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Clifford Tori and the singularly perturbed Cahn–Hilliard equation

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Abstract

In this paper we construct entire solutions u_ε to the Cahn–Hilliard equation $-\varepsilon^2\Delta(-\varepsilon^2\Delta u + W'(u)) + W''(u)(-\varepsilon^2\Delta u + W'(u)) = \varepsilon^4\lambda_\varepsilon(1-u_\varepsilon)$, under the volume constraint $\int_{\mathbb{R}^3}(1-u_\varepsilon)^2 dx = 8\sqrt{2}\pi^2 c_\varepsilon$, with $c_\varepsilon \to 1$ as $\varepsilon \to 0$, whose nodal set approaches the Clifford Torus, that is the Torus with radii of ratio $1/\sqrt{2}$ embedded in $\mathbb{R}^3$, as $\varepsilon \to 0$. It is crucial that the Clifford Torus is a Willmore hypersurface and it is non-degenerate, up to conformal transformations. The proof is based on the Lyapunov–Schmidt reduction and on careful geometric expansions of the Laplacian.

Keywords: Lyapunov–Schmidt reduction; Cahn–Hilliard equation; Willmore surface; Clifford Torus

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1. Introduction

The Allen–Cahn equation

$$-\varepsilon^2 \Delta u = u - u^3, \tag{1}$$

arises in several physical contexts, such as the study of the stable configurations of two different fluids confined in a bounded container Ω. If $u(x)$ is the density of one of the two fluids at a point $x \in \Omega$ and the energy per unit volume is given by a function W of u, it looks reasonable to obtain stable configurations by minimizing the energy functional

$$E(u) = \int_\Omega W(u)dx$$

among all distributions fulfilling the volume constraint

$$\int_\Omega u dx = m. \tag{2}$$

If, for instance, $W(u) = (1-u^2)^2$, and $m \in (-1, 1)$, any piecewise constant function taking only the values ± 1 and satisfying (2) is a minimizer, irrespectively of the shape of the interface. Therefore this model is unsatisfactory, since it is very far from the reasonable physical assumption that the interfaces are area minimizers, so one replaces the energy by

$$E_\varepsilon(u) = \int_\Omega \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{(1-u^2)^2}{4\varepsilon} \right) dx.$$

We can see that there is a competition between the potential energy, that forces u to be close to ± 1, and the gradient term that penalizes the phase transition. By minimizing this functional, we are looking for the physical interfaces in which the phase transition can occur.

The minimizers u_ε of E_ε are solutions to the Euler Lagrange equation, that is (1). In order to see if the interfaces are actually minimal surfaces, it is interesting to study the asymptotic behaviour of the level sets $\{u_\varepsilon = c\}$ as the parameter $\varepsilon \to 0$. It is useful to exploit the variational structure of the problem. It was shown by Modica and Mortola that the energy E_ε, seen as a functional on $L^1(\Omega)$ and extended to be $+\infty$ when the integrand is not an L^1 function, Γ-converges to the functional

$$E(u) = \begin{cases} cPer_\Omega(\{u=1\}) & \text{if } u = \pm 1 \text{ a.e. in } \Omega \\ +\infty & \text{otherwise in } L^1(\Omega) \end{cases}$$

in the strong topology of $L^1(\Omega)$ (see [21]), where $c > 0$ is a suitable constant.

Moreover, Modica showed that, if u_ε are minimizers of F_ε under the volume constraint

$$\int_\Omega u_\varepsilon dx = m,$$

for some $m \in (-1, 1)$, then there exists a sequence $\varepsilon_k \to 0$ such that u_{ε_k} converges to some function u in $L^1(\Omega)$ (see proposition 3 of [20]). Furthermore, Theorem 1 of [20] asserts that $u = \pm 1$ a.e. in Ω, and the set

$$E = \{x \in \Omega : u(x) = 1\}$$

is actually a perimeter minimizer between all the subsets $F \subset \Omega$ satisfying the volume constraint

$$|F| = \frac{|\Omega| + m}{2}.$$

Further results about the relation between the minimizers of E_ε and the minimizers of the perimeter can be found in [20] and in [6], where Choksi and Sternberg also described the relation between phase transition theory and the study of a certain kind of polymers.

Conversely, it is an interesting problem to understand if any minimal hypersurface can be achieved as the limit of nodal sets of minimizers of the Ginzburg–Landau energy E_ε.

The first result in this direction is due to Kohn and Sternberg (see [15]). They considered a smooth bounded domain $\Omega \subset \mathbb{R}^2$ and, as an interface, a disjoint union of segments l_i meeting the boundary $\partial\Omega$ orthogonally. They defined u_0 to be locally constant on $\Omega \setminus \cup_i l_i$, taking the values ± 1, and constructed a sequence of minimizers u_ε converging to u_0 in $L^1(\Omega)$.

In [24], Pacard and Ritoré proved a more general result, that holds true for a larger class of interfaces. They started from a minimal hypersurface Σ in a compact Riemannian manifold M and, under suitable assumptions, they showed that it can be achieved as the limit as $\varepsilon \to 0$ of nodal sets (that is 0-level sets) of solutions u_ε of the rescaled Allen–Cahn equation (1). These solutions u_ε were constructed with techniques such as fixed point theorems and the Lyapunov–Schmidt reduction, and are not necessarily minimizers.

As regards the hypersurface Σ, they imposed some restrictions. They required it to be *admissible*, that is the nodal set of a smooth function $f : M \to \mathbb{R}$. In the sequel, we will set

$$M^+(\Sigma) = \{p \in M : f(p) > 0\} \text{ and } M^-(\Sigma) = \{p \in M : f(p) < 0\}.$$

Moreover, Σ has to be *non-degenerate*. In order to explain the notion of non-degeneracy, let us give the variational characterization of minimal hypersurfaces. A hypersurface Σ in a compact Riemannian manifold M is said to be minimal if it is a minimizer for the area functional, whose critical points are characterized by the Euler equation $H = 0$, where H denotes the mean curvature of Σ. In the sequel, the mean curvature H of a hypersurface Σ embedded in $\mathbb{R}^N$ will always be

$$H = k_1 + \cdots + k_{N-1},$$

where the k_j's are the principal curvatures.

The second variation of the area functional is given by

$$A''(\Sigma)[\phi, \psi] = \int_\Sigma L_0\phi(y)\psi(y)d\sigma(y),$$

where the self-adjoint operator

$$L_0\phi = -\Delta_\Sigma\phi - |A|^2\phi$$

is called the Jacobi operator of Σ and

$$|A|^2 = k_1^2 + \cdots + k_{N-1}^2$$

is the squared norm of its second fundamental form. By definition, a minimal hypersurface Σ is said to be non-degenerate if its Jacobi operator

$$L_0 : C^{2,\alpha}(\Sigma) \to C^{0,\alpha}(\Sigma)$$

is an isomorphism. For an introduction to these topics, see also [8].

Moreover, the results in [24] hold even if the potential $W(t) = (1 - t^2)^2/4$ is replaced by a more general double-well potential, that is a smooth function W such that

$$\begin{cases} W(t) \geq 0 & \text{for any } t, \\ W(t) = 0 & \text{if and only if } t = \pm 1, \\ W''(\pm 1) > 0. \end{cases} \tag{3}$$

To sum up, they proved the following Theorem.

Theorem 1 *([24]). Let W be as in (3). Let Σ be an admissible non-degenerate minimal hypersurface in a compact Riemannian manifold M. Then there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ there exists a solution u_ε to the rescaled Allen–Cahn equation*

$$-\varepsilon^2\Delta u_\varepsilon + W'(u_\varepsilon) = 0$$

such that $u_\varepsilon \to \pm 1$ on compact subsets of $M^\pm(\Sigma)$.

Anyway, despite several results lead to think that, in some sense, the nodal sets of the solutions to the Allen–Cahn equation resemble minimal surfaces, there are also solutions for which the nodal set is far from being minimal. For instance, Agudelo, Del Pino and Wei constructed axially symmetric solutions $u = u(|x'|, x_3)$ in $\mathbb{R}^3$ such that the components of the nodal set, for $|x'|$ large enough, look like a catenoid (see [3]).

The Lyapunov–Schmidt reduction was also applied to the non-compact case, to construct entire solutions to the Allen–Cahn equation in $\mathbb{R}^9$ that are monotone in one variable but not one-dimensional, since their nodal set resembles the Bombieri–De Giorgi–Giusti graph, that is a minimal graph over $\mathbb{R}^8$ that is not affine (see [5,7]). This solutions are related to a famous conjecture of De Giorgi, that asserts that, at least for $N \leq 8$, any entire bounded solution $|u| < 1$ to the Allen–Cahn equation

$$-\Delta u = u - u^3$$

satisfying $\partial_N u > 0$ in the whole $\mathbb{R}^N$ must be one-dimensional, that is it must depend just on one Euclidean variable, in other words $u(x) = u(< a, x >)$, for some unit vector $a \in S^{N-1}$. The result by Del Pino, Kowalczyk and Wei shows that de Giorgi's conjecture is sharp about the upper bound on the dimension. Up to now it is known that the conjecture is true in dimension $N = 2$ (see [11,10]) and $N = 3$ (see [2,10]). The conjecture is still open in dimension $4 \leq N \leq 8$, although notable progress was made by Savin (see [26]), that proved that the conjecture is true in dimension $4 \leq N \leq 8$ under the reasonable assumption that, for any $x' \in \mathbb{R}^{N-1}$,

$$\lim_{x_N \to \pm\infty} u(x', x_N) = \pm 1,$$

that yields that these solutions are minimizers of the energy

$$E(u) = \int_{\mathbb{R}^3} \Big(\frac{1}{2}|\nabla u|^2 + \frac{1}{4}(1-u^2)^2\Big) dx.$$

We are interested here in analogues of these results for the Cahn–Hilliard equation

$$-\varepsilon^2 \Delta(-\varepsilon^2 \Delta u + W'(u)) + W''(u)(-\varepsilon^2 \Delta u + W'(u)) = 0, \tag{4}$$

with W satisfying (3). Note that, as in the case of Allen–Cahn, we rescale the equation in order to treat Γ-convergence. If, for instance, we study the equation in a bounded domain $\Omega \subset \mathbb{R}^N$, it is possible to see that it is the Euler equation of the functional

$$\mathcal{W}_\varepsilon(u) = \begin{cases} \frac{1}{2\varepsilon} \int_\Omega \Big(\varepsilon \Delta u - \frac{W'(u)}{\varepsilon}\Big)^2 dx & \text{if } u \in L^1(\Omega) \cap H^2(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases}$$

As in the case of the functionals E_ε related to the Allen–Cahn equation, some Γ-convergence results are known about $\mathcal{W}_\varepsilon$. More precisely, the asymptotic behaviour of $\mathcal{W}_\varepsilon$ as $\varepsilon \to 0$ is related to the Willmore functional

$$\mathcal{W}(u) = c \int_{\partial E \cap \Omega} H^2_{\partial E}(y) d\mathcal{H}^{N-1},$$

where $E = \{u = 1\}$, if $u = \pm 1$ a.e., defined when the interface ∂E is smooth enough. The nodal sets of the critical points u of $\mathcal{W}$ are called *Willmore hypersurfaces*. The Euler equation satisfied by this kind of hypersurfaces is

$$-\Delta_\Sigma H = \frac{1}{2}H^3 - 2HK,$$

where H is the mean curvature and K is the Gauss curvature of $\Sigma = \partial E$. In the sequel, the Gauss curvature K of hypersurface Σ embedded in $\mathbb{R}^N$ will always be

$$K = k_1 \dots k_{N-1}.$$

An equivalent form of the Willmore equation is

$$-\Delta_\Sigma H + \frac{1}{2}H(H^2 - 2|A|^2) = 0. \tag{5}$$

The Willmore functional arises naturally in general relativity, since it is related to the Hawking mass, that is

$$m_H(\Sigma) = \sqrt{\frac{Area(\Sigma)}{16\pi}}(1 - \frac{1}{16\pi}\mathcal{W}(\Sigma)).$$

Here Σ can be interpreted as the surface of a body whose mass has to be measured. Furthermore, this functional also appears in biology, under the name of *Helfrich energy*, and it is used to describe the behaviour of some lipid bilayer cell membranes. For further details and references, we suggest to see [16,13,14].

In [4] Bellettini and Paolini proved the $\Gamma - \limsup$ inequality for smooth Willmore hypersurfaces, while the $\Gamma - \liminf$ inequality is much harder to prove. Up to now it has been proved in dimension $N = 2, 3$ by Röger and Schätzle in [25], and, independently, in dimension $N = 2$, by Nagase and Tonegawa in [23]. The problem is still open in higher dimension, while it is known that the approximation does not hold, in general, for non-smooth sets, even in dimension $N = 2$.

In view of these Γ-convergence results that establish a link between the Cahn–Hilliard functional and the Willmore functional, it is interesting to see if also the above counter-part is true. In other words, we try to answer the following question: given a Willmore hypersurface Σ, is it possible to construct a sequence of solutions u_ε of the Cahn–Hilliard equation (4) whose nodal sets approach Σ as $\varepsilon \to 0$? In the paper, we show that this result holds true up to a Lagrange multiplier if, for instance, Σ is the standard Clifford Torus, that is the zero level set of the function

$$f(x) = \left(\sqrt{2} - \sqrt{x_1^2 + x_2^2}\right)^2 + x_3^2 - 1. \tag{6}$$

It has been recently proved in [17] that the Clifford Torus is the unique minimizer of the Willmore energy (up to conformal transformations) among surfaces of genus greater or equal than 1. The Lagrange multiplier is due to a volume constraint, as it will be clear from the statement of Theorem 1 and Remark 3.

It is interesting to see that it is possible to construct these solutions in such a way that they respect the symmetries of the Torus, that is the symmetry with respect to the x_1x_2-plane and with respect to any rotation that fixes the x_3-axis. A Lagrange multiplier λ_ε appears due to this volume constraint.

Theorem 2. *Let W be an even double-well potential satisfying (3). Let Σ be the Clifford Torus. Then there exists ε_0 such that for any $0 < \varepsilon < \varepsilon_0$ there exists a solution $(u_\varepsilon, \lambda_\varepsilon) \in C^{4,\alpha}(\mathbb{R}^3) \times \mathbb{R}$ to*

$$-\varepsilon^2\Delta(-\varepsilon^2\Delta u_\varepsilon + W'(u_\varepsilon)) + W''(u_\varepsilon)(-\varepsilon^2\Delta u_\varepsilon + W'(u_\varepsilon)) = \varepsilon^4\lambda_\varepsilon(1-u_\varepsilon), \tag{7}$$

where $\lambda_\varepsilon = O(\varepsilon)$ is a Lagrange multiplier, under the volume constraint

$$\int_{\mathbb{R}^3} (1-u_\varepsilon)^2 dx = 8\sqrt{2}\pi^2 c_\varepsilon,\ c_\varepsilon \to 1 \text{ as } \varepsilon \to 0. \tag{8}$$

Moreover, $u_\varepsilon \to \pm 1$ and $\partial_k u_\varepsilon \to 0$ uniformly on compact subsets of $\Sigma^\pm$, for $1 \le k \le 4$, and u respects the symmetries of the Torus, that is $u_\varepsilon(x_1, x_2, x_3) = u_\varepsilon(x_1, x_2, -x_3)$ and $u_\varepsilon(x) = u_\varepsilon(Rx)$, for any $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and for any rotation $R \in SO(3)$ such that $R(0,0,1) = (0,0,1)$.

In the statement of the Theorem, we denoted

$$\Sigma^+ := \{x \in \mathbb{R}^3 : f(x) > 0\} \text{ and } \Sigma^- := \{x \in \mathbb{R}^3 : f(x) < 0\},$$

and

$$c_\varepsilon := 1 + \varepsilon \int_0^\infty (v_\star^2 - 1)dt.$$

This result is a fourth order analogue of Theorem 1 by Pacard and Ritoré (see [24]). The proof is based on the Lyapunov–Schmidt reduction, that is we split equation (4) into a system of two equations. The auxiliary equation will be solved by using the spectral decomposition of the linearized Allen–Cahn operator and the bifurcation equation will be solved thanks to the *nondegeneracy* of the Clifford Torus, up to conformal maps. For a more detailed introduction to the techniques developed in the proof, see section 2.

In order to explain what we mean by nondegeneracy, we go back to the variational definition of Willmore hypersurface and we consider the second variation of the Willmore functional, that is

$$\mathcal{W}''(\Sigma)[\phi, \psi] = \int_\Sigma \tilde{L}_0 \phi \psi \, d\sigma,$$

where $\tilde{L}_0$ is the self-adjoint operator given by

$$\tilde{L}_0\phi = L_0^2\phi + \frac{3}{2}H^2 L_0\phi - H(\nabla_\Sigma\phi, \nabla_\Sigma H) + 2(A\nabla_\Sigma\phi, \nabla_\Sigma H) + 2H < A, \nabla^2\phi > +\phi(2 < A, \nabla^2 H > +|\nabla_\Sigma H|^2 + 2H\mathrm{tr}A^3). \tag{9}$$

Here we have denoted by $(\cdot,\cdot)$ the scalar product induced by the metric g on Σ, indeed, for instance $(\nabla\phi, \nabla H) = g^{ij}H_i\phi_j$, and by $<\cdot,\cdot>$ the trace of the product of two matrices, so for instance $< A, \nabla^2\phi >= A^{ij}\nabla^2_{ij}\phi$, and $A^{ij} = g^{ik}g^{jl}A_{kl}$. It is possible to find the explicit computation of the first and the second variation of the Willmore functional $\mathcal{W}$ in [16], section 3. This is the analogue of the Jacobi operator in the case of minimal hypersurfaces. In view of a result by White [28], the Willmore functional is invariant under conformal transformations of the Euclidean space, that is homotheties, isometries and Möbius transformations, i.e. inversions with respect to spheres. On the other hand, by Corollary 2, page 34, of [27], we know that its second variation is positive definite on the orthogonal complement of the space of conformal transformations, hence the kernel of $\tilde{L}_0$ exactly consists of these transformations.

Remark 3. In view of the above discussion, $\tilde{L}_0$ is injective if restricted to the space of functions with zero average and fulfilling the symmetries of the Torus, that is the symmetry with respect to the x_1x_2-plane and with respect to all rotations of $\mathbb{R}^3$ that fix the x_3 axis. In fact, the variation of internal volume under a normal variation ϕ is given by $\int_\Sigma \phi$. Maintaining constant volume corresponds to variations ϕ with zero average. Working in this class we then exclude non-trivial homotheties. When considering sharp interfaces, this constraint is equivalent to prescribe the integral of $(1-u_\varepsilon)^2$, that is to impose

$$\int_{\mathbb{R}^3} (1-u_\varepsilon)^2 dx = 8\sqrt{2}\pi^2 c_\varepsilon$$

$$= 4|\Sigma^+|_3\left(1 + \varepsilon\int_0^\infty (v_\star^2 - 1)dt\right),$$

where $|\Sigma^+|_3 = 2\sqrt{2}\pi^2$ is the volume of the interior of the Clifford Torus, that is its 3-dimensional Lebesgue measure.

2. Some useful facts in differential geometry

For $0 < \varepsilon \le 1$, we define the rescaled Clifford Torus as $\Sigma_\varepsilon := \{\varepsilon^{-1}\zeta : \zeta \in \Sigma\}$. In other words, $\Sigma_\varepsilon = \{y \in \mathbb{R}^3 : f_\varepsilon(y) = 0\}$, where $f_\varepsilon(y) := \varepsilon^{-2} f(\varepsilon y)$ and f is defined in (6).

For $0 < \tau < \sqrt{2} - 1$ and $0 < \varepsilon \le 1$, we define the tubular neighbourhood of width τ/ε of Σ_ε as

$$V_{\tau/\varepsilon} = \{x \in \mathbb{R}^3 : dist(x, \Sigma_\varepsilon) < \tau/\varepsilon\}.$$

On this neighbourhood of Σ_ε, we introduce a new system of coordinates, known as Fermi coordinates. First we define

$$Z_\varepsilon : \Sigma_\varepsilon \times (-\tau/\varepsilon, \tau/\varepsilon) \to V_{\tau/\varepsilon}$$

by the relation

$$Z_\varepsilon(y,z) = \exp_y(z\nu(\varepsilon y)) = y + z\nu(\varepsilon y), \tag{10}$$

where $\nu(\varepsilon y)$ is the outward-pointing unit normal to the original Torus Σ at εy, that coincides with the outward-pointing unit normal to Σ_ε at y, and $\exp_y$ is the exponential map of $\mathbb{R}^3$ at y seen as a point of $\mathbb{R}^3$. If τ is small enough, that is $0 < \tau < \sqrt{2}-1$ in the case of the Clifford Torus, Z_ε is a diffeomorphism. In other words, Z_ε is a change of coordinates on $V_{\tau/\varepsilon}$, and the coordinates $(y,z) = Z_\varepsilon^{-1}(x)$ are known as Fermi coordinates of the rescaled torus Σ_ε, or stretched Fermi coordinates of the Torus.

Remark 4. Any function $u : V_{\tau/\varepsilon} \to \mathbb{R}$ can be seen as a function of (y,z). More precisely, we can consider the composition $u^\star(y,z) = u(Z_\varepsilon(y,z))$. In the sequel, with a slight abuse of notation, we will write $u = u(y,z)$.

Let us fix a point $\zeta_0 \in \Sigma$ and a parametrization onto a neighbourhood $V \subset \Sigma$ of ζ_0, that is a smooth function

$$Y : U \to V$$

on an open set $U \subset \mathbb{R}^2$ such that $Y(\xi_0) = \zeta_0$, for some $\xi_0 \in U$. Then, setting $U_\varepsilon = \varepsilon^{-1}U$ and $V_\varepsilon = \varepsilon^{-1}V$, the function

$$Y_\varepsilon : U_\varepsilon \to V_\varepsilon$$

given by $Y_\varepsilon(\mathrm{y}) := \varepsilon^{-1}Y(\varepsilon \mathrm{y})$ is a parametrization of Σ_ε. In the sequel, we will denote by y the points in U_ε and by $y = Y_\varepsilon(\mathrm{y})$ the points in V_ε. For any $|z| < \tau/\varepsilon$, we consider the surface

$$\Sigma_{\varepsilon,z} := \{y + z\nu(\varepsilon y),\, y \in \Sigma_\varepsilon\}. \tag{11}$$

On this surface, we consider the parametrization

$$X_\varepsilon(\mathrm{y}, z) := Y_\varepsilon(\mathrm{y}) + z\nu(\varepsilon Y_\varepsilon(\mathrm{y})). \tag{12}$$

In particular, $X := X_1$ is a parametrization of $\Sigma_z := \Sigma_{1,z}$, the homothetic surface to Σ at distance z. It is known that the tangent vectors $\{\partial_i X_\varepsilon(\mathrm{y},z)\}_{i=1,2}$ constitute a basis of the tangent space $T_{y+z\nu(\varepsilon y)}\Sigma_{\varepsilon,z}$, that will be referred to as the standard basis. We define the coefficients of the metric of $\Sigma_{\varepsilon,z}$ at $y + z\nu(\varepsilon y)$ as follows

$$\tilde{g}_{\varepsilon,ij}(y,z) := < \partial_i X_\varepsilon(\mathrm{y}), \partial_j X_\varepsilon(\mathrm{y}) > = \tilde{g}_{ij}(\varepsilon y, \varepsilon z), \tag{13}$$

where $<\cdot,\cdot>$ denotes the scalar product of $\mathbb{R}^3$ and $i,j = 1,2$. The Laplacian on $\Sigma_{\varepsilon,z}$ is given by

$$\Delta_{\Sigma_{\varepsilon,z}} = \frac{1}{\sqrt{\det \tilde{g}_\varepsilon(y,z)}}\partial_j\left(\sqrt{\det \tilde{g}_\varepsilon(y,z)}\,\tilde{g}_\varepsilon^{ij}(y,z)\partial_i\right) = \tilde{g}_\varepsilon^{ij}(y,z)\partial_{ij} + \tilde{b}_\varepsilon^i(y,z)\partial_i, \tag{14}$$

where

$$\tilde{b}^i_\varepsilon(y,z) := \partial_j \tilde{g}^{ij}_\varepsilon(y,z) + \frac{1}{2}\partial_j\big(\log\det\tilde{g}_\varepsilon(y,z)\big)\tilde{g}^{ij}_\varepsilon(y,z) \tag{15}$$

and $\tilde{g}^{ij}_\varepsilon := (\tilde{g}^{-1}_\varepsilon)_{ij}$ are the elements of the inverse of the metric. These quantities are related to the ones of Σ_z through the relations

$$\tilde{g}^{ij}_\varepsilon(y,z) = \tilde{g}^{ij}(\varepsilon y, \varepsilon z),$$
$$\tilde{b}^i_\varepsilon(y,z) = \varepsilon \tilde{b}^i(\varepsilon y, \varepsilon z),$$

with $\tilde{g}^{ij} := \tilde{g}^{ij}_1$ and $\tilde{b}^i := \tilde{b}^i_1$. We define the second fundamental form at $y + z\nu(\varepsilon y) \in \Sigma_{\varepsilon,z}$ to be the linear application of the tangent space $T_{y+z\nu(\varepsilon y)}\Sigma_{\varepsilon,z}$ into itself that, in the standard basis $\{\partial_i X_\varepsilon(\mathrm{y},z)\}_{i=1,2}$, is represented by the matrix

$$\tilde{A}_{\varepsilon,ij}(y,z) := - < \partial_i \nu(\varepsilon y), \partial_j X_\varepsilon(\mathrm{y},z) > . \tag{16}$$

We introduce the mean curvature $\tilde{H}_\varepsilon(y,z)$ of $\Sigma_{\varepsilon,z}$ at $y + z\nu(\varepsilon y)$ as follows

$$\tilde{H}_\varepsilon(y,z) := (\tilde{A}_\varepsilon)^i_i(y,z) = \tilde{g}^{ij}_\varepsilon(y,z)\tilde{A}_{\varepsilon,ij}(y,z).$$

In other words

$$\tilde{H}_\varepsilon(y,z) = \tilde{k}_{\varepsilon,1}(y,z) + \tilde{k}_{\varepsilon,2}(y,z),$$

where $\tilde{k}_{\varepsilon,i}(y,z)$ are the *principal curvatures* of $\Sigma_{\varepsilon,z}$, that is eigenvalues of the matrix $\tilde{g}^{-1}_\varepsilon(y,z)\tilde{A}_\varepsilon(y,z)$. Therefore we can see that the metric $\tilde{g}_{\varepsilon,ij}(y,z)$ and the matrix representing the second fundamental $\tilde{A}_{\varepsilon,ij}(y,z)$ form depend on the parametrization, while this is not the case for $\tilde{H}_\varepsilon(y,z)$. Setting, as above $\tilde{A}_{ij} := \tilde{A}_{1,ij}$ and $\tilde{H} := \tilde{H}_1$, we have $\tilde{A}_{\varepsilon,ij}(y,z) = \varepsilon\tilde{A}_{ij}(\varepsilon y, \varepsilon z)$ and $\tilde{H}_\varepsilon(y,z) = \varepsilon\tilde{H}(\varepsilon y, \varepsilon z)$.

