Cohomology of Lie Superalgebras: Forms, Integral Forms and Coset Superspaces

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Abstract

We study Chevalley-Eilenberg cohomology of physically relevant Lie superalgebras related to supersymmetric theories, providing explicit expressions for their cocycles in terms of their Maurer-Cartan forms. We include integral forms in the picture by defining the notions of constant densities and integral forms related to a Lie superalgebra. We develop a suitable generalization of Chevalley-Eilenberg cohomology extended to integral forms and we prove that it is isomorphic via a Poincar duality-type pairing to the ordinary Chevalley-Eilenberg cohomology of the Lie superalgebra. Next, we study equivariant Chevalley-Eilenberg cohomology for coset superspaces, which plays a crucial role in supergravity and superstring models. Again, we treat explicitly several examples, providing cocycles' expressions and revealing a characteristic infinite-dimensional cohomology.

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1 Introduction

The mathematical development of cohomology of Lie algebras [28] [41] has been prompted and characterized by a twofold reason in relation to the theory of Lie groups.
On one hand, in a diverging direction with respect to Lie groups, Lie algebra cohomology unties the representation theory of Lie algebras from the corresponding representation theory of Lie groups, by allowing a completely algebraic proof of the *Weyl theorem* [56], which was originally of analytic nature. On the other hand, in a converging direction with respect to Lie groups, in many important instances Lie algebra cohomology makes computations of the de Rham cohomology of the corresponding Lie groups easier. Nowadays, applications of Lie algebra cohomology range from representation theory in pure mathematics to modern physics - let us just recall that *Kac-Moody* and *Virasoro algebras*, which play a central role in string theory, are central extensions of the *polynomial loop-algebra* and the *Witt algebra* respectively, and, as such, they are related to Lie algebra's 2-cohomology group. While it is quite natural to generalize a cohomology theory from Lie algebra to Lie superalgebra [42] [45] (recent reviews and computations can be

found in [3, 4, 5, 6]) both from a derived-functorial point of view and, more concretely, via cochain complexes, it can be seen that the two directions sketched above are doomed to breakdown as one moves to the super setting. Indeed, in the representation-theoretic direction, there is no Weyl theorem for Lie superalgebras, initially leading to the opinion that the cohomology theory is rather empty and meaningless. Further, in the topological direction, when working with Lie supergroups and their related Lie superalgebras, Cartan theorem resists a naive "super" generalization, as it only encodes topological information. On the other hand, a different point of view is possible, namely one can look at the failure of the Weyl theorem in the supersymmetric setting as an opportunity, rather than a pathological feature of the theory, for it suggests that the cohomology groups of Lie superalgebras might have a much richer structure than the one that can be guessed by analogy with the ordinary theory. Remarkably, physics is paving this way: cocycles arising from cohomology of Lie superalgebras - in particular, *Poincaré superalgebras* - are getting related to higher *Wess-Zumino-Witten* (*WZW*) terms in supersymmetric Lagrangians (the so-called *brane scan* and its recent higher-version, the *brane bouquet*, which promotes Lie superalgebras to L_∞-superalgebras and consider their cohomology), see [2] and the more recently [7] [9] [29] [30] [49]. It is fair to observe, though, that even the cohomology of a finite-dimensional Lie superalgebra does not vanish in general for degrees greater than the dimension of the algebra - as it happens in the ordinary case instead -: this makes the actual computation of the cohomology of Lie superalgebras in general into a difficult task. Accordingly, results can be found in the literature for specific choices of superalgebras - in particular in low-degree [54] -, but only very few results encompassing the whole framework are available [32], even just for the *Betti numbers* of Lie superalgebras. Even less is known regarding the cohomology and the structure of cocycles of coset or homogeneous superspaces, which play a fundamental role in many superstring and supergravity models. If on one hand it is likely that a detailed knowledge of these equivariant cohomologies would help to understand the geometric nature and invariant structure of convoluted supergravity Lagrangians [39] [40], it is also fair to notice that - once again - computations are difficult even in the most basic examples.

On a different note, getting back to the relations between algebras and groups, as mentioned above, it is a well-known fact that the de Rham cohomology of a Lie group can be formulated in terms of its underlying Lie algebra, thus making feasible computations, which otherwise would be very difficult. Trying to generalize this to Lie supergroups, one would run into an issue, which is deeply ingrained in the theory of forms and the related integration theory in supergeometry. Indeed, in order to formulate a coherent notion of geometric integration on supermanifolds [46], besides differential forms, one also needs to take into account *integral forms*, a notion which is crucial, though not widely known and understood: for example, a supergeometric analogue of Stokes' theorem [47] [57] is proved using integral forms. On the other hand, it needs to be remarked that Lie superalgebra cohomology is nothing but a "$\mathbb{Z}_2$-graded generalization" of the ordinary Lie algebra cohomology, and, as such, it is not capable to account for objects other than differential forms on supermanifolds, such as in particular, integral forms, which simply do not enter the picture [58]. It is natural to ask if it is possible to provide a formulation of Lie superalgebra cohomology capturing properties of integral forms as well, and, in turn, what are the relations between the ordinary Lie superalgebra cohomology and this newly defined cohomology.

Extensive treatment and exhaustive discussion on Lie superalgebras cohomology have been given in [32]: many specific examples are studied, and it is shown that the $\mathbb{R}$-valued cohomology of some Lie superalgebras does not coincide with the cohomology of the reduced bosonic Lie algebras. An interesting example of this peculiar phenomenology is given by the family of orthosymplectic Lie superalgebras $\mathfrak{osp}(n|m)$, whose Lie superalgebra cohomology reads

$$H^p_{CE}\left(\mathfrak{osp}(n|m)\right) \cong \begin{cases} H^p_{CE}\left(\mathfrak{so}(n)\right) & n \geq 2m \\ H^p_{CE}\left(\mathfrak{sp}(m)\right) & n < 2m. \end{cases} \tag{1.1}$$

The above result is rather surprising. It can be interpreted by saying that, depending on the dimensions, only a fraction of the algebraic invariants, which are built from the invariant tensors of the bosonic subalgebra - the so-called Casimir's -, contributes to the cohomology of the Lie superalgebra. However, by the very definition of Lie algebra cohomology, one would expect that *all* of the invariants have to appear and are to be kept into account.
In this regard, we will see that all of the invariants coming from both $\mathfrak{so}(n)$ and $\mathfrak{sp}(m)$ appear in the module of constant densities (or Haar Berezinian) of the Lie superalgebra. This does not define a new cocycle of the original Chevalley-Eilenberg cohomology of the Lie superalgebra, because the Berezinian does not belong to the Chevalley-Eilenberg complex in the first place. On the other hand, the Berezinian is the pivotal construction that allows the introduction of a different complex, the (algebraic analogue of the) complex of integral forms, together with a related notion of cohomology. Given this, it is relevant to consider the relation between the cohomology of this complex and the original Chevalley-Eilenberg cohomology. Finally, it is worth mentioning that, besides differential and integral forms, it is possible to introduce another kind of form, namely *pseudoforms*: these can be again arranged into complexes and a related cohomology can be introduced

- a relevant example of these cohomologies of pseudoforms has been studied in the recent [25].

In the present work, after a brief review of Chevalley-Eilenberg cohomology of Lie algebras and superalgebras and a basic introduction to integral forms - which aims at making the paper as self-consistent as possible -, we extend the notion of integral forms to the Lie superalgebraic context and we define a related cohomology theory. On the way, we introduce the pivotal construction of module of constant densities or Haar Berezinian, by adapting to left invariant forms and field the relevant Koszul complex construction and its cohomology. We then establish an isomorphism between the Chevalley-Eilenberg cohomology of integral forms and the ordinary Chevalley-Eilenberg cohomology of the superalgebra in question and we comment the result, pointing out differences and similarities with respect to the known quasi-isomorphism of differential and integral forms on supermanifolds. We provide explicit computations of these cohomologies in several cases of physical interest, by looking at the Lie superalgebra of symmetries of relevant superspaces - concrete expressions for the cocycles are given in the Appendix. However, it is fair to remark that, even if Lie supergroups and their associated Lie superalgebras appear in several physical applications and allowed to establish important results, coset supermanifolds actually offer to the most interesting and rich scenarios, providing several ways to take into account different amount of symmetries. For this reason, the last part of the paper is dedicated to the computations of equivariant Chevalley-Eilenberg cohomology for coset superspaces: several examples are discussed and the typical phenomenology is pointed out.

2 Chevalley-Eilenberg Cohomology: Main Definitions

2.1 Lie Algebras, Lie Superalgebras and Left Invariant Forms

We start recalling the basic definitions, first in the usual setting, then in the *super* one. Let $\mathfrak{g}$ be an ordinary finite dimensional Lie algebra defined over the field k (we will only deal with $k = \mathbb{R}$ or $\mathbb{C}$), and let V be a $\mathfrak{g}$-module or a representation space for $\mathfrak{g}$. We define the (Chevalley-Eilenberg) p-cochains of $\mathfrak{g}$ with values in V to be alternating k-linear maps from $\mathfrak{g}$ to V [28],

$$C^p_{CE}(\mathfrak{g}, V) := Hom_k\left(\wedge^p \mathfrak{g}, V\right), \tag{2.2}$$

where, in taking the exterior power $\mathfrak{g}$ is looked at as a vector space. The above (2.2) can be lifted into a cochain complex by introducing the (Chevalley-Eilenberg) differential $d^p_{\mathfrak{g}} : C^p_{CE}(\mathfrak{g}, V) \to C^{p+1}_{CE}(\mathfrak{g}, V)$, defined as

$$\begin{aligned} d^p_{\mathfrak{g}} f(x_1 \wedge \ldots \wedge x_{p+1}) \coloneqq & \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} f([x_i, x_j] \wedge x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge \hat{x}_j \wedge \ldots \wedge x_{p+1}) + \\ & + \sum_{i=1}^{p+1} (-1)^{i+1} x_i \cdot f(x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge x_{p+1}), \end{aligned} \tag{2.3}$$

for $f \in Hom_k(\wedge^p \mathfrak{g}, V)$ and where the hatted entries are omitted. It is not too hard to prove that $d^{p+1} \circ d^p = 0$, so that one can define the *Chevalley-Eilenberg complex* of $\mathfrak{g}$ taking values values in V as the pair $(C^\bullet_{CE}(\mathfrak{g}, V), d^\bullet)$.

In this paper we will only deal with the case the cochains take values in the trivial $\mathfrak{g}$-module, that is $V = k$ and we will denote the k-valued cochains of $\mathfrak{g}$ simply as

$$C^p_{CE}(\mathfrak{g}) \coloneqq \bigwedge^p \mathfrak{g}^*. \tag{2.4}$$

Notice that in this case the second summand of the differential (2.3) vanishes identically, since $x \cdot f = 0$ for any $x \in \mathfrak{g}$ and $f \in \bigwedge^\bullet \mathfrak{g}^*$.
Given these definitions, the cohomology is defined in the usual fashion. We call *Chevalley-Eilenberg cocycles* the elements of the vector space

$$Z^p_{CE}(\mathfrak{g}, V) \coloneqq \{f \in C^p_{CE}(\mathfrak{g}, V) : d^p f = 0\}, \tag{2.5}$$

and *Chevalley-Eilenberg coboundaries* the elements in the vector space

$$B^p_{CE}(\mathfrak{g}, V) \coloneqq \{f \in C^p_{CE}(\mathfrak{g}, V) : \exists g \in C^{p-1}_{CE}(\mathfrak{g}, V) : f = d^{p-1} g\}, \tag{2.6}$$

and we define the *Chevalley-Eilenberg p-cohomology group* of $\mathfrak{g}$ valued in V as the quotient vector space

$$H^p_{CE}(\mathfrak{g}, V) \coloneqq Z^p_{CE}(\mathfrak{g}, V) \big/ B^p_{CE}(\mathfrak{g}, V). \tag{2.7}$$

Denoting now $\mathfrak{g}$ a Lie *super*algebra with $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ its *even* and *odd* components in the $\mathbb{Z}_2$-grading, one can easily generalize the above construction just by taking care of the signs related to the $\mathbb{Z}_2$-grading (*parity*) of $\mathfrak{g}$. In particular, the definition of cochains and cohomology groups is unchanged and the previous differential in (2.3) modifies to [45]

$$\begin{aligned} d^p f(x_1 \wedge \ldots \wedge x_{p+1}) \coloneqq & \sum_{1 \leq i < j \leq p+1} (-1)^{i+j+\delta_{i,j}+\delta_{i-1,j}} f([x_i, x_j] \wedge x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \hat{x}_j \wedge \ldots \wedge x_{p+1}) + \\ & + \sum_{i=1}^{p+1} (-1)^{i+1+\delta_{r-1,r}} x_i \cdot f(x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge x_{p+1}), \end{aligned} \tag{2.8}$$

where $\delta_{i,j} := |x_i|(|f| + \sum_{k=0}^{i} |x_k|)$ for any $f \in \bigwedge^p \mathfrak{g}^* \otimes V$, $x_i \in \mathfrak{g}$, in order to take into account the parity, *i.e.* the $\mathbb{Z}_2$-grading of the elements. Also, notice whenever the odd dimension of the Lie superalgebra is greater than zero, *i.e.* if $\mathfrak{g}_1 \neq \{0\}$, the Chevalley-Eilenberg cochain complex is not bounded from above as in the ordinary case.

In physics literature, and in particular in supergravity, Chevalley-Eilenberg cohomology is introduced via *Cartan geometry*, better than via the above purely algebraic framework. Focusing on the trivial coefficient case, one starts with an ordinary Lie group G, with $\ell_g : G \to G$ the *left translation* by $g \in G$ so that $\ell_g(h) = g \cdot h \in G$ for any $h \in G$. Then, the vector space of *left invariant p-forms G* is defined as

$$\Omega^p_L(G) := \{\omega \in \Omega^p(G) : \ell^*_g \omega = \omega\}. \tag{2.9}$$

It is easy to see that any left-invariant p-form is determined by its value at the origin $e \in G$, leading to the vector spaces isomorphism $\Omega^p_L(G) \cong \bigwedge^p \mathfrak{g}^* = C^p_{CE}(\mathfrak{g})$. Further, it is not hard to see that if $\omega^{(\ell)}$ is a left-invariant p-form, then $(d\omega^{(\ell)})_e = d_\mathfrak{g}\omega$, where d is the de Rham differential, $d_\mathfrak{g}$ is the Chevalley-Eilenberg differential introduced above and $\omega \in C^p_{CE}(\mathfrak{g})$ is such that $\omega = \omega^{(\ell)}_e$. This shows that the Lie algebra cohomology can be described in terms of the de Rham cohomology of left invariant differential forms on the Lie group whose the Lie algebra is associated to, *i.e.*

$$H^p_{CE}(\mathfrak{g}, V) \cong H^p(\Omega^p_L(G), d), \tag{2.10}$$

thus making contact between two seemingly different cohomologies and making possible to compute Lie algebra cohomology via *forms*, see for example [43].
In this context let us consider a k-basis of left-invariant forms $\omega^i \in \Omega^1_L(G)$ together with its dual basis of left-invariant vector field $X_i \in (\Omega^1_L(G))^*$, with $\omega^i_g(X_{j,g}) = \delta^i_j$ for any $g \in G$. Then, the $\omega^i \in \Omega^1_L(G)$ satisfy the *Maurer-Cartan structure equation*

$$d\omega^i = -\frac{1}{2} C^i{}_{jk}\, \omega^j \wedge \omega^k, \tag{2.11}$$

where the $C^i{}_{jk}$ are the *structure constants* relative to the basis ω^i. The sums over repeated indices are understood. These equations are equivalent to the Lie bracket relations for the basis X^i of the algebra of left-invariant vector fields, $[X_j, X_k] = C^i_{jk} X_i$. Also it can be easily checked that $d \circ d = 0$ is equivalent to *Jacobi identity*, as

$$d(d\omega^k) = -\frac{1}{2} C^k{}_{ij}\, d\omega^i \wedge \omega^j + \frac{1}{2} C^k{}_{ij}\, \omega^i \wedge d\omega^j = \frac{1}{2} C^k{}_{i[j} C^i{}_{lm]}\, \omega^l \wedge \omega^m \wedge \omega^j = 0, \tag{2.12}$$

where $\omega^i \in \Omega_L(G)$ and where $C^k{}_{i[j} C^i{}_{lm]} = 0$ is indeed the Jacobi identity.

In the present paper we will deal only with *matrix Lie (super)groups*, *i.e.* Lie (super)groups which admit an embedding into some GL-(super)group. In this case, the elements of the basis of $\Omega^1_L(G)$ can be taken of forms $\mathcal{V} = dgg^{-1}$, where $g := (g_{ij})$ is matrix-valued. We call them *Maurer-Cartan forms*, as they satisfy Maurer-Cartan equations (2.11) by construction. In turns, we will take the cochains to be generated starting from the basis of Maurer-Cartan forms $\{\mathcal{V}^i\}$, *i.e.* the *vielbeins* in the physics literature, so that

$$C^p_{CE}(\mathfrak{g}) = \Omega^p_L(G) = \left\{ c_{i_1 \ldots i_p} \mathcal{V}^{i_1} \wedge \ldots \wedge \mathcal{V}^{i_p} \right\} \quad \text{for} \quad c_{i_1 \ldots i_p} \in k. \tag{2.13}$$

Notice that the above discussion is readily generalizable to the $\mathbb{Z}_2$-graded setting of a Lie *super*group $\mathcal{G}$, but a remark about the *parity* is in order. Indeed, in supergeometry one takes $\Omega^1(\mathcal{G}) := \Pi \mathcal{T}^*(\mathcal{G})$, where Π is the parity changing functor. In this convention the de Rham differential d is an *odd* derivation. This leads to consider *even* and *odd* vielbeins $\{\psi^\alpha | \mathcal{V}^i\}$ generating the $\mathbb{Z}_2$-graded vector space $\Omega^1_L(\mathcal{G})$, where the even ψ^α's arise from odd coordinates and the odd $\mathcal{V}^i$'s arise from even coordinates. What it is crucial to observe is that, accordingly, this should be related to the *parity changed dual* of the Lie superalgebra $\Pi\mathfrak{g}^*$, that is at the level of the cochains one has

$$C^\bullet(\Pi\mathfrak{g}) := S^\bullet \Pi\mathfrak{g}^* \cong \Omega^\bullet_L(\mathcal{G}), \tag{2.14}$$

where $S^\bullet$ is the supersymmetric product functor [47]. Likewise, at the level of the algebra, commutators $[\cdot,\cdot]$ become *supercommutators* $[\cdot,\cdot\}$. In particular, on the parity reversed algebra $\Pi\mathfrak{g}$, if πX and $\pi Y \in \Pi\mathfrak{g}$ one poses $[\pi X, \pi Y\} := [X, Y\}$ for X and Y in $\mathfrak{g}$.

2.2 Integral Forms and Chevalley-Eilenberg Cohomology

2.2.1 Defining Chevalley-Eilenberg Cohomology of Integral Forms

As briefly reviewed in the previous section, Chevalley-Eilenberg cohomology can be introduced as the cohomology of the vector space of the left-invariant differential forms of the associated Lie group. On supermanifolds, though, differential forms need to be supplemented by another kind of forms in order to obtain a coherent notion of integration: these are the so-called *integral forms*, which are briefly reviewed in the Appendix in the hope of making this paper as self-contained as possible. See also classical and recent literature on the topic [11, 14, 15, 17, 18, 19, 20, 21, 23, 24, 26, 27, 36, 47, 50].

It can therefore be expected that on a supermanifold, such as a Lie supergroup, also a notion of cohomology of *left invariant integral* - better than differential - *forms* is possible, as to encode different information about the Lie group and the associated Lie algebra.

Left invariant integral forms on a Lie supergroup $\mathcal{G}$ can be introduced following the two different approaches - see also the Appendix.

2.2.2 Integral Forms as Generalized Functions

To introduce integral forms as *generalized functions* on $\mathcal{G}$, one starts from a basis of Maurer-Cartan forms $\{\psi^\alpha|\mathcal{V}^i\}$ with even ψ's and odd $\mathcal{V}$'s and restrict to consider only integral forms (A.162) written in terms of them:

$$\omega_{\mathfrak{g}}(\psi|\mathcal{V}) = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{a_i\in\{0,1\},r_j\geq 0}\omega_{[a_1\dots a_m r_1\dots r_m]}(\mathcal{V}^1)^{a_1}\dots(\mathcal{V}^n)^{a_m}\delta^{(r_1)}(\psi^1)\dots\delta^{(r_m)}(\psi^m), \quad (2.15)$$

where all indices are antisymmetric and $\omega_{[a_1\dots a_m r_1\dots r_m]}\in k$. We will call these integral forms on $\mathfrak{g}$ for short. The indices r_i in $\delta^{(r_i)}(\psi^i)$ denote the r_i-th derivatives of the Dirac delta distributions. Notice that, if $\mathcal{Y}_{A=i|\alpha} \coloneqq \{\mathcal{P}_i|\mathcal{Q}_\alpha\}$ is the basis[1] of generators of the Lie superalgebra $\mathfrak{g}$ which is dual (up to a parity shift) to the basis of the Maurer-Cartan forms above, so that $\psi^\alpha(\pi\mathcal{Q}_\beta)=\delta^\alpha_\beta$ and $\mathcal{V}^i(\pi\mathcal{P}_j)=\delta^i_j$, then the most general integral form on $\mathfrak{g}$ of degree $n-\ell$, see (A.165), will be written as

$$\omega_{\mathfrak{g}}^{n-\ell} = \omega^{i_1\dots i_\ell}\iota_{\mathcal{Y}^{i_1}}\dots\iota_{\mathcal{Y}^{i_\ell}}\omega_{\mathfrak{g}}^{top}, \quad (2.16)$$

for $\mathcal{Y}$ spanning both even and odd dimensions of $\mathfrak{g}$ and the indices of the tensor $\omega^{i_1\dots i_\ell}$ symmetrized or anti-symmetrized according to the parity of the related contraction (the sum over repeated indices is understood). Notice that contractions act as derivatives on the delta's. In the above expression one fixes the integral top form up to a multiplicative constant to be

$$\begin{aligned}\omega_{\mathfrak{g}}^{top} &= \mathcal{V}^1\dots\mathcal{V}^n\delta(\psi^1)\dots\delta(\psi^m)\\ &= \prod_I \mathcal{C}^{(I)}_{i_1\dots i_p}\mathcal{V}^{i_1}\dots\mathcal{V}^{i_p}\delta(\psi^1)\dots\delta(\psi^m)\ ,\end{aligned} \quad (2.17)$$

that is $\omega_{\mathfrak{g}}^{top}$ is again expressed only in terms of the Maurer-Cartan forms, which makes it formally left-invariant. In the second line we have written the same expression in terms of all the invariant tensors $\mathcal{C}^{(I)}_{i_1\dots i_p}$ of the corresponding bosonic subalgebras, with I running over the different invariant tensors.

