Bollettino dell'Unione Matematica Italiana
https://doi.org/10.1007/s40574-025-00469-4

Projective techniques in twistor geometry

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Received: 6 December 2024 / Accepted: 4 February 2025

Abstract
This survey explores the interplay between twistor geometry and projective geometry, focusing on their applications to algebraic surfaces. We explore two main topics: the inclusion of twistor fibers and lines in these surfaces, and the behavior of twistor discriminant loci, with a particular focus on degree-2 surfaces. The study highlights contributions from Ballico and collaborators, comparing the twistor spaces of $\mathbb{CP}^3$ and the flag threefold $\mathbb{F}$, which reveal fascinating contrasts and parallels. Key findings include a detailed analysis of surfaces in $\mathbb{CP}^3$ and $\mathbb{F}$ that either contain finite or infinite twistor fibers. The survey also touches on cubic surfaces in $\mathbb{CP}^3$ and their counterparts in $\mathbb{F}$, where the configurations of twistor fibers lead to intriguing results. Special attention is given to surfaces of twistor degree 2, including how their geometry and singularities interact with twistor projections. In particular, we discuss smooth and singular surfaces in $\mathbb{F}$ of bidegree $(1, 1)$ and $(0, 2)$, as well as their discriminant loci.

Keywords Twistor geometry · Flag manifolds · Bidegree surfaces · Discriminant loci · Conic classification · Unitary transformations

Mathematics Subject Classification 32L25 · 14J10 · 14M15 · 53C15 · 51F25

Contents

To Edoardo Ballico on the occasion of his 70th birthday.

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Published online: 18 February 2025

1 Introduction

Since its introduction [28], Twistor theory has proven to be a powerful tool for studying various geometric aspects of even-dimensional manifolds. Although the original goals of the theory have not been fully realized, partly due to physical inconsistencies (see e.g. [13] for a recent review), substantial efforts continue to modify its more problematic features (see Palatial Twistor Spaces [29]). In any case, the lasting interest in twistor theory can likely be attributed to its many intriguing mathematical properties, as evidenced by the steady growth of scientific literature on the subject.

In fact [18], twistor spaces have demonstrated significant effectiveness in the study of complex structures, harmonic maps, minimal surfaces, special metrics, and, more broadly, any topic related to conformal geometry or harmonic analysis (see again [13] and the references therein).

From a mathematical point of view, given an even-dimensional Riemannian manifold (M, g), its twistor space $\mathcal{Z}$ is a bundle parameterizing almost complex structures defined on M and compatible with g, i.e., "orthogonal almost complex structures." The results achieved in this area are so enlightening and the underlying ideas so intuitive that numerous generalizations have emerged, even extending into odd-dimensional geometry (such as CR-twistor spaces [25], Sasakian twistor spaces [17], hypercomplex twistor spaces [24], and others).

Notably, the most striking results are often encountered when the focus is on studying four-dimensional manifolds (see e.g. [26]). In the four-dimensional case [14], the twistor space $\mathcal{Z}$ of a Riemannian manifold (M, g) is a complex manifold if and only if M is anti-self-dual, meaning that the anti-self-dual part of its Weyl tensor is identically zero. Moreover, a complex 3-manifold $\mathcal{Z}$ is the twistor space of some four-dimensional Riemannian manifold if and only if $\mathcal{Z}$ admits a foliation by rational curves (called "twistor lines" or "twistor fibers") with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ that are invariant under an anti-holomorphic involution $j : \mathcal{Z} \to \mathcal{Z}$.

Despite this rich theoretical framework, the only two three-dimensional algebraic twistor spaces are $\mathbb{CP}^3$, projecting onto the 4-sphere $(\mathbb{S}^4, g_{\text{rnd}})$ identified with the left quaternionic projective line $\mathbb{HP}^1$, and the flag threefold $\mathbb{F}$ projecting onto $(\mathbb{CP}^2, g_{\text{FS}})$ [23], where

$$\mathbb{F} := \{(p, \ell) = ([p_0 : p_1 : p_2], [\ell_0 : \ell_1 : \ell_2]) \in \mathbb{CP}^2 \times \mathbb{CP}^2 \mid p_0\ell_0 + p_1\ell_1 + p_2\ell_2 = 0\}.$$

While significant progress has been made in studying these spaces (see, for instance, [5, 30]), a lack of dedicated studies on the projective aspects of the twistor fibration remains. This gap in the literature was the motivation for our first paper [2].

In this survey, we focus on the contributions by Edoardo Ballico and collaborators on the interplay between the projective and twistor geometry of both $\mathbb{CP}^3$ and the flag threefold. Their works address key questions about the behavior of twistor fibers in algebraic surfaces, the invariance of geometric structures under conformal automorphisms, and the classification of surfaces containing twistor lines.

Furthermore, an important theme of this survey is the comparison between the two primary three-dimensional twistor spaces: $\mathbb{CP}^3$, associated with the 4-sphere, and the flag threefold, associated with the complex projective plane. Both spaces exhibit unique properties and parallels that make their study particularly illuminating. This comparative perspective will serve as a unifying thread throughout the sections that follow. Most of the proofs of the results presented in this article are only hinted at. However, the precise bibliographical references are always given.

Outline of the paper

This survey is organized as follows:

- In Sect. 2, we provide the necessary preliminaries, including definitions of twistor spaces, the twistor projection, the anti-holomorphic map j, and the explicit construction of orthogonal complex structures (OCS) for $\mathbb{CP}^3$ and the flag threefold. We also discuss the equations of twistor fibers, their parameter spaces, conformal automorphisms, the twistor discriminant locus, and invariance results.
- Sect. 3 is devoted to the study of twistor fibers contained in algebraic surfaces. We discuss results on surfaces containing infinitely many twistor fibers and provide bounds for non-j-invariant surfaces. Key findings from [4, 7] are compared.
- Sect. 4 compares the results from [2, 5], highlighting differences in methodologies and their implications for twistor geometry. In the last part of this section, we analyze surfaces of bidegree $(1, 2)$ in the flag threefold, as discussed in [6], and compare these findings with known studies of cubic surfaces in $\mathbb{CP}^3$ [10, 11].
- Sect. 5 explores the behavior of quadric surfaces in $\mathbb{CP}^3$ under the twistor projection, based on [19, 30]. These results are compared with those for bidegree $(1, 1)$ surfaces in the flag threefold from [7]. This section also includes original discussions on singular surfaces of bidegree $(1, 1)$ and $(0, 2)$ and their twistor discriminant locus.

2 Preliminaries

This section introduces the foundational concepts and tools. Recall that an algebraic surface is said to be integral if it is reduced and irreducible.

2.1 Twistor spaces and twistor projections

Twistor spaces are key objects in twistor theory, providing a geometric link between conformal geometry and complex geometry. For a Riemannian manifold (M, g), the twistor space $\mathcal{Z}$ is a bundle over M whose fibers parameterize almost complex structures on M that are orthogonal and compatible with g. The bundle map is denoted by $\pi : \mathcal{Z} \to M$ and is usually referred as *twistor projection*.

In four dimensions, the twistor space $\mathcal{Z}$ is a complex manifold if and only if (M, g) is anti-self-dual, meaning that the anti-self-dual part of the Weyl curvature tensor vanishes [14]. In such cases, $\mathcal{Z}$ is equipped with a natural complex structure, and its geometry is closely tied to that of the base manifold. A key feature of twistor spaces is their foliation by rational curves, known as *twistor fibers*, which have a normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ and are invariant under an anti-holomorphic involution $j : \mathcal{Z} \to \mathcal{Z}$.

In this setting the only algebraic cases are:

- $\mathbb{CP}^3$: The twistor space of the 4-sphere $(\mathbb{S}^4, g_{\mathrm{rnd}})$. The twistor projection, denoted $\pi_{\mathbb{CP}^3} : \mathbb{CP}^3 \to \mathbb{HP}^1 \simeq \mathbb{S}^4$, is explicitly defined as:

$$\pi_{\mathbb{CP}^3}([z_0, z_1, z_2, z_3]) = [z_0 + z_1 j, z_2 + z_3 j],$$

where $[z_0, z_1, z_2, z_3]$ are homogeneous coordinates in $\mathbb{CP}^3$ and j is a quaternionic unit [2].

- The Flag Threefold $\mathbb{F}$: The twistor space of $(\mathbb{CP}^2, g_{FS})$, where g_{FS} is the Fubini–Study metric. The Segre embedding maps the product space $\mathbb{CP}^2 \times \mathbb{CP}^2$ into $\mathbb{CP}^8$:

$$\text{Seg}: \ \mathbb{CP}^2 \times \mathbb{CP}^2 \to \mathbb{CP}^8, \quad ([p_0 : p_1 : p_2], [\ell_0 : \ell_1 : \ell_2]) \\ \mapsto [p_0\ell_0 : p_0\ell_1 : p_0\ell_2 : p_1\ell_0 : \cdots : p_2\ell_2].$$

The flag manifold $\mathbb{F}$ is then realized as a linear section of $\mathbb{CP}^8$ via the incidence relation $p_0\ell_0+p_1\ell_1+p_2\ell_2 = 0$, making $\mathbb{F}$ a hypersurface (a linear section, in fact) of $\mathbb{CP}^2 \times \mathbb{CP}^2$. The Picard group of $\mathbb{F}$, $\text{Pic}(\mathbb{F})$, is isomorphic to $\mathbb{Z}^2$ and is generated by H_1, the pullback of the hyperplane class from the first $\mathbb{CP}^2$, and by H_2, the pullback of the hyperplane class from the second $\mathbb{CP}^2$. Thus, the Picard group is isomorphic to $\mathbb{Z}^2$ and generated by H_1 and H_2:

$$\text{Pic}(\mathbb{F}) = \langle H_1, H_2 \rangle \sim \mathbb{Z}^2.$$

Let C be an integral curve in $\mathbb{F}$; we define the bidegree of C as the pair of positive integers (d_1, d_2), where

$$d_i = 0 \quad \text{if} \quad \pi_i(C) = \{x\}, \quad \text{otherwise} \quad d_i = \deg(\pi_i(C)) \deg(\pi_{i|C}).$$

