}_\mathbb{Z} \lesssim \mathcal{U}(\mathcal{H})$. In this notation, $\mathbb{P}_\mathbb{Z} \lesssim \mathbb{O}_\mathbb{Z} \leq \mathbb{U}_\mathbb{Z} \lesssim \mathcal{U}(\mathcal{H})$, and all these groups act through automorphisms on $\mathfrak{B}$, as restrictions of the action (5.1.2). Since $\mathcal{H}_0$ is an invariant subspace, $\mathcal{U}(\mathcal{H})$ acts by automorphisms on $\mathfrak{B}_0$, as well as any of its subgroup. In particular, if $U \in \mathbb{U}_\mathbb{Z}$ and $x = \sum_{k\in\mathbb{Z}} z_k e_k \in \mathcal{H}_0$, then

$$\alpha_U\left(W\left(\sum_{k\in\mathbb{Z}} z_k e_k\right)\right) = W\left(\sum_{k\in\mathbb{Z}} z_k U e_k\right) = W\left(\sum_{k\in\mathbb{Z}} z_k \sum_{l=-n}^{n} U_{l,k} e_l\right) \tag{5.1.4}$$

where n is the least positive integer such that $U \in \mathbb{U}[n]$.

In the next, we adhere to the consolidated tradition in the noncommutative probability literature.

Definition 5.1.1. *Let $G \subseteq \mathcal{U}(\mathcal{H})$ be a semigroup of unitary operators over $\mathcal{H}$. If $\varphi \in \mathcal{S}^G(\mathcal{H})$, then we call φ: stationary, if $G = \mathbb{Z}$; spreadable, if $G = \mathbb{J}_\mathbb{Z}$; exchangeable, if $G = \mathbb{P}_\mathbb{Z}$; and rotatable, if $G = \mathbb{O}_\mathbb{Z}$. Further, we call unitarily invariant each $\varphi \in \mathcal{S}^{\mathbb{U}_\mathbb{Z}}(\mathfrak{B})$.*

We plan to study the C*-dynamical systems $(\mathfrak{B}_0, G, \alpha\restriction_G)$ for G equal to any of the groups $\mathbb{P}_\mathbb{Z}$, $\mathbb{O}_\mathbb{Z}$ and $\mathbb{U}_\mathbb{Z}$, where $\mathfrak{B}_0$ is the proxy subalgebra introduced in Section 4.5. Let QF be the set of quasi-free states Definition 4.6.2. Then, clearly $\mathrm{QF}(\mathrm{CCR}(\mathcal{H})) \subseteq \mathcal{S}^{\mathbb{O}_\mathbb{Z}}(\mathfrak{B})$, while more generally $\mathrm{QF}(\mathrm{CCR}(\mathcal{H})) \subseteq \mathcal{S}^{\mathrm{Isom}(\mathcal{H})}(\mathfrak{B})$.

Since $\mathfrak{B}_0$ is an infinite tensor product, strategies relying on C*-dynamical systems, specifically on asymptotical abelianness (Section 3.2), can be applied giving us insights into the structure of invariant states of Definition 3.2.2. We write the action (5.1.4) of $\mathbb{U}_\mathbb{Z}$ on the infinite tensor product $\otimes_\mathbb{Z}\mathfrak{A}$. Firstly, we need a small calculation.

Lemma 5.1.2. *For any $k \in \mathbb{N}$ and $x_1, \ldots, x_k \in \mathcal{H}$,*

$$W(x_1 + \ldots + x_k) = \exp\left(-\sum_{i<j, i,j\in[k]} \sigma(x_i, x_j)\right) \prod_{l=1}^{k} W(x_l)\,. \tag{5.1.5}$$

Proof. By induction on k. The case $k = 2$, is exactly (4.1.11). If $k \geq 3$, for

all vectors $x_1, \ldots, x_k \in \mathcal{H}_0$,

$$\begin{aligned}
&W(x_1 + \ldots + x_k) = W((x_1 + \ldots + x_{k-1}) + x_k) \\
&= e^{-\sigma(x_1+\ldots+x_{k-1},x_k)} W(x_1 + x_2 + \ldots + x_{k-1}) W(x_k) \\
&= e^{-\sum_{i=1}^{k-1} \sigma(x_i,x_k)} W(x_1 + x_2 + \ldots + x_{k-1}) W(x_k) \\
&= e^{-\sum_{i=1}^{k-1} \sigma(x_i,x_k)} e^{-\sum_{j=1}^{k-2} \sum_{i=1}^{j-1} \sigma(x_i,x_j)} W(x_1) W(x_2) \cdot \ldots \cdot W(x_{k-1}) W(x_k) \qquad (5.1.6) \\
&= e^{-\sum_{j=1}^{k-1} \sum_{i=1}^{j-1} \sigma(x_i,x_j)} W(x_1) W(x_2) \cdot \ldots \cdot W(x_{k-1}) W(x_k) \\
&= \exp\left(- \sum_{i<j, \{i,j\} \subseteq [k]} \sigma(x_i, x_j) \right) W(x_1) W(x_2) \cdot \ldots \cdot W(x_{k-1}) W(x_k)
\end{aligned}$$

□

In light of (4.5.4), we refine (5.1.4) finding the effect of the action the coordinates of the tensor product $\mathfrak{B}_0$.

Lemma 5.1.3. *For any $U = (U_{i,j})_{i,j \in \mathbb{Z}}$ in $\mathbb{U}_\mathbb{Z}$,*

$$\alpha_U(i_k(W(z))) = \prod_j i_j(W(U_{j,k} z)), \quad k \in \mathbb{Z}, \ z \in \mathbb{C}. \qquad (5.1.7)$$

As an immediate consequence of Lemma 5.1.3, for every $U \in \mathbb{U}_\mathbb{Z}$, one has $\alpha_U(\mathfrak{B}_0) \subseteq \mathfrak{B}_0$. Therefore, the dynamical system $(\mathfrak{B}, G, \alpha)$, with α as in (5.1.2), can be restricted to a dynamical system $(\mathfrak{B}_0, G, \alpha\restriction_G)$, for all $G \leq \mathbb{U}_\mathbb{Z}$. In the particular case of $G = \mathbb{U}_\mathbb{Z}$ (and then $G = \mathbb{O}_\mathbb{Z}$) this is explicited in (5.1.4).

Proof of Lemma 5.1.3. For all $k \in \mathbb{Z}$, $z \in \mathbb{C}$,

$$\alpha_U(i_k(W(z))) = \alpha_U(W(z e_k))$$

from (4.5.4) for the product evaluated on the singleton $\{k\}$.

Thus,

$$\alpha_U(i_k(W(z))) = W(U z e_k) = W\Big(\sum_{j=-n}^{n} U_{j,k} z e_j \Big)$$

because, for some $n \in \mathbb{N}$, $U z e_k = \sum_{j=-n}^{n} U_{j,k} z e_j$, for all $k \in \mathbb{Z}, z \in \mathbb{C}$.

Writing $x_j = U_{j,k} z e_j$ for each $j \in \mathbb{Z}$ and fixed $k \in \mathbb{Z}$, Lemma 5.1.2 implies

$$\begin{aligned}
\alpha_U W(z e_k) &= \exp\left(- \sum_{i<j, i,j \in [-n:n]} \sigma(U_{i,k} z e_i, U_{j,k} z e_j) \right) \prod_{l=-n}^{n} W(U_{l,k} z e_l) \\
&= \exp\left(- \sum_{i<j, i,j \in [-n:n]} \sigma(U_{i,k} z e_i, U_{j,k} z e_j) \right) \prod_{l=-n}^{n} i_l(W(U_{l,k} z))
\end{aligned}$$

but σ is bilinear and antisymmetric, thus

$$\sum_{i<j,i,j\in[-n:n]} \sigma(U_{i,k}ze_i, U_{j,k}ze_j) = \sum_{i,j\in[-n:n]} \sigma(U_{i,k}ze_i, U_{j,k}ze_j)$$
$$= \sigma\left(\sum_{i\in[-n:n]} U_{i,k}ze_i, \sum_{j\in[-n:n]} U_{j,k}ze_j\right) = \sigma(Uze_k, Uze_k) = 0\,.$$

□

5.2 $\mathbb{O}_{\mathbb{Z}}$-invariance

By the strategy illustrated in Section 3.2 of Chapter 3, it sufficient to check that there is a sequence along which transformed elements commute with any other element in $\mathfrak{B}_0$. Underlying to the works on T-deformed second quantization scheme models, there are considered variations of the sequence of bijections $g_n : \mathbb{Z} \to \mathbb{Z}$ given by, for all $n, k \in \mathbb{N}$,

$$g_n(k) = \begin{cases} 2^n - 1 + k & 1 \le k \le 2^{n-1} \\ k - 2^n - 1 & 2^{n-1} \le k \le 2^n \\ k & k > 2^n \end{cases} \tag{5.2.1}$$

which was first considered in [Stø69] to prove the theorems reported in Section 3.3 of Chapter 3.