[1] We conventionally denote vectors with lower indices and forms with upper indices.

In this context, the form (2.17) of the Berezinian explicitly constructed via left-invariant forms has a useful interpretation: the presence of the delta's formally set to zero the fermionic contributions that one would get applying the Chevalley-Eilenberg differential to the vielbeins, thus reducing the Maurer-Cartan equations to those of the bosonic subalgebra. It follows that, if on one hand invariants of a part of the bosonic subalgebra are encoded in Chevalley-Eilenberg cohomology as in Fuks' theorem, on the other hand, the remaining invariants of the bosonic subalgebra are encoded inside integral forms, since the Berezinian is built in terms of all invariants of the bosonic subalgebra under examinations.

Finally, observe that the Maurer-Cartan differential can be generalized as to act on integral forms on $\mathfrak{g}$ in the following way

$$\Omega^{n-\ell}_{int}(\mathfrak{g}) \ni \omega^{n-\ell}_{\mathfrak{g}} \longmapsto d(\omega^{n-\ell}_{\mathfrak{g}}) := \frac{1}{2} C^{A}{}_{BC} (\pi \mathcal{Y}^*)^B (\pi \mathcal{Y}^*)^C \iota_{\mathcal{Y}^A} \left(\omega^{i_1 \ldots i_\ell} \iota_{\mathcal{Y}^{i_1}} \ldots \iota_{\mathcal{Y}^{i_\ell}} \omega^{top}_{\mathfrak{g}} \right) \tag{2.18}$$

where $C^{A}{}_{BC}$ are the structure constants of the Lie superalgebra $\mathfrak{g}$ with A, B and C are the cumulative indices for $i|\alpha$. Notice that the right-hand side of (2.18) defines indeed an integral form of degree $n - \ell + 1$, *i.e.* $d(\omega^{n-l}_{\mathfrak{g}}) \in \Omega^{n-l+1}_{int}(\mathfrak{g})$ and, once again, that the differential is indeed nilpotent thanks to Jacobi identity for the Lie superalgebra $\mathfrak{g}$. It then does make sense to define the integral Chevalley-Eilenberg cohomology as

$$H^{n-\ell}_{CE,int}(\mathfrak{g}) := \frac{\{\omega_{\mathfrak{g}} \in \Omega^{n-\ell}_{int}(\mathfrak{g}) : d\omega_{\mathfrak{g}} = 0\}}{\{\omega_{\mathfrak{g}} \in \Omega^{n-l}_{int}(\mathfrak{g}) : \exists \eta_{\mathfrak{g}} \in \Omega^{n-\ell-1}_{int}(\mathfrak{g}) : d\eta_{\mathfrak{g}} = \omega_{\mathfrak{g}}\}}. \tag{2.19}$$

2.2.3 Integral Forms as Berezinian-valued Polyfields

Whereas the above description of integral forms has the merit of being intuitive and suitable for computations, it of course breakdown in any category that does not admit the notion of generalized functions or Dirac delta distributions. A second definition suitable to any geometric category is given by introducing integral forms as sections of the tensor product of sheaves

$$\Sigma^p(\mathcal{M}) := \mathcal{B}er(\mathcal{M}) \otimes_{\mathcal{O}_{\mathcal{M}}} (\Omega^{n-p}(\mathcal{M}))^*, \tag{2.20}$$

where $\mathcal{M}$ is a generic supermanifold. Notice that an integral form is a much more complicated object than a differential form - whose definition appears as more natural, being an obvious generalization of the ordinary notion. Nonetheless, integral forms are in some sense more useful objects than differential forms, as they control integration theory on supermanifolds. The same is true for Lie supergroups, where is to be expected that (suitably defined) left-invariant integral forms would also play a role in the corresponding representation theory, in particular when it comes to introduce a notion of unitary representations - something which has so far been neglected.

In the following we will see how (2.20) specializes to the case of a Lie supergroup. Finally, we will see how left-invariant differential forms and left-invariant integral forms can be seen as arising from a unique construction, that only makes only use of natural objects defined on the Lie superalgebra of the supergroup, such as its universal enveloping algebra.

2.2.4 Haar Berezinian and Left Invariant Integral Forms

The first problem to be addressed in order to introduce a notion of left-invariant integral forms is how to intrinsically define a *left invariant Berezinian*, or module of *constant densities*. In the ordinary case of a Lie group G, the *Haar determinant* - which integrated yields the "volume" of the Lie group - is constructed as the top exterior power of the left-invariant 1-forms, *i.e.* $\det(G) = \mathbb{R} \cdot \omega^1 \wedge \ldots \wedge \omega^n$ for $\mathrm{Span}_{\mathbb{R}}\{\omega^1, \ldots, \omega^n\} = \Omega^1_L(G)$. There is no generalization of this construction since there are no top-exterior forms on a superspace if the odd dimension is greater or equal to 1: in particular, the Berezinian cannot be realized this way. Nonetheless, the determinant module of an ordinary vector space can be intrinsically constructed also via cohomology of the dual of the *Koszul complex* of the vector space [51]: this construction, instead, admits a non-trivial generalization to supergeometry.

Namely, for a real or complex Lie supergroup $\mathcal{G}$ of dimension $n|m$ let us consider the following dg k-algebra (where k is a characteristic-zero field) given by the tensor product

$$\mathcal{K}^{\bullet}_{\mathcal{G}} := \left(\bigoplus_{i=0}^{\infty} S^i \Omega^1_L(\mathcal{G}) \right) \otimes_k S^{\bullet} \mathcal{T}_L(\mathcal{G}) \tag{2.21}$$

where $\Omega_L(\mathcal{G})$ and $\mathcal{T}_L(\mathcal{G})$ are left invariant forms and fields respectively. Notice that $S^{\bullet}\mathcal{T}_L(G)$ is a k-algebra generated by the elements in degree 1

$$\mathcal{T}_L(G) = \mathrm{span}_k\{X_1 \ldots, X_n, |\xi_1, \ldots, \xi_m\} \cong \mathfrak{g}, \tag{2.22}$$

with $X_i = \pi(\mathcal{V}^i)^*$ and $\xi_j = \pi(\psi^j)^*$, for $\{\psi^j | \mathcal{V}^i\}$ generating $\Omega^1_L(\mathcal{G})$.

The homological operator acting on $\mathcal{K}^{\bullet}_{\mathcal{G}}$ is given by the multiplication by the element

$$\delta := \sum_{i=1}^{n} \mathcal{V}^i \otimes X_i + \sum_{\alpha=1}^{m} \psi^{\alpha} \otimes \xi_{\alpha}. \tag{2.23}$$

in the dg k-algebra $\mathcal{K}^\bullet$. Notice that since δ is odd, the multiplication by δ is nilpotent. Also, the $\mathbb{Z}$-grading is inherited by that of the polyfields $S^\bullet \mathcal{T}_L(\mathcal{G})$.

Definition 1 (Super Koszul Complex of $\mathcal{G}$)**.** Let $\mathcal{G}$ be a Lie supergroup. We call the pair $(\mathcal{K}, \delta)$ the Koszul complex of $\mathcal{G}$.

Let us define

$$\mathcal{D} := \mathcal{V}^1 \dots \mathcal{V}^n \otimes \xi_1 \dots \xi_m. \tag{2.24}$$

It is immediate to see that one has the following inclusion of ideals $(\delta, D) \subseteq \ker \delta$. Notice that, by abuse of notation, $\delta \in (\delta, \mathcal{D})$ is seen as an element generating the ideal and then as a homological operator when we consider its kernel.

Theorem 1. *The cohomology $H_\delta(\mathcal{K}^\bullet_{\mathcal{G}})$ of the super Koszul complex $(\mathcal{K}^\bullet_{\mathcal{G}}, \delta)$ is concentrated in degree m and isomorphic to $\Pi^{n+m}k$. In particular, the cohomology class is generated by the element $\mathcal{D}$.*

Proof. Just rearrange the definitions to write

$$\mathcal{K}^\bullet_{\mathcal{G}} = k[X_i, \psi^\alpha | \mathcal{V}^i, \xi_\alpha]. \tag{2.25}$$

Then, if $N = n + m$ one put

$$(u_1, \dots, u_N) := (X_1, \dots, \psi^m), \qquad (\epsilon_1, \dots, \epsilon_N) := (\mathcal{V}^1, \dots, \xi_m). \tag{2.26}$$

Then $\delta = \sum_{i=1}^N u_i \epsilon_i$ and $\mathcal{D} = \prod_i^N \epsilon_i$. Upon defining $B := k[u_1, \dots, u_n]$ one has $k[X_i, \psi^\alpha | \mathcal{V}^i, \xi_\alpha] = B[X_i, \psi^\alpha]$. By anticommutativity it follows that

$$B[X_i, \psi^\alpha] = \bigwedge_B^\bullet (X^i, \psi_\alpha). \tag{2.27}$$

Then this is a standard Koszul complex with Koszul differential given by $\delta = \sum_i u_i \epsilon_i$. Its cohomology is concentrated in degree N and generated over k by the element $\mathcal{D}$. □

A direct computation leads to the following corollary.

Corollary 1. *Let $f \in Aut_k(\Omega^1_L(G))$, then the induced automorphism on the module generated by $\mathcal{D}$ is given by the Berezinian of the automorphism f.*

The previous result in turn justifies the following definition.

Definition 2 (Haar Berezinian / Module of Constant Densities of $\mathcal{G}$)**.** Let $\mathcal{G}$ be a real or complex Lie supergroup. We define the Haar Berezinian or module of constant densities of $\mathcal{G}$ to be the cohomology module $H_\delta(\mathcal{K}^\bullet_{\mathcal{G}})$ and we denote it with $\mathrm{Ber}^{\mathcal{H}}(\mathcal{G})$.

Note in particular that one has

$$\mathrm{Ber}^{\mathcal{H}}(\mathcal{G}) \cong k \cdot [\mathcal{D}] \cong k \cdot [\mathcal{V}^1 \dots \mathcal{V}^n \otimes \xi_1 \dots \xi_m], \tag{2.28}$$

where the $\{\mathcal{V}^i\}$ are left invariant odd forms generating $\Pi \mathfrak{g}_0$ and the ξ_j are left invariant odd vector fields generating $\mathfrak{g}_1$.

Using the above notion of Haar Berezinian for a Lie supergroup $\mathcal{G}$ of dimension $n|m$, *left invariant integral forms* can be introduced into this Lie-theoretic framework as

$$C^p_{CE,int}(\mathfrak{g}) := \mathrm{Ber}^{\mathcal{H}}(\mathfrak{g}) \otimes S^{n-p}\Pi\mathfrak{g}, \tag{2.29}$$

see 2.20 to compare the definition with the ordinary sheaf-theoretic definition on a supermanifold $\mathcal{M}$. Notice that we have used the isomorphism between left invariant vector fields or Lie supergroup $\mathcal{G}$ and elements of its Lie superalgebra $\mathfrak{g}$. A notion of differential acting on left invariant integral forms as $\delta^p : C^p_{CE,int}(\mathfrak{g}) \to C^{p+1}_{CE,int}(\mathfrak{g})$ can be introduced as follows. First one extends the notion of supercommutator - or Lie derivative - to the whole supersymmetric product $S^n\Pi\mathfrak{g}$ recursively. For $\mathcal{X} \in \mathfrak{g}$, having already defined $\mathcal{L}_\mathcal{X} : S^h\Pi\mathfrak{g} \to S^h\Pi\mathfrak{g}$ for $h < p$ we uniquely define the action of $\mathcal{L}_\mathcal{X}$ on $S^p\Pi\mathfrak{g}$ via the relation

$$\mathcal{L}_\mathcal{X}(\langle \omega, \tau \rangle) = \langle \mathcal{L}_\mathcal{X}(\omega), \tau \rangle + (-1)^{|\omega||\mathcal{X}|} \langle \omega, \mathcal{L}_\mathcal{X}(\tau) \rangle \tag{2.30}$$

for any $\omega \in S^{i>0}\Pi\mathfrak{g}^*$ and $\tau \in S^p\Pi\mathfrak{g}$, and where $\langle \cdot, \cdot \rangle$ is the duality pairing between $\Pi\mathfrak{g}^*$ and $\Pi\mathfrak{g}$, extended to higher tensor powers, see [11]. Using this, we introduce the map

$$\begin{aligned} \delta^p : C^p_{CE,int}(\mathfrak{g}) &\longrightarrow C^{p+1}_{CE,int} \\ \mathcal{D} \otimes \tau &\longmapsto \delta^p(\mathcal{D} \otimes \tau) = \mathcal{D} \otimes \textstyle\sum_A \iota_{\pi\mathcal{X}_A^*} \mathcal{L}_{\mathcal{X}_A}(\tau) \end{aligned} \tag{2.31}$$

where the index A runs over both even and odd coordinates, $\mathcal{D}$ is a Haar Berezinian tensor density in $\mathrm{Ber}^{\mathcal{H}}(\mathfrak{g})$ and $\{\mathcal{X}_A\}$ are left-invariant vector fields generating $\mathfrak{g}$, so that $\{\pi\mathcal{X}_A^*\}$ are generators for $\Pi\mathfrak{g}^*$. Here $\iota_{\pi\mathcal{X}_A^*}$ is the contraction with the form $\pi\mathcal{X}_A^*$: in this respect the above can be re-written as $\delta^p(\mathcal{D} \otimes \tau) = \mathcal{D} \otimes \sum_A \langle \pi\mathcal{X}_A^*, \mathcal{L}_{\mathcal{X}_A}(\tau) \rangle$. Nilpotency can be checked formally as

$$\begin{aligned} \frac{1}{2}\{\delta, \delta\} &= \sum_{A,B} (\iota_{\pi\mathcal{X}_A^*} \mathcal{L}_{\mathcal{X}_A} \iota_{\pi\mathcal{X}_B^*} \mathcal{L}_{\mathcal{X}_B} + \iota_{\pi\mathcal{X}_B^*} \mathcal{L}_{\mathcal{X}_B} \iota_{\pi\mathcal{X}_A^*} \mathcal{L}_{\mathcal{X}_A}) \\ &= \sum_{A,B} \left((-1)^{|\mathcal{X}_A||\mathcal{X}_B| + |\mathcal{X}_A|} + (-1)^{|\mathcal{X}_A||\mathcal{X}_B| + |\mathcal{X}_A| + 1} \right) \iota_{\pi\mathcal{X}_A^*} \iota_{\pi\mathcal{X}_B^*} \mathcal{L}_{\mathcal{X}_A} \mathcal{L}_{\mathcal{X}_B} = 0. \end{aligned} \tag{2.32}$$

The above discussion proves that the pair $(C^{\bullet}_{CE,int}(\mathfrak{g}), \delta^{\bullet})$ defines a dg-superspace, and therefore justifies the following definitions.

Definition 3. Let $\mathfrak{g}$ be a Lie superalgebra. We call the pair $(C^{\bullet}_{CE,int}(\mathfrak{g}), \delta^{\bullet})$ integral Chevalley-Eilenberg complex of the Lie superalgebra $\mathfrak{g}$ and the modules $C^{p}_{CE,int}(\mathfrak{g})$ the integral Chevalley-Eilenberg p-cochains. Accordingly, we define its cohomology as

$$H^{p}_{CE,int}(\mathfrak{g}) := \frac{\ker\left(\delta^p : C^{p}_{CE,int}(\mathfrak{g}) \to C^{p+1}_{CE,int}(\mathfrak{g})\right)}{\operatorname{im}\left(\delta^{p-1} : C^{p-1}_{CE,int}(\mathfrak{g}) \to C^{p}_{CE,int}(\mathfrak{g})\right)}. \tag{2.33}$$

It is to be noted that the above differential (2.31) only acts on $S^{\bullet}\Pi\mathfrak{g}$: one can therefore alternatively define the cohomology (2.33) starting from the (co)chains $\widehat{C}^p(\mathfrak{g}) := S^p\Pi\mathfrak{g}$ on which δ acts and look at the Haar Berezinian as a *twist* by degree n. This gives *per se* an indication on the cohomology content of this complex: this will be expanded in the next sections.
As a remark on notation, let us stress that we will henceforth call *differential* Chevalley-Eilenberg p-cochains the elements in the vector superspace $C^{p}_{CE,dif}(\mathfrak{g}) := S^p\Pi\mathfrak{g}^*$ and *integral* Chevalley-Eilenberg p-cochains the elements in the vector superspace $C^{p}_{CE,int}(\mathfrak{g}) := \mathrm{Ber}^{\mathcal{H}}(\mathfrak{g}) \otimes S^{n-p}\Pi\mathfrak{g}$, as above.

Finally, in order to convince the reader that the two (a priori different) formalisms introduced above - integral Chevalley-Eilenberg cochains via generalized functions and via Berezinian-valued polyfields -, are indeed equivalent, we explicitly consider the following computations. We start with a $(n-1)$-integral form

$$\omega^{(n-1)} = \mathcal{D} \otimes \sum_{A=1}^{m+n} T^A \left(\pi\mathcal{Y}_A\right) \equiv T^A \iota_{\mathcal{Y}_A} \omega^{top} \ . \tag{2.34}$$

By applying the operator $\delta^{(1)} \equiv d$ to $\omega^{(n-1)}$ we obtain

$$\begin{aligned} \delta^{(1)}\omega^{(n-1)} &= \mathcal{D} \otimes \sum_B \sum_A \iota_{(\pi\mathcal{Y}^*_B)} T^A \mathcal{L}_{\mathcal{Y}_B}\left(\pi\mathcal{Y}_A\right) = \\ &= \mathcal{D} \otimes \sum_B \sum_{A,C} \iota_{(\pi\mathcal{Y}^*_B)} T^A f^C_{BA}\left(\pi\mathcal{Y}_C\right) = \mathcal{D} \otimes \sum_B \sum_{A,C} T^A f^C_{BA} \delta_{BC} \ , \end{aligned} \tag{2.35}$$

where we have used $\iota_{(\pi\mathcal{Y}^*_A)}\left(\pi\mathcal{Y}_B\right) = \delta^A_B$. The previous expression is zero if we assume the property of the structure constants $\sum_B f^B_{BA} = 0$, *i.e.*, if we assume the Lie superalgebra to be *unimodular* (see, for example [44] where this terminology is introduced). Throughout the paper, we assume that any Lie superalgebra we are dealing with satisfies this property.
On the other hand we have

$$d\omega^{(n-1)} = \frac{1}{2} f^A_{BC}\left(\pi\mathcal{Y}^*\right)^B \left(\pi\mathcal{Y}^*\right)^C \iota_{\mathcal{Y}_A} T^D \iota_{\mathcal{Y}_D} \omega^{top} = f^A_{BC} \delta_A{}^B \delta_D{}^C T^D \omega^{top} = 0 \ , \tag{2.36}$$

in virtue of the assumed unimodularity.

Notice that the previous example is two-folded: first, it is an example of calculation in both of the realisations together with a check of their equivalence (see also App. A in this regard). Second, it shows that the Haar Berezinian $\mathcal{D} \equiv \omega^{top}$, which is obviously closed with respect to $\delta^{(\bullet)} \equiv d$, *is not exact* (for unimodular Lie superalgebras), thus showing that it is always a cohomology class. We summarize this in the following lemma.

Lemma 1 (The Haar Berezinian is Cohomology Class)**.** *Let $\mathfrak{g}$ be a unimodular Lie superalgebra and let the integral Chevalley-Eilenberg cohomology $H^{\bullet}_{CE,int}(\mathfrak{g})$ be defined as in* (2.19) *or in* (2.33)*. Then the Haar Berezinian is a non-zero cohomology class, and in particular $H^{n}_{CE,int}(\mathfrak{g}) \cong k \cdot [\mathcal{D}]$.*

2.2.5 Isomorphism Between Differential and Integral Form Cohomologies.

Moving on from the previous section, we will now show an explicit isomorphism between differential and integral Chevalley-Eilenberg cohomology. We focus on the case of *classical basic Lie superalgebras* (see [31]), *i.e.*, the classical Lie superalgebras with a non-degenerate invariant bilinear form, that is those admitting a supermetric g_{AB}, generalization of the ordinary Killing-Cartan form. To do so, we will use the formalism where the Haar Berezinian is treated as a differential form as in (2.17) and the nilpotent operator is actually the de Rham differential in its Cartan form realization. The proof for integral forms written as in (2.29) with respect to the differential (2.31) follows from the "dictionary" between the two formalisms, see App. A.