If a curve D has irreducible components $C_1, \ldots, C_s$, then the bidegree of D is the sum of the bidegrees of $C_1, \ldots, C_s$. Let us pass now to surfaces. Let $d_1, d_2 \in \mathbb{N}$ be any pair of natural numbers. We define:

$$\mathcal{O}_{\mathbb{F}}(d_1, d_2) = \pi_1^* \mathcal{O}_{\mathbb{CP}^2}(d_1) \otimes \pi_2^* \mathcal{O}_{\mathbb{CP}^2}(d_2),$$

where $\pi_1 : \mathbb{F} \to \mathbb{CP}^2$ and $\pi_2 : \mathbb{F} \to \mathbb{CP}^2$ are the projections to the first and second factors, respectively. Let $|\mathcal{O}_{\mathbb{F}}(d_1, d_2)|$ denote the projective space $\mathbb{P}(H^0(\mathbb{F}, \mathcal{O}_{\mathbb{F}}(d_1, d_2)))$ and let $S \subset \mathbb{F}$ be an algebraic surface in $|\mathcal{O}(d_1, d_2)|$. We say that S has bidegree (d_1, d_2), and we write $\text{bdeg}(S) = (d_1, d_2)$.
The twistor projection, denoted $\pi_{\mathbb{F}} : \mathbb{F} \to \mathbb{CP}^2$, is defined as:

$$\pi_{\mathbb{F}}(p, \ell) = p^* \times \ell,$$

where $(p, \ell) \in \mathbb{F}$, $p = [p_0 : p_1 : p_2]$ is a point in $\mathbb{CP}^2$, $\ell = [\ell_0 : \ell_1 : \ell_2]$ is a line in $\mathbb{CP}^2$, and p^* is the dual of p. So that $\pi_{\mathbb{F}}(p, \ell)$ represent the point at infinity on ℓ with respect to p. [20]
The explicit cross-product is given by [7]:

$$\pi_{\mathbb{F}}(p, \ell) = [\overline{p}_1\ell_2 - \overline{p}_2\ell_1 : \overline{p}_2\ell_0 - \overline{p}_0\ell_2 : \overline{p}_0\ell_1 - \overline{p}_1\ell_0].$$

The anti-holomorphic involution j is a central concept in the study of twistor spaces, providing the real structure that ties the complex geometry of the twistor space $\mathcal{Z}$ to the conformal geometry of its base manifold M. It is a map $j : \mathcal{Z} \to \mathcal{Z}$ that satisfies the following key properties:

- j is anti-holomorphic;
- j is an involution, satisfying $j^2 = \text{id}$;
- j interacts naturally with the twistor projection $\pi : \mathcal{Z} \to M$, preserving the correspondence between points in $\mathcal{Z}$ and the conformal geometry of M.

The specific form of j depends on the choice of twistor space and the underlying base manifold. Below, we define j explicitly for $\mathbb{CP}^3$ and $\mathbb{F}$.

For $\mathbb{CP}^3$ the anti-holomorphic involution $j_{\mathbb{CP}^3}$ is explicitly defined as:

$$j_{\mathbb{CP}^3}([z_0, z_1, z_2, z_3]) = [-\bar{z}_1, \bar{z}_0, -\bar{z}_3, \bar{z}_2].$$

The anti-holomorphic involution $j_{\mathbb{F}}$ for the flag manifold $\mathbb{F}$, is given by:

$$j_{\mathbb{F}}(p, \ell) = (\bar{\ell}, \bar{p}),$$

We now present the explicit construction of OCS for $\mathbb{CP}^3$ and the flag manifold $\mathbb{F}$.

For the twistor space $\mathbb{CP}^3$ of the 4-sphere $\mathbb{S}^4$, the OCS J at any point $[1, z_1, z_2, z_3] \in \mathbb{CP}^3$ is explicitly constructed as follows [30, Section 2]:

$$J = -\frac{1}{1+|z_1|^2}\begin{pmatrix} 0 & 1-|z_1|^2 & 2\mathrm{Im}(z_1) & 2\mathrm{Re}(z_1) \\ -1+|z_1|^2 & 0 & -2\mathrm{Re}(z_1) & -2\mathrm{Im}(z_1) \\ -2\mathrm{Im}(z_1) & 2\mathrm{Re}(z_1) & 0 & 1-|z_1|^2 \\ -2\mathrm{Re}(z_1) & -2\mathrm{Im}(z_1) & -1+|z_1|^2 & 0 \end{pmatrix}.$$

In [3], Proposition 1.10 establishes a fundamental link between the algebraicity of the entries of J and its embedding into an algebraic surface. Specifically, if the entries of the matrix J are algebraic functions defined on a Zariski-open subset of $\mathbb{C}^2 \subset \mathbb{HP}^1$, then J lies on an algebraic hypersurface in the twistor space. The proof relies on the observation that the entries of J satisfy some polynomial equations. By showing that these polynomials define an algebraic locus in the twistor space, it is possible to prove that the graph of J corresponds to an algebraic surface. The Zariski-openness ensures that the hypersurface is well-defined globally outside a lower-dimensional exceptional set.

For the twistor space $\mathbb{F}$ of $\mathbb{CP}^2$, the OCS J' at $q \in \mathbb{CP}^2$ is defined on the base manifold $\mathbb{CP}^2$ as follows:

$$T_q\mathbb{CP}^2 \cong L \oplus L^{\perp},$$

where L is tangent to a line ℓ passing through a point q, and $L^{\perp}$ is orthogonal to L with respect to the Fubini–Study metric.

The almost complex structure J' acts on this decomposition by:

$$J' = \begin{cases} J & \text{on } L, \\ -J & \text{on } L^{\perp}, \end{cases}$$

where J is the restriction of standard complex structure of $\mathbb{CP}^2$.

We now pass to discuss twistor fibers. On the affine subset $\{[p, q] \in \mathbb{HP}^1 \mid p \neq 0\}$, where $q = q_1 + q_2 j$, the fibers of the projection $\pi : \mathbb{HP}^1 \to \mathbb{CP}^3$ are explicitly given by:

$$\begin{cases} z_2 = z_0 q_1 - z_1 \bar{q_2}, \\ z_3 = z_0 q_2 + z_1 \bar{q_1}. \end{cases}$$

The parameter space of lines in $\mathbb{CP}^3$ is given by the Grassmannian $\mathrm{Gr}(2, 4)$ identified with the Klein quadric $K = \{t_1 t_6 - t_2 t_3 + t_4 t_5 = 0\} \subset \mathbb{CP}^5$. The map $j_{\mathbb{CP}^3}$ induces a map on $\mathbb{CP}^5$ (also denoted by $j_{\mathbb{CP}^3}$) defined as:

$$j_{\mathbb{CP}^3}(t_1 : t_2 : t_3 : t_4 : t_5 : t_6) = (\bar{t}_1 : \bar{t}_5 : -\bar{t}_4 : -\bar{t}_3 : \bar{t}_2 : \bar{t}_6),$$

which is an anti-holomorphic involution. Then, twistor lines in $\mathrm{Gr}(2, 4)$, correspond to fixed points under the action of $j_{\mathbb{CP}^3}$.

Passing now to the other case of our study, given $q \in \mathbb{CP}^2$, the fiber $\pi_{\mathbb{F}}^{-1}(q)$ is given by:

$$\pi_{\mathbb{F}}^{-1}(q) = \{(p, \ell) \in \mathbb{F} \mid p^* \times \ell = q\} = \{(p, \ell) \in \mathbb{F} \mid q\ell = 0,\ pq^* = 0\}.$$

This defines a curve in the flag manifold $\mathbb{F}$ that has bidegree (1, 1).

The parameter space for these bidegree (1, 1) curves is the space $\mathbb{CP}^2 \times \mathbb{CP}^2 \setminus \mathbb{F}$ (points in $\mathbb{F}$ correspond to reducible curves). In this context, twistor fibers are the fixed points under the map $j_{\mathbb{F}}$ extended canonically to $\mathbb{CP}^2 \times \mathbb{CP}^2$.

The set of twistor fibers is a Zariski dense subset of the related parameter space, whether it is the space of lines in $\mathbb{CP}^3$ or of bidegree (1, 1) curves in $\mathbb{F}$.

A general surface in $\mathbb{F}$ of bidegree (d_1, d_2) is expected to intersect a generic twistor fiber in exactly $d_1 + d_2$ points. Thus, given the intersection properties described above, it makes sense to define the twistor degree of a surface S in $\mathbb{F}$ as the sum of its bidegree components:

$$d = d_1 + d_2.$$

2.2 Conformal automorphisms; twistor discriminant locus and related invariances

A conformal automorphism of the twistor space $\mathcal{Z}$ is the lift of a conformal transformation of the base space M. More precisely, it is an automorphism of the twistor space that commutes with the map j. This commutation property is justified by the fact that the twistor fibers are fixed points under the action of j.

Let us now explicitly describe conformal automorphisms for $\mathbb{CP}^3$. Consider $[1, q = q_1 + q_2 j] \in \mathbb{HP}^1$; the projective transformation associated with the Möbius transformation $(qc + d)^{-1}(qa + b)$ can be written in matrix form as follows:

$$\begin{pmatrix} a_1 & -\bar{a}_2 & b_1 & -\bar{b}_2 \\ a_2 & \bar{a}_1 & b_2 & \bar{b}_1 \\ c_1 & -\bar{c}_2 & d_1 & -\bar{d}_2 \\ c_2 & \bar{c}_1 & d_2 & \bar{d}_1 \end{pmatrix}.$$

Thus, a projective transformation arises from a conformal transformation of $\mathbb{S}^4$ if and only if it exhibits the complex conjugation symmetries shown in the matrix above (see, e.g. [9, 30]).

The automorphisms of $\mathbb{F}$ are generated by matrices $B \in \mathrm{SL}(3, \mathbb{C})$ that act on the flag manifold as:

$$B \cdot (p, \ell) = (pB^{-1}, B\ell),$$

along with the involution $\kappa : (p, \ell) \mapsto (\ell, p)$. This group of automorphisms is isomorphic to the projective general linear group $\mathrm{PGL}(3, \mathbb{C}) = \mathrm{SL}(3, \mathbb{C})/\mathbb{Z}_3$, where the central element $e^{2\pi i/3} I$ acts trivially.