We prove that any rotatable state $\omega \in \mathcal{S}^{\mathbb{O}_{\mathbb{Z}}}(\mathfrak{B}_0)$ is an infinite tensor product state. To this aim, we will need to consider the following sequence of bijections $\mathbb{Z}$,

$$g_n(k) = \begin{cases} k + 2^{n-1}, & 0 \le k < 2^{n-1} \text{ or } -2^n \le k < -2^{n-1} \\ k - 2^{n-1}, & 2^{n-1} \le k < 2^n \text{ or } -2^n \le k \le -1 \\ k, & k \ge 2^n \text{ or } k < -2^n \end{cases} \tag{5.2.2}$$

for all $n \in \mathbb{N}$, $k \in \mathbb{Z}$. In particular, g_n is represented by an infinite matrix such that for each row $j \in \mathbb{N}$ all entries are zero except the entry $(j, g_n(j)) = 1$, for all $n \in \mathbb{N}$. A direct inspection shows that mappings in (5.2.2) are finitary permutations, and then $g_n \in \mathbb{O}_{\mathbb{Z}}$, for all $n \in \mathbb{N}$. Thus, the sequence $\{g_n : n \in \mathbb{N}\}$ is contained in $\mathbb{O}_{\mathbb{Z}}$, where we kept in writing g_n for these permutation matrices. As stated by Theorem 3.3.5 in Section 3.3 of Chapter 3, the sequence g_n (5.2.1) implies

Proposition 5.2.1. *Both $\mathcal{S}^{\mathbb{Z}}(\mathfrak{B}_0)$, $\mathcal{S}^{\mathbb{P}_{\mathbb{Z}}}(\mathfrak{B}_0)$ are a Choquet simplices.*

As noted in [CF16, Section 4.2., p.16, (ii)], one has that

$$\mathcal{S}^{\mathbb{Z}}(\mathfrak{B}_0) \supsetneq \mathcal{S}^{\mathbb{P}_{\mathbb{Z}}}(\mathfrak{B}_0)\,. \tag{5.2.3}$$

Lemma 5.2.2. *The restricted C^*-dynamical system $(\mathfrak{B}_0, \mathbb{O}_{\mathbb{Z}}, \alpha\restriction_{\mathbb{O}_{\mathbb{Z}}})$ is asymptotically abelian.*

Proof. We will prove that, for any $a, b \in \bigotimes_{\mathbb{Z}} \mathfrak{A}$, one has

$$\lim_{n\to\infty} \|\rho_{g_n}(a)b - b\rho_{g_n}(a)\| = 0\,.$$

A standard density argument shows it is enough to deal with a, b lying in a dense subalgebra of $\bigotimes_{\mathbb{Z}} \mathfrak{A}$. With this in mind, we can consider localized simple tensors a, b. In other words, there is no lack of generality if we take $a = \prod_{|k|\leq m} i_k(W(\lambda_k))$ and $b = \prod_{|l|\leq m} i_l(W(\mu_l))$ for some $m > 0$ and $\lambda_k, \mu_l \in \mathbb{C}$ for all $k, l \in \{-m, \ldots, 0, \ldots, m\}$. Applying (5.1.3), one finds

$$\rho_{g_n}(a) = \prod_j \prod_{|k|\leq m} i_j(W(\lambda_k(g_n)_{j,k})\,.$$

Now if n satisfies $2^{n-1} > m$, the definition of g_n implies that the entries $(g_n)_{j,k}$ are 0 for all j with $|j| > m$. Phrased differently, for n such that $2^{n-1} > m$ one has that $\rho_{g_n}(a)$ is a simple tensor whose j-th factors are different from the identity I only when $|j| > m$, hence $\rho_{g_n}(a)$ commutes with b. □

Referring to the strategy hinged on Theorem 3.2.7 in Section 3.2 of Chapter 3, Lemma 5.2.2 implies that $\mathcal{S}^{\mathbb{O}_{\mathbb{Z}}}(\mathfrak{B}_0)$ is a Choquet simplex. Thus, we have that rotatable states appear as unique mixtures of extreme states.

We give the strategy for the description of the extreme rotatable states. The infinite tensor product states are to be interpreted as "quantum iid sequences" with respect to tensor independence [Mur11]. Theorem 3.3.5 from [Stø69] gives the strong clustering property as the characterizing property for a state to be extreme or equivalently, an infinite product state. In this way, it is known that each $\varphi \in \mathcal{S}(\mathfrak{A})$ is such that $\otimes_{\mathbb{Z}}\varphi \in \mathcal{S}^{\mathbb{P}_{\mathbb{Z}}}(\mathfrak{B}_0)$, by Section 3.3, because $\mathfrak{B}_0$ is an infinite tensor product. We show that each extreme rotatable $\omega_0 \in \mathcal{S}^{\mathbb{O}_{\mathbb{Z}}}(\mathfrak{B}_0)$ factors into an infinite product $\omega_0 = \otimes_{\mathbb{Z}}\varphi$, for a single state on the sample algebra, $\varphi \in \mathcal{S}(\mathfrak{A})$. In a similar way, we show that if $\varphi \in \mathcal{ES}^{\mathbb{O}_{\mathbb{Z}}}(\mathfrak{B}_0)$ then φ is strongly clustering. We divide the proof in a couple of lemmata.

Lemma 5.2.3. *For every localized simple tensor $a \in \otimes_{\mathbb{Z}}\mathfrak{A}$ and every $O \in \mathbb{O}_{\mathbb{Z}}$, one has*

$$\alpha_{Og_n}(a) = \alpha_{g_n}(a)$$

eventually for all n.

Proof. By hypothesis a can be taken as $\prod_{|k|\leq m} i_k(W(\lambda_k))$ for some $m > 0$ and $\lambda_k \in \mathbb{C}$ for all k's in $\{-m, \ldots, m\}$. In addition, there exists $d > 0$ such that $O_{i,j} = \delta_{i,j}$ for all i, j with $|i|, |j| > d$. The statement holds if n satisfies $2^{n-1} \geq \max\{m, d\}$. Indeed, ρ_O acts as the identity on simple tensors whose factors different from I lie outside the discrete interval $\{-d, \ldots, d\}$, and $\rho_{g_n}(a)$ is a tensor of this form. □

Lemma 5.2.4. *If $\omega \in \mathcal{ES}^{\mathbb{O}_{\mathbb{Z}}}(\mathfrak{B}_0)$, then for every a in $\mathfrak{B}_0$ one has*

$$\lim_{n\to\infty} \pi_\omega(\alpha_{g_n}(a))\xi_\omega = \omega(a)\xi_\omega$$

in the weak topology of $\mathcal{H}_\omega$.

Proof. As usual, we will identify $\mathfrak{B}_0$ with the infinite tensor product $\bigotimes_{\mathbb{Z}} \mathfrak{A}$. We do not harm the generality of the proof if we suppose that a is localized. Let ω be an extreme rotatable state on $\mathfrak{B}_0$, and localized simple tensor $a \in \mathfrak{B}_0$. By weak compactness of the unit ball $(\mathcal{H}_\omega)_1$, the bounded sequence $\{\pi_\omega(\alpha_{g_n}(a))\xi_\omega : n \in \mathbb{N}\} \subset \mathcal{H}_\omega$ must have at least one accumulation point, say ξ. Any accumulation point is $\mathbb{O}_{\mathbb{Z}}$-invariant. Indeed, if ξ is an accumulation point, then, by definition, there exists a subnet $\{\pi_\omega(\alpha_{g_{n_\alpha}}(a))\xi_\omega : \alpha \in I\}$ such that $\xi = \lim_\alpha \pi_\omega(\alpha_{g_{n(\alpha)}}(a))\xi_\omega$ (in the weak topology of $\mathcal{H}_\omega$). Then, for any $O \in \mathbb{O}_{\mathbb{Z}}$,

$$\begin{aligned} U_O^\omega \xi &= \lim_\alpha U_O^\omega \pi_\omega(\alpha_{g_{n_\alpha}}(a))\xi_\omega = \lim_\alpha U_O^\omega \pi_\omega(\alpha_{g_{n_\alpha}}(a))(U_O^\omega)^* \xi_\omega \\ &= \lim_\alpha \pi_\omega(\alpha_{Og_{n_\alpha}}(a))\xi_\omega = \lim_\alpha \pi_\omega(\alpha_{g_{n_\alpha}}(a))\xi_\omega = \xi \end{aligned}$$

where the second-last equality is a consequence of Lemma 5.2.3. Since ω is extreme, ξ is such that $\xi = \lambda\xi_\omega$ for some complex scalar $\lambda \in \mathbb{C}$, due to Lemma 5.2.2 and [Sak98, Proposition 3.1.12.]. Note that λ is necessarily equal to $\omega(a)$ irrespective of the chosen subsequence. In other words, $\omega(a)\xi_\omega$ is the only accumulation point of the set $\{\pi_\omega(\alpha_{g_n}(a))\xi_\omega : n \in \mathbb{N}\}$. Therefore, again by weak compactness of the unit ball of $\mathcal{H}_\omega$, the whole sequence $\{\pi_\omega(\alpha_{g_n}(a))\xi_\omega : n \in \mathbb{N}\}$ converges to $\omega(a)\xi_\omega$.

