Carleman estimates, observability inequalities and null controllability for interior degenerate non smooth parabolic equations

Genni Fragnelli
Dipartimento di Matematica
Università di Bari "Aldo Moro"
Via E. Orabona 4
70125 Bari - Italy
email: genni.fragnelli@uniba.it
Dimitri Mugnai
Dipartimento di Matematica e Informatica
Università di Perugia
Via Vanvitelli 1, 06123 Perugia - Italy
email: dimitri.mugnai@unipg.it

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CHAPTER 1

Introduction

In the last recent years an increasing interest has been devoted to degenerate parabolic equations. Indeed, many problems coming from physics (boundary layer models in [**13**], models of Kolmogorov type in [**7**], ...), biology (Wright-Fisher models in [**50**] and Fleming-Viot models in [**29**]), and economics (Black-Merton-Scholes equations in [**23**]) are described by degenerate parabolic equations, whose linear prototype is

$$\begin{cases} u_t - \mathcal{A}u = h(t,x), & (t,x) \in (0,T) \times (0,1), \\ u(0,x) = u_0(x) \end{cases} \tag{1.1}$$

with the associated desired boundary conditions, where $\mathcal{A}u = \mathcal{A}_1 u := (au_x)_x$ or $\mathcal{A}u = \mathcal{A}_2 u := au_{xx}$.

In this paper we concentrate on a special topic related to this field of research, i.e. Carleman estimates for the adjoint problem to (1.1). Indeed, they have so many applications that a large number of papers has been devoted to prove some forms of them and possibly some applications. For example, it is well known that they are a fundamental tool to prove observability inequalities, which lead to global null controllability results for (1.1) also in the non degenerate case: for all $T > 0$ and for all initial data $u_0 \in L^2((0,T) \times (0,1))$ there is a suitable control $h \in L^2((0,T) \times (0,1))$, supported in a subset ω of $[0,1]$, such that the solution u of (1.1) satisfies $u(T,x) = 0$ for all $x \in [0,1]$ (see, for instance, [**1**] - [**6**], [**13**] - [**19**], [**26**], [**27**], [**30**], [**31**], [**33**], [**40**], [**41**], [**42**], [**47**] and the references therein).

Moreover, Carleman estimates are also extremely useful for several other applications, especially for unique continuation properties (for example, see [**25**], [**36**] and [**40**]), for inverse problems, in parabolic, hyperbolic and fractional settings, e.g. see [**8**], [**21**], [**38**], [**48**], [**49**], [**53**], [**54**] and their references.

The common point of all the previous papers dealing with degenerate equations, is that the function a degenerates at the boundary of the domain. For example, as a, one can consider the double power function

$$a(x) = x^k(1-x)^\alpha, \quad x \in [0,1],$$

where k and α are positive constants. For related systems of degenerate equations we refer to [**1**], [**2**] and [**14**].

However, the papers cited above deal with a function a that degenerates at the boundary of the spatial domain. To our best knowledge, [**51**] is the first paper treating the existence of a solution for the Cauchy problem associated to a parabolic equation which degenerates in the interior of the spatial domain, while degenerate parabolic problems modelling biological phenomena and related optimal control problems are later studied in [**43**] and [**9**]. Recently, in [**32**] the authors analyze in detail the degenerate operator $\mathcal{A}$ in the space $L^2(0,1)$, with or without weight, proving that it is nonpositive and selfadjoint, hence it generates a cosine family and, as a consequence, an analytic semigroup. In [**32**] the well-posedness of (1.1) with Dirichlet boundary conditions is also treated, but nothing is said about other properties, like Carleman estimates or controllability results. Indeed, these arguments are the subject of the recent paper [**33**], where only the divergence case is considered and the function a is assumed to be of class C^1 far from the degenerate point, which belongs to the interior of the spatial domain.

In this paper we consider both problems in divergence and in non divergence form with a non smooth coefficient (for the precise assumptions see below), and we first prove Carleman estimates for the adjoint problem of the parabolic equation with interior degeneracy

$$\begin{cases} u_t - \mathcal{A}u = h, & (t,x) \in Q_T := (0,T) \times (0,1), \\ u(t,0) = u(t,1) = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{1.2}$$

that is for solutions of the problem

$$\begin{cases} v_t + \mathcal{A}v = h, & (t,x) \in Q_T, \\ v(t,1) = v(t,0) = 0, & t \in (0,T). \end{cases} \tag{1.3}$$

Here u_0 belongs to a suitable Hilbert space X ($L^2(0,1)$ in the divergence case and $L^2_{\frac{1}{a}}(0,1)$ in the non divergence case, see the following chapters), and the control $h \in L^2(0,T;X)$ acts on a nonempty subinterval ω of $(0,1)$ which is allowed to contain the degenerate point x_0.

We underline the fact that in the present paper we consider both equations *in divergence* and *in non divergence form*, since the last one *cannot* be recast from the equation in divergence form, in general: for example, the simple equation

$$u_t = a(x)u_{xx}$$

can be written in divergence form as

$$u_t = (au_x)_x - a'u_x,$$

only if a' does exist; in addition, even if a' exists, considering the well-posedness for the last equation, additional conditions are necessary: for instance, for the prototype $a(x) = x^K$, well-posedness is guaranteed if $K \geq 2$ ([**44**]). However, in [**16**] the authors prove that if $a(x) = x^K$ the global null controllability fails exactly when $K \geq 2$. For this reason, already in [**18**], [**19**] and [**31**] the authors consider parabolic problems in non divergence form proving directly that, under suitable conditions for which well-posedness holds, the problem is still globally null controllable, that is the solution vanishes identically at the final time by applying a suitable localized control. In particular, while in [**18**] or [**19**] Dirichlet boundary conditions are considered, in [**31**] Neumann boundary conditions are assumed.

The question of controllability of partial differential systems with *non smooth* coefficients, i.e. the coefficient a is not of class C^1 (or even with higher regularity, as sometimes it is required), and its dual counterpart, observability inequalities, is not fully solved yet. In fact, the presence of a non smooth coefficient introduces several complications, and, in fact, the literature in this context is quite poor. We are only aware of the following few papers in which Carleman estimate are proved always in the *non degenerate case*, but in the case in which the coefficient of the operator is somehow *non smooth*. In [**22**] and [**41**] the non degenerate coefficient is actually assumed smooth apart from across an interface where it may jump, with or without some monotonicity condition ([**22**] and [**41**], respectively), while in [**10**] the non degenerate coefficient is assumed to be piecewise smooth. Carleman estimates for a non degenerate BV coefficient were proved in [**39**], but however, the coefficient was supposed to be of class C^1 in an open subset of $(0,1)$, and then controllability for (1.2) and semilinear extension are given. Finally, in [**37**] a is supposed to be of class $W^{1,\infty}(0,1)$, but again it does not degenerate at any point. For completeness, we also quote [**28**], where boundary controllability result for non degenerate BV coefficients are proved using Russel's method and not Carleman estimates.

As far as we know, no Carleman estimates for (1.3) are known when a is *globally non smooth* and degenerates at an interior point x_0, nor when a is non degenerate and non smooth. For this reason, the object of this paper is twofold: first, we prove Carleman estimates in the non degenerate case when a is not smooth. In particular we treat the case of an absolutely continuous coefficient, and thus *globally* of class BV, though with some restrictions, and the case of a $W^{1,\infty}$ coefficient. This case was already considered in [**37**], but they proved a version of Carleman estimates with all positive integrals in the righ-hand-side, while in our version we include a negative one, which is needed for the subsequent applications (see Theorem 3.1 and Theorem 3.2). Second, we prove Carleman estimates in degenerate non smooth cases. Such estimates are then used to prove observability inequalities (and hence null controllability results).

To our best knowledge, this paper is the first one where, in the case of an absolutely continuous coefficient - which is even allowed to degenerate - non smoothness is assumed in the whole domain, though with some restrictions.

Concerning the non smooth non degenerate case, in the spatial domain $(0,1)$ we assume that

(a_1) $a \in W^{1,1}(0,1)$, $a \geq a_0 > 0$ in $(0,1)$ and there exist two functions $\mathfrak{g} \in L^1(0,1)$, $\mathfrak{h} \in W^{1,\infty}(0,1)$ and two strictly positive constants $\mathfrak{g}_0$, $\mathfrak{h}_0$ such that $\mathfrak{g}(x) \geq \mathfrak{g}_0$ for a.e. x in $[0,1]$ and

$$-\frac{a'(x)}{2\sqrt{a(x)}}\left(\int_x^1 \mathfrak{g}(t)dt + \mathfrak{h}_0\right) + \sqrt{a(x)}\mathfrak{g}(x) = \mathfrak{h}(x) \quad \text{for a.e. } x \in [0,1],$$

in the divergence case,

$$\frac{a'(x)}{2\sqrt{a(x)}}\left(\int_x^1 \mathfrak{g}(t)dt + \mathfrak{h}_0\right) + \sqrt{a(x)}\mathfrak{g}(x) = \mathfrak{h}(x) \quad \text{for a.e. } x \in [0,1],$$

in the non divergence case; or

(a_2) $a \in W^{1,\infty}(0,1)$ and $a \geq a_0 > 0$ in $(0,1)$.

However, in Chapter 3 we shall present the precise setting and the related Carleman estimate in a general interval (A,B), since we shall not use it in the whole $(0,1)$ but in suitable subintervals.

Concerning the degenerate case, we shall admit two types of degeneracy for a, namely weak and strong degeneracy. More precisely, we shall handle the two following cases:

HYPOTHESIS 1.1. **Weakly degenerate case (WD):** there exists $x_0 \in (0,1)$ such that $a(x_0) = 0$, $a > 0$ on $[0,1]\setminus\{x_0\}$, $a \in W^{1,1}(0,1)$ and there exists $K \in (0,1)$ such that $(x - x_0)a' \leq Ka$ a.e. in $[0,1]$.

HYPOTHESIS 1.2. **Strongly degenerate case (SD):** there exists $x_0 \in (0,1)$ such that $a(x_0) = 0$, $a > 0$ on $[0,1]\setminus\{x_0\}$, $a \in W^{1,\infty}(0,1)$ and there exists $K \in [1,2)$ such that $(x - x_0)a' \leq Ka$ a.e. in $[0,1]$.

Typical examples for weak and strong degeneracies are $a(x) = |x - x_0|^\alpha$, $0 < \alpha < 1$ and $a(x) = |x - x_0|^\alpha$, $1 \leq \alpha < 2$, respectively.

For the proof of the related Carleman estimates and observability inequalities a fundamental rôle is played by the following general weighted Hardy-Poincaré inequality for functions which may *not be* globally absolutely continuous in the domain, but whose irregularity point is "controlled" by the fact that the weight degenerates exactly there. Such an inequality, of independent interest, was proved in [**33**, Proposition 2.3], and reads as follows.

PROPOSITION 1.1 (Hardy–Poincaré inequality). *Assume that $p \in C([0,1])$, $p > 0$ on $[0,1] \setminus \{x_0\}$, $p(x_0) = 0$ and there exists $q \in (1,2)$ such that the function*

$$x \mapsto \frac{p(x)}{|x - x_0|^q} \text{ is nonincreasing on the left of } x = x_0$$
$$\text{and nondecreasing on the right of } x = x_0.$$

Then, there exists a constant $C_{HP} > 0$ such that for any function w, locally absolutely continuous on $[0, x_0) \cup (x_0, 1]$ and satisfying

$$w(0) = w(1) = 0 \text{ and } \int_0^1 p(x)|w'(x)|^2\,dx < +\infty\,,$$

the following inequality holds:

$$\int_0^1 \frac{p(x)}{(x - x_0)^2} w^2(x)\,dx \leq C_{HP} \int_0^1 p(x)|w'(x)|^2\,dx. \tag{1.4}$$

Actually, such a proposition is valid without requiring $q < 2$.

Applying the Carleman estimate (and other tools) to any solution v of the adjoint problem (1.3), we derive the observability inequalities

$$\int_0^1 v^2(0,x)dx \leq C \int_0^T \int_\omega v^2(t,x)dxdt,$$

in the divergence case and

$$\int_0^1 v^2(0,x)\frac{1}{a}dx \leq C \int_0^T \int_\omega v^2(t,x)\frac{1}{a}dxdt,$$

in the non divergence one. The proof of these last inequalities are obtained by studying some auxiliary problems, introduced with suitable cut-off functions and reflections (see Lemmas 5.1, 5.2 and 5.4), and is the content of the long Chapter 5, where, using a standard technique in this framework, one can also prove null controllability results for (1.2).

Finally, such results are extended to the semilinear problem

$$\begin{cases} u_t - \mathcal{A}u + f(t,x,u) = h(t,x)\chi_\omega(x), & (t,x) \in (0,T) \times (0,1),\\ u(t,1) = u(t,0) = 0, & t \in (0,T),\\ u(0,x) = u_0(x), & x \in (0,1), \end{cases} \tag{1.5}$$

in the weakly degenerate case using the fixed point method developed in [**27**] for nondegenerate problems. We note that, as in the nondegenerate case, our method relies on a compactness result for which the fact that $\frac{1}{a} \in L^1(0,1)$ is an essential assumption, and it forces us to consider only the weakly degenerate case. However, in the complete linear case, i.e $f(t,x,u) = c(t,x)u(t,x)$, the null controllability result holds also for the strongly degenerate case, since in this case it is a consequence of the results proved for (1.2), see Corollary 6.1.