The role of conformal automorphisms of the twistor space is taken by the special unitary group $\mathrm{SU}(3)$ acting on $\mathbb{F}$ as $\mathrm{SL}(3, \mathbb{C})$. For practical calculations, it is more convenient to allow $B \in \mathrm{U}(3)$, as the center of $\mathrm{U}(3)$ acts trivially on the flag manifold. In this context, we can further restrict to the maximal torus $T \subset \mathrm{SU}(3)$, which consists of diagonal matrices.

Many conformal invariances arise from the so-called twistor discriminant locus

Definition 1 Let $X \subset \mathcal{Z}$ be an algebraic surface of (twistor) degree d. The twistor discriminant locus of X, is defined as follows

$$\mathrm{Disc}(X) := \{q \in M \mid |\pi^{-1}(q) \cap X| \neq d\}.$$

In other words, the twistor discriminant locus consists of those points $q \in M$ where the intersection between the surface X and the twistor fiber $\pi^{-1}(q)$ does not contain exactly d points. Thus, the restriction of the twistor projection π to $X \setminus \pi^{-1}(\mathrm{Disc}(X))$ defines an unramified degree d covering map over $M \setminus \mathrm{Disc}(X)$. In particular the number of twistor fibers contained in X is invariant under conformal transformations.

For $\mathbb{CP}^3$, some general results about the twistor discriminant locus are already known. In particular, as shown in [3, Corollary 3.4], the intersections of a general algebraic surface with the family of twistor lines can only result in the following configurations: d distinct points; one double point and $d-2$ distinct points; two distinct double points and $d-4$ distinct points; or one triple point and $d-3$ distinct points.

Theorem 1 [3, Theorem 2.3] *Let $X \subset \mathbb{CP}^3$ be an integral algebraic surface of degree d. Then, one of the following mutually exclusive statements hold*:

1. *$Disc(X)$ is finite if and only if $|Disc(X)| = 1$, if and only if $d = 1$;*
2. $\dim(Disc(X)) = 1$ *if and only if X is conformally equivalent to the Segre quadric Q and in this case $Disc(X)$ is conformally equivalent to $S^1 \subset S^4$;*
3. *In all other cases $Disc(X)$ is a (possibly singular) real algebraic compact set of pure dimension 2.*

3 Surfaces containing infinitely many twistor fibers

In this section, we explore some first relationship between twistor fibers and algebraic surfaces. We will discuss general results on surfaces that contain infinitely many twistor fibers and provide a first simple bound for non-j-invariant surfaces. In particular, we focus on the degree constraints for these surfaces in $\mathbb{CP}^3$ and the flag threefold $\mathbb{F}$, comparing the corresponding constructions for both spaces. The key findings from [4, 7] will be compared throughout (see also [16]).

First of all, surfaces containing infinitely many twistor fibers must be j-invariant. In fact, if an integral surface X is not j-invariant, then $X \cap j(X)$ is a finite union of (possibly non-reduced) curves. For $\mathbb{CP}^3$, if $\deg(X) = d$, then, thanks to Bézout, this number is at most d^2. For $\mathbb{F}$, we recall the multiplication rules in the Chow ring: by [7, Proposition 3.11], we have that

$$\begin{aligned} \mathcal{O}_F(1,0)\cdot\mathcal{O}_F(1,0)\cdot\mathcal{O}_F(1,0) &= 0 \\ \mathcal{O}_F(1,0)\cdot\mathcal{O}_F(0,1)\cdot\mathcal{O}_F(1,0) &= 1 \\ \mathcal{O}_F(0,1)\cdot\mathcal{O}_F(1,0)\cdot\mathcal{O}_F(0,1) &= 1 \\ \mathcal{O}_F(0,1)\cdot\mathcal{O}_F(0,1)\cdot\mathcal{O}_F(0,1) &= 0 \end{aligned} \tag{1}$$

hence, the maximum number of bidegree $(1, 1)$ curves contained in a surface X of bidegree (a, b) is at most $a^2 + ab + b^2$ which lower to $a^2 + ab + b^2 - 1$, if one considers only twistor fibers (see [7, Section 8]). Moreover, due to the peculiar nature of $j_{\mathbb{F}}$, it is clear that if the surface $X \subset \mathbb{F}$ is j-invariant, then it has bidegree of type (d, d). This fact has a sort of interpretation in $\mathbb{CP}^3$. In fact, in [4] it is proved the following result

Proposition 2 [4, Theorem 1.2] *Let $X \subset \mathbb{CP}^3$ be an integral surface containing infinitely many twistor lines. Then* $\deg(X)$ *is even.*

Sketch of the proof Here we will present a sketch of the proof in the case in which X is rational. The general case exploit a detailed analysis of the normalisation of X.

Assume $d = \deg(X)$, and let $u : Y \to X$ be the normalization map, where $Y = \mathbb{P}(E)$ for some rank-2 vector bundle E over a smooth curve C. By [4, Theorem 3.1], E must be semistable. Since X is rational, the curve C must be $\mathbb{CP}^1$ and hence has genus zero. But for genus zero, semistability implies Y is the Hirzebruch surface $\mathbb{F}_0$. Therefore, the degree $d = \deg(X)$ must be divisible by 2 because the Hirzebruch surface $\mathbb{F}_0$ corresponds to even-degree surfaces in $\mathbb{CP}^3$. Therefore, the bundle E decomposes as $\mathcal{O}(d/2) \oplus \mathcal{O}(d/2)$, ensuring that d is even. □

About explicit examples, in [4, Proposition 1.3] are given two methods for constructing integral rational ruled surfaces of even degree that contain infinitely many twistor lines. The first construction exploits the theory of quaternionic slice regularity [1, 8, 22]. A quaternionic slice regular function f defined on a subset $\Omega \subset \mathbb{H}$ of the quaternions $\mathbb{H}$ can be lifted to a holomorphic function $\tilde{f} : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^3$; here $\mathbb{CP}^1 \times \mathbb{CP}^1$ represent the lift of the *slice* complex structure on $\mathbb{R}^4$ minus a line [21].

The explicit map described for constructing these surfaces involves complex components $g, h, \hat{g}, \hat{h}$ of f as follows:

$$\tilde{f} : \mathbb{CP}^1 \times \mathbb{CP}^1 \to X, \quad ([s, u], [1, v]) \mapsto [s, u, sg(v) - u\hat{h}(v), sh(v) + u\hat{g}(v)].$$

This method produces ruled surfaces parameterized by the twistor lift of a slice regular function, thus containing infinitely many twistor fibers.

The second method is shared for both twistor spaces under examination.

Theorem 3 ([4, Theorem 1.4] and [5, Theorem 5.4]) *Let C be a smooth, connected complex projective curve defined over $\mathbb{R}$ with $C(\mathbb{R}) \neq \emptyset$. Then,*

1. *For any integer d_0, there exists an integer $d \geq d_0$ and a degree d integral surface $X \subset \mathbb{CP}^3$ such that: X contains infinitely many twistor lines and the normalization of X is a $\mathbb{CP}^1$-bundle over C.*
2. *For any integer $a \geq g + 2$, where g is the genus of C, there exists an integral surface $X \in |\mathcal{O}_F(a, a)|$ such that X contains infinitely many twistor fibers and X is birational to $C \times \mathbb{CP}^1$.*

We provide a sketch of the proof. For the details, see [4, 5].

Sketch of the proof The key idea behind both cases is that the smooth curve C, defined over $\mathbb{R}$, serves as a kind of "skeleton" around which we build the surfaces. The fact that C has real points is crucial because it allows us to construct and parameterize the infinite families of twistor fibers in a controlled and geometric way. The constructions in $\mathbb{CP}^3$ and the flag manifold $\mathbb{F}$ are different in detail but very similar in spirit.

For the $\mathbb{CP}^3$ case, we start with C. The normalization of C defines a $\mathbb{CP}^1$-bundle over C, that essentially represents a ruled surface. By carefully embedding this ruled surface into $\mathbb{CP}^3$, we can ensure that X contains infinitely many twistor lines. The degree d of X can be adjusted to meet the given lower bound d_0, and the existence of infinitely many twistor lines comes from the fact that the real points of C give a dense set of parameters for these lines.

In the case of the flag manifold $\mathbb{F}$, the process is similar. The strategy starts by embedding C into $\mathbb{CP}^2 \times \mathbb{CP}^2$ using a very ample line bundle L. The choice of a real subspace of global sections of L allows us to map C into $\mathbb{F}$, where its image T becomes the locus parameterizing twistor fibers. By taking the union of these twistor fibers, we construct a surface X of bidegree (a, a) with $a \geq g + 2$. The surface X is birational equivalent to $C \times \mathbb{CP}^1$, ensuring a ruled structure that inherently accommodates infinitely many twistor fibers. □

4 Surfaces containing finitely many twistor fibers

The first nontrivial bound on the number of twistor fibers on a smooth surface can be given in terms of the Miyaoka Theorem [27].

4.1 Miyaoka-type bounds

The following classical bound was proved by Miyaoka as a consequence of [27, Proposition 2.1.1] stating that if $\tilde{X}$ is a minimal surface of non-negative Kodaira dimension, then the number k of disjoint smooth rational curves contained in $\tilde{X}$ is such that

$$k \leq \frac{2}{3}c_2(\tilde{X}) - \frac{1}{3}c_1^2(\tilde{X}),$$

where $c_1(\tilde{X})$ and $c_2(\tilde{X})$ are the first and second Chern classes of the surface $\tilde{X}$. Therefore, we have

Proposition 4 *Let $X \subset \mathbb{CP}^3$ be a smooth degree $d \geq 4$ projective surface. Then, the number k of disjoint lines contained in X is such that $k \leq 2d(d-2)$.*

Clearly, since in $\mathbb{CP}^3$ twistor fibers are (skew) lines, the same bound apply to them.