□

Proposition 5.2.5. *If ω is an extreme state in $\mathcal{S}^{\mathbb{O}_{\mathbb{Z}}}(\mathfrak{B}_0)$, then there exists a state φ on $\mathfrak{A}$ such that $\omega = \otimes_{\mathbb{Z}}\varphi$. We term φ the factorizing state of ω.*

Proof. We start by showing that any extreme state is strongly clustering, that is

$$\lim_n \omega(\rho_{g_n}(a)b) = \omega(a)\omega(b)\,,$$

for all $a, b \in \mathfrak{B}_0$. This is a consequence of Lemma 5.2.4, for

$$\begin{aligned}\lim_n \omega(\rho_{g_n}(a)b) &= \lim_n \langle \pi_\omega(\rho_{g_n}(a))\pi_\omega(b)\xi_\omega, \xi_\omega\rangle \\ &= \lim_n \langle \pi_\omega(b)\xi_\omega, \pi_\omega(\rho_{g_n}(a^*))\xi_\omega\rangle = \omega(a)\omega(b).\end{aligned}$$

We next define φ as a state on $\mathfrak{A}$ by setting $\varphi(a) := \omega(i_j(a))$, $a \in \mathfrak{A}$, $j \in \mathbb{Z}$. The definition is well posed since it does not depend on j, for ω is exchangeable being even rotatable. In order to prove the thesis, we need to show that

$$\omega(i_{j_1}(a_1)\cdots i_{j_n}(a_n)) = \varphi(a_1)\cdots\varphi(a_n)\,,$$

for every $n \in \mathbb{N}$, for every $j_1, \ldots, j_n$ in $\mathbb{Z}$, and $a_1, \ldots, a_n$ in $\mathfrak{A}$. This can be done by induction on n exactly as in the proof of [Stø69, Theorem 2.7.], to which the reader is referred for the missing details. □

Remark 5.2.6. Clearly, Lemma 5.2.4 also applies to $\mathbb{U}_\mathbb{Z}$-invariant states on $\mathfrak{B}_0$. The properties of the C*-dynamical system $(\mathfrak{B}_0, \mathbb{O}_\mathbb{Z}, \alpha\restriction_{\mathbb{O}_\mathbb{Z}})$ are valid for $(\mathfrak{B}_0, \mathbb{U}_\mathbb{Z}, \alpha\restriction_{\mathbb{U}_\mathbb{Z}})$. In particular, Lemma 5.2.4 and Proposition 5.2.5 also apply to $\mathcal{ES}^{\mathbb{U}_\mathbb{Z}}(\mathfrak{B}_0)$.

Similar to the case of the CAR algebra considered in [CVR23], we find that the invariant state sets under discussion are Choquet simplices and each one is a face of the other.

Proposition 5.2.7. *Each of* $\mathcal{S}^{\mathbb{U}_\mathbb{Z}}(\mathfrak{B}_0), \mathcal{S}^{\mathbb{O}_\mathbb{Z}}(\mathfrak{B}_0), \mathcal{S}^{\mathbb{P}_\mathbb{Z}}(\mathfrak{B}_0)$ *and* $\mathcal{S}^{\mathbb{Z}}(\mathfrak{B}_0)$ *is a Choquet simplex. Moreover,*

$$\mathcal{S}^{\mathbb{U}_\mathbb{Z}}(\mathfrak{B}_0) \subset_F \mathcal{S}^{\mathbb{O}_\mathbb{Z}}(\mathfrak{B}_0) \subset_F \mathcal{S}^{\mathbb{P}_\mathbb{Z}}(\mathfrak{B}_0) \subset_F \mathcal{S}^{\mathbb{Z}}(\mathfrak{B}_0)$$

where $S \subseteq_F T$ *denotes that* $S \subseteq T$*, and* S *is a face of* T*, for the convex sets* S, T*.*

Proof. We can argue as in the proof of [CVR23, Theorem 3.5.] and reduce the problem to showing that any extreme state in one of the first three sets (say $\mathcal{S}^{\mathbb{U}_\mathbb{Z}}(\mathfrak{B}_0)$) remains extreme in the next set to the right ($\mathcal{S}^{\mathbb{O}_\mathbb{Z}}(\mathfrak{B}_0)$). But this is trivial for the first two inclusions because extreme $\mathbb{U}_\mathbb{Z}$-invariant/$\mathbb{O}_\mathbb{Z}$-invariant/symmetric states are product states, and product states are extreme in $\mathcal{S}^{\mathbb{P}_\mathbb{Z}}(\mathfrak{B}_0)$ as follows from [Stø69, Theorem 2.7.]. As for $\mathcal{S}^{\mathbb{P}_\mathbb{Z}}(\mathfrak{B}_0)$ being a face of $\mathcal{S}^{\mathbb{Z}}(\mathfrak{B}_0)$, one can proceed in the same way as done in [CVR23, Theorem 3.5.]. □

A functional equation

We see how the rotatability on $\mathfrak{B}_0$ allows for a generalization of Freedman's theorem 1.3.2. Let $G_\varphi : \mathbb{C} \to \mathbb{C}$ be the characteristic function of the state $\varphi \in \mathcal{S}(\mathfrak{A})$.

Lemma 5.2.8. *Let φ be a state on $\mathfrak{A}$. Then, any product $\omega = \otimes_{\mathbb{Z}}\varphi$ is a rotatable state on $\mathfrak{B}_0$ if and only if G_φ satisfies the equations*

$$G_\varphi(z) = \prod_j G_\varphi(O_{j,k}z)\,, \quad z \in \mathbb{C}\,,\ k \in \mathbb{Z}\,, \tag{5.2.4}$$

for any given matrix $O = (O_{k,j})_{k,j\in\mathbb{Z}}$ in $\mathbb{O}_{\mathbb{Z}}$.

Proof. It is a matter of direct computation. If ω is rotatable, then thanks to Lemma 5.1.3 we have

$$\begin{aligned} G_\varphi(z) =& \varphi(W(z)) = \omega(i_k(W(z)) = \omega(\rho_O(i_k(W(z))) \\ =& \omega\Big(\prod_j i_j(W(O_{j,k}z))\Big) = \prod_j \varphi(W(O_{j,k}z)) \\ =& \prod_j G_\varphi(O_{j,k}z)\,, \end{aligned}$$

for all z in $\mathbb{C}$. On the other hand, if the equation is satisfied, then similar calculations show that ω is rotatable. □

We see that (5.2.4) characterizes rotatable states on $\mathfrak{B}_0$. In the sequel, we read it as a functional equation and write $G \vDash \mathrm{FE}$ to denote that G is any complex function, $G : \mathbb{C} \to \mathbb{C}$ and solves (5.2.4). Moreover, $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ stands for the extended real line, and $\overline{\mathbb{R}_+} = [0, +\infty]$ denotes its positive part (including zero).

Proposition 5.2.9. *Let φ be a state on $\mathfrak{A}$ such that $\otimes_{\mathbb{Z}}\varphi \in \mathcal{S}^{\mathbb{O}_{\mathbb{Z}}}(\mathfrak{B}_0)$. Then, either $G_\varphi = \mathbf{1}_{\{0\}}$, the indicator function of $\{0\}$, or there exists $\sigma^2 \geq 0$ such that*

$$G_\varphi\restriction_{\mathbb{R}}(t) = e^{-\frac{\sigma^2 t^2}{2}}, \quad t \in \mathbb{R}\,.$$

Proof. Since $G_\varphi \vDash \mathrm{FE}$ in particular we find $G_\varphi(z) = G_\varphi(-z)$ for all $z \in \mathbb{C}$. As a consequence, the restriction of G_φ to the real line is completely determined by its restriction to the positive half-line. Furthermore, by Proposition 4.6.3, the restriction of G_φ to the real line must be a definite-positive function.
Let now $z = x + iy$ be any given complex number. By considering a 2×2 orthogonal matrix given by a rotation by an angle α the functional equation yields

$$G_\varphi(|z|) = G_\varphi(\cos\alpha|z|)G_\varphi(\sin\alpha|z|)\,.$$

If we now choose α any argument of z, we find $G_\varphi(\sqrt{x^2+y^2}) = G_\varphi(x)G_\varphi(y)$. Henceforth we will denote by g the restriction of G_φ to the real line. From what we have seen, g is even and

$$g(\sqrt{x^2+y^2}) = g(x)g(y)\,, \tag{5.2.5}$$

for all $x, y \geq 0$. Since g is even, we may as well work with the restriction of g to the positive half line. Let us define $h : [0,\infty) \to \mathbb{C}$ as $h(t) = g(\sqrt{t})$, $t \in \mathbb{R}$. From (5.2.5) one has that h satisfies $h(t+s) = h(t)h(s)$, for $t, s \geq 0$. Let ρ be the absolute value of h, *i.e.* $\rho(t) = |h(t)|$, $t \geq 0$. We clearly have $\rho(t+s) = \rho(t)\rho(s)$, for all $t, s \geq 0$. There are two cases to deal with depending on whether ρ may vanish at a point or not. If it does, that is if there exists $t_0 > 0$ with $\rho(t_0) = 0$, then $\rho = \mathbb{1}_{\{0\}}$. Indeed, if $t \geq t_0$, then $\rho(t) = \rho(t-t_0)\rho(t_0) = 0$. Furthermore, $\rho(\frac{t_0}{n}) = \rho(t_0)^{\frac{1}{n}} = 0$ for every $n \in \mathbb{N}$ with $n \geq 1$. In conclusion $\rho(t) = 0$ for all $t > 0$, hence $\rho = \mathbb{1}_{\{0\}}$.
If ρ never vanishes, then there exists $\Psi : [0,\infty) \to \mathbb{R}$ such that $\rho(t) = e^{\Psi(t)}$ and $\Psi(t+s) = \Psi(t) + \Psi(s)$ for all $t, s \geq 0$. Clearly, any such Ψ is the restriction of a function $\widetilde{\Psi}$ defined on the whole real line that satisfies Cauchy's functional equation, namely $\widetilde{\Psi}(t+s) = \widetilde{\Psi}(t) + \widetilde{\Psi}(s)$, t, s in $\mathbb{R}$. If $\widetilde{\Psi}$ is continuous, then $\widetilde{\Psi}(t) = ct$, $t \in \mathbb{R}$, for some real constant c. In addition, for G_φ to be bounded, it is necessary to take only non-positive values, say $c = -\frac{\sigma^2}{2}$ for some σ in $\mathbb{R}$. Discontinuous (non-measurable) solutions of Cauchy's equation are ruled out because their graph is well known to be dense in $\mathbb{R}^2$, thus giving rise to an unbounded ρ.
From now on we can work under the hypothesis that $\rho(t) > 0$ for all $t \geq 0$, which allows us to decompose h as $h(t) = \rho(t)\chi(t)$, for all $t \geq 0$, with $\chi(t) \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ for all $t \geq 0$. Now χ satisfies the same equation as h, that is $\chi(t+s) = \chi(t)\chi(s)$. The conclusion will be fully reached once we have shown that χ must be identically 1 (when $\rho > 0$). To this end, note that χ cannot take non-real values. Suppose on the contrary there exists $t_0 > 0$ such that $\chi(t_0)$ is not real. The function $G_\varphi(t) = \chi(t^2)\rho(t^2)$, $t \in \mathbb{R}$, then turns out not to be positive definite. Indeed, the 2×2 matrix $(G_\varphi(t_i - t_j))_{i,j\in[2]}$ with $t_1 = 0$ and $t_2 = t_0$ is