The paper is organized as follows. First of all, we underline the fact that all chapters, except for the final Chapters 6 and 7, are divided into two subsections that deal with the divergence case and the non divergence one separately. In Chapter 2 we give the precise setting for the weakly and the strongly degenerate cases and some general tools we shall use several times. In Chapter 3 we prove Carleman estimates for the adjoint problem of (1.2) with a non smooth non degenerate coefficient. In Chapter 4 we provided one of the main results of this paper, i.e. Carleman estimates in the degenerate (non smooth) case. In Chapter 5 we apply the previous Carleman estimates to prove observability inequalities which, together with Caccioppoli type inequalities, let us derive new null controllability results for degenerate problems. In particular, in the divergence case, we handle both the cases in which the degeneracy point is *inside* or *outside* the control region ω; on the contrary, in the non divergence case we consider only the case of a degeneracy point being *outside* ω (see Comment 2 in Chapter 7 for the reason of this fact). In Chapter 6 we extend the previous results to complete linear and semilinear problems. Finally, in Chapter 7 we conclude the paper with some general remarks, which we consider fundamental.

CHAPTER 2

Mathematical tools and preliminary results

We begin this chapter with a lemma that is crucial for the rest of the paper:

LEMMA 2.1 ([**33**], Lemma 2.1). *Assume that Hypothesis* 1.1 *or* 1.2 *is satisfied.*

(1) *Then for all* $\gamma \geq K$ *the map*

$$x \mapsto \frac{|x-x_0|^\gamma}{a} \text{ is nonincreasing on the left of } x = x_0$$
$$\text{and nondecreasing on the right of } x = x_0,$$
$$\text{so that } \lim_{x\to x_0} \frac{|x-x_0|^\gamma}{a} = 0 \text{ for all } \gamma > K.$$

(2) *If* $K < 1$, *then* $\frac{1}{a} \in L^1(0,1)$.

(3) *If* $K \in [1,2)$, *then* $\frac{1}{\sqrt{a}} \in L^1(0,1)$ *and* $\frac{1}{a} \notin L^1(0,1)$.

REMARK 1. We underline the fact that if $\frac{1}{a} \in L^1(0,1)$, then $\frac{1}{\sqrt{a}} \in L^1(0,1)$. On the contrary, if $a \in W^{1,\infty}([0,1])$ and $\frac{1}{\sqrt{a}} \in L^1(0,1)$, then $\frac{1}{a} \notin L^1(0,1)$ (see [**33**, Remark 2]).

1. Well-posedness in the divergence case

In order to study the well-posedness of problem (1.2), we introduce the operator

$$\mathcal{A}_1 u := (au_x)_x$$

and we consider two different classes of weighted Hilbert spaces, which are suitable to study two different situations, namely the *weakly degenerate* (WD) and the *strongly degenerate* (SD) cases. We remark that we shall use the standard notation H for Sobolev spaces with non degenerate weights and the calligraphic notation $\mathcal{H}$ for spaces with degenerate weights.

CASE (WD): if Hypothesis 1.1 holds, we consider

$$\mathcal{H}^1_a(0,1) := \big\{u \text{ is absolutely continuous in } [0,1],$$
$$\sqrt{a}u' \in L^2(0,1) \text{ and } u(0) = u(1) = 0\big\},$$

and

$$\mathcal{H}^2_a(0,1) := \big\{u \in \mathcal{H}^1_a(0,1) |\, au' \in H^1(0,1)\big\};$$

CASE (SD): if Hypothesis 1.2 holds, we consider

$$\mathcal{H}^1_a(0,1) := \big\{u \in L^2(0,1) \ \ |\, u \text{ locally absolutely continuous in } [0,x_0)\cup(x_0,1],$$
$$\sqrt{a}u' \in L^2(0,1) \text{ and } u(0) = u(1) = 0\big\}$$

and

$$\mathcal{H}^2_a(0,1) := \big\{u \in \mathcal{H}^1_a(0,1) \,|\, au' \in H^1(0,1)\big\}.$$

In both cases we consider the norms

$$\|u\|^2_{\mathcal{H}^1_a(0,1)} := \|u\|^2_{L^2(0,1)} + \|\sqrt{a}u'\|^2_{L^2(0,1)},$$

and

$$\|u\|^2_{\mathcal{H}^2_a(0,1)} := \|u\|^2_{\mathcal{H}^1_a(0,1)} + \|(au')'\|^2_{L^2(0,1)}$$

and we set

$$D(\mathcal{A}_1) = \mathcal{H}^2_a(0,1).$$

Thanks to lemma 2.1 one can prove the following characterizations for the (SD) case which are already given in [**33**, Propositions 2.1 and 2.2].

PROPOSITION 2.1 ([**33**], Proposition 2.1). *Let*

$$\begin{aligned} X := \Big\{u \in L^2(0,1) \,|\, & \ u \textit{ locally absolutely continuous in } [0,1]\setminus\{x_0\}, \\ & \sqrt{a}u' \in L^2(0,1), au \in H^1_0(0,1) \textit{ and} \\ & (au)(x_0) = u(0) = u(1) = 0\Big\}. \end{aligned}$$

Then, under Hypothesis 1.2 *we have*

$$\mathcal{H}^1_a(0,1) = X.$$

Using the previous result, one can prove the following additional characterization.

PROPOSITION 2.2 ([**33**], Proposition 2.2). *Let*

$$\begin{aligned} D := \Big\{u \in L^2(0,1) \,|\, & \ u \textit{ locally absolutely continuous in } [0,1]\setminus\{x_0\}, \\ & au \in H^1_0(0,1), au' \in H^1(0,1), au \textit{ is continuous at } x_0 \textit{ and} \\ & (au)(x_0) = (au')(x_0) = u(0) = u(1) = 0\Big\}. \end{aligned}$$

Then, under Hypothesis 1.2 *we have*

$$\mathcal{H}^2_a(0,1) = D(\mathcal{A}_1) = D.$$

Now, let us go back to problem (1.2), recalling the following

DEFINITION 2.1. If $u_0 \in L^2(0,1)$ and $h \in L^2(Q_T) := L^2(0,T;L^2(0,1))$, a function u is said to be a (weak) solution of (1.2) if

$$u \in C([0,T];L^2(0,1)) \cap L^2(0,T;\mathcal{H}^1_a(0,1))$$

and

$$\begin{aligned} &\int_0^1 u(T,x)\varphi(T,x)\,dx - \int_0^1 u_0(x)\varphi(0,x)\,dx - \int_{Q_T} u\varphi_t\,dxdt = \\ &-\int_{Q_T} au_x\varphi_x\,dxdt + \int_{Q_T} h\varphi\chi_\omega\,dxdt \end{aligned}$$

for all $\varphi \in H^1(0,T;L^2(0,1)) \cap L^2(0,T;\mathcal{H}^1_a(0,1))$.

As proved in [**32**] (see Theorems 2.2, 2.7 and 4.1), problem (1.2) is well-posed in the sense of the following theorem:

THEOREM 2.1. *Assume Hypothesis* 1.1 *or* 1.2. *For all* $h \in L^2(Q_T)$ *and* $u_0 \in L^2(0,1)$, *there exists a unique weak solution* $u \in C([0,T]; L^2(0,1)) \cap L^2(0,T; \mathcal{H}^1_a(0,1))$ *of* (1.2) *and there exists a universal positive constant* C *such that*

$$\sup_{t\in[0,T]} \|u(t)\|^2_{L^2(0,1)} + \int_0^T \|u(t)\|^2_{\mathcal{H}^1_a(0,1)} dt \le C(\|u_0\|^2_{L^2(0,1)} + \|h\|^2_{L^2(Q_T)}). \tag{2.1}$$

Moreover, if $u_0 \in \mathcal{H}^1_a(0,1)$, *then*

$$u \in H^1(0,T; L^2(0,1)) \cap C([0,T]; \mathcal{H}^1_a(0,1)) \cap L^2(0,T; \mathcal{H}^2_a(0,1)), \tag{2.2}$$

and there exists a universal positive constant C *such that*

$$\begin{aligned} &\sup_{t\in[0,T]} \left(\|u(t)\|^2_{\mathcal{H}^1_a(0,1)}\right) + \int_0^T \left(\|u_t\|^2_{L^2(0,1)} + \|(au_x)_x\|^2_{L^2(0,1)}\right) dt \\ &\qquad \le C\left(\|u_0\|^2_{\mathcal{H}^1_a(0,1)} + \|h\|^2_{L^2(Q_T)}\right). \end{aligned} \tag{2.3}$$

In addition, $\mathcal{A}_1$ *generates an analytic contraction semigroup of angle* $\pi/2$.

2. Well-posedness in the non divergence case

We start proving some preliminary results concerning the well-posedness of problem (1.2) in the non divergence case. For this, we consider the operator

$$\mathcal{A}_2 u := au_{xx},$$

which is related to the following weighted Hilbert spaces:

$$L^2_{\frac{1}{a}}(0,1) := \left\{u \in L^2(0,1) \ \mid \int_0^1 \frac{u^2}{a} dx < \infty\right\},$$

$$\mathcal{H}^1_{\frac{1}{a}}(0,1) := L^2_{\frac{1}{a}}(0,1) \cap H^1_0(0,1),$$

and

$$\mathcal{H}^2_{\frac{1}{a}}(0,1) := \left\{u \in \mathcal{H}^1_{\frac{1}{a}}(0,1) \,\middle|\, u' \in H^1(0,1)\right\},$$

endowed with the associated norms

$$\|u\|^2_{L^2_{\frac{1}{a}}(0,1)} := \int_0^1 \frac{u^2}{a} dx, \quad \forall\, u \in L^2_{\frac{1}{a}}(0,1),$$

$$\|u\|^2_{\mathcal{H}^1_{\frac{1}{a}}} := \|u\|^2_{L^2_{\frac{1}{a}}(0,1)} + \|u'\|^2_{L^2(0,1)}, \quad \forall\, u \in \mathcal{H}^1_{\frac{1}{a}}(0,1),$$

and

$$\|u\|^2_{\mathcal{H}^2_{\frac{1}{a}}(0,1)} := \|u\|^2_{\mathcal{H}^1_{\frac{1}{a}}(0,1)} + \|au''\|^2_{L^2_{\frac{1}{a}}(0,1)}, \quad \forall\, u \in \mathcal{H}^2_{\frac{1}{a}}(0,1).$$

Indeed, it is a trivial fact that, if $u' \in H^1(0,1)$, then $au'' \in L^2_{\frac{1}{a}}(0,1)$, so that the norm for $\mathcal{H}^2_{\frac{1}{a}}(0,1)$ is well defined, and we can also write in a more appealing way

$$\mathcal{H}^2_{\frac{1}{a}}(0,1) := \left\{u \in \mathcal{H}^1_{\frac{1}{a}}(0,1) \,\middle|\, u' \in H^1(0,1) \text{ and } au'' \in L^2_{\frac{1}{a}}(0,1)\right\}.$$

Finally, we take

$$D(\mathcal{A}_2) = \mathcal{H}^2_{\frac{1}{a}}(0,1).$$

Using Lemma 2.1, also the following characterization in the (WD) case is straightforward:

Proposition 2.3 ([**32**], Corollary 3.1). *Assume Hypothesis* 1.1. *Then,* $\mathcal{H}^1_{\frac{1}{a}}(0,1)$ *and* $H^1_0(0,1)$ *coincide algebraically. Moreover the two norms are equivalent. As a consequence,* $\mathcal{H}^2_{\frac{1}{a}}(0,1) = H^2(0,1) \cap H^1_0(0,1)$.

Hence, in the (WD) case, $C_c^\infty(0,1)$ is dense in $\mathcal{H}^1_{\frac{1}{a}}(0,1)$.

We also have the following characterization for the (SD) case:

Proposition 2.4 ([**32**], Propositions 3.6). *Suppose that Hypothesis* 1.2 *holds and set*

$$X := \{u \in \mathcal{H}^1_{\frac{1}{a}}(0,1) \ \mid u(x_0) = 0\}.$$

Then

$$\mathcal{H}^1_{\frac{1}{a}}(0,1) = X,$$

and, for all $u \in X$, $\|u\|_{\mathcal{H}^1_{\frac{1}{a}}(0,1)}$ *is equivalent to* $\left(\int_0^1 (u')^2 dx\right)^{\frac{1}{2}}$.

We remark that [**32**, Propositions 3.6] was proved assuming that

"there exists $x_0 \in (0,1)$ *such that* $a(x_0) = 0$, $a > 0$ *on* $[0,1] \setminus \{x_0\}$, $a \in W^{1,\infty}(0,1)$, $\dfrac{1}{a} \notin L^1(0,1)$ *and there exists* $C > 0$ *such that* $\frac{1}{a(x)} \le \frac{C}{|x-x_0|^2}$, *for all* $x \in [0,1] \setminus \{x_0\}$*".*

However, the last assumption is clearly satisfied under Hypothesis 1.2, thanks to Lemma 2.1.1, Lemma 2.1.2 and Remark 1.

We shall also need the following characterization:

Proposition 2.5. *Suppose that Hypothesis* 1.2 *holds and set*

$$D := \{u \in \mathcal{H}^2_{\frac{1}{a}}(0,1) \ \mid \ au' \in H^1(0,1) \text{ and } u(x_0) = (au')(x_0) = 0\}.$$

Then $D(\mathcal{A}_2) = \mathcal{H}^2_{\frac{1}{a}}(0,1) = D$.

Proof. Since it is clear that $D \subseteq \mathcal{H}^2_{\frac{1}{a}}(0,1)$, we take $u \in \mathcal{H}^2_{\frac{1}{a}}(0,1)$ and we prove that $u \in D$.