Lemma 5. *For a function $u : V_{\tau/\varepsilon} \to \mathbb{R}$ of class C^2, the Laplacian in Fermi coordinates is given by*

$$\Delta u(y,z) = \Delta_{\Sigma_{\varepsilon,z}} u(y,z) - \varepsilon\tilde{H}(\varepsilon y, \varepsilon z)\partial_z u(y,z) + \partial_{zz}u(y,z). \tag{17}$$

For the notation, see Remark 4.

Proof. For any $y \in \Sigma_\varepsilon$ and $|z| < \tau/\varepsilon$, $\mathbb{R}^3$ splits into the direct sum of the tangent space to $\Sigma_{\varepsilon,z}$ and the one dimensional subspace generated by the unit normal $\nu(\varepsilon y)$, that is $\mathbb{R}^3 = T_{y+z\nu(\varepsilon y)}\Sigma_{\varepsilon,z} + \mathbb{R}$. The vectors $\{\partial_i X_\varepsilon(y,z), \nu(\varepsilon y)\}_{i=1,2}$ constitute a basis of $\mathbb{R}^3 = T_{y+z\nu(\varepsilon y)}\mathbb{R}^3$. The metric in this basis is given by

$$G_\varepsilon(y,z) = \begin{bmatrix} \tilde{g}_\varepsilon(y,z) & 0 \\ 0 & 1 \end{bmatrix}. \tag{18}$$

The inverse is

$$G_\varepsilon^{-1}(y,z) = \begin{bmatrix} \tilde{g}_\varepsilon^{-1}(y,z) & 0 \\ 0 & 1 \end{bmatrix}. \tag{19}$$

Here $1 \le I, J \le 3$ and $1 \le i, j \le 2$. The Laplacian on $\mathbb{R}^3$ in the metric G_ε is given by

$$\Delta u = \frac{1}{\sqrt{\det G_\varepsilon(y,z)}} \partial_J (\sqrt{\det G_\varepsilon(y,z)} G_\varepsilon^{IJ}(y,z)\partial_I) =$$

$$G_\varepsilon^{IJ}(y,z)\partial_{IJ}u(y,z) + \partial_J G_\varepsilon^{IJ}(y,z)\partial_I u(y,z) + \frac{1}{2}\partial_J(\log\det G_\varepsilon(y,z))G_\varepsilon^{IJ}(y,z)\partial_I u(y,z).$$

Thus

$$G_\varepsilon^{IJ}(y,z)\partial_{IJ}u(y,z) = \tilde{g}_\varepsilon^{ij}(y,z)\partial_{ij}u(y,z) + \partial_{zz}u(y,z)$$

$$\partial_J G_\varepsilon^{IJ}(y,z)\partial_I u(y,z) = \partial_j \tilde{g}_\varepsilon^{ij}(y,z)\partial_i u(y,z)$$

$$\frac{1}{2}\partial_J(\log\det G_\varepsilon(y,z))G_\varepsilon^{IJ}(y,z)\partial_I u(y,z) =$$

$$\frac{1}{2}\partial_j(\log\det \tilde{g}_\varepsilon(y,z))\tilde{g}_\varepsilon^{ij}(y,z)\partial_i u(y,z) + \frac{1}{2}\partial_z(\log\det \tilde{g}_\varepsilon(y,z))\partial_z u(y,z).$$

To conclude, we point out that

$$\frac{1}{2}\partial_z(\log\det \tilde{g}_\varepsilon(y,z)) = -\tilde{H}_\varepsilon(y,z) = -\varepsilon \tilde{H}(\varepsilon y, \varepsilon z). \quad \square$$

Exploiting the Taylor expansion of $\tilde{H}$ of the mean curvature of a given hypersurface provided by Del Pino, Kowalczyk and Wei (see [7]), we have that

$$\tilde{H}(\varepsilon y, \varepsilon z) = \sum_{i=1}^{2} \frac{k_i(\varepsilon y)}{1 - \varepsilon z k_i(\varepsilon y)} = \sum_{j \ge 1} (\varepsilon z)^{j-1} H_j(\varepsilon y), \;\; H_j(\varepsilon y) := \sum_{i=1}^{2} k_i^j(\varepsilon y) \tag{20}$$

Here $k_i(\varepsilon y) := \tilde{k}_{\varepsilon,i}(y,0)$ are the principal curvatures of the Clifford Torus Σ at εy. Therefore the Taylor expansions of the first and the second derivatives of $\tilde{H}$ are

$$\begin{cases} \tilde{H}_z(\varepsilon y, \varepsilon z) = \sum_{j \ge 1} j(\varepsilon z)^{j-1} H_{j+1}(\varepsilon y), \\ \tilde{H}_{zz}(\varepsilon y, \varepsilon z) = \sum_{j \ge 1} j(j+1)(\varepsilon z)^{j-1} H_{j+2}(\varepsilon y). \end{cases} \tag{21}$$

In the sequel, we will set $H(\varepsilon y) := H_1(\varepsilon y)$, $|A(\varepsilon y)|^2 := H_2(\varepsilon y)$ and $\mathrm{tr}A^3(y) := H_3(\varepsilon y)$.

Now we need the Taylor expansion in εz of $\Delta_{\Sigma_{\varepsilon,z}}$. For our purposes, it is enough to know the terms of order zero and one, while we also need the term of order two in the expansion of $\tilde{H}$. For this reason, we prefer not to expand the full Laplacian on $\mathbb{R}^3$. In fact, an expansion up to order one would not be enough, because we cannot neglect the terms involving $\mathrm{tr}A^3$, while an expansion up to order two would be a useless effort, in fact it would involve the terms of order two of $\Delta_{\Sigma_{\varepsilon,z}}$, that will always simplify in our forthcoming calculations. Before stating next Lemma, we recall that

$$\Delta_{\Sigma_\varepsilon} = \frac{1}{\sqrt{\det g_\varepsilon(y)}}\partial_j\left(\sqrt{\det g_\varepsilon(y)}g_\varepsilon^{ij}(y)\partial_i\right) = g_\varepsilon^{ij}(y)\partial_{ij} + b_\varepsilon^i(y)\partial_i, \tag{22}$$

where

$$\begin{aligned} g_\varepsilon^{ij}(y) &:= \tilde{g}_\varepsilon^{ij}(y,0) = \tilde{g}^{ij}(\varepsilon y, 0) = g^{ij}(\varepsilon y) \\ b_\varepsilon^i(y) &:= \tilde{b}_\varepsilon^i(y,0) = \varepsilon\tilde{b}^i(\varepsilon y, 0) = \varepsilon b^i(\varepsilon y). \end{aligned} \tag{23}$$

It is possible to find similar computations in [19], where Mahmoudi, Sànchez and Yao treat the more general case of a k-dimensional submanifold in an N-dimensional manifold.

Lemma 6. *For a function $u : V_{\tau/\varepsilon} \to \mathbb{R}$ of class C^2, for any $y \in \Sigma_\varepsilon$, for any $|z| \le \tau/\varepsilon$,*

$$\Delta_{\Sigma_{\varepsilon,z}} u = \Delta_{\Sigma_\varepsilon} u + \varepsilon z(a_1^{ij}(\varepsilon y)\partial_{ij} + \varepsilon b_1^i(\varepsilon y)\partial_i)$$
$$+ (\varepsilon z)^2(a_2^{ij}(\varepsilon y)\partial_{ij} + \varepsilon b_2^i(\varepsilon y)\partial_i) + \overline{a}^{ij}(\varepsilon y, \varepsilon z)\partial_{ij} + \varepsilon\overline{b}_i(\varepsilon y, \varepsilon z)\partial_i,$$

where

$$a_1^{ij} := 2A^{ij}, b_1^i := 2\partial_j A^{ij} + 2\Gamma_{kj}^k A^{ij} - g^{ij}H_j,$$
$$a_2^{ij} := \frac{1}{2}\partial_{zz}\tilde{g}^{ij}(\varepsilon y, 0), b_2^i := \frac{1}{2}\partial_{zz}\tilde{b}^i(\varepsilon y, 0),$$

everything evaluated at εy, and the remainders satisfy $|\overline{a}^{ij}(\varepsilon y, \varepsilon z)|, |\overline{b}^i(\varepsilon y, \varepsilon z)| \le c\varepsilon^3|z|^3$, for some constant $c > 0$ depending on Σ.

Let $\phi, \psi : \Sigma \to \mathbb{R}$ be C^2 functions. Let us set $\phi_i := \partial_i\phi$. We recall that, by the properties of the covariant derivative,

$$\nabla_k A^{ij} = \partial_k A^{ij} + \Gamma_{kl}^i A^{lj} + \Gamma_{kl}^j A^{li},$$
$$\nabla_{ij}^2\phi = \phi_{ij} - \Gamma_{ij}^k\phi_k,$$

where everything is evaluated at εy. Moreover, by Codazzi's equation, $\nabla_j A^{ij} = g^{ik}\nabla_k A_j^j$, so in particular,

$$\begin{aligned} a_1^{ij}\phi_i\psi_j &= 2(A\nabla\phi, \nabla\psi) \\ a_1^{ij}\psi_{ij} + b_1^i\psi_i &= 2A^{ij}\psi_{ij} - 2\Gamma_{ji}^k A^{ij}\psi_k + 2\nabla_j A^{ij}\psi_i \\ -(\nabla_\Sigma H, \nabla_\Sigma\psi) &= 2 < A, \nabla^2\psi > +(\nabla_\Sigma\psi, \nabla_\Sigma H), \end{aligned} \tag{24}$$

where we have set

$$< A, \nabla^2\psi >:= A^{ij}\nabla^2_{ij}\psi = A^{ij}\psi_{ij} + \Gamma^k_{ij}\psi_k.$$

Proof. By (12) and (13), we can see that

$$\tilde{g}_{\varepsilon,ij}(y,z) = g_{ij} + \varepsilon z(< \partial_i Y, \partial_j \nu > + < \partial_j Y, \partial_i \nu >) + (\varepsilon z)^2 < \partial_i \nu, \partial_j \nu > .$$

In the proof, it is understood that the geometric quantities of Σ are evaluated at εy. In view of (16) with $z = 0$, we have

$$\partial_i \nu = -A_i^k \partial_k Y$$

therefore

$$\tilde{g}_{\varepsilon,ij}(y,z) = g_{ij} - \varepsilon z(g_{ik}A^k_j + g_{jk}A^k_i) + (\varepsilon z)^2 < \partial_i \nu, \partial_j \nu >= \\ g_{ij} - 2\varepsilon z A_{ij} + (\varepsilon z)^2 < \partial_i \nu, \partial_j \nu > . \tag{25}$$

In order to expand the Laplacian, we need the expansion of the inverse of the metric. It is useful to write it as

$$\tilde{g}_\varepsilon = L + M,$$

with $L_{ij} = g_{ij}$ and $M = -2\varepsilon z A_{ij} + (\varepsilon z)^2 < \partial_i \nu, \partial_j \nu >$. Equivalently, $\tilde{g}_\varepsilon = L(I + L^{-1}M)$, hence

$$\tilde{g}_\varepsilon^{-1} = (I + L^{-1}M)^{-1}L^{-1} = (I - L^{-1}M + O((\varepsilon z)^2))L^{-1} = L^{-1} - L^{-1}ML^{-1} + O((\varepsilon z)^2),$$

thus

$$\tilde{g}_\varepsilon^{ij}(y,z) = g^{ij} + 2\varepsilon z A^{ij} + O((\varepsilon z)^2),$$

where $A^{ij} = g^{ik}g^{jl}A_{kl}$. Moreover

$$\log\det\tilde{g}_\varepsilon(y,z) = \log\det g_\varepsilon(y) + \mathrm{tr}(L^{-1}M) + O((\varepsilon z)^2) = \log\det g_\varepsilon - 2\varepsilon z H + O((\varepsilon z)^2),$$

so, since $\frac{1}{2}\partial_j(\log\det g)A^{ij} = \Gamma^k_{kj}A^{ij}$,

$$\Delta_{\Sigma_{\varepsilon,z}} = (g^{ij} + 2\varepsilon z A^{ij})\partial_{ij} + \varepsilon(\partial_j g^{ij} + 2\varepsilon z \partial_j A^{ij})\partial_i \\ + \varepsilon\Big(\frac{1}{2}\partial_j(\log\det g) - \varepsilon z H_j\Big)(g^{ij} + 2\varepsilon z A^{ij})\partial_i + O((\varepsilon z)^2) = \\ \Delta_{\Sigma_\varepsilon} + \varepsilon z\Big\{2A^{ij}\partial_{ij} + \varepsilon(2\partial_j A^{ij} + 2\Gamma^k_{kj}A^{ij} - g^{ij}H_j)\partial_i\Big\} + O((\varepsilon z)^2). \quad \square$$

As a consequence, we have the following expansion of the Laplacian

$$\Delta = \partial_{zz} - \varepsilon \tilde{H}(\varepsilon y, \varepsilon z)\partial_z + \Delta_{\Sigma_\varepsilon} + \varepsilon z(a_1^{ij}(\varepsilon y)\partial_{ij} + \varepsilon b_1^i(\varepsilon y)\partial_i) \tag{26}$$
$$+ (\varepsilon z)^2(a_2^{ij}(\varepsilon y)\partial_{ij} + \varepsilon b_2^i(\varepsilon y)\partial_i) + \overline{a}^{ij}(\varepsilon y, \varepsilon z)\partial_{ij} + \varepsilon \overline{b}^i(y, z)\partial_i.$$

Although (26) looks nice, we prefer to look for the expression of the Laplacian in a slightly different system of coordinates. We fix a C^2 function $\phi : \Sigma \to \mathbb{R}$ whose $L^\infty(\Sigma)$-norm is less than $1/4$ and we introduce a new change of variables, that is we put

$$t = z - \phi(\varepsilon y). \tag{27}$$

The expression of the Laplacian will be more complicated than (26), but more appropriate for our purposes. The reason is that we know the kernel of the operator $-(\Delta_{\Sigma_\varepsilon} + \partial_{tt}) + W''(v_\star(t))$, that is the one dimensional space generated by $v_\star'(t)$, while we do not know exactly the kernel (if any) of $-(\Delta_{\Sigma_\varepsilon} + \partial_{zz}) + W''(v_\star(z - \phi(\varepsilon y)))$.

Given a function

$$f : \Sigma_\varepsilon \times \mathbb{R} \to \mathbb{R}$$

of class C^2, it is possible to define

$$\mathrm{f} : \Sigma_\varepsilon \times \mathbb{R} \to \mathbb{R}$$

by setting $\mathrm{f}(y, t) := f(y, z - \phi(\varepsilon y))$. A computation shows that

$$\begin{aligned}
\mathrm{f}_t(y,t) &= f_z(y, z-\phi)\\
\mathrm{f}_i(y,t) &= f_i(y, z-\phi) - \varepsilon\phi_i f_z(y, z-\phi)\\
\mathrm{f}_{ij}(y,t) &= f_{ij}(y, z-\phi) - \varepsilon\phi_i f_{zj}(y, z-\phi) - \varepsilon\phi_i f_{zj}(y, z-\phi)\\
&\quad + \varepsilon^2\phi_{ij} f_z(y, z-\phi) + \varepsilon\phi_i\phi_j f_{zz}(y, z-\phi),
\end{aligned}$$

where ϕ and its derivatives are evaluated at εy, thus, in these coordinates, the expression of the Laplacian of a function u defined in $V_{\tau/\varepsilon}$ of class C^2 is given by

$$\Delta = \partial_{tt} + g^{ij}\partial_{ij} + \varepsilon b^i\partial_i + \mathrm{D} = \partial_{tt} + \Delta_{\Sigma_\varepsilon} + \mathrm{D}, \tag{28}$$

where the operator D is given by

$$\mathrm{D} := -\varepsilon\hat{H}(\varepsilon y, \varepsilon(t+\phi))\partial_t - \varepsilon^2\Delta_\Sigma\phi\partial_t - 2\varepsilon g^{ij}\phi_i\partial_{tj} + \varepsilon^2|\nabla_\Sigma\phi|^2\partial_{tt} \tag{29}$$
$$+ \varepsilon(t+\phi)\left\{a_1^{ij}\partial_{ij} + \varepsilon b_1^i\partial_i - \varepsilon^2(a_1^{ij}\phi_{ij} + b_1^i\phi_i)\partial_t - 2\varepsilon a_1^{ij}\phi_i\partial_{tj} + \varepsilon^2 a_1^{ij}\phi_i\phi_j\partial_{tt}\right\}$$
$$+ \varepsilon^2(t+\phi)^2\left\{a_2^{ij}\partial_{ij} + \varepsilon b_2^i\partial_i - \varepsilon^2(a_2^{ij}\phi_{ij} + b_2^i\phi_i)\partial_t - 2\varepsilon a_2^{ij}\phi_i\partial_{tj} + \varepsilon^2 a_2^{ij}\phi_i\phi_j\partial_{tt}\right\}$$
$$+ \hat{a}^{ij}\partial_{ij} + \varepsilon\hat{b}^i\partial_i - \varepsilon^2(\hat{a}^{ij}\phi_{ij} + \hat{b}^i\phi_i)\partial_t - 2\varepsilon\hat{a}^{ij}\phi_i\partial_{tj} + \varepsilon^2\hat{a}^{ij}\phi_i\phi_j\partial_{tt}.$$

Here we have set $\hat{H}(\varepsilon y, \varepsilon(t+\phi)) := \tilde{H}(\varepsilon y, \varepsilon z)$, $\hat{a}^{ij}(\varepsilon y, \varepsilon(t+\phi)) = \overline{a}^{ij}(\varepsilon y, \varepsilon z)$, $\hat{b}^i(\varepsilon y, \varepsilon(t+\phi)) = \overline{b}^i(\varepsilon y, \varepsilon z)$ and all the geometric quantities of Σ are evaluated at εy.

3. Functional setting

3.1. Functions on Σ

As first we define, for $0 < \alpha < 1$, the space $C^{k,\alpha}(\Sigma)$ as the set of functions $\phi : \Sigma \to \mathbb{R}$ that are k times differentiable and whose k-th partial derivatives are Hölder continuous with exponent α. We endow these spaces with the norms

$$|\phi|_{C^{k,\alpha}(\Sigma)} := \sum_{j=0}^{k} ||\nabla^j \phi||_\infty + \sup_{p \neq q} \frac{|\nabla^k \phi(p) - \nabla^k \phi(q)|}{d(p,q)^\alpha}.$$

In order to treat $\tilde{L}_0$, we define the spaces of functions that respect the symmetries of the Torus, that is the symmetry with respect to the x_1x_2-plane and with respect to any rotation that keeps the x_3-axis fixed. To be precise, we set $T(x_1, x_2, x_3) := (x_1, x_2, -x_3)$ and

$$SO_{x_3}(3) := \{R \in SO(3) : Re_3 = e_3\},$$

where $e_3 = (0, 0, 1)$, and we define

$$\begin{aligned} C^{k,\alpha}(\Sigma)_s := \{\phi \in C^{k,\alpha}(\Sigma) : \phi(\zeta) = \phi(T\zeta) \text{ for any } \zeta \in \Sigma, \\ \phi(\zeta) = \phi(R\zeta) \text{ for any } R \in SO_{x_3}(3)\}. \end{aligned}$$

We note that $SO_{x_3}(2) \simeq SO(2)$, in the sense that any matrix $R \in SO_{x_3}(3)$ has the form

$$R = \begin{bmatrix} \tilde{R} & 0 \\ 0 & 1 \end{bmatrix},$$

for some rotation of the x_1x_2-plane $\tilde{R} \in SO(2)$.

Equivalently, we can see the Torus as the quotient of the square $[0, 2\pi]^2$ by the equivalence relation that identifies the opposite sides. In this way, our spaces will become

$$C^{k,\alpha}(\Sigma)_s := \{\phi \in C^{k,\alpha}([0, 2\pi]) : \phi(\vartheta_1) = \phi(2\pi - \vartheta_1) \text{ for any } \vartheta_1 \in [0, \pi]\}. \tag{30}$$

In other words, functions respecting these symmetries are actually periodic functions of one real variable, symmetric with respect to $\vartheta_1 = \pi$. In the sequel, we will also be interested in the spaces

$$\begin{aligned} C^{k,\alpha}(\Sigma)_{s,0} := \{\phi \in C^{k,\alpha}([0, 2\pi]) : \phi(\vartheta_1) = \phi(2\pi - \vartheta_1) \text{ for any } \vartheta_1 \in [0, \pi], \\ \phi'(0) = \phi'(2\pi) = \phi^{(3)}(0) = \phi^{(3)}(2\pi) = 0\}. \end{aligned} \tag{31}$$

By the symmetries of the Laplacian, the gradient and the geometric quantities of Σ, one can show that $\tilde{L}_0$ preserves the symmetries of functions $\phi \in C^{4,\alpha}(\Sigma)_s$, that is it maps $C^{4,\alpha}(\Sigma)_s$ into $C^{0,\alpha}(\Sigma)_s$.

Let us introduce the operator

$$\mathcal{L} : C^{4,\alpha}(\Sigma)_{s,0} \times \mathbb{R} \to C^{0,\alpha}(\Sigma)_s \times \mathbb{R}$$

defined by

$$\mathcal{L}(\phi, \lambda) := \left(\tilde{L}_0\phi + \lambda, \int_\Sigma \phi(\zeta) d\sigma(\zeta) \right).$$

When we solve the equation $\mathcal{L}(\phi, \lambda) = (h, a)$, with h symmetric with respect to y_1, the solution ϕ will satisfy the same symmetry property, thus also its second and fourth derivative will do, while the first and third will be antisymmetric. In order to extend it by periodicity, we need $\phi'(0) = \phi'(2\pi)$ and $\phi^{(3)}(0) = \phi^{(3)}(2\pi)$, hence we need zero Neumann boundary conditions.