Sketch of the proof The proof is a consequence of [27, Proposition 2.1.1] where

$$c_1^2 = d(d-4)^2, \quad c_2 = d(d^2 - 4d + 6), \quad c_2 - \frac{1}{3}c_1^2 = \frac{2}{3}d(d-1)^2,$$

where $c_i := c_i(X)$. Then, if $L_1, \ldots, L_k$ are disjoint lines on X. They have self-intersection number $-(d-2)$. Hence [27, Formula (6)] implies that

$$\frac{(d-1)^2}{3(d-2)} \leq k \leq \frac{2}{3}d(d-1)^2, \quad \text{that is,} \ \ k \leq 2d(d-2).$$

□

The same idea applies to $\mathbb{F}$. In fact, it holds:

Theorem 5 [5, Theorem 3.1] *Let $S \subset \mathbb{F}$ be a smooth surface of bidegree (a, b). The maximum number of disjoint twistor fibers contained in S satisfies*:

$$k \leq \frac{2(a+b-2)(3a^2b - a^2 + 3ab^2 - 4ab + 3a - b^2 + 3b)}{(a+b-1)^2}.$$

Clearly, the proof [5] is just a matter of computing Chern Classes, that, thanks to the multiplication tables in Formula (1) are given by

$$c_1(S) = 2(a+b-2), \quad c_2(S) = 6(a+b) + 3a^2b - 2a^2 + 3ab^2 - 8ab - 2b^2.$$

4.2 Surfaces with prescribed number of twistor fibers

We now pass to more articulated results contained in [2, 5]. Many of the results contained in these two papers are quite technical and hence, in this survey, we will only recall the main results and techniques.

Both papers establish bounds on the maximum number of twistor fibers that can exist on certain types of surfaces. Both works employ classical tools of algebraic geometry, including intersection theory, cohomology, and the analysis of ideal sheaves. Both paper discuss the smoothness of surfaces containing a certain amount of twistor fibers.

We begin the discussion for $\mathbb{CP}^3$. Let us start with some notation.

Notation 1 For any closed subscheme $E \subseteq \mathbb{CP}^3$, the *ideal sheaf* of E, denoted by $\mathcal{I}_E \subseteq \mathcal{O}_{\mathbb{CP}^3}$, is defined as the kernel of the natural morphism $\mathcal{O}_{\mathbb{CP}^3} \to i_*\mathcal{O}_E$, where i is the inclusion map of E into $\mathbb{CP}^3$.

When $E \subset E'$, the ideal sheaf of E in E' is written as $\mathcal{I}_{E,E'} \subseteq \mathcal{O}_{E'}$. For a closed subscheme $E \subseteq \mathbb{CP}^3$, the *residual scheme* of E with respect to E', denoted by $\mathrm{Res}_{E'}(E)$, is the closed subscheme of $\mathbb{CP}^3$ whose ideal sheaf is given by the conductor $\mathcal{I}_E : \mathcal{I}_{E'}$. It follows that $\mathrm{Res}_{E'}(E) \subseteq E$, and if E is a reduced algebraic set, $\mathrm{Res}_{E'}(E)$ represents the closure in $\mathbb{CP}^3$ of the difference $E \backslash (E \cap E')$, which corresponds to the union of irreducible components of E that are not contained in E'. Assuming $\deg(E') = d$, there is a natural exact sequence of coherent sheaves on $\mathbb{CP}^3$:

$$0 \to \mathcal{I}_{\mathrm{Res}_{E'}(E)}(t-d) \to \mathcal{I}_E(t) \to \mathcal{I}_{E\cap E',E'}(t) \to 0. \tag{2}$$

For any subscheme $E \subseteq E'$ and any curve $C \subset E'$, the *residual scheme* of E with respect to C is denoted by $\mathrm{Res}_C(E)$ and defined via the conductor $\mathcal{I}_{E,E'} : \mathcal{I}_{C,E'}$. If E' is smooth, C is an effective Cartier divisor of E', and the following residual exact sequence of coherent sheaves on E' holds:

$$0 \to \mathcal{I}_{\mathrm{Res}_C(E)}(t)(-C) \to \mathcal{I}_{E,E'}(t) \to \mathcal{I}_{E\cap C,C}(t) \to 0. \tag{3}$$

If E is the disjoint union of two closed subschemes of $\mathbb{CP}^3$, say $E = E_1 \sqcup E_2$ with E_1, E_2 closed in $\mathbb{CP}^3$, then the residual scheme satisfies $\mathrm{Res}_{E'}(E) = \mathrm{Res}_{E'}(E_1) \cup \mathrm{Res}_{E'}(E_2)$, and $E' \cap E = (E' \cap E_1) \cup (E' \cap E_2)$. Moreover, since $\mathrm{Res}_{E'}(E_i) \subseteq E_i$, the intersection $\mathrm{Res}_{E'}(E_1) \cap \mathrm{Res}_{E'}(E_2)$ is empty.

To discuss singularities, we recall the concept of the *first infinitesimal neighborhood of a point* (often referred to as a "fat point"). For any $q \in \mathbb{CP}^3$, let $2q$ denote the first infinitesimal neighborhood of q, which is a closed subscheme of $\mathbb{CP}^3$ whose ideal sheaf is $\mathcal{I}^2_{2q}$. The scheme $2q$ is zero-dimensional, with reduced scheme $(2q)_{\mathrm{red}} = \{q\}$. If q lies in an affine or projective n-dimensional space, then $\deg(2q) = \binom{n+2}{2}$, and for $n = 3, 2, 1$, we have $\deg(2q) = 4, 3, 2$, respectively. If $q \notin E'$, it follows that $E \cap E' = \emptyset$ and $\mathrm{Res}(2q) = 2q$. On the other hand, if $q \in E'$, let $(2q, E')$ denote the closed subscheme of E' with $\mathcal{I}^2_{(2q,E')}$ as its ideal sheaf. Then $(2q, E') = 2q \cap E'$ (in the scheme-theoretic sense), and the scheme $(2q, E')$ is zero-dimensional with $(2q, E')_{\mathrm{red}} = \{q\}$. If E' is smooth at q, we have $\deg((2q, E')) = 3$ and $\mathrm{Res}_{E'}(2q) = \{q\}$.

Let us now pass to the main results in [2]. Define the numbers $\nu(d)$, $\nu_n(d)$, $\nu_s(d)$, and $\nu_j(d)$ as follows:

$$\nu(d) := \left\lfloor \frac{\binom{d+3}{3} - 1}{d+1} \right\rfloor, \quad \nu_n(d) = \begin{cases} \nu(d-1) & \text{if } d \geq 2, \\ 0 & \text{if } d \leq 1 \end{cases} \quad \nu_s(d) = \begin{cases} \nu(d-3) & \text{if } d \geq 4, \\ 0 & \text{if } d \leq 3. \end{cases}$$

and

$$\nu_j(d) = \begin{cases} \nu_n(d-8) = \nu(d-9) & \text{if } d \geq 9, \\ 0 & \text{if } d \leq 8. \end{cases}$$

The integer $\nu(d)$ represents the maximum number of disjoint lines in $\mathbb{CP}^3$, in general position, such that the space of degree d hypersurfaces containing these lines, denoted as $|\mathcal{I}_E(d)|$, has positive dimension. Here, E is the set of those lines. The other two integers (ν_n and ν_s) are smaller versions of the original one, chosen to ensure that the dimension of the space of surfaces with some specific constraint (normality or smoothness) containing that specified number of lines is positive. The subtle details of these ideas are explained in details in [2], here we will only describe the general strategies of the proofs.

Theorem 6 [2, Theorem 1.4] *For $d \geq 4$,*

1. *There exists an irreducible degree d surface $X \subset \mathbb{CP}^3$ containing $\nu(d)$ general twistor lines.*
2. *There exists an irreducible degree d surface $X \subset \mathbb{CP}^3$ containing $\nu_n(d)$ twistor lines, none intersecting the singular locus of X, where* $\mathrm{Sing}(X)$ *is finite.*
3. *There exists a smooth degree d surface $Y \subset \mathbb{CP}^3$ containing $\nu_s(d)$ general twistor lines.*

Sketch of the proof 1. First of all, use the Density Lemmas for twistor fibers [2, Lemma 3.2] to select a general configuration of $\nu(d)$ twistor fibers: a key property is that twistor fibers are skew so they avoid pathological arrangements. Then prove that $h^1(\mathcal{I}_E(d)) = 0$, where E is the set of twistor lines. This ensures that the space of surfaces containing E has the expected dimension. This claim follows from the general result that for disjoint lines $E \subset \mathbb{CP}^3$, $h^1(\mathcal{I}_E(t)) = 0$ for $t \geq \#E - 1$ (see [2, Remark 1.2]). Then calculate the dimension of the space $|\mathcal{I}_E(d)|$ as $\binom{d+3}{3} - (d+1)\nu(d) - 1$. This ensures the parameter space is non-empty, showing surfaces containing these lines exist. To ensure smoothness, consider the short exact sequence involving "fat points" $2q$:

$$0 \to \mathcal{I}_{\mathrm{Res}(2q,E)}(d-1) \to \mathcal{I}_{2q\cup E}(d) \to \mathcal{I}_{2q,E}(d) \to 0.$$

The next step is to show that $h^1(\mathcal{I}_{2q\cup E}(d)) = 0$, ensuring that a general surface is smooth at q. Then apply Bertini's theorem to ensure that singular surfaces lie in a lower-dimensional subset of the parameter space.

2. Start with $\nu(d)$ lines and reduce to $\nu_n(d)$ by removing lines while maintaining generality. Consider the residual exact sequences:

$$0 \to \mathcal{I}_{\mathrm{Res}(2q\cup E)}(d-1) \to \mathcal{I}_{2q\cup E}(d) \to \mathcal{I}_{2q\cap E}(d) \to 0.$$

The idea here is to use this sequence to show that $h^1(\mathcal{I}_{2q\cup E}(d)) = 0$, ensuring that singularities are finite and occur only in lower-dimensional subsets of $|\mathcal{I}_E(d)|$. Then, the next step is to prove that $h^1(\mathcal{I}_E(d-1)) = 0$ ensuring the residual schemes $\mathrm{Res}(2q, E)$ behave well. Now, with fewer lines ($\nu_n(d)$), the dimension of $|\mathcal{I}_E(d)|$ increases, ensuring that surfaces are irreducible and accommodate $\nu_n(d)$ lines smoothly.