By Proposition 2.4, $u(x_0) = 0$, so that it is sufficient to prove that $au' \in H^1(0,1)$ and $(au')(x_0) = 0$. Since $u' \in H^1(0,1)$ and $a \in W^{1,\infty}(0,1)$, we immediately have that $au' \in L^2(0,1)$. Moreover, $(au')' = a'u' + au'' \in L^2(0,1)$ since $au'' \in L^2_{1/a}(0,1) \subset L^2(0,1)$, and thus $au' \in H^1(0,1) \subset C([0,1])$. Thus there exists $\lim_{x\to x_0}(au')(x) = (au')(x_0) = L \in \mathbb{R}$. Assume by contradiction that $L \neq 0$, then there exists $c > 0$ such that

$$|(au')(x)| \ge c$$

for all x in a neighborhood of x_0. Thus

$$|\left(a(u')^2\right)(x)| \ge \frac{c^2}{a(x)},$$

for all x in a neighborhood of x_0, $x \neq x_0$. But $\dfrac{1}{a} \notin L^1(0,1)$, thus we would have $\sqrt{a}u' \notin L^2(0,1)$, while $\sqrt{a}u' \in L^2(0,1)$, since a is bounded and $u' \in L^2(0,1)$. Hence $L = 0$, that is $(au')(x_0) = 0$. $\square$

For the rest of the paper, a crucial tool is also the following Green formula:

LEMMA 2.2. *For all* $(u,v) \in \mathcal{H}^2_{\frac{1}{a}}(0,1) \times \mathcal{H}^1_{\frac{1}{a}}(0,1)$ *one has*

$$\int_0^1 u''v\,dx = -\int_0^1 u'v'\,dx. \tag{2.4}$$

PROOF. It is trivial, since $u' \in H^1(0,1)$ and $v \in H^1_0(0,1)$. □

Finally, we will use the following

LEMMA 2.3 ([**32**], Lemma 3.7). *Assume Hypothesis* 1.2. *Then, there exists a positive constant* C *such that*

$$\int_0^1 v^2 \frac{1}{a} dx \le C \int_0^1 (v')^2 dx$$

for all $v \in \mathcal{H}^1_{\frac{1}{a}}(0,1)$.

We also recall the following definition:

DEFINITION 2.2. Assume that $u_0 \in L^2_{\frac{1}{a}}(0,1)$ and $h \in L^2_{\frac{1}{a}}(Q_T) := L^2(0,T;L^2_{\frac{1}{a}}(0,1))$. A function u is said to be a (weak) solution of (1.2) if

$$u \in C([0,T];L^2_{\frac{1}{a}}(0,1)) \cap L^2(0,T;\mathcal{H}^1_{\frac{1}{a}}(0,1))$$

and satisfies

$$\int_0^1 \frac{u(T,x)\varphi(T,x)}{a(x)}dx - \int_0^1 \frac{u_0(x)\varphi(0,x)}{a(x)}dx - \int_{Q_T} \frac{\varphi_t(t,x)u(t,x)}{a(x)}dxdt =$$
$$-\int_{Q_T} u_x(t,x)\varphi_x(t,x)dxdt + \int_{Q_T} h(t,x)\frac{\varphi(t,x)}{a(x)}dxdt$$

for all $\varphi \in H^1(0,T;L^2_{\frac{1}{a}}(0,1)) \cap L^2(0,T;\mathcal{H}^1_{\frac{1}{a}}(0,1))$.

Problem (1.2) is well-posed in the sense of the following theorem:

THEOREM 2.2. *Assume Hypothesis* 1.1 *or* 1.2. *Then, the operator* $\mathcal{A}_2 : D(\mathcal{A}_2) \to L^2_{\frac{1}{a}}(0,1)$ *is self–adjoint, nonpositive on* $L^2_{\frac{1}{a}}(0,1)$ *and it generates an analytic contraction semigroup of angle* $\pi/2$. *Moreover, for all* $h \in L^2_{\frac{1}{a}}(Q_T)$ *and* $u_0 \in L^2_{\frac{1}{a}}(0,1)$, *there exists a unique solution* $u \in C\big([0,T];L^2_{\frac{1}{a}}(0,1)\big) \cap L^2\big(0,T;\mathcal{H}^1_{\frac{1}{a}}(0,1)\big)$ *of* (1.2) *such that*

$$\sup_{t\in[0,T]} \|u(t)\|^2_{L^2_{\frac{1}{a}}(0,1)} + \int_0^T \|u(t)\|^2_{\mathcal{H}^1_{\frac{1}{a}}(0,1)}dt \le C_T \left(\|u_0\|^2_{L^2_{\frac{1}{a}}(0,1)} + \|h\|^2_{L^2_{\frac{1}{a}}(Q_T)} \right), \tag{2.5}$$

for some positive constant C_T. *In addition, if* $h \in W^{1,1}(0,T;L^2_{\frac{1}{a}}(0,1))$ *and* $u_0 \in \mathcal{H}^1_{\frac{1}{a}}(0,1)$, *then*

$$u \in C^1\big([0,T];L^2_{\frac{1}{a}}(0,1)\big) \cap C\big([0,T];D(\mathcal{A}_2)\big), \tag{2.6}$$

and there exists a positive constant C *such that*

$$\begin{aligned} &\sup_{t\in[0,T]} \left(\|u(t)\|^2_{\mathcal{H}^1_{\frac{1}{a}}(0,1)} \right) + \int_0^T \left(\|u_t\|^2_{L^2_{\frac{1}{a}}(0,1)} + \|au_{xx}\|^2_{L^2_{\frac{1}{a}}(0,1)} \right) dt \\ &\qquad \le C \left(\|u_0\|^2_{\mathcal{H}^1_{\frac{1}{a}}(0,1)} + \|h\|^2_{L^2(Q_T)} \right). \end{aligned} \tag{2.7}$$

PROOF. In the (WD) case the existence part is proved in [**32**, Theorems 3.3 and 4.3]. For the (SD) case, under different assumptions on the domain of the operator $\mathcal{A}_2 u := a u_{xx}$, it was proved in [**32**, Theorems 3.4 and 4.3], but here we must prove the theorem again, since the domain of $\mathcal{A}_2$ is different.

First, $D(\mathcal{A}_2)$ being dense in $L^2_{\frac{1}{a}}(0,1)$, in order to show that $\mathcal{A}_2$ generates an analytic semigroup, it is sufficient to prove that $\mathcal{A}_2$ is nonpositive and self-adjoint, hence $m-$dissipative by [**20**, Corollary 2.4.8].

Thus: $\mathcal{A}_2$ is nonpositive, since by (2.4), it follows that, for any $u \in D(\mathcal{A}_2)$

$$\langle \mathcal{A}_2 u, u\rangle_{L^2_{\frac{1}{a}}(0,1)} = \int_0^1 u'' u\, dx = -\int_0^1 (u')^2 dx \le 0.$$

Let us show that $\mathcal{A}_2$ is self-adjoint. First of all, observe that $\mathcal{H}^1_{\frac{1}{a}}(0,1)$ is equipped with the natural inner product

$$(u,v)_1 := \int_0^1 \left(\frac{uv}{a} + u'v'\right) dx$$

for any $u, v \in \mathcal{H}^1_{\frac{1}{a}}(0,1)$ and thanks to Lemma 2.3, the norm $\sqrt{(u,u)_1}$ is equivalent to $\|u'\|_{L^2(0,1)}$, for all $u \in \mathcal{H}^1_{\frac{1}{a}}(0,1)$.

Now, consider the function $F : L^2_{\frac{1}{a}}(0,1) \to L^2_{\frac{1}{a}}(0,1)$ defined as $F(f) := u \in \mathcal{H}^1_{\frac{1}{a}}(0,1)$ where u is the unique solution of

$$\int_0^1 u'v' dx = \int_0^1 \frac{f}{a} v dx$$

for all $u \in \mathcal{H}^1_{\frac{1}{a}}(0,1)$. Note that F is well-defined by the Lax–Milgram Theorem, which also implies that F is continuous. Now, easy calculations show that F is symmetric and injective. Hence, F is self-adjoint. As a consequence, $\mathcal{A}_2 = F^{-1} : D(\mathcal{A}_2) \to L^2_{\frac{1}{a}}(0,1)$ is self-adjoint by [**52**, Proposition A.8.2].

At this point, since $\mathcal{A}_2$ is a nonpositive, self–adjoint operator on a Hilbert space, it is well known that $(\mathcal{A}_2, D(\mathcal{A}_2))$ generates a cosine family and an analytic contractive semigroup of angle $\frac{\pi}{2}$ on $L^2_{\frac{1}{a}}(0,1)$ (see [**4**, Example 3.14.16 and 3.7.5]) or [**35**, Theorem 6.12]).

Finally, let us prove (2.5)–(2.7). First, being $\mathcal{A}_2$ the generator of a strongly continuous semigroup on $L^2_{\frac{1}{a}}(0,1)$, if $u_0 \in L^2_{\frac{1}{a}}(0,1)$, then the solution u of (1.2) belongs to $C\big([0,T]; L^2_{\frac{1}{a}}(0,1)\big) \cap L^2\big(0,T; \mathcal{H}^1_{\frac{1}{a}}(0,1)\big)$, while, if $u_0 \in D(\mathcal{A}_2)$ and $h \in W^{1,1}(0,T; L^2_{\frac{1}{a}}(0,1))$, then $u \in C^1\big([0,T]; L^2_{\frac{1}{a}}(0,1)\big) \cap C\big([0,T]; \mathcal{H}^2_{\frac{1}{a}}(0,1)\big)$ by [**20**, Lemma 4.1.5 and Proposition 4.1.6].

Now, we shall prove (2.7).

First, take $u_0 \in D(\mathcal{A}_2)$ and multiply the equation by u/a; by the Cauchy–Schwarz inequality we obtain for every $t \in (0,T]$,

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2_{\frac{1}{a}}(0,1)} + \|u_x(t)\|^2_{L^2(0,1)} \le \frac{1}{2}\|u(t)\|^2_{L^2_{\frac{1}{a}}(0,1)} + \frac{1}{2}\|h(t)\|^2_{L^2_{\frac{1}{a}}(0,1)}, \tag{2.8}$$

from which we easily get

$$\|u(t)\|^2_{L^2_{\frac{1}{a}}(0,1)} \le e^T \left(\|u(0)\|^2_{L^2_{\frac{1}{a}}(0,1)} + \|h\|^2_{L^2_{\frac{1}{a}}(Q_T)} \right) \tag{2.9}$$

for every $t \leq T$. Integrating (2.8), from (2.9) we also find

$$\int_0^T \|u_x(t)\|^2_{L^2(0,1)} dt \leq C_T \left(\|u(0)\|^2_{L^2_{\frac{1}{a}}(0,1)} + \|h\|^2_{L^2_{\frac{1}{a}}(Q_T)} \right) \tag{2.10}$$

for every $t \leq T$ and some universal constant $C_T > 0$. Thus, by (2.9) and (2.10), (2.5) follows if $u_0 \in D(\mathcal{A}_2)$. Since $D(\mathcal{A}_2)$ is dense in $L^2_{\frac{1}{a}}(0,1)$, the same inequality holds if $u_0 \in L^2_{\frac{1}{a}}(0,1)$.

Now, we multiply the equation by $-u_{xx}$, we integrate on $(0,1)$ and, using the Cauchy–Schwarz inequality, we easily get

$$\frac{d}{dt}\|u_x(t)\|^2_{L^2(0,1)} + \|au_{xx}(t)\|^2_{L^2_{\frac{1}{a}}(0,1)} \leq \|h\|^2_{L^2_{\frac{1}{a}}(0,1)}$$

for every $t \leq T$, so that we find $C'_T > 0$ such that

$$\|u_x(t)\|^2_{L^2(0,1)} + \int_0^T \|au_{xx}(t)\|^2_{L^2_{\frac{1}{a}}(0,1)} dt \leq C'_T \left(\|u_x(0)\|_{L^2(0,1)} + \|h\|^2_{L^2_{\frac{1}{a}}(Q_T)} \right) \tag{2.11}$$

for every $t \leq T$.

Finally, from $u_t = au_{xx} + h$, squaring and integrating, using the fact that $a^2 \leq ca$ for some $c > 0$, we find

$$\int_0^T \|u_t(t)\|^2_{L^2(0,1)} \leq C \left(\int_0^T \|au_{xx}\|^2_{L^2_{\frac{1}{a}}(0,1)} + \|h\|^2_{L^2_{\frac{1}{a}}(Q_T)} \right),$$

and together with (2.11) we find

$$\int_0^T \|u_t(t)\|^2_{L^2(0,1)} \leq C \left(\|u_x(0)\|_{L^2(0,1)} + \|h\|^2_{L^2_{\frac{1}{a}}(Q_T)} \right). \tag{2.12}$$

In conclusion, (2.9), (2.11) and (2.12) give (2.7). Clearly, (2.6) and (2.7) hold also if $u_0 \in \mathcal{H}^1_{\frac{1}{a}}(0,1)$, since $\mathcal{H}^2_{\frac{1}{a}}(0,1)$ is dense in $\mathcal{H}^1_{\frac{1}{a}}(0,1)$. $\square$

CHAPTER 3

Carleman estimate for non degenerate parabolic problems with non smooth coefficient

1. Preliminaries

In this chapter we prove Carleman estimates in the non degenerate case, but in the case in which the coefficient of the operator is *globally non* smooth, in the stream of [**33**], thus improving [**10**], [**22**], [**37**], [**39**] and [**41**].