It is known that $\tilde{L}_0$ is self-adjoint with respect to the $L^2(\Sigma)$-scalar product (see [16], section 3), thus, $\mathcal{L}$ is self adjoint with respect to the scalar product

$$< (\phi, \lambda), (\psi, \mu) > := \int_\Sigma \phi(\zeta)\psi(\zeta) d\sigma(\zeta) + \lambda\mu.$$

Remark 3 can be rephrased by saying that the operator $\mathcal{L}$ is injective. In fact, if $\mathcal{L}(\phi, \lambda) = 0$, multiplying by ϕ and integrating over Σ we get

$$\int_\Sigma \tilde{L}_0\phi(\zeta)\phi(\zeta) d\sigma(\zeta) = -\lambda \int_\Sigma \phi(\zeta) d\sigma(\zeta) = 0.$$

By the result of Weiner ([27], Corollary 2, page 34), the elements of the Kernel of $\tilde{L}_0$ are the normal components of the vector fields generated by conformal transformations. The only ones that preserve the symmetries of the Torus are dilations, that are excluded by the volume constraint. This is equivalent to say that $X \cap N = 0$, where

$$N := \{\phi \in C^4(\Sigma) : \tilde{L}_0\phi = 0\}, \; X := \left\{ \phi \in C^{4,\alpha}(\Sigma)_{s,0} : \int_\Sigma \phi(\zeta) d\zeta = 0 \right\}. \tag{32}$$

Moreover, once again by [27], Corollary 2, page 34, $\tilde{L}_0$ is positive definite on

$$N^\perp := \left\{ \phi \in C^{4,\alpha}(\Sigma) : \int_\Sigma \phi n d\sigma(\zeta) = 0, \forall n \in N \right\} \tag{33}$$

thus we conclude that $\phi = 0$, so $\lambda = 0$.

In order to show the solvability of the linear problem

$$\begin{cases} \tilde{L}_0\phi(\zeta) + \lambda = h(\zeta) & \forall \zeta \in \Sigma, \\ \int_\Sigma \phi = a, \end{cases} \tag{34}$$

also surjectivity is needed. For this purpose, we will use the Fredholm theory. First we note that, if $\phi \in C^{4,\alpha}(\Sigma)_{s,0}$, then

$$\tilde{L}_0\phi = \phi^{(4)} + L_1\phi,$$

for some linear operator of order 3, that will be denoted by L_1, that can be computed explicitly exploiting the parametrization (74) of Σ, $\phi^{(4)}$ denotes the fourth derivative with respect to ϑ_1 and

$$\int_\Sigma \phi d\sigma(\zeta) = \int_0^{2\pi} \phi(\vartheta_1)(\sqrt{2} + \cos\vartheta_1)d\vartheta_1,$$

therefore, since $h \in C^{0,\alpha}(\Sigma)_s$ (34) becomes

$$\begin{cases} \phi^{(4)} + L_1\phi + \lambda = h & \forall\zeta \in \Sigma, \\ \int_0^{2\pi} \phi(\vartheta_1)d\vartheta_1 = \frac{a}{\sqrt{2}} - \frac{1}{\sqrt{2}}\int_0^{2\pi} \phi(\vartheta_1)\cos\vartheta_1 d\vartheta_1, \end{cases} \tag{35}$$

or equivalently

$$(\mathcal{L}_1 + \mathcal{L}_2)(\phi, \lambda) = (h, a/\sqrt{2}), \tag{36}$$

where

$$\mathcal{L}_1(\phi, \lambda) := \left(\phi^{(4)} + \lambda, \int_0^{2\pi} \phi(\vartheta_1)d\vartheta_1\right) \tag{37}$$

and

$$\mathcal{L}_2(\phi, \lambda) := \left(L_1\phi, \frac{1}{\sqrt{2}}\int_0^{2\pi} \phi(\vartheta_1)\cos\vartheta_1 d\vartheta_1\right). \tag{38}$$

In the next Proposition, we will prove the invertibility of $\mathcal{L}_1$, whose inverse will enable us to apply the Fredholm theory.

Proposition 7. *For any $(h, a) \in C^{0,\alpha}(\Sigma)_s \times \mathbb{R}$, there exists a unique solution $(\phi, \lambda) \in C^{4,\alpha}(\Sigma)_{s,0} \times \mathbb{R}$ to the problem*

$$\begin{cases} \phi^{(4)} + \lambda = h & \forall\vartheta_1 \in [0, 2\pi] \\ \int_0^{2\pi} \phi(\vartheta_1)d\vartheta_1 = a, \end{cases} \tag{39}$$

satisfying the estimate

$$|\phi|_{C^{4,\alpha}(\Sigma)} + |\lambda| \le c(|h|_{C^{0,\alpha}(\Sigma)} + |a|), \tag{40}$$

for some constant $c > 0$.

Proof. First we write $\phi = \phi^{||} + \phi^{\perp}$, where $\phi^{||}$ is a constant and $\phi^{\perp}$ has zero-average. Rephrasing (39) in this way, we get $\phi^{||} = a/2\pi$ and $\phi^{\perp}$ has to satisfy

$$(\phi^{\perp})^{(4)} + \lambda = h \; \forall \vartheta_1 \in [0, 2\pi]$$
$$\int_0^{2\pi} \phi^{\perp} = 0, \tag{41}$$
$$(\phi^{\perp})'(0) = (\phi^{\perp})'(2\pi) = (\phi^{\perp})^{(3)}(0) = (\phi^{\perp})^{(3)}(2\pi) = 0. \tag{42}$$

As a consequence, our problem is solved if we set

$$\lambda = \frac{1}{2\pi}\int_0^{2\pi} h, \phi^{\perp}(\vartheta_1) = \int_0^1 G(\vartheta_1, s)(h(s) - \lambda)ds + \phi_0,$$

where G is the Grenn function of $\frac{d^4}{d\vartheta_1^4}$ with double Neumann boundary conditions and $\phi_0 \in \mathbb{R}$ is chosen in such a way that $\phi^{\perp}$ has zero average. Since h is symmetric with respect to $\vartheta_1 = \pi$, then, by uniqueness, the same is true for $\phi^{\perp}$. The regularity of $\phi^{\perp}$ and the estimates of the norms follow by construction. □

Proposition 8. *Problem (34) is uniquely solvable for any* $(h, a) \in C^{0,\alpha}(\Sigma)_s \times \mathbb{R}$, *and the solution* $(\phi, \lambda) \in C^{4,\alpha}(\Sigma)_{s,0} \times \mathbb{R}$ *fulfills the estimate*

$$|\phi|_{C^{4,\alpha}(\Sigma)} + |\lambda| \le c(|h|_{C^{0,\alpha}(\Sigma)} + |a|), \tag{43}$$

for some constant $c > 0$.

Proof. It is enough to write (34) in the form

$$(\phi, \lambda) + \mathcal{L}_1^{-1}\mathcal{L}_2(\phi, \lambda) = \mathcal{L}_1^{-1}(h, a)$$

and to apply the Fredholm alternative theorem. Notice that $\mathcal{L}_1^{-1}\mathcal{L}_2$ is compact and $I + \mathcal{L}_1^{-1}\mathcal{L}_2$ is injective on the space $C^{4,\alpha}(\Sigma)_{s,0} \times \mathbb{R}$, since $\mathcal{L}$ is, as we observed above. □

In the sequel, we will often use the notation

$$B_k(1/4) := \{\phi \in C^{k,\alpha}(\Sigma)_{s,0} : |\phi|_{C^{k,\alpha}(\Sigma)} \le 1/4\}.$$

In constructing approximate solutions, we will generate error terms with high powers of ε but with high derivatives of ϕ. To deal with these terms, we introduce some smoothing operators, as in [24]. Let us recall that Alinhac and Gérard (see [1]) constructed a family of smoothing operators $\{R_\theta\}_{\theta\geq 1}$ such that

$$|R_\theta\phi|_{C^{k,\alpha}(\Sigma)} \leq c|\phi|_{C^{k',\alpha'}(\Sigma)} \text{ if } k+\alpha \leq k'+\alpha' \tag{44}$$

$$|R_\theta\phi|_{C^{k,\alpha}(\Sigma)} \leq c\theta^{k+\alpha-k'-\alpha'}|\phi|_{C^{k',\alpha'}(\Sigma)} \text{ if } k+\alpha \geq k'+\alpha' \tag{45}$$

$$|\phi - R_\theta\phi|_{C^{k,\alpha}(\Sigma)} \leq c\theta^{k+\alpha-k'-\alpha'}|\phi|_{C^{k',\alpha'}(\Sigma)} \text{ if } k+\alpha \leq k'+\alpha'. \tag{46}$$

We point out that, since the construction of the R_θ's relies on cutting-off high Fourier modes, we have

$$\int_\Sigma \phi = \int_\Sigma R_\theta\phi, \ \forall\phi \in C^{4,\alpha}(\Sigma)_s.$$

It is possible to find further details in [9], where the periodic (compact) case is specifically treated.

These operators are useful to gain regularity. For instance, we will start our construction of the approximate solution from a function $\phi \in C^{4,\alpha}(\Sigma)$, since the operator $\tilde{L}_0$ is a fourth-order one, but some higher order derivatives will appear in the construction. As a consequence, we replace ϕ by $\phi_\star := R_{1/\varepsilon}\phi$ in the expansion of the Laplacian (29), that is we set $t = z - \phi_\star(\varepsilon y)$, for any $y \in \Sigma_\varepsilon$. We need to estimate the derivatives of $\phi_\star$ up to order six, which will be done by means of the properties of the operators R_θ.

3.2. *Exponentially decaying functions on* $\mathbb{R}^3$

For any $\delta > 0$ and for any $x \in \mathbb{R}^N$, we define

$$\varphi_\delta(x) := \zeta(|x|) + (1-\zeta(|x|))e^{\delta|x|},$$

where $\chi : \mathbb{R} \to \mathbb{R}$ is a C^∞ cutoff function such that

$$\zeta(t) = \begin{cases} 1 & \text{for } t < 1 \\ 0 & \text{for } t > 2. \end{cases}$$

Moreover, we introduce the weighted spaces

$$C^{k,\alpha}_\delta(\mathbb{R}^3) := \{u \in C^{k,\alpha}(\mathbb{R}^3) : ||\tilde{u}_\delta||_{C^{k,\alpha}(\mathbb{R}^3)} < \infty\},$$

where $\tilde{u}_\delta := u\varphi_\delta$ and $C^{k,\alpha}(\mathbb{R}^3)$ is the space of $C^k(\mathbb{R}^3)$ functions whose kth-derivatives are Hölder continuous with exponent α. We point out that functions belonging $C^{k,\alpha}_\delta(\mathbb{R}^3)$ decay exponentially with rate δ, and the same is true for their derivatives.

This spaces are endowed with the norm

$$||u||_{C^{k,\alpha}(\mathbb{R}^3)} := \sum_{j=0}^{k} ||\nabla^j u||_\infty + [\nabla^k u]_\alpha.$$

In order to construct solutions to (4) that respect the symmetry of the Torus, we need to introduce the spaces of functions fulfilling these symmetries, that is

$$C^{k,\alpha}_{\delta,s}(\mathbb{R}^3) := \{u \in C^{k,\alpha}_{\delta}(\mathbb{R}^3) : u(Tx) = u(x)\,,\, u(Rx) = u(x) \text{ for any } R \in SO_{x_3}(3)\}.$$

Remark 9. We note that, for instance, if $u \in C^{2,\alpha}_{\delta,s}(\mathbb{R}^3)$, then $\Delta u \in C^{0,\alpha}_{\delta,s}(\mathbb{R}^3)$. In fact, by definition, any $u \in C^{2,\alpha}_{\delta,s}(\mathbb{R}^3)$ satisfies $u(x) = u_T(x)$, where $u_T(x) := u(Tx)$. Taking the Laplacian, we can see that $\Delta u(x) = \Delta u_T(x) = \Delta u(Tx)$, and similarly, if $R \in SO_{x_3}(3)$ and we set $u_R(x) = u(Rx)$, then $\Delta u(x) = \Delta u_R(x) = \Delta u(Rx)$.

3.3. *Functions on $\Sigma_\varepsilon \times \mathbb{R}$*

First we will show existence and uniqueness of the heteroclinic solution to the ODE $-v_\star'' + W'(v_\star) = 0$. The result is known, but since the proof is quite short, we report it for completeness.

Lemma 10. *Let W be an even double well potential satisfying (3). Then there exists a unique solution $v_\star$ to the problem*

$$\begin{cases} -v_\star'' + W'(v_\star) = 0 \\ v_\star(0) = 0 \\ v_\star \to \pm 1 & \text{as } t \to \pm\infty, \end{cases} \tag{47}$$

and this solution is odd.

It is known that, if $W(t) = \frac{1}{4}(1 - t^2)^2$ is the classical double-well potential, then $v_\star(t) = \tanh(t/\sqrt{2})$.

Proof. Let $v_\star$ be the unique solution to the Cauchy Problem

$$\begin{cases} -v_\star'' + W'(v_\star) = 0 \\ v_\star(0) = 0 \\ v_\star'(0) = \sqrt{2W(0)}. \end{cases}$$

Let (a, b) be its maximal interval of definition, with $a < 0 < b$. Since the function $w(t) = -v_\star(-t)$ is still a solution to the same Cauchy Problem, $v_\star$ is an odd function, so it is enough to study $v_\star$ in the positive half line and $a = -b$. Multiplying the ODE by $v_\star'$ and integrating we have

$$\frac{1}{2}(v_\star')^2 = W(v_\star) + c. \tag{48}$$

Evaluating at $t = 0$, it is possible to see that $c = 0$. As a consequence, $v_\star' > 0$ in $(0, b)$. In fact, if we assume by contradiction that there exists a first t_0 such that $v_\star'(t_0) = 0$, then $W(v_\star(t_0)) = 0$, so in particular $v_\star(t_0) = 1$, but, by the uniqueness Cauchy Theorem, this implies that $v_\star \equiv 1$ in a neighbourhood of t_0, a contradiction. As a consequence, it is possible to define

$$l := \lim_{t \to b} v_\star(t).$$

By monotonicity, we know that $l > 0$. Now we want to rule out the case $l = \infty$. In fact, if this were true, we would have $v_\star'' < 0$ near 0 and $v_\star'' > 0$ near b, so there should exist $t_1 > 0$ such that $v_\star''(t_1) = 0$. Therefore, using the equation and (48), we can see that $v_\star(t_1) = 1$ and $v_\star'(t_1) = 0$, which is not possible.

Since $l < \infty$, we have $b = \infty$. Now, always by (48), we get that $v_\star' \to \sqrt{2W(l)}$ as $t \to \infty$. Since u is bounded, $W(l) = 0$, hence $l = 1$.

Uniqueness follows from the Cauchy Theorem. □

It is known that $v_\star$ converges exponentially to ± 1 as $t \to \pm\infty$ at a rate which is given by $\sqrt{W''(1)} = \sqrt{W''(-1)}$, since W is even. More precisely, for any $k \in \mathbb{N}$, there exists a constant c_k such that

$$|\partial_t^k (v_\star - 1)| \le c_k e^{-t\sqrt{W''(1)}} \text{ for any } t \ge 0 \tag{49}$$

and

$$|\partial_t^k (v_\star + 1)| \le c_k e^{t\sqrt{W''(1)}} \text{ for any } t \le 0. \tag{50}$$

For instance, in the classical case $W(t) = \frac{1}{4}(1 - t^2)^2$, we have $\sqrt{W''(\pm 1)} = \sqrt{2}$.

For $0 < \delta < \sqrt{W''(1)}$, we define the function

$$\psi_\delta(t) = (1 + e^t)^\delta (1 + e^{-t})^\delta.$$

For $0 < \varepsilon \le 1$ and $0 < \alpha < 1$, we define the space $C_\delta^{k,\alpha}(\Sigma_\varepsilon \times \mathbb{R})$ as the set of functions $U : \Sigma_\varepsilon \times \mathbb{R} \to \mathbb{R}$ that are k times differentiable and whose k-th partial derivatives are Hölder continuous with exponent α. This space is endowed with the norm

$$||U||_{C_\delta^{k,\alpha}(\Sigma_\varepsilon \times \mathbb{R})} = ||U\psi_\delta||_{C^{k,\alpha}(\Sigma_\varepsilon \times \mathbb{R})},$$

where

$$||U||_{C^{k,\alpha}(\Sigma_\varepsilon \times \mathbb{R})} = \sum_{j=0}^{k} ||\nabla^j U||_{L^\infty(\Sigma_\varepsilon \times \mathbb{R})} + \sup_{x \neq y} \frac{|\nabla^k u(x) - \nabla^k u(y)|}{|x - y|^\alpha}.$$

Given the heteroclinic solution $v_\star$, we can define the spaces

$$\mathcal{E}_\delta^{k,\alpha}(\Sigma_\varepsilon \times \mathbb{R}) := \left\{ U \in C_\delta^{k,\alpha}(\Sigma_\varepsilon \times \mathbb{R}) : \int_{-\infty}^{\infty} U(y,t) v_\star'(t) dt = 0 \text{ for any } y \in \Sigma_\varepsilon \right\}$$

of functions that orthogonal, for any $y \in \Sigma_\varepsilon$, to $v_\star'$.

Moreover, as above, we will be interested in the spaces of functions that respect the symmetries of the Torus, thus we define

$$C^{k,\alpha}_{\delta,s}(\Sigma_\varepsilon \times \mathbb{R}) := \{U \in C^{k,\alpha}_{\delta}(\Sigma_\varepsilon \times \mathbb{R}) : U_T = U,\ U_R = U \text{ for any } R \in SO_{x_3}(3)\},$$

where we have set $U_T(y,z) := U(Ty,z)$ and $U_R(y,z) := U(Ry,z)$. Furthermore, we set $\mathcal{E}^{k,\alpha}_{\delta,s}(\Sigma_\varepsilon \times \mathbb{R}) := \mathcal{E}^{k,\alpha}_{\delta}(\Sigma_\varepsilon \times \mathbb{R}) \cap C^{k,\alpha}_{\delta,s}(\Sigma_\varepsilon \times \mathbb{R})$. These spaces consist of functions that are both symmetric and orthogonal to $v'_\star$.

In the sequel, we will often mention the operator

$$\mathcal{L}_\varepsilon[U] := -(\Delta_{\Sigma_\varepsilon} + \partial_{tt})U(y,t) + W''(v_\star(t))U(y,t),$$

defined for any $U \in C^{4,\alpha}_{\delta}(\Sigma_\varepsilon \times \mathbb{R})$.

4. Idea of the proof: Lyapunov–Schmidt reduction

By a rescaling argument, it is enough to construct solutions to

$$-\Delta(-\Delta u + W'(u)) + W''(u)(-\Delta u + W'(u)) = \varepsilon^4 \lambda(1-u),$$

whose nodal set is close to Σ_ε, since we can obtain the required solutions to (4) by setting $\tilde{u}(x) := u(x/\varepsilon)$. Thus we set

$$F(u) = -\Delta(-\Delta u + W'(u)) + W''(u)(-\Delta u + W'(u)). \tag{51}$$

A computation shows that

$$\begin{aligned} F'(u)[v] = -\Delta(-\Delta v + W''(u)v) + W''(u)(-\Delta v + W''(u)v) \\ + W'''(u)(-\Delta u + W'(u))v \end{aligned} \tag{52}$$

and

$$\begin{aligned} F''(u)[v,w] = -\Delta(W'''(u)vw) + (W'''(u)W''(u) + W^{(4)}(u)(-\Delta u + W'(u)))vw + \\ W'''(u)[w(-\Delta v + W''(u)v) + v(-\Delta w + W''(u)w)]. \end{aligned} \tag{53}$$

In order to produce the required solutions we fix $\varepsilon > 0$ small and a small function $\phi \in C^{4,\alpha}(\Sigma)_{s,0}$, in the sense that $|\phi|_{C^{4,\alpha}(\Sigma)} < 1/4$, and we define the approximate solution $v_{\varepsilon,\phi}$ in such a way that its nodal set is close to

$$\Sigma_{\varepsilon,\phi} := \{y + \phi(\varepsilon y)\nu(\varepsilon y) : y \in \Sigma_\varepsilon\},$$

for ε small enough and $v_{\varepsilon,\phi} \equiv \pm 1$ outside a sufficiently small neighbourhood of $\Sigma_{\varepsilon,\phi}$. More precisely, we set $\phi_\star := R_{1/\varepsilon}\phi$,

$$\Sigma_{\varepsilon,\phi_\star} := \{y + \phi_\star(\varepsilon y)\nu(\varepsilon y) : y \in \Sigma_\varepsilon\},$$

$$\mathbb{H}(x) := \begin{cases} 1 & \text{if } f_\varepsilon(x) > 0 \\ 0 & \text{if } f_\varepsilon(x) = 0 \\ -1 & \text{if } f_\varepsilon(x) < 0 \end{cases}$$

and, for any $\varepsilon > 0$ and for any integer $m > 0$,

$$\chi_m(x) := \begin{cases} \zeta(|t| - \frac{\tau}{2\varepsilon} - m) & \text{if } x = Z_\varepsilon(y, t + \phi_\star(\varepsilon y)) \text{ and } d(x, \Sigma_{\varepsilon,\phi_\star}) < \tau/\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

We look for an approximate solution of the form

$$v_{\varepsilon,\phi}(x) = \chi_5(x)\tilde{v}_{\varepsilon,\phi}(y,t) + (1 - \chi_5(x))\mathbb{H}(x), \tag{54}$$

where $t = z - \phi_\star(\varepsilon y)$ and $v_{\varepsilon,\phi}$ is understood to coincide with $\mathbb{H}$ outside the support of χ_5. From now on, we will consider the expansion of the Laplacian (29) with ϕ replaced by $\phi_\star$. This is more convenient for the forthcoming computations. Moreover $v_{\varepsilon,\phi}$ will vanish close to $\Sigma_{\varepsilon,\phi}$ and it will respect the symmetries of the Torus. We stress that these cutoff functions actually depend on ϕ, but we prefer not to put the subscript ϕ to simplify the notation. However, we will see that the error $F(v_{\varepsilon,\phi})$ is small, but not zero, therefore we have to add a correction $w = w_{\varepsilon,\phi,\lambda}$ depending on ε, ϕ and λ in order to obtain a real solution, that is $F(v_{\varepsilon,\phi} + w) = 0$. Rephrasing our problem in this way, the unknowns are ϕ and w, for any $\varepsilon > 0$ small but fixed. Expanding F in Taylor series, our equation becomes

$$F(v_{\varepsilon,\phi}) + F'(v_{\varepsilon,\phi})w + Q_{\varepsilon,\phi}(w) = 0, \tag{55}$$

where

$$Q_{\varepsilon,\phi}(w) = \int_0^1 dt \int_0^t F''(v_{\varepsilon,\phi} + sw)[w,w]ds. \tag{56}$$

However, we are not able to solve it directly, because of the lack of coercivity of $F'(v_{\varepsilon,\phi})$.