Let us start by considering a closed differential form $\omega^{(1)}$, such that $d\omega^{(1)} = 0$. We define its *Berezinian complement* $\star\omega^{(1)}$ as

$$\begin{aligned} \star : \Omega^{1}_{CE,dif}(\mathfrak{g}) &\longrightarrow \Omega^{n-1}_{CE,int}(\mathfrak{g}) \\ \omega^{(1)} &\longmapsto \star\omega^{(1)} := \iota_{\mathcal{Y}}\omega^{top}_{\mathfrak{g}} \ , \end{aligned} \tag{2.37}$$

where $\pi\mathcal{Y}$ is the vector field dual to $\omega^{(1)}$. Using the (super)metric g_{AB}, one can write $\omega^{(1)} = C_A \pi\mathcal{Y}^{*A}, \mathcal{Y} = \dfrac{g^{AB}C_A}{||C||^2}\mathcal{Y}_B$, where $\mathcal{Y}^A(\mathcal{Y}_B) = \delta^A_B$ and $||C||^2 = C_A g^{AB} C_B$, and analogously for higher forms. Then we have $d \star \omega^{(1)} = d\iota_{\mathcal{Y}}\omega^{top}_{\mathfrak{g}} = 0$, as we have shown in (2.36). For a p-superform the generalization follows by extending (2.37) as

$$\begin{aligned} \star : \Omega^{p}_{CE,dif}(\mathfrak{g}) &\longrightarrow \Omega^{n-p}_{CE,int}(\mathfrak{g}) \\ \omega^{(p)} &\longmapsto \star\omega^{(p)} = (\star\omega)^{(n-p)} := \iota_{\mathcal{Y}_1} \dots \iota_{\mathcal{Y}_p}\omega^{top}_{\mathfrak{g}} \ , \end{aligned} \tag{2.38}$$

where $\omega^{(p)}(\mathcal{Y}_1, \ldots, \mathcal{Y}_p) = 1$. Given $\omega^{(p)} \in H^p_{CE,dif}(\mathfrak{g})$, we have

$$d\left(\omega_{A_1 \ldots A_p} (\pi\mathcal{Y}^*)^{A_1} \wedge \ldots \wedge (\pi\mathcal{Y}^*)^{A_p}\right) = p\omega_{A_1\ldots A_p} f^{A_1}{}_{RS} (\pi\mathcal{Y}^*)^R \wedge (\pi\mathcal{Y}^*)^S \wedge (\pi\mathcal{Y}^*)^{A_2} \wedge \ldots \wedge (\pi\mathcal{Y}^*)^{A_p} = 0$$

$$\iff \quad \omega_{A_1\ldots A_p} f^{A_1}{}_{RS} = 0 \ . \tag{2.39}$$

Recall that capital Latin indices represent both bosonic and fermionic indices, thus expressions like (2.39) do not have definite (graded) skew-symmetrisation of indices. We now show that this condition implies $d\star\omega^{(p)} = 0$. First of all, we observe that the integral form $\star\omega^{(p)}$ reads

$$\star\omega^{(p)} = T^{A_1\ldots A_p} \iota_{\mathcal{Y}^{A_1}} \ldots \iota_{\mathcal{Y}^{A_p}} \omega^{top}_{\mathfrak{g}} \ , \quad \text{such that} \quad T^{A_1\ldots A_p}\omega_{A_1\ldots A_p} = 1 \ . \tag{2.40}$$

It is easy to see that

$$d\star\omega^{(p)} = 0 \quad \iff \quad T^{A_1 A_2 \ldots A_p} f^R{}_{A_1 A_2} = 0 \ . \tag{2.41}$$

Recalling that every basic classical Lie superalgebra admits a non-degenerate bilinear form, see *e.g.* [31], we can use it in order to write the coefficients T of the integral form in terms of the coefficients ω of the superform as

$$T^{A_1 A_2 \ldots A_p} = \frac{1}{||\omega||^2} g^{A_1 B_1} \ldots g^{A_p B_p} \omega_{B_1 \ldots B_p} \ , \ \text{where} \ ||\omega||^2 = \omega_{A_1\ldots A_p} g^{A_1 B_1} \ldots g^{A_p B_p} \omega_{B_1\ldots B_p} \ . \tag{2.42}$$

By substituting (2.42) in (2.41), we obtain

$$\frac{1}{||\omega||^2} g^{A_1 B_1} \ldots g^{A_p B_p} \omega_{B_1\ldots B_p} f^R{}_{A_1 A_2} = \frac{1}{||\omega||^2} g^{A_3 B_3} \ldots g^{A_p B_p} \omega_{B_1 \ldots B_p} f^{R B_1 B_2} =$$

$$= \pm \frac{1}{||\omega||^2} g^{A_3 B_3} \ldots g^{A_p B_p} g^{RL} g^{B_2 M} \omega_{B_1\ldots B_p} f^{B_1}{}_{LM} = 0 \ , \tag{2.43}$$

as a consequence of (2.39) and of the non-degeneracy of the (super)metric (the $\pm$ sign comes from the exchange of indices in the structure constants and it depends on their bosonic/fermionic nature). Hence it follows that $d\omega^{(p)} = 0$ implies $d\star\omega^{(p)} = 0$.

From the previous argument we can now infer the isomorphism between the cohomologies of differential and integral forms. In particular, let $\omega^{(p)} \in H^p_{CE,dif}(\mathfrak{g})$, then

$$\omega^{(p)} \wedge \star\omega^{(p)} = \omega^{top}_{\mathfrak{g}} \in \mathrm{Ber}^{\mathcal{H}}(\mathfrak{g}) \ . \tag{2.44}$$

By contradiction, let us now assume $(\star\omega)^{(n-p)} = d\Lambda^{(n-p-1)}$, we get

$$\omega^{top}_{\mathfrak{g}} = d\left(\omega^{(p)} \wedge \Lambda^{(n-p-1)}\right) \ , \tag{2.45}$$

contradicting that $\omega^{top}_{\mathfrak{g}}$ is a cohomology representative as shown at the end of the previous section. This shows that the operator $\star$ descends to an isomorphism in cohomology. The result is summarized in the following.

Theorem 2. *Let $\mathfrak{g}$ be a classical basic Lie superalgebra. Then the mapping $\star$ defined in* (2.38) *descends to an isomorphism in cohomology,* i.e.

$$\star : H^i_{CE,dif}(\mathfrak{g}) \xrightarrow{\cong} H^{n-i}_{CE,int}(\mathfrak{g}) \ , \tag{2.46}$$

where n is the even dimension of $\mathfrak{g}$ and $i > 0$. For $i = 0$ the isomorphism $H^0_{CE,dif}(\mathfrak{g}) \cong H^n_{CE,int}(\mathfrak{g})$ reads $k \mapsto k \cdot [\mathcal{D}]$.

Remark. The above result can be seen as a sort of *Poincaré duality* in the context of Lie superalgebras. To make sense out of this statement one should in the first place recall that in the context of supermanifolds, the complexes of differential and integral forms are quasi-isomorphism, *i.e.* there is a "direct" isomorphism of vector (super)spaces

$$H^i_{d\mathcal{R}}(\mathcal{M}) \cong H^i_{\mathcal{S}p}(\mathcal{M}), \tag{2.47}$$

where we have denoted with $H^i_{d\mathcal{R}}(\mathcal{M})$ the de Rham cohomology of differential forms on $\mathcal{M}$ - isomorphic to the de Rham cohomology of the reduced space of $\mathcal{M}$ - and with $H^i_{\mathcal{S}p}(\mathcal{M})$ the Spencer cohomology of integral forms on $\mathcal{M}$ - which is the analog of the de Rham cohomology adapted to the notion of integral forms, see [11, 52] for modern accounts, based on homological methods. In physical literature (see, *e.g.*, [57]), this quasi-isomorphism is realised by means of the *Picture Changing Operator*, a formal multiplicative operator, defined via the embedding of the reduced manifold $\mathcal{M}_{red}$ into the supermanifold $\mathcal{M}$, that allows to lift differential forms to integral forms. Of course, the two realisations are analogous, and the representatives match perfectly in two formalisms.
The previous quasi-isomorphism of complexes suggests that, homologically (or, more precisely, in a derived setting) differential and integral forms can be used interchangeably. On the other hand, in the ordinary setting the situation is very different, and differential and integral forms account for very different needs in relation to integration theory: namely, while differential forms can only be integrated on ordinary submanifolds of codimension $\ell|m$ in a given $n|m$-dimensional supermanifold $\mathcal{M}$, integral forms can be integrated on codimension $\ell|0$ sub-supermanifold of $\mathcal{M}$, for $\ell \leq n$. The peculiar integration theory on supermanifolds is mirrored in the statement of Poincar duality that proves the existence of a perfect pairing between (cohomology class of) differential and integral forms and it reads

$$H^i_{d\mathcal{R}}(\mathcal{M}) \cong H^{n-i}_{\mathcal{S}p,c}(\mathcal{M}), \tag{2.48}$$

where now $H^{n-i}_{\mathcal{S}p,c}(\mathcal{M})$ is the Spencer cohomology of compactly supported integral forms.

The above situation can be pictorially represented in the following diagram.

$$
\begin{array}{ccc}
H^i_{d\mathcal{R}}(\mathcal{M}) & \dots & H^{i+j}_{d\mathcal{R}(\mathcal{M})} \\
\big\downarrow {\scriptstyle q.iso.} & & \big\downarrow {\scriptstyle q.iso.} \\
H^i_{\mathcal{S}p}(\mathcal{M}) & \dots & H^{i+j}_{\mathcal{S}p}(\mathcal{M})
\end{array}
\qquad\qquad
\begin{array}{ccc}
H^i_{d\mathcal{R}}(\mathcal{M}) & \dots & H^{i+j}_{d\mathcal{R}}(\mathcal{M}) \\
 & \overset{\mathcal{P.D.}\quad\mathcal{P.D.}}{\times} & \\
H^{n-i-j}_{\mathcal{S}p,c(\mathcal{M})} & \dots & H^{n-i}_{\mathcal{S}p,c}(\mathcal{M})
\end{array}
\tag{2.49}
$$

Here, on the left, is represented the direct quasi-isomorphism of complexes as in (2.47) and on the right the Poincar duality for supermanifold as in (2.48).

In the case of Lie superalgebras and the related of Chevalley-Eilenberg cohomology we have a different pattern. Indeed the direct isomorphism (2.47) is in general lost (see also comments in the Appendices B.3 and B.4 in this regard). On the other hand, Theorem 2 shows that there exists an isomorphism of the Poincar duality-type. Namely, for n the even dimension of the basic classic Lie superalgebra $\mathfrak{g}$, one has

$$
\begin{array}{ccc}
H^i_{CE,dif}(\mathfrak{g}) & \dots & H^{i+j}_{CE,dif}(\mathfrak{g}) \\
 & \overset{\star\quad\star}{\times} & \\
H^{n-i-j}_{CE,int}(\mathfrak{g}) & \dots & H^{n-i}_{CE,int}(\mathfrak{g})
\end{array}
\tag{2.50}
$$

where the perfect pairing is realized via the operator $\star$ defined above. Equivalently, this can be interpreted as a homology-cohomology duality for Lie superalgebras, as the notion of compact support and integration loose their significance in this algebraic context.
In [25], two of the authors studied Chevalley-Eilenberg cohomology for the complexes of *pseudoforms*, showing that it is not isomorphic to the one of differential forms or integral forms, neither directly nor via shifts. This cohomology might be richer, giving more classes that keep into account the existence of other invariants (or sub-invariants) related to the Lie superalgebra.

3 Poincaré Polynomials and Series

Before we move to compute examples of Chevalley-Eilenberg cohomology, we review the definition of Poincaré series and Poincaré polynomials. In a purely bosonic setting these provide useful tools for the computation of the dimension of the cohomology groups, as they can be computed in a completely abstract fashion, *e.g.* by Molien-Weyl formula [33]. However, the extension of such techniques to the super-setting is non-trivial and not available in the literature. As a consequence, in the present work, we will limit ourselves to employing the Poincaré polynomial/series mostly as a bookkeeping device: only in some cases, we will build the Poincaré polynomial/series according to some simple rules, that will be explained in the following section. We will also use them directly in the computation of the dimension of cohomology groups for the equivariant cohomology.

For X a *graded* k-vector space with direct decomposition into p-degree homogeneous subspaces given by $X = \bigoplus_{p\in\mathbb{Z}} X_p$, we call the formal series

$$\mathcal{P}(t) = \sum_p (\dim_k X_p)(-t)^p \tag{3.51}$$

the *Poincaré series* of X. Notice that we have implicitly assumed that X is of *finite type*, *i.e.* its homogeneous subspaces X_p are finite dimensional for every p. The unconventional sign in $(-t)^p$ takes into account the *parity* of X_p, which is given by $p \bmod 2$. If also $\dim_k X$ is finite, then $\mathcal{P}_X(t)$ becomes a polynomial $\mathcal{P}_X[t]$, called *Poincaré polynomial* of X.

In the algebraic setting of this paper, we have that $X = \bigoplus_{p=0}^{\infty} H^p_{CE,dif}(\mathfrak{g})$ and $b_p(\mathfrak{g}) = \dim_k H^p_{CE}(\mathfrak{g})$, so that the Poincaré series of the Lie (super)algebra $\mathfrak{g}$ is denoted as

$$\mathcal{P}_{\mathfrak{g}}(t) = \sum_p b_p(\mathfrak{g})(-t)^p = \prod_i (1-t^{c_i})^{\alpha_i} \ , \tag{3.52}$$

where the c_i are the corresponding exponents in the factorized form of the polynomial (see [33] or [38]) and the α_i take care of the multiplicity of the different factor with power c_i. The exponent α_i could be also negative, indicating a power series expansion in t.

Notice that we used the notation of series $\mathcal{P}(t)$ on purpose: indeed, as we shall see, $H^\bullet_{CE}(\mathfrak{g})$ is in general not finite dimensional for a generic Lie superalgebra $\mathfrak{g}$. In this context, we can retrieve some useful results using the Poincaré series. For example, Künneth theorem, which computes the cohomology of products of spaces, can simply be written as

$$\mathcal{P}_{X\times Y}(t) = \mathcal{P}_X(t)\cdot\mathcal{P}_Y(t). \tag{3.53}$$

Finally, following [38], if $\mathfrak{g}$ is a Lie algebra with Poincaré polynomial $\mathcal{P}_{\mathfrak{g}}(t) = \prod_l (1-t^{c_l^{\mathfrak{g}}})$ and $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, of the same rank (*Cartan pairs*, see [38]) with Poincaré polynomial $\mathcal{P}_{\mathfrak{h}}(t) = \prod_m (1-t^{c_m^{\mathfrak{h}}})$, then the Poincaré polynomial of the coset space will be given by

$$\mathcal{P}_{\mathfrak{g}/\mathfrak{h}}(t) = \frac{\prod_l (1-t^{c_l^{\mathfrak{g}}+1})}{\prod_m (1-t^{c_m^{\mathfrak{h}}+1})} \ . \tag{3.54}$$

Although we are not aware of a general proof of this fact for Lie superalgebras, we will see that this applies to the examples in consideration. A remark is in order though. Indeed, whereas for ordinary coset spaces arising from finite-dimensional Lie algebras $\mathfrak{h} \subset \mathfrak{g}$, (3.54) always yields a polynomial, in the context of Lie superalgebras we can get a series, as we will explicitly see in section 5.

4 Chevalley-Eilenberg Cohomology: Computations

4.1 "Flat" Supertranslation Algebras

We now briefly present examples of cohomology of "flat" supertranslation algebras, *i.e.* Lie superalgebras of translations of superspaces of the kind $\mathbb{R}^{n|m}$, which we will denote $\mathit{susy}(\mathbb{R}^{n|m})$.

4.1.1 Dimension 1

Written in terms of the "flat coordinates" of the superspace, the vielbeins generating Chevalley-Eilenberg cochains read

$$\mathcal{V} := dx - \theta^1 d\theta^2 - \theta^2 d\theta^1, \quad \psi^\alpha = d\theta^\alpha, \tag{4.55}$$

for $\alpha = 1, 2,$. The Maurer-Cartan equations are (up to a rescaling)

$$d\mathcal{V} = -2\psi^1\psi^2, \qquad d\psi^\alpha = 0. \tag{4.56}$$

The cohomology is thus given by

$$H^p_{CE,\mathit{diff}}(\mathit{susy}(\mathbb{R}^{1|2})) \cong \Pi H^{1-p}_{CE,\mathit{int}}(\mathit{susy}(\mathbb{R}^{1|2})) \cong \begin{cases} \mathbb{R} & p = 0 \\ \mathbb{R}^2 & p > 0, \end{cases} \tag{4.57}$$

where the integral forms are parity reversed because of the Berezinian being odd and in particular $H^1_{CE,\mathit{int}}(\mathit{susy}(\mathbb{R}^{1|2})) \cong \mathbb{R} \cdot \mathcal{D}_{\mathit{susy}(\mathbb{R}^{1|2})}$, for $\mathcal{D}_{\mathit{susy}(\mathbb{R}^{1|2})}$ the Haar Berezinian. The explicit expressions for the cocycles representatives of these classes are deferred to the appendix B.1. Notice that this easy example shows a remarkable difference between the Chevalley-Eilenberg cohomology of an ordinary Lie algebra and that of a Lie superalgebra, namely the fact that even the cohomology of finite-dimensional Lie superalgebras can be *infinite*, whereas clearly, every finite-dimensional Lie algebra has a finite-dimensional Chevalley-Eilenberg cohomology.

4.1.2 Dimension 2

This is a physically relevant example, as it corresponds to the superstranslation algebra of the superspace $\mathbb{R}^{1,1|2}$, which is called $D = 2$, $\mathcal{N} = 1$ superspace. Just like above, in terms of the flat coordinates of the superspace, the vielbeins generating the Chevalley-Eilenberg cochains reads

$$\mathcal{V}^i = dx^i - \theta^\alpha \Gamma^i_{\alpha\beta} d\theta^\beta, \quad \psi^\alpha = d\theta^\alpha \tag{4.58}$$

for $i = 0, 1$ and $\alpha = 1, 2$, where the gamma matrices are

$$\Gamma^1_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \Gamma^2_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.59}$$

The Maurer-Cartan equations reads

$$d\mathcal{V}^i = \psi^\alpha \Gamma^i_{\alpha\beta} \psi^\beta, \qquad d\psi^\alpha = 0, \tag{4.60}$$

again for for $i = 0, 1$ and $\alpha = 1, 2$. The cohomology is then computed to be

$$H^p_{CE,\mathit{diff}}(\mathit{susy}(\mathbb{R}^{1,1|2})) \cong H^{2-p}_{CE,\mathit{int}}(\mathit{susy}(\mathbb{R}^{1,1|2})) \cong \begin{cases} \mathbb{R} & p = 0, 2 \\ \mathbb{R}^2 & p = 1 \\ 0 & p > 2. \end{cases} \tag{4.61}$$

As above, explicit expressions for representatives of these cocycles can be found in the appendix B.2.

4.1.3 Dimension 3

This case corresponds to the supertranslation algebra of the superspace $\mathbb{R}^{1,2|2}$, called $D = 3$, $\mathcal{N} = 1$ superspace. The vielbeins generating the Chevalley-Eilenberg cochains read

$$\mathcal{V}^a := dx^a - \theta^\alpha \gamma^a_{\alpha\beta} d\theta^\beta, \qquad \psi^\alpha := d\theta^\alpha. \tag{4.62}$$

for $a = 0, \ldots, 2$ and $\alpha = 1, 2$, where we are using real and symmetric gamma matrices $\gamma^a_{\alpha\beta}$, which are defined via *charge conjugation*, given by the Pauli matrix $C := -i\sigma_2 = \epsilon_{\alpha\beta}$

$$\gamma^0_{\alpha\beta} := (C\Gamma^0)_{\alpha\beta} = -\mathbf{1}, \qquad \gamma^1_{\alpha\beta} := (C\Gamma^1)_{\alpha\beta} = \sigma^3, \qquad \gamma^2_{\alpha\beta} := (C\Gamma^2)_{\alpha\beta} = -\sigma^1 \,, \tag{4.63}$$

where

$$\Gamma^0 := i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \Gamma^1 := i\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \Gamma^2 := \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.64}$$

generate the related Clifford algebra. The Maurer-Cartan equations in turn are given by

$$d\mathcal{V}^a = \psi^\alpha \gamma^a_{\alpha\beta} \psi^\beta, \qquad d\psi^\alpha = 0. \tag{4.65}$$

Out of these, the cohomology is computed to be

$$H^p_{CE,\mathit{diff}}(\mathit{susy}(\mathbb{R}^{1,2|2})) \cong \Pi H^{3-p}_{CE,\mathit{int}}(\mathit{susy}(\mathbb{R}^{1,2|2})) \cong \begin{cases} \mathbb{R} & p = 0\ , \\ \mathbb{R}^2 & p = 1\ , \\ \Pi\mathbb{R}^2 & p = 2\ , \\ \Pi\mathbb{R} & p = 3\ , \\ 0 & p > 3, \end{cases} \tag{4.66}$$

where the integral forms are parity reversed because of the Berezinian being odd. Explicit representatives for these cocycles are given in the appendix B.3; there we comment about the connection with the superspace $\mathfrak{siso}(1,2|N=1)/\mathfrak{so}(1,2)$ and the physical relevance of the cohomology groups we found. In particular, we comment on the Lorentz-invariant $(0|2)-$integral form and its interpretation as Picture Changing Operator.

4.1.4 Dimension 4

This case corresponds to the superspace $\mathbb{R}^{1,3|4}$ based upon the 4-dimensional Minkowski space $\mathbb{R}^{1,3}$, which is the usual superspace for flat rigid supersymmetry $\mathcal{N}=1$ models, and therefore the first step toward supergravity models. Some of the results of the present section have been also discussed, e.g., in [8].