Fix two real numbers $A < B$ and consider the problem

$$\begin{cases} v_t + \mathcal{A}v = h, & (t,x) \in (0,T) \times (A,B), \\ v(t,A) = v(t,B) = 0, & t \in (0,T). \end{cases} \tag{3.1}$$

Here we suppose that in a case a is of class $W^{1,1}(A,B) \subset BV(A,B)$, but no additional smoothness condition is required in some subsets, as in the previous related papers, and in the other case we assume that a is of class $W^{1,\infty}(A,B)$. More precisely, we assume to deal with a non degenerate problem with a coefficient a satisfying one of the two conditions below:

HYPOTHESIS 3.1.

(a_1) $a \in W^{1,1}(A,B)$, $a \geq a_0 > 0$ in (A,B) and there exist two functions $\mathfrak{g} \in L^1(A,B)$, $\mathfrak{h} \in W^{1,\infty}(A,B)$ and two strictly positive constants $\mathfrak{g}_0$, $\mathfrak{h}_0$ such that $\mathfrak{g}(x) \geq \mathfrak{g}_0$ for a.e. x in $[A,B]$ and

$$-\frac{a'(x)}{2\sqrt{a(x)}}\left(\int_x^B \mathfrak{g}(t)dt + \mathfrak{h}_0\right) + \sqrt{a(x)}\mathfrak{g}(x) = \mathfrak{h}(x) \quad \text{for a.e. } x \in [A,B],$$

in the divergence case,

$$\frac{a'(x)}{2\sqrt{a(x)}}\left(\int_x^B \mathfrak{g}(t)dt + \mathfrak{h}_0\right) + \sqrt{a(x)}\mathfrak{g}(x) = \mathfrak{h}(x) \quad \text{for a.e. } x \in [A,B],$$

in the non divergence one, or

(a_2) $a \in W^{1,\infty}(A,B)$ and $a \geq a_0 > 0$ in (A,B).

REMARK 2. Of course, the first equality in (a_1) can be written as

$$-\left[\sqrt{a(x)}\left(\int_x^B \mathfrak{g}(t)dt + \mathfrak{h}_0\right)\right]' = \mathfrak{h}(x),$$

and the second one as

$$-a(x)\left(\frac{\int_x^B \mathfrak{g}(t)dt + \mathfrak{h}_0}{\sqrt{a(x)}}\right)' = \mathfrak{h}(x).$$

Example 3.1. Let us fix $(A,B) = (0,1)$. In the divergence case, if $a(x) = 2-\sqrt{1-x}$, we can choose $\mathfrak{h}_0 = 1$, $\mathfrak{h} = 0$ and

$$\mathfrak{g}(x) = \frac{\sqrt{2}}{4\sqrt{1-x}} a^{-3/2} \geq \frac{1}{8} := \mathfrak{g}_0;$$

in the non divergence case, if $a(x) = \sqrt{2-x}$, we can choose $\mathfrak{h}_0 = 1$, $\mathfrak{h} = 0$ and

$$\mathfrak{g}(x) = \frac{1}{4a^{3/2}} \geq \frac{1}{8\sqrt{2}} := \mathfrak{g}_0.$$

Now, let us introduce the function $\Phi(t,x) := \Theta(t)\psi(x)$, where

$$\Theta(t) := \frac{1}{[t(T-t)]^4} \tag{3.2}$$

and

$$\psi(x) := \begin{cases} -r\left[\int_A^x \frac{1}{\sqrt{a(t)}}\int_t^B \mathfrak{g}(s)dsdt + \int_A^x \frac{\mathfrak{h}_0}{\sqrt{a(t)}}dt\right] - \mathfrak{c}, & \text{if } (a_1) \text{ holds,}\\ e^{r\zeta(x)} - \mathfrak{c}, & \text{if } (a_2) \text{ holds.}\end{cases} \tag{3.3}$$

Here

$$\zeta(x) = \mathfrak{d}\int_x^B \frac{1}{a(t)}dt,$$

where $\mathfrak{d} = \|a'\|_{L^\infty(A,B)}$, $r > 0$ and $\mathfrak{c} > 0$ is chosen in the second case in such a way that $\max_{[A,B]} \psi < 0$.

Remark 3. Hypothesis 3.1 lets us treat *non smooth* coefficients in the whole spatial domain. To our best knowledge, this is the first case in which such a situation is studied, and for this we need a technical assumption, precisely represented by our hypothesis. However, we believe that, since non smooth coefficients are present, some conditions must be imposed, otherwise it would be impossible to differentiate and obtain the desired Carleman estimates.

2. The divergence case.

Our related Carleman estimate for the problem in divergence form is the following:

Theorem 3.1. *Assume Hypothesis* 3.1. *Then, there exist three positive constants* C, s_0 *and* r *such that every solution* v *of* (3.1) *in*

$$\mathcal{V}_1 := L^2\big(0,T;H^2_a(A,B)\big) \cap H^1\big(0,T;H^1_a(A,B)\big)$$

satisfies, for all $s \geq s_0$,

$$\begin{aligned}&\int_0^T\int_A^B \left(s\Theta(v_x)^2 + s^3\Theta^3v^2\right)e^{2s\Phi}dxdt\\ &\leq C\left(\int_0^T\int_A^B h^2e^{2s\Phi}dxdt - sr\int_0^T\left[a^{3/2}e^{2s\Phi}\Theta\left(\int_x^B \mathfrak{g}(\tau)d\tau + \mathfrak{h}_0\right)(v_x)^2\right]_{x=A}^{x=B}dt\right),\end{aligned} \tag{3.4}$$

if (a_1) *holds and*

$$\begin{aligned}(3.5)\qquad &\int_0^T\int_A^B \left(s\Theta e^{r\zeta}(v_x)^2 + s^3\Theta^3 e^{3r\zeta}v^2\right)e^{2s\Phi}dxdt\\ &\le C\left(\int_0^T\int_A^B h^2 e^{2s\Phi}dxdt - sr\int_0^T \left[ae^{2s\Phi}\Theta e^{r\zeta}(v_x)^2\right]_{x=A}^{x=B}dt\right),\end{aligned}$$

if (a_2) *is in force.*

Here the non degenerate Sobolev spaces are defined as

$$\begin{aligned}H^1_a(A,B) := \big\{&u \text{ is absolutely continuous in } [A,B],\\ &\sqrt{a}u' \in L^2(A,B) \text{ and } u(A)=u(B)=0\big\},\end{aligned}$$

and

$$H^2_a(A,B) := \big\{u \in H^1_a(A,B) |\, au' \in H^1(A,B)\big\},$$

with the related norms.

Observe that, since the function a is non degenerate, $H^1_a(A,B)$ and $H^2_a(A,B)$ coincide with $H^1_0(A,B)$ and $H^2(A,B)\cap H^1_0(A,B)$, respectively.

Remark 4. Obviously, in (3.5) we can delete all factors $e^{r\zeta}$ and $e^{3r\zeta}$, since ζ is non negative and bounded. Indeed, in Chapter 5, we will use such a version. However, we think that inequality (3.5) is more interesting due to the presence of the weights.

Let us proceed with the proof of Theorem 3.1. For $s>0$, define the function

$$w(t,x) := e^{s\Phi(t,x)}v(t,x),$$

where v is any solution of (3.1) in $\mathcal{V}_1$; observe that, since $v\in\mathcal{V}_1$ and $\psi<0$, then $w\in\mathcal{V}_1$. Of course, w satisfies

$$(3.6)\qquad \begin{cases}(e^{-s\Phi}w)_t + \left(a(e^{-s\Phi}w)_x\right)_x = h, & (t,x)\in(0,T)\times(A,B),\\ w(t,A)=w(t,B)=0, & t\in(0,T),\\ w(T^-,x)=w(0^+,x)=0, & x\in(A,B).\end{cases}$$

The previous problem can be recast as follows. Set

$$Lv := v_t + (av_x)_x \quad\text{and}\quad L_s w = e^{s\Phi}L(e^{-s\Phi}w), \quad s>0.$$

Then (3.6) becomes

$$(3.7)\qquad \begin{cases}L_s w = e^{s\Phi}h,\\ w(t,A)=w(t,B)=0, & t\in(0,T),\\ w(T^-,x)=w(0^+,x)=0, & x\in(A,B).\end{cases}$$

Computing $L_s w$, one has

$$L_s w = L_s^+ w + L_s^- w,$$

where

$$L_s^+ w := (aw_x)_x - s\Phi_t w + s^2 a(\Phi_x)^2 w,$$

and

$$L_s^- w := w_t - 2sa\Phi_x w_x - s(a\Phi_x)_x w.$$

Moreover,

$$
\begin{aligned}
2\langle L_s^+ w, L_s^- w\rangle &\le 2\langle L_s^+ w, L_s^- w\rangle + \|L_s^+ w\|^2_{L^2(\tilde{Q}_T)} + \|L_s^- w\|^2_{L^2(\tilde{Q}_T)} \\
&= \|L_s w\|^2_{L^2(\tilde{Q}_T)} = \|he^{s\Phi}\|^2_{L^2(\tilde{Q}_T)},
\end{aligned} \tag{3.8}
$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $L^2(\tilde{Q}_T)$ and $\tilde{Q}_T = (0,T) \times (A,B)$. As usual, we will separate the scalar product $\langle L_s^+ w, L_s^- w\rangle$ in distributed terms and boundary terms.

The following lemma is the crucial starting point, which will be used also in the degenerate cases; for this reason, some comments refer to the degenerate situation.

LEMMA 3.1. *The following identity holds:*

(3.9)

$$
\left.
\begin{aligned}
&\langle L_s^+ w, L_s^- w\rangle \\
&= \frac{s}{2}\int_0^T\int_A^B \Phi_{tt} w^2 dxdt + s^3 \int_0^T\int_A^B \big(2a\Phi_{xx} + a'\Phi_x\big) a(\Phi_x)^2 w^2 dxdt \\
&- 2s^2 \int_0^T\int_A^B a\Phi_x\Phi_{tx} w^2 dxdt + s\int_0^T\int_A^B (2a\Phi_{xx} + a'\Phi_x) a(w_x)^2 dxdt \\
&+ s\int_0^T\int_A^B a(a\Phi_x)_{xx} w w_x dxdt
\end{aligned}
\right\} \{D.T.\}
$$

$$
\{B.T.\}
\left\{
\begin{aligned}
&+\int_0^T [a w_x w_t]_{x=A}^{x=B} dt - \frac{s}{2}\int_A^B [w^2\Phi_t]_{t=0}^{t=T} dx + \frac{s^2}{2}\int_A^B [a(\Phi_x)^2 w^2]_{t=0}^{t=T} dt \\
&-\frac{1}{2}\int_A^B [a(w_x)^2]_{t=0}^{t=T} dx + \int_0^T [-sa(a\Phi_x)_x w w_x]_{x=A}^{x=B} dt \\
&+\int_0^T [-s\Phi_x a^2 (w_x)^2 + s^2 a\Phi_t\Phi_x w^2 - s^3 a^2(\Phi_x)^3 w^2]_{x=A}^{x=B} dt.
\end{aligned}
\right.
$$

PROOF. It *formally* reminds the proof of [**3**, Lemma 3.4] in $(0,1)$, but therein all the calculations were immediately motivated due to the choice of the domain of the operator: in particular, a was assumed to be of class C^1 with the unique possible exception of the degeneracy point $x = 0$, where Dirichlet boundary conditions were imposed in the (WD) case and the condition $(au_x)(0) = 0$ was assumed in the (SD) case, thus making all integration by parts possible.

Now integrations by parts are not immediately justified, since, at least in the (SD) case - or if (a_1) holds -, the unknown function is *not* in a Sobolev space of the whole interval (A,B), and so different motivations are necessary; moreover, the boundary condition for the (SD) case chosen in [**3**] corresponds exactly to the one which characterizes the domain of the operator, and of course this fact makes life easier.

Here, we start noticing that all integrals appearing in $\langle L_s^+ w, L_s^- w\rangle$ are well defined both in the non degenerate case and in the degenerate case by Lemma 2.1, as simple calculations show, recalling that $w = e^{s\Phi}v$. Then, in the following, we perform *formal* calculations, that we will justify accurately in Appendix A.