4.1. The auxiliary equation: a gluing procedure

We look for a solution of the form

$$w(x) = \chi_2(x)U(y,t) + V(x), \tag{57}$$

where V is defined in the whole $\mathbb{R}^3$, U is defined in the entire $\Sigma_\varepsilon \times \mathbb{R}$. Since we want our solutions u_ε to respect the symmetries of the Torus, we look for solutions U and V such that

$$\begin{aligned} U(y,t) = U(Ty,t),\ \ U(y,t) = U(Ry,t),\ \ &\text{for any } R \in SO_{x_3}(3) \text{ and } (y,t) \in \Sigma_\varepsilon \times \mathbb{R} \\ V(x) = V(Tx), \qquad V(x) = V(Rx), \qquad &\text{for any } R \in SO_{x_3}(3) \text{ and } x \in \mathbb{R}^3. \end{aligned}$$

Now we observe that the potential

$$\Gamma_{\varepsilon,\phi}(x) := (1 - \chi_1(x))W''(v_{\varepsilon,\phi}) + \chi_1(x)W''(1) \tag{58}$$

is positive and bounded away from 0 in the whole $\mathbb{R}^3$, that is, for any $0 < \delta < \sqrt{W''(1)}$, $0 < \delta^2 < \Gamma_{\varepsilon,\phi}(x) < W''(1)$ provided ε is small enough, the estimate is uniform in ϕ. Moreover, using that $\chi_2\chi_1 = \chi_1$, (51) becomes

$$\begin{aligned} 0 = \chi_2\Big\{ & F(\tilde{v}_{\varepsilon,\phi}) + F'(\tilde{v}_{\varepsilon,\phi})[U] + \chi_1 Q_{\varepsilon,\phi}(U+V) + \chi_1 \mathrm{M}_{\varepsilon,\phi}(V) \\ & - \varepsilon^4\chi_1\lambda(1-\tilde{v}_{\varepsilon,\phi} - V) + \varepsilon^4\lambda U\Big\} \\ & + (-\Delta + \Gamma_{\varepsilon,\phi})^2 V + (1-\chi_2)F(v_{\varepsilon,\phi}) + (1-\chi_1)Q_{\varepsilon,\phi}(\chi_2 U + V) \\ & + \mathrm{N}_{\varepsilon,\phi}(U) + \mathrm{P}_{\varepsilon,\phi}(V) - \varepsilon^4\lambda(1-\chi_1)(1-v_{\varepsilon,\phi} - V), \end{aligned} \tag{59}$$

where

$$\begin{aligned} \mathrm{M}_{\varepsilon,\phi}(V) &:= (W''(\tilde{v}_{\varepsilon,\phi}) - W''(1))(-\Delta V + \Gamma_{\varepsilon,\phi}V) \\ &\quad + (-\Delta + W''(\tilde{v}_{\varepsilon,\phi}))\big[(W''(\tilde{v}_{\varepsilon,\phi}) - W''(1))V\big] \end{aligned} \tag{60}$$

$$\begin{aligned} \mathrm{N}_{\varepsilon,\phi}(U) &:= -2 < \nabla\chi_2, \nabla(-\Delta U + W''(\tilde{v}_{\varepsilon,\phi})U) > -\Delta\chi_2(-\Delta U + W''(\tilde{v}_{\varepsilon,\phi})U) \\ &\quad + (-\Delta + W''(\tilde{v}_{\varepsilon,\phi}))(-2 < \nabla\chi_2, \nabla U > -\Delta\chi_2 U) \end{aligned} \tag{61}$$

$$\begin{aligned} \mathrm{P}_{\varepsilon,\phi}(V) &:= -2 < \nabla\chi_1, \nabla((W''(\tilde{v}_{\varepsilon,\phi}) - W''(1))V) > -\Delta\chi_1(W''(\tilde{v}_{\varepsilon,\phi}) - W''(1))V \\ &\quad + W'''(v_{\varepsilon,\phi})(-\Delta v_{\varepsilon,\phi} + W'(v_{\varepsilon,\phi}))V. \end{aligned} \tag{62}$$

Hence we have reduced our problem to finding a solution (V, U) to the system

$$\begin{aligned} &(-\Delta + \Gamma_{\varepsilon,\phi})^2 V + (1-\chi_2)F(v_{\varepsilon,\phi}) + (1-\chi_1)Q_{\varepsilon,\phi}(\chi_2 U + V) \\ &\quad + \mathrm{N}_{\varepsilon,\phi}(U) + \mathrm{P}_{\varepsilon,\phi}(V) = \varepsilon^4\lambda(1-\chi_1)(1-v_{\varepsilon,\phi} - V) \qquad \text{in } \mathbb{R}^3 \end{aligned} \tag{63}$$

$$\begin{aligned} &F(\tilde{v}_{\varepsilon,\phi}) + F'(\tilde{v}_{\varepsilon,\phi})[U] + \chi_1 Q_{\varepsilon,\phi}(U+V) \\ &\quad + \chi_1\mathrm{M}_{\varepsilon,\phi}(V) = \varepsilon^4\lambda\chi_1(1-\tilde{v}_{\varepsilon,\phi} - V) - \varepsilon^4\lambda U \text{ for } |t| \le \tau/2\varepsilon + 4. \end{aligned} \tag{64}$$

The system of equations (63) and (64) is known as *auxiliary equation*. First we solve equation (63) for any fixed U, thanks to coercivity, due to the fact that $\Gamma_{\varepsilon,\phi}$ is bounded away from 0 uniformly in ε and ϕ. We will see that our solution also depends on the data U and ϕ in a Lipschitz way.

Proposition 11. *For any $\varepsilon > 0$ small enough, for any $U \in C^{4,\alpha}_{\delta,s}(\Sigma_\varepsilon \times \mathbb{R})$ satisfying $||U||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \le 1$, for any $\phi \in B_4(1/4)$ and for any $|\lambda| < 1$, equation (63) admits a unique solution $V_{\varepsilon,\phi,\lambda,U} \in C^{4,\alpha}_{\delta,s}(\mathbb{R}^3)$ satisfying*

$$\begin{cases} ||V_{\varepsilon,\phi,\lambda,U}||_{C^{4,\alpha}(\mathbb{R}^3)} \le C_1 e^{-a/\varepsilon} \\ ||V_{\varepsilon,\phi,\lambda,U_1} - V_{\varepsilon,\phi,\lambda,U_2}||_{C^{4,\alpha}(\mathbb{R}^3)} \le C_1 e^{-a/\varepsilon}||U_1 - U_2||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \\ ||V_{\varepsilon,\phi_1,\lambda_1,U} - V_{\varepsilon,\phi_2,\lambda_2,U}||_{C^{4,\alpha}(\mathbb{R}^3)} \le C_1 e^{-a/\varepsilon}(|\phi_1 - \phi_2|_{C^{4,\alpha}(\Sigma)} + |\lambda_1 - \lambda_2|) \end{cases} \tag{65}$$

$a := \delta\tau/2$, for any U_1, U_2 satisfying $||U_1||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})}, ||U_2||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \leq 1$, for any $\phi_1, \phi_2 \in B_4(1/4)$, for any $|\lambda_1|, |\lambda_2| < 1$, for some constants $a, C_1 > 0$ independent of U, ε and ϕ.

The proof of Proposition 11 is based on a fixed point argument (see section 6).

Now we consider equation (64). In order to solve it, we need to extend it to the whole $\Sigma_\varepsilon \times \mathbb{R}$. First we observe that

$$F'(v_{\varepsilon,\phi})[U] = \mathcal{L}^2_\varepsilon[U] + \mathrm{R}_{\varepsilon,\phi}[U],$$

where

$$\begin{aligned}\mathrm{R}_{\varepsilon,\phi}[U] &:= \mathcal{L}_\varepsilon(\mathrm{D} + W''(\tilde{v}_{\varepsilon,\phi}) - W''(v_\star))[U] + (\mathrm{D} + W''(\tilde{v}_{\varepsilon,\phi}) - W''(v_\star))\mathcal{L}_\varepsilon[U] \\ &\quad + (\mathrm{D} + W''(\tilde{v}_{\varepsilon,\phi}) - W''(v_\star))^2[U] + W'''(\tilde{v}_{\varepsilon,\phi})(-\Delta\tilde{v}_{\varepsilon,\phi} + W'(\tilde{v}_{\varepsilon,\phi}))U,\end{aligned}$$

D is defined in (29), with ϕ replaced by $\phi_\star$. Therefore we reduced ourselves to consider

$$\begin{aligned}\mathcal{L}^2_\varepsilon[U] &= -F(\tilde{v}_{\varepsilon,\phi}) - \chi_1 Q_{\varepsilon,\phi}(U+V) - \mathrm{R}_{\varepsilon,\phi}[U] \\ &\quad - \chi_1\mathrm{M}_{\varepsilon,\phi}(V) + \varepsilon^4\lambda\chi_1(1 - \tilde{v}_{\varepsilon,\phi} - V) - \varepsilon^4\lambda U\end{aligned} \tag{66}$$

in the entire $\Sigma_\varepsilon \times \mathbb{R}$. We would like to solve this equation with a fixed point argument, but, in order to do so, the right-hand side must be orthogonal to the Kernel of $\mathcal{L}^2_\varepsilon$, that is the one dimensional space generated by $v'_\star(t)$, hence we can solve the problem

$$\begin{aligned}\mathcal{L}^2_\varepsilon[U] &= -F(\tilde{v}_{\varepsilon,\phi}) + \varepsilon^4\lambda\chi_1(1-\tilde{v}_{\varepsilon,\phi}) - \mathrm{T}(U, V_{\varepsilon,\phi,\lambda,U}, \phi) + p(y)v'_\star(t) \\ &\int_{-\infty}^{\infty} U(y,t)v'_\star(t)dt = 0 \text{ for any } y \in \Sigma_\varepsilon,\end{aligned} \tag{67}$$

where we have set, for the sake of simplicity,

$$\mathrm{T}(U, V, \phi) := \chi_1 Q_{\varepsilon,\phi}(U+V) + \mathrm{R}_{\varepsilon,\phi}(U) + \chi_1\mathrm{M}_{\varepsilon,\phi}(V) + \varepsilon^4\lambda(\chi_1 V + U)$$

$$p(y) := \frac{1}{c_\star}\int_{-\infty}^{\infty}\big(F(\tilde{v}_{\varepsilon,\phi}) - \varepsilon^4\lambda\chi_1(1-\tilde{v}_{\varepsilon,\phi}) + \mathrm{T}(U, V_{\varepsilon,\phi,\lambda,U}, \phi)\big)(y,t)v'_\star(t)dt$$

and $c_\star := \int_{-\infty}^{\infty}(v'_\star(t))^2dt$.

Before stating the next proposition, let us observe that any function $U : \Sigma_\varepsilon \times \mathbb{R} \to \mathbb{R}$ can be written as the sum of an even part and an odd part, the even part being $U_e(y,t) := \frac{1}{2}(U(y,t) + U(y,-t))$ and the odd part being $U_o(y,t) := \frac{1}{2}(U(y,t) + U(y,-t))$.

Proposition 12. *For any $\varepsilon > 0$ small enough, for any $\phi \in B_4(1/4)$ and for any $|\lambda| < 1$, we can find a solution $U_{\varepsilon,\phi,\lambda} \in \mathcal{E}^{4,\alpha}_{\delta,s}(\Sigma_\varepsilon \times \mathbb{R})$ to equation (67) satisfying*

$$\begin{cases} ||U_{\varepsilon,\phi,\lambda}||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \leq C_2\varepsilon^3 \\ ||(U_{\varepsilon,\phi,\lambda})_o||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \leq C_2\varepsilon^4 \\ ||U_{\varepsilon,\phi_1,\lambda_1} - U_{\varepsilon,\phi_2,\lambda_2}||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \leq C_2\varepsilon^3(|\phi_1-\phi_2|_{C^{4,\alpha}(\Sigma)} + |\lambda_1-\lambda_2|), \end{cases} \tag{68}$$

for any $\phi_1, \phi_2 \in B_4(1/4)$, *for any* $|\lambda_1|, |\lambda_2| < 1$, *for some constant* $C_2 > 0$ *independent of* ε.

The proof of Proposition 12 will be given in section 6.

4.2. The bifurcation equation

In conclusion, we will show that it is possible to find ϕ that solves

$$\int_{-\infty}^{\infty} \big(F(\tilde{v}_{\varepsilon,\phi}) - \varepsilon^4\lambda\chi_1(1-\tilde{v}_{\varepsilon,\phi}) + \mathrm{T}(U, V_{\varepsilon,\phi,U}, \phi)\big)(y,t)v_\star'(t)dt = 0 \tag{69}$$

for any $y \in \Sigma_\varepsilon$ and such that the real solution $u_\varepsilon(x) := v_{\varepsilon,\phi}(x/\varepsilon) + w_{\varepsilon,\phi,\lambda}(x/\varepsilon)$ satisfies the volume constraint (8). We will show in Proposition 17 and in the proof of Proposition 12 in Section 6.4 that equation (69) is equivalent to

$$\tilde{L}_0\phi(y) + \frac{2}{c_\star}\lambda = \varepsilon q_\varepsilon^1(\phi,\lambda)(y) + \varepsilon^2 q_\varepsilon^2(\phi,\lambda)(y), \tag{70}$$

where q_ε^1 satisfies

$$\begin{cases} |q_\varepsilon^1(\phi,\lambda)|_{C^{0,\alpha}(\Sigma)} \leq c \\ |q_\varepsilon^1(\phi_1,\lambda_1) - q_\varepsilon^1(\phi_2,\lambda_2)|_{C^{0,\alpha}(\Sigma)} \leq c(|\phi_1-\phi_2|_{C^{0,\alpha}(\Sigma)} + |\lambda_1-\lambda_2|), \end{cases} \tag{71}$$

for any $(\phi,\lambda) \in C^{4,\alpha}(\Sigma)_{s,0} \times \mathbb{R}$ fulfilling $|\phi|_{C^{4,\alpha}(\Sigma)} + |\lambda| < 1$, and q_ε^2 satisfies

$$\begin{cases} |q_\varepsilon^2(\phi,\lambda)|_{C^{0,\alpha}(\Sigma)} \leq C \\ |q_\varepsilon^2(\phi_1,\lambda_1) - q_\varepsilon^2(\phi_2,\lambda_2)|_{C^{0,\alpha}(\Sigma)} \leq C(|\phi_1-\phi_2|_{C^{0,\alpha}(\Sigma)} + |\lambda_1-\lambda_2|), \end{cases} \tag{72}$$

for any $(\phi,\lambda) \in C^{4,\alpha}(\Sigma)_{s,0} \times \mathbb{R}$ fulfilling $|\phi|_{C^{4,\alpha}(\Sigma)} + |\lambda| < \tilde{c}\varepsilon$, for some $\tilde{c} > 0$. We stress that the constant C appearing in (72) may depend on $\tilde{c}$. As regards the volume constraint, we note that, by the change of variables $x' = x/\varepsilon$,

$$8\pi^2\sqrt{2}c_\varepsilon = \int_{\mathbb{R}^3} (1-u_\varepsilon(x))^2 dx = \varepsilon^3 \int_{\mathbb{R}^3} 1 - (v_{\varepsilon,\phi}(x) + w_{\varepsilon,\phi,\lambda}(x))^2 dx,$$

$$c_\varepsilon := 1 + \varepsilon\int_0^\infty (v_\star^2 - 1)dt.$$

The latter integral can be calculated exploiting the natural change of variables

$$\begin{cases} x_1 = \varepsilon^{-1}\cos(\varepsilon y_2)\big((z+\varepsilon^{-1})\cos(\varepsilon y_1) + \varepsilon^{-1}\sqrt{2}\big), \\ x_2 = \varepsilon^{-1}\sin(\varepsilon y_2)\big((z+\varepsilon^{-1})\cos(\varepsilon y_1) + \varepsilon^{-1}\sqrt{2}\big), \\ x_3 = \varepsilon^{-1}(z+\varepsilon^{-1})\sin(\varepsilon y_1), \end{cases} \tag{73}$$

on $V_{\tau/\varepsilon}$, induced by the parametrization $Y_\varepsilon(y) = \varepsilon^{-1}Y(\varepsilon y)$, where

$$Y(\vartheta_1, \vartheta_2) := (\cos\vartheta_2(\cos\vartheta_1 + \sqrt{2}), \sin\vartheta_2(\cos\vartheta_1 + \sqrt{2}), \sin\vartheta_2) \tag{74}$$

and $(\vartheta_1, \vartheta_2) = \varepsilon(y_1, y_2) \in [0, 2\pi)^2$.

Proposition 13. *For any $\varepsilon > 0$ small enough, for any $\phi \in C^{4,\alpha}(\Sigma)_{s,0}$, we have*

$$\int_{\mathbb{R}^3} 1 - (v_{\varepsilon,\phi}(x) + w_{\varepsilon,\phi,\lambda}(x))^2 dx = \varepsilon^{-3} 8\pi^2\sqrt{2}c_\varepsilon + 4\varepsilon^{-2}\int_\Sigma \phi(\zeta)d\sigma(\zeta)$$

$$+ 16\sqrt{2}\pi^2\varepsilon^{-1}\int_0^\infty t(1 - v_\star(t))dt + 4G_\varepsilon(\phi, \lambda),$$

with G_ε fulfilling

$$\begin{cases} |G_\varepsilon(\phi, \lambda)| \le c, \\ |G_\varepsilon(\phi_1, \lambda_1) - G_\varepsilon(\phi_2, \lambda_2)| \le c(|\phi_1 - \phi_2|_{C^{4,\alpha}(\Sigma)} + |\lambda_1 - \lambda_2|) \end{cases}$$

if $\phi, \phi_1, \phi_2 \in C^{4,\alpha}(\Sigma)_s$ satisfy $|\phi|_{C^{4,\alpha}(\Sigma)}, |\phi_1|_{C^{4,\alpha}(\Sigma)}, |\phi_2|_{C^{4,\alpha}(\Sigma)} \le \tilde{c}\varepsilon$, for some $\tilde{c} > 0$, for any $\lambda, \lambda_1, \lambda_2$ such that $|\lambda|, |\lambda_1|, |\lambda_2| < 1$.

The proof of this Proposition will be given in Section 7. Therefore, in terms of ϕ and λ, equation (8) is equivalent to equation

$$\int_\Sigma \phi(\zeta)d\sigma(\zeta) = -4\sqrt{2}\pi^2\varepsilon\int_0^\infty t(1 - v_\star(t))dt - \varepsilon^2 G_\varepsilon(\phi, \lambda). \tag{75}$$

In order to solve the bifurcation equation (70), we need to couple it with (75), due to the properties of $\tilde{L}_0$ (see Section 4.1), and we solve the system by a fixed point argument, that will be explained in this Proposition, whose proof will be carried out in Section 7.

Proposition 14. *For any $\varepsilon > 0$ small enough, the bifurcation equation*

$$\tilde{L}_0\phi + \frac{2}{c_\star}\lambda = \varepsilon q_\varepsilon^1(\phi, \lambda) + \varepsilon^2 q_\varepsilon^2(\phi, \lambda) \tag{76}$$

$$\int_\Sigma \phi(\zeta)d\sigma(\zeta) = -4\sqrt{2}\pi^2\varepsilon\int_0^\infty t(1 - v_\star(t))dt - \varepsilon^2 G_\varepsilon(\phi, \lambda), \tag{77}$$

admits a solution $(\phi, \lambda) \in C^{4,\alpha}(\Sigma)_{s,0} \times \mathbb{R}$ *satisfying*

$$|\phi|_{C^{4,\alpha}(\Sigma)} + |\lambda| \le C_3 \varepsilon, \tag{78}$$

for some constant $C_3 = C_3(W, \tau) > 0$.

Remark 15. (i) In the statement of Proposition 14, we use the term bifurcation equation for a system. This is a slight abuse of language, however it is convenient since it is consistent with the bifurcation theory.

(ii) As we will see in the proof of Proposition 17 below, the Willmore equation will appear at order ε^3, while the linearized operator

$$\begin{aligned}\tilde{L}_0 \phi = L_0^2 \phi + \frac{3}{2} H^2 L_0 \phi - H(\nabla_\Sigma \phi, \nabla_\Sigma H) + 2(A \nabla_\Sigma \phi, \nabla_\Sigma H) + \\ 2H < A, \nabla^2 \phi > + \phi(2 < A, \nabla^2 H > + |\nabla_\Sigma H|^2 + 2H \mathrm{tr} A^3),\end{aligned} \tag{79}$$

will appear at order ε^4, thus it is crucial for the remainder to be smaller in order to apply a contraction mapping principle. This is actually the case thanks to the fact that the odd part of $U_{\varepsilon,\phi,\lambda}$ is of order ε^4.

5. The approximate solution

5.1. Construction

First one can try to take $v_\star(z - \phi(\varepsilon y))$ as an approximate solution, where $\phi \in B_4(1/4) = \{\phi \in C^{4,\alpha}(\Sigma)_{s,0} : |\phi|_{C^{4,\alpha}} < 1/4\}$ is some small function that respects the symmetries of Σ and (y, z) are the Fermi coordinates of Σ. We will see that these symmetries will be inherited by our approximate solution (see Remark 16 below). In this way, our approximate solution vanishes exactly on

$$\Sigma_{\varepsilon,\phi} := \{y + \phi(\varepsilon y)\nu(\varepsilon y) : y \in \Sigma_\varepsilon\} \tag{80}$$

and goes from -1 to 1 in a monotone way. The speed of the phase transition is independent of ε, but it becomes faster and faster when ε becomes small if compared to the scale of Σ_ε, that increases as $1/\varepsilon$.

With the change of variables $t = z - \phi_\star(\varepsilon y)$, we have

$$v_\star(z - \phi(\varepsilon y)) = v_\star(t + \phi_\star(\varepsilon y) - \phi(\varepsilon y)).$$

We recall that $\phi_\star := R_{1/\varepsilon}\phi$, where the smoothing operators $R_{1/\varepsilon}$ are introduced in Section 3.1 and satisfy (44), (45), (46). We write

$$v_\star(t + \phi_\star(\varepsilon y) - \phi(\varepsilon y)) = v_\star(t) + v_{1,\varepsilon,\phi}(y, t),$$

where

$$v_{1,\varepsilon,\phi}(y, t) := v_\star(t + \phi_\star(\varepsilon y) - \phi(\varepsilon y)) - v_\star(t).$$

Since the Fermi coordinates are just defined in a neighbourhood of the Torus, our approximate solution is not defined everywhere. For our purposes, it is enough to consider it in the set

$$B = \{x = Z_\varepsilon(y, t + \phi_\star(\varepsilon y)) \in \mathbb{R}^3 : |t| < \tau/2\varepsilon + 7\}, \tag{81}$$

that is a tubular neighbourhood of

$$\Sigma_{\varepsilon,\phi_\star} = \{y + \phi_\star(\varepsilon y)\nu(\varepsilon y) : y \in \Sigma_\varepsilon\}$$

of width $\tau/2\varepsilon + 7$. Then it will be extended to the whole $\mathbb{R}^3$ with the aid of a cutoff function.

Now let us first compute $F(v_\star(t))$. In the sequel, by a Taylor expansion, we will also consider the contribution of $v_{1,\varepsilon,\phi}$. In the forthcoming computations, $v_\star$ and its derivatives will always be evaluated at t, the geometric quantities, ϕ and its derivatives will always be evaluated at εy. By (26),

$$\begin{aligned}
-\Delta v_\star + W'(v_\star) = &-v_\star'' + W'(v_\star) + \varepsilon \hat{H}(\varepsilon y, \varepsilon(t+\phi_\star))v_\star' \\
&+ \varepsilon^2 \Delta_\Sigma \phi_\star v_\star' - \varepsilon^2 |\nabla\phi_\star|^2 v_\star'' + \varepsilon^3 (t+\phi_\star)(a_1^{ij}(\phi_\star)_{ij} + b_1^i(\phi_\star)_i)v_\star' \\
&- \varepsilon^3 (t+\phi_\star) a_1^{ij}(\phi_\star)_i(\phi_\star)_j v_\star'' \\
&+ \varepsilon^4 (t+\phi_\star)^2 (a_2^{ij}(\phi_\star)_{ij} + b_2^i(\phi_\star)_i)v_\star' - \varepsilon^4 (t+\phi_\star)^2 a_2^{ij}(\phi_\star)_i(\phi_\star)_j v_\star'' \\
&\varepsilon^2 (\overline{a}^{ij}(\phi_\star)_{ij} + \overline{b}^i(\phi_\star)_i)v_\star' - \varepsilon^2 \overline{a}^{ij}(\phi_\star)_i(\phi_\star)_j v_\star''.
\end{aligned} \tag{82}$$

The term of order 0 in ε vanishes since $v_\star$ satisfies the ODE $-v_\star'' + W'(v_\star) = 0$. Thus, in order to compute $F(v_\star)$, we need to apply the linear operator $-\Delta + W''(v_\star)$ to the remaining terms. We will write down all terms of order less or equal than 4, the other ones being lower order terms, in some sense that will be clear soon. Let us set, for any function $v \in C^2(\mathbb{R})$, $L_\star v := -v'' + W''(v_\star)v$. Differentiating the ODE satisfied by $v_\star$, we get $L_\star v_\star' = 0$, thus using the Taylor expansion of $\tilde{H}$, the first term of (82) gives

$$\begin{aligned}
T^1_{\varepsilon,\phi}(y,t) = &\big(-\Delta + W''(v_\star)\big)(\varepsilon \hat{H}(\varepsilon y, \varepsilon(t+\phi_\star))v_\star') = \varepsilon^2 (H^2 - 2|A|^2)v_\star'' \\
&+ \varepsilon^3 \Big\{ (2H|A|^2 - 4\mathrm{tr}A^3)(t+\phi_\star)v_\star'' + (H|A|^2 - 2\mathrm{tr}A^3)v_\star' - \Delta_\Sigma H v_\star' \\
&+ 2(\nabla_\Sigma H, \nabla_\Sigma \phi_\star)v_\star'' - H|\nabla_\Sigma\phi_\star|^2 v_\star''' + H\Delta_\Sigma\phi_\star v_\star'' \Big\} \\
&+ \varepsilon^4 \Big\{ (|A|^4 - 6H_4 + 2H\mathrm{tr}A^3)((t+\phi_\star)^2 v_\star'' + (t+\phi_\star)v_\star') - \Delta_\Sigma |A|^2 (t+\phi_\star)v_\star' \\
&+ 2(\nabla_\Sigma |A|^2, \nabla_\Sigma\phi_\star)(t+\phi_\star)v_\star'' - |A|^2 |\nabla_\Sigma\phi_\star|^2 (t+\phi_\star)v_\star''' + \Delta_\Sigma\phi_\star |A|^2 (t+\phi_\star)v_\star'' \\
&- (a_1^{ij}H_{ij} + b_1^i H_i)(t+\phi_\star)v_\star' + 2a_1^{ij}H_i(\phi_\star)_j(t+\phi_\star)v_\star'' \\
&+ H(a_1^{ij}(\phi_\star)_{ij} + b_1^i(\phi_\star)_i)(t+\phi_\star)v_\star'' - H a_1^{ij}(\phi_\star)_i(\phi_\star)_j v_\star''' \Big\} + \varepsilon^5 F^1_{\varepsilon,\phi}(y,t),
\end{aligned} \tag{83}$$

with $F^1_{\varepsilon,\phi}$ is small and Lipschitzian in ϕ, in the sense that, in view of (44),

$$\begin{cases} |S_{\varepsilon,\phi}F^1_{\varepsilon,\phi}|_{C^{0,\alpha}(\Sigma)} \le c \\ |S_{\varepsilon,\phi_1}F^1_{\varepsilon,\phi_1} - S_{\varepsilon,\phi_2}F^1_{\varepsilon,\psi}|_{C^{0,\alpha}(\Sigma)} \le c|\phi_1 - \phi_2|_{C^{4,\alpha}(\Sigma)}, \end{cases} \tag{84}$$

for any $\phi, \phi_1, \phi_2 \in B_4(1/4)$, for some constant $c = c(W,\tau) > 0$ independent of ε and ϕ.