$$\begin{pmatrix} 1 & \chi(t_0^2)\rho(t_0^2) \\ \chi(t_0^2)\rho(t_0^2) & 1 \end{pmatrix},$$

which even fails to be Hermitian.
In other terms, the only possible values for χ are 1 and -1. But, again, the values -1 is ruled out. Indeed, if there existed t_0 with $\chi(t_0) = -1$, then from the very equation satisfied by χ we would have $\chi(\frac{t_0}{2}) = \sqrt{\chi(t_0)}$, and both square roots of -1 are not real, being $\pm i$.

□

Remark 5.2.10. Proposition 5.2.5 along with Lemma 5.2.2 provides a novel proof of Freedman's theorem for commutative random variables by considerations made in [BR81, p.24] which we expound here. Freedman's theorem 1.3.2 is obtained by restricting our dynamics to the commutative subalgebra

$$\mathfrak{R} = C^*(\{W(te_j) \,|\, j \in \mathbb{Z}, t \in \mathbb{R}\}) \cap \mathfrak{B}_0 \,. \tag{5.2.6}$$

This is

$$\mathfrak{R} = \otimes_{\mathbb{Z}} \mathrm{CCR}(\mathbb{R}, \mathbf{0}) \tag{5.2.7}$$

where we recognize the commutative CCR algebra of Example 4.3.2, for which $\mathrm{CCR}(\mathbb{R}, \mathbf{0}) \simeq C(\beta\mathbb{R})$. We recall that $C(\beta\mathbb{R}) \cong \mathrm{AP}(\mathbb{R})$, where $\mathrm{AP}(\mathbb{R})$ denotes the commutative C*-algebra of almost-periodic functions. The application of Proposition 5.2.5 to the case of $\mathfrak{R} = \otimes_{\mathbb{Z}} C(\beta\mathbb{R})$ reads in this way, as a statement for iid real-random variables, and a generalization of Freedman's theorem 1.3.2.

We identify all rotatable states, by finding all the $G \vDash \mathrm{FE}$.

Lemma 5.2.11. *If $G \vDash \mathrm{FE}$, then, there exists a positive function*

$$\sigma^2 : [0, 2\pi[\to [0, +\infty] \tag{5.2.8}$$

such that

$$G(z) = \exp\left(-\tfrac{\sigma^2(\arg(z))}{2}|z|^2\right) \tag{5.2.9}$$

for all $z \in \mathbb{C}$.

Proof. It is an application of Proposition 5.2.9, thanks to which the solutions of our equations can be determined by considering their restrictions to lines through the origin. More precisely, let $z_0 = x_0 + iy_0$ be a fixed complex number. Let us consider the line in the complex plane $\{tz_0 = tx_0 + ity_0 : t \in \mathbb{R}\}$ and denote by $g : \mathbb{R} \to \mathbb{C}$ the restriction of G to this line, that is $g(t) := G(tz_0)$, $t \in \mathbb{R}$. Now the function g can be determined in the same way as in the proof of Proposition 5.2.9, which means we must have $g(t) = e^{-\sigma^2(z_0)t^2} = e^{-\frac{\sigma^2(z_0)t^2|z_0|^2}{|z_0|^2}}$ for all real t. Setting $z = tz_0$, one finally sees that $G(z) = e^{-\sigma^2(z)|z|^2}$, where $\sigma^2(z) = \frac{\sigma^2(z_0)}{|z_0|^2}$ depends only on the argument of $z = x + iy$, say on the ratio $\frac{y}{x}$. □

Example 5.2.12. *For all reals $\lambda, \mu \geq 0$, define $t_{\lambda,\mu}(z) = \sqrt{\lambda}\,\mathrm{Re}(z) + \sqrt{\mu}\, i\mathrm{Im}(z)$, $t_{\lambda,\mu} : \mathbb{C} \to \mathbb{C}$. Then, the bilinear form $s_{\lambda,\mu}(z, w) = t_{\lambda,\mu}(z)t_{\lambda,\mu}(w)$ defines the functional $\varphi_{\lambda,\mu} : \mathfrak{A} \to \mathbb{C}$, $\varphi(W(z)) = e^{-\frac{1}{2}s_{\lambda,\mu}(z,z)}$. More explicitly,*

$$\varphi_{\lambda,\mu}(W(z)) := e^{-\frac{1}{2}(\lambda x^2 + \mu y^2)}\,, \quad z = x + iy\,.$$

Proposition 4.6.3 tells us that for $\varphi(W(z)) = e^{-\frac{1}{2}s(z,z)}$, z *in* $\mathbb{C}$, *to define a state, it is necessary and sufficient that*

$$s(z,z)s(w,w) \geq s(z,w)^2\,,$$

for all z, w *in* $\mathbb{C}$, *where* $s(z,z) = \lambda x^2 + \mu y^2$ *with* $z = x + iy$.
If the inequality holds for complex numbers $z = x_1 + iy_1$ *and* $w = x_2 + iy_2$, *then* $(\lambda x_1^2 + \mu y_1^2)(\lambda x_2^2 + \mu y_2^2) \geq (y_1 x_2 - x_1 y_2)^2$. *In particular, for* $x_2 = 0$, *we find* $(\lambda\mu - 1)x_1^2 + \mu^2 y_1^2 \geq 0$, *for all* $x_1, y_1 \in \mathbb{R}$, *from which* $\lambda\mu \geq 1$ *follows.*
Conversely, if $\lambda\mu \geq 1$, *then*

$$\begin{aligned} s_{\lambda,\mu}(z,z)s_{\lambda,\mu}(w,w) &= (\lambda x_1^2 + \mu y_1^2)(\lambda x_2^2 + \mu y_2^2) \\ &= \lambda^2 x_1^2 x_2^2 + \lambda\mu x_1^2 y_2^2 + \lambda\mu y_1^2 x_2^2 + \mu^2 y_1^2 y_2^2 \\ &\geq -2\lambda\mu x_1 x_2 y_1 y_2 + \lambda\mu x_1^2 y_2^2 + \lambda\mu y_1^2 x_2^2 \\ &= \lambda\mu(x_1 y_2 - y_1 x_2)^2 \\ &\geq (x_1 y_2 - y_1 x_2)^2 = s_{\lambda,\mu}(z,w)^2\,. \end{aligned}$$

Accordingly, for any choice of λ, μ *with* $\lambda\mu \geq 1$, *the infinite tensor product* $\otimes_{\mathbb{Z}}\varphi$ *provides a rotatable state on* $\mathfrak{B}_0$, *which is also* $\mathbb{U}_{\mathbb{Z}}$*-invariant and extreme.*

5.3 Unitary invariance

As said in Remark 5.2.6, the C*-dynamical system $(\mathfrak{B}_0, \mathbb{U}_{\mathbb{Z}}, \alpha\restriction_{\mathbb{U}_{\mathbb{Z}}})$ inherits its asymptotic abelianness and extreme unitary invariant states are product states. In addition, the more restrictive condition imposed by $\mathbb{U}_{\mathbb{Z}}$ yields a more precise functional form for the characteristic function.

Proposition 5.3.1. *If* $\varphi \in \mathcal{S}(\mathfrak{A})$ *is such that* $\otimes_{\mathbb{Z}}\varphi \in \mathcal{S}^{\mathbb{U}_{\mathbb{Z}}}(\mathfrak{B}_0)$, *then, either* $G_\varphi = \mathbf{1}_{\{0\}}$ *or*

$$G_\varphi(z) = e^{-\frac{\sigma^2|z|^2}{2}}\,, \quad z \in \mathbb{C}\,,$$

where σ^2 *is a constant with* $\sigma^2 \geq 1$.