3. Start with $\nu_n(d)$ lines and further reduce to $\nu_s(d)$, ensuring generality at every step. Then use Bertini's Theorem to ensure general smoothness. Use the following residual exact sequences:

$$0 \to \mathcal{I}_{\mathrm{Res}(H)}(d-1) \to \mathcal{I}_E(d) \to \mathcal{I}_{E\cap H,H}(d) \to 0,$$

where H is a plane intersecting E to show $h^1(\mathcal{I}_E(d)) = 0$. This can be done by controlling contributions from $\mathcal{I}_{E\cap H,H}(d)$.

□

With some more effort, it is possible to prove the following existence result.

Theorem 7 [2, Theorem 1.5] *For $d \geq 10$ and $0 \leq k \leq \nu_j(d)$, there exists a smooth degree d surface $X \subset \mathbb{CP}^3$ containing k general twistor lines and no other line not intersecting any of these k lines.*

Sketch of the proof As in the previous proof, choose $k \leq \nu_j(d)$ general twistor lines such that they satisfy the conditions for generality. Here the number $\nu_j(d)$ is derived from cohomological computations of lower bounds.

Then, by analyzing the space $|\mathcal{I}_E(d)|$, it is possible to ensure that the parameter space of surfaces containing k lines has the expected dimension. After that, consider the following residual exact sequences: □

$$0 \to \mathcal{I}_{\mathrm{Res}(E \cup L)}(d-1) \to \mathcal{I}_{E \cup L}(d) \to \mathcal{I}_{E \cap L}(d) \to 0,$$

to prove that $h^1(\mathcal{I}_{E \cup L}(d)) = 0$ to ensure configurations with extra lines L lie in lower-dimensional subsets. Finally, use Bertini's Theorem to exclude singular configurations. □

The last result of this series prove the existence of smooth and j-invariant surfaces containing a prescribed number of twistor fibers.

Theorem 8 [2, Theorem 1.6] *Take d, $\nu_j(d)$ and $k \leq \nu_j(d)$ as in Theorem 7 with d even; if $d \equiv 2 \pmod 4$ assume k odd; if $d \equiv 0 \pmod 4$ assume k even. Then there exists a degree d smooth j-invariant surface $X \subset \mathbb{CP}^3$ such that X contains exactly k twistor lines.*

Sketch of the proof By [2, Corollary 3.5], we have $h^1(\mathcal{I}_E(d)) = 0$ and so $h^0(\mathcal{I}_E(d)) = \binom{d+3}{3} - k(d+1)$. The integer $\binom{d+3}{3} = \frac{(d+3)(d+2)(d+1)}{6}$ is even if and only if $d \equiv 0 \pmod 4$ or $d \equiv 2 \pmod 4$. The integer $d+1$ is odd if and only if d is even. Thus, $\binom{d+3}{3} - k(d+1)$ is odd, i.e., $\dim |\mathcal{I}_E(d)|$ is even, if and only if either $d \equiv 2 \pmod 4$ and k is odd, or $d \equiv 0 \pmod 4$ and k is even, i.e., in the cases needed to prove the result.

Fix any union $E \subset \mathbb{CP}^3$ of finitely many twistor fibers. For each $t \in \mathbb{Z}$, the projective space $|\mathcal{I}_E(t)|$ is j-invariant. Now we claim that, if $\dim |\mathcal{I}_E(t)|$ is even, then the set of all j-invariant elements of $|\mathcal{I}_E(t)|$ is Zariski dense in $|\mathcal{I}_E(t)|$ (for the proof of this claim see [2]). Then (by the proof of Theorem 7) there is a smooth $X \in \Gamma$ containing no line L with $L \cap E = \emptyset$. □

We now pass to recover the analogous results for the $\mathbb{F}$ contained in [5]. As for $\mathbb{CP}^3$ recall that if $T \subset \mathbb{F}$ is a union of disjoint curves of bidegree $(1, 1)$ contained in $\mathbb{F}$, the sheaf $\mathcal{I}_{T,\mathbb{F}}(a, b)$ is the ideal sheaf associated to T in $\mathbb{F}$, twisted by the line bundle $\mathcal{O}_{\mathbb{F}}(a, b)$. Explicitly:

$$\mathcal{I}_{T,\mathbb{F}}(a, b) = \mathcal{I}_T \otimes \mathcal{O}_{\mathbb{F}}(a, b),$$

where $\mathcal{I}_T$ is the sheaf of ideals vanishing on T. The space of global sections $H^0(\mathbb{F}, \mathcal{I}_{T,\mathbb{F}}(a, b))$ parametrizes sections of $\mathcal{O}_{\mathbb{F}}(a, b)$ that vanish on T.

First of all we give a strong rigidity result.

Proposition 9 [5, Proposition 4.1] *For positive integers a, b, let $T \subset \mathbb{F}$ be any union of $x \geq a^2 + ab + b^2$ pairwise disjoint smooth curves of bidegree $(1, 1)$. Then, there exists at most one integral surface of bidegree (a, b) that contains T.*

Sketch of the proof Let S, S' be integral surfaces of bidegree (a, b) containing T with $S \neq S'$. Since $S \cap S'$ is an ample Cartier divisor of degree a^2+ab+b^2, it must be connected. However, T consists of $x \geq a^2 + ab + b^2$ disjoint components, which contradicts the assumption that $S \neq S'$. Hence, S is unique. □

Then we have a useful vanishing theorem.

Theorem 10 [5, Theorem 4.4] *For integers* $b \geq a \geq 1$ *and* $0 \leq x \leq a(a-1)/2$, *let* $T \subset \mathbb{F}$ *be a general union of* x *smooth curves of bidegree* $(1, 1)$. *Then*:

$$h^1(\mathcal{I}_{T,\mathbb{F}}(a, b)) = 0,$$

where $h^1(\mathcal{I}_{T,\mathbb{F}}(a, b)) = dim H^1(\mathcal{I}_{T,\mathbb{F}}(a, b))$.

Sketch of the proof This is proved using induction on the bidegree and residual exact sequences. Let $x = a(a-1)/2$ be a general union T of x smooth curves of bidegree $(1, 1)$. Then, it is possible to prove by induction that the cohomology group $h^1(\mathcal{I}_{T,\mathbb{F}}(a, b))$ vanishes:

- For $a = 1$, as $x = 0$ then $h^1(\mathcal{I}_{T,\mathbb{F}}(a, a)) = 0$.
- For $a+1$, the argument involves restricting to general surfaces X and Y of bidegree $(1, 0)$ and $(0, 1)$, using the Hirzebruch surface structure and vanishing of higher cohomology.

This reduction helps establish the vanishing result for $h^1(\mathcal{I}_{T,F}(a, a))$, which is then generalized to $h^1(\mathcal{I}_{T,F}(a, b))$ through an additional induction on the difference $b - a$ (see [5] for further details). □

Then we have the main result of [5, Section 4] and then a Corollary.

Theorem 11 [5, Theorem 1.1] *Let* $T \subset \mathbb{F}$ *be a union of disjoint curves of bidegree* $(1, 1)$. *Then*, *a general surface* $S \in |\mathcal{I}_{T,\mathbb{F}}(a, b)|$ *does not contain any additional smooth curve of bidegree* $(1, 1)$ *disjoint from* T. *Moreover*, *a general surface* $S \in |\mathcal{I}_{T,\mathbb{F}}(a, b)|$ *is irreducible and smooth.*

Sketch of the proof For the first part, assume by contradiction that $S \in |\mathcal{I}_{T,\mathbb{F}}(a, b)|$ contains an additional smooth curve of bidegree $(1, 1)$ C disjoint from T. Such C would restrict the possible parameter space of S, but since C is not general, S would contradict the generality of T. For the second part, the vanishing of $h^1(\mathcal{I}_{T,\mathbb{F}}(a, b)) = 0$ (from Theorem 10) implies that $\mathcal{I}_{T,\mathbb{F}}(a, b)$ is globally generated. Using Bertini's theorem, a general surface is irreducible and smooth outside T. A further local analysis shows S is smooth at T. □

Corollary 12 [5, Corollary 4.6] *If* $T \subset \mathbb{F}$ *is a general union of* x *twistor fibers*, *then*:

1. *A general* $S \in |\mathcal{I}_{T,\mathbb{F}}(a, b)|$ *is irreducible*, *smooth*, *and contains exactly* x *twistor fibers.*
2. *There exists a smooth irreducible surface* S *of bidegree* (a, b) *containing exactly* x *twistor fibers.*
3. *Near* S, *the family of surfaces of bidegree* (a, b) *is a differentiable manifold of real dimension*

$$2\left(\binom{a+2}{2}\binom{b+2}{2} - \binom{a+1}{2}\binom{b+1}{2} - x(a+b+1)\right) + 4x.$$

4.3 Cubic surfaces in $\mathbb{CP}^3$ and surfaces of bidegree $(1, d)$ in $\mathbb{F}$

We now specialize our discussion to the family of surfaces of (twistor) degree 3. Given a non-singular cubic surface in $\mathbb{CP}^3$, this contains exactly 27 lines (Cayley–Salmon theorem). In [10] the authors prove that among these 27 lines, at most 5 can be twistor lines. The configuration of these 5 twistor lines depends on a set of 5 points on a 2-sphere in S^4, such that no 4 points lie on a circle. Any set of 5 points on a 2-sphere in S^4, with no 4 points on a circle, determines a one-parameter family of non-singular cubic surfaces with 5 twistor lines. Notice that cubics in this family are projectively equivalent but not conformally equivalent. Other results related to the twistor geometry of Cubics surfaces in $\mathbb{CP}^3$ can be found in [9, 11].

In [6] we studied similar problems for bidegree $(1, d)$ surfaces in $\mathbb{F}$. Let us start with some notation. Let $\mathcal{C}(n)$ be the set of n smooth pairwise disjoint curves of bidegree $(1, 1)$ and $\mathcal{C}^*(n)$ be the subset of $\mathcal{C}(n)$ consisting of n disjoint twistor fibers that are not collinear, i.e., no curve of bidegree $(1, 0)$ intersects more than two of the twistor fibers in the set. Then define $\mathcal{T}(n)$ and $\mathcal{T}^*(n)$ as the analogous sets of twistor fibers. It is useful to recall that surfaces of bidegree $(1, 0)$ and $(0, 1)$ contain exactly one twistor fiber (that, we recall, is a smooth curve of bidegree $(1, 1)$) and are Hirzebruch surfaces of type 1.