Let us start with

$$
\begin{aligned}
&\int_0^T\int_A^B L_s^+ w w_t dxdt = \int_0^T\int_A^B \{(aw_x)_x - s\Phi_t w + s^2 a(\Phi_x)^2 w\} w_t dxdt \\
&= \int_0^T [aw_x w_t]_{x=A}^{x=B} dt - \int_0^T \frac{1}{2}\frac{d}{dt}\left(\int_A^B a(w_x)^2 dx\right) dt \\
&- \frac{s}{2}\int_A^B dx \int_0^T \Phi_t (w^2)_t dt + \frac{s^2}{2}\int_A^B dx \int_0^T a(\Phi_x)^2 (w^2)_t dt \\
&= \int_0^T [aw_x w_t]_{x=A}^{x=B} dt - \frac{s}{2}\int_A^B [w^2\Phi_t]_{t=0}^{t=T} dx + \frac{s^2}{2}\int_A^B [a(\Phi_x)^2 w^2]_{t=0}^{t=T} dt \\
&- \frac{1}{2}\int_A^B [a(w_x)^2]_{t=0}^{t=T} dx + \frac{s}{2}\int_0^T\int_A^B \Phi_{tt} w^2 dxdt \\
&- s^2 \int_0^T\int_A^B a\Phi_x \Phi_{xt} w^2 dxdt.
\end{aligned}
\tag{3.10}
$$

In addition, we have

$$
\begin{aligned}
&\int_0^T\int_A^B L_s^+ w(-2sa\Phi_x w_x) dxdt = -s\int_0^T\int_A^B \Phi_x \left[(aw_x)^2\right]_x dxdt \\
&+ s^2\int_0^T\int_A^B a\Phi_t\Phi_x \left(w^2\right)_x dxdt - s^3\int_0^T\int_A^B a^2(\Phi_x)^3 \left(w^2\right)_x dxdt \\
&= \int_0^T [-s\Phi_x(aw_x)^2 + s^2 a\Phi_t\Phi_x w^2 - s^3 a^2(\Phi_x)^3 w^2]_{x=A}^{x=B} dt \\
&+ s\int_0^T\int_A^B \Phi_{xx}(aw_x)^2 dxdt - s^2\int_0^T\int_A^B (a\Phi_x)_x \Phi_t w^2 \\
&- s^2\int_0^T\int_A^B a\Phi_x\Phi_{tx} w^2 dxdt + s^3\int_0^T\int_A^B [a^2(\Phi_x)^3]_x w^2 dxdt.
\end{aligned}
\tag{3.11}
$$

Finally,

$$
\begin{aligned}
&\int_0^T\int_A^B L_s^+ w(-s(a\Phi_x)_x w) dxdt = \int_0^T [-saw_x w(a\Phi_x)_x]_{x=A}^{x=B} dt \\
&+ s\int_0^T\int_A^B a(a\Phi_x)_{xx} w w_x dxdt + s\int_0^T\int_A^B a(a\Phi_x)_x (w_x)^2 dxdt \\
&+ s^2\int_0^T\int_A^B (a\Phi_x)_x \Phi_t w^2 dxdt - s^3\int_0^T\int_A^B a(\Phi_x)^2 (a\Phi_x)_x w^2 dxdt.
\end{aligned}
\tag{3.12}
$$

Adding (3.10)–(3.12), writing $[a^2(\Phi_x)^3]_x = [a(\Phi_x)^2]_x a\Phi_x + a(\Phi_x)^2(a\Phi_x)_x$, (3.9) follows immediately. □

Now, the crucial step is to prove the following estimates:

LEMMA 3.2. *Assume that Hypothesis* 3.1.(a_1) *holds. Then there exist two positive constants* s_0 *and* C *such that for all* $s \geq s_0$ *the distributed terms of* (3.9) *satisfy the estimate*

$$\begin{aligned}
&\frac{s}{2}\int_0^T\int_A^B \Phi_{tt}w^2dxdt + s^3\int_0^T\int_A^B \left(2a\Phi_{xx}+a'\Phi_x\right)a(\Phi_x)^2w^2dxdt\\
&-2s^2\int_0^T\int_A^B a\Phi_x\Phi_{tx}w^2dxdt + s\int_0^T\int_A^B (2a\Phi_{xx}+a'\Phi_x)a(w_x)^2dxdt\\
&+s\int_0^T\int_A^B a(a\Phi_x)_{xx}ww_xdxdt\\
&\geq Cs\int_0^T\int_A^B \Theta(w_x)^2dxdt + Cs^3\int_0^T\int_A^B \Theta^3w^2dxdt.
\end{aligned}$$

PROOF. Using the definition of Φ, the distributed terms of $\int_0^T\int_0^1 L_s^+wL_s^-wdxdt$ take the form

$$\begin{aligned}
&\frac{s}{2}\int_0^T\int_A^B \ddot{\Theta}\psi w^2dxdt + 2s^3r^3\int_0^T\int_A^B \Theta^3\sqrt{a}\mathfrak{g}\left(\int_x^B \mathfrak{g}(\tau)d\tau+\mathfrak{h}_0\right)^2 w^2dxdt\\
&-2s^2r^2\int_0^T\int_A^B \Theta\dot{\Theta}\left(\int_x^B \mathfrak{g}(\tau)d\tau+\mathfrak{h}_0\right)^2 w^2dxdt\\
&-sr\int_0^T\int_A^B \Theta a\left(\frac{a'}{2\sqrt{a}}\left(\int_x^B \mathfrak{g}(\tau)d\tau+\mathfrak{h}_0\right)-\sqrt{a}g\right)_x ww_xdxdt\\
&+2sr\int_0^T\int_A^B \Theta a\sqrt{a}\mathfrak{g}(w_x)^2dxdt.
\end{aligned}\tag{3.13}$$

Hence, since, by Hypothesis 3.1.(a_1), $\mathfrak{g}\geq\mathfrak{g}_0$ and $a\geq a_0$, we can estimate (3.13) from below in the following way:

$$\begin{aligned}
(3.13) &\geq \frac{s}{2}\int_0^T\int_A^B \ddot{\Theta}\psi w^2dxdt + 2s^3r^3\sqrt{a_0}\mathfrak{g}_0\mathfrak{h}_0^2\int_0^T\int_A^B \Theta^3w^2dxdt\\
&-2s^2r^2\int_0^T\int_A^B \Theta\dot{\Theta}\left(\int_x^B \mathfrak{g}(\tau)d\tau+\mathfrak{h}_0\right)^2 w^2dxdt\\
&+2sr\mathfrak{g}_0a_0\sqrt{a_0}\int_0^T\int_A^B \Theta(w_x)^2dxdt + sr\int_0^T\int_A^B \Theta a\mathfrak{h}'ww_xdxdt.
\end{aligned}$$

Observing that

$$|\Theta\dot{\Theta}|\leq c\Theta^{9/4}, \Theta^\mu\leq c\Theta^\nu \text{ if } 0<\mu<\nu \text{ and } |\ddot{\Theta}|\leq c\Theta^{3/2}\leq c\Theta^3 \tag{3.14}$$

for some positive constants c, we conclude that, for s large enough,

$$\begin{aligned}
&\left|-2s^2r^2\int_0^T\int_A^B \Theta\dot{\Theta}\left(\int_x^B \mathfrak{g}(\tau)d\tau+\mathfrak{h}_0\right)^2 w^2\right|\\
&\leq 2r^2s^2c\left(\int_A^B \mathfrak{g}(\tau)d\tau+\mathfrak{h}_0\right)^2\int_0^T\int_A^B \Theta^3w^2dxdt\\
&\leq \frac{C}{6}s^3\int_0^T\int_A^B \Theta^3w^2dxdt,
\end{aligned}$$

for some $C > 0$ and $s \geq \dfrac{12r^2c\left(\int_A^B \mathfrak{g}(\tau)d\tau + \mathfrak{h}_0\right)^2}{C}$. Moreover, we have

$$\left|\frac{s}{2}\int_0^T\int_A^B \ddot{\Theta}\psi w^2 dxdt\right| \leq sc\max_{[A,B]}|\psi|\int_0^T\int_A^B \Theta^3 w^2 dxdt$$
$$\leq \frac{C}{6}s^3\int_0^T\int_A^B \Theta^3 w^2 dxdt$$

for $s \geq \sqrt{\dfrac{6c\max_{[A,B]}|\psi|}{C}}$ and

$$\left|sr\int_0^T\int_A^B \Theta a\mathfrak{h}' w w_x dxdt\right| \leq \frac{1}{\varepsilon}sr\int_0^T\int_A^B \Theta a^2|\mathfrak{h}'|^2 w^2 dxdt$$
$$+\varepsilon sr\int_0^T\int_A^B \Theta(w_x)^2 dxdt$$
$$\leq \frac{1}{\varepsilon}src\max_{[A,B]}a^2\|\mathfrak{h}'\|^2_{L^\infty(A,B)}\int_0^T\int_A^B \Theta^3 w^2 dxdt + \varepsilon sr\int_0^T\int_A^B \Theta(w_x)^2 dxdt$$
$$\leq \frac{C}{6}s^3\int_0^T\int_A^B \Theta^3 w^2 dxdt + \varepsilon sr\int_0^T\int_A^B \Theta(w_x)^2 dxdt,$$

for $s \geq \sqrt{\dfrac{6\varepsilon^{-1}rc\max_{[A,B]}a^2\|\mathfrak{h}'\|^2_{L^\infty(A,B)}}{C}}$. In conclusion, by the previous inequalities, we find

$$(3.13) \geq s^3\left(2r^3\sqrt{a_0}\mathfrak{g}_0\mathfrak{h}_0^2 - \frac{C}{2}\right)\int_0^T\int_A^B \Theta^3 w^2 dxdt$$
$$+ sr\left(2\mathfrak{g}_0 a_0\sqrt{a_0} - \varepsilon\right)\int_0^T\int_A^B \Theta(w_x)^2 dxdt.$$

Finally, choosing $\varepsilon = \mathfrak{g}_0 a_0\sqrt{a_0}$ and r such that

$$2r^3\sqrt{a_0}\mathfrak{g}_0\mathfrak{h}_0^2 - \frac{C}{2} > 0,$$

the claim follows. □

The counterpart of the previous inequality in the $W^{1,\infty}$ case is the following

LEMMA 3.3. *Assume that Hypothesis* 3.1.(a_2) *holds. Then there exist two positive constants* s_0 *and* C *such that for all* $s \geq s_0$ *the distributed terms of* (3.9) *satisfy the estimate*

$$
\begin{aligned}
&\frac{s}{2}\int_0^T\int_A^B \Phi_{tt}w^2dxdt + s^3\int_0^T\int_A^B \big(2a\Phi_{xx}+a'\Phi_x\big)a(\Phi_x)^2w^2dxdt\\
&-2s^2\int_0^T\int_A^B a\Phi_x\Phi_{tx}w^2dxdt + s\int_0^T\int_A^B (2a\Phi_{xx}+a'\Phi_x)a(w_x)^2dxdt\\
&+s\int_0^T\int_A^B a(a\Phi_x)_{xx}ww_xdxdt\\
&\geq Cs\int_0^T\int_A^B \Theta e^{r\zeta}(w_x)^2dxdt + Cs^3\int_0^T\int_A^B \Theta^3e^{3r\zeta}w^2dxdt.
\end{aligned}
$$

Proof. We proceed as in the proof of the previous Lemma. In this case the distributed terms of $\int_0^T\int_0^1 L_s^+wL_s^-wdxdt$ take the form

$$
\begin{aligned}
&\frac{s}{2}\int_0^T\int_A^B \ddot{\Theta}\psi w^2dxdt + s^3r^3\mathfrak{d}^3\int_0^T\int_A^B \Theta^3\frac{e^{3r\zeta}}{a^2}[2r\mathfrak{d}-a']w^2dxdt\\
&-2s^2r^2\int_0^T\int_A^B \Theta\dot{\Theta}\frac{e^{2r\zeta}}{a}w^2dxd + sr\mathfrak{d}\int_0^T\int_A^B \Theta e^{r\zeta}[2r\mathfrak{d}-a'](w_x)^2dxdt\\
&-sr^2\mathfrak{d}^2\int_0^T\int_A^B \frac{1}{a}\Theta e^{r\zeta}[r\mathfrak{d}+a']ww_xdxdt.
\end{aligned}
\tag{3.15}
$$

By Hypothesis 3.1.(a_2), choosing $r>1$, one has

$$
\mathfrak{d}^3(2r\mathfrak{d}-a') \geq \|a'\|^4_{L^\infty(A,B)} \quad\text{and}\quad \mathfrak{d}(2r\mathfrak{d}-a')\geq \|a'\|^2_{L^\infty(A,B)},
$$

thus

$$
\begin{aligned}
(3.15) \geq\ &\frac{s}{2}\int_0^T\int_A^B \ddot{\Theta}\psi w^2dxdt + \frac{s^3r^3\|a'\|^4_{L^\infty(A,B)}}{\max a^2}\int_0^T\int_A^B \Theta^3e^{3r\zeta}w^2dxdt\\
&-\frac{2s^2r^2}{a_0}\int_0^T\int_A^B |\Theta\dot{\Theta}|e^{2r\zeta}w^2dxdt + sr\|a'\|^2_{L^\infty(A,B)}\int_0^T\int_A^B \Theta e^{r\zeta}(w_x)^2dxdt\\
&-sr^2\mathfrak{d}^2\int_0^T\int_A^B \frac{1}{a}\Theta e^{r\zeta}[r\mathfrak{d}+a']ww_xdxdt.
\end{aligned}
\tag{3.16}
$$

Using the estimates in (3.14), we conclude that, for s large enough,

$$
\begin{aligned}
&\left|-\frac{2s^2r^2}{a_0}\int_0^T\int_A^B |\Theta\dot{\Theta}|e^{2r\zeta}w^2dxdt\right| \leq \frac{2r^2c}{a_0\min e^{r\zeta}}s^2\int_0^T\int_A^B \Theta^3e^{3r\zeta}w^2dxdt\\
&\leq \frac{C}{6}s^3\int_0^T\int_A^B \Theta^3e^{3r\zeta}w^2dxdt,
\end{aligned}
\tag{3.17}
$$

for some $C>0$ and $s\geq \dfrac{12r^2c}{Ca_0\min e^{r\zeta}}$. Moreover, we have

$$
\begin{aligned}
\left|\frac{s}{2}\int_0^T\int_A^B \ddot{\Theta}\psi w^2dxdt\right| &\leq \frac{s}{2}c\max_{[A,B]}|\psi|\int_0^T\int_A^B \Theta^3w^2dxdt\\
&\leq \frac{cs\max_{[A,B]}|\psi|}{\min e^{3r\zeta}}\int_0^T\int_A^B \Theta^3e^{3r\zeta}w^2dxdt\\
&\leq \frac{C}{6}s^3\int_0^T\int_A^B \Theta^3e^{3r\zeta}w^2dxdt
\end{aligned}
\tag{3.18}
$$

for $s \geq \sqrt{\dfrac{6c\max_{[A,B]}|\psi|}{C\min e^{3r\zeta}}}$ and

$$\left|-sr^2\mathfrak{d}^2\int_0^T\int_A^B\frac{1}{a}\Theta e^{r\zeta}[r\mathfrak{d}+a']ww_xdxdt\right|$$
$$\leq 2sr^3\|a'\|^3_{L^\infty(A,B)}\frac{1}{a_0}\int_0^T\int_A^B\Theta e^{r\zeta}|ww_x|dxdt$$
$$\leq sr^3\frac{\|a'\|^3_{L^\infty(A,B)}}{\varepsilon a_0}\int_0^T\int_A^B\Theta e^{r\zeta}w^2dxdt$$
$$+ sr^3\frac{\varepsilon\|a'\|^3_{L^\infty(A,B)}}{a_0}\int_0^T\int_A^B\Theta e^{r\zeta}(w_x)^2dxdt$$
$$\leq sr^3c\frac{\|a'\|^3_{L^\infty(A,B)}}{\min e^{2r\zeta}a_0\varepsilon}\int_0^T\int_A^B\Theta^3e^{3r\zeta}w^2dxdt+$$
$$sr^3\frac{\varepsilon\|a'\|^3_{L^\infty(A,B)}}{a_0}\int_0^T\int_A^B\Theta e^{r\zeta}(w_x)^2dxdt$$
$$\leq \frac{C}{6}s^3\int_0^T\int_A^B\Theta^3e^{3r\zeta}w^2dxdt + sr^3\frac{\varepsilon\|a'\|^3_{L^\infty(A,B)}}{a_0}\int_0^T\int_A^B\Theta e^{r\zeta}(w_x)^2dxdt$$

for $s \geq \sqrt{\dfrac{6r^3c\frac{\|a'\|^3_{L^\infty(A,B)}}{\min e^{2r\zeta}a_0\varepsilon}}{C}}$.