Similarly, the second term of (82) gives

$$\begin{aligned} T^2_{\varepsilon,\phi}(y,t) &= \big(-\Delta + W''(v_\star)\big)(\varepsilon^2\Delta_\Sigma\phi_\star v_\star') = \varepsilon^3 H\Delta_\Sigma\phi_\star v_\star'' \\ &+ \varepsilon^4\Big\{-(\Delta_\Sigma)^2\phi_\star v_\star' + |A|^2\Delta_\Sigma\phi_\star(t+\phi_\star)v_\star'' + 2(\nabla_\Sigma\Delta_\Sigma\phi_\star, \nabla_\Sigma\phi_\star)v_\star'' \\ &+ (\Delta_\Sigma\phi_\star)^2 v_\star'' - |\nabla_\Sigma\phi_\star|^2\Delta_\Sigma\phi_\star v_\star'''\Big\} + \varepsilon^5 F^2_{\varepsilon,\phi}(y,t), \end{aligned} \tag{85}$$

with $F^2_{\varepsilon,\phi}$ fulfilling (84).

The third term of (82) is already quadratic in ϕ, but, for the sake of completeness, we prefer to write it down.

$$\begin{aligned} T^3_{\varepsilon,\phi}(y,t) &= \big(-\Delta + W''(v_\star)\big)(-\varepsilon^2|\nabla_\Sigma\phi_\star|^2 v_\star'') = \varepsilon^2|\nabla_\Sigma\phi_\star|^2(v_\star^{(4)} - W''(v_\star)v_\star'') \\ &- \varepsilon^3 H|\nabla_\Sigma\phi_\star|^2 v_\star''' + \varepsilon^4\Big\{-|A|^2|\nabla_\Sigma\phi_\star|^2(t+\phi_\star)v_\star''' + \Delta_\Sigma|\nabla_\Sigma\phi_\star|^2 v_\star'' \\ &- 2(\nabla_\Sigma|\nabla_\Sigma\phi_\star|^2, \nabla_\Sigma\phi_\star)v_\star''' + |\nabla_\Sigma\phi_\star|^4 v_\star^{(4)} - |\nabla_\Sigma\phi_\star|^2\Delta_\Sigma\phi_\star v_\star'''\Big\} + \varepsilon^5 F^3_{\varepsilon,\phi}(y,t) \end{aligned} \tag{86}$$

The fourth term of (82) gives

$$\begin{aligned} T^4_{\varepsilon,\phi}(y,t) &= \big(-\Delta + W''(v_\star)\big)\big(\varepsilon^3(a_1^{ij}(\phi_\star)_{ij} + b_1^i(\phi_\star)_i)(t+\phi_\star)v_\star'\big) = \\ &- 2\varepsilon^3(a_1^{ij}(\phi_\star)_{ij} + b_1^i(\phi_\star)_i)v_\star'' \\ &+ \varepsilon^4 H(a_1^{ij}(\phi_\star)_{ij} + b_1^i(\phi_\star)_i)(v_\star' + (t+\phi_\star)v_\star'') + \varepsilon^5 F^4_{\varepsilon,\phi}(y,t). \end{aligned} \tag{87}$$

The fifth term of (82) gives

$$\begin{aligned} T^5_{\varepsilon,\phi}(y,t) &= \big(-\Delta + W''(v_\star)\big)(-\varepsilon^3 a_1^{ij}(\phi_\star)_i(\phi_\star)_j(t+\phi_\star)v_\star'') = \\ &\varepsilon^3 a_1^{ij}(\phi_\star)_i(\phi_\star)_j((t+\phi_\star)v_\star^{(4)} - (t+\phi_\star)W''(v_\star)v_\star'' + 2v_\star''') \\ &- \varepsilon^4 H a_1^{ij}(\phi_\star)_i(\phi_\star)_j(v_\star'' + (t+\phi_\star)v_\star''') + \varepsilon^5 F^5_{\varepsilon,\phi}(y,t), \end{aligned}$$

with $F^3_{\varepsilon,\phi}$, $F^4_{\varepsilon,\phi}$, $F^5_{\varepsilon,\phi}$ fulfilling (84).

Now we consider the terms involving a_2^{ij} and b_2^i. We will see that all the contributions of order ε^4 coming from these terms will simplify, therefore we do not need to know the explicit expression of a_2^{ij} and b_2^i.

$$\begin{aligned}
T^6_{\varepsilon,\phi}(y,t) = \Big\{ &\big(-\Delta + W''(v_\star)\big)\big(\varepsilon^4(a_2^{ij}(\phi_\star)_{ij} + b_2^i(\phi_\star)_i)(t+\phi_\star)^2 v_\star' \\
&- \varepsilon^4 a_2^{ij}(\phi_\star)_i(\phi_\star)_j(t+\phi_\star)^2 v_\star''\big) \\
&+ \varepsilon^2(\overline{a}^{ij}(\phi_\star)_{ij} + \overline{b}^i(\phi_\star)_i)v_\star' - \varepsilon^2 \overline{a}^{ij}(\phi_\star)_i(\phi_\star)_j v_\star'' \Big\} \\
= &-\varepsilon^4(a_2^{ij}(\phi_\star)_{ij} + b_2^i(\phi_\star)_i)(2v_\star' + 4(t+\phi_\star)v_\star'') \\
&- \varepsilon^4 a_2^{ij}(\phi_\star)_i(\phi_\star)_j(2v_\star'' + 4(t+\phi_\star)v_\star''' + (t+\phi_\star)^2 v_\star^{(4)} + W''(v_\star)v_\star'') \\
&+ \big(-\Delta + W''(v_\star)\big)\Big\{\varepsilon^2(\overline{a}^{ij}(\phi_\star)_{ij} + \overline{b}^i(\phi_\star)_i)v_\star' - \varepsilon^2\overline{a}^{ij}(\phi_\star)_i(\phi_\star)_j v_\star''\Big\} + \varepsilon^5 F^6_{\varepsilon,\phi}(y,t),
\end{aligned} \tag{88}$$

with $F^6_{\varepsilon,\phi}$ fulfilling (84).

It turns out that, in the expansion of $F(v_\star(t))$, the term of order ε^2 is $(H^2 - 2|A|^2)v_\star'' + |\nabla\phi_\star|^2(v_\star^{(4)} - W''(v_\star)v_\star'')$. Since it is too large for our purposes, we add a correction to the approximate solution in order to cancel it. Moreover, there is also a quadratic term appearing at order ε^3, that is $-2H|\nabla\phi_\star|^2 v_\star'''$. Although it is smaller if, for instance, $|\phi|_{C^{4,\alpha}(\Sigma)} \le c\varepsilon$, for some constant $c > 0$, we would like to get rid of it in order to improve the approximation.

In order to do so, we set

$$\eta(t) := -v_\star'(t)\int_0^t (v_\star'(s))^{-2} ds \int_{-\infty}^s \frac{\tau(v_\star'(\tau))^2}{2} d\tau,$$

$$\tilde\eta(t) := -v_\star'(t)\int_0^t (v_\star'(s))^{-2} ds \int_{-\infty}^s v_\star'(\tau)v_\star''(\tau) d\tau.$$

This functions are exponentially decaying, odd and solve

$$L_\star\eta(t) = -\eta''(t) + W''(v_\star(t))\eta(t) = \frac{1}{2}tv_\star'(t),\ L_\star\tilde\eta(t) = v_\star''(t),$$

$$\int_{-\infty}^{\infty}\eta(t)v_\star'(t)dt = \int_{-\infty}^{\infty}\tilde\eta(t)v_\star'(t)dt = 0.$$

Differentiating this relation once more, it is possible to see that $L_\star^2\eta(t) = -v_\star''(t)$ and $L_\star^2\tilde\eta = -v_\star^{(4)} + W''(v_\star)v_\star''$. Our new approximate solution will be

$$\tilde v_{\varepsilon,\phi}(y,t) := v_\star(t) + v_{1,\varepsilon,\phi}(y,t) + v_{2,\varepsilon,\phi}(y,t),$$

where

$$v_{2,\varepsilon,\phi}(y,t) = \varepsilon^2(\psi(\varepsilon y) + \varepsilon L\phi_\star(\varepsilon y))\eta(t) + \varepsilon^2|\nabla\phi_\star(\varepsilon y)|^2\tilde\eta(t), \tag{89}$$

with $\psi : \Sigma \to \mathbb{R}$ and L linear in ϕ to be determined later. In the sequel, η, $\tilde\eta$ and its derivatives are evaluated at t, the geometric quantities, ϕ and its derivatives will be evaluated at εy.

Taking the Taylor expansion of F,

$$\begin{aligned}F(\tilde{v}_{\varepsilon,\phi})(y,t) &= F(v_\star) + F'(v_\star)[v_{1,\varepsilon,\phi}] + F'(v_\star)[v_{2,\varepsilon,\phi}]\\ &+ \frac{1}{2}F''(v_\star)[v_{1,\varepsilon,\phi}, v_{1,\varepsilon,\phi}] + F''(v_\star)[v_{1,\varepsilon,\phi}, v_{2,\varepsilon,\phi}]\\ &+ \frac{1}{2}F''(v_\star)[v_{2,\varepsilon,\phi}, v_{2,\varepsilon,\phi}] + C_{\varepsilon,\phi}[v_{1,\varepsilon,\phi} + v_{2,\varepsilon,\phi}],\end{aligned}$$

where

$$C_{\varepsilon,\phi}[w] = \int_0^1 dt \int_0^t ds \int_0^s F'''\big(v_\star + \tau w\big)[w,w,w]d\tau.$$

It follows from the expansion of the Laplacian and the properties of the smoothing operators ((44), (45), (46)) that

$$F'(v_\star)[v_{1,\varepsilon,\phi}](y,t) = \varepsilon^4 \Delta_\Sigma^2(\phi_\star - \phi) + \varepsilon^5 F^{12}_{\varepsilon,\phi}(y,t), \tag{90}$$

with $F^{12}_{\varepsilon,\phi}$ satisfying (84). We point out that this extra term does not give rise to terms of order ε^2 and ε^3.

Now we compute $F'(v_\star)[v_{2,\varepsilon,\phi}]$. As first we note that

$$\begin{aligned}T^7_{\varepsilon,\phi}(y,z) &= W'''(v_\star)(-\Delta v_\star + W'(v_\star))(\varepsilon^2\psi\eta + \varepsilon^3 L\phi_\star\eta + \varepsilon^2|\nabla\phi_\star|^2\tilde{\eta})\\ &= \varepsilon^3 H\psi W'''(v_\star)\eta v'_\star + \varepsilon^3 H|\nabla\phi_\star|^2 W'''(v_\star)\tilde{\eta}v'_\star\\ &+ \varepsilon^4\Big\{(\psi\Delta_\Sigma\phi_\star + HL\phi_\star + (t+\phi_\star)\psi|A|^2)W'''(v_\star)\eta v'_\star - |\nabla\phi_\star|^2\psi W'''(v_\star)\eta v''_\star\\ &+ |\nabla\phi_\star|^2 W'''(v_\star)\tilde{\eta}(|A|^2(t+\phi_\star)v'_\star + \varepsilon^2\Delta_\Sigma\phi_\star v'_\star - |\nabla\phi_\star|^2 v''_\star)\Big\}\\ &+ \varepsilon^5 F^7_{\varepsilon,\phi}(y,t),\end{aligned}$$

with $F^7_{\varepsilon,\phi}$ fulfilling (84).

After that, we have to compute $\big(-\Delta + W''(v_\star)\big)^2(\varepsilon^2(\psi + \varepsilon L\phi_\star)\eta + \varepsilon^2|\nabla\phi_\star|^2\tilde{\eta})$. We obtain

$$\begin{aligned}&\big(-\Delta + W''(v_\star)\big)(\varepsilon^2(\psi + \varepsilon L\phi_\star)\eta + \varepsilon^2|\nabla\phi_\star|^2\tilde{\eta}) =\\ &\varepsilon^2(\psi L_\star\eta + |\nabla\phi_\star|^2 L_\star\tilde{\eta}) + \varepsilon^3(H\psi\eta' + L\phi_\star L_\star\eta + H|\nabla\phi_\star|^2\tilde{\eta}')\\ &+ \varepsilon^4\Big\{-\Delta_\Sigma\psi\eta + \big(|A|^2\psi(t+\phi_\star) + HL\phi_\star + 2(\nabla_\Sigma\psi, \nabla_\Sigma\phi_\star) + \psi\Delta_\Sigma\phi_\star\big)\eta'\\ &- \psi|\nabla_\Sigma\phi_\star|^2\eta'' - \Delta|\nabla\phi_\star|^2\tilde{\eta} - |\nabla\phi_\star|^4\tilde{\eta}''\\ &\big(|A|^2(t+\phi_\star)|\nabla\phi_\star|^2 + \Delta_\Sigma\phi_\star|\nabla\phi_\star|^2 + 2(\nabla\phi_\star, \nabla|\nabla\phi_\star|^2)\big)\tilde{\eta}'\Big\} + \varepsilon^5\tilde{F}_{\varepsilon,\phi}(y,t),\end{aligned}$$

with $\tilde{F}_{\varepsilon,\phi}$ satisfying (84).

Applying the operator once more, we obtain

$$T^8_{\varepsilon,\phi}(y,t) = \big(-\Delta + W''(v_\star)\big)(\varepsilon^2(\psi + \varepsilon L\phi_\star)L_\star\eta + \varepsilon^2|\nabla\phi_\star|^2 L_\star\tilde\eta) = \tag{91}$$

$$\varepsilon^2(\psi L_\star^2\eta + |\nabla\phi_\star|^2 L_\star^2\tilde\eta) + \varepsilon^3\Big\{L\phi_\star L_\star^2\eta + H\psi(L_\star\eta)' + H|\nabla\phi_\star|^2(L_\star\tilde\eta)'\Big\}$$

$$+\varepsilon^4\Big\{-\Delta_\Sigma\psi L_\star\eta + \big(|A|^2\psi(t+\phi_\star) + HL\phi_\star + 2(\nabla_\Sigma\psi, \nabla_\Sigma\phi_\star) + \psi\Delta_\Sigma\phi_\star\big)(L_\star\eta)'$$

$$-\psi|\nabla_\Sigma\phi_\star|^2(L_\star\eta)'' - \Delta|\nabla\phi_\star|^2 L_\star\tilde\eta - |\nabla\phi_\star|^4(L_\star\tilde\eta)''$$

$$\big(|A|^2(t+\phi_\star)|\nabla\phi_\star|^2 + \Delta_\Sigma\phi_\star|\nabla\phi_\star|^2 + 2(\nabla\phi_\star, \nabla|\nabla\phi_\star|^2)\big)(L_\star\tilde\eta)'\Big\} + \varepsilon^5 F^8_{\varepsilon,\phi}(y,t),$$

with $F^8_{\varepsilon,\phi}$ satisfying (84).

Moreover,

$$T^9_{\varepsilon,\phi}(y,t) = \big(-\Delta + W''(v_\star)\big)(\varepsilon^3 H\psi\eta' + \varepsilon^3 H|\nabla\phi_\star|^2\tilde\eta') = \tag{92}$$

$$\varepsilon^3(\psi HL_\star(\eta') + H|\nabla\phi_\star|^2 L_\star(\tilde\eta')) + \varepsilon^4(H^2\psi\eta'' + H^2|\nabla\phi_\star|^2\tilde\eta'') + \varepsilon^5 F^9_{\varepsilon,\phi}(y,t),$$

with $F^9_{\varepsilon,\phi}$ satisfying (84).

As regards the term of order ε^4 of (91), we note that

$$T^{10}_{\varepsilon,\phi}(y,t) = \varepsilon^4\big(-\Delta + W''(v_\star)\big)\Big\{-\Delta_\Sigma\psi\eta + \big(|A|^2\psi(t+\phi_\star) + HL\phi_\star \tag{93}$$

$$+2(\nabla_\Sigma\psi, \nabla_\Sigma\phi_\star) + \psi\Delta_\Sigma\phi_\star\big)\eta' - \psi|\nabla_\Sigma\phi_\star|^2\eta'' - \Delta|\nabla\phi_\star|^2\tilde\eta - |\nabla\phi_\star|^4\tilde\eta''$$

$$\big(|A|^2(t+\phi_\star)|\nabla\phi_\star|^2 + \Delta_\Sigma\phi_\star|\nabla\phi_\star|^2 + 2(\nabla\phi_\star, \nabla|\nabla\phi_\star|^2)\big)\tilde\eta'\Big\}$$

$$= \varepsilon^4\Big\{-\Delta_\Sigma\psi L_\star\eta + \big(HL\phi_\star + 2(\nabla_\Sigma\psi, \nabla_\Sigma\phi_\star) + \psi\Delta_\Sigma\phi_\star\big)L_\star(\eta')$$

$$+|A|^2\psi L_\star((t+\phi_\star)\eta') - \psi|\nabla_\Sigma\phi_\star|^2 L_\star(\eta'') - \Delta|\nabla\phi_\star|^2\tilde\eta - |\nabla\phi_\star|^4(L_\star\tilde\eta)''$$

$$\big(|A|^2(t+\phi_\star)|\nabla\phi_\star|^2 + \Delta_\Sigma\phi_\star|\nabla\phi_\star|^2 + 2(\nabla\phi_\star, \nabla|\nabla\phi_\star|^2)\big)(L_\star\tilde\eta)'\Big\} + \varepsilon^5 F^{10}_{\varepsilon,\phi}(y,t),$$

with $F^{10}_{\varepsilon,\phi}$ satisfying (84). To conclude, also

$$F^{11}_{\varepsilon,\phi}(y,t) = \big(-\Delta + W''(v_\star)\big)\tilde F_{\varepsilon,\phi}(y,t)$$

is negligible, that is it satisfies (84), since $\tilde F_{\varepsilon,\phi}$ does.

Now we have to considered the contribution of $F''(v_\star)[v_{2,\varepsilon,\phi}, v_{2,\varepsilon,\phi}]$, since it gives rise to a term of order ε^4. However, we will see that this contribution will cancel after projection, since the term of order ε^4 is orthogonal to $v_\star'$, indeed

$$F''(v_\star)[v_{2,\varepsilon,\phi}, v_{2,\varepsilon,\phi}] \tag{94}$$
$$= \varepsilon^4\psi^2\big(W'''(v_\star)W''(v_\star)\eta^2 - (W'''(v_\star)\eta^2)'' + 2W'''(v_\star)\eta L_\star\eta\big) + \varepsilon^5 F^{13}_{\varepsilon,\phi}(y,t),$$

with $F^{13}_{\varepsilon,\phi}$ satisfying (84). By the properties of the smoothing operators R_θ (see (44), (45), (46)), we can see that

$$||F''(v_\star)[v_{1,\varepsilon,\phi}, v_{2,\varepsilon,\phi}]||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \le c\varepsilon^6|\phi|_{C^{4,\alpha}(\Sigma)} \tag{95}$$

and the same is true for $F''(v_\star)[v_{1,\varepsilon,\phi}, v_{1,\varepsilon,\phi}]$ and $C_{\varepsilon,\phi}[v_{1,\varepsilon,\phi}+v_{2,\varepsilon,\phi}]$.

We point out that, with this choice of approximate solution, the term of order ε^2 of $F(\tilde v_{\varepsilon,\phi})$ is precisely

$$(H^2-2|A|^2)v_\star'' + \psi L_\star^2\eta + |\nabla\phi_\star|^2(v_\star^{(4)} - W''(v_\star)v_\star'' + L_\star^2\tilde\eta),$$

which vanishes if we set $\psi := H^2-2|A|^2$, since $L_\star^2\eta = -v_\star''$ and $L_\star^2\tilde\eta = -v_\star^{(4)} + W''(v_\star)v_\star''$. Moreover, if we set $L\phi = \phi(2H|A|^2 - 4\mathrm{tr}A^3) + 2H\Delta_\Sigma\phi - 4(A,\nabla^2\phi)$, the term of order ε^3 is

$$\begin{aligned}
&(H|A|^2-2\mathrm{tr}A^3)(2tv_\star''+v_\star) + |\nabla\phi_\star|^2(-2v_\star''' + 2(L_\star\tilde\eta)') - \Delta_\Sigma H v_\star'\\
&\quad + (\phi_\star(2H|A|^2-4\mathrm{tr}A^3) + 2H\Delta_\Sigma\phi_\star - 4(A,\nabla^2\phi_\star))v_\star'' + L\phi L_\star^2\eta\\
&2H(H^2-2|A|^2)(L_\star\eta)' + 2(A\nabla\phi_\star,\nabla\phi_\star)\big((t+\phi_\star)v_\star^{(4)} - (t+\phi_\star)W''(v_\star)v_\star'' + 2v_\star'''\big) =\\
&\quad (H|A|^2-2\mathrm{tr}A^3)(2tv_\star''+v_\star') - \Delta_\Sigma H v_\star' + 2H(H^2-2|A|^2)(L_\star\eta)'\\
&\quad + 2(A\nabla\phi_\star,\nabla\phi_\star)\big((t+\phi_\star)v_\star^{(4)} - (t+\phi_\star)W''(v_\star)v_\star'' + 2v_\star'''\big),
\end{aligned}$$

since $(L_\star\tilde\eta)' = v_\star'''$ and $L_\star^2\eta = -v_\star''$.

We recall that $\tilde v_{\varepsilon,\phi}$ is just defined in B, while our global approximate solution is $v_{\varepsilon,\phi}(x) = \chi_5(x)\tilde v_{\varepsilon,\phi}(y,t) + (1-\chi_5(x))\mathbb{H}(x)$ (see (54)).

Remark 16. It follows from the construction that our approximate solution respects the symmetries of the Torus, that is $v_{\varepsilon,\phi}(x) = v_{\varepsilon,\phi}(Tx)$ and $v_{\varepsilon,\phi}(x) = v_{\varepsilon,\phi}(Rx)$, for any $R\in SO_{x_3}(3)$.

5.2. Projection onto the Kernel

As we noticed in section 4.2, we need to consider the projection of the error $F(\tilde v_{\varepsilon,\phi})$.

Proposition 17. *Let us set, for any $\phi\in B_4(1/4) = \{\phi\in C^{4,\alpha}(\Sigma)_{s,0} : |\phi|_{C^{4,\alpha}(\Sigma)} < 1/4\}$,*

$$L\phi := -4 < A, \nabla^2\phi > +2H\Delta_\Sigma\phi + \phi(2H|A|^2 - 4trA^3), \tag{96}$$

where $c_\star := \int_{-\infty}^{\infty}(v_\star'(t))^2dt$. Then, for any $y\in\Sigma_\varepsilon$, the projection of $F_\varepsilon(\tilde v_{\varepsilon,\phi})$ satisfies

$$\int_{-\infty}^{\infty}(F(\tilde v_{\varepsilon,\phi}) - \varepsilon^4\chi_1\lambda(1-\tilde v_{\varepsilon,\phi}))(y,t)v_\star'(t)dt = -\varepsilon^4c_\star\tilde{L_0}\phi - 2\varepsilon^4\lambda + \varepsilon^5\mathcal{F}_{\varepsilon,\phi,\lambda}(\varepsilon y), \tag{97}$$

with $\mathcal{F}_{\varepsilon,\phi}$ *uniformly bounded and Lipschitzian in* $\phi \in B_4(1/4)$ *and in* ε, *that is there exists a constant* $c = c(W, \tau) > 0$ *such that*

$$\begin{cases} |\mathcal{F}_{\varepsilon,\phi,\lambda}|_{C^{0,\alpha}(\Sigma)} \le c, \\ |\mathcal{F}_{\varepsilon,\phi_1,\lambda_1} - \mathcal{F}_{\varepsilon,\phi_2,\lambda_2}|_{C^{0,\alpha}(\Sigma)} \le c(|\phi_1 - \phi_2|_{C^{4,\alpha}(\Sigma)} + |\lambda_1 - \lambda_2|), \end{cases} \tag{98}$$

for any $\phi, \phi_1, \phi_2 \in C^{4,\alpha}(\Sigma)_s$ *with* $|\phi|_{C^{4,\alpha}(\Sigma)}, |\phi_1|_{C^{4,\alpha}(\Sigma)}, |\phi_2|_{C^{4,\alpha}(\Sigma)} \le \tilde{c}\varepsilon$, *for any* $|\lambda|, |\lambda_1|, |\lambda_2| < \tilde{c}\varepsilon$, *for some* $C > 0$, *and for any* $\varepsilon > 0$ *small enough.*

Remark 18. It follows from the computations that $\mathcal{F}_{\varepsilon,\phi,\lambda}(y) = \mathcal{H}(y) + \varepsilon\mathcal{G}_{\varepsilon,\phi,\lambda}(y)$, where $|\mathcal{H}|_{C^{0,\alpha}(\Sigma)} < \infty$ is bounded and

$$|\mathcal{G}_{\varepsilon,\phi,\lambda}|_{C^{0,\alpha}(\Sigma)} \le C, \text{ if } |\phi|_{C^{4,\alpha}(\Sigma)} + |\lambda| < \tilde{c}\varepsilon,$$

where $\tilde{c} > 0$ is arbitrary and C may depend on $\tilde{c}$, but not on ε.