The generators of the 1-cochains of the Lie superalgebra satisfies Maurer-Cartan equations analogous to (4.65)

$$d\mathcal{V}^a = \psi^\alpha \gamma^a_{\alpha\beta} \psi^\beta, \qquad d\psi^\alpha = 0\,, \tag{4.67}$$

but clearly now the gamma's are 4-dimensional Dirac matrices, instead of 2-dimensional. To make contact with the Minkowskian model, it is convenient to use the irreducible chiral components, the (left) Weyl spinors $\chi^\alpha \in (1/2,0)$ and (right) anti-Weyl spinors $\bar\lambda^{\dot\alpha} \in (0,1/2)$ for $\alpha,\dot\alpha = 1,2$ of the above reducible Dirac representation $\psi \in (1/2,0)\oplus(0,1/2)$, so that $\psi = (\chi^\alpha, \bar{\lambda}^{\dot{\alpha}})$. In this representation, the above Maurer-Cartan equations modify to

$$d\mathcal{V}^{\alpha\dot{\alpha}} = \chi^\alpha \bar{\lambda}^{\dot{\alpha}}, \qquad d\chi^\alpha = 0, \qquad d\bar{\lambda}^{\dot{\alpha}} = 0. \tag{4.68}$$

Notice that the odd 1-forms $\mathcal{V}^a$ is represented as a *bi-spinor* $\mathcal{V}^{\alpha\dot{\alpha}} = \bar{\sigma}_a^{\alpha\dot{\alpha}} \mathcal{V}^a$, via the matrices $\bar{\sigma}$ of the $(0, 1/2)$ irreducible component.

This representation makes the task of computing cohomology class representatives easier; for example, we immediately see that any form containing only left (or right) Weyl spinors is closed, non-exact:

$$\omega^{(p|0)}(\chi) = \chi^{\alpha_1} \dots \chi^{\alpha_p} \in H^p_{CE,dif}(susy(\mathbb{R}^{1,3|4})) \ , \ \forall p \in \mathbb{N} \ . \tag{4.69}$$

This immediately shows that the dimension of any cohomology group is different from zero, hence the cohomology is infinite-dimensional. In the appendix, we collect explicit representatives for the first cohomology groups.

A compact expression that gives an easy way to calculate the dimension of each cohomology group is given by the Poincaré series. By looking at (4.68), we see that we can consistently assign weight $1/2$ to χ^α, $\bar{1}/2$ to $\bar{\lambda}^{\dot{\alpha}}$ and consequently $1/2 + \bar{1}/2$ to the bispinors $\mathcal{V}^{\alpha\dot{\alpha}}$. The $\mathcal{V}$s are odd, so we can take at most four of them, while the χs and the $\bar{\lambda}$ are even, so we can take any number of them, thus obtaining the following expression of the Poincaré series:

$$\begin{aligned} \mathcal{P}^{dif}_{susy(\mathbb{R}^{1,3|4})}(\sqrt{t}, \sqrt{\bar{t}}) &= \frac{\left(1 - \sqrt{t}\sqrt{\bar{t}}\right)^4}{\left(1 - \sqrt{t}\right)^2 \left(1 - \sqrt{\bar{t}}\right)^2} = \\ &= 1 + 2(\sqrt{t} + \sqrt{\bar{t}}) + 3(t + \bar{t}) + 4(t\sqrt{t} + t\sqrt{\bar{t}}) - 2(\bar{t}\sqrt{t} + t\sqrt{\bar{t}}) + \dots \ . \end{aligned} \tag{4.70}$$

Each monomial describes a generator of the cohomology and the signs keep track of the parity. For example, the first cohomology groups are

$$H^p_{CE,dif}(susy(\mathbb{R}^{1,3|4})) = H^{4-p}_{CE,int}(susy(\mathbb{R}^{1,3|4})) = \begin{cases} \mathbb{R} \ , \ \text{if } p = 0 \ , \\ \mathbb{R}^4 \ , \ \text{if } p = 1 \ , \\ \mathbb{R}^6 \oplus \Pi\mathbb{R}^4 \ , \ \text{if } p = 2 \ , \\ \mathbb{R}^8 \oplus \Pi\mathbb{R}^9 \ , \ \text{if } p = 3 \ , \\ \mathbb{R}^{12} \oplus \Pi\mathbb{R}^{12} \ , \ \text{if } p = 4 \ , \\ \dots \end{cases} \ . \tag{4.71}$$

4.2 "Curved" Lie Superalgebras

We now study the cohomology of different Lie superalgebras, which we refer to as "curved" with respect to the flat cases introduced in the previous sections. These are the local models of some physically interesting rigid backgrounds where to study supergravity or gauge theories.

4.2.1 Dimension 2: $\mathfrak{u}(1|1)$

In the easiest case $\mathfrak{u}(1|1)$, one has a 2|2-dimensional Lie superalgebra, whose general element can be given in the following form

$$X = \left(\begin{array}{c|c} ia & \theta + i\psi \\ \hline -\psi - i\theta & ib \end{array}\right), \tag{4.72}$$

for $a, b \in \mathbb{R}$ and $\theta, \psi \in \Pi\mathbb{R}$, so that the even and odd generators can be chosen to be the (super)matrices

$$X_1 = \left(\begin{array}{c|c} i & \\ \hline & 0 \end{array}\right), \quad X_2 = \left(\begin{array}{c|c} 0 & \\ \hline & i \end{array}\right), \quad \Psi_1 = \left(\begin{array}{c|c} & 1 \\ \hline -i & \end{array}\right), \quad \Psi_2 = \left(\begin{array}{c|c} & i \\ \hline -1 & \end{array}\right), \tag{4.73}$$

together with the commutation relations

$$\begin{gathered} [X_i, X_j] = 0, \quad [X_1, \Psi_1] = \Psi_2, \quad [X_1, \Psi_2] = -\Psi_1, \quad [X_2, \Psi_1] = -\Psi_2, \quad [X_2, \Psi_2] = \Psi_1 \\ \{\Psi_1, \Psi_1\} = -2X_1 - 2X_2, \quad \{\Psi_2, \Psi_2\} = -2X_1 - 2X_2, \quad \{\Psi_1, \Psi_2\} = 0. \end{gathered} \tag{4.74}$$

Introducing the dual (up to parity) basis of Maurer-Cartan forms of $\Pi\mathfrak{u}(1|1)^*$, defined so that $\Pi\mathfrak{u}(1|1)^* = \mathrm{Span}_{\mathbb{R}}\{\mathcal{V}^i|\psi^\alpha\}$ for $i = 1, 2$ and $\alpha = 1, 2$, with $V^i(\pi X_j) = \delta^i_j$ and $\psi^\alpha(\pi\Psi_\beta) = \delta^\alpha_\beta$, one sees from (4.74) and (4.74) that the Maurer-Cartan equations read

$$dV^1 = dV^2 = -\sum_{\alpha=1}^{2}(\psi^\alpha)^2, \quad d\psi^1 = \psi^2\frac{(V^1 - V^2)}{2}, \quad d\psi^2 = \psi^1\frac{(-V^1 + V^2)}{2}. \tag{4.75}$$

Changing the basis to $U \coloneqq \frac{V^1 - V^2}{2}$ and $W \coloneqq \frac{V^1 + V^2}{2}$, the Maurer-Cartan equations simplify to

$$dU = 0, \qquad dW = -\sum_{\alpha=1}^{2}(\psi^\alpha)^2, \qquad d\psi^1 = U\psi^2, \qquad d\psi^2 = -U\psi^1. \tag{4.76}$$

Using the above Maurer-Cartan equations (4.76), it is not hard to compute the related Chevalley-Eilenberg cohomology:

$$H^0_{CE,dif}(\mathfrak{u}(1|1)) \cong \mathbb{R}\cdot 1, \qquad H^1_{CE,dif}(\mathfrak{u}(1|1)) \cong \mathbb{R}\cdot\{U\}, \quad H^{p>1}_{CE,dif}(\mathfrak{u}(1|1)) = 0. \tag{4.77}$$

The previous (4.77) encodes the result, proved for any $\mathfrak{gl}(m|n)$-type Lie superalgebra in [32], that only a fraction of the bosonic subalgebra $\mathfrak{u}(1)\times\mathfrak{u}(1)$ contributes to the cohomology. The Poincaré polynomial reads

$$\mathcal{P}^{dif}_{\mathfrak{u}(1|1)}[t] = \mathcal{P}^{dif}_{\mathfrak{u}(1)}[t] = 1-t. \tag{4.78}$$

In the case of integral Chevalley-Eilenberg cohomology, using the isomorphism described in the previous chapter one finds

$$H^2_{CE,int}(\mathfrak{u}(1|1)) \cong \mathbb{R}\cdot\mathcal{D}_{\mathfrak{u}(1|1)}, \qquad H^1_{CE,int}(\mathfrak{u}(1|1)) \cong \mathbb{R}\cdot\{\iota_{\pi U^*}\mathcal{D}_{\mathfrak{u}(1|1)}\},$$
$$H^{p<1}_{CE,int}(\mathfrak{u}(1|1)) = 0. \tag{4.79}$$

where we have posed again $\mathcal{D}_{\mathfrak{u}(1|1)} = UW\delta(\psi^1)\delta(\psi^2)$. This explicit realization of the Berezinian allows one to see that the first cohomology class $H^1_{CE,int}(\mathfrak{u}(1|1))$ is represented by $W\delta(\psi^1)\delta(\psi^2)$, thus suggesting the that the second abelian factor of the bosonic subalgebra $\mathfrak{u}(1)\times\mathfrak{u}(1)$, represented by W, is accounted in the cohomology of the integral forms better than the differential forms - which account instead for U. Notice that the presence of the delta's, formally set to zero the ψ's, so that the Maurer-Cartan equation becomes simply $dW = 0$, making W into a cohomology class. Finally, the corresponding expression for the Poincaré polynomial reads

$$\mathcal{P}^{int}_{\mathfrak{u}(1|1)}[t] = -t + t^2 = -t(1-t) \tag{4.80}$$

The factor $-t$ in the second expression denotes the additional cohomology class related to the second invariant of the Lie superalgebra.

4.2.2 A Remark on Cartan Theorem on Compact Lie Groups

A crucial result in Lie algebra cohomology theory is a theorem due to Cartan, which states that under the topological assumptions of *compactness* and *connectedness*, the de Rham cohomology of a Lie group G is isomorphic to the cohomology of its Lie algebra (valued in the real numbers), *i.e.* $H^p_{dR}(G) \cong H^p_{CE}(\mathfrak{g})$; clearly, the result is remarkable not only from a conceptual point of view but also from a computational point of view, for it allows to get topological information on large interesting classes of Lie groups via linear algebra. The

above result on $\mathfrak{u}(1|1)$ shows that the result does not hold in the supersetting, whereas one naively substitutes the ordinary compact Lie group G with a compact Lie supergroup $\mathcal{G}$ and the Lie algebra $\mathfrak{g}$ with its Lie superalgebra.
Let us look indeed at the Lie supergroup $U(1|1)$ related to $\mathfrak{u}(1|1)$. Especially in this context, it is convenient to introduce the unitary supergroup $U(1|1)$ as the *super Harish-Chandra pair* $(U(1) \times U(1), \mathfrak{u}(1|1))$, since the categories of Lie supergroups and super Harish-Chandra pairs are indeed equivalent [12]. As it is well-known [11], the de Rham cohomology of a supermanifold only depends on its underlying topological space, and as such it is completely determined by the first entry, *i.e.* the ordinary Lie group, of the super Harish-Chandra pair. In our case, we obtain the cohomology of a 2-torus $S^1 \times S^1 \cong U(1) \times U(1)$:

$$H^p_{dR}(U(1|1)) \cong \begin{cases} \mathbb{R} & p = 0 \\ \Pi\mathbb{R}^2 & p = 1 \\ \mathbb{R} & p = 2. \end{cases} \tag{4.81}$$

This shows that the de Rham cohomology of *compact* Lie supergroups, such as $U(1|1)$ which is topologically a 2-torus, is not isomorphic to the Chevalley-Eilenberg cohomology of superforms of their related Lie superalgebras.

Notice by the way, that the isomorphism is restored once one reduces to deal with the *even* - or *topological* - part of a Lie superalgebra. In other words, if, as a vector space, a Lie superalgebra is such that $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, and its related (*e.g.* via Harish-Chandra pair) Lie supergroup $\mathcal{G}$ is topologically compact as a (super)manifold, then one finds that for any p

$$H^p_{dR}(\mathcal{G}) \cong H^p_{CE,dif}(\mathfrak{g}_0). \tag{4.82}$$

This is readily seen in the above case for the Lie superalgebra $\mathfrak{u}(1|1)$, where modding out the odd part of the underlying vector space, one is left with Maurer-Cartan equations of the form $dU = 0$ and $dW = 0$, which indeed lead to the same cohomology of the 2-torus. Once again, it has, therefore, to be stressed that whilst fermions play really no role when computing de Rham cohomology of a supermanifold as nilpotents do not modify topology, in the case of Chevalley-Eilenberg cohomology of a Lie superalgebra, which is ultimately determined by the structure of commutators or, equivalently, by the Maurer-Cartan equations, fermions play a crucial role and they do indeed determine the cohomology structure, which might be very different - either richer or poorer - from the cohomology of the topological even part of the superalgebra. In other words, it is to be stressed once again that the cohomology of a Lie superalgebra, in general, is different from the de Rham cohomology of supergroups.

4.2.3 Dimension 3: $\mathfrak{osp}(1|2)$ and its İnönü-Wigner Contraction to $\mathit{susy}(\mathbb{R}^{1,2|2})$

In this section we consider $\mathfrak{osp}(1|2) = B(0,1)$ and compute its cohomology. Secondly, we relate the computation with the "flat" case of the Lie superalgebra $\mathit{susy}(\mathbb{R}^{1,2|2})$ considered in Sec. 4.1.3.

A convenient choice of basis for $\mathfrak{osp}(1|2)$ is provided as follows:

$$\mathcal{P}_0 = \frac{1}{2}\left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & -1 & 0 \end{array}\right), \qquad \mathcal{P}_1 = \frac{1}{2}\left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & -1 \end{array}\right), \qquad \mathcal{P}_2 = \frac{1}{2}\left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right), \tag{4.83}$$

$$\mathcal{Q}_1 = \left(\begin{array}{c|cc} 0 & 1 & 1 \\ \hline 1 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right), \qquad \mathcal{Q}_2 = \left(\begin{array}{c|cc} 0 & 1 & -1 \\ \hline -1 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right). \tag{4.84}$$

Making use of the previously introduced (real and symmetric) gamma matrices $\gamma^i_{\alpha\beta}$ the commutation relations can be written in the following very convenient way

$$[\mathcal{P}_a, \mathcal{P}_b] = -\epsilon_{abc}\mathcal{P}_c, \qquad \{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} = -2\gamma^a_{\alpha\beta}\mathcal{P}_a, \qquad [\mathcal{Q}_\alpha, \mathcal{P}_a] = -\gamma_\alpha{}^a{}_\beta\mathcal{Q}_\beta \tag{4.85}$$

where ϵ_{abc} is the Levi-Civita symbol and where we observe that the first commutation relation follows by the isomorphism $\mathfrak{sp}(2,\mathbb{R}) \cong \mathfrak{so}(2,1,\mathbb{R}) \cong \mathfrak{su}(1,1,\mathbb{R})$. The Maurer-Cartan forms, which are dual to the above generators of the Lie superalgebra $\mathfrak{osp}(1|2)$ up to parity, are given by $C^1_{CE,dif}(\mathfrak{osp}(1|2)) = \Pi\mathfrak{osp}(1|2)^* = \mathrm{Span}_\mathbb{R}\{\psi^\alpha|\mathcal{V}^a\}$ for $a = 0,1,2$ and $\alpha = 1,2$ with $\mathcal{V}^a(\pi\mathcal{P}_b) = \delta^a_b$ and $\psi^\alpha(\pi\mathcal{Q}_\beta) = \delta^\alpha_\beta$. The above commutation relations lead to the following set of Maurer Cartan equations (up to a sign redefinition):

$$d\mathcal{V}^a = \epsilon_{bc}{}^a\mathcal{V}^b\mathcal{V}^c + \psi^\alpha\gamma^a_{\alpha\beta}\psi^\beta, \qquad d\psi^\alpha = \mathcal{V}^a\gamma_{a,\alpha\beta}\psi^\beta. \tag{4.86}$$

The cohomology reads

$$\begin{aligned}
&H^0_{CE,dif}(\mathfrak{osp}(1|2)) \cong \mathbb{R}\cdot 1,\\
&H^1_{CE,dif}(\mathfrak{osp}(1|2)) \cong 0,\\
&H^2_{CE,dif}(\mathfrak{osp}(1|2)) \cong 0\\
&H^3_{CE,dif}(\mathfrak{osp}(1|2)) \cong \mathbb{R}\cdot\left\{\frac{1}{2}\mathcal{V}^a(\psi\gamma_a\psi) - \frac{1}{6}\epsilon_{abc}\mathcal{V}^a\mathcal{V}^b\mathcal{V}^c\right\}
\end{aligned} \tag{4.87}$$

and $H^{p>3}_{CE,dif}(\mathfrak{osp}(1|2)) \cong 0$. Notice that this result is confirmed by the theorem of Fuks, which states that the cohomology of $\mathfrak{osp}(1|2)$ is isomorphic to that of its bosonic subalgebra $\mathfrak{sp}(2,\mathbb{R})$, thus leading to the Poincaré polynomial

$$\mathcal{P}^{dif}_{\mathfrak{osp}(1|2)}[t] = \mathcal{P}_{\mathfrak{sp}(2,\mathbb{R})} = 1 - t^3. \tag{4.88}$$

Notice, though, that with respect to the bosonic Lie algebra $\mathfrak{sp}(2,\mathbb{R})$ the representative of the 3-cohomology of the Lie superalgebra $\mathfrak{osp}(1|2)$ is shifted in the fermionic directions as can be seen directly by the above expression.

Quite similarly, the integral Chevalley-Eilenberg cohomology reads

$$\begin{aligned}
&H^3_{CE,int}(\mathfrak{osp}(1|2)) \cong \mathbb{R} \cdot \epsilon_{abc}\mathcal{V}^a\mathcal{V}^b\mathcal{V}^c\epsilon_{\alpha\beta}\delta(\psi^\alpha)\delta(\psi^\beta),\\
&H^2_{CE,int}(\mathfrak{osp}(1|2)) \cong 0,\\
&H^1_{CE,int}(\mathfrak{osp}(1|2)) \cong 0,\\
&H^0_{CE,int}(\mathfrak{osp}(1|2)) \cong \mathbb{R} \cdot \left\{ \frac{1}{2}\mathcal{V}^a\mathcal{V}^b(\iota_{\pi\mathcal{Q}_\alpha}\gamma_{[ab],\alpha\beta}\iota_{\pi\mathcal{Q}_\beta})\epsilon_{\alpha\beta}\delta(\psi^\alpha)\delta(\psi^\beta) - \frac{1}{6}\epsilon_{\alpha\beta}\delta(\psi^\alpha)\delta(\psi^\beta) \right\}.
\end{aligned} \tag{4.89}$$

Again we can represent these classes using the Poincaré polynomial (in the case of "curved" superalgebras the only consistent weight assignment is given by the form number),

$$\mathcal{P}^{int}_{\mathfrak{osp}(1|2)}[t] = -t^3(1 - t^{-3}) = (1 - t^3). \tag{4.90}$$

The factor $-t^3$ takes into account the form number carried by the Berezinian, and the term t^{-3} accounts for the contractions on the Berezinian, as in general described in Section 2 in order to establish the isomorphism between the integral forms and the differential form classes.

It is worth to observe the relation between the "curved" and "flat" 3-dimensional case. Indeed, simply redefining the generators of the superalgebra $\mathfrak{osp}(1|2)$ by a constant parameter λ as follows,

$$\mathcal{Q}^\lambda_\alpha := \frac{1}{\sqrt{\lambda}}\mathcal{Q}_\alpha, \qquad \mathcal{P}^\lambda_a := \frac{1}{\lambda}\mathcal{P}_a, \tag{4.91}$$

one finds that the new Maurer-Cartan equations for $\mathcal{V}^a_\lambda$ and ψ^α_λ read

$$d\mathcal{V}^a_\lambda = \lambda\epsilon_{bc}{}^a\mathcal{V}^b_\lambda\mathcal{V}^c_\lambda + \psi^\alpha_\lambda\gamma^a_{\alpha\beta}\psi^\beta_\lambda, \qquad d\psi^\alpha_\lambda = \lambda\mathcal{V}^a\gamma_{a,\alpha\beta}. \tag{4.92}$$

The limit $\lambda \to 0$ is called *İnönü-Wigner contraction* and it is immediate to see that it gives back the Maurer-Cartan equations for the superalgebra $\mathit{susy}(\mathbb{R}^{1,2|2})$: in this sense $\mathit{susy}(\mathbb{R}^{1,2|2})$ can be seen as the "flat" limit of the orthosymplectic superalgebra $\mathfrak{osp}(1|2)$.

4.2.4 Dimension 4: $\mathfrak{osp}(2|2)$

We now consider a "curved" 4 dimensional case, studying the cohomology of the Lie superalgebra $\mathfrak{osp}(2|2) = C(2)$. Before we start, though, it is useful to stress that the related Lie supergroup $OSp(2|2)$ cannot be given an interpretation from a Minkowskian point of view, since it breaks the $SO(1,3)$-invariance to the bosonic subgroup $SO(2) \times Sp(2)$. However, the example provides a useful comparison with the remarkable "flat superspace" case above.