In conclusion, by the previous inequalities, we obtain

$$(3.15) \geq \left(\frac{r^3\|a'\|^4_{L^\infty(A,B)}}{\max a^2}-\frac{C}{2}\right)s^3\int_0^T\int_A^B\Theta^3e^{3r\zeta}w^2dxdt$$
$$+ sr\|a'\|^2_{L^\infty(A,B)}\left(1-r^2\frac{\varepsilon\|a'\|_{L^\infty(A,B)}}{a_0}\right)\int_0^T\int_A^B\Theta e^{r\zeta}(w_x)^2dxdt.$$

Finally, choosing $\varepsilon = \dfrac{a_0}{2r^2\|a'\|_{L^\infty(A,B)}}$ and $r > 1$ such that

$$\frac{r^3\|a'\|^4_{L^\infty(A,B)}}{\max a^2}-\frac{C}{2} > 0,$$

the claim follows.

□

Concerning the boundary terms in (3.9), we have

Lemma 3.4. *The boundary terms in* (3.9) *reduce to*

$$sr\int_0^T\left[a^{3/2}\Theta\left(\int_x^B\mathfrak{g}(\tau)d\tau+\mathfrak{h}_0\right)(w_x)^2\right]_{x=A}^{x=B}dt$$

if (a_1) *holds and*

$$sr\|a'\|_{L^\infty(A,B)}\int_0^T[a\Theta e^{r\zeta}(w_x)^2]_{x=A}^{x=B}dt$$

if (a_2) *holds.*

Proof. First of all, since $w \in \mathcal{V}_1$, then $w \in C\big([0,T]; H^1(A,B)\big)$. Thus $w(0,x)$, $w(T,x)$, $w_x(0,x)$, $w_x(T,x)$, $(w_x)^2(t,B)$ and $(w_x)^2(t,A)$ are indeed well defined. Moreover, we have that $w_t(t,A)$ and $w_t(t,B)$ make sense and are actually 0.

But also $w_x(t,A)$ and $w_x(t,B)$ are well defined. In fact $w(t,\cdot) \in H^2(A,B)$ and $w_x(t,\cdot) \in W^{1,2}(A,B) \subset C([A,B])$. Thus $\int_0^T [a w_x w_t]_{x=A}^{x=B} dt$ is well defined and actually equals 0, as we get using the boundary conditions on w. Thus, using the boundary conditions of $w = e^{s\Phi} v$ with $v \in \mathcal{V}_1$, we get

$$[w^2\Phi_t]_{t=0}^{t=T} = [e^{2s\Phi} v^2 \Phi_t]_{t=0}^{t=T} = 0,$$

$$[a(\Phi_x)^2 w^2]_{t=0}^{t=T} = \begin{cases} [\Theta^2 r^2 \left(\int_x^B \mathfrak{g}(\tau)d\tau + \mathfrak{h}_0\right)^2 e^{2s\Phi} v^2]_{t=0}^{t=T} = 0, & \text{if } (a_1) \text{ holds},\\ [\Theta^2 e^{2r\zeta} r^2 e^{2s\Phi} v^2]_{t=0}^{t=T} = 0, & \text{if } (a_2) \text{ holds}, \end{cases}$$

$$[a(w_x)^2]_{t=0}^{t=T} = \begin{cases} \left[a e^{2s\Phi} \left(-\frac{sr}{\sqrt{a}} \Theta \left(\int_x^B \mathfrak{g}(\tau) d\tau + \mathfrak{h}_0 \right) v + v_x \right)^2 \right]_{t=0}^{t=T} = 0, & \text{if } (a_1) \text{ holds},\\ \left[a e^{2s\Phi} \left(-\dfrac{sr\|a'\|_{L^\infty(A,B)} \Theta e^{r\zeta}}{a} v + v_x \right)^2 \right]_{t=0}^{t=T} = 0, & \text{if } (a_2) \text{ holds}. \end{cases}$$

Finally, all integrals involving $[w]_{x=A}^{x=B}$ are obviously 0, so that the boundary terms in (3.9) reduce to

$$\begin{aligned} &- s \int_0^T [\Phi_x a^2 (w_x)^2]_{x=A}^{x=B} dt \\ &= \begin{cases} sr \int_0^T \left[a^{3/2} \Theta \left(\int_x^B \mathfrak{g}(\tau) d\tau + \mathfrak{h}_0 \right) (w_x)^2 \right]_{x=A}^{x=B} dt, & \text{if } (a_1) \text{ holds},\\ sr \|a'\|_{L^\infty(A,B)} \int_0^T [a\Theta e^{r\zeta} (w_x)^2]_{x=A}^{x=B} dt, & \text{if } (a_2) \text{ holds}. \end{cases} \end{aligned}$$

□

From Lemmas 3.1 - 3.4, we deduce immediately that there exist two positive constants C and s_0, such that all solutions w of (3.6) satisfy, for all $s \geq s_0$,

$$\begin{aligned} \int_0^T \int_A^B L_s^+ w L_s^- w dx dt \geq Cs &\int_0^T \int_A^B \Theta (w_x)^2 dx dt \\ &+ Cs^3 \int_0^T \int_A^B \Theta^3 w^2 dx dt \\ &+ sr \int_0^T \left[a^{3/2} \Theta \left(\int_x^B \mathfrak{g}(\tau) d\tau + \mathfrak{h}_0 \right) (w_x)^2 \right]_{x=A}^{x=B} dt, \end{aligned} \tag{3.19}$$

if (a_1) holds, and

$$\begin{aligned} \int_0^T \int_A^B L_s^+ w L_s^- w dx dt \geq Cs &\int_0^T \int_A^B \Theta e^{r\zeta} (w_x)^2 dx dt \\ &+ Cs^3 \int_0^T \int_A^B \Theta^3 e^{3r\zeta} w^2 dx dt \\ &+ sr \|a'\|_{L^\infty((A,B)} \int_0^T [a\Theta e^{r\zeta} (w_x)^2]_{x=A}^{x=B} dt, \end{aligned} \tag{3.20}$$

if (a_2) holds.

Thus, a straightforward consequence of (3.8), (3.19) and (3.20) is the next result.

PROPOSITION 3.1. *Assume Hypothesis* 3.1. *Then, there exist three positive constants* C, s_0 *and* r *such that all solutions* w *of* (3.6) *in* $\mathcal{V}_1$ *satisfy, for all* $s \geq s_0$,

$$
\begin{aligned}
&s\int_0^T\int_A^B \Theta(w_x)^2dxdt + s^3\int_0^T\int_A^B \Theta^3w^2dxdt\\
&\leq C\left(\int_0^T\int_A^B h^2e^{2s\Phi}dxdt - sr\int_0^T\left[a^{3/2}\Theta\left(\int_x^B \mathfrak{g}(\tau)d\tau + \mathfrak{h}_0\right)(w_x)^2\right]_{x=A}^{x=B}dt\right),
\end{aligned}
$$

if (a_1) *holds and*

$$
\begin{aligned}
&s\int_0^T\int_A^B \Theta e^{r\zeta}(w_x)^2dxdt + s^3\int_0^T\int_A^B \Theta^3e^{3r\zeta}w^2dxdt\\
&\leq C\left(\int_0^T\int_A^B h^2e^{2s\Phi}dxdt - sr\int_0^T\left[a\Theta e^{r\zeta}(w_x)^2\right]_{x=A}^{x=B}dt\right),
\end{aligned}
$$

if (a_2) *does.*

Now, we are ready to conclude the

PROOF OF THEOREM 3.1. Recalling the definition of w, we have $v = e^{-s\Phi}w$ and $v_x = -s\Theta\psi' e^{-s\Phi}w + e^{-s\Phi}w_x$. Thus, recalling that ψ' is bounded, since a is non degenerate, we have that, if (a_1) holds, there exist some $c > 0$ such that

$$
\begin{aligned}
\left(s\Theta(v_x)^2 + s^3\Theta^3v^2\right)e^{2s\Phi} &\leq c\left[s\Theta(s^2\Theta^2e^{-2s\Phi}w^2 + e^{-2s\Phi}(w_x)^2) + s^3\Theta^3e^{-2s\Phi}w^2\right]e^{2s\Phi}\\
&\leq c\left(s^3\Theta^3w^2 + s\Theta(w_x)^2\right).
\end{aligned}
$$

Analogously, if (a_2) holds, we have

$$
\begin{aligned}
\left(s\Theta e^{r\zeta}(v_x)^2 + s^3\Theta^3e^{3r\zeta}v^2\right)e^{2s\Phi} &\leq c\left[s\Theta e^{r\zeta}(s^2\Theta^2w^2 + (w_x)^2) + s^3\Theta^3e^{3r\zeta}w^2\right]\\
&\leq c\left[s\Theta e^{r\zeta}(w_x)^2 + s^3\Theta^3e^{3r\zeta}w^2\right],
\end{aligned}
$$

since $\zeta > 0$. Therefore, there exists a constant $C > 0$ such that

$$
\begin{aligned}
&\int_0^T\int_A^B \left(s\Theta(v_x)^2 + s^3\Theta^3v^2\right)e^{2s\Phi}dxdt\\
&\leq C\int_0^T\int_A^B \left(s\Theta(w_x)^2dxdt + s^3\Theta^3w^2\right)dxdt,
\end{aligned}
$$

if (a_1) holds, and

$$
\begin{aligned}
&\int_0^T\int_A^B \left(s\Theta e^{r\zeta}(v_x)^2 + s^3\Theta^3e^{3r\zeta}v^2\right)e^{2s\Phi}dxdt\\
&\leq C\int_0^T\int_A^B \left(s\Theta e^{r\zeta}(w_x)^2dxdt + s^3\Theta^3e^{3r\zeta}w^2\right)dxdt,
\end{aligned}
$$

if (a_2) holds. By Proposition 3.1, Theorem 3.1 follows at once. □

3. The non divergence case

For the problem in non divergence form Theorem 3.1 becomes

THEOREM 3.2. *Assume Hypothesis* 3.1. *Then, there exist three positive constants* C, s_0 *and* r *such that every solution* $v \in \mathcal{V}_2 := L^2\big(0,T;H^2_{\frac{1}{a}}(A,B)\big) \cap H^1\big(0,T;H^1_{\frac{1}{a}}(A,B)\big)$ *of*

$$\begin{cases} v_t + av_{xx} = h, & (t,x) \in (0,T)\times(A,B),\\ v(t,A) = v(t,B) = 0, & t \in (0,T), \end{cases} \tag{3.21}$$

satisfies, for all $s \geq s_0$,

$$\int_0^T \int_A^B \left(s\Theta(v_x)^2 + s^3\Theta^3 v^2\right) e^{2s\Phi} dxdt$$
$$\leq C\left(\int_0^T \int_A^B h^2 e^{2s\Phi} dxdt - sr\int_0^T \Theta(t)\left[\sqrt{a}\left(\int_x^B \mathfrak{g}(\tau)d\tau + \mathfrak{h}_0\right)(v_x)^2 e^{2s\Phi}\right]_{x=A}^{x=B} dt\right),$$

if (a_1) *holds and* (3.5) *if* (a_2) *is in force.*

Here $H^1_{\frac{1}{a}}(A,B)$ and $H^2_{\frac{1}{a}}(A,B)$ are formally defined as in the degenerate case.

REMARK 5. Remark 4 still holds also for the previous theorem.