Proof. Since $\mathbb{O}_{\mathbb{Z}} \leq \mathbb{U}_{\mathbb{Z}}$, then $G_\varphi \vDash \mathrm{FE}$, with the extra condition that $G_\varphi(z) = G_\varphi(e^{i\theta}z)$, $z \in \mathbb{C}$, which holds for all $\theta \in \mathbb{R}$. But then $G_\varphi(z)$ only depends on $|z|$, which means it is uniquely determined by its restriction to the real line. Proposition 5.2.9 gives that either $G_\varphi = \mathbf{1}_{\{0\}}$ or $G_\varphi(z) = e^{-\frac{\sigma^2|z|^2}{2}}$, $z \in \mathbb{C}$, for some $\sigma^2 > 0$. Finally, the condition $\sigma^2 \geq 1$ comes from Example 5.2.12 with $\lambda = \mu = \sigma^2$. □

Let $\varphi \in \mathcal{S}(\mathfrak{A})$. Then $\otimes_{\mathbb{Z}}\varphi \in \mathcal{ES}^{\mathbb{O}_{\mathbb{Z}}}(\mathfrak{B}_0)$. By Proposition 5.2.9, its characteristic function is a quasi-free with coefficients $\lambda = \mu = \sigma^2 \geq 0$ that can be also infinite. On the other hand, by Example 5.2.12, that any quasi-free functional induces a state only if $\lambda\mu \geq 1$, thus, $\sigma^2 \geq 1$. The lower bound has a nice interpretation also in physical terms. Consider the complex plane as the phase-space for a one-dimensional boson particle $W(z) \in \mathfrak{B}_0$. Then, the real and imaginary axis represents position and momentum of the observable $W(z)$, and σ^2 is the product of the error on their measurements. We move on towards a clearer geometrical description of the extreme $\mathbb{U}_{\mathbb{Z}}$ invariant states. We noted in Remark 4.6.4 that the tracial state on $\mathfrak{B}_0$ is an infinite tensor product state, denoted by $\omega_\infty \in \otimes_{\mathbb{Z}}\mathcal{S}(\mathfrak{A})$. We observe that $\omega_{\sigma^2} = \otimes_{\mathbb{Z}}\varphi_{\sigma^2}$ is extreme rotatable, by direct observation, while the tracial state is rotatable. We recall that ω_{σ^2} denotes the infinite tensor product $\otimes_{\mathbb{Z}}\varphi_{\sigma^2}$, where φ_{σ^2} is the state on $\mathfrak{A}$ determined by gaussian characteristic function of parameter $2^{-1}\sigma^2$, and $[0, +\infty]$ for the one-point compactification of the positive real half-line. The mapping

$$\begin{aligned} \Psi :&[1, +\infty] \to \mathcal{ES}^{\mathbb{U}_{\mathbb{Z}}}(\mathfrak{B}_0) \\ &\Psi(\sigma^2) = \omega_{\sigma^2} \end{aligned} \tag{5.3.1}$$

is well-defined and injective. It is also continuous by continuity of the Gaussian function in the variable σ^2.

Finally, (5.3.1) is surjective by the discussion in Remark 5.2.6. We summarize the observations made in the next

Corollary 5.3.2. *The map* $\Psi : [1, +\infty] \to \mathcal{E}(\mathcal{S}^{\mathbb{U}_{\mathbb{Z}}}(\mathfrak{B}_0))$ *given by*

$$\Psi(\sigma^2) = \begin{cases} \omega_{\sigma^2}\,, & \text{if } \sigma^2 < \infty \\ \omega_\infty\,, & \text{if } \sigma^2 = \infty \end{cases}$$

is a homeomorphism of topological spaces. In particular, $\mathcal{E}(\mathcal{S}^{\mathbb{U}_{\mathbb{Z}}}(\mathfrak{B}_0))$ *is a compact subset of* $\mathcal{S}^{\mathbb{U}_{\mathbb{Z}}}(\mathfrak{B}_0)$

Proposition 5.3.3. *If* ω *is a* $\mathbb{U}_{\mathbb{Z}}$*-invariant state on* $\mathfrak{B}_0$*, then there exists a unique Borel probability measure* μ *on* $[1, +\infty]$ *such that*

$$\omega = \int_{[1,+\infty]} \omega_{\sigma^2} \mathrm{d}\mu(\sigma^2)\,.$$

More explicitly, for every x *in* $\mathcal{H}_0$ *one has*

$$\omega(W(x)) = \int_1^{+\infty} e^{-\frac{\sigma^2\|x\|^2}{2}} \mathrm{d}\mu(\sigma^2) + \mu(\{\infty\})\omega_\infty(W(x))\,.$$

Proof. From the first part of Proposition 5.2.7, it follows uniqueness of the Borel probability measure of the barycentric decomposition, while Corollary make precise that the support is $[1,+\infty]$. The second formula is implied by the barycentric decomposition so that one can compute as follows.

If $x = \sum_{i\in F}\lambda_i e_i$, for any finite subset $F \subset \mathbb{Z}$, where $\lambda_j \in \mathbb{C}$, for all $j \in F$, then

$$\begin{aligned}\omega_{\sigma^2}(W(x)) &= \omega_{\sigma^2}\left(\prod_{i\in F} W(\lambda_i e_i)\right) = \prod_{i\in F}\varphi_{\sigma^2}(W(\lambda_i)) \\ &= \prod_{i\in F} e^{-\frac{\sigma^2|\lambda_i|^2}{2}} = e^{-\frac{\sigma^2\|x\|^2}{2}}\,.\end{aligned}$$

Thus, $\omega_{\sigma^2}(W(x)) = e^{-\frac{\sigma^2\|x\|^2}{2}}$, for all $x \in \mathcal{H}_0$. □

Proposition 5.3.3 sets the basis upon which we prove our main contribution to distributional symmetries in quantum probability. Besides that, it allows some considerations on regularity of CCR states.

Proposition 5.3.4. *If μ is a Borel probability measure on $[1,+\infty]$ with $\mu(\{\infty\}) = 0$, then the corresponding $\mathbb{U}_{\mathbb{Z}}$-invariant state $\omega := \int_1^{+\infty}\omega_{\sigma^2}\mathrm{d}\mu(\sigma^2)$ is regular.*

Proof. For any x in $\mathcal{H}_0$ we have

$$\omega(W(tx)) = \int_1^{+\infty} e^{-\frac{\sigma^2 t^2\|x\|^2}{2}}\mathrm{d}\mu(\sigma^2)\,.$$

The continuity in t of the expression in the above equation is a consequence of Lebesgue's dominated convergence theorem since $e^{-\frac{\sigma^2 t^2\|x\|^2}{2}} \leq 1$, for all $t \in \mathbb{R}$ and $\sigma^2 \in [1,+\infty)$. □

Let $\omega \in \mathcal{S}(\mathfrak{B}_0)$ be a regular state. Then, $\mu_\omega(\{\infty\}) = 0$, for its barycentric measure μ_ω. Therefore, regular states are also analytic, and these are the only ones among the $\mathbb{U}_{\mathbb{Z}}$-invariant states.

Remark 5.3.5. If ρ is any non-regular state on $\mathfrak{A} = \mathrm{CCR}(\mathbb{C},\sigma)$, then $\otimes_{\mathbb{Z}}\rho$ is certainly an extreme symmetric state on $\mathfrak{B}_0$, which by construction fails to be regular. In this way it is possible to exhibit a good many non-regular extreme symmetric states, in stark contrast with $\mathbb{U}_{\mathbb{Z}}$-invariant states, whose only non-regular extreme state is the canonical trace.

Theorem 5.3.6. *The compact convex $\mathcal{S}^{\mathcal{U}(\mathcal{H})}(\mathfrak{B})$ is a Bauer simplex whose extreme point set is $\{\omega_{\sigma^2} : \sigma^2 \in [1,+\infty]\}$. Explicitly, if ω is a $\mathcal{U}(\mathcal{H})$-invariant state on $\mathfrak{B}$, then there exists a unique Borel probability measure μ on $[1,+\infty]$ such that*

$$\omega = \int_{[1,+\infty]}\omega_{\sigma^2}\mathrm{d}\mu(\sigma^2)\,.$$

Proof. Let $\omega \in \mathcal{S}(\mathfrak{B})$ be a $\mathcal{U}(\mathcal{H})$-invariant state. Then, the functional $\omega\restriction_{\mathfrak{B}_0}$ is a state on $\mathfrak{B}_0$ that is $\mathbb{U}_{\mathbb{Z}}$-invariant. Then the map $\mathcal{S}^{\mathcal{U}(\mathcal{H})}(\mathfrak{B}) \to \mathcal{S}^{\mathbb{U}_{\mathbb{Z}}}(\mathfrak{B}_0)$ given by $\omega \mapsto \omega\restriction_{\mathfrak{B}_0}$ is well-defined. We show that it is an affine homeomorphism between compact convex sets. Injectivity of Ψ follows from transitivity of $\mathcal{U}(\mathcal{H})$ on $\mathcal{H}$: if x lies in $\mathcal{H} \setminus \mathcal{H}_0$, it is possible to find $y \in \mathcal{H}_0$ and $U \in \mathcal{U}(\mathcal{H})$ such that $\|x\| = \|y\|$ and $y = Ux$. But then, $\omega(W(x)) = \omega(W(Ux)) = \omega(W(y))$. In other words, ω is completely determined by its restriction to $\mathfrak{B}_0$. The map Ψ is clearly surjective thanks to Proposition 5.3.3. The continuity of Ψ is trivial. By compactness of $\mathcal{S}^{\mathcal{U}(\mathcal{H})}(\mathfrak{B})$ it follows that Ψ is a closed map, which means its inverse is continuous as well. □

Remark 5.3.7. The hypothesis of Theorem 5.3.6 that requires the $\mathcal{U}(\mathcal{H})$-invariance cannot be dropped. This can be seen by looking at the so-called flat extensions of a state defined on $\mathfrak{B}_0$. To this end, pick any $\mathbb{U}_{\mathbb{Z}}$-invariant state ω on $\mathfrak{B}_0$. Then one can define a state Ω on the whole $\mathfrak{B}$ by setting

$$\Omega(W(x)) = \begin{cases} \omega(W(x)), & x \in \mathcal{H}_0 \\ 0, & x \in \mathcal{H} \setminus \mathcal{H}_0 \end{cases}$$

The proof that the definition above yields a state on $\mathfrak{B}$ can be found in [AMS93, Proposition 2.1]. However, it remains to show that Ω is invariant under the action of $\mathbb{U}_{\mathbb{Z}}$ on $\mathfrak{B}$. But this easily follows from the inclusions $U\mathcal{H}_0 \subset \mathcal{H}_0$ and $U(\mathcal{H} \setminus \mathcal{H}_0) \subset \mathcal{H} \setminus \mathcal{H}_0$, which hold true for any unitary U in $\mathbb{U}_{\mathbb{Z}}$. Phrased differently, the map that restricts $\mathbb{U}_{\mathbb{Z}}$-invariant states on $\mathfrak{B}$ to $\mathfrak{B}_0$ fails to be injective.