The Maurer-Cartan forms are given by $\mathcal{V}^0$, $\mathcal{V}^a = \gamma^a_{\alpha\beta}\mathcal{V}^{\alpha\beta}$, for $a = 1,2,3$, ψ^α_I, for $I = 1,2$, having kept separated a "time" direction with $\mathcal{V}^0$. They satisfy the Maurer-Cartan equations

$$\begin{aligned} d\mathcal{V}^{\alpha\beta} &= (\mathcal{V} \wedge \mathcal{V})^{\alpha\beta} + \psi^\alpha_I \eta^{IJ} \psi^\beta_J \,, \\ d\mathcal{V}^0 &= -\epsilon_{\alpha\beta}\psi^\alpha_I \epsilon^{IJ}\psi^\beta_J \,, \\ d\psi^\alpha_I &= (V \wedge \psi)^\alpha_I + \epsilon_I^J \mathcal{V}^0 \psi^\alpha_J \,. \end{aligned} \tag{4.93}$$

Notice that in the suitable "flat" limit, one retrieves the flat model discussed in the previous section. The cohomology is computed to be given by the following representatives:

$$\begin{aligned} H^0_{CE,dif}(\mathfrak{osp}(2|2)) &= \mathbb{R} \cdot 1 \,, \\ H^3_{CE,dif}(\mathfrak{osp}(2|2)) &= \mathbb{R} \cdot \left\{ \psi^\alpha_I \eta^{IJ} \psi^\beta_J \mathcal{V}_{\alpha\beta} + \psi^\alpha_I \epsilon^{IJ} \psi^\beta_J \epsilon_{\alpha\beta} \mathcal{V}^0 + \mathcal{V} \wedge \mathcal{V} \wedge \mathcal{V} \right\} , \end{aligned} \tag{4.94}$$

with related Poincaré polynomial given by

$$\mathcal{P}^{dif}_{\mathfrak{osp}(2|2)}[t] = (1 - t^3). \tag{4.95}$$

This matches the result by Fuks, claiming that the cohomology of $\mathfrak{osp}(2|2)$ is non-zero in degree 0 and 3 and isomorphic to that of its bosonic subalgebra $\mathfrak{sp}(2,\mathbb{R})$. For the integral form cohomology, one finds

$$\begin{aligned} H^1_{CE,int}(\mathfrak{osp}(2|2) &= \mathbb{R} \cdot \left\{ \iota^I_\alpha \eta_{IJ} \iota^J_\beta \mathcal{V}^0 (\mathcal{V} \wedge \mathcal{V})^{\alpha\beta} \delta^4(\psi) + \iota^I_\alpha \epsilon_{IJ} \iota^J_\beta \epsilon^{\alpha\beta} (\mathcal{V} \wedge \mathcal{V} \wedge \mathcal{V}) \delta^4(\psi) + \mathcal{V}^0 \delta^4(\psi) \right\} \\ H^4_{CE,int}(\mathfrak{osp}(2|2) &= \mathbb{R} \cdot \left\{ \mathcal{V}^0 \mathcal{V} \wedge \mathcal{V} \wedge \mathcal{V} \delta^4(\psi) \right\} , \end{aligned} \tag{4.96}$$

together with the related Poincaré polynomial as in (4.90)

$$\mathcal{P}^{int}_{\mathfrak{osp}(2|2)}[t] = t^4(1 - t^{-3}) = -t\left(1 - t^3\right) . \tag{4.97}$$

The "$-t$" factor in (4.97) can be interpreted as a manifestation of the $\mathfrak{so}(2)$ subalgebra invariant which is not taken into account among superforms.

5 Coset Superspaces and Equivariant Chevalley-Eilenberg Cohomology

In this section we briefly introduce *equivariant* Chevalley-Eilenberg cohomology, a crucial tool to study the cohomology of *coset* or *homogeneous superspaces* $\mathcal{G}/\mathcal{H}$ where $\mathcal{G}$ is a Lie supergroup and $\mathcal{H}$ is a Lie sub-supergroup of $\mathcal{G}$.

Very few examples of Lie supergroup, or *group supermanifolds*, are indeed solutions of supergravity/string equations of motion, for example AdS_3 in the case of non-critical strings and a few others. Nonetheless, the space of geometric backgrounds modelled on coset spaces is much richer, the most remarkable example being the case of supersymmetric backgrounds built on *coset supermanifolds*. In this context, the most important instance is that of a coset supermanifold realized by modding out a certain *bosonic subgroup*: the infamous examples of $AdS_5 \times S^5$ and $AdS_4 \times \mathbb{CP}^3$ belong this category [48] [34]. Furthermore, a less explored instance is that obtained by modding out a true *Lie sub-supergroup*. In any of these cases, it is interesting to compute their (equivariant) cohomology, as it can uncover insights into the physics related to the model.

Given a Lie supergroup $\mathcal{G}$ and a Lie sub-supergroup $\mathcal{H}$ of $\mathcal{G}$ we define the related Lie superalgebras by $\mathfrak{g}$ and $\mathfrak{h}$. Then, attached to the coset superspace $\mathcal{G}/\mathcal{H}$ we will have, correspondingly, the quotient $\mathfrak{g}/\mathfrak{h}$, whose elements are equivalence classes $\mathfrak{g} \bmod \mathfrak{h}$. As a vector superspace, there always exists a direct linear decomposition of $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{C}, \tag{5.98}$$

but the choice of $\mathfrak{C}$ is ambiguous and different compatibility conditions between this direct linear decomposition and the Lie algebra structures can be imposed. More in details, the coset superspace $\mathcal{G}/\mathcal{H}$ is said to be *reductive* if there exists an $\mathrm{ad}(\mathfrak{h})$-invariant choice of $\mathfrak{C}$, *i.e.*

$$\mathrm{ad}(\mathfrak{h}) \cdot \mathfrak{C} = [\mathfrak{h}, \mathfrak{C}] \subset \mathfrak{C}. \tag{5.99}$$

Further, imposing that $[\mathfrak{C}, \mathfrak{C}] \subset \mathfrak{h}$ we get that the coset $\mathcal{G}/\mathcal{H}$ is a *symmetric superspace*, but in general the commutators close as

$$[\mathfrak{C}, \mathfrak{C}] \subset \mathfrak{g}. \tag{5.100}$$

As in the ordinary setting, a left-translation in the coset superspace induces a map $(\ell_{[g^{-1}]})_* : \mathcal{T}_{[g]}\mathcal{G}/\mathcal{H} \to \mathcal{T}_{[e]}\mathcal{G}/\mathcal{H} \cong \mathfrak{g}/\mathfrak{h}$ which can be seen as $\mathfrak{g}/\mathfrak{h}$-valued 1-forms, the so-called

Maurer-Cartan forms. As above, we will always deal with *matrix* Lie superalgebras. The Maurer-Cartan forms are usually written starting from the coset superspace elements as $\omega^g_{MC} = [g^{-1}dg]$. Notice that, choosing another representative gh for $h \in \mathcal{H}$ instead of g, we get

$$\omega^{gh}_{MC} = [\mathrm{ad}(h)(g^{-1}dg)] = \mathrm{ad}(h) \cdot \omega^g_{MC}, \tag{5.101}$$

since $[h^{-1}dh] = 0$ in the quotient $\mathfrak{g}/\mathfrak{h}$. Passing from the above coordinate-invariant formalism to a particular choice of coordinates, in line with the general philosophy of the paper of finding explicit expressions, we choose a certain direct linear decomposition of $\mathfrak{g}$ as above and, in turn, a basis $\{h_i\}$ for $i = 1, \ldots, \dim \mathfrak{h}$ of generators for $\mathfrak{h}$ and a basis $\{k_J\}$ for $J = 1, \ldots, \dim \mathfrak{C}$ of generators for $\mathfrak{C}$. Notice that the parametrization of the elements of the coset superspace $[g] \in \mathcal{G}/\mathcal{H}$ is far from being unique. The Maurer-Cartan form related to this decomposition and choice of basis can be computed as to get

$$\omega_{MC} = \mathcal{V}^J k_J + \omega^i h_i, \tag{5.102}$$

where the $\mathcal{V}^J$'s are the supervielbein forms and the ω^i's are interpreted as the connection forms associated with the action of the sub-superalgebra $\mathfrak{h}$. The vielbein and connection forms satisfy the following Maurer-Cartan equations that can be read off the commutation relations of $\mathfrak{g}$

$$\begin{aligned} d\mathcal{V}^I &= f^I_{JK}\,\mathcal{V}^J \wedge \mathcal{V}^K + f^I_{iJ}\,\omega^i \wedge \mathcal{V}^J, \\ d\omega^i &= f^i_{jk}\omega^j \wedge \omega^k + f^i_{IJ}\mathcal{V}^I \wedge \mathcal{V}^K. \end{aligned} \tag{5.103}$$

The second one can be re-written as

$$\mathcal{R}^i := d\omega^i - f^i_{jk}\omega^j \wedge \omega^k = f^i_{IJ}\mathcal{V}^I \wedge \mathcal{V}^K. \tag{5.104}$$

Here the structure constants are written with respect to the above decomposition of the Lie superalgebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{C}$ and $\mathcal{R}^i$ is referred to as the "curvature" of the gauge connection ω^i related to the subalgebra $\mathfrak{h}$. The form of the first Maurer-Cartan equation in (5.103) in turn makes convenient to introduce a *covariant differential* defined as

$$\mathcal{D}\mathcal{V}^I := d\mathcal{V}^I - f^I_{iJ}\,\omega^i \wedge \mathcal{V}^J. \tag{5.105}$$

Notice that this differential is *not* nilpotent, indeed one has

$$\mathcal{D}^2\mathcal{V}^I = -\mathcal{R}^i f^I_{iJ}\mathcal{V}^J \tag{5.106}$$

using Jacobi identity. This can be re-written as $\mathcal{D}^2\mathcal{V}^I = -\mathcal{L}_{\mathcal{R}} V^I$, where we have denoted $\mathcal{L}_{\mathcal{R}}$ the action of the Lie derivative on the vielbeins $\mathcal{V}^I$ along the (vertical, form-valued) vector $\mathcal{R}^i h_i$.

The above expression makes clear that in order to have a well-defined differential cochain complex for coset superspaces, we need to impose further conditions on the forms to take into account. Namely, we need the Maurer-Cartan forms, call them Ω's, to be *basic*, this means that we require

$$\mathcal{L}_H \Omega = 0, \qquad \iota_H \Omega = 0, \tag{5.107}$$

for any vector H coming from the subalgebra $\mathfrak{h}$. Roughly speaking, one can visualize these requirements thinking about a principal $\mathcal{H}$-bundle $\pi : \mathcal{P} \to \mathcal{G}/\mathcal{H}$: in this respect a *basic form* Ω is a form defined on the principal bundle $\Omega = \pi^*(\mathcal{V})$ such that it has *no vertical components* (it is horizontal) and *no vertical variation* (it stays horizontal), *i.e.* basic forms are in the intersection $\ker(\iota_H) \cap \ker(\mathcal{L}_H)$. Calling $C^p_{\mathcal{B}asic}(\mathfrak{g}/\mathfrak{h})$ the (vector) superspace of the basic p-forms, we accordingly define the equivariant (Chevalley-Eilenberg) cohomology to be the cohomology of the basic forms with respect to the differential $\mathcal{D}$ introduced above.

$$H^p_{\mathcal{EQ}}(\mathfrak{g}/\mathfrak{h}) := \frac{\{\Omega \in C^p_{\mathcal{B}asic}(\mathfrak{g}/\mathfrak{h}) : \mathcal{D}\Omega = 0\}}{\{\Omega \in C^p_{\mathcal{B}asic}(\mathfrak{g}/\mathfrak{h}) : \exists \eta \in C^{p-1}_{\mathcal{B}asic}(\mathfrak{g}/\mathfrak{h}) \; \Omega = \mathcal{D}\eta\}}. \tag{5.108}$$

In absence of encompassing "structure theorems", different methods are possible to compute the cohomology of coset superspaces. Our strategy will be to supplement brute force computations with the indications coming from the Poincaré polynomial of coset superspaces. This will tell, for example, when a cohomology space is expected to be infinite-dimensional, as we shall see.

Following [38], if $\mathfrak{g}$ is a Lie superalgebra with Poincaré series given by $\mathcal{P}_{\mathfrak{g}}(t) = \sum_i b_i^{\mathfrak{g}} t^i = \prod_{i=1}^{\mathrm{rank}(\mathfrak{g})} \left(1 - t^{c_i^{\mathfrak{g}}}\right)$, for appropriate $c_i^{\mathfrak{g}}$ and $\mathfrak{h}$ is a Lie sub-superalgebra of $\mathfrak{g}$, of the same rank (*Cartan pairs*, see [38]) having Poincaré series given by $\mathcal{P}_{\mathfrak{h}}(t) = \sum_j b_i^{\mathfrak{h}} t^j = \prod_{i=1}^{\mathrm{rank}(\mathfrak{h})} \left(1 - t^{c_i^{\mathfrak{h}}}\right)$, then the Poincaré series for the coset will be given by the following formula

$$\mathcal{P}_{\mathfrak{g}/\mathfrak{h}}(t) = \frac{\prod_l (1 - t^{c_l^{\mathfrak{g}}+1})}{\prod_m (1 - t^{c_m^{\mathfrak{h}}+1})} \; . \tag{5.109}$$

This product formula is very helpful since it provides information regarding the dimensions of the cohomology groups.

5.1 Lower Dimensional Cosets of $\mathfrak{osp}(1|2)$ and $\mathfrak{u}(1|1)$

Let us now consider the Lie superalgebra $\mathfrak{osp}(1|2)$ introduced above. In agreement with [32], we have seen that $H^{\bullet}_{CE}(\mathfrak{osp}(1|2)) \cong H^{\bullet}_{CE}(\mathfrak{sp}(2,\mathbb{R}))$ and in particular, its Poincaré polynomial reads $\mathcal{P}_{\mathfrak{osp}(1|2)}(t) = 1 - t^3$ with the 3-cohomology class generated by $\omega^{(3)} = \psi\gamma_a\psi\mathcal{V}^a + \frac{1}{3}\epsilon_{abc}\mathcal{V}^a\mathcal{V}^b\mathcal{V}^c$, where the vielbeins ψ's and $\mathcal{V}$'s have been introduced above together with the gamma matrices $\gamma^i_{\alpha\beta}$.
Looking at the Lie supergroup $OSp(1|2)$[2] related to the Lie superalgebra $\mathfrak{osp}(1|2)$ it is natural to consider two coset manifolds. The first one is the coset $OSp(1|2)/SO(1,1)$, which is known as the *anti de Sitter superspace* $AdS^{2|2}$. The second one is a purely fermionic superspace, actually a "fat point", given by the coset $OSp(1|2)/Sp(2,\mathbb{R})$, which is a $0|2$-dimensional superspace. Here we consider their algebraic counterparts.
Let us start from $\mathfrak{osp}(1|2)/\mathfrak{so}(1,1)$: the related Poincaré polynomial reads

$$\mathcal{P}_{\mathfrak{osp}(1|2)/\mathfrak{so}(1,1)}[t] = \frac{1-t^4}{1-t^2} = 1 + t^2, \tag{5.110}$$

upon using the so-called *Weyl trick*, in order to identify the Chevalley-Eilenberg cohomology of $\mathfrak{so}(1,1)$ with that of $\mathfrak{so}(2) \cong \mathfrak{u}(1)$. This suggests that besides the constants, there is a single cohomology class at degree two. Explicitly, introducing the Maurer-Cartan vielbeins $\{\mathcal{V}^0, \mathcal{V}^{\ddagger}, \mathcal{V}^{=}|\psi^{\pm}\}$, one gets the following Maurer-Cartan equations:

$$\mathcal{D}\mathcal{V}^0 = \mathcal{R} = i\mathcal{V}^{\ddagger}\wedge\mathcal{V}^{=} + \psi^+\wedge\psi^- \,, \quad \mathcal{D}\mathcal{V}^{\ddagger} = i\psi^+\wedge\psi^+ \,, \quad \mathcal{D}V^{=} = -i\psi^-\wedge\psi^- \,,$$
$$\mathcal{D}\psi^+ = \mathcal{V}^{\ddagger}\wedge\psi^- \,, \quad \mathcal{D}\psi^- = \mathcal{V}^{=}\wedge\psi^+ \,. \tag{5.111}$$

The infinitesimal action of the subgroup is given by

$$\mathcal{L}_{\mathcal{T}}V^{\ddagger} = 2iV^{\ddagger} \,, \quad \mathcal{L}_{\mathcal{T}}V^{=} = -2iV^{=} \,, \quad \mathcal{L}_{\mathcal{T}}\psi^{\pm} = \pm i\psi^{\pm}. \tag{5.112}$$

Note that the 2-form

$$\mathcal{R} = i\mathcal{V}^{\ddagger}\wedge\mathcal{V}^{=} + \psi^+\wedge\psi^- \tag{5.113}$$

is (real) basic and closed. It is not exact because $\mathcal{R} = \mathcal{D}\mathcal{V}^0$, but $\mathcal{V}^0$ is not a basic form. Therefore, the equivariant superform cohomology is described by $\{1, \mathcal{R}\}$:

$$H^p_{\mathcal{EQ}}(\mathfrak{osp}(1|2)/\mathfrak{so}(2)) = \begin{cases} \mathbb{R} & p = 0, 2 \\ 0 & \text{else.} \end{cases} \tag{5.114}$$

[2]Recall that, in analogy to the group $SO(3)$ which can be seen as the isometry of a three-sphere, the reduced subgroup of $OSp(1|2)$ is $SO(1,2)$, which can be seen as the isometry of the anti-de Sitter space AdS^3.

In the case of the fermionic coset $\mathfrak{osp}(1|2)/\mathfrak{sp}(2)$ the Poincaré polynomial reads

$$\mathcal{P}_{\mathfrak{osp}(1|2)/\mathfrak{sp}(2)}(t) = \frac{1-t^4}{1-t^4} = 1. \tag{5.115}$$

We expect therefore only the constants to be in the cohomology, which is indeed the case since now $\mathcal{R}$ is *not* basic as now the forms $\mathcal{V}^{\ddagger}$ and $\mathcal{V}^{=}$ are not vielbeins, but connections instead, coming from the subalgebra $\mathfrak{sp}(2)$:

$$H^p_{\mathcal{EQ}}(\mathfrak{osp}(1|2)/\mathfrak{sp}(2,\mathbb{R})) = \begin{cases} \mathbb{R} & p=0 \\ 0 & \text{else.} \end{cases} \tag{5.116}$$

We now consider the case of $\mathfrak{u}(1|1)$, whose Chevalley-Eilenberg cohomology has been discussed above. Namely, we consider the coset $\mathfrak{u}(1|1)/\mathfrak{u}(1)$ of dimension $1|2$ and $\mathfrak{u}(1|1)/\mathfrak{u}(1)\oplus \mathfrak{u}(1)$ of dimension $0|2$.

Let us start from the first coset superspace. A subtle point is that we have to choose how to embed the abelian factor $\mathfrak{u}(1)$ inside $\mathfrak{u}(1|1)$: indeed the automorphism of $\mathfrak{u}(1|1)_0 = \mathfrak{u}(1) \oplus \mathfrak{u}(1)$ that exchange the factors does not lift to $\mathfrak{u}(1|1)$ (see, *e.g.*, [31]). With reference to the previous section, we can embed $\mathfrak{u}(1)$ in such a way that its Maurer-Cartan form (the connection, in view of the equivariant cohomology) is associated either to U or to W. In the case it is associated with U, then the cohomology trivializes as can be readily observed from the Maurer-Cartan equations: the only non-zero equivariant cohomology group is the zeroth-cohomology group:

$$H^p_{\mathcal{EQ}}(\mathfrak{u}(1|1)/\mathfrak{u}_U(1)) = \begin{cases} \mathbb{R} & p=0 \\ 0 & \text{else,} \end{cases} \tag{5.117}$$

having called $\mathfrak{u}(1|1)/\mathfrak{u}_U(1)$ the related coset.

Things changes drastically if $\mathfrak{u}(1)$ is embedded in a way such that its Maurer-Cartan forms correspond with W. In this case U is the generator of a cohomology class, indeed it is closed and basic. Moreover, also the bilinears $(\psi^1\psi^2)^p$ for any p are in the equivariant cohomology: indeed they can be seen to be exact with respect to the non-basic term W. The cohomology is therefore infinite-dimensional and generated by the elements $\{1, U\} \otimes \{(\psi^1\psi^2)^p\}$ for any $p \geq 0$. We thus have

$$H^p_{\mathcal{EQ}}(\mathfrak{u}(1|1)/\mathfrak{u}_W(1)) = \begin{cases} \mathbb{R} & p \text{ even} \\ \Pi\mathbb{R} & p \text{ odd,} \end{cases} \tag{5.118}$$

having called $\mathfrak{u}(1|1)/\mathfrak{u}_W(1)$ the coset.

Finally, considering the coset $\mathfrak{u}(1|1)/\mathfrak{u}(1) \oplus \mathfrak{u}(1)$, we have that in this case both U and

W correspond to Maurer-Cartan forms for the subalgebra. From the Maurer-Cartan equations, it follows that the cohomology is generated by the representatives given by the fermionic bilinears $\{(\psi^1\psi^2)^p\}$, for any $p \geq 0$ so that the equivariant cohomology is non-zero in any even degree.

$$H^p_{\mathcal{EQ}}(\mathfrak{u}(1|1)/\mathfrak{u}(1) \oplus \mathfrak{u}(1)) = \begin{cases} \mathbb{R} & p \text{ even} \\ 0 & p \text{ odd}, \end{cases} \tag{5.119}$$

Using the formula (5.109), the corresponding Poincaré series read

$$\mathcal{P}_{\mathfrak{u}(1|1)/\mathfrak{u}(1)\oplus\mathfrak{u}(1)}(t) = \frac{1-t^2}{(1-t^2)^2} = \frac{1}{1-t^2} = 1 + t^2 + t^4 + \dots \, . \tag{5.120}$$

which match the computations above (5.119).

5.2 Higher Dimensional Cosets of $\mathfrak{osp}(2|2)$, $\mathfrak{osp}(3|2)$ and $\mathfrak{osp}(4|2)$

We now consider higher-dimensional cosets of $\mathfrak{osp}(n|2)$ for $n = 2, 3, 4$. We start with some general considerations, and then we specialize to the single cases together with their cosets.