Proof. Above we computed $F(\tilde{v}_{\varepsilon,\phi})$ using (28), now we just project it term by term.

Integrating by parts we can show that

$$\int_{-\infty}^{\infty} t v_\star''(t) v_\star'(t) dt = -\frac{1}{2} c_\star \tag{99}$$

$$\int_{-\infty}^{\infty} (L_\star \eta(t))' v_\star'(t) dt = \frac{1}{4} c_\star \tag{100}$$

$$\int_{-\infty}^{\infty} L_\star(\eta'(t)) v_\star'(t) dt = 0, \tag{101}$$

so in particular

$$\int_{-\infty}^{\infty} W'''(v_\star(t))\eta(t)(v_\star'(t))^2 dt = \int_{-\infty}^{\infty} \left\{L_\star\eta(t) - L_\star(\eta'(t))\right\} v_\star'(t) dt = \frac{1}{4} c_\star.$$

Moreover, setting $b_\star := \int_{-\infty}^{\infty} (v_\star''(t))^2 dt = -\int_{-\infty}^{\infty} v_\star'''(t) v_\star'(t) dt$, we can see that

$$\int_{-\infty}^{\infty} \left\{t v_\star^{(4)}(t) - t W''(v_\star(t)) v_\star''(t) + 2 v_\star'''(t)\right\} v_\star'(t) dt = \tag{102}$$

$$-\int_{-\infty}^{\infty} t L_\star(v_\star''(t)) v_\star'(t) dt - 2b_\star = -\int_{-\infty}^{\infty} t v_\star''(t) L_\star(v_\star'(t)) dt + 2b_\star - 2b_\star = 0$$

because $L_\star(v_\star') = 0$.

First we show that the term of order ε^3 vanishes after projection. In fact, by (99), (100), (102) and the Willmore equation,

$$(H|A|^2 - 2\mathrm{tr}A^3)\int_{-\infty}^{\infty}(2tv_\star'' + v_\star)v_\star' dt - \Delta_\Sigma H\int_{-\infty}^{\infty}(v_\star')^2 dt + 2H(H^2 - 2|A|^2)\int_{-\infty}^{\infty}(L_\star\eta)'v_\star' dt$$
$$+ 2(A\nabla\phi_\star, \nabla\phi_\star)\int_{-\infty}^{\infty}\big((t+\phi_\star)v_\star^{(4)} - (t+\phi_\star)W''(v_\star)v_\star'' + 2v_\star'''\big)v_\star' dt$$
$$= c_\star(-\Delta_\Sigma H + \frac{1}{2}H(H^2 - 2|A|^2)) = 0$$

In the forthcoming calculations, we will only consider the terms of order at least ε^4. For notational convenience, we define $\tilde{T}^k_{\varepsilon,\phi}$ to be $\tilde{T}^k_{\varepsilon,\phi}$ minus the terms of order ε^2 and ε^3. For instance

$$\tilde{T}^1_{\varepsilon,\phi}(y,t) = T^1_{\varepsilon,\phi} - \varepsilon^2(H^2 - 2|A|^2)v_\star''$$
$$- \varepsilon^3\Big\{(2H|A|^2 - 4\mathrm{tr}A^3)(t+\phi_\star)v_\star'' + (H|A|^2 - 2\mathrm{tr}A^3)v_\star' - \Delta_\Sigma H v_\star'$$
$$+ 2(\nabla_\Sigma H, \nabla_\Sigma\phi_\star)v_\star'' - H|\nabla_\Sigma\phi_\star|^2 v_\star''' + H\Delta_\Sigma\phi_\star v_\star''\Big\}.$$

Moreover, the right-hand side will always be evaluated at εy. By (99) and (24),

$$\int_{-\infty}^{\infty}\tilde{T}^1_{\varepsilon,\phi}(y,t)v_\star'(t)dt$$
$$= \varepsilon^4 c_\star\Big\{-\phi_\star\Delta_\Sigma|A|^2 - (\nabla_\Sigma|A|^2, \nabla_\Sigma\phi_\star) - \frac{1}{2}|A|^2\Delta_\Sigma\phi_\star - \phi_\star(2 < A, \nabla^2 H > + |\nabla_\Sigma H|^2)$$
$$- 2(A\nabla_\Sigma H, \nabla_\Sigma\phi_\star) - \frac{1}{2}H(2 < A, \nabla^2\phi_\star > + (\nabla_\Sigma H, \nabla_\Sigma\phi_\star))\Big\} + \varepsilon^5\mathcal{F}^1_{\varepsilon,\phi},$$

$$\int_{-\infty}^{\infty}\tilde{T}^2_{\varepsilon,\phi}(y,t)v_\star'(t)dt = \varepsilon^4 c_\star\Big\{-(\Delta_\Sigma)^2\phi_\star - \frac{1}{2}|A|^2\Delta_\Sigma\phi_\star\Big\} + \varepsilon^5\mathcal{F}^2_{\varepsilon,\phi},$$

$$\int_{-\infty}^{\infty}\tilde{T}^3_{\varepsilon,\phi}(y,t)v_\star'(t)dt = \varepsilon^5\mathcal{F}^3_{\varepsilon,\phi},$$

$$\int_{-\infty}^{\infty}\tilde{T}^4_{\varepsilon,\phi}(y,t)v_\star'(t)dt = \varepsilon^4\frac{1}{2}c_\star H(2 < A, \nabla^2\phi_\star > + (\nabla_\Sigma H, \nabla_\Sigma\phi_\star)) + \varepsilon^5\mathcal{F}^4_{\varepsilon,\phi},$$

with $\mathcal{F}^1_{\varepsilon,\phi}, \mathcal{F}^2_{\varepsilon,\phi}, \mathcal{F}^3_{\varepsilon,\phi}, \mathcal{F}^4_{\varepsilon,\phi}$ satisfying (98).

By (102),

$$\int_{-\infty}^{\infty} \tilde{T}^5_{\varepsilon,\phi}(y,t)v_\star'(t)dt = \varepsilon^5 \mathcal{F}^5_{\varepsilon,\phi},$$

with $\mathcal{F}^5_{\varepsilon,\phi}$ satisfying (98). Once again by (99), we can see that

$$\int_{-\infty}^{\infty} \tilde{T}^6_{\varepsilon,\phi}(y,t)v_\star'(t)dt = \varepsilon^5 \mathcal{F}^6_{\varepsilon,\phi},$$

with $\mathcal{F}^6_{\varepsilon,\phi}$ satisfying (98).

Now let us consider the terms coming from the correction.

$$\int_{-\infty}^{\infty} \tilde{T}^7_{\varepsilon,\phi}(y,t)v_\star'(t)dt$$

$$= \varepsilon^4 c_\star \frac{1}{4}\Big\{(H^2-2|A|^2)\Delta_\Sigma\phi_\star + HL\phi_\star + (H^2-2|A|^2)|A|^2\phi_\star\Big\} + \varepsilon^5\mathcal{F}^7_{\varepsilon,\phi},$$

$$\int_{-\infty}^{\infty} \tilde{T}^8_{\varepsilon,\phi}(y,t)v_\star'(t)dt$$

$$= \varepsilon^4 c_\star \Big\{\frac{1}{4}(H^2-2|A|^2)|A|^2\phi_\star + \frac{1}{4}HL\phi_\star + H(\nabla_\Sigma H, \nabla_\Sigma\phi_\star)$$
$$- (\nabla_\Sigma|A|^2, \nabla_\Sigma\phi_\star) + \frac{1}{4}(H^2-2|A|^2)\Delta_\Sigma\phi_\star\Big\} + \varepsilon^5\mathcal{F}^8_{\varepsilon,\phi},$$

with $\mathcal{F}^7_{\varepsilon,\phi}$, $\mathcal{F}^8_{\varepsilon,\phi}$ satisfying (98). To conclude, also

$$\mathcal{F}^9_{\varepsilon,\phi} = \int_{-\infty}^{\infty} \Big\{\tilde{T}^9_{\varepsilon,\phi}(y,t) + \tilde{T}^{10}_{\varepsilon,\phi}(y,t)\Big\}v_\star'(t)dt \tag{103}$$

fulfills (98). The contribution of the term $F'(v_\star)[v_{2,\varepsilon,\phi}]$ is given by

$$\int_{-\infty}^{\infty} F'(v_\star)[v_{2,\varepsilon,\phi}](y,t)v_\star'(t)dt = -\varepsilon^4 c_\star \Delta^2_\Sigma(\phi-\phi_\star) + \mathcal{F}^{10}_{\varepsilon,\phi}, \tag{104}$$

with $\mathcal{F}^{10}_{\varepsilon,\phi}$ satisfying (98). In conclusion, by the choice of L (see (96)) we obtain exactly $\tilde{L}_0 R_{1/\varepsilon} + \Delta^2(Id - R_{1/\varepsilon})$ as a linear term at order ε^4. Moreover, by (95), the terms involving $F''(v_\star)$ and $C_{\varepsilon,\phi}$ do not give rise to terms of order ε^4 in the projection. In conclusion, we have

$$\int_{-\infty}^{\infty} F(\tilde{v}_{\varepsilon,\phi})(y,t)v'_\star(t)dt = -\varepsilon^4 c_\star(\tilde{L_0}\phi_\star(\varepsilon y) + \Delta_\Sigma^2(\phi-\phi_\star)(\varepsilon y)) + \varepsilon^5 \mathcal{F}^{11}_{\varepsilon,\phi}(\varepsilon y),$$

where $\mathcal{F}^{11}_{\varepsilon,\phi}$ satisfies (98). We can see that

$$\tilde{L_0}\phi_\star + \Delta_\Sigma^2(\phi-\phi_\star) = \tilde{L_0}\phi + (\tilde{L_0} - \Delta_\Sigma^2)(\phi_\star - \phi),$$

and $\varepsilon^{-1}(\tilde{L_0} - \Delta_\Sigma^2)(\phi_\star - \phi)$ satisfies (98), by the property (46), because $\tilde{L_0} - \Delta_\Sigma^2$ is a second order operator, thus, for instance

$$|\varepsilon^{-1}(\tilde{L_0} - \Delta_\Sigma^2)(\phi_\star - \phi)|_{C^{0,\alpha}(\Sigma)} \le c\varepsilon^{-1}|\phi_\star - \phi|_{C^{2,\alpha}(\Sigma)} \le c\varepsilon|\phi|_{C^{2,\alpha}(\Sigma)}.$$

Using the oddness of $\tilde{v}_{\varepsilon,\phi}$ in t, we directly compute

$$-\int_{\mathbb{R}} \varepsilon^4\lambda\chi_1(1-\tilde{v}_{\varepsilon,\phi}(y,t))v'_\star(t)dt = -2\varepsilon^4\lambda + \varepsilon^4\lambda\mathcal{F}^{12}_{\varepsilon,\phi}(y),$$

with $\mathcal{F}^{12}_{\varepsilon,\phi}$ satisfying (98) □

6. Solving the auxiliary equation

This Section will be devoted to the proofs of Propositions 11 and 12. In both cases, we will first study the linear problem associated to our equation and then we will apply a contraction mapping principle.

6.1. Solvability far away from Σ_ε: the linear problem

We will prove the following Proposition.

Proposition 19. *Let $0 < \delta < \sqrt{W''(1)}$. Then, for any $\varepsilon > 0$ small enough, for any $\phi \in B_4(\tau/4)$, and for any $f \in C^{0,\alpha}_{\delta,s}(\mathbb{R}^3)$, the equation*

$$(-\Delta + \Gamma_{\varepsilon,\phi})^2 V = f \tag{105}$$

admits a unique solution $V = \Psi_{\varepsilon,\phi}(f)$ in $C^{4,\alpha}_{\delta,s}(\mathbb{R}^3)$ satisfying $||V||_{C^{4,\alpha}(\mathbb{R}^3)} \le c||f||_{C^{0,\alpha}(\mathbb{R}^3)}$, for some constant $c > 0$ independent of ε and ϕ.

Remark 20. (i) The symmetries of the solution follow for free from the symmetries of the Laplacian and of $\Gamma_{\varepsilon,\phi}$. In fact, if $f \in C^{0,\alpha}_{\delta,s}(\mathbb{R}^3)$, and V is a solution to $(-\Delta + \Gamma_{\varepsilon,\phi})^2 V = f$, then also $u_T(x) := u(Tx)$ is a solution, thus, by uniqueness, $u = u_T$. The same argument also shows that $u = u_R$, for any $R \in SO_{x_3}(3)$, hence $u \in C^{4,\alpha}_{\delta,s}(\mathbb{R}^3)$.

(ii) In particular, if $|f| \le ce^{-\sqrt{W''(1)}|x|}$, then the absolute value of the solution is bounded by $e^{-\delta|x|}$, for any $0 < \delta < \sqrt{W''(1)}$.

We split the proof into some lemmas and a proposition, with the aid of some remarks. First we reduce ourselves to consider a second order PDE, then, by a bootstrap argument, we will solve our forth order equation.

Proposition 21. *Let $0 < \delta < \sqrt{W''(1)}$. Then, for any $\varepsilon > 0$ small enough, for any $\phi \in B_4(\tau/4)$, and for any $f \in C^{0,\alpha}_{\delta}(\mathbb{R}^3)$, the equation*

$$-\Delta u + \Gamma_{\varepsilon,\phi} u = f \tag{106}$$

admits a unique solution $u = \tilde{\Psi}_{\varepsilon,\phi}(f)$ in $C^{2,\alpha}_{\delta}(\mathbb{R}^3)$ satisfying $||u||_{C^{2,\alpha}(\mathbb{R}^3)} \le c||f||_{C^{0,\alpha}(\mathbb{R}^3)}$, for some constant $c > 0$ independent of ε and ϕ.

Proof. *Step (i): existence, uniqueness and local Hölder regularity.*

Existence and uniqueness of the weak solution follow from the Riesz representation theorem. Since $f \in C^{0,\alpha}_{loc}(\mathbb{R}^3)$, then $u \in C^{2,\alpha}_{loc}(\mathbb{R}^3)$.

Step (ii): decay of the solution: $u\varphi_\delta \in L^\infty(\mathbb{R}^3)$.

We will use the function $e^{-\delta|x|}$ as a barrier. More precisely, we fix $\rho > 0$ and $|z| > \rho$. Then we fix $\sigma > 0$ and $R > |z|$ so large that $u(x) < \sigma$ for $|x| \ge R$. Therefore u fulfills

$$\begin{cases} u < \max_{\partial B_\rho} u < \lambda e^{-\delta\rho} < \lambda e^{-\delta\rho} + \sigma & \text{for } |x| = \rho \\ u < \sigma < \lambda e^{-\delta R} + \sigma & \text{for } |x| = R \\ (-\Delta + \Gamma_{\varepsilon,\phi})(u - (\lambda e^{-\delta|x|} + \sigma)) \le \left(c - \lambda\delta\frac{N-1}{R}\right)e^{-\delta r} \le 0 & \text{for } \rho < |x| < R, \end{cases}$$

provided $\lambda \ge \lambda_0$, with λ_0 independent of σ. By the maximum principle we get that $u(z) < \lambda e^{-\delta|z|} + \sigma$, for any $|z| \ge \rho$ and for any $\sigma > 0$. In the same way, one can prove that $u(z) > -\lambda e^{-\delta|z|} - \sigma$ Letting $\sigma \to 0$, we get that $u\varphi_\delta \in L^\infty(\mathbb{R}^3)$.

Step (iii): estimate of the L^∞-norm of the solution.

Since $u\varphi_\delta \in L^\infty(\mathbb{R}^3)$, then it exists a point $y \in \mathbb{R}^3$ such that $|u(y)| = ||u||_\infty$. If $u(y) > 0$, then y is a maximum point, thus

$$\delta^2 u(y) \le -\Delta u(y) + \Gamma_{\varepsilon,\phi}(y)u(y) = f(y) \le ||f||_\infty,$$

that is

$$||u||_{L^\infty(\mathbb{R}^3)} \le c||f||_{L^\infty(\mathbb{R}^3)}.$$

A similar argument shows that the same estimate is true if $u(y) < 0$ (a minimum).

Step (iv): continuity of the right inverse.

By [12] (chapter 6.1, Corollary 6.3), we have, for any $x \in \mathbb{R}^3$,

$$||u||_{C^{2,\alpha}(B_1(x))} \le c(||f||_{C^{0,\alpha}(B_2(x))} + ||u||_{L^\infty(B_2(x))}). \tag{107}$$

Since x is arbitrary, we conclude that

$$||u||_{C^{2,\alpha}(\mathbb{R}^3)} \le c(||u||_{L^\infty(\mathbb{R}^3)} + ||f||_{C^{0,\alpha}(\mathbb{R}^3)}) \le c||f||_{C^{0,\alpha}(\mathbb{R}^3)}. \tag{108}$$

Step (v): decay of the derivatives.

By the decay of u, we already know that $\tilde{u}_\delta \in L^\infty(\mathbb{R}^3)$. Moreover, $\tilde{u}_\delta$ satisfies the equation

$$-\Delta\tilde{u}_\delta + \Gamma_{\varepsilon,\phi}\tilde{u}_\delta = \tilde{f}_\delta - 2 < \nabla u, \nabla\varphi_\delta > -u\Delta\varphi_\delta = \tag{109}$$
$$\tilde{f}_\delta - 2\varphi_{-\delta} < \nabla\tilde{u}_\delta, \nabla\varphi_\delta > +\tilde{u}_\delta(2(\varphi_{-\delta})^2|\nabla\varphi_\delta|^2 - \varphi_{-\delta}\Delta\varphi_\delta),$$

thus, once again by [12] (chapter 6.1, Corollary 6.3),

$$||\tilde{u}_\delta||_{C^{2,\alpha}(B_1(x))} \le c(||\tilde{u}_\delta||_{L^\infty(\mathbb{R}^3)} + ||\tilde{f}_\delta||_{C^{0,\alpha}(\mathbb{R}^3)}) < \infty, \tag{110}$$

for any $x \in \mathbb{R}^3$, thus $u \in C^{2,\alpha}_\delta(\mathbb{R}^3)$. □

Now we can conclude the proof of Proposition 19.

Proof. Given $f \in C^{0,\alpha}_\delta(\mathbb{R}^3)$, we have to find $V \in C^{4,\alpha}_\delta(\mathbb{R}^3)$ fulfilling

$$\begin{cases} (-\Delta + \Gamma_{\varepsilon,\phi})^2 V = f \\ ||V||_{C^{4,\alpha}(\mathbb{R}^3)} \le c||f||_{C^{0,\alpha}(\mathbb{R}^3)}. \end{cases}$$

In order to do so, we use Proposition 21 twice to find $u \in C^{2,\alpha}_\delta(\mathbb{R}^3)$ and $V \in C^{2,\alpha}_\delta(\mathbb{R}^3)$, such that

$$\begin{cases} (-\Delta + \Gamma_{\varepsilon,\phi})u = f \\ (-\Delta + \Gamma_{\varepsilon,\phi})V = u, \end{cases}$$

and

$$\begin{cases} ||u||_{C^{2,\alpha}(\mathbb{R}^3)} \le c||f||_{C^{0,\alpha}(\mathbb{R}^3)} \\ ||V||_{C^{2,\alpha}(\mathbb{R}^3)} \le c||u||_{C^{0,\alpha}(\mathbb{R}^3)}. \end{cases}$$

Now it remains to estimate the higher order derivatives of u. For this purpose, we differentiate the equation satisfied by u and we get

$$(-\Delta + \Gamma_{\varepsilon,\phi})V_j = u_j - (\Gamma_{\varepsilon,\phi})_j V \tag{111}$$

for $j = 1, \dots, 3$, hence, applying the regularity estimates for $(-\Delta + \Gamma_{\varepsilon,\phi})$,

$$||V_j||_{C^{2,\alpha}(\mathbb{R}^3)} \le c(||u_j||_{C^{0,\alpha}(\mathbb{R}^3)} + ||f||_{C^{0,\alpha}(\mathbb{R}^3)}) \le c||f||_{C^{0,\alpha}(\mathbb{R}^3)}$$

and, since the right-hand side of (111) behaves like $e^{\delta|x|}$, then, arguing as in the proof of Proposition 21, step (ii), we can see that $V_j \in C^{2,\alpha}_\delta(\mathbb{R}^3)$, that is $V \in C^{3,\alpha}_\delta(\mathbb{R}^3)$.

Similarly, differentiating the equation once again, we see that

$$(-\Delta + \Gamma_{\varepsilon,\phi})V_{ij} = u_{ij} - (\Gamma_{\varepsilon,\phi})_i V_j - (\Gamma_{\varepsilon,\phi})_j V_i - (\Gamma_{\varepsilon,\phi})_{ij} V,$$

for $i, j = 1, \dots, 3$, so in particular

$$||V_{ij}||_{C^{2,\alpha}(\mathbb{R}^3)} \le c(||u_{ij}||_{C^{0,\alpha}(\mathbb{R}^3)} + ||f||_{C^{0,\alpha}(\mathbb{R}^3)}) \le c||f||_{C^{0,\alpha}(\mathbb{R}^3)},$$

and $V \in C^{4,\alpha}_\delta(\mathbb{R}^3)$. □

6.2. The proof of Proposition 11: solving equation (63) by a fixed point argument

Equation (63) is equivalent to the fixed point problem

$$V = T_1(V) := -\Psi_{\varepsilon,\phi}\Big\{(1-\chi_2)F(v_{\varepsilon,\phi}) + (1-\chi_1)Q_{\varepsilon,\phi}(\chi_2 U + V) + \mathrm{N}_{\varepsilon,\phi}(U) + \mathrm{P}_{\varepsilon,\phi}(V) - \varepsilon^4\lambda(1-\chi_1)(1 - v_{\varepsilon,\phi} - V)\Big\},$$

that we will solve by showing that T_1 is a contraction on the ball

$$\Lambda_1 := \{V \in C^{4,\alpha}_{\delta,s}(\mathbb{R}^3) : ||V||_{C^{4,\alpha}(\mathbb{R}^3)} \le C_1 e^{-a/\varepsilon}\},$$

provided the constant C_1 is large enough. In fact, by the exponential decay of U far from Σ_ε, we get that

$$||\mathrm{N}_{\varepsilon,\phi}(U)||_{C^{4,\alpha}(\mathbb{R}^3)} \le \tilde{c}e^{-a/\varepsilon},$$

$a := \delta\tau/2$, for some constant $\tilde{c} > 0$ independent of ε and ϕ. By (49) and (50), the same is true for $(1-\chi_2)F(v_{\varepsilon,\phi})$. Moreover, by (82), (49) and (50),

$$||\mathrm{P}_{\varepsilon,\phi}(V)||_{C^{4,\alpha}_\delta(\mathbb{R}^3)} \le c\varepsilon||V||_{C^{4,\alpha}_\delta(\mathbb{R}^3)} \le c\varepsilon e^{-a/\varepsilon},$$

with $c > 0$ depending on W, τ, δ but not of ε and ϕ. Moreover, using that

$$||(1-\chi_1)V||_{C^{4,\alpha}(\mathbb{R}^3)} \le c||V||_{C^{4,\alpha}(\mathbb{R}^3)}$$

and

$$||(1-\chi_1)\chi_2 U||_{C^{4,\alpha}(\mathbb{R}^3)} \le ce^{-a/\varepsilon},$$

where $(1-\chi_1)\chi_2 U$ is understood to be 0 outside the support of χ_2, and the definition of $Q_{\varepsilon,\phi}$ (see (56)), we get

$$||(1-\chi_1)Q_{\varepsilon,\phi}(\chi_2 U + V)||_{C^{4,\alpha}(\mathbb{R}^3)} \le ce^{-2a/\varepsilon}.$$

Furthermore, we can see that

$$||\lambda(1-\chi_1)(1-\tilde{v}_{\varepsilon,\phi}-V)||_{C^{0,\alpha}(\mathbb{R}^3)} \leq c|\lambda|(e^{-a/\varepsilon}+||V||_{C^{4,\alpha}(\mathbb{R}^3)}). \tag{112}$$

Up to now, we have just proved that T_1 maps Λ_1 in itself, provided C_1 is large enough. In order to show that it is actually a contraction, we need to estimate its Lipschitz constant. The terms depending on V are $P_{\varepsilon,\phi}$, that fulfills

$$||P_{\varepsilon,\phi}(V_1)-P_{\varepsilon,\phi}(V_2)||_{C^{4,\alpha}(\mathbb{R}^3)} \leq c\varepsilon||V_1-V_2||_{C^{4,\alpha}(\mathbb{R}^3)}$$

for some constant $c>0$ independent of ε and ϕ, $(1-\chi_1)Q_{\varepsilon,\phi}(\chi_2 U+V)$, that fulfills

$$||(1-\chi_1)(Q_{\varepsilon,\phi}(\chi_2 U+V)-Q_{\varepsilon,\phi}(\chi_2 U+V))||_{C^{4,\alpha}_\delta(\mathbb{R}^3)} \leq ce^{-a/\varepsilon}||V_1-V_2||_{C^{4,\alpha}(\mathbb{R}^3)},$$

and $\varepsilon^4\lambda(1-\chi_1)V$, that satisfies a similar estimate.