In the next proposition we show that the states of Theorem 5.3.6 belong to the folium of all normal states in the CCR Fock representation, provided they have zero mass at infinity.

Proposition 5.3.8. *Any regular $\mathcal{U}(\mathcal{H})$-invariant state on $\mathfrak{B}$ is normal in the CRR Fock representation.*

Proof. By Theorem 5.3.6 any $\omega \in \mathcal{S}^{\mathcal{U}(\mathcal{H})}(\mathfrak{B})$ can be written as

$$\omega = \int_{[1,+\infty]} \omega_{\sigma^2} \mathrm{d}\mu(\sigma^2),$$

for a unique probability measure on $[1, +\infty]$ with $\mu(\{\infty\}) = 0$.

We recall that for each $\sigma^2 \geq 1$, the state ω_{σ^2} is normal in the CRR Fock representation $\pi_{\mathcal{F}}$.

By [Pit13, Lemma 6.4.1.], the operator T_{σ^2} is equal to

$$\frac{\Gamma(A(I+A)^{-1})}{\mathrm{Tr}(\Gamma(A(I+A)^{-1}))}$$

with $A = \frac{\sigma^2-1}{2} I$. Note that

$$\widetilde{\omega}(S) := \int_1^{+\infty} \mathrm{Tr}(ST_{\sigma^2})\mathrm{d}\mu(\sigma^2)\,, \quad S \in \mathrm{B}(\mathcal{F}_+(\mathcal{H}))$$

defines a state on $\mathrm{B}(\mathcal{F}_+(\mathcal{H}))$ such that $\omega(a) = \widetilde{\omega}(\pi_{\mathcal{F}}(a))$ for every $a \in \mathfrak{B}$. Thus, it is sufficient to prove that $\widetilde{\omega}$ is a normal state.

Let $\{S_n : n \in \mathbb{N}\} \subset \mathrm{B}(\mathcal{F}_+(\mathcal{H}))$ be a bounded increasing sequence of positive operator, and let S be its sup. As is known, S is the strong limit of S_n, hence for every $\sigma^2 \geq 1$, $\lim_n \mathrm{Tr}(S_n T_{\sigma^2}) = \mathrm{Tr}(ST_{\sigma^2})$. Applying the monotone convergence theorem,

$$\lim_n \widetilde{\omega}(S_n) = \lim_n \int_1^{+\infty} \mathrm{Tr}(S_n T_{\sigma^2})\mathrm{d}\mu(\sigma^2) = \int_1^{+\infty} \mathrm{Tr}(ST_{\sigma^2})\mathrm{d}\mu(\sigma^2) = \widetilde{\omega}(S)\,.$$

□

We would like to remark that a $\mathcal{U}(\mathcal{H})$-invariant state ω on $\mathfrak{B}$ can be thought of as a quantum stochastic process of a particular kind. Indeed, any such state is uniquely determined by its restriction ω_0 to $\mathfrak{B}_0$, and the tensor product structure of $\mathfrak{B}_0$ allows us to recover a stochastic process with sample algebra $\mathfrak{A} = \mathrm{CCR}(\mathbb{C}, \sigma)$ from ω_0, see [CRZ23, Section 3]. In particular, one may consider the tail algebra of the process, as defined in *e.g.* [CF15; CR22b], and ask what it looks like. The question is very easily answered for extreme states of $\mathcal{S}^{\mathbb{U}_\mathbb{Z}}(\mathfrak{B}_0)$, $\mathcal{S}^{\mathbb{O}_\mathbb{Z}}(\mathfrak{B}_0)$ or $\mathcal{S}^{\mathbb{P}_\mathbb{Z}}(\mathfrak{B}_0)$. In fact, extreme states of both $\mathcal{S}^{\mathbb{U}_\mathbb{Z}}(\mathfrak{B}_0)$ and $\mathcal{S}^{\mathbb{O}_\mathbb{Z}}(\mathfrak{B}_0)$ are all instances of extreme symmetric states, being product states. In [CF15, Proposition 2.9.], it is shown that extreme symmetric have a trivial tail algebra, in the wider context of local action on quasi-local C^*-algebras.

The tail algebra of non-extreme states can of course fail to be trivial. Nevertheless, it will always be commutative. See [CRZ23, Proposition 2.10] and the references therein. Furthermore, there is only one conditional expectation from $\pi_{\omega_0}(\mathfrak{B}_0)''$ onto the tail algebra, and, more importantly, the process itself is conditionally independent and identically distributed (with centered Gaussian distribution for states in $\mathcal{S}^{\mathbb{U}_\mathbb{Z}}(\mathfrak{B}_0)$ or $\mathcal{S}^{\mathbb{O}_\mathbb{Z}}(\mathfrak{B}_0)$) with respect to the conditional expectation.

Appendix A

Appendix

We are concerned about probabilities, that is, finite measures with normalized total mass. We call any measure on a Borel measurable space a Borel measure. Let T be a topological space and $\mathcal{B}(T)$ the σ-algebra of the Borel (measurable) subsets.

A Borel measure μ is said to be σ-finite, if there exists a countable family $F_k \in \mathcal{B}(T), k \in \mathbb{N}$ such that $\mu(F_k) < \infty$, for all $k \in \mathbb{N}$ and $\cup_k F_k = T$; μ is said locally finite, if it is finite on each member of some topological base for T.

Let $E \in \mathcal{B}(T)$. Then, E is called inner regular for μ, if

$$\mu(E) = \sup_{O \in \mathcal{O}_E} (\mu(O)) \tag{A.0.1}$$

where $\mathcal{O}_E$ is the family of open subsets $O \supseteq E$. Further, E is called outer regular, if

$$\mu(E) = \inf_{K \in \mathcal{K}_E} \mu(K) \tag{A.0.2}$$

where $\mathcal{K}_E$ that of compact subsets $K \subseteq E$.

Any σ-finite measure is called inner (outer resp.) regular, if all measurable subsets are inner (outer resp.). A σ-finite measure μ on T is said regular, if it is both inner and outer regular, which means $\sup_{O \in \mathcal{O}_E}(\mu(O)) = \inf_{K \in \mathcal{K}_E} \mu(K)$, for all $E \in \mathcal{B}(T)$.

For a measurable space $(M, \mathscr{A})$ a map $\phi : \mathscr{A} \to \mathbb{R}$ that it is σ-additive i.e. maps countable disjoint unions of members of $\mathscr{A}$ into the series of their image, is said to be a signed measure. For signed measures, there exists a unique measurable subset on which they assume non-negative real values, by the Hahn-decomposition theorem. Such decomposition is then a pair of measures ϕ_+, ϕ_- such that $\phi(F) = \phi_+(F) + \phi_-(F)$, for all $F \in \mathscr{A}$.

Further, the functional on the real vector space of signed measure given by

$$\|\phi\|_{\mathrm{TV}} = \sup \{\phi(F) - \phi(T \setminus F) \,|\, F \in \mathcal{B}(T)\} \tag{A.0.3}$$

is a norm called the total variation norm. For a signed measure ϕ on measurable space $(M, \mathscr{A})$, $\|\phi\|_{\mathrm{TV}} = \phi_+(M) + \phi_-(M)$. Therefore, for a finite measure μ on $(M, \mathscr{A})$, $\|\mu\| = \mu(M)$ and $\|\mu\| = 1$ whenever μ is a probability.

Beyond compact Hausdorff spaces, the most general measurable spaces we are concerned with are the locally compact and Hausdorff spaces e.g. non-unital C*-algebra spectra. In these spaces, a Borel measure μ on T is equivalently locally finite or finite on each compact subset. Any Borel measure μ is said to be Radon, if it "concentrates" on compact subsets, i.e. for all $B \in \mathcal{B}(T)$, $\varepsilon > 0$, there exists a compact $K \subseteq B$ such that $\mu(B \setminus K) < \varepsilon$. Thus, a measure is equivalently Radon, or locally finite and inner regular. A probability measure on T is a Borel measure which is finite and normalized i.e. $\mu(T) = 1$. A finite measure is either regular, or alternatively, neither inner nor outer regular. Therefore, a regular Borel probability measure is regular Radon and viceversa. We denote by $\mathcal{M}(T)$ the set of regular Borel measures that are finite, and with $\mathcal{M}_1(T)$ the probability measures.