On a general ground, the Maurer-Cartan equations for $\mathfrak{osp}(n|2)$ read

$$\begin{aligned} d\mathcal{V}_{(\alpha\beta)} &= \psi^I_\alpha \psi^J_\beta \eta_{IJ} + (\mathcal{V} \wedge \mathcal{V})_{(\alpha\beta)} \, , \\ d\mathcal{T}^{[IJ]} &= -\psi^I_\alpha \psi^J_\beta \Omega^{\alpha\beta} + (\mathcal{T} \wedge \mathcal{T})^{[IJ]} \, , \\ d\psi^I_\alpha &= \mathcal{V}_{\alpha\beta}\Omega^{\beta\gamma}\psi^I_\gamma + \mathcal{T}^{[IJ]}\eta_{JK}\psi^K_\alpha \, , \end{aligned} \tag{5.121}$$

where the Maurer-Cartan forms are given by $\{\psi^I_\alpha | \mathcal{V}^{(\alpha\beta)}, \mathcal{T}^{[IJ]}\}$ for $\alpha, \beta = 1, 2$ and $I, J, K, \ldots = 1, \ldots, n$. We have $(\mathcal{V} \wedge \mathcal{V})_{(\alpha\beta)} = \mathcal{V}_{(\alpha\gamma)}\Omega^{\gamma\delta}\mathcal{V}_{(\delta\beta)}$ and $(\mathcal{T} \wedge \mathcal{T})^{[IJ]} = \mathcal{T}^{[IK]}\eta_{KL}\mathcal{T}^{[LJ]}$, where $\Omega^{\alpha\beta}$ is the 2-symplectic invariant tensor (from $\mathfrak{sp}(2)$) and η_{IJ} is the Euclidean rotation-invariant metric (from $\mathfrak{so}(n)$).

It is not difficult to verify that for any n the 3-form

$$\omega^{(3)} = \psi^I_\alpha \psi^J_\beta \eta_{IJ} \mathcal{V}^{(\alpha\beta)} + \psi^I_\alpha \psi^J_\beta \Omega^{\alpha\beta} \mathcal{T}_{IJ} + (\mathcal{V} \wedge \mathcal{V} \wedge \mathcal{V})_{\alpha\beta}\Omega^{\alpha\beta} + (\mathcal{T} \wedge \mathcal{T} \wedge \mathcal{T})_{IJ}\eta^{IJ} \tag{5.122}$$

gives an invariant which is a representative of the 3-cohomology group $H^3_{CE}(\mathfrak{osp}(n|2))$, shared by all of the $\mathfrak{osp}(n|2)$. Through direct computation, we find that there is a unique cohomology class up to $n = 3$ (besides the constants in the 0-cohomology), indeed, as in [32], we have that

$$\mathcal{P}_{\mathfrak{osp}(2|2)}[t] = \mathcal{P}_{\mathfrak{osp}(3|2)}[t] = \mathcal{P}_{\mathfrak{osp}(1|2)}[t] = \mathcal{P}_{\mathfrak{sp}(2,\mathbb{R})}[t] = 1 - t^3. \tag{5.123}$$

Things start changing in the case of $\mathfrak{osp}(4|2)$, where one has that

$$\mathcal{P}_{\mathfrak{osp}(4|2)}[t] = \mathcal{P}_{\mathfrak{so}(4)}[t] = (1-t^3)^2, \tag{5.124}$$

where we recall that $D_2 \cong A_1 \otimes A_1$ for the complexified algebras and the Poincaré polynomial for A_1 is indeed $1 - t^3$. We therefore expect a further 3-form in the cohomology $H^3_{CE}(\mathfrak{osp}(4|2))$. This is indeed the case and the extra cohomology representative is given by

$$\widetilde{\omega}^{(3)} = \epsilon_{IJKL}\psi^I_\alpha \psi^J_\beta \Omega^{\alpha\beta}\mathcal{T}^{KL} + \epsilon_{IJK[M}\eta_{N]L}\mathcal{T}^{IJ}\mathcal{T}^{KL}\mathcal{T}^{MN}. \tag{5.125}$$

Cohomology classes for higher dimensional $\mathfrak{osp}(n|2)$ for $n > 4$ can be constructed in similar way.

5.2.1 $\mathfrak{osp}(2|2)$

Let us now get back to the specific case of the Lie superalgebra $\mathfrak{osp}(2|2)$. This is a Lie superalgebra of dimension $4|4$, whose reduced algebra is given by $\mathfrak{osp}(2|2)_0 = \mathfrak{so}(2) \oplus \mathfrak{sp}(2)$. We consider its cosets $\mathfrak{osp}(2|2)/\mathfrak{so}(1,1)$ and $\mathfrak{osp}(2|2)/\mathfrak{so}(2) \oplus \mathfrak{so}(1,1)$, respectively of dimension $3|4$ and $2|4$. While the Poincaré series for the first coset can not be guessed by (5.109) (because the two superalgebras have different rank), it can be immediately written for the second coset:

$$\mathcal{P}_{\mathfrak{osp}(2|2)/(\mathfrak{so}(2)\oplus\mathfrak{so}(1,1))}(t) = \frac{(1-t^4)}{(1-t^2)^2} = \frac{1+t^2}{1-t^2}. \tag{5.126}$$

Let us calculate explicitly the cohomology of the two cosets: by looking at the Maurer-Cartan equations one finds

$$\begin{aligned} d\mathcal{V}^{(\alpha\beta)} &= (\mathcal{V}\wedge\mathcal{V})^{(\alpha\beta)} + \psi^\alpha_i \wedge \psi^\beta_j \eta^{ij}, \\ d\mathcal{W} &= \psi^\alpha_i \wedge \psi^\beta_j \epsilon^{ij}\epsilon_{\alpha\beta}, \\ d\psi^\alpha_i &= \mathcal{V}^{(\alpha\gamma)}\epsilon_{\gamma\beta}\wedge\psi^\beta_i - \mathcal{W}\epsilon_{ij}\wedge\psi^{\alpha i}\,. \end{aligned} \tag{5.127}$$

for $\alpha = 1, 2$ and $i = 1, 2$, where η^{ij} is the Minkowski metric.
In the case of the first coset $\mathfrak{osp}(2|2)/\mathfrak{so}(1,1)$, we consider two ways to embed $\mathfrak{so}(1,1)$ in $\mathfrak{osp}(2|2)$: in the first case we embed it in the $\mathfrak{sp}(2)$ part, in the second one we embed it in the $\mathfrak{so}(2)$ part (after a suitable signature redefinition via unitary trick). In the first case, one finds the cohomology class

$$\mathcal{R} = (\gamma^0)_{\alpha\beta}((\mathcal{V}\wedge\mathcal{V})^{(\alpha\beta)} + \psi^\alpha_i \wedge \psi^\beta_j \eta^{ij})\ , \tag{5.128}$$

where γ^0 is the 0-th Dirac gamma matrix: it is easy to check that this is indeed a basic closed and not exact form. To do this it is convenient to decompose the vielbeins as $\mathcal{V}^{(\alpha\beta)} = \gamma_{a(\alpha\beta)} V^a, a \in \{0, \pm\}$ (as in the previous section for $\mathfrak{osp}(1|2)$), then we quotient with respect to $\mathcal{V}^0$. Hence (5.128) represents a form which is closed by construction, being exact with respect to a non-basic object, basic since it does not depend on $\mathcal{V}^0$ and non-exact. In the second case, we have that the $\mathfrak{so}$-algebra is embedded in the $\mathfrak{so}$-subalgebra of $\mathfrak{osp}$, hence in this case we are doing the quotient with respect to $\mathcal{W}$. In this case we immediately see from the Maurer-Cartan equations (5.127) that the bilinear

$$(\psi \cdot \psi) = \psi_i^\alpha \wedge \psi_j^\beta \epsilon^{ij} \epsilon_{\alpha\beta} = \mathcal{D}\mathcal{W} \ , \tag{5.129}$$

together with its powers, is a basic, closed, non-exact form.

On the other hand, we can study the second coset $\mathfrak{osp}(2|2)/\left(\mathfrak{so}(1,1) \oplus \mathfrak{so}(2)\right)$. In this case, either $\mathcal{R}$ or the bilinear (and all its powers) $(\psi \cdot \psi)$ are part of the equivariant cohomology, making the cohomology infinite dimensional, generated by $\{1, \mathcal{R}\} \otimes \{(\psi \cdot \psi)^p\}$ for any p. The cohomologies that we have studied explicitly then read

$$H^p_{\mathcal{EQ}}(\mathfrak{osp}(2|2)/\mathfrak{so}(1,1))_{\mathcal{V}^0} = \begin{cases} \mathbb{R} & p = 0, 2 \\ 0 & \text{else,} \end{cases} \tag{5.130}$$

$$H^p_{\mathcal{EQ}}(\mathfrak{osp}(2|2)/\mathfrak{so}(1,1))_{\mathcal{W}} = \begin{cases} \mathbb{R} & p \text{ even} \\ 0 & \text{else,} \end{cases} \tag{5.131}$$

$$H^p_{\mathcal{EQ}}(\mathfrak{osp}(2|2)/\left(\mathfrak{so}(1,1) \oplus \mathfrak{so}(2)\right)) = \begin{cases} \mathbb{R} & p = 0, \\ \mathbb{R}^2 & p = 2, 4, \ldots \\ 0 & \text{else.} \end{cases} \tag{5.132}$$

5.2.2 $\mathfrak{osp}(3|2)$

Let us look at the case of the cosets of $\mathfrak{osp}(3|2)$. Cosets by $\mathfrak{so}(2)$ or $\mathfrak{so}(1,1)$ works in pretty the same way as the above case of $\mathfrak{osp}(2|2)$. On the other hand, it is interesting to deal with the case $\mathfrak{osp}(3|2)/\mathfrak{so}(2) \oplus \mathfrak{so}(1,1)$. First, observe that algebra and subalgebra have the same rank, so by (5.109) the Poincaré series reads

$$\mathcal{P}_{\mathfrak{osp}(3|2)/\mathfrak{so}(2)\oplus\mathfrak{so}(1,1)}(t) = \frac{1+t^2}{1-t^2}, \tag{5.133}$$

which is the same as in the case of the coset $\mathfrak{osp}(2|2)/\mathfrak{so}(2) \oplus \mathfrak{so}(1,1)$. In particular, two equivariant cohomology classes can be singled out. We have

$$\begin{aligned} \mathcal{D}\mathcal{V}_0 &= \psi^I_\alpha \psi^J_\beta \eta_{IJ} \gamma_0^{\alpha\beta} + \mathcal{V}_+ \wedge \mathcal{V}_-, \\ \mathcal{D}\mathcal{T}^0 &= -\psi^I_\alpha \psi^J_\beta \Omega^{\alpha\beta} s_{IJ} + \mathcal{T}_+ \wedge \mathcal{T}_-, \end{aligned} \tag{5.134}$$

where $s_{IJ} = -s_{JI}$, which select a "direction" $\mathfrak{so}(3)$ denoted as $\mathcal{T}_0$ - in pretty much the same way as the γ^0 allows to select a "direction" in the Lie algebra $\mathfrak{sp}(2,\mathbb{R})$. Notice that the above elements are not (equivariantly) exact as $\mathcal{V}^0$ and $\mathcal{T}_0$ are not basic since they are the generators of the subgroup. These two forms correspond to the $2t^2$ term of (5.133), obtained by expanding the denominator. Notice that to take into account the two independent cohomology classes above, it would be more convenient to have a term of the kind $(1+t^2)^2$ in the numerator. If we multiply and divide the above series by $1+t^2$ we get indeed

$$\mathcal{P}_{\mathfrak{osp}(3|2)/\mathfrak{so}(2)\oplus\mathfrak{so}(1,1)}(t) = \frac{(1+t^2)^2}{1-t^4}. \tag{5.135}$$

This suggests that the infinite cohomology can be obtained in terms of the two forms (5.134) and of a 4-form, which can be found to be

$$\mathcal{X}^{(4)} = \psi^I_\alpha \gamma_a^{\alpha\beta} \psi^J_\beta \eta^{ab} \epsilon_{RIJ} \eta^{RS} \epsilon_{SKL} \psi^K_{\alpha'} \gamma_b^{\alpha'\beta'} \psi^L_{\beta'}, \tag{5.136}$$

This is basic, closed and not exact. The cohomology therefore reads

$$H^p_{\mathcal{EQ}}(\mathfrak{osp}(3|2)/\mathfrak{so}(2) \oplus \mathfrak{so}(1,1)) = \begin{cases} \mathbb{R} & p = 0 \\ \mathbb{R}^2 & p \text{ even} \\ 0 & p \text{ odd.} \end{cases} \tag{5.137}$$

5.2.3 $\mathfrak{osp}(4|2)$

Finally, let us take a brief look at an interesting coset of $\mathfrak{osp}(4|2)$, namely $\mathfrak{osp}(4|2)/\mathfrak{u}(2)$. In this case the problem can be studied by considering the spinor representation of $\mathfrak{so}(1,3) \sim \mathfrak{so}(4) \cong \mathfrak{su}(2) \times \mathfrak{su}(2)$. In this formulation we have

$$\mathcal{T}^{[IJ]} \to \begin{cases} \mathcal{T}^{(AB)} = (\sigma_{[IJ]})^{(AB)} \mathcal{T}^{[IJ]} \\ \tilde{\mathcal{T}}^{(\dot{A}\dot{B})} = (\sigma_{[IJ]})^{(\dot{A}\dot{B})} \mathcal{T}^{[IJ]} \end{cases} , \ A, B = 1, 2 \ , \tag{5.138}$$

$$\psi^I_\alpha \ \to \ \psi_{\alpha A\dot{A}} = (\sigma_I)_{A\dot{A}} \psi^I_\alpha \ , \tag{5.139}$$

where $(\sigma^I)_{AB}$ and $(\sigma^I)_{\dot{A}\dot{B}}$ are the two copies of the Pauli matrices of $\mathfrak{su}(2) \times \mathfrak{su}(2)$, $(\sigma_{[IJ]})^{(AB)} = \left[(\sigma_I)^{A\dot{A}}, (\sigma_J)_{\dot{A}}^{\ B}\right]$ and $(\sigma_{[IJ]})^{(\dot{A}\dot{B})} = \left[(\sigma_I)^{A\dot{A}}, (\sigma_J)_A^{\ \dot{B}}\right]$. The Maurer-Cartan equations (5.121) then become

$$\begin{aligned}
& d\mathcal{V}_{(\alpha\beta)} = (\psi \cdot \psi)_{(\alpha\beta)} + (\mathcal{V} \wedge \mathcal{V})_{(\alpha\beta)} \ , \\
& d\mathcal{T}^{(AB)} = -\left(\psi \cdot \psi\right)^{(AB)} + (\mathcal{T} \wedge \mathcal{T})^{(AB)} \ , \\
& d\tilde{\mathcal{T}}^{(\dot{A}\dot{B})} = -\left(\psi \cdot \psi\right)^{(\dot{A}\dot{B})} + (\tilde{\mathcal{T}} \wedge \tilde{\mathcal{T}})^{(\dot{A}\dot{B})} \ , \\
& d\psi_{\alpha A\dot{A}} = \mathcal{V}_{\alpha\beta} \Omega^{\beta\gamma} \psi_{\gamma A\dot{A}} + \sigma_{IA\dot{A}} \left((\sigma^{[IJ]})_{(CD)} \mathcal{T}^{(CD)} + (\sigma^{[IJ]})_{(\dot{C}\dot{D})} \mathcal{T}^{(\dot{C}\dot{D})} \right) \eta_{JK} \sigma^{KA\dot{A}} \psi_{\alpha A\dot{A}} \ .
\end{aligned} \tag{5.140}$$

Let us consider the coset $\mathfrak{osp}(4|2)/\mathfrak{su}(2)$: we can quotient out one of the two $\mathfrak{su}(2)$, for example the one generated by $\tilde{\mathcal{T}}$. We immediately see that the bilinears $(\psi \cdot \psi)^{(\dot{A}\dot{B})} = -\nabla \tilde{\mathcal{T}}^{(\dot{A}\dot{B})}$ become exact, with respect to non basic objects, hence they are cohomology representatives of the coset algebra. The same holds for any power and product of these bilinears. Moreover, we have another cohomology representative which is given by (5.122) but with only the set of non-modded out $\mathcal{T}$'s:

$$\omega^{(3)} = (\psi \cdot \psi)_{(\alpha\beta)} \mathcal{V}^{(\alpha\beta)} + (\psi \cdot \psi)^{(AB)} \mathcal{T}_{(AB)} + (\mathcal{V} \wedge \mathcal{V} \wedge \mathcal{V})_{\alpha\beta} \Omega^{\alpha\beta} + (\mathcal{T} \wedge \mathcal{T} \wedge \mathcal{T})_{AB} \eta^{AB} \ . \quad (5.141)$$

It follows that the cohomology is generated by $\left\{1, \omega^{(3)}\right\} \otimes \left\{ \left[(\psi \cdot \psi)^{(\dot{A}\dot{B})} \right]^n \right\}, \forall n \in \mathbb{N}$. The computation of the dimensions of the cohomology spaces is not difficult, but tedious since it heavily relies on the Fierz identities, hence it is not written here. Notice that the "finite part" of the cohomology (to be precise, its bosonic restriction) is exactly what is left from the bosonic quotient $\mathfrak{so}(4)/\mathfrak{su}(2) \cong \mathfrak{su}(2)$. We will comment further on this in the next paragraph, relying on this observation.

We could now proceed further by modding out another $\mathfrak{u}(1)$, in order to study the coset space $\mathfrak{osp}(4|2)/\mathfrak{u}(2)$. Given the previous results, the calculation is straightforward: we can either embed the $\mathfrak{u}(1)$ into the remaining $\mathfrak{su}(2)$, which is generated by the $\mathcal{T}^{(AB)}$'s, or into $\mathfrak{sp}(2)$, which is generated by the $\mathcal{V}^{(\alpha\beta)}$'s. However, the two embeddings are equivalent since $\mathfrak{sp}(2) \cong \mathfrak{su}(2)$ at the level of complex algebras. If we perform the quotient in the $\mathfrak{sp}(2)$-part, hence we immediately see that (5.141) is no longer basic, so it does not contribute to the cohomology. However, a 2-form as the one in (5.128) appears, making contribution to the cohomology. It follows that the cohomology of the coset $\mathfrak{osp}(4|2)/\mathfrak{u}(2)$ is generated by $\left\{1, \mathcal{R}^{(2)}\right\} \otimes \left\{ \left[(\tilde{\psi} \cdot \psi)^{(\dot{A}\dot{B})} \right]^n \right\}, \forall n \in \mathbb{N}$.
A comment is now mandatory: as we already noticed at the end of the previous paragraph, the "finite part" of the cohomology (again, its bosonic restriction) corresponds to what is left from the bosonic quotient $\mathfrak{so}(4)/\mathfrak{u}(2) \cong \mathfrak{su}(2)/\mathfrak{u}(1)$. We can interpret this result - and the previous one - as follows. Fuks' theorem states that $H^p_{CE,\mathit{dif}}(\mathfrak{osp}(4|2)) = H^p_{CE,\mathit{dif}}(\mathfrak{so}(4))$: when modding out the subalgebra, the bosonic restriction of the finite part is actually given by the coset of the (purely bosonic) subalgebra which is selected by Fuks' theorem. Notice that this holds true as long as we are embedding the subalgebra in the part which actually contributes to the full cohomology of the algebra. Indeed, we have seen in the $\mathfrak{u}(1|1)/\mathfrak{u}(1)$ example that if we embed the divisor subalgebra in the subalgebra not contributing to the cohomology, we obtain a different finite part. Hence, under the discussed assumption, it can be conjectured that, for example, if we consider the superalgebra

$\mathfrak{osp}(n|m)$, given a subalgebra $\mathfrak{h}$, one will find that

$$\left[H^p_{CE}\left(\frac{\mathfrak{osp}(n|m)}{\mathfrak{h}}\right)\right]_{FP} \cong \begin{cases} H^p_{CE}\left(\frac{\mathfrak{so}(n)}{\mathfrak{h}}\right) & \text{, if } n \geq 2m \\ H^p_{CE}\left(\frac{\mathfrak{sp}(m)}{\mathfrak{h}}\right) & \text{, if } n < 2m \end{cases}, \tag{5.142}$$

where the subscript "FP" denotes the finite part of the cohomology. An evidence supporting this claim is provided by the Poincaré polynomials, which can be computed combining Fuks' results with [38], in the case of equal rank pairs as follows

$$\mathcal{P}_{\mathfrak{osp}(n|m)/\mathfrak{h}}(t) = \frac{\mathcal{P}'_{\mathfrak{osp}(n|m)}(t)}{\mathcal{P}'_{\mathfrak{h}}(t)} \tag{5.143}$$

where the prime denotes the augmented power by one for all powers in the Poincaré series. It would be interesting to verify if this holds for cosets of the other basic Lie superalgebras as well as to improve general results comprising also quotient spaces, as in [38].

5.3 Cosets of $\mathfrak{osp}(1|4)$: $D = 4$, $\mathcal{N} = 1$ Anti de Sitter Superspace

It is well-known that the ordinary anti de Sitter spacetimes AdS_D in D-dimensions can be obtained starting from the Lie groups $SO(2, D-1)$ and $SO(1, D-1)$ as the coset manifold $SO(2, D-1)/SO(1, D-1)$, for example the AdS_4 is obtained by taking the quotient of the Lie group $SO(2,3)$ by the Lorentz group $SO(1,3)$ (see [13] for a discussion in relation to supergravity, with Chevalley-Eilenberg cohomology in sight). This construction can be generalized to a coset superspace as to obtain the superspace extension of the anti-de Sitter spacetimes. In this section we will compute the equivariant cohomology of one such construction, namely the $D = 4$, $\mathcal{N} = 1$ anti-de Sitter superspace $AdS_{4|4}$ realized as the quotient supermanifold $OSp(1|4)/SO(1,3)$. At the level of the Lie superalgebras one starts from $\mathfrak{osp}(1|4)$, which is of dimension $10|4$ and whose reduced Lie algebra is $\mathfrak{sp}(4,\mathbb{R})$. Using that $\mathfrak{sp}(4,\mathbb{R}) \cong \mathfrak{so}(2,3)$ one can recover the quotient that yields the anti de Sitter 4-space AdS_4 at the level of the groups. Notice that the quotient manifold $OSp(1|4)/SO(1,3)$ has dimension $4|4$, therefore it is $\mathcal{N} = 1$ (minimal) supersymmetric extension for the AdS_4 and we call it $AdS_{4|4}$. We will denote the corresponding coset of Lie superalgebras $\mathfrak{ads}_{4|4} := \mathfrak{osp}(1|4)/\mathfrak{so}(1,3)$.