3.1. Proof of Theorem 3.2 when (a_1) holds:

We proceed as in Chapter 2. For $s > 0$, define the function

$$w(t,x) := e^{s\Phi(t,x)}v(t,x),$$

where v is any solution of (3.1) in $\mathcal{V}_2$; observe that, since $v \in \mathcal{V}_2$ and $\psi < 0$, then $w \in \mathcal{V}_2$. Of course, w satisfies

$$\begin{cases} (e^{-s\Phi}w)_t + a(e^{-s\Phi}w)_{xx} = h, & (t,x) \in (0,T)\times(A,B),\\ w(t,A) = w(t,B) = 0, & t \in (0,T),\\ w(T^-,x) = w(0^+,x) = 0, & x \in (A,B). \end{cases} \tag{3.22}$$

Setting

$$Lv := v_t + av_{xx} \quad \text{and} \quad L_s w = e^{s\Phi}L(e^{-s\Phi}w), \quad s > 0,$$

then (3.22) becomes

$$\begin{cases} L_s w = e^{s\Phi}h,\\ w(t,A) = w(t,B) = 0, & t \in (0,T),\\ w(T^-,x) = w(0^+,x) = 0, & x \in (A,B). \end{cases} \tag{3.23}$$

Computing $L_s w$, one has

$$L_s w = L_s^+ w + L_s^- w,$$

where

$$L_s^+ w := aw_{xx} - s\Phi_t w + s^2 a(\Phi_x)^2 w,$$

and

$$L_s^- w := w_t - 2sa\Phi_x w_x - sa\Phi_{xx} w.$$

Moreover, similarly to (3.8), we have

$$2\langle L_s^+ w, L_s^- w\rangle_{L^2_{\frac{1}{a}}(\tilde{Q}_T)} \leq \|he^{s\Phi}\|^2_{L^2_{\frac{1}{a}}(\tilde{Q}_T)}, \tag{3.24}$$

where $\langle \cdot, \cdot \rangle_{L^2_{\frac{1}{a}}(\tilde{Q}_T)}$ denotes the usual scalar product in $L^2_{\frac{1}{a}}(\tilde{Q}_T)$ and $\tilde{Q}_T = (0,T) \times (A,B)$. As usual, separating the scalar product $\langle L_s^+ w, L_s^- w\rangle_{L^2_{\frac{1}{a}}(\tilde{Q}_T)}$ in distributed terms and boundary terms, we obtain

LEMMA 3.5. *The following identity holds:*
(3.25)

$$
\begin{aligned}
< L_s^+ w, L_s^- w >_{L^2_{\frac{1}{a}}(\tilde{Q}_T)} \;=\; & \left.\begin{aligned} & s\int_{\tilde{Q}_T} (a\Phi_{xx} + (a\Phi_x)_x)(w_x)^2 dxdt \\ & + s^3 \int_{\tilde{Q}_T} (\Phi_x)^2 (a\Phi_{xx} + (a\Phi_x)_x) w^2 dxdt \\ & - 2s^2 \int_{\tilde{Q}_T} \Phi_x \Phi_{xt} w^2 dxdt + \frac{s}{2}\int_{\tilde{Q}_T} \frac{\Phi_{tt}}{a} w^2 dxdt \\ & + s\int_{\tilde{Q}_T} (a\Phi_{xx})_x w w_x dxdt \end{aligned}\right\} \{D.T.\} \\
\{B.T.\} & \left\{\begin{aligned} & -\frac{1}{2}\int_A^B \Big[(w_x)^2\Big]_0^T dx + \int_0^T \Big[w_x w_t\Big]_A^B dt \\ & -s\int_0^T \Big[a\Phi_x (w_x)^2\Big]_A^B dt - s\int_0^T \Big[a\Phi_{xx} w w_x\Big]_A^B dt \\ & +\frac{1}{2} s \int_A^B \Big[\Big(s(\Phi_x)^2 - \frac{\Phi_t}{a}\Big) w^2\Big]_0^T dx \\ & -s^2 \int_0^T \Big[\big(sa(\Phi_x)^3 - \Phi_x\Phi_t\big) w^2\Big]_A^B dt. \end{aligned}\right.
\end{aligned}
$$

PROOF. It is an adaptation of the proof of [**16**, Lemma 3.8] to which we refer. Let us simply remark that in our case all integrals and integrations by parts are justified by the definition of $H^2_{\frac{1}{a}}(A,B)$ and by the regularity of the functions $\mathfrak{g}$ and $\mathfrak{h}$. □

Now, the crucial step is to prove the following estimates:

LEMMA 3.6. *Assume that Hypothesis* 3.1.(a_1) *holds. Then there exist two positive constants* s_0 *and* C *such that for all* $s \ge s_0$ *the distributed terms of* (3.25) *satisfy the estimate*

$$
\begin{aligned}
& s\int_0^T\int_A^B (a\Phi_{xx} + (a\Phi_x)_x)(w_x)^2 dxdt + s^3\int_0^T\int_A^B (\Phi_x)^2 (a\Phi_{xx} + (a\Phi_x)_x) w^2 dxdt \\
& - 2s^2 \int_0^T\int_A^B \Phi_x\Phi_{xt} w^2 dxdt + \frac{s}{2}\int_0^T\int_A^B \frac{\Phi_{tt}}{a} w^2 dxdt + s\int_{\tilde{Q}_T} (a\Phi_{xx})_x w w_x dxdt \\
& \ge Cs\int_0^T\int_A^B \Theta (w_x)^2 dxdt + Cs^3 \int_0^T\int_A^B \Theta^3 w^2 dxdt.
\end{aligned}
$$

PROOF. Using the definition of Φ, the distributed terms of $\int_{Q_T} \frac{1}{a} L_s^+ w L_s^- w dxdt$ take the form
(3.26)

$$\frac{s}{2}\int_0^T\int_A^B \frac{1}{a}\ddot{\Theta}\psi w^2 dxdt + 2s^3r^3\int_0^T\int_A^B \frac{1}{\sqrt{a}}\Theta^3\mathfrak{g}\left(\int_x^B \mathfrak{g}(\tau)d\tau+\mathfrak{h}_0\right)^2 w^2 dxdt$$
$$-2s^2r^2\int_0^T\int_A^B \frac{1}{a}\Theta\dot{\Theta}\left(\int_x^B \mathfrak{g}(\tau)d\tau+\mathfrak{h}_0\right)^2 w^2dxdt + 2sr\int_0^T\int_A^B \Theta\sqrt{a}\mathfrak{g}(w_x)^2dxdt$$
$$+ sr\int_0^T\int_A^B \Theta\mathfrak{h}' w w_x dxdt.$$

Hence, since, by Hypothesis 3.1.(a_1), $\mathfrak{g}\geq\mathfrak{g}_0$ and $a\geq a_0$, we can estimate (3.26) from below in the following way:

$$(3.26) \geq \frac{s}{2}\int_0^T\int_A^B \frac{1}{a}\ddot{\Theta}\psi w^2 dxdt + 2s^3r^3\frac{1}{\max_{[A,B]}\sqrt{a}}\mathfrak{g}_0\mathfrak{h}_0^2\int_0^T\int_A^B \Theta^3 w^2dxdt$$
$$-2s^2r^2\frac{1}{a_0}\int_0^T\int_A^B \Theta|\dot{\Theta}|\left(\int_x^B \mathfrak{g}(\tau)d\tau+\mathfrak{h}_0\right)^2 w^2dxdt + 2sr\mathfrak{g}_0\sqrt{a_0}\int_0^T\int_A^B \Theta(w_x)^2dxdt$$
$$+ sr\int_0^T\int_A^B \Theta\mathfrak{h}' w w_x dxdt.$$

By the estimates in (3.14), we conclude that, for s large enough,

$$\begin{aligned} &2s^2r^2\frac{1}{a_0}\int_0^T\int_A^B \Theta|\dot{\Theta}|\left(\int_x^B \mathfrak{g}(\tau)d\tau+\mathfrak{h}_0\right)^2 w^2 \\ &\leq 2r^2s^2\frac{1}{a_0}c\left(\int_A^B \mathfrak{g}(\tau)d\tau+\mathfrak{h}_0\right)^2\int_0^T\int_A^B \Theta^3w^2dxdt \\ &\leq \frac{C}{6}s^3\int_0^T\int_A^B \Theta^3w^2dxdt, \end{aligned}$$

for some $C>0$ and $s\geq \dfrac{12r^2c\left(\int_A^B \mathfrak{g}(\tau)d\tau+\mathfrak{h}_0\right)^2}{Ca_0}$. Moreover, we have

$$\begin{aligned} \left|\frac{s}{2}\int_0^T\int_A^B \frac{1}{a}\ddot{\Theta}\psi w^2dxdt\right| &\leq sc\frac{1}{a_0}\max_{[A,B]}|\psi|\int_0^T\int_A^B \Theta^3w^2dxdt \\ &\leq \frac{C}{6}s^3\int_0^T\int_A^B \Theta^3w^2dxdt \end{aligned}$$

for $s \geq \sqrt{\dfrac{6c\max_{[A,B]}|\psi|}{Ca_0}}$, and

$$\left|sr\int_0^T\int_A^B \Theta\mathfrak{h}'ww_x dxdt\right| \leq \frac{1}{\varepsilon}sr\int_0^T\int_A^B\Theta|\mathfrak{h}'|^2w^2dxdt + \varepsilon sr\int_0^T\int_A^B\Theta(w_x)^2dxdt$$
$$\leq \frac{1}{\varepsilon}src\|\mathfrak{h}'\|^2_{L^\infty(A,B)}\int_0^T\int_A^B\Theta^3w^2dxdt + \varepsilon sr\int_0^T\int_A^B\Theta(w_x)^2dxdt$$
$$\leq \frac{C}{6}s^3\int_0^T\int_A^B\Theta^3w^2dxdt + \varepsilon sr\int_0^T\int_A^B\Theta(w_x)^2dxdt$$

for $s \geq \sqrt{\dfrac{6\varepsilon^{-1}rc\|\mathfrak{h}'\|^2_{L^\infty(A,B)}}{C}}$. In conclusion, by the previous inequalities, we find

$$(3.26) \geq s^3\left(2r^3\frac{1}{\max_{[A,B]}\sqrt{a}}\mathfrak{g}_0\mathfrak{h}_0^2 - \frac{C}{2}\right)\int_0^T\int_A^B\Theta^3w^2dxdt$$
$$+ sr\left(2\mathfrak{g}_0\sqrt{a_0}-\varepsilon\right)\int_0^T\int_A^B\Theta(w_x)^2dxdt$$

for some $C>0$ and s large enough.

Finally, choosing $\varepsilon = \mathfrak{g}_0\sqrt{a_0}$ and r such that

$$2r^3\frac{1}{\max_{[A,B]}\sqrt{a}}\mathfrak{g}_0\mathfrak{h}_0^2 - \frac{C}{2} > 0,$$

the claim follows. □

Concerning the boundary terms in (3.25), we have

Lemma 3.7. *The boundary terms in* (3.25) *reduce to*

$$sr\int_0^T\Theta(t)\left[\sqrt{a}\left(\int_x^B\mathfrak{g}(\tau)d\tau + \mathfrak{h}_0\right)(w_x)^2\right]_{x=A}^{x=B}dt.$$

Proof. Using the definition of Φ we have that the boundary terms become

$$(3.27)\quad \begin{aligned}\{B.T.\} \;=\; & -\frac{1}{2}\int_A^B\Big[(w_x)^2\Big]_{t=0}^{t=T}dx + \int_0^T\Big[w_xw_t\Big]_{x=A}^{x=B}dt\\ & + \frac{1}{2}\int_A^B\Big[\Big(s^2\Theta^2(\psi')^2 - \frac{s}{a}\dot{\Theta}\psi\Big)w^2\Big]_{t=0}^{t=T}dx - s\int_0^T\Theta(t)\Big[a\psi'(w_x)^2\Big]_{x=A}^{x=B}dt\\ & - s\int_0^T\Theta(t)\Big[a\psi''ww_x\Big]_{x=A}^{x=B}dt - s^3\int_0^T\Theta^3(t)\Big[a(\psi')^3w^2\Big]_{x=A}^{x=B}dt\\ & + s^2\int_0^T\Theta(t)\dot{\Theta}(t)\Big[\psi\psi'w^2\Big]_{x=A}^{x=B}dt.\end{aligned}$$

Since $w \in \mathcal{V}_2$, $w(0,x)$, $w(T,x)$, $w_x(0,x)$, $w_x(T,x)$ and $\int_A^B\left[w_x^2\right]_{t=0}^{t=T}dx$ are well defined, using the boundary conditions and the definition of w itself, we get

$$\int_A^B\Big[-\frac{1}{2}(w_x)^2 + \frac{1}{2}\Big(s^2\Theta^2(\psi')^2 - \frac{s}{a}\dot{\Theta}\psi\Big)w^2\Big]_{t=0}^{t=T}dx = 0.$$

Moreover, since $w \in \mathcal{V}_2$, we have that $w_t(t,A)$ and $w_t(t,B)$ make sense. Therefore, also $w_x(t,A)$ and $w_x(t,B)$ are well defined, since $w(t,\cdot) \in H^2_{\frac{1}{a}}(A,B)$. Thus $\int_0^T [w_x w_t]_{x=A}^{x=B} dt$ is well defined and actually equals 0. Indeed, by the boundary conditions, we find

$$|w_t(t,x)| \le \left(\int_A^B w_{tx}(t,y)^2 dy \right)^{1/2} \max\{\sqrt{x-A}, \sqrt{B-x}\} \to 0$$

as $x \to A$ or $x \to B$, the integral being finite.