Lipschitz dependence on U, ϕ and λ.

Given $\phi\in B_4(1/4)$ and $U_1,U_2\in C^{4,\alpha}(\Sigma_\varepsilon\times\mathbb{R})$, the difference between the solutions V_{ε,ϕ,U_1} and $V_{\varepsilon,\phi,\lambda,U_1}$ fulfills

$$\begin{aligned}(-\Delta+\Gamma_{\varepsilon,\phi})^2&(V_{\varepsilon,\phi,\lambda,U_1}-V_{\varepsilon,\phi,\lambda,U_2})\\ &=(1-\chi_1)(Q_{\varepsilon,\phi}(\chi_2 U_2+V_{\varepsilon,\phi,\lambda,U_2})-Q_{\varepsilon,\phi}(\chi_2 U_1+V_{\varepsilon,\phi,\lambda,U_1}))\\ &\quad+\mathrm{N}_{\varepsilon,\phi}(U_2)-\mathrm{N}_{\varepsilon,\phi}(U_1)+\mathrm{P}_{\varepsilon,\phi}(V_{\varepsilon,\phi,\lambda,U_2})-\mathrm{P}_{\varepsilon,\phi}(V_{\varepsilon,\phi,\lambda,U_1}).\end{aligned}$$

By (61), the terms involving $\mathrm{N}_{\varepsilon,\phi}$ satisfy

$$||\mathrm{N}_{\varepsilon,\phi}(U_1)-\mathrm{N}_{\varepsilon,\phi}(U_2)||_{C^{0,\alpha}(\mathbb{R}^3)} \leq ce^{-a/\varepsilon}||U_2-U_1||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})}.$$

By (62), the terms involving $\mathrm{N}_{\varepsilon,\phi}$ can be estimated with the difference between the solutions, that is

$$||\mathrm{P}_{\varepsilon,\phi}(V_{\varepsilon,\phi,\lambda,U_1})-\mathrm{P}_{\varepsilon,\phi}(V_{\varepsilon,\phi,\lambda,U_2})||_{C^{0,\alpha}(\mathbb{R}^3)} \leq ce^{-a/\varepsilon}||V_{\varepsilon,\phi,\lambda,U_1}-V_{\varepsilon,\phi,\lambda,U_2}||_{C^{4,\alpha}(\mathbb{R}^3)}, \tag{113}$$

and

$$\begin{aligned}||(1-\chi_1)(Q_{\varepsilon,\phi}(\chi_2 U_1+V_{\varepsilon,\phi,\lambda,U_1})-Q_{\varepsilon,\phi}(\chi_2 U_2+V_{\varepsilon,\phi,\lambda,U_2}))||_{C^{0,\alpha}(\mathbb{R}^3)} \leq\\ ce^{-a/\varepsilon}(||V_{\varepsilon,\phi,\lambda,U_1}-V_{\varepsilon,\phi,\lambda,U_2}||_{C^{4,\alpha}(\mathbb{R}^3)}+||U_1-U_2||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})}).\end{aligned}$$

Therefore, applying $\Psi_{\varepsilon,\phi}$ to the right-hand side of (113), we obtain

$$\begin{aligned}||V_{\varepsilon,\phi,\lambda,U_1}-V_{\varepsilon,\phi,\lambda,U_2}||_{C^{4,\alpha}(\mathbb{R}^3)} \leq\\ ce^{-a/\varepsilon}(||V_{\varepsilon,\phi,\lambda,U_1}-V_{\varepsilon,\phi,\lambda,U_2}||_{C^{4,\alpha}(\mathbb{R}^3)}+||U_1-U_2||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})}),\end{aligned}$$

thus, reabsorbing the norm of the difference between the solutions,

$$\frac{1}{2}||V_{\varepsilon,\phi,\lambda,U_1} - V_{\varepsilon,\phi,\lambda,U_2}||_{C^{4,\alpha}(\mathbb{R}^3)} \leq (1 - ce^{-a/\varepsilon})||V_{\varepsilon,\phi,\lambda,U_1} - V_{\varepsilon,\phi,\lambda,U_2}||_{C^{4,\alpha}(\mathbb{R}^3)}$$
$$\leq ce^{-a/\varepsilon}||U_1 - U_2||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})}.$$

The Lipschitz dependence on ϕ and λ can be treated with a similar argument. It is worth to point out that also the potential $\Gamma_{\varepsilon,\phi}$ actually depends on ϕ, through the approximate solution and the cutoff function. However, this dependence is mild enough for our purposes, in fact the difference of the potentials $\Gamma_{\varepsilon,\phi_1} - \Gamma_{\varepsilon,\phi_2}$ is exponentially small in ε.

6.3. Invertibility in a neighbourhood of Σ_ε: the linear problem

Now we look for a solution to equation (64) respecting the symmetries of the Torus. First we study the linear operator $\mathcal{L}^2_\varepsilon$.

Proposition 22. *Let $0 < \delta < \sqrt{W''(\pm 1)}$ and $\phi \in B_4(1/4)$. For any $f \in \mathcal{E}^{0,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R})$, there exists a unique solution $U = G_\varepsilon(f)$ in $\mathcal{E}^{4,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R})$ to $\mathcal{L}^2_\varepsilon U = f$ such that*

$$||U||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \leq C||f||_{C^{0,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})},$$

for some constant $C > 0$ which is independent of ε.

If f respects the symmetries of the Torus, then also the solution $U = G_\varepsilon f$ does. In other words, G_ε maps $\mathcal{E}^{4,\alpha}_{\delta,s}(\Sigma_\varepsilon \times \mathbb{R})$ into $\mathcal{E}^{0,\alpha}_{\delta,s}(\Sigma_\varepsilon \times \mathbb{R})$. This fact follows from uniqueness.

It is useful to see that we can control the odd part of the solution with the odd part (in t) of f and the same is true for the even parts.

Lemma 23. *Let $0 < \delta < \sqrt{W''(\pm 1)}$ and $f \in C^{0,\alpha}_{\delta,s}(\Sigma_\varepsilon \times \mathbb{R})$. Let $U \in C^{4,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R})$ be the solution to $\mathcal{L}^2_\varepsilon U = f$. Then*

$$\begin{cases} ||U_o||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \leq c||f_o||_{C^{0,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \\ ||U_e||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \leq c||f_e||_{C^{0,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})}, \end{cases}$$

where c is the constant found in Proposition 22.

Proof. We set, for any $(y,t) \in \Sigma_\varepsilon \times \mathbb{R}$, $\tilde{U}(y,t) := U(y,-t)$ and $\tilde{f}(y,t) := f(y,-t)$. Using that W'' is even and $v_\star$ is odd, we can see that $\mathcal{L}^2_\varepsilon\tilde{U} = \tilde{f}$. Therefore, subtracting and multiplying by $1/2$, we get

$$\mathcal{L}^2_\varepsilon\left(\frac{U(y,t) - \tilde{U}(y,t)}{2}\right) = \frac{f(y,t) - \tilde{f}(y,t)}{2},$$

that is $\mathcal{L}^2_\varepsilon U_o(y,t) = f_o$. In addition,

$$\int_{-\infty}^{\infty} U_o(y,t)v'_\star(t)dt = \int_{-\infty}^{\infty} f_o(y,t)v'_\star(t)dt = 0,$$

for any $y \in \Sigma_\varepsilon$, hence $U_o = G_\varepsilon(f_o)$, so in particular the first estimate holds true. The second one can be proved by a similar argument. $\square$

Now we prove Proposition 22, with the aid of some Lemmas and Remarks.

First we consider the spectral decomposition of $\mathcal{L}_\varepsilon$. We will denote by $(\lambda_j, \phi_j)_{j\geq 0}$ the eigendata of $-\Delta_\Sigma$. We observe that $\lambda_0 = 0$, $\lambda_j \geq \lambda_1 > 0$, ϕ_0 is constant and, without loss of generality, we can assume that $||\phi_j||_{L^2(\Sigma)} = 1$ (see [24]). Similarly, we will denote by $\{\mu_k\}_{k\geq 0}$ the eigenvalues of $L_\star = -\partial_{tt} + W''(v_\star(t))$. In [22], Müller proved that $\mu_0 = 0$, and the corresponding eigenspace, that is the Kernel, is generated by $v'_\star(t)$, while $\mu_k \geq \mu_1 > 0$ (see also [18]).

Remark 24. The eigenvalues of $\mathcal{L}_\varepsilon$ are $\{\mu_k + \varepsilon^2\lambda_j\}_{j,k\geq 0}$, thus all non-zero eigenvalues are positive and bounded away from 0, indeed $\mu_k + \varepsilon^2\lambda_j \geq \varepsilon^2\lambda_1 > 0$.

Lemma 25. *Let*

$$\mathcal{L}_\varepsilon : H^1(\Sigma_\varepsilon \times \mathbb{R}) \to H^{-1}(\Sigma_\varepsilon \times \mathbb{R})$$

be defined by the duality relation

$$\Big\langle \mathcal{L}_\varepsilon U_1, U_2 \Big\rangle = \int_{\Sigma_\varepsilon \times \mathbb{R}} \Big\{ (\nabla_{\Sigma_\varepsilon} U_1, \nabla_{\Sigma_\varepsilon} U_2) + \partial_t U_1 \partial_t U_2 + W''(v_\star(t)) U_1 U_2 \Big\} d\sigma(y) dt,$$

for any $U_1, U_2 \in C^{k,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R})$. Then

$$Ker(\mathcal{L}_\varepsilon) = span(v'_\star(t)).$$

Proof. It is possible to see that $(\lambda_{\varepsilon,j}, \phi_{\varepsilon,j})_{j\geq 0} := (\varepsilon^2\lambda_j, \varepsilon^2\phi_j(\varepsilon y))_{j\geq 0}$ are eigendata of Σ_ε and $\phi_{\varepsilon,j}$ are orthonormal in $L^2(\Sigma_\varepsilon)$. Any function $w \in H^1(\Sigma_\varepsilon \times \mathbb{R})$ can be expanded in Fourier series as follows

$$U(y,t) = \sum_{j\geq 0} U_j(t)\phi_{\varepsilon,j}(y)$$

where

$$U_j(t) = \int_{\Sigma_\varepsilon} U(y,t)\phi_{\varepsilon,j}(y) d\sigma(y).$$

If $\mathcal{L}_\varepsilon w = 0$, applying the operator to each term in the series, we get

$$-\partial_{tt} U_j(t) + \lambda_{\varepsilon,j} U_j(t) + W''\big(v_\star(t)\big) U_j(t) = 0$$

for any $j \geq 0$, so $U_0(t) = c v'_\star(t)$ and $w_j = 0$ for $j \geq 1$. $\square$

Let

$$\mathcal{O} := \left\{ U \in H^1(\Sigma_\varepsilon \times \mathbb{R}) : \int\limits_{\Sigma_\varepsilon \times \mathbb{R}} U(y,t) v_\star'(t) d\sigma(y) dt = 0 \right\},$$

be the orthogonal to $v_\star'(t)$ in $H^1(\Sigma_\varepsilon \times \mathbb{R})$.

Lemma 26. *For any $f \in L^2(\Sigma_\varepsilon \times \mathbb{R})$ satisfying*

$$\int\limits_{-\infty}^{\infty} f(y,t) v_\star'(t) dt = 0 \textit{ for any } y \in \Sigma_\varepsilon,$$

there exists a unique $U \in H^1(\Sigma_\varepsilon \times \mathbb{R})$ such that

$$\begin{cases} \mathcal{L}_\varepsilon U = f \\ \int_{-\infty}^{\infty} U(y,t) v_\star'(t) dt = 0 \quad \textit{for any } y \in \Sigma_\varepsilon. \end{cases}$$

Proof. At first we observe that

$$||U|| = \int\limits_{\Sigma_\varepsilon \times \mathbb{R}} |\nabla_{\Sigma_\varepsilon} U(y,z)|^2 + (\partial_{tt} U(y,t))^2 + W''(v_\star'(z)) U^2(y,z) d\sigma(y) dt \tag{114}$$

is an equivalent norm on $\mathcal{O}$, that is, for any $U \in X$, we have

$$c_{\varepsilon,1} ||U||_{H^1(\Sigma_\varepsilon \times \mathbb{R})} \le ||U|| \le c_{\varepsilon,2} ||U||_{H^1(\Sigma_\varepsilon \times \mathbb{R})},$$

for some constants $c_{\varepsilon,1}, c_{\varepsilon,2} > 0$. In fact, by the spectral decomposition of $\mathcal{L}_\varepsilon$, (see Remark 24),

$$\int\limits_{\Sigma_\varepsilon \times \mathbb{R}} \mathcal{L}_\varepsilon U U d\sigma(y) dt \ge \varepsilon^2 \lambda_1 \int\limits_{\Sigma_\varepsilon \times \mathbb{R}} U^2 d\sigma(y) dt.$$

Since $W''(v_\star(t))$ is bounded, a pointwise estimate yields that

$$\int\limits_{\Sigma_\varepsilon \times \mathbb{R}} \mathcal{L}_\varepsilon U U d\sigma(y) dt \ge \int\limits_{\Sigma_\varepsilon \times \mathbb{R}} |\nabla_{\Sigma_\varepsilon} U|^2 + (\partial_{tt} U)^2 d\sigma(y) dt - c \int\limits_{\Sigma_\varepsilon \times \mathbb{R}} U^2 d\sigma(y) dt, \tag{115}$$

for some constant $c > 0$. Now we point out that, for any $0 < \lambda < 1$, we have

$$\int\limits_{\Sigma_\varepsilon \times \mathbb{R}} \mathcal{L}_\varepsilon U U d\sigma(y) dt = \lambda \int\limits_{\Sigma_\varepsilon \times \mathbb{R}} \mathcal{L}_\varepsilon U U d\sigma(y) dt + (1-\lambda) \int\limits_{\Sigma_\varepsilon \times \mathbb{R}} \mathcal{L}_\varepsilon U U d\sigma(y) dt \ge$$

$$\lambda \left(\int\limits_{\Sigma_\varepsilon \times \mathbb{R}} |\nabla_{\Sigma_\varepsilon} U|^2 + (\partial_{tt} U)^2 d\sigma(y) dt - c \int\limits_{\Sigma_\varepsilon \times \mathbb{R}} U^2 d\sigma(y) dt \right) + (1-\lambda)\varepsilon^2 \lambda_1 \int\limits_{\Sigma_\varepsilon \times \mathbb{R}} U^2 d\sigma(y) dt,$$

so, in order to prove the lower bound, it is enough to choose $\lambda < \varepsilon^2\lambda_1/(c+\varepsilon^2\lambda_1)$. As a consequence, by the Riesz representation theorem, for any $f \in L^2(\Sigma_\varepsilon \times \mathbb{R})$ such that

$$\int_{\Sigma_\varepsilon \times \mathbb{R}} f(y,t)v'_\star(t)d\sigma(y)dt = 0, \tag{116}$$

the equation $\mathcal{L}_\varepsilon U = f$ admits a unique solution $U \in \mathcal{O}$. We observe that orthogonality condition (116) is necessary for solvability, since

$$\int_{\Sigma_\varepsilon \times \mathbb{R}} f(y,t)v'_\star(t)d\sigma(y)dt = \int_{\Sigma_\varepsilon \times \mathbb{R}} \mathcal{L}_\varepsilon U(y,t)v'_\star(t)d\sigma(y)dt = \int_{\Sigma_\varepsilon \times \mathbb{R}} U(y,t)\mathcal{L}_\varepsilon v'_\star(t)d\sigma(y)dt = 0.$$

If in particular f satisfies (114), then, by proposition 8, 4 of [24], also w satisfies (114). □

Now we are ready to conclude the proof of Proposition 22.

Proof. There are two more steps. As first we need some regularity theory to estimate the $C^{2,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R})$ norm of the solution U if $f \in \mathcal{E}^{0,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R})$, then we have to iterate the estimates to deal with the operator $\mathcal{L}^2_\varepsilon$. For the first step, see Proposition 8, 3 of [24]. As regards the second one, we argue as follows.

If $f \in \mathcal{E}^{0,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R})$, the above discussion yields that we can find $\tilde{U} \in \mathcal{E}^{2,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R})$ such that

$$\begin{cases} \mathcal{L}_\varepsilon \tilde{U} = f \\ ||\tilde{U}||_{C^{2,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R})} \le C||f||_{C^{0,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R})}, \end{cases}$$

for some constant $C > 0$ independent of ε. Now, by the same argument, we can find $U \in \mathcal{E}^{2,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R})$ satisfying

$$\begin{cases} \mathcal{L}_\varepsilon U = \tilde{U} \\ ||U||_{C^{2,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R})} \le C||\tilde{U}||_{C^{0,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R})} \le C||f||_{C^{0,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R})}, \end{cases}$$

for some constant $C > 0$ independent of ε. To conclude the proof, we have to show that $U \in C^{4,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R})$ and

$$||U||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R})} \le C||f||_{C^{0,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R})}. \tag{117}$$

In order to do so we apply a bootstrap argument. We differentiate (117) with respect to y_j and we get

$$\mathcal{L}_\varepsilon U_j = \tilde{U}_j.$$

By (117), we get that $U_j \in C^{2,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R})$ and

$$||U_j||_{C^{2,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \le C||\tilde{U}_j||_{C^{0,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \le C||\tilde{U}||_{C^{2,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \le C||f||_{C^{0,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})}.$$

In the same way, taking the derivative with respect to t, we get

$$\mathcal{L}_\varepsilon U_t = \tilde{U}_t - \frac{1}{\varepsilon}W'''(v_\star(t))v'_\star(t)U.$$

Exactly as before, we have

$$||U_t||_{C^{2,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \le C(||\tilde{U}_t||_{C^{0,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} + ||W'''(v_\star(z))v'_\star(t)U||_{C^{0,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})}) \le$$
$$C(||\tilde{U}||_{C^{2,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} + ||U||_{C^{0,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})}) \le C(||f||_{C^{0,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} + ||\tilde{U}||_{C^{2,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})}) \le$$
$$C||f||_{C^{0,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})}.$$

Therefore we have

$$||\nabla^3(U\psi_\delta)||_\infty \le C||\nabla U||_{C^{2,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \le C||f||_{C^{0,\alpha}_{\varepsilon,\delta}(\Sigma\times\mathbb{R})}.$$

Differentiating the equation once again, we get

$$||\nabla^4(U\psi_\delta)||_\infty + [\nabla^4(U\psi_\delta)]_\alpha \le C||f||_{C^{0,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})}.$$

In conclusion, we have (117). □

6.4. *The proof of Proposition 12: solving equation (67) by a fixed point argument*

Equation (67) is equivalent to the fixed point problem

$$U = T_2(U) := G_\varepsilon\left\{ -F(\tilde{v}_{\varepsilon,\phi}) + \varepsilon^4\chi_1\lambda(1-\tilde{v}_{\varepsilon,\phi}) - \mathrm{T}(U, V_{\varepsilon,\phi,U},\phi) + p(y)v'_\star(t)\right\}.$$

Once again, we will solve it by showing that T_2 is a contraction on the ball

$$\Lambda_2 := \{U \in \mathcal{E}^{4,\alpha}_{\delta,s}(\Sigma_\varepsilon\times\mathbb{R}) : ||U||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \le C_2\varepsilon^3\},$$

provided $C_2 > 0$ is large enough. First we observe that, by definition of p, the right hand side is orthogonal to $v'_\star(t)$ for any $y \in \Sigma_\varepsilon$, thus we can actually apply the operator G_ε. Moreover, if U respects the symmetries of the Torus, then also the right-hand side does, thus, applying G_ε, we get once again something that respects these symmetries. Now we show that, if $||U||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \le C_2\varepsilon^3$, then also $T_2(U)$ satisfies the same upper bound, for some large constant C_2.

We note that

$$||F(\tilde{v}_{\varepsilon,\phi})||_{C^{0,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \le C\varepsilon^3,$$

for some constant C depending just on W, τ and the geometric quantities of Σ, and the same is true for $p(y)v_\star'(t)$. The other terms are smaller. For instance, using (56) and the fact that V is exponentially small,

$$||\chi_1 Q_{\varepsilon,\phi}(U+V)||_{C^{0,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \le c\varepsilon^6.$$

Similarly, we can see that $||\mathrm{M}_{\varepsilon,\phi}(V)||_{C^{0,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \le ce^{-a/\varepsilon}$. In addition, since all the coefficients of $\mathrm{R}_{\varepsilon,\phi}$ are at least of order ε, we get that

$$||\mathrm{R}_{\varepsilon,\phi}(U)||_{C^{0,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \le c\varepsilon||U||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \le c\varepsilon^4.$$

Moreover,

$$||\varepsilon^4\lambda(\chi_1 U+V)||_{C^{0,\alpha}_\delta(\mathbb{R}^3)} \le c\varepsilon^4|\lambda|(1+||U||_{C^{4,\alpha}_\delta(\mathbb{R}^3)}).$$

As regards the Lipschitz dependence on U, we observe that

$$||\chi_1(Q_{\varepsilon,\phi}(U_1+V)-Q_{\varepsilon,\phi}(U_2+V))||_{C^{0,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \le c\varepsilon^3||U_1-U_2||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})}$$

and

$$||\mathrm{R}_{\varepsilon,\phi}(U_1)-\mathrm{R}_{\varepsilon,\phi}(U_2)||_{C^{0,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \le c\varepsilon||U_1-U_2||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})}.$$

Estimate of the odd part of the solution $U_{\varepsilon,\phi,\lambda}$.

Up to now we have proved the existence of a solution $U_{\varepsilon,\phi,\lambda}$ to equation (67) satisfying $||U_{\varepsilon,\phi,\lambda}||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \le c\varepsilon^3$. However, we point out that the only terms of order ε^3 in the right-hand side come from $F(\tilde{v}_{\varepsilon,\phi})$. In fact, as we observed above, $||\mathrm{T}(U, V_{\varepsilon,\phi,\lambda,U}, \phi)||_{C^{0,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \le c\varepsilon^4$, so in particular the same is true for

$$\frac{1}{c_\star}\left(\int_{-\infty}^{\infty} \mathrm{T}(U, V_{\varepsilon,\phi,\lambda,U}, \phi)v_\star'(t)dt\right)v_\star'(t).$$

Moreover, by Proposition 17,

$$\int_{-\infty}^{\infty} (F(\tilde{v}_{\varepsilon,\phi}) - \varepsilon^4\lambda\chi_1(1-\tilde{v}_{\varepsilon,\phi}))(y,t)v_\star'(t)dt$$

is of order ε^4. Going back to Section 5, it is possible to see that the only terms of order ε^3 in $F(\tilde{v}_{\varepsilon,\phi})$ are even in t, thus the odd part of the right-hand side is of order ε^4, and therefore, by Lemma 23, the same is true for $U_{\varepsilon,\phi,\lambda}$, namely $||(U_{\varepsilon,\phi,\lambda})_o||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \le c\varepsilon^4$.

Lipschitz dependence on ϕ and λ.

Let us fix $\phi_1, \phi_2 \in C^{4,\alpha}(\Sigma)_{s,0}$ with $|\phi_k|_{C^{4,\alpha}(\Sigma)} < 1$ and $|\lambda| < 1$. To simplify the notation, we set, for $k = 1, 2$, $\tilde{V}_k := \tilde{v}_{\varepsilon,\phi_k,\lambda}$, $U_k := U_{\varepsilon,\phi_k,\lambda}$, $V_k := V_{\varepsilon,\phi_k,\lambda,U_k}$ and so on. In this proof, ε will always be small but fixed, and we will be interested in the dependence on ϕ.