Let us start analyzing the Chevalley-Eilenberg cohomology of $\mathfrak{osp}(1|4)$. At the level of the Poincaré polynomial we have

$$\mathcal{P}_{\mathfrak{osp}(1|4)}[t] = \mathcal{P}_{\mathfrak{sp}(4,\mathbb{R})}[t] = 1 - t^3 - t^7 + t^{10}. \tag{5.144}$$

Introducing a set of gamma matrices $\gamma^a_{\alpha\beta}$ for $a = 0, \ldots, 9$ and $\alpha, \beta = 1, \ldots, 4$ we represent the Maurer-Cartan odd forms by bi-spinors as follows

$$\mathcal{V}^a = \gamma^a_{\alpha\beta} \mathcal{V}^{\alpha\beta} \ , \ a = 1, \ldots, 10, \quad \alpha, \beta = 1, \ldots, 4. \tag{5.145}$$

Notice that this is consistent as long as the indices α, β are symmetrized, *i.e.* $\mathcal{V}^{\alpha\beta} = \mathcal{V}^{(\alpha\beta)}$, as to yield 10 components. Further, we use the (standard) symplectic matrix $\Omega_{\alpha\beta}$ and its inverse $\Omega^{\alpha\beta}$ to lower and raise indices. This representation is particularly convenient, as the Maurer-Cartan equations read

$$\begin{aligned} d\mathcal{V}_{\alpha\beta} &= \psi_\alpha \psi_\beta + \left(\mathcal{V}\Omega\mathcal{V}\right)_{\alpha\beta} , \\ d\psi_\alpha &= \left(\mathcal{V}\Omega\psi\right)_\alpha , \end{aligned} \tag{5.146}$$

having introduced also the (even) vielbeins ψ^α. Here we have made use of the notation $\left(\mathcal{V}\Omega\mathcal{V}\right)_{\alpha\beta} = \mathcal{V}_{\alpha\gamma}\Omega^{\gamma\delta}\mathcal{V}_{\delta\beta}$ and $\left(\mathcal{V}\Omega\psi\right)_\alpha = \mathcal{V}_{\alpha\beta}\Omega^{\beta\gamma}\psi_\gamma$. Let us look for the 3-form explicitly: the most general odd 3-form reads

$$\omega^{(3|0)} = c_1 \left(\mathcal{V}^{\alpha_1\beta_1} \Omega_{\beta_1\alpha_2} \mathcal{V}^{\alpha_2\beta_2} \Omega_{\beta_2\alpha_3} \mathcal{V}^{\alpha_3\beta_3} \Omega_{\beta_3\alpha_1} \right) + c_2 \mathcal{V}^{\alpha\beta} \psi_\alpha \psi_\beta , \tag{5.147}$$

where c_1 and c_2 are constants coefficient. We shorten the previous expression by $\omega^{(3)} := c_1 \mathcal{V}^{(3)} + c_2 \mathcal{V}\psi^{(2)}$. By compatibility with the cohomology of the reduced algebra $\mathfrak{sp}(4, \mathbb{R})$ we conclude that $c_1 \neq 0$, and in particular, we put $c_1 = 1$. Imposing the closure condition $d\omega^{(3)} = 0$ we fix the second coefficient:

$$\begin{aligned} 0 = d\omega^{(3)} = 3 \left[\left(\psi^{\alpha_1} \psi^{\beta_1} + \left(\mathcal{V}\Omega\mathcal{V}\right)^{\alpha_1\beta_1} \right) \Omega_{\beta_1\alpha_2} V^{\alpha_2\beta_2} \Omega_{\beta_2\alpha_3} \mathcal{V}^{\alpha_3\beta_3} \Omega_{\beta_3\alpha_1} \right] + \\ + c_2 \left[\psi^\alpha \psi^\beta \psi_\alpha \psi_\beta - 2 \mathcal{V}^{\alpha\beta} \mathcal{V}_{\alpha\gamma} \Omega^{\gamma\delta} \psi_\delta \psi_\beta \right] . \end{aligned} \tag{5.148}$$

Let us look at the terms in this expression: the second term, namely the one proportional to $\mathcal{V}^4$ is zero by trace identity, indeed we can write $\mathcal{V}^3\mathcal{V} = -\mathcal{V}\mathcal{V}^3$, but on the other hand, by cyclicity we have $\mathcal{V}^3\mathcal{V} = \mathcal{V}\mathcal{V}^3$. The third term, namely the one proportional to ψ^4, is zero since $\psi^\alpha\psi_\alpha = \psi^\alpha\Omega_{\alpha\beta}\psi^\beta = 0$, being the ψ's even and Ω antisymmetric. This allows us to fix $c_2 = 3/2$ as to get that the first cancel the last term and obtaining a closed form. Further, in order to show that $\omega^{(3)}$ is not exact, we have to consider the most general even 2-form and show that its Chevalley-Eilenberg differential cannot generate $\omega^{(3)}$. However a crucial observation simplifies the job: we cannot construct a (non-zero) 2-form which is a *singlet*, *i.e.* having all of the indices contracted (the only case would be $V^{\alpha\beta}V_{\alpha\beta} + \psi^\alpha\psi_\alpha$ which is equal to zero, as shown above). Hence we have (after multiplying by an overall factor)

$$H^3_{CE}\left(\mathfrak{osp}\left(1|4\right)\right) = \left\{ \frac{1}{3} \left(V^{\alpha_1\beta_1} \Omega_{\beta_1\alpha_2} V^{\alpha_2\beta_2} \Omega_{\beta_2\alpha_3} V^{\alpha_3\beta_3} \Omega_{\beta_3\alpha_1} \right) + \frac{1}{2} V^{\alpha\beta} \psi_\alpha \psi_\beta \right\} . \tag{5.149}$$

With completely analogous arguments we can construct the most general odd 7-form as

$$\begin{aligned}\omega^{(7)} &= c_1 \left(\mathcal{V}^{\alpha_1\beta_1}\Omega_{\beta_1\alpha_2}\mathcal{V}^{\alpha_2\beta_2}\Omega_{\beta_2\alpha_3}\mathcal{V}^{\alpha_3\beta_3}\Omega_{\beta_3\alpha_4}V^{\alpha_4\beta_4}\Omega_{\beta_4\alpha_5}\mathcal{V}^{\alpha_5\beta_5}\Omega_{\beta_5\alpha_6}\mathcal{V}^{\alpha_6\beta_6}\Omega_{\beta_6\alpha_7}V^{\alpha_7\beta_7}\Omega_{\beta_7\alpha_1}\right) + \\ &+ c_2 \left(V^{\alpha_1\beta_1}\Omega_{\beta_1\alpha_2}\mathcal{V}^{\alpha_2\beta_2}\Omega_{\beta_2\alpha_3}\mathcal{V}^{\alpha_3\beta_3}\Omega_{\beta_3\alpha_4}\mathcal{V}^{\alpha_4\beta_4}\Omega_{\beta_4\alpha_5}\mathcal{V}^{\alpha_5\beta_5}\right)\psi_{\alpha_1}\psi_{\alpha_5} + \\ &+ c_3 \left(\mathcal{V}s^{\alpha_1\beta_1}\Omega_{\beta_1\alpha_2}\mathcal{V}^{\alpha_2\beta_2}\Omega_{\beta_2\alpha_3}\mathcal{V}^{\alpha_3\beta_3}\Omega_{\beta_3\alpha_1}\right)\left(\mathcal{V}^{\alpha_1\mu}\Omega_{\mu\nu}\mathcal{V}^{\nu\alpha_2}\right)\psi_{\alpha_1}\psi_{\alpha_2} + \\ &+ c_4 \left(\mathcal{V}^{\alpha_1\beta}\Omega_{\beta\gamma}\mathcal{V}^{\gamma\alpha_2}\right)\mathcal{V}^{\alpha_3\alpha_4}\psi_{\alpha_1}\psi_{\alpha_2}\psi_{\alpha_3}\psi_{\alpha_4}. \end{aligned} \tag{5.150}$$

We note that we do not have a term of the form $\mathcal{V}\psi^6$ since it would be trivially 0, as can be checked. We can write $\omega^{(7)}$ in a more compact way as

$$\omega^{(7)} = c_1\mathcal{V}^7 + c_2\left(\mathcal{V}^5\right)^{\alpha\beta}\psi_\alpha\psi_\beta + c_3\mathcal{V}^3\left(\mathcal{V}^2\right)^{\alpha\beta}\psi_\alpha\psi_\beta + c_4\left(\mathcal{V}^2\right)^{\alpha\beta}\mathcal{V}^{\gamma\delta}\psi_\alpha\psi_\beta\psi_\gamma\psi_\delta\ , \tag{5.151}$$

where the contractions are omitted. Again by compatibility with the reduced algebra cohomology, we need to have $c_1 \neq 0$. The remaining coefficients c_2, c_3, c_4 can be easily fixed imposing $d\omega^{(7)} = 0$: again, as above, the resulting form will not be exact since it is not possible to have a non-trivial singlet represented by an even 6-form.
Finally the top representative in the cohomology, the form $\omega^{(10)}$ is simply given given by the multiplication

$$\omega^{(10)} = \omega^{(3)} \wedge \omega^{(7)}, \tag{5.152}$$

exploiting the ring structure of the cohomology. Notice that $\omega^{(10)}$ is non-zero since, for example, the term of the form $\mathcal{V}^3 \wedge \mathcal{V}^7$ is non-vanishing, and since either $\omega^{(3)}$ or $\omega^{(7)}$ are closed and non-exact it follows that $\omega^{(10)}$ is closed and non-exact as well.

We now study the equivariant cohomology of the coset superspace $\mathfrak{ads}_{4|4} = \mathfrak{osp}(1|4)/\mathfrak{so}(1,3)$. In order to do so, we have to "split" the Maurer-Cartan forms $\mathcal{V}^{\alpha\beta}$ coming from the $\mathfrak{sp}(4,\mathbb{R}) \subset \mathfrak{osp}(1|4)$ into the *coset* Maurer-Cartan forms (vielbeins) and those coming from $\mathfrak{so}(1,3)$ (connections). Again, making use of the gamma matrices, *i.e.* of the spin structure, we can decompose the vielbeins as

$$\mathcal{V}_{(\alpha\beta)} = \gamma^a_{(\alpha\beta)}\mathcal{V}_a + \gamma^{[ab]}_{(\alpha\beta)}\mathcal{V}_{[ab]}, \tag{5.153}$$

for $a = 1, \ldots, 4$ and $\alpha = 1, \ldots, 4$, where the $\mathcal{V}^a$ are the four vierbein of the coset space that lifts to AdS_4 and $\mathcal{V}_{[ab]}$ are the six vielbeins of $\mathfrak{so}(1,3)$. The Poincaré polynomial can be computed using the result by [38] - notice that both the algebras involved have the same rank, actually 2 - and it reads

$$\mathcal{P}_{\mathfrak{ads}_{4|4}}[t] = \frac{\left(1-t^4\right)\left(1-t^8\right)}{\left(1-t^4\right)^2} = 1 + t^4. \tag{5.154}$$

We therefore expect a single equivariant cohomology class at degree 4, besides the constants. In particular, we expect this to be related to the "volume form" $\omega^{(4)}_{\mathfrak{ads}_4}$ coming from the AdS_4 space. Using the above decomposition (5.153) and the previously obtained Maurer-Cartan equations (5.146) one gets the following Maurer-Cartan equations

$$\begin{aligned} \mathcal{D}\mathcal{V}_a &= \psi\gamma_a\psi\,, \\ \mathcal{D}\psi_\alpha &= \mathcal{V}_a\gamma^a\psi\,, \\ R_{ab} \equiv d\mathcal{V}_{[ab]} + (\mathcal{V}\wedge\mathcal{V})_{[ab]} &= \psi\gamma_{[ab]}\psi, \end{aligned} \tag{5.155}$$

where the covariant derivative $\mathcal{D}$ is with respect to the connection $\mathcal{V}_{[ab]}$ of the subgroup $\mathfrak{so}(1,3)$.

Working as above, we have that the most general even 4-singlet reads

$$\omega^{(4|0)} = c_1\epsilon_{abcd}\mathcal{V}^a\mathcal{V}^b\mathcal{V}^c\mathcal{V}^d + c_2\mathcal{V}^a\mathcal{V}^b\left(\psi\gamma_{ab}\psi\right). \tag{5.156}$$

Notice that there cannot be terms of the form $\psi^4 = \left(\psi\gamma^{ab}\psi\right)\left(\psi\gamma_{ab}\psi\right)$, since they vanish because of the Fierz identities. As above, we have that $c_1 \neq 0$ by compatibility with the cohomology of the reduced algebra $\omega^{(4)}_{\mathfrak{ads}_4} = \epsilon_{abcd}V^aV^bV^cV^d$. Hence we can fix $c_1 = 1$ without loss of generality. The coefficient c_2 is fixed by imposing that $\mathcal{D}\omega^{(4)} = 0$:

$$0 = \mathcal{D}\omega^{(4)} = 4\epsilon_{abcd}\left(\psi\gamma^a\psi\right)\mathcal{V}^b\mathcal{V}^c\mathcal{V}^d + 2c_2\left[\psi\gamma^a\psi\mathcal{V}^b\left(\psi\gamma_{ab}\psi\right) + \mathcal{V}^a\mathcal{V}^b\left(\left(\mathcal{V}^c\gamma_c\psi\right)\gamma_{ab}\psi\right)\right]\,. \tag{5.157}$$

The second term in the sum vanishes because of Fierz identities, while the last term, after using γ matrices properties, cancels the first one upon fixing $c_2 = -2$. Finally, we can conclude that $\omega^{(4)}$ is not exact, since, once again, it is not possible to write an odd 3-singlet that generates the term $\mathcal{V}^4$. Hence we have

$$H^4_{\mathcal{EQ}}\left(\mathfrak{ads}_{4|4}\right) = \mathbb{R}\cdot\left\{\epsilon_{abcd}\mathcal{V}^a\mathcal{V}^b\mathcal{V}^c\mathcal{V}^d - 2\mathcal{V}^a\mathcal{V}^b\left(\psi\gamma_{ab}\psi\right)\right\}. \tag{5.158}$$

All in all we have:

$$H^p_{\mathcal{EQ}}\left(\mathfrak{ads}_{4|4}\right) = \begin{cases} \mathbb{R} & p = 0,4 \\ 0 & \text{else.} \end{cases} \tag{5.159}$$

We conclude with the integral form Chevalley-Eilenberg cohomology. As discussed in the previous section, by the isomorphism, we have two cohomology classes at picture four, the maximal picture degree. They have the explicit expressions

$$\begin{aligned} H^{(0|4)}\left(\mathfrak{ads}_{4|4}\right) &= \mathbb{R}\cdot\left\{2\mathcal{V}^a\mathcal{V}^b\iota_{\pi Q}\gamma_{ab}\iota_{\pi Q}\delta^4(\psi) + \delta^4(\psi)\right\}, \\ H^{(4|4)}\left(\mathfrak{ads}_{4|4}\right) &= \mathbb{R}\cdot\left\{\epsilon_{abcd}\mathcal{V}^a\mathcal{V}^b\mathcal{V}^c\mathcal{V}^d\delta^4(\psi)\right\}. \end{aligned} \tag{5.160}$$

6 Conclusions and Outlook

The present work spawns from the observation that since Lie superalgebra cohomology is nothing but a straightforward generalization of ordinary Lie algebra cohomology, it is not capable to account for objects different than differential forms on the corresponding Lie supergroup. On the other hand, it is well-known that in order to make a meaningful connection with integration theory, when working on supermanifolds, differential forms have to be supplemented by integral forms, whose geometry is not at all captured by Chevalley-Eilenberg cohomology.

To this end, after reviewing Chevalley-Eilenberg cohomology for ordinary Lie algebras and Lie superalgebras and their relations to forms on the corresponding Lie groups or Lie supergroups, we extend the notion of Chevalley-Eilenberg cochains to include also *integral forms* and we define a corresponding cohomology. We thus show a duality between the ordinary Chevalley-Eilenberg cohomology for a certain Lie superalgebra - which looks at forms on the corresponding Lie supergroup - and this newly defined (Chevalley-Eilenberg) cohomology accounting for integral forms instead. We observe that most notably - and differently from de Rham cohomology -, this cohomology always features the true analogue of a top-form, a Berezinian form appearing in the integral form complex.

Nonetheless, besides general results, a great deal of focus in this paper is on explicit direct computations: in particular, we provide explicit expressions for cocycles of Lie superalgebras of physical interest, namely supertranslations of flat superspaces and classical Lie superalgebras, up to dimension 4, in terms of their Maurer-Cartan forms.

The second part of the paper is devoted to equivariant Chevalley-Eilenberg cohomology, which is related to the (super)symmetries of coset supermanifolds, which provides very important backgrounds for supergravity and superstring theories. Again, several examples up to dimension 4 are studied and explicit expressions for their cocycles are provided, culminating with the case of super anti-de Sitter space $AdS_{4|4}$. Here, a mixture of techniques has been exploited, spanning from Poincaré polynomials computations for equal rank pairs to brute force computations.

We remark that our analysis has uncovered new cocycles spawning from fermionic generators - both in ordinary and equivariant Chevalley-Eilenberg cohomology - and several characteristic examples of infinite-dimensional cohomology. In hoping that the present results might come useful to understand the geometry of supergravity and string backgrounds and the mathematics behind it, we stress though, that this research scenario looking at relating Chevalley-Eilenberg cohomology and the extended geometry of forms on supermanifolds is far from being exhausted. Indeed, in the present paper, the so-called pseudoforms [25] have not been addressed, even if nonetheless they play an important role both in the integration theory on superspaces and in its applications: it is legit to ask if they can be fitted in the picture we have presented and which role they play.

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A A Primer of Integral Forms on Supermanifolds

Given a supermanifold $\mathcal{M}$, say of dimension $n|m$, differential forms in $\Omega^\bullet(\mathcal{M})$ are not enough to define a coherent notion of integration on $\mathcal{M}$. This leads to the introduction of *integral forms*, which are geometrically as important as differential forms, see [47] and the recent papers [11, 14, 15, 17, 18, 19, 20, 21, 23, 24, 26, 27, 36, 50]. Loosely speaking, whereas differential forms lead to a consistent geometric integration on ordinary bosonic submanifolds (*i.e.* sub-manifolds of codimension $p|m$) in $\mathcal{M}$, integral forms play the same role on sub-supermanifolds of codimension $p|0$ in $\mathcal{M}$, and in particular, they control integration on $\mathcal{M}$ itself. Notice that, even if it is often left understood or not stated, integral forms are ubiquitous in theoretical high energy physics: for example, the Lagrangian density of a supersymmetric theory in superspace is indeed a *top* integral form. There are (at least) two ways to introduce integral forms, which we now briefly recall.

The first approach is to define integral forms as *generalized functions* on $\mathrm{Tot}\,\Pi\mathcal{T}(\mathcal{M})$ [57], that is elements $\omega(x^1, \ldots, x^n, d\theta^1, \ldots, d\theta^m | \theta^1, \ldots, \theta^m, dx^1, \ldots dx^n) \in \Pi\mathcal{T}(\mathcal{M})$, where $x^i|\theta^\alpha$ are local coordinate for $\mathcal{M}$, which only allows a *distributional dependence* supported in $d\theta^1 = \ldots = d\theta^m = 0$. Algebraically, integral forms can be (roughly) described as $\Omega^\bullet(\mathcal{M})$-modules generated over the set (of *Dirac delta* distributions and their derivatives) $\{\delta^{(r_1)}(d\theta^1) \wedge \ldots \wedge \delta^{(r_m)}(d\theta^m)\}$, for $r_i \geq 0$, together with the defining relations

$$d\theta^\alpha \delta^{(k)}(d\theta^\alpha) = -k\delta^{(k-1)}(d\theta_\alpha) \quad \text{for} \quad k \geq 0 \tag{A.161}$$

for any $\alpha = 1, \dots, m$ and any $k \geq 0$, which are deduced analytically by integration by parts. Notice that the case $k = 0$ tells that the expressions $d\theta^\alpha \delta^{(0)}(d\theta^\alpha)$ vanishes, so that the presence of the delta's can be seen as a localization in the locus $d\theta^\alpha = 0$ in $\mathrm{Tot}\,\Pi\mathcal{T}(\mathcal{M})$. Locally, an integral form ω_{int} is written as a (generalized) tensor

$$\omega_{int}(x, d\theta|\theta, dx) = \\ = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{a_i\in\{0,1\}, r_j\geq 0} \omega_{[a_1\dots a_m r_1 \dots r_m]}(x|\theta)(dx^1)^{a_1}\dots(dx^n)^{a_m}\delta^{(r_1)}(d\theta^1)\dots\delta^{(r_m)}(d\theta^m), \tag{A.162}$$

where all indices are antisymmetric (recalling that two delta's anticommute with each other), and where we note that there cannot be $d\theta$'s thanks to the above relations (A.161). We will say that an integral forms has *picture* m, to mean that we are considering expressions that admits only a distributional dependence on *all* of the m coordinates $d\theta^1, \dots, d\theta^m$ on $\mathrm{Tot}\,\Pi\mathcal{T}(\mathcal{M})$. Further, with reference to the previous expression A.162, we assign a *degree* to an integral form according to the definition

$$\deg(\omega_{int}) := \sum_{i=1}^{n} a_j - \sum_{j=1}^{m} r_j, \tag{A.163}$$

so that we will say that an integral form has picture m and degree $p \leq n$. In particular, a *top* integral form is an integral form of degree n,

$$\omega_{int}^{top} = \omega(x|\theta)dx^1\dots dx^n\delta(d\theta^1)\dots\delta(d\theta^m), \tag{A.164}$$

and it can be checked that this expression has the transformation properties of a section of the Berezinian line bundle $\mathcal{B}er(\mathcal{M}) := \mathcal{B}er(\Pi\mathcal{T}^*(\mathcal{M}))$ of the supermanifold $\mathcal{M}$. Notice that all of the integral forms as in A.162 can be generated from the above A.164 by repeatedly acting with *contractions* along (coordinate) vector fields, *i.e.*

$$\omega_{int}^{n-\ell} = \iota_{X_1}\dots\iota_{X_\ell}\,\omega_{int}^{top}, \tag{A.165}$$

where we recall that in particular, for the coordinate vector fields $\partial_{x^i}|\partial_{\theta^\alpha}$ one has that $|\iota_{\partial_{x^i}}| = 1$ and $|\iota_{\partial_{\theta^\alpha}}| = 0$. The modules of integral forms are then structured into a complex letting d operate as the usual de Rham differential on $\Omega^\bullet(\mathcal{M})$ and declaring that its action on the delta's, is trivial *i.e.* posing $d(\delta(d\theta^\alpha)) = 0$ for any α.