Let S be a locally compact Polish space. Then, S can be partitioned by relatively compact subsets i.e. subsets whose closure in S is compact. From this fact, any Borel measure on S which is finite on relatively compact subsets is σ-finite i.e. the measure is finite on each of the members of whichever family of measurable subsets whose union is S. Let $\mathcal{M}_{\mathrm{lf}}(S)$ be the locally finite measures on S. $\mathcal{M}_{\mathrm{lf}}(S)$ is itself a measurable space for the σ-algebra generated by subsets of the form

$$\{\mu \in \mathcal{M}_{\mathrm{lf}}(S) \,|\, \mu(B) \leq x\}\,, B \in \mathcal{B}(S), x \geq 0\,. \tag{A.0.4}$$

Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space. A random variable $\nu : \Omega \to \mathcal{M}_{\mathrm{lf}}(S)$ is called a random measure. Equivalently, any map $\nu : \Omega \to \mathcal{M}_{\mathrm{lf}}(S)$ is a random measure iff the integral $\nu(f) = \int f(x)\mathrm{d}\nu(x)$ of each positive Borel measurable function $f : S \to \mathbb{R}_+$ is a real random variables. The distribution of a random measure is the probability on $\mathcal{M}_{\mathrm{lf}}(S)$ given by $\mathcal{L}(\nu) = \mathbb{P} \circ \nu^{-1}$, specifically, $\mathcal{L}(\nu)(B) = \mathbb{P}\{\omega \in \Omega \,|\, \nu(\omega) \in B\}$, for all $B \subseteq \mathcal{M}_1(S)$ and measurable according to the σ-algebra generated by (A.0.4). A random measure ν on the space of probability measures on S (i.e. $\nu \in \mathcal{M}_1(S)$ almost-surely) is called a random probability measure on S.

A.1 Convex analysis

This section collects results in convex analysis that are necessary to discuss Bishop-de Leeuw [BL59] and Choquet's [CM63] theorems. The references are [Phe66; Wea13]. We shall fix some nomenclature as in the literature there

are slight variations. The reference space is a locally convex TVS denoted V, within which all theorems in this section are framed.

A topological vector space (TVS) is a vector space V equipped with a topology that makes vector addition and scalar multiplication continuous functions i.e. $x \mapsto x + y$ is continuous, for all y, and $x \mapsto \lambda x$ is continuous, for all $\lambda \in \mathbb{C}$. Linear combinations where all the coefficients are positive are called conical combination; while if the coefficients sums up to 1, they are called affine combinations; conic affine combinations are called convex combinations. Let V be a vector space. A subset $X \subseteq V$ is said to be a convex subset, if $tx + (1-t)y \in X$, for all $x, y \in X$, $t \in [0,1]$.

Definition A.1.1 (Locally convex space, faces, extreme points)**.** *Let V a TVS. If any point of a TVS has a neighborhood base whose members are convex subsets, then the TVS is called locally convex.*

For a subset $S \subseteq V$, let $S+S = \{s_1 + s_2 \,|\, s_1, s_2 \in S\}$ and $RS = \{rs \,|\, s \in R, s \in S\}$ for each $R \subseteq \mathbb{C}$. A subset $X \subseteq V$ is said to be a cone, if $\mathbb{R}_+ X \subseteq X$. Intersections of convex subsets are again convex. The convex hull of any subset $S \subseteq V$ is defined as the smallest convex subset containing S, or equivalently, the intersection of all such convex subsets, or more explicitly and conveniently, the set $\operatorname{conv} S = \cap_{n=0}^{\infty} \{\sum_{k=1}^{n} t_k x_k \,|\, x_k \in S, t_k \in [0,1], k \in [n]\}$. As a fact, the closure of $\operatorname{conv} S$, $\overline{\operatorname{conv} S}$, can also be obtained by the intersection of all convex subset containing S that are closed subsets, usually denoted by $\overline{\operatorname{conv}}(S)$. A function between convex subsets $f : C \to D$ is also called convex, if $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$, for all $x, y \in C$, $t \in [0,1]$, while it is called affine, if the equality holds.

In TVS, cones order the space, and the definition of what is a simplex arises. Let V be a locally convex TVS. The partial order relation is defined by, for all $x, y \in V$, $x \leq y$, if $y - x \in P$. In $P - P$ there are defined upper bounds for each pair $x, y \in P$, that are, $z \in P$ such that $z \geq x$ and $z \geq y$. Denote by $x \vee y$ the least upper bound when it exists. Clearly, $\vee$ distributes over the vector summation when it is defined i.e. $z \vee (x + y) = (z \vee x) + (z \vee y)$, whenever $z \vee x, z \vee y$ are defined. A poset (i.e. partially ordered set) L is called lattice, if it is partially ordered and the partial order is such that any pair of points x, y have a least common upper bound denoted $x \vee y$ with respect to that order. Equivalently, any poset L is a lattice whenever there is (well-defined) the map $(x, y) \mapsto x \vee y$. In abstract lattice theory parlance, the binary operation $(x, y) \mapsto x \vee y$ is called the join operation. An example of a lattice is the subset of orthogonal projections $\mathsf{P}(\mathcal{H})$ in a concrete C*-algebra $\mathrm{B}(\mathcal{H})$, for $\mathcal{H}$ a Hilbert space.

A subset $X \subseteq V$ is a base for the cone P, if $P = \{\alpha x \,|\, x \in X\}$. Let $\tilde{X}$ be the cone with base X.

Definition A.1.2 (Simplex). *If $\widetilde{X}-\widetilde{X}$ is a lattice, then X is called a simplex.*

Theorem A.1.3 (Riesz-Markov, [RS72], Theorem IV.14). *If X is a compact Hausdorff space. Then, any positive linear functional $\varphi \in (C(X))^*_+$ has a unique regular Borel measure on X, μ_φ such that $\varphi(f) = \int f(x)\mathrm{d}\mu_\varphi(x)$.*

It results that, if X is a compact Hausdorff space and φ a positive linear functional $\varphi \in (C(X))^*_+$, then $\|\varphi\|_{C(X)^*} = \|\mu_\varphi\|_{\mathrm{TV}}$ where $\|\cdot\|_{\mathrm{TV}}$ is the total variation norm (A.0.3) of μ as a positive signed measure. In particular, $\varphi(1) = \mu(X)$.

Theorem A.1.3 tells us that the state space of the commutative C*-algebra $C(X)$, $\mathcal{S}(C(X))$, is isometrically *-isomorphic to the convex set of probability measures on X, $\mathcal{M}_1(X)$. By [Phe66, Proposition 1.2., p. 4], in any compact convex $X \subseteq V$, every point $x \in \overline{\mathrm{conv}}(X)$, the barycenter (3.2.2) of some probability measure $\mu \in \mathcal{M}_1(X)$

$$\varphi(x) = \int f(y)\,\mathrm{d}\mu(y)$$

Theorem A.1.4 (Krein-Milman, [DS58], Theorem V.8.4). *Let V a locally convex TVS and $X \subseteq V$ a compact subset. Then, $\overline{\mathrm{conv}}\mathcal{E}X \supseteq X$. Consequently, $\overline{\mathrm{conv}}\mathcal{E}X = \overline{\mathrm{conv}}(X)$. In particular, $\overline{\mathrm{conv}(X)} = X$, if X is convex.*

The Krein-Milman theorem is equivalent to the existence of a decomposition (3.2.2) for points in a compact convex X that is made with respect to a probability with pseudosupported on $\mathcal{E}(X)$. See [Phe66, p.5]. In this sense, Theorem A.1.4 is generalized by the

Theorem A.1.5 (Bishop-de Leeuw theorem, [Phe66], p.17).
Let V be a locally convex TVS and $X \subseteq V$ be a compact convex subset. If $x \in X$, then there exists a probability measure μ on X such that $x = \int y\,\mathrm{d}\mu(y)$ and $\mu(B) = 0$, for any regular Borel subset $B \subseteq X$ not intersecting $B \cap \mathcal{E}X = \varnothing$.

Recall that compact convex subsets are closed convex but need not to be metrizable. The next theorem generalizes slightly the validity of (3.2.2) with respect to Theorem A.1.4, referring still to usual Borel probability measures.

Theorem A.1.6 (Choquet's existence theorem, [Phe66], p.14). *Let $X \subseteq V$ be a metrizable compact and convex subset. Then, for each $x \in X$ there exists a probability measure μ supported on $\mathcal{E}X$ such that (3.2.2) holds.*

The result is refined upon the Choquet-Meyer theorem [Phe66, p.56], which characterizes simplexes, Definition A.1.2, and it proves that

Theorem A.1.7 (Choquet, [Phe66], p.60). *Let $X \subseteq V$ be a closed, convex and metrizable. Then, X is a simplex iff for all $x \in X$ there exists a unique probability μ supported on the extreme points $\mathcal{E}X$ such that (3.2.2) holds.*

A.2 Functional analysis

The continuous functional calculus reviewed in Chapter 2 is known to be extended to any Borel functions applied to selfadjoint bounded linear operator of any Hilbert space.

Let $\mathcal{H}$ be a Hilbert space, a selfadjoint operator $A \in \mathrm{B}(\mathcal{H})$ and $u \in \mathcal{H}$. The map $C(\sigma(A)) \to \mathbb{C}$ such that $f \mapsto \langle f(A)u, u\rangle$ is a positive linear functional. Theorem A.1.3 ensures that

$$\langle f(A)u, u\rangle = \int_{\sigma(A)} f(x)\mathrm{d}\mu_u(x)$$

for a unique measure μ_u, called the spectral measure of the vector $u \in \mathcal{H}$. In this way, if $g : \mathbb{R} \to \mathbb{R}$ is a bounded Borel-measurable function, one has that $\langle g(A)u, u\rangle = \int_{\sigma(A)} g(x)\mathrm{d}\mu_u(x)$ is defined; using the polarization identity, it follows $\sum_{\varepsilon\in\{\pm1,\pm i\}}\langle g(A)(u+\varepsilon v), u+\varepsilon v\rangle = \langle g(A)u, v\rangle$, for all $v \in \mathcal{H}$. Therefore, Riesz's lemma for sesquilinear forms yields an operator, which it is still denoted by $g(A)$ of A. The operator $g(A)$ is what is called the Borel functional calculus.