Lemma 13 [6, Lemma 2.12] *Fix $d \geq 0$, $n \geq 1$, and $A \in \mathcal{C}(n)$. Then we have*:

$$h^0(\mathcal{I}_A(0, d)) = \frac{(d-n+2)(d-n+1)}{2}$$

and

$$h^1(\mathcal{I}_A(0, d)) = \begin{cases} \frac{n(n-1)}{2} & \text{if } n \leq d+1, \\ n(d+1) - \frac{(d+2)(d+1)}{2} & \text{if } n \geq d+1. \end{cases}$$

Proof Recall that $\mathcal{O}_{\mathbb{F}}(0, d) = \pi_1^*(\mathcal{O}_{\mathbb{CP}^2}(d))$, and $\mathcal{I}_A(0, d) = \pi_1^*(\mathcal{I}_{T,\mathbb{CP}^2}(d))$, where $T = \pi_1(A)$ is a union of n distinct lines in $\mathbb{CP}^2$. In general, we have that:

$$h^0(\mathcal{O}_{\mathbb{F}}(0, d)) = \binom{d+2}{2}, \quad h^0(\mathcal{I}_A(0, d)) = \binom{d+2-n}{2}, \quad h^0(\mathcal{O}_A(0, d)) = n(d+1),$$

hence, using the exact sequence

$$0 \to \mathcal{I}_A(0, d) \to \mathcal{O}_{\mathbb{F}}(0, d) \to \mathcal{O}_A(0, d) \to 0,$$

and since $h^1(\mathcal{O}_{\mathbb{F}}(0, d)) = 0$ and $\binom{d+2-n}{2} = 0$ if $n \geq d+1$, we get the result. □

Let $A \in \mathcal{C}(n)$ and let C be any connected component of A. Set $B := A \setminus C$. Then, for any $a, b \geq 0$, if $Y \in |\mathcal{O}_C(0, 1)|$, we have the following residual exact sequence:

$$0 \to \mathcal{I}_{\mathrm{Res}_Y(A)}(a, b-1) \to \mathcal{I}_A(a, b) \to \mathcal{I}_{A\cap Y, Y}(a, b) \to 0,$$

but since $\mathrm{Res}_Y(A) = B$ and $A \cap Y = (B \cap Y) \cup C$, we have:

$$0 \to \mathcal{I}_B(a, b-1) \to \mathcal{I}_A(a, b) \to \mathcal{I}_{(B\cap Y)\cup C, Y}(a, b) \to 0. \tag{2}$$

The set $\mathcal{C}^*(n)$ can be characterized as follows [6].

Theorem 14 [6, Theorem 2.19] *Let $d \geq 0$ and $A \in \mathcal{C}(d+1)$. Then $A \in \mathcal{C}^*(d+1)$ if and only if $h^1(\mathcal{I}_A(1, d)) = 0$.*

Proof Suppose firstly that $A \notin \mathcal{C}^*(d+1)$, i.e., there exists a $(1, 0)$ curve L intersecting three or more components of A. Consider the exact sequence:

$$0 \to \mathcal{I}_A(1, d) \to \mathcal{O}_{\mathbb{F}}(1, d) \to \mathcal{O}_A(1, d) \to 0.$$

The restriction map $H^0(\mathcal{O}_{\mathbb{F}}(1, d)) \to H^0(\mathcal{O}_A(1, d))$ is not surjective because $\mathcal{O}_A(1, d)$ includes higher multiplicities along L. Thus, $h^1(\mathcal{I}_A(1, d)) > 0$.

Assume now that $A \in \mathcal{C}^*(d+1)$. We use induction on $d \geq 0$. The case $d = 0$ is true by Lemma 13. So we can assume $d > 0$. Let C be a connected component of A, set $B := A \backslash C$ and call Y the unique element of $|\mathcal{O}_C(0,1)|$. Consider the residual exact sequence in Formula (2), with $a = 1$ and $b = d$. Since $A \in \mathcal{C}(d+1)$, $C \cap B = \emptyset$ and $B \cap Y$ is a set of d different points, up to the identification of D with F_1 we have

$$\mathcal{I}_{(B\cap Y)\cup C, F_1}(1,d) \cong \mathcal{I}_{(B\cap Y)\cup F_1}(1,d)(h+(d+1)f) \cong \mathcal{I}_{B\cap Y, F_1}(df).$$

Using the exact sequence in Formula (2) and induction, we are left to prove that $h^1(\mathcal{I}_{B\cap Y, F_1}(df)) = 0$ if and only if $A \in \mathcal{C}^*(d+1)$.
Consider now the following exact sequence:

$$0 \to \mathcal{I}_{B\cap Y, F_1}(df) \to \mathcal{O}_{F_1}(df) \to \mathcal{O}_{B\cap Y}(df) \to 0. \tag{11}$$

Since $h^0(\mathcal{O}_{F_1}(df)) = d+1$ and $h^0(\mathcal{O}_{B\cap Y}(df)) = d$, we have the following inequality:

$$h^1(\mathcal{I}_{B\cap Y, F_1}(df)) > 0 \quad \text{if and only if} \quad h^0(\mathcal{I}_{B\cap Y, F_1}(df)) \geq 2.$$

The last inequality means that there are at least two different sets of d fibers containing the set of d points $B \cap Y$. This is equivalent to the fact that there exists a fiber $L \in |f|$ such that $\#(B \cap L) \geq 2$. Since L is a curve of bidegree $(1,0)$ in $\mathbb{F}$, then $L \cdot Y = 0$ in the intersection ring of $\mathbb{F}$, so $L \subset Y$, and we get $L \cap C \neq \emptyset$. Thus $\#(L \cap A) \geq 3$, which means that $A \notin \mathcal{C}^*(d+1)$. □

The latter result is crucial for the following theorems [6].

Theorem 15 [6, Theorem 1.1] *For any* $d \in \mathbb{N}$ *and* $A \in \mathcal{T}^*(d+2)$, *there is no irreducible surface of bidegree* $(1,d)$ *containing* A.

Sketch of the proof Assume there exists $S \in |\mathcal{O}_{\mathbb{F}}(1,d)|$ irreducible and containing $A \in \mathcal{T}^*(d+2)$. By Theorem 14, $A \notin \mathcal{C}^*(d+2)$, as collinearity would occur. Then use the residual exact sequence constructed as in the proof of Theorem 14 for $\mathcal{I}_A(1,d)$:

$$0 \to \mathcal{I}_{B\cap Y}(1,d-1) \to \mathcal{I}_A(1,d) \to \mathcal{I}_{A\cap Y,Y}(1,d) \to 0.$$

The last part of the proof is devoted to show that $h^1(\mathcal{I}_A(1,d)) > 0$, contradicting the existence of S. □

Besides this last result it is also possible to prove an existence result [6].

Theorem 16 [6, Theorem 1.2] *Fix* $d \geq 1$ *and* $0 \leq n \leq d+2$. *There exists an irreducible surface* $S \in |\mathcal{O}_{\mathbb{F}}(1,d)|$ *containing exactly n twistor fibers.*

With some more effort, these results can be specified to lower degree.

Theorem 17 [6, Theorems 1.3, 1.4 and 1.5] *There is no irreducible* $S \in |\mathcal{O}_{\mathbb{F}}(1,2)|$ *containing at least* 5 *twistor fibers and there is no irreducible* $S \in |\mathcal{O}_{\mathbb{F}}(1,3)|$ *containing at least* 6 *twistor fibers.*

Fix $0 \leq n \leq 3$. *There exists a smooth* $S \in |\mathcal{O}_{\mathbb{F}}(1,2)|$ *containing exactly n twistor fibers. Additionally, there exists a bidegree* $(1,2)$ *irreducible surface containing exactly* 4 *twistor fibers.*

By recalling that the twistor degree of a bidegree $(1,2)$ surface is equal to 3, this last theorem says that the situation between $\mathbb{CP}^3$ and $\mathbb{F}$ is quite different. In fact, as we said before there exist cubics in $\mathbb{CP}^3$ that contain 5 twistor fibers.

5 Surfaces of twistor degree 2

In this last section, which contains the only original material of this survey, we discuss results related to surfaces of twistor degree 2.

Surfaces of twistor degree 2 provide a compelling family of examples, though their geometry is already non-trivial. In $\mathbb{CP}^3$, their conformal classification is discussed in [19, 30]. Specifically, any smooth quadric hypersurface in $\mathbb{CP}^3$ is conformally equivalent to the zero set of

$$e^{\lambda+i\nu}X_0^2 + e^{\mu-i\nu}X_1^2 + e^{-\lambda+i\nu}X_2^2 + e^{-\mu-i\nu}X_3^2, \tag{3}$$

or

$$i(X_0^2 + X_2^2) - kX_0X_1 + kX_2X_3 - X_0X_3 + X_1X_2, \tag{4}$$

where (X_0, X_1, X_2, X_3) are homogeneous coordinates in $\mathbb{CP}^3$, and the parameters $\lambda, \mu, \nu \in \mathbb{R}$ satisfy $0 \leq \lambda \leq \mu$ and $\nu \in [0, \pi/2)$, with $k \in [0, 1)$.

The geometry of these smooth quadrics under twistor projection is classified by their *discriminant loci*.

Proposition 18 [30] *Let X be in canonical form* (3)*, with $0 \leq \lambda \leq \mu$, and $0 \leq \nu < \pi/2$.*

- *The corresponding quadric X contains exactly two twistor lines if and only if $\lambda = \mu \neq 0$ and $\nu = 0$. In this case, $Disc(X)$ is a real torus pinched at two points.*
- *The quadric X contains a family of twistor lines over a circle if and only if $\lambda = \mu = \nu = 0$.*
- *Otherwise, X contains no twistor lines, and $Disc(X)$ is a smooth unknotted torus.*

In the case of (4)*, with $k \in [0, 1)$, the corresponding quadric X contains exactly one twistor line, and $Disc(X)$ is a real torus pinched at one point.*

The classification of the singular cases was completed in [19]. For singular quadrics, we have:

- All hyperplanes of multiplicity 2 are conformally equivalent, and $Disc(X) = \mathbb{S}^4$.
- If X is the union of two distinct hyperplanes that intersect on a twistor fiber, then X is conformally equivalent to $X(r) = X_0(X_1 - rX_0)$, where $r > 0$, and $Disc(X) = \pi_{\mathbb{CP}^3}(\{X_0 = 0\} \cap \{X_1 - rX_0 = 0\})$. Quadrics with different r are pairwise conformally inequivalent.
- If X is the union of two distinct hyperplanes that do not intersect on a twistor fiber, then X is conformally equivalent to $\{X_0X_3 = 0\}$.
- If X has a unique singular point and contains twistor lines, then it is conformally equivalent to the quadric $\{X_1^2 = X_0X_2\}$, and $Disc(X)$ is a round 2-sphere in $\mathbb{S}^4$.
- If X has a unique singular point and does not contain twistor lines, then it is conformally equivalent to the quadric

  $$Q(r) = X_2^2 + X_0(X_1 + rX_1),$$

  where $r > 0$, and $Disc(X)$ is homeomorphic to a 2-sphere with two identified points. Different values of r correspond to conformally inequivalent quadrics.