First we note that, by construction,

$$||F(\tilde{v}_1) - F(\tilde{v}_2)||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \leq c\varepsilon^3|\phi_1 - \phi_2|_{C^{4,\alpha}(\Sigma)},$$

(see Section 5.1). Using the Lipschitz dependence of V on the data proved in Proposition 11 and the definitions of $\mathrm{M}_{\varepsilon,\phi}$, $Q_{\varepsilon,\phi}$ and $\mathrm{R}_{\varepsilon,\phi}$, it is possible to see that

$$||\mathrm{M}_1(V_1) - \mathrm{M}_2(V_2)||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \leq ce^{-a/\varepsilon}(||U_1 - U_2||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} + |\phi_1 - \phi_2|_{C^{4,\alpha}(\Sigma)}),$$

$$||\chi_1(Q_1(U_1 + V_1) - Q_2(U_2 + V_2))||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \leq c\varepsilon^3(||U_1 - U_2||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} + |\phi_1 - \phi_2|_{C^{4,\alpha}(\Sigma)}),$$

$$||\mathrm{R}_1(U_1) - \mathrm{R}_2(U_2)||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \leq c\varepsilon||U_1 - U_2||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} + c\varepsilon^4|\phi_1 - \phi_2|_{C^{4,\alpha}(\Sigma)}.$$

Now it remains to deal with $p(y)$, that also depends on ε, ϕ and λ. We write, for any $y \in \Sigma_\varepsilon$ and $\phi \in B_4(1/4)$,

$$p(y) = \frac{1}{c_\star}\int_{-\infty}^{\infty}(F(\tilde{v}_{\varepsilon,\phi}) - \varepsilon^4\lambda(1 - \tilde{v}_{\varepsilon,\phi}))(y,t)v'_\star(t)dt$$
$$+ p_1(\phi,\lambda)(y) + p_2(\phi,\lambda)(y) + p_3(\phi,\lambda)(y) + p_4(\phi,\lambda)(y),$$

where we have set

$$p_1(\phi,\lambda)(y) := \frac{\varepsilon^4\lambda}{c_\star}\int_{-\infty}^{\infty}((1 - \tilde{v}_{\varepsilon,\phi})(1 - \chi_1) + \chi_1 U + V)(y,t)v'_\star(t)dt, \tag{118}$$

$$p_2(\phi,\lambda)(y) := \frac{1}{c_\star}\int_{-\infty}^{\infty}\chi_1 Q_{\varepsilon,\phi}(U + V)(y,t)v'_\star(t)dt, \tag{119}$$

$$p_3(\phi,\lambda)(y) := \frac{1}{c_\star}\int_{-\infty}^{\infty}\chi_1\mathrm{M}_{\varepsilon,\phi}(V)(y,t)v'_\star(t)dt \tag{120}$$

$$p_4(\phi,\lambda)(y) := \frac{1}{c_\star}\int_{-\infty}^{\infty}\mathrm{R}_{\varepsilon,\phi}(U)(y,t)v'_\star(t)dt \tag{121}$$

and $U := U_{\varepsilon,\phi,\lambda}$, $V := V_{\varepsilon,\phi,\lambda,U}$. Since we want to deal with functions defined on Σ, we will set, for any $y \in \Sigma_\varepsilon$, $\tilde{p}_i(\phi,\lambda)(\varepsilon y) := p_i(\phi,\lambda)(y)$, for $i = 1,\dots,4$. It follows from Proposition 17 and that

$$\left|\int_{-\infty}^{\infty}(F(\tilde{v}_{\varepsilon,\phi_1})-\varepsilon^4\lambda_1(1-\tilde{v}_{\varepsilon,\phi_1}))(y,t)v'_\star(t)dt-\int_{-\infty}^{\infty}(F(\tilde{v}_{\varepsilon,\phi_2})-\varepsilon^4\lambda_2(1-\tilde{v}_{\varepsilon,\phi_2}))(y,t)v'_\star(t)dt\right|$$
$$\leq c\varepsilon^3(|\phi_1-\phi_2|_{C^{4,\alpha}(\Sigma)}+|\lambda_1-\lambda_2|).$$

In addition, by the previous discussion,

$$\begin{cases}|\tilde{p}_1(\phi,\lambda)|_{C^{0,\alpha}(\Sigma)}\leq c\varepsilon^7\\ |\tilde{p}_1(\phi_1,\lambda_1)-\tilde{p}_1(\phi_2,\lambda_2)|_{C^{0,\alpha}(\Sigma)}\leq c\varepsilon^7(|\phi_1-\phi_2|_{C^{4,\alpha}(\Sigma)}+|\lambda_1-\lambda_2|).\end{cases} \tag{122}$$

Furthermore, by the Lipschitz dependence of V on the data, proved in Proposition 11, and by the fact that $||U||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})}\leq C_2\varepsilon^3$, we have

$$\begin{cases}|\tilde{p}_2(\phi,\lambda)|_{C^{0,\alpha}(\Sigma)}\leq c\varepsilon^6\\ |\tilde{p}_2(\phi_1,\lambda_1)-\tilde{p}_2(\phi_2,\lambda_2)|_{C^{0,\alpha}(\Sigma)}\leq c\varepsilon^3(e^{-a/\varepsilon}|\phi_1-\phi_2|_{C^{4,\alpha}(\Sigma)}\\ +||U_1-U_2||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})}+|\lambda_1-\lambda_2|),\end{cases} \tag{123}$$

and, similarly

$$\begin{cases}|\tilde{p}_3(\phi,\lambda)|_{C^{0,\alpha}(\Sigma)}\leq ce^{-a/\varepsilon}\\ |\tilde{p}_3(\phi_1,\lambda_1)-\tilde{p}_3(\phi_2,\lambda_2)|_{C^{0,\alpha}(\Sigma)}\leq ce^{-a/\varepsilon}(|\phi_1-\phi_2|_{C^{4,\alpha}(\Sigma)}\\ +||U_1-U_2||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})}+|\lambda_1-\lambda_2|).\end{cases} \tag{124}$$

As regards $\tilde{p}_4$, we give a first, rough estimate that is enough to prove the Lipschitz dependence of U on ϕ. However, we will see later that this estimate is actually not enough to solve the bifurcation equation, thus we will improve it in Lemma 27, using the estimate of the odd part of U (see section 7).

$$\begin{cases}|\tilde{p}_4(\phi,\lambda)|_{C^{0,\alpha}(\Sigma)}\leq c\varepsilon^4\\ |\tilde{p}_4(\phi_1,\lambda_1)-\tilde{p}_4(\phi_2,\lambda_2)|_{C^{0,\alpha}(\Sigma)}\leq c\varepsilon||U_1-U_2||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})}.\end{cases} \tag{125}$$

In conclusion, the equation satisfied by the difference of the solutions U_1-U_2 is of the form

$$\mathcal{L}^2_\varepsilon(U_1-U_2)=g(\phi_1,\lambda_1)(y,t)-g(\phi_2,\lambda_2)(y,t),$$

where $g(\phi_i,\lambda_i)$ and U_i satisfy

$$\int_{-\infty}^{\infty}(g(\phi_1,\lambda_1)-g(\phi_2,\lambda_2))(y,t)v'_\star(t)dt=\int_{-\infty}^{\infty}(U_1-U_2)(y,t)v'_\star(t)dt=0,$$

thus, by Proposition 22,

$$||U_1 - U_2||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \leq c\varepsilon||U_1 - U_2||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} + c\varepsilon^3(|\phi_1 - \phi_2|_{C^{4,\alpha}(\Sigma)} + |\lambda_1 - \lambda_2|),$$

and hence, reabsorbing the first term of the right-hand side,

$$\frac{1}{2}||U_1 - U_2||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \leq c\varepsilon^3(|\phi_1 - \phi_2|_{C^{4,\alpha}(\Sigma)} + |\lambda_1 - \lambda_2|).$$

7. Solving the bifurcation equation

7.1. The proof of Proposition 13

First we note that, setting $w := w_{\varepsilon,\phi,\lambda}$,

$$\int_{\mathbb{R}^3}(1 - v_{\varepsilon,\phi} - w)^2 dx = \int_{\mathbb{R}^3}(1 - v_{\varepsilon,\phi})^2 dx - 2\int_{\mathbb{R}^3}(1 - v_{\varepsilon,\phi})w dx + \int_{\mathbb{R}^3} w^2 dx.$$

Now let us fix some notation. For any $\phi \in C^{4,\alpha}(\Sigma)_s$ and $0 < \varepsilon \leq 1$, $|\Sigma_{\varepsilon,\phi_\star}|_3$ will be the volume of the interior of $\Sigma_{\varepsilon,\phi_\star}$, that is its 3-Lebesgue measure. Moreover, we set

$$B_1 := \{x = Z_\varepsilon(y, t + \phi_\star(\varepsilon y)) : -7 - \tau/2\varepsilon < t < 0\}$$
$$B_2 := \{x = Z_\varepsilon(y, z) : 0 < t < 7 + \tau/2\varepsilon\},$$

V_i will be the volume of B_i, for $i = 1, 2$, and $A := \mathbb{R}^3 \backslash B$. Now we note that

$$\int_{\mathbb{R}^3}(1 - v_{\varepsilon,\phi}(x))^2 dx = \int_A (1 - v_{\varepsilon,\phi}(x))^2 dx + \int_B (1 - v_{\varepsilon,\phi}(x))^2 dx$$

and

$$\int_A (1 - v_{\varepsilon,\phi}(x))^2 dx = 4(|\Sigma_{\varepsilon,\phi_\star}|_3 - V_1), \tag{126}$$

where the expression of the volume of $\Sigma_{\varepsilon,\phi_\star}$ is

$$|\Sigma_{\varepsilon,\phi_\star}|_3 = \varepsilon^{-3}2\pi^2\sqrt{2} + \varepsilon^{-2}\int_\Sigma \phi_\star(\zeta)d\sigma(\zeta) \tag{127}$$
$$+ 2\pi\varepsilon^{-1}\int_0^{2\pi} \phi_\star^2(\vartheta)(\cos\vartheta + \sqrt{2}/2)d\vartheta + \frac{2\pi}{3}\int_0^{2\pi} \phi_\star^3(\vartheta)\cos(\vartheta)d\vartheta.$$

In order to compute this integral, we used the change of variables induced by the parametrization (74) of the Torus. Moreover,

$$\int_B (1-v)^2 = \int_B 1 - 2\int_B v + \int_B v^2.$$

We can see that

$$\int_B 1 = V_1 + V_2 \tag{128}$$

and

$$-2\int_B v = -8\pi \int_0^{2\pi} (2\phi_\star(\vartheta)\cos\vartheta + \varepsilon^{-1}\sqrt{2})d\vartheta \int_0^{7+\tau/2\varepsilon} t v_\star(t)dt + G_\varepsilon^1(\phi) \tag{129}$$

$$\int_B v^2 = 4\pi \int_0^{2\pi} (\phi_\star^2(\vartheta)\cos\vartheta + \varepsilon^{-1}\phi_\star(\vartheta)(2\cos\vartheta + \sqrt{2}) + \varepsilon^{-2})d\vartheta \int_0^{6+\tau/2\varepsilon} v_\star^2(t)dt + G_\varepsilon^2(\phi). \tag{130}$$

Moreover, using that

$$\int_\Sigma (\phi_\star - \phi)(\zeta)d\zeta = 0,$$

we can see that the term $v_\star(t + \phi_\star - \phi) - v_\star(t)$ does not contribute to the main term of our integral. Taking the sum of (126), (128), (129), (130),

$$\begin{aligned}\int_{\mathbb{R}^3} (1-v)^2 &= \varepsilon^{-3} 8\pi^2\sqrt{2}\Big(1 + \varepsilon\int_0^\infty (v_\star^2 - 1)dt\Big) - 8\pi^2\sqrt{2}\int_{t>6+\tau/\sqrt{2}} (v_\star^2 - 1)dt \\ &+ 4\int_\Sigma \phi_\star + 8\pi\varepsilon^{-1}\int_0^{2\pi} \phi_\star^2(\vartheta)(\cos\vartheta + \sqrt{2}/2)d\vartheta + \frac{8\pi}{3}\int_0^{2\pi} \phi_\star^3(\vartheta)\cos(\vartheta)d\vartheta \\ &+ 8\pi\int_0^{2\pi} (2\phi(\vartheta)\cos\vartheta + \varepsilon^{-1}\sqrt{2}) \int_0^{6+\tau/2\varepsilon} t(1 - v_\star(t))dt \\ &+ 4\pi\int_0^{2\pi} (\phi^2(\vartheta)\cos\vartheta + \varepsilon^{-1}\phi(\vartheta)(2\cos\vartheta + \sqrt{2}))d\vartheta \int_0^{6+\tau/2\varepsilon} (v_\star^2 - 1)dt \\ &+ 2\pi\int_0^{2\pi} (\phi^2(\vartheta)\cos\vartheta + \varepsilon^{-1}\phi(\vartheta)(2\cos\vartheta + \sqrt{2}))(\phi_\star - \phi)(\vartheta)d\vartheta \int_{-6-\tau/2\varepsilon}^{6+\tau/2\varepsilon} v_\star'(t)dt\end{aligned} \tag{131}$$

$$+ 2\pi \int_0^{2\pi} \cos\vartheta (\phi_\star - \phi)(\vartheta) d\vartheta \int_{-6-\tau/2\varepsilon}^{6+\tau/2\varepsilon} v_\star'(t) t^2 dt + G_\varepsilon^3(\phi) =$$

$$\varepsilon^{-3} 8\pi^2 \sqrt{2} \Big(1 + \varepsilon \int_0^\infty (v_\star^2 - 1) dt \Big) + 4 \int_\Sigma \phi_\star(\zeta) d\zeta$$

$$+ 16\pi^2 \sqrt{2} \varepsilon^{-1} \int_0^\infty t(1 - v_\star(t)) dt + G_\varepsilon^4(\phi)$$

$$= \varepsilon^{-3} 8\pi^2 \sqrt{2} c_\varepsilon + 4 \int_\Sigma \phi(\zeta) d\zeta + 16\pi^2 \sqrt{2} \varepsilon^{-1} \int_0^\infty t(1 - v_\star(t)) dt + G_\varepsilon^4(\phi),$$

where G_ε^i satisfy (75), $i = 1, \dots, 4$.

It remains to deal with the terms involving w. First we note that, if $f \in C_\delta^{0,\alpha}(\mathbb{R}^3)$, then the unique solution u of

$$(-\Delta + 2)^2 u = f \text{ in } \mathbb{R}^3$$

is given by the convolution between f and some suitable Green function $G : \mathbb{R}^3 \backslash \{0\} \to \mathbb{R}$, namely

$$u(x) = \int_{\mathbb{R}^3} G(x - y) f(y) dy.$$

Since w satisfies

$$(-\Delta + 2)^2 w = -F(v_{\varepsilon,\phi}) - Q_{\varepsilon,\phi}(w) + ((-\Delta + 2)^2 - F'(v_{\varepsilon,\phi}))[w], \tag{132}$$

then it can be written as

$$w(x) = \int_{\mathbb{R}^3} G(x - y)(-F(v_{\varepsilon,\phi}) - Q_{\varepsilon,\phi}(w) + ((-\Delta + 2)^2 - F'(v_{\varepsilon,\phi}))[w])(y) dy.$$

Here we are interested in showing the estimate

$$\int_{\mathbb{R}^3} |w(x)| dx \le c, \tag{133}$$

for some constant $c > 0$. We already know by construction that $w \in C_\delta^{4,\alpha}(\mathbb{R}^3)$ and $||w||_{C^{4,\alpha}(\mathbb{R}^3)} \le c\varepsilon^3$. We note that, by Fubini's theorem,

$$\int_{\mathbb{R}^3} |w(x)|dx =$$

$$\int_{\mathbb{R}^3} dx \left| \int_{\mathbb{R}^3} G(x-y)(-F(v_{\varepsilon,\phi}) - Q_{\varepsilon,\phi}(w) + ((-\Delta+2)^2 - F'(v_{\varepsilon,\phi}))[w])(y)dy \right| \leq$$

$$\int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} G(x-y)| - F(v_{\varepsilon,\phi}) - Q_{\varepsilon,\phi}(w) + ((-\Delta+2)^2 - F'(v_{\varepsilon,\phi}))[w]|(y)dy =$$

$$\int_{\mathbb{R}^3} | - F(v_{\varepsilon,\phi}) - Q_{\varepsilon,\phi}(w) + ((-\Delta+2)^2 - F'(v_{\varepsilon,\phi}))[w]|(y)dy \int_{\mathbb{R}^3} G(x-y)dx.$$

With a change of variables, we get that, for any $y \in \mathbb{R}^3$

$$\int_{\mathbb{R}^3} G(x-y)dx = \int_{\mathbb{R}^3} G(z)dz$$

is a positive constant, thus it remains to estimate the other integral. We can see that, by construction

$$\begin{aligned} F(v_{\varepsilon,\phi}) &= \chi_5 F(\tilde{v}_{\varepsilon,\phi}) - 2(\nabla\chi_5, \nabla\tilde{v}_{\varepsilon,\phi}) - \Delta\chi_5\tilde{v}_{\varepsilon,\phi} \\ &+ (-\Delta + W''(v_{\varepsilon,\phi}))\big(W'(v + (1-\chi_5)(\mathbb{H} - v_{\varepsilon,\phi})) - \chi_5 W'(v_{\varepsilon,\phi})\big), \end{aligned}$$

therefore it is supported in a ball of radius c/ε, for some constant $c > 0$, and it is bounded by a constant times ε^3, hence

$$\int_{\mathbb{R}^3} |F(v_{\varepsilon,\phi})|dy = \int_{B_{c/\varepsilon}(0)} |F(v_{\varepsilon,\phi})|dy \leq c|B_{c/\varepsilon}(0)|\varepsilon^3 \leq c.$$

A similar estimate holds for the linear term, since

$$\begin{aligned} ((-\Delta+2)^2 - F'(v_{\varepsilon,\phi}))w &= 2(W''(v)-2)\Delta w + 2(\nabla(W''(v)-2), \nabla w) + \Delta(W''(v)-2)w \\ &+ (4 - W''(v)^2)w - W^{(3)}(v)(-\Delta v + W'(v))w, \end{aligned}$$

is supported in the same ball and it is bounded, in $L^\infty(\mathbb{R}^3)$ norm, by a constant times ε^3. The term involving $Q_{\varepsilon,\phi}(w)$ is the most tricky. It is explicitly given by

$$\begin{aligned} \int_{\mathbb{R}^3} |Q_{\varepsilon,\phi}(w)(y)|dy \leq \int_{\mathbb{R}^3} \int_0^1 dt \int_0^t &|\Delta(W''(v_{\varepsilon,\phi} + sw)w^2) \\ &+ 2W'''(v_{\varepsilon,\phi} + sw)(-\Delta w + W''(v_{\varepsilon,\phi} + sw)w) + \\ &+ (W''(v_{\varepsilon,\phi} + sw)W'''(v_{\varepsilon,\phi} + sw) \end{aligned}$$

$$+ W^{(4)}(v_{\varepsilon,\phi} + sw)(-\Delta(v_{\varepsilon,\phi} + sw) + W'(v_{\varepsilon,\phi} + sw)))w^2|ds \le c\varepsilon^3\Big(\int_{\mathbb{R}^3}(|w(y)| + |\nabla w(y)|)dy\Big).$$

In conclusion,

$$\int_{\mathbb{R}^3} |w(x)|dx \le c\Big(1 + \varepsilon^3 \int_{\mathbb{R}^3} (|w(x)| + |\nabla w(x)|)dx\Big) \tag{134}$$

Similarly, the first derivatives of w satisfy

$$\partial_j w(x) = \int_{\mathbb{R}^3} \partial_j G(x-y)(-F(v_{\varepsilon,\phi}) - Q_{\varepsilon,\phi}(w) + ((-\Delta+2)^2 - F'(v_{\varepsilon,\phi}))[w])(y)dy,$$

hence the gradient satisfies an estimate like (134), thus

$$\int_{\mathbb{R}^3} (|w(x)| + |\nabla w(x)|)dx \le c\Big(1 + \varepsilon^3 \int_{\mathbb{R}^3} (|w(x)| + |\nabla w(x)|)dx\Big)$$

that is

$$\int_{\mathbb{R}^3} (|w(x)| + |\nabla w(x)|)dx \le c. \tag{135}$$

In particular, (135) yields that

$$\Big|\int_{\mathbb{R}^3} (1 - v_{\varepsilon,\phi})w dx\Big| \le c \int_{\mathbb{R}^3} |w|dx \le c$$

$$\Big|\int_{\mathbb{R}^3} w^2 dx\Big| \le c\varepsilon^3 \int_{\mathbb{R}^3} |w|dx \le c\varepsilon^3.$$

The Lipschitz dependence on the data follows from similar arguments.

7.2. *The proof of Proposition 14*

Before giving the proof, we state a technical Lemma, in which we prove that the term $\tilde{p}_4$ is small enough.

Lemma 27. *For any $\varepsilon > 0$ small enough, for any ϕ, ϕ_1, ϕ_2 satisfying $|\phi|_{C^{4,\alpha}(\Sigma)}, |\phi_1|_{C^{4,\alpha}(\Sigma)}, |\phi_2|_{C^{4,\alpha}(\Sigma)} < 1$, for any $|\lambda|, |\lambda_1|, |\lambda_2| < 1$, we have*

$$\begin{cases} |\tilde{p}_4(\phi,\lambda)|_{C^{0,\alpha}(\Sigma)} \le c\varepsilon^5 \\ |\tilde{p}_4(\phi_1,\lambda_1) - \tilde{p}_4(\phi_2,\lambda_2)|_{C^{0,\alpha}(\Sigma)} \le c\varepsilon^5(|\phi_1-\phi_2|_{C^{0,\alpha}(\Sigma)} + |\lambda_1-\lambda_2|), \end{cases}$$

for some constant $c > 0$.

Proof. We write $U = U_o + U_e$, where we have set, for the sake of simplicity, $U := U_{\varepsilon,\phi,\lambda}$. By Proposition 12, we know that $||U_o||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \le c\varepsilon^4$, therefore $||\mathrm{R}_{\varepsilon,\phi}(U_o)||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \le c\varepsilon^5$, since all the coefficients of $\mathrm{R}_{\varepsilon,\phi}$ are at least of order ε. It remains to deal with the even part U_e. We will see that all the terms of order ε^4 in the expression of $\mathrm{R}_{\varepsilon,\phi}(U_e)$ will vanish after projection. This can be seen by a direct computation

$$\begin{aligned}\mathrm{R}_{\varepsilon,\phi}(U_e) = \varepsilon\big\{&HW'''(v_\star)v_\star' U_e - \Delta_{\Sigma_\varepsilon}(\partial_t U_e + a_1^{ij}\partial_{ij}U_e t) \\ &+ H\partial_{ttt}U_e + W''(v_\star)(H\partial_t U_e + a_1^{ij}\partial_{ij}U_e t) \\ &+ (H\partial_t + ta_1^{ij}\partial_{ij})\mathcal{L}_\varepsilon U_e + \tilde{R}_{\varepsilon,\phi}(U_e)\big\},\end{aligned}$$

where $\tilde{R}_{\varepsilon,\phi}(U_e)$ is some linear operator with coefficients of order at least ε^2. All the terms of order ε are odd, thus they vanish when we multiply by $v_\star'$ and integrate, the other ones give rise to terms of order ε^5, being $U_{\varepsilon,\phi}$ of order ε^3. □

Remark 28. Before proving Proposition 14, we point out that

$$q_\varepsilon^1(\phi,\lambda)(y) = (\frac{1}{c_\star}\mathcal{H} + \varepsilon^{-5}\tilde{p}_4(\phi,\lambda))(y)$$

$$q_\varepsilon^2(\phi,\lambda)(y) = \left(\frac{1}{c_\star}\mathcal{G}_{\varepsilon,\phi,\lambda} + \varepsilon^{-6}(\tilde{p}_1(\phi,\lambda) + \tilde{p}_2(\phi,\lambda) + \tilde{p}_3(\phi,\lambda))\right)(y)$$

actually satisfy (71) and (72). For the notations, see Remark 18 and Proposition 17.

Now we are ready to prove Proposition 14.

Proof. In view of Proposition 17, the system of equations (70) and (75) is equivalent to the fixed point problem

$$\begin{aligned}(\phi,\lambda) = T_3(\phi,\lambda) := \mathcal{L}^{-1}\Big(&\varepsilon q_\varepsilon^1(\phi, c_\star\lambda/2)(y) + \varepsilon^2 q_\varepsilon^2(\phi, c_\star\lambda/2)(y), \\ &4\sqrt{2}\pi^2\varepsilon\int_0^\infty t(1-v_\star(t))dt + \varepsilon^2 G_\varepsilon(\phi, c_\star\lambda/2)\Big).\end{aligned}$$

Using the properties (71), (72), Proposition 13 and (75), we can show that T_3 is a contraction on the ball

$$\Lambda_3 := \{(\phi,\lambda) \in C^{4,\alpha}(\Sigma)_{s,0} \times \mathbb{R} : |\phi|_{C^{4,\alpha}(\Sigma)} + |\lambda| < C_3\varepsilon\},$$

provided C_3 is large enough. □

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