For any measurable space $(M, \mathscr{A})$, a projection-valued measure on $(M, \mathscr{A})$ is any map $\xi : \mathscr{A} \to \mathrm{B}(\mathcal{H})$ such that $\xi(\varnothing) = 0$, $\xi((-x, x)) = 1_{\mathrm{B}(\mathcal{H})}$, for all $x \in \mathbb{R}$, and $\xi(F_1 \mathscr{A} p F_2) = \xi(F_1)\xi(F_2)$, for all $F_1, F_2 \in \mathscr{A}$, and $\xi(F) = \lim_{n\to\infty}\xi(F_n)$, if the $F \in \mathscr{A}$ is the disjoint union of the family of $F_n \in \mathscr{A}$, $n \in \mathbb{N}$. For each $S \in \mathcal{B}(\mathbb{R})$, the indicator function is Borel measurable, and if $A \in \mathrm{B}(\mathcal{H})$ is selfadjoint, then $\mathbf{1}_S(A)$ is a bounded linear operator of $\mathcal{H}$, and, in fact, a projection. For any selfadjoint $A \in \mathrm{B}(\mathcal{H})$, the family of projections $\{\mathbf{1}_S(A) \,|\, S \in \mathcal{B}(\mathbb{R})\}$ defines a projection-valued measure $\xi^A(S) = \mathbf{1}_S(A)$, for all $S \in \mathscr{A}$. See [RS72, Section VII.3].

Theorem A.2.1 (Spectral decomposition, [RS72], Theorem VII.8)**.** *Let $\mathcal{H}$ be a Hilbert space. Then, there is a one-to-one correspondence between selfadjoint operators $A \in \mathrm{B}(\mathcal{H})$ and projection-valued measure*

$$\xi^A = \{\mathbf{1}_S(A) \,|\, S \text{ Borel subset of } \mathbb{R}\}$$

such that

$$A = \int_{\mathbb{R}} x\,\xi^A(\mathrm{d}x)$$

In Section 4.1, we considered the Bose field for Weyl operators that is a mapping that assigns to each $x \in \mathcal{H}$ a selfadjoint linear operator $B(x)$ as done in [EL77].

Let $\mathcal{H}$ be a Hilbert space, $A \in \mathrm{B}(\mathcal{H})$ a selfadjoint operator and $u \in \mathcal{H}$. Then, the series $\sum_{n=0}^{\infty}\frac{(itA)^n}{n!}u$ strongly converges in $\mathcal{H}$ and e^{itA} is defined

as this strong limit. In other words, the expression e^{itA} defines a bounded linear operator of $\mathcal{H}$ whenever A is bounded. Due to [Sto32], this is possible also for unbounded selfadjoint linear operators.

By a linear operator T in $\mathcal{H}$, we mean a map T such that, if $D(T)$ denotes its domain, then $D(T)$ is a subspace of $\mathcal{H}$, $Tu \in \mathcal{H}$ and $T(\lambda u + \mu v) = \lambda Tu + \mu Tv$, for all $u, v \in D(T)$, $\lambda, \mu \in \mathbb{C}$. We denote by $D(T)$ the domain of any linear map T. If $D(T)$ is dense in $\mathcal{H}$, then T is said to be densely defined. If T, S are linear operators in $\mathcal{H}$, we say that T extends S, if $D(S) \subseteq D(T)$ and $Su = Tu$, for all $u \in D(S)$. The adjoint of a densely defined operator T is the map that sends each $v \in D(T)$ to $w \in \mathcal{H}$ for which $\langle Tu, v\rangle = \langle u, w\rangle$, for all $u \in D(T)$, holds. It is denoted with T^* and it is a linear operator in $\mathcal{H}$. A linear operator is said to be symmetric, if it is extended by its adjoint. It is said to be self-adjoint, if it is symmetric and it extends its adjoint. See [RS72, Section VIII.2] for further details.

Definition A.2.2. *Let $\mathcal{H}$ be a Hilbert space. A map $\mathbb{R} \to \mathcal{U}(\mathcal{H})$ $t \mapsto U_t$ such that $U_0 = 1$, $U_{t+s} = U_t U_s$, for all $t, s \in \mathbb{R}$ is said to be a one-parameter unitary group (of $\mathcal{H}$) and usually indicated by the family of its images $\{U_t\}_{t\in\mathbb{R}}$. A one-parameter unitary group $\{U_t\}_{t\in\mathbb{R}}$ is is said to be strongly continuous, if $U_t u \to U_s u$ in $\mathcal{H}$, as $t \to s$, for all $u \in \mathcal{H}$. $\{U_t\}_{t\in\mathbb{R}}$ is it said weakly measurable, if $t \mapsto \langle U_t u, v\rangle$ is a measurable function $\mathbb{R} \to \mathbb{C}$ for all $u, v \in \mathcal{H}$.*

The next theorems are [Sto32, Theorem A, B], as they appear in [RS72].

Theorem A.2.3 (Theorem VIII.7, [RS72])**.** *Let T be a selfadjoint linear operator in $\mathcal{H}$. Then $U_t = e^{itT}$ defines a strongly-continuous one-parameter group. Moreover, $\{U_t\}_{t\in\mathbb{R}}$ has strong derivative at each $u \in D(T)$,*

$$\lim_{t\to 0} t^{-1}(U_t u - u) \text{ exists iff } u \in D(T)\,.$$

All strongly continuous one-parameter groups on a Hilbert space are of this form.

Theorem A.2.4 (Stone's theorem, VIII.8, [RS72])**.** *If $\{U_t\}_{t\in\mathbb{R}}$ is a strongly-continuous one-parameter group on $\mathcal{H}$, then there exists a self-adjoint linear operator T on $\mathcal{H}$ such that $U_t = e^{itT}$, for all $t \in \mathbb{R}$.*

Theorem A.2.5 (Theorem D, [Sto32])**.** *Let $\{U_t\}_{t\in\mathbb{R}}$ with Stone generator T. Then,*

$$Tu = -i \lim_{t\to 0} t^{-1}(U_t u - u) \tag{A.2.1}$$

for each $u \in \mathcal{H}$ for which the strong limit exists.

Definition A.2.6 (Stone generator)**.** *Let $\{U_t\}_{t\in\mathbb{R}}$ be a strongly continuous one-parameter unitary group on $\mathcal{H}$. Then, the self-adjoint linear operator T such that $U_t = e^{itT}$ is said to be its Stone generator.*

[Sto32, Theorem C] shows that the conclusion of Theorem A.2.4, can be obtained also by replacing the strong continuity of any one-parameter unitary group $\{U_t\}_{t\in\mathbb{R}}$ by its weak measurability. As the CCR Fock representation 4.1 is weakly continuous, it is weakly measurable.

For completeness, it is worth to point out that Stone [Sto51] and von Neumann [Neu55] extend the theory including relationships for strongly continuous one-parameter unitary groups and projection-valued measures. These are highlighted by the following two theorems found in [Par92].

Theorem A.2.7 (Stone's theorem, Theorem 10.8 [Par92])**.** *Let $k \in \mathbb{N}$, $\mathcal{H}$ a Hilbert space and $\{U_x\}_{x\in\mathbb{R}^k}$ a strongly continuous one-parameter unitary group of $\mathcal{H}$. Then, there exists a unique projection-valued measure $\xi : \mathcal{B}(\mathbb{R}^k) \to \mathsf{P}(\mathcal{H})$ such that*

$$U_x = \int e^{ix\cdot y}\xi(\mathrm{d}y), \quad x \in \mathbb{R}^k \tag{A.2.2}$$

where $x\cdot y = \sum_{j=1}^k x_j y_j$ is the Euclidean product. Conversely, if ξ is projection-valued measure as above, then, (A.2.2) defines a strongly continuous one-parameter unitary group $\{U_x\}_{x\in\mathbb{R}^k}$ of $\mathcal{H}$.

Theorem A.2.8 (Von Neumann's spectral theorem, unbounded case, Theorem 12.4, [Par92])**.** *Let $\mathcal{H}$ be a Hilbert space. If T is a linear selfadjoint operator in $\mathcal{H}$, then there exists a unique projection-valued measure $\xi : \mathcal{B}(\mathbb{R}) \to \mathsf{P}(\mathcal{H})$ such that*

$$T = \int x\xi(\mathrm{d}x) \tag{A.2.3}$$

Conversely, any projection-valued measure ξ is such that (A.2.3) is a self-adjoint linear operator on $\mathcal{H}$.

Theorem A.2.8 was first proved in [Neu55] and refines his Theorem A.2.1 to the case of unbounded selfadjoint operators. It allows, together with the properties of the spectral integration, the generalization of the Stone's theorem A.2.7, also proved in [Sto32, Theorem C].

Theorem A.2.9 (Stone's theorem, generalized version, Theorem 13.1 [Par92])**.** *Let $\mathcal{H}$ be a Hilbert space, $\{U_t\}_{t\in\mathbb{R}}$ a strongly continuous one-parameter unitary group and $\xi : \mathcal{B}(\mathbb{R}) \to \mathsf{P}(\mathcal{H})$ its spectral projection-valued measure. Then, the*

strong derivative at each $u \in \mathcal{H}$ $\lim_{t\to 0} t^{-1}(U_t u - u)$ *exists iff* $u \in D(T)$*, where* $T = \int x\xi(\mathrm{d}x)$*. Moreover,* T *is selfadjoint and*

$$\frac{\mathrm{d}}{\mathrm{d}t} U_t u = iTU_t u,\ u \in D(T)\,. \tag{A.2.4}$$

In this strong sense, it is written $U_t = e^{itT}$*.*

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