For the particular case of cone surfaces, we also refer to [15] and to [3, Theorem 4.2] for a generalization of what is described here.

In the case of the flag threefold $\mathbb{F}$, twistor degree 2 corresponds to surfaces of bidegree (2, 0), (1, 1), and (0, 2). Clearly, bidegrees (2, 0) and (0, 2) are completely analogous.

Smooth surfaces of bidegree (1, 1) were studied in [7]. We summarize some main facts and then discuss the remaining cases.

Any surface of bidegree (1, 1) can be represented as

$$S_A := \{(p, \ell) \in \mathbb{F} \mid pA\ell = 0\},$$

where A is a 3×3 complex non-scalar matrix. These surfaces are parametrized by the so-called Kronecker pencil of the form $sA + tI$, where $s, t \in \mathbb{C}$ with $s \neq 0$. There are five projective classes, summarized as follows (see [7, Lemma 4.1]):

- S_{A_1}, where $A_1 = \mathrm{diag}(0, 1, \lambda)$ and $\lambda \in \mathbb{C}\backslash\{0, 1\}$, is smooth and corresponds to a del Pezzo surface of degree 6, unique up to biholomorphism.
- S_{A_2}, where $A_2 = \mathrm{diag}(0, 0, 1)$, is reducible and singular along the curve of bidegree (1, 1),

$$\{([p_0 : p_1 : 0], [\ell_0 : \ell_1 : 0]) \in \mathbb{F} \mid p_0\ell_0 + p_1\ell_1 = 0\}.$$

Let A_3, A_4, and A_5 be the following matrices:

$$A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

- S_{A_3} has a single singular point ([1 : 0 : 0], [0 : 1 : 0]).
- S_{A_4} is reducible and singular along the reducible curve of bidegree (1, 1),

$$\{([p_0 : 0 : p_2], [0 : \ell_1 : \ell_2] \in \mathbb{F}) \mid p_2\ell_2 = 0\}.$$

- S_{A_5} has a single singular point ([1 : 0 : 0], [0 : 0 : 1]).

To obtain a unitary (i.e., conformal) classification, it is convenient to use the so-called QR-factorization. As already stated, any surface of bidegree (1, 1) can be written as S_A, and it is equivalent under projective transformation to S_{A_i} for a suitable $i = 1, 2, 3, 4, 5$. That is, there exists an invertible C such that $A_i = C^{-1}AC$. Now, any such C can be written as $C = QR$, with $Q \in U(3)$ and R upper triangular. Hence, $Q^{-1}AQ = R^{-1}A_iR$. This reduction can be further specified: the conjugacy class of A does not change under the action (by conjugacy) of the matrix $X = \mathrm{diag}(e^{i(\vartheta_1+\vartheta_2)}, e^{i\vartheta_2}, 1)$ (see [7, Section 4.3]). Therefore, any smooth surface of bidegree (1, 1) can be written as S_{A_1}, with

$$A_1 = \begin{pmatrix} 0 & a & b \\ 0 & 1 & c \\ 0 & 0 & \lambda \end{pmatrix},$$

where $a, b, c \in \mathbb{C}$, and $a' = ae^{i\vartheta_1}$, $b' = be^{i(\vartheta_1+\vartheta_2)}$, and $c' = ce^{i\vartheta_2}$ define the same surface. With these tools, the behavior of smooth surfaces of bidegree (1, 1) can be discussed (see [7, Sections 7 and 9]).

If S is a smooth j-invariant surface of bidegree (1, 1) (i.e., $S = S_{A_1}$, with $A_1 = \mathrm{diag}(0, 1, \lambda)$ and $\lambda \in \mathbb{R}\backslash\{0, 1\}$), it contains infinitely many twistor fibers parameterized by a circle, corresponding to the orbit of a maximal torus in $SU(3)$.

If S is a smooth "diagonal" non-j-invariant surface of bidegree (1, 1) (i.e., $S = S_{A_1}$, with $A_1 = \mathrm{diag}(0, 1, \lambda)$ and $\lambda \in \mathbb{C}\backslash\mathbb{R}$), it does not contain any twistor fibers, and $Disc(S)$ is a smooth 2-dimensional real torus.

Let us now assume $S = S_{A_1}$, with

$$A_1 = \begin{pmatrix} 0 & a & b \\ 0 & 1 & c \\ 0 & 0 & \lambda \end{pmatrix}.$$

Under certain conditions on a, b, c, λ, the surface S contains exactly one or two twistor fibers. In this case, $Disc(S)$ is defined as the zero set of a quadratic expression dependent on the coefficients of A. In particular, under specific real conditions ($a \neq 0$, $b = c = 0$, $\lambda = 2$), the branch locus of π is homeomorphic to a torus that may have 0, 1, or 2 singular points. The exact number of singular points depends on $|a|$.

We now complete this discussion by analyzing singular surfaces of bidegree (1, 1) and the whole family of surfaces of bidegree (0, 2).

5.1 Singular surfaces of bidegree (1, 1)

A reducible surface of bidegree (1, 1) is the union of a surface of bidegree (1, 0) with a surface of bidegree (0, 1). As mentioned earlier, these are Hirzebruch surfaces and contain exactly one twistor fiber. Moreover, the intersection of two such surfaces is always a (possibly reducible) curve of bidegree (1, 1).

First, let us find all possible unitary classes for each projective class S_{A_i}, $i = 2, 3, 4, 5$. The approach follows [7, Section 4.3] and utilizes the so-called QR-factorization: up to similarities, any matrix A representing a bidegree (1, 1) surface is equivalent to one of the A_i's, i.e., there exists an invertible C such that $A = C^{-1}A_iC$. Any such C can be decomposed as $C = QR$, with $Q \in U(3)$ and R upper triangular. Thus, $Q^{-1}AQ = RA_iR^{-1}$. If R is given as

$$R = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix},$$

with $ADF \neq 0$, then:

$$RA_2R^{-1} = \begin{pmatrix} 0 & 0 & c/f \\ 0 & 0 & e/f \\ 0 & 0 & 1 \end{pmatrix}, \quad RA_3R^{-1} = \begin{pmatrix} 0 & a/d & c/f - ae/df \\ 0 & 0 & e/f \\ 0 & 0 & 1 \end{pmatrix},$$

$$RA_4R^{-1} = \begin{pmatrix} 0 & a/d & -ae/df \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad RA_5R^{-1} = \begin{pmatrix} 0 & a/d & b/f - ae/df \\ 0 & 0 & d/f \\ 0 & 0 & 0 \end{pmatrix}.$$

Moreover, the unitary class does not change under the conjugacy action of the matrix $X = \mathrm{diag}(\exp(i\vartheta_1), \exp(i\vartheta_2), 1)$. Hence, all possible unitary classes are summarized in the following proposition.

Proposition 19 *Let $S_A \subset \mathbb{F}$ be a singular surface of bidegree* (1, 1). *Then, if A is in the same projective class as A_i, then, up to unitary transformation, A can be represented by the* *following matrix* A'_i:

$$A'_2 = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad A'_3 = \begin{pmatrix} 0 & 1 & t \\ 0 & 0 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad A'_4 = \begin{pmatrix} 0 & 1 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A'_5 = \begin{pmatrix} 0 & 1 & t \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \tag{5}$$

where $t \in \mathbb{C}$ *and* $x, y \in \mathbb{R}$. *Surfaces with different values of the parameters* t, x, y *are inequivalent.*

Proof Starting from the previous considerations about A_i, $i = 2, 3, 4, 5$, it is sufficient to compute $XRA_iR^{-1}X^{-1}$ for each $i = 2, 3, 4, 5$ and to choose appropriately the angles ϑ_1 and ϑ_2. The fact that, for each $i = 2, 3, 4, 5$, different values of the parameters yield inequivalent surfaces can be verified by direct computation. □

Proposition 20 *Let* $S_A \subset \mathbb{F}$ *be a reducible surface of bidegree* $(1, 1)$. *If* A *is of the form* A'_2, *then*:

$$Disc(S_A) = \left\{ z \in \mathbb{CP}^2 \,\middle|\, [z_0 : z_1 : z_2] \begin{pmatrix} 1 & 0 & -x \\ 0 & 1 & -y \\ 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} \bar{z}_0 \\ \bar{z}_1 \\ \bar{z}_2 \end{bmatrix} = 0 \right\},$$

and S_A *contains the twistor fibers over* $[0, 0, 1]$ *and* $[x, y, 1]$.

If A *is of the form* A'_4, *then*:

$$Disc(S_A) = \left\{ z \in \mathbb{CP}^2 \,\middle|\, z_0(z_1 + yz_2) = 0 \right\},$$

and S_A *contains the twistor fibers over* $[1, 0, 0]$ *and* $[0, 1, y]$.

Proof The explicit equations of $S_{A'_2}$ and $S_{A'_4}$ are given by:

$$S_{A'_2} : (xp_0 + yp_1 + p_2)\ell_2 = 0, \quad S_{A'_4} : p_0(\ell_1 + y\ell_2) = 0,$$

therefore $S_{A'_2} = \{xp_0 + yp_1 + p_2 = 0\} \cdot \{\ell_2 = 0\}$ and $S_{A'_4} = \{p_0 = 0\} \cdot \{\ell_1 + y\ell_2 = 0\}$. It is straightforward to verify that $\{xp_0 + yp_1 + p_2 = 0\}$ contains the fiber over $[x : y : 1]$, $\pi_{\mathbb{F}}^{-1}[0 : 0 : 1] \subset \{\ell_2 = 0\}$, $\pi_{\mathbb{F}}^{-1}[1 : 0 : 0] \subset \{p_0 = 0\}$, and $\pi_{\mathbb{F}}^{-1}[0 : 1 : y] \subset \{\ell_1 + y\ell_2 = 0\}$.