In the second approach one defines integral forms of degree p as sections of the vector bundle on $\mathcal{M}$

$$\Sigma^p(\mathcal{M}) := \mathcal{B}er(\mathcal{M}) \otimes_{\mathcal{O}_\mathcal{M}} (\Omega^{n-p}(\mathcal{M}))^* = \mathcal{B}er(\mathcal{M}) \otimes_{\mathcal{O}_\mathcal{M}} S^{n-p}(\Pi\mathcal{T}(\mathcal{M})). \tag{A.166}$$

where $\mathcal{B}er(\mathcal{M})$ is the Berezinian line bundle of $\mathcal{M}$ and $\Pi\mathcal{T}(\mathcal{M})$ the parity-reversed tangent bundle. The correspondence between integral forms in the different representations reads

$$\omega^{(n-\ell)} = \mathcal{D} \otimes (\pi X_1 \odot \ldots \odot \pi X_\ell) \;\; \leftrightsquigarrow \;\; \omega^{(n-\ell)} = \iota_{X^1} \ldots \iota_{X^\ell} \omega^{top}_{int} \tag{A.167}$$

where $\mathcal{D}$ is a section of $\mathcal{B}er(\mathcal{M})$ and $\pi X_1 \odot \ldots \odot \pi X_\ell$ is a section of $S^\ell \Pi\mathcal{T}(\mathcal{M})$, together with the correspondence of sections of Berezinian line bundle, or integral top forms, mentioned above, *i.e.* $\omega^{top}_{int} \leftrightsquigarrow \mathcal{D}$. Clearly, given the above tensor product structure, defining a nilpotent differential acting as $\delta^p : \Sigma^p(\mathcal{M}) \to \Sigma^{p+1}(\mathcal{M})$ is not at all trivial matter, as originally discussed in [47] and recently realized in [11], but this can be done as getting a complex which will in general be *unbounded from below*

$$\ldots \longrightarrow \mathcal{B}er(\mathcal{M}) \otimes S^{n-p}(\Pi\mathcal{T}(\mathcal{M})) \longrightarrow \ldots \longrightarrow \mathcal{B}er(\mathcal{M}) \otimes \Pi\mathcal{T}(\mathcal{M}) \longrightarrow \mathcal{B}er(\mathcal{M}) \longrightarrow 0. \tag{A.168}$$

Remarkably, these different approaches, which agree in terms of general results, complement each other. If on one hand, this second approach is probably more suitable when it comes to dealing with mathematical and foundational issues where well-definiteness is crucial, on the other hand, the first approach proves quite efficient when it comes to actual computations. The different nature of these two approaches is mirrored, for example, in the proof of which is probably the most important result in the theory, *i.e.* the (natural) isomorphism between the cohomology of differential forms $H^p_d(\Omega^\bullet(\mathcal{M}))$ and integral forms $H^p_\delta(\Sigma^\bullet(\mathcal{M}))$ on supermanifolds, namely introducing in the first approach the crucial notion of *Picture Changing Operators* (see, e.g., [20]), which maps differential to integral forms and vice-versa, and via a spectral sequence argument in the second approach [11].

B Explicit Expressions

B.1 Dimension 1

In this section, we collect explicit expressions for cohomology representatives of $\mathit{susy}(\mathbb{R}^{1|2})$ as described in Sec. 4.1.1:

$$H^p_{CE,dif}(\mathit{susy}(\mathbb{R}^{1|2})) \cong \mathbb{R} \cdot \left\{ \left(\psi^1\right)^p, \left(\psi^2\right)^p \right\} \;, \; \forall p \in \mathbb{N} \; . \tag{B.169}$$

The dual integral form cohomology groups are generated by

$$\begin{aligned} H^{1-p}_{CE,int}(\mathit{susy}(\mathbb{R}^{1|2})) &\cong \mathrm{Ber}^{\mathcal{H}} \left(\mathit{susy}(\mathbb{R}^{1|2}) \right) \cdot \{\pi\chi_1^p, \pi\chi_2^p\} \equiv \\ &\equiv \mathbb{R} \cdot \left\{ \iota^p_{\chi_1}, \iota^p_{\chi_2} \right\} \mathcal{V}\delta\left(\psi^1\right)\delta\left(\psi^2\right) \;, \; \forall p \in \mathbb{N} \;, \end{aligned} \tag{B.170}$$

where χ_α are the vector fields dual to $\pi\psi^\alpha$.

B.2 Dimension 2

In this section we collect explicit expressions for cohomology representatives of $\mathit{susy}(\mathbb{R}^{1,1|2})$ as described in Sec. 4.1.2:

$$H^0_{CE,dif}(\mathit{susy}(\mathbb{R}^{1,1|2})) \cong \mathbb{R}\cdot 1, \qquad H^1_{CE,dif}(\mathit{susy}(\mathbb{R}^{1,1|2})) \cong \mathbb{R}\cdot\{\psi^1,\psi^2\}$$
$$H^2_{CE,dif}(\mathit{susy}(\mathbb{R}^{1,1|2})) \cong \mathbb{R}\cdot\left\{\sum_{\alpha=1}^{2}(\psi^\alpha)^2\right\}, \qquad H^{p>2}_{CE,dif}(\mathit{susy}(\mathbb{R}^{1,1|2})) \cong 0. \tag{B.171}$$

Analogously, the integral form cohomology representatives are given by

$$\begin{aligned}
&H^2_{CE,int}(\mathit{susy}(\mathbb{R}^{1,1|2})) \cong \mathbb{R}\cdot \mathcal{V}^1\mathcal{V}^2\delta(\psi^1)\delta(\psi^2)\ ,\\
&H^1_{CE,int}(\mathit{susy}(\mathbb{R}^{1,1|2})) \cong \mathbb{R}\cdot\{\iota_{\chi_1}\mathcal{V}^1\mathcal{V}^2\delta(\psi^1)\delta(\psi^2), \iota_{\chi_2}\mathcal{V}^1\mathcal{V}^2\delta(\psi^1)\delta(\psi^2)\}\\
&H^0_{CE,int}(\mathit{susy}(\mathbb{R}^{1,1|2})) \cong \mathbb{R}\cdot\left\{\left(\sum_{\alpha=1}^{2}(\iota_{\chi_\alpha})^2\right)\mathcal{V}^1\mathcal{V}^2\delta(\psi^1)\delta(\psi^2)\right\}\ ,\\
&H^{p<0}_{CE,int}(\mathit{susy}(\mathbb{R}^{1,1|2})) \cong 0\ ,
\end{aligned} \tag{B.172}$$

where χ_α are the vector fields dual to $\pi\psi^\alpha$.

B.3 Dimension 3

In this section we collect explicit expressions for cohomology representatives of $\mathit{susy}(\mathbb{R}^{1,2|2})$ as described in Sec. 4.1.3:

$$\begin{aligned}
&H^0_{CE,dif}(\mathit{susy}(\mathbb{R}^{1,2|2})) \cong \mathbb{R}\cdot 1,\\
&H^1_{CE,dif}(\mathit{susy}(\mathbb{R}^{1,2|2})) \cong \mathbb{R}\cdot\{\psi^\alpha\},\\
&H^2_{CE,dif}(\mathit{susy}(\mathbb{R}^{1,2|2})) \cong \mathbb{R}\cdot\left\{\mathcal{V}^a\gamma_{a,\alpha\beta}\psi^\beta\right\},\\
&H^3_{CE,dif}(\mathit{susy}(\mathbb{R}^{1,2|2})) \cong \mathbb{R}\cdot\left\{\mathcal{V}^a\psi^\alpha\gamma_{a,\alpha\beta}\psi^\beta\right\},\\
&H^{p>3}_{CE,dif}(\mathit{susy}(\mathbb{R}^{1,2|2})) \cong 0\ .
\end{aligned} \tag{B.173}$$

In particular, we want to stress that the $(3|0)$-form generating $H^3_{CE,dif}(\mathit{susy}(\mathbb{R}^{1,2|2}))$ is the well-known (see, e.g., [4, 7, 9, 10, 29]) Lorentz-invariant $(3|0)$-form of the coset $\mathfrak{siso}(1,2|N=1)/\mathfrak{so}(1,2)$. This class is of particular interest as it can be related to the construction of Wess-Zumino terms for the superstring (see, e.g., [37]).

The integral form cohomology representatives are given by

$$
\begin{aligned}
&H^3_{CE,int}(\mathit{susy}(\mathbb{R}^{1,2|2})) \cong \mathbb{R}\cdot\{\mathcal{V}^0\mathcal{V}^1\mathcal{V}^2\delta(\psi^1)\delta(\psi^2)\},\\
&H^2_{CE,int}(\mathit{susy}(\mathbb{R}^{1,1|2})) \cong \mathbb{R}\cdot\{\mathcal{V}^0\mathcal{V}^1\mathcal{V}^2\iota_{\chi_\alpha}\delta(\psi^1)\delta(\psi^2)\}\\
&H^1_{CE,int}(\mathit{susy}(\mathbb{R}^{1,1|2})) \cong \mathbb{R}\cdot\left\{\mathcal{V}^a\mathcal{V}^b\gamma_{ab,\alpha\beta}\iota_{\chi_\beta}\delta(\psi^1)\delta(\psi^2)\right\},\\
&H^0_{CE,int}(\mathit{susy}(\mathbb{R}^{1,1|2})) \cong \mathbb{R}\cdot\left\{\mathcal{V}^a\mathcal{V}^b\iota_{\chi_\alpha}\gamma^{\alpha\beta}_{ab}\iota_{\chi_\beta}\delta(\psi^1)\delta(\psi^2)\right\},\\
&H^{p<0}_{CE,int}(\mathit{susy}(\mathbb{R}^{1,1|2})) \cong 0,
\end{aligned}
\tag{B.174}
$$

where χ_α are the vector fields dual to $\pi\psi^\alpha$. Again, the Lorentz-invariant $(0|2)$-integral form is of particular interest, as it represents the supersymmetric Picture Changing Operator of the flat, rigid superspace $\mathfrak{siso}(1,2|N=1)/\mathfrak{so}(1,2)$. In [14, 23, 24, 36], some applications of this class are described.

B.4 Dimension 4

In this section, we collect explicit expressions for cohomology representatives of $\mathit{susy}(\mathbb{R}^{1,3|4})$ as described in Sec. 4.1.4. We will restrict to the first cohomology groups, general expressions for any cohomology groups can be easily calculated analogously:

$$
\begin{aligned}
&H^0_{CE,dif}(\mathit{susy}(\mathbb{R}^{1,3|4})) \cong \mathbb{R}\cdot 1,\\
&H^1_{CE,dif}(\mathit{susy}(\mathbb{R}^{1,3|4})) \cong \mathbb{R}\cdot\{\chi^\alpha,\ \bar\lambda^{\dot\alpha}\}\\
&H^2_{CE,dif}(\mathit{susy}(\mathbb{R}^{1,3|4})) \cong \mathbb{R}\cdot\{\chi^\alpha\chi^\beta,\ \bar\lambda^{\dot\alpha}\bar\lambda^{\dot\beta},\ \chi^\alpha\epsilon_{\alpha\beta}\mathcal{V}^{\beta\dot\beta},\ \bar\lambda^{\dot\alpha}\epsilon_{\dot\alpha\dot\beta}\mathcal{V}^{\beta\dot\beta}\},\\
&H^3_{CE,dif}(\mathit{susy}(\mathbb{R}^{1,3|4})) \cong \mathbb{R}\cdot\{\chi^\alpha\chi^\beta\chi^\gamma,\ \bar\lambda^{\dot\alpha}\bar\lambda^{\dot\beta}\bar\lambda^{\dot\gamma},\ \chi^\gamma\chi^\alpha\epsilon_{\alpha\beta}\mathcal{V}^{\beta\dot\beta},\ \bar\lambda^{\dot\gamma}\bar\lambda^{\dot\alpha}\epsilon_{\dot\alpha\dot\beta}\mathcal{V}^{\beta\dot\beta},\ \chi^\alpha\bar\lambda^{\dot\alpha}\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\mathcal{V}^{\beta\dot\beta}\},\\
&H^4_{CE,dif}(\mathit{susy}(\mathbb{R}^{1,3|4})) \cong \mathbb{R}\cdot\{\chi^\alpha\chi^\beta\chi^\gamma\chi^\delta,\ \bar\lambda^{\dot\alpha}\bar\lambda^{\dot\beta}\bar\lambda^{\dot\gamma}\bar\lambda^{\dot\delta},\ \mathcal{V}^{\alpha\dot\alpha}\epsilon_{\alpha\beta}\mathcal{V}^{\beta\dot\beta}\epsilon_{\dot\alpha\dot\gamma}\epsilon_{\dot\beta\dot\delta}\bar\lambda^{\dot\gamma}\bar\lambda^{\dot\delta},\\
&\qquad \mathcal{V}^{\alpha\dot\alpha}\epsilon_{\dot\alpha\dot\beta}\mathcal{V}^{\beta\dot\beta}\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}\chi^\gamma\chi^\delta,\ \chi^\delta\chi^\gamma\chi^\alpha\epsilon_{\alpha\beta}\mathcal{V}^{\beta\dot\beta},\ \bar\lambda^{\dot\delta}\bar\lambda^{\dot\gamma}\bar\lambda^{\dot\alpha}\epsilon_{\dot\alpha\dot\beta}\mathcal{V}^{\beta\dot\beta}\},\ \ldots
\end{aligned}
\tag{B.175}
$$

The $(3|0)$-form $\chi^\alpha\bar\lambda^{\dot\alpha}\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\mathcal{V}^{\beta\dot\beta}$ and the two $(4|0)$-forms $\omega^{(4|0)}_{chir} = \mathcal{V}^{\alpha\dot\alpha}\epsilon_{\alpha\beta}\mathcal{V}^{\beta\dot\beta}\epsilon_{\dot\alpha\dot\gamma}\epsilon_{\dot\beta\dot\delta}\bar\lambda^{\dot\gamma}\bar\lambda^{\dot\delta}$ and ${}^{(4|0)}_{antichir} = \mathcal{V}^{\alpha\dot\alpha}\epsilon_{\dot\alpha\dot\beta}\mathcal{V}^{\beta\dot\beta}\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}\chi^\gamma\chi^\delta$[3] are the Lorentz-invariant cocycles of the superspace $\mathfrak{siso}(1,3|N=1)/\mathfrak{so}(1,3)$. These cocycles are used, e.g., to construct higher WZ terms and the corresponding branes (see, e.g., [1, 4, 29]). The subscripts for the two $(4|0)$-forms indicate that they are *chiral* and *antichiral*, respectively:

$$
\mathcal{L}_{\bar D_{\dot\alpha}}\omega^{(4|0)}_{chir} = 0\ ,\ \mathcal{L}_{D_\alpha}\omega^{(4|0)}_{antichir} = 0\ ,\ \forall\alpha,\dot\alpha\ ,
\tag{B.176}
$$

[3]Usually, one considers only the real linear combination of the two $(4|0)$-forms.

where D_α is the dual of $\pi\chi^\alpha$ and $\bar{D}_{\dot{\alpha}}$ is the dual of $\pi\bar{\lambda}^{\dot{\alpha}}$. We will comment further on this in a while.

The integral form representatives for the cohomology groups are given by a twist via the Berezinian, as explained in Sect. 2.2.5. The Berezinian can be explicitly realised as

$$\mathcal{D}_{susy(\mathbb{R}^{1,3|4}))} = \mathcal{V}^{\alpha\dot{\alpha}}\epsilon_{\alpha\beta}\mathcal{V}^{\beta\dot{\beta}}\epsilon_{\dot{\beta}\dot{\gamma}}\mathcal{V}^{\gamma\dot{\gamma}}\epsilon_{\gamma\delta}\mathcal{V}^{\delta\dot{\delta}}\epsilon_{\dot{\delta}\dot{\alpha}}\delta(\chi^\mu)\epsilon_{\mu\nu}\delta(\chi^\nu)\delta(\bar{\lambda}^{\dot{\mu}})\epsilon_{\dot{\mu}\dot{\nu}}\delta(\bar{\lambda}^{\dot{\nu}}) \ . \tag{B.177}$$

We can immediately introduce Lorentz-invariant integral forms, corresponding to the Lorentz-invariant superforms as

$$\begin{aligned} \omega^{(1|4)} &= (\mathcal{V}\mathcal{V}\mathcal{V})^{\alpha\dot{\alpha}}\,\iota_\alpha\left(\delta(\chi^\mu)\epsilon_{\mu\nu}\delta(\chi^\nu)\right)\bar{\iota}_{\dot{\alpha}}\left(\delta(\bar{\lambda}^{\dot{\mu}})\epsilon_{\dot{\mu}\dot{\nu}}\delta(\bar{\lambda}^{\dot{\nu}})\right) \ , \\ \omega^{(0|4)}_{antichir} &= (\mathcal{V}\mathcal{V})^{\alpha\beta}\,\iota_\alpha\iota_\beta\left(\delta(\chi^\mu)\epsilon_{\mu\nu}\delta(\chi^\nu)\right)\left(\delta(\bar{\lambda}^{\dot{\mu}})\epsilon_{\dot{\mu}\dot{\nu}}\delta(\bar{\lambda}^{\dot{\nu}})\right) \ , \\ \omega^{(0|4)}_{chir} &= (\mathcal{V}\mathcal{V})^{\dot{\alpha}\dot{\beta}}\left(\delta(\chi^\mu)\epsilon_{\mu\nu}\delta(\chi^\nu)\right)\bar{\iota}_{\dot{\alpha}}\bar{\iota}_{\dot{\beta}}\left(\delta(\bar{\lambda}^{\dot{\mu}})\epsilon_{\dot{\mu}\dot{\nu}}\delta(\bar{\lambda}^{\dot{\nu}})\right) \ . \end{aligned} \tag{B.178}$$

The subscripts on the two (0|4)-forms stand, as for their duals, for chiral and anti-chiral:

$$\mathcal{L}_{D_\alpha}\omega^{(0|4)}_{antichir} = 0 \ , \ \mathcal{L}_{\bar{D}_{\dot{\alpha}}}\omega^{(0|4)}_{chir} = 0 \ , \ \forall \alpha, \dot{\alpha} \ . \tag{B.179}$$

These forms and their duals are related to the Weyl / anti-Weyl decomposition of the $N = 1, D = 4$ superspace and, in particular, they can be related to the notions of *(anti-)chiral Berezinian* and *(anti-)chiral Picture Changing Operator*. Really, if one considers only the chiral or the anti-chiral sector of even differential forms, one can find four picture-2 cohomology classes:

$$\omega^{(4|2)}_{chir} \equiv \mathcal{B}er_{chir} = \mathcal{V}^{\alpha\dot{\alpha}}\epsilon_{\alpha\beta}\mathcal{V}^{\beta\dot{\beta}}\epsilon_{\dot{\beta}\dot{\gamma}}\mathcal{V}^{\gamma\dot{\gamma}}\epsilon_{\gamma\delta}\mathcal{V}^{\delta\dot{\delta}}\epsilon_{\dot{\delta}\dot{\alpha}}\delta(\chi^\mu)\epsilon_{\mu\nu}\delta(\chi^\nu) \ , \tag{B.180}$$

$$\omega^{(4|2)}_{antichir} \equiv \mathcal{B}er_{antichir} = \mathcal{V}^{\alpha\dot{\alpha}}\epsilon_{\alpha\beta}\mathcal{V}^{\beta\dot{\beta}}\epsilon_{\dot{\beta}\dot{\gamma}}\mathcal{V}^{\gamma\dot{\gamma}}\epsilon_{\gamma\delta}\mathcal{V}^{\delta\dot{\delta}}\epsilon_{\dot{\delta}\dot{\alpha}}\delta(\bar{\lambda}^{\dot{\mu}})\epsilon_{\dot{\mu}\dot{\nu}}\delta(\bar{\lambda}^{\dot{\nu}}) \ , \tag{B.181}$$

$$\omega^{(0|2)}_{antichir} = \delta(\bar{\lambda}^{\dot{\mu}})\epsilon_{\dot{\mu}\dot{\nu}}\delta(\bar{\lambda}^{\dot{\nu}}) \ , \ \omega^{(0|2)}_{chir} = \delta(\chi^\mu)\epsilon_{\mu\nu}\delta(\chi^\nu) \ . \tag{B.182}$$

The forms constructed with a non-zero and non-maximal number of δs are called *pseudo-forms* (see, e.g., [57]); the rigorous construction of pseudoforms and their cohomology on Lie superalgebras is studied in [25]. These forms have a wide in many physical contexts: in $N = 1, D = 4$ supergravity, the action is naturally written as a sum of a chiral and an anti-chiral action (see, e.g., [55]); they have been studied for gauge theories and WZ theory, e.g., in [16].

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