The discriminant loci $Disc(S_{A'_2})$ and $Disc(S_{A'_4})$ are obtained as $\pi_{\mathbb{F}}(\{xp_0 + yp_1 + p_2 = 0\} \cap \{\ell_2 = 0\})$ and $\pi_{\mathbb{F}}(\{p_0 = 0\} \cap \{\ell_1 + y\ell_2 = 0\})$, respectively. These computations follow the argument in [7, Lemma 6.5]. □

Let us now turn to the singular irreducible cases. For these instances, we will not compute the twistor discriminant locus but only identify the possible twistor fibers contained in the surface. As shown in [7], the full discriminant locus can be quite complex and computationally challenging.

Proposition 21 *Let* $S_A \subset \mathbb{F}$ *be a singular irreducible surface of bidegree* $(1, 1)$.

- *If* A *is of the form* A'_3, *then* S_A *contains the twistor fiber* $\pi_{\mathbb{F}}^{-1}[0 : y : 1]$ *if and only if* $ty = 1$.
- *If* A *is of the form* A'_5, *then* S_A *contains the twistor fiber* $\pi_{\mathbb{F}}^{-1}[0 : 1 : 0]$ *if and only if* $t = 0$.

Proof The idea of the proof follows the argument presented in [7, Theorem 9.3]. Fix $q \in \mathbb{CP}^2$. Recall that the twistor fiber of q is given by:

$$\pi^{-1}(q) = \{(p, \ell) \mid p\ell = 0,\ q\ell = 0,\ pq^* = 0\} = qH \cap H_{q^*},$$

where qH and H_{q^*} are the Hirzebruch surfaces of bidegree $(0, 1)$ and $(1, 0)$, respectively, defined by

$$qH := \{(p, \ell) \in \mathbb{F} \mid q\ell = 0\}, \quad H_{q^*} := \{(p, \ell) \in \mathbb{F} \mid pq^* = 0\}.$$

Now consider:

$$S_A \cap \{q\ell = 0\} = \{(p, \ell) \mid p\ell = 0,\ pA\ell = 0,\ q\ell = 0\}.$$

This intersection is non-empty if and only if:

$$\det(p \mid pA \mid q) = 0.$$

Let us start with $S_{A_3'}$. The determinant equation becomes:

$$\begin{aligned} 0 = \det(p \mid pA_3' \mid q) &= (q_2 - tq_1)p_0^2 + (q_0t - q_1y)p_0p_1 - (q_0 + q_1)p_0p_2 \\ &\quad + yq_0p_1^2 + q_0p_1p_2. \end{aligned}$$

This defines a conic $\mathcal{C}$, represented by the following matrix:

$$C = \begin{pmatrix} q_2 - tq_1 & \frac{tq_0 - q_1y}{2} & -\frac{q_0+q_1}{2} \\ \frac{tq_0 - q_1y}{2} & yq_0 & \frac{q_0}{2} \\ -\frac{q_0+q_1}{2} & \frac{q_0}{2} & 0 \end{pmatrix}.$$

The conic $\mathcal{C}$ is reducible only if: $0 = \det(C) = -\frac{q_0^2}{4}((t + y)q_0 + yq_1 + q_2)$. In particular, if $q_0 = 0$, then $\mathcal{C}$ becomes $p_0((q_2 - tq_1)p_0 - q_1yp_1 - q_1p_2)$; if $q_2 = -(t + y)q_0 - yq_1$, then $\mathcal{C}$ splits as:

$$(-(q_0 + q_1)p_0 + q_0p_1)((t + y)p_0 + yp_1 + p_2).$$

Following the proof in [7, Theorem 9.3], if $q_0 = 0$, $S_{A_3'}$ contains the fiber over $q = [q_0 : q_1 : q_2]$ if and only if:

$$\text{rank}\begin{pmatrix} 0 & \bar{q}_1 & \bar{q}_2 \\ 1 & 0 & 0 \end{pmatrix} = 1, \quad \text{or} \quad \text{rank}\begin{pmatrix} 0 & \bar{q}_1 & \bar{q}_2 \\ q_2 - tq_1 & -yq_1 & -q_1 \end{pmatrix} = 1.$$

The first matrix always has rank 2. The second matrix has rank 1 if and only if:

$$\bar{t}y = 1 \quad (\text{so } t \in \mathbb{R}), \quad \text{and} \quad q_2 = tq_1.$$

Repeating the analysis for A_5', the matrix of the conic $\mathcal{C}$ is:

$$C = \begin{pmatrix} q_2 - tq_1 & \frac{tq_0 - q_1}{2} & -\frac{q_0}{2} \\ \frac{tq_0 - q_1}{2} & q_0 & 0 \\ -\frac{q_0}{2} & 0 & 0 \end{pmatrix}.$$

The determinant of C simplifies to $\det(C) = -\frac{q_0^3}{4}$. For $q_0 = 0$, the conic $\mathcal{C}$ splits as:

$$\mathcal{C}:\ p_0[(tq_2 - q_1)p_0 - q_1p_1] = 0.$$

As before, the fiber is contained in $S_{A'_5}$ if:

$$\text{rank}\begin{pmatrix} 0 & \bar{q}_1 & \bar{q}_2 \\ 1 & 0 & 0 \end{pmatrix} = 1, \quad \text{or} \quad \text{rank}\begin{pmatrix} 0 & \bar{q}_1 & \bar{q}_2 \\ q_2 - tq_1 & -q_1 & 0 \end{pmatrix} = 1.$$

The first matrix always has rank 2, while the second matrix has rank 1 if and only if $t = 0$ and $q_2 = 0$. □

5.2 Singular surfaces of bidegree (0, 2)

As explained in [7, Section 3.3], if $S \subset \mathbb{F}$ is a surface of bidegree $(0, d)$, then $S = \pi_1^{-1}(C)$, where $C \subset \mathbb{CP}^2$ is a curve of degree d and π_1 denotes the natural projection onto the first component of $\mathbb{F} \subset \mathbb{CP}^2 \times \mathbb{CP}^2$. Moreover, if $S' = \pi_1^{-1}(C')$, then S and S' are projectively/unitarily equivalent if and only if the curves C and C' are equivalent. Therefore, to classify surfaces of bidegree (0, 2) under unitary transformations, it suffices to classify conics in $\mathbb{CP}^2$ up to unitary equivalence.

This classification is well known, and a recent account can be found in [12]. Specifically, the moduli space of smooth conics is given by:

$$C_{r,s} := \{[p_0 : p_1 : p_2] \in \mathbb{CP}^2 \mid p_0^2 + r^2 p_1^2 + s^2 p_2^2 = 0\},$$

where $0 < s \leq r \leq 1$. The reducible cases are obtained when $s = 0$. Clearly, for a smooth conic C, if $S = \pi_1^{-1}(C)$, then S does not contain any twistor fiber (otherwise $\pi_i(S)$, $i = 1, 2$, would contain a line).

Theorem 22 *Let $S = \pi_1^{-1}(C) \subset \mathbb{F}$ be a smooth surface of bidegree* (0, 2). *Then,* $Disc(S) = C^\vee$, *where $C^\vee$ denotes the dual conic of C.*

Proof Any smooth surface S of bidegree (0, 2) is of the form:

$$S_{r,s} := \{(p, \ell) \in \mathbb{F} \mid p_0^2 + r^2 p_1^2 + s^2 p_2^2 = 0\}.$$

For such an S, the twistor discriminant locus is the set of points $q \in \mathbb{CP}^2$ such that the curve:

$$\{(p, \ell) \in \mathbb{F} \mid p\bar{q} = 0,\ q\ell = 0\}$$

is tangent to $S_{r,s}$. This condition is satisfied if and only if $q = [q_0 : q_1 : q_2]$ lies on the dual conic, i.e., the line $p_0\bar{q}_0 + p_1\bar{q}_1 + p_2\bar{q}_2 = 0$ is tangent to C. The locus of such points q is given by:

$$\bar{q}_0^2 + \frac{\bar{q}_1^2}{r^2} + \frac{\bar{q}_2^2}{s^2} = 0.$$

Since $r, s \in \mathbb{R}$, this describes the dual conic $C^\vee$.

If $r = s = 0$, the surface corresponds to a double plane. These are all equivalent, and their twistor discriminant locus is the entire $\mathbb{CP}^2$. Additionally, such surfaces contain exactly one twistor fiber.

We conclude this section by analyzing the remaining reducible case.

Proposition 23 *Let $S = \pi_1^{-1}(L_1 \cup L_2) \subset \mathbb{F}$ be a reduced and reducible surface of bidegree* (0, 2). *Then*:

$$Disc(S) = \pi_{\mathbb{F}}(\pi_1^{-1}(L_1 \cap L_2)),$$

and if $S = S_{r,0}$, then it contains the twistor fibers over $[1 : \pm ir : 0]$.

Proof The proof follows from direct inspection and the fact that if a surface is reducible, $S = S_1 \cup S_2$, then:

$$Disc(S) = Disc(S_1) \cup Disc(S_2) \cup \pi_{\mathbb{F}}(S_1 \cap S_2).$$

For $S = S_{r,0}$, we have:

$$S = \{p_0 + irp_1 = 0\} \cup \{p_0 - irp_1 = 0\}.$$

These components determine the twistor fibers over $[1 : \pm ir : 0]$, as claimed.

Acknowledgements The author would like to express his sincere gratitude to the anonymous referee for its comments and suggestions, which have improved the quality of this work. The author was partially supported by PRIN 2022MWPMAB—“Interactions between Geometric Structures and Function Theories” and by GNSAGA of INdAM.

Funding Open access funding provided by Università degli Studi di Bari Aldo Moro within the CRUI-CARE Agreement.

Data availability There is no data to be made available.

Declarations

Conflict of interest The corresponding author states that there is no conflict of interest.

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