Now, $w(t,A)$ and $w(t,B)$ being well defined, by the boundary conditions on w, the other terms of (3.25) reduce to

$$-s \int_0^T \Theta(t) \Big[a\psi'(w_x)^2 \Big]_{x=A}^{x=B} dt = sr \int_0^T \Theta(t) \left[\sqrt{a} \left(\int_x^B \mathfrak{g}(\tau) d\tau + \mathfrak{h}_0 \right) (w_x)^2 \right]_{x=A}^{x=B} dt.$$

□

3.2. Proof of Theorem 3.2 **when** (a_2) **holds:** Now, assume that Hypothesis 3.1.(a_2) holds. Then inequality (3.5) in the non divergence case is a simple consequence of Theorem 3.1:
rewrite the equation of (3.1) as $v_t + (av_x)_x = \bar{h}$, where $\bar{h} := h + a'v_x$. Then, applying Theorem 3.1, there exists two positive constants C and $s_0 > 0$, such that for all $s \ge s_0$,

$$\begin{aligned} &\int_0^T \int_A^B \left(s\Theta e^{r\zeta}(v_x)^2 + s^3\Theta^3 e^{3r\zeta} v^2 \right) e^{2s\Phi} dxdt \\ &\le C \left(\int_0^T \int_A^B \bar{h}^2 e^{2s\Phi} dxdt - sr \int_0^T \left[a\Theta e^{r\zeta}(v_x)^2 e^{2s\Phi} \right]_{x=A}^{x=B} dt \right). \end{aligned} \tag{3.28}$$

Using the definition of $\bar{h}$, the term $\int_{Q_T} \bar{h}^2 e^{2s\Phi(t,x)} dxdt$ can be estimated in the following way

$$\begin{aligned} \int_0^T \int_A^B \bar{h}^2 e^{2s\Phi} dxdt &\le 2 \int_0^T \int_A^B h^2 e^{2s\Phi} dxdt + 2\|a'\|^2_{L^\infty(\tilde{Q}_T)} \int_0^T \int_A^B e^{2s\Phi} (v_x)^2 dxdt \\ &\le 2 \int_0^T \int_A^B h^2 e^{2s\Phi} dxdt + 2\|a'\|^2_{L^\infty(\tilde{Q}_T)} c \int_0^T \int_A^B \Theta e^{r\zeta} e^{2s\Phi} (v_x)^2 dxdt, \end{aligned} \tag{3.29}$$

where $c := \max_{[0,T]} (t(T-t))^4 = \left(\frac{T}{2} \right)^8$. Thus, by (3.28) and (3.29), one has

$$\begin{aligned} &\int_0^T \int_A^B \left(s\Theta e^{r\zeta}(v_x)^2 - 2\|a'\|^2_{L^\infty(\tilde{Q}_T)} c\Theta e^{r\zeta}(v_x)^2 + s^3\Theta^3 e^{3r\zeta} v^2 \right) e^{2s\Phi} dxdt \\ &\le C \left(\int_0^T \int_A^B h^2 e^{2s\Phi} dxdt - sr \int_0^T \left[a\Theta e^{r\zeta}(v_x)^2 e^{2s\Phi} \right]_{x=A}^{x=B} dt \right). \end{aligned}$$

Now, let $s_1 > 0$ be such that $\dfrac{s_1}{2} \geq 2\|a'\|^2_{L^\infty(\tilde{Q}_T)}c$. Then, for all $s \geq s_1$

$$\int_0^T \int_A^B \left(s\Theta e^{r\zeta}(v_x)^2 - 2\|a'\|^2_{L^\infty(\tilde{Q}_T)} c\Theta e^{r\zeta}(v_x)^2 \right) e^{2s\Phi} dxdt$$
$$\geq \frac{s}{2} \int_0^T \int_A^B \Theta e^{r\zeta}(v_x)^2 e^{2s\Phi} dxdt.$$

Hence the claim follows for all $s \geq \max\{s_0, s_1\}$.

CHAPTER 4

Carleman estimate for degenerate non smooth parabolic problems

In this chapter we prove crucial estimates of Carleman type in presence of a degenerate coefficient. Such inequalities will be used, for example, to prove observability inequalities for the adjoint problem of (1.2) in both the weakly and the strongly degenerate cases.

1. Carleman estimate for the problem in divergence form

Let us consider again problem (3.1) in divergence form, where now a satisfies one of the assumptions describing the (WD) or the (SD) case, which we briefly recollect in the following

Hypothesis 4.1. The function a satisfies Hypothesis 1.1 or Hypothesis 1.2. Moreover, if Hypothesis 1.2 holds with $K > \dfrac{4}{3}$, we suppose that there exists a constant $\vartheta \in (0, K]$ such that the function

$$x \mapsto \frac{a(x)}{|x - x_0|^{\vartheta}} \quad \begin{cases} \text{is nonincreasing on the left of } x = x_0, \\ \text{is nondecreasing on the right of } x = x_0. \end{cases} \tag{4.1}$$

In addition, when $K > \dfrac{3}{2}$, the previous map is bounded below away from 0 and there exists a constant $\Sigma > 0$ such that

$$|a'(x)| \leq \Sigma |x - x_0|^{2\vartheta - 3} \text{ for a.e. } x \in [0, 1]. \tag{4.2}$$

Remark 6. Condition (4.1) is more general than the corresponding one for $x_0 = 0$ required in [**3**] for the (SD) case. Indeed, in this paper we require it only in the sub-case $K > \dfrac{4}{3}$ of the (SD) case. On the other hand, let us note that requiring (4.1), also with $x_0 = 0$ as in [**3**], together with Hypothesis 1.1, implies $\vartheta a \leq (x - x_0)a' \leq Ka$ in $(0, 1)$, so that a' is automatically bounded away from 0 far from x_0. Similar situations were considered in [**10**], [**22**], [**39**], or [**41**], where a certain regularity was assumed somewhere, even in the non degenerate non smooth case.

To our best knowledge, this paper is the first one where non smoothness is assumed globally in the case of an absolutely continuous coefficient, besides degenerate.

Remark 7. The additional requirements for the sub-case $K > 3/2$ are technical ones, which are used just to guarantee the convergence of some integrals (see the Appendix). Of course, the prototype $a(x) = |x - x_0|^K$ satisfies such a condition with $\vartheta = K$.

Now, in order to state our Carleman estimate in presence of a degenerate non smooth coefficient, we start similarly to the previous chapter; but, being such an inspiration only formal, the result is completely different. In particular the sign of the boundary term will have a different and crucial role in the two cases.

Let us introduce the function $\varphi(t,x) := \Theta(t)\psi(x)$, where Θ is defined as in (3.2) and

$$\psi(x) := c_1 \left[\int_{x_0}^{x} \frac{y-x_0}{a(y)} dy - c_2 \right], \tag{4.3}$$

with $c_2 > \max\left\{ \dfrac{(1-x_0)^2}{a(1)(2-K)}, \dfrac{x_0^2}{a(0)(2-K)} \right\}$ and $c_1 > 0$. A more precise restriction on c_1 will be needed later for the observability inequalities of Chapter 5. Observe that $\Theta(t) \to +\infty$ as $t \to 0^+, T^-$, and by Lemma 2.1 we have that, if $x > x_0$,

$$\begin{aligned} \psi(x) &\le c_1 \left[\int_{x_0}^{x} \frac{(y-x_0)^K}{a(y)} \frac{1}{(y-x_0)^{K-1}} dy - c_2 \right] \\ &\le c_1 \left[\frac{(x-x_0)^K}{a(x)} \frac{(x-x_0)^{2-K}}{2-K} - c_2 \right] \\ &\le c_1 \left[\frac{(1-x_0)^K}{a(1)} \frac{(1-x_0)^{2-K}}{2-K} - c_2 \right] \\ &= c_1 \left[\frac{(1-x_0)^2}{(2-K)a(1)} - c_2 \right] < 0. \end{aligned} \tag{4.4}$$

In the same way one can treat the case $x \in [0, x_0)$, so that

$$\psi(x) < 0 \quad \text{for every } x \in [0,1].$$

Moreover, it is also easy to see that $\psi \ge -c_1 c_2$.

Our main result is the following.

THEOREM 4.1. *Assume Hypothesis* 4.1*. Then, there exist two positive constants* C *and* s_0 *such that every solution* v *of* (3.1) *in divergence form in*

$$\mathcal{S}_1 := L^2\big(0,T;\mathcal{H}^2_a(0,1)\big) \cap H^1\big(0,T;\mathcal{H}^1_a(0,1)\big) \tag{4.5}$$

satisfies, for all $s \ge s_0$,

$$\begin{aligned} &\int_{Q_T} \left(s\Theta a (v_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a} v^2 \right) e^{2s\varphi} dxdt \\ &\le C \left(\int_{Q_T} h^2 e^{2s\varphi} dxdt + s c_1 \int_0^T \left[a\Theta e^{2s\varphi}(x-x_0)(v_x)^2 dt \right]_{x=0}^{x=1} \right), \end{aligned}$$

where c_1 *is the constant introduced in* (4.3).

1.1. Proof of Theorem 4.1**.** We start as in the proof of Theorem 3.1: for $s > 0$, define the function

$$w(t,x) := e^{s\varphi(t,x)} v(t,x),$$

where v is any solution of (3.1) in $\mathcal{S}_1$, so that $w \in \mathcal{S}_1$ and w satisfies (3.6), which we re–write as (3.7), with Φ replaced by φ. Moreover, Lemma 3.1 still holds also in this case, again with Φ replaced by φ. Thus we start with the analogue of Lemma 3.3, which now gives the following estimate:

LEMMA 4.1. *Assume Hypothesis 4.1. Then there exist two positive constants s_0 and C such that for all $s \geq s_0$ the distributed terms of (3.9) satisfy the estimate*

$$\begin{aligned} &s^3 \int_{Q_T} \left([a^2(\varphi_x)^3]_x - a(\varphi_x)^2(a\varphi_x)_x\right) w^2 dxdt \\ &+ \frac{s}{2}\int_{Q_T} \varphi_{tt} w^2 dxdt - 2s^2 \int_{Q_T} a\varphi_x\varphi_{tx} w^2 dxdt \\ &+ s\int_{Q_T} (\varphi_{xx}a^2 + a(a\varphi_x)_x)(w_x)^2 dxdt \\ &\geq \frac{C}{2} s \int_{Q_T} \Theta a (w_x)^2 dxdt + \frac{C^3}{2} s^3 \int_{Q_T} \Theta^3 \frac{(x-x_0)^2}{a} w^2 dxdt. \end{aligned}$$

PROOF. Using the definition of φ and recalling that

$$a\varphi_x = c_1\Theta(x-x_0) \text{ and } (a\varphi_x)_x = c_1\Theta, \tag{4.6}$$

so that $(a\varphi_x)_{xx} = 0$, the distributed terms of $\int_0^T\int_0^1 L_s^+ w L_s^- w dxdt$ take the form

$$\begin{aligned} &\frac{s}{2}\int_{Q_T} \ddot{\Theta}\psi w^2 dxdt + s^3 c_1^3 \int_0^T\int_0^1 \Theta^3 \left[\left(\frac{(x-x_0)^3}{a}\right)_x + \frac{(x-x_0)^2}{a}\right] w^2 dxdt \\ &- 2s^2c_1^2 \int_0^T\int_0^1 \Theta\dot{\Theta}\frac{(x-x_0)^2}{a} w^2 dxdt \\ &+ sc_1 \int_0^T\int_0^1 \Theta\left[a\left(\frac{x-x_0}{a}\right)_x + 1\right] a(w_x)^2 dxdt. \end{aligned} \tag{4.7}$$

Now, by Lemma 2.1, we immediately have that $\left(\frac{(x-x_0)^3}{a}\right)_x \geq 0$. Moreover,

$$a\left(\frac{x-x_0}{a}\right)_x = \frac{a-(x-x_0)a'}{a} \geq 1-K \quad \text{for every } x \in (0,1).$$

Hence, in the (WD) case it is immediately positive, while in the (SD) case we have

$$a\left(\frac{x-x_0}{a}\right)_x + 1 \geq 2-K > 0 \quad \text{for every } x \in (0,1).$$

Hence, we can estimate (4.7) from below in the following way:

$$\begin{aligned} &\geq \frac{s}{2}\int_0^T\int_0^1 \ddot{\Theta}\psi w^2 dxdt + s^3c_1^3\int_0^T\int_0^1 \Theta^3\frac{(x-x_0)^2}{a}w^2 dxdt \\ &- 2s^2c_1^2\int_0^T\int_0^1 \Theta\dot{\Theta}\frac{(x-x_0)^2}{a}w^2 dxdt + s\bar{C}\int_{Q_T}\Theta a(w_x)^2 dxdt \end{aligned} \tag{4.8}$$

for some universal positive constant $\bar{C} > 0$.

Using (3.14), we conclude that, for s large enough,

$$\begin{aligned} &\left| -2s^2c_1^2\int_0^T\int_0^1 \Theta\dot{\Theta}\frac{(x-x_0)^2}{a}w^2 dxdt \right| \leq 2cc_1^2 s^2 \int_0^T\int_0^1 \Theta^3\frac{(x-x_0)^2}{a}w^2 dxdt \\ &\leq \frac{c_1^3}{4}s^3\int_0^T\int_0^1 \Theta^3\frac{(x-x_0)^2}{a}w^2 dxdt, \end{aligned} \tag{4.9}$$

as soon as $s \geq \dfrac{8c}{c_1}$.

Moreover, Θ being convex and since (3.14) holds, by the very definition of ψ, we have

$$\frac{s}{2}\int_{Q_T}\ddot{\Theta}\psi w^2dxdt \geq -s\frac{c_1c_2}{2}c\int_0^T\int_0^1\Theta^{3/2}w^2dxdt, \tag{4.10}$$

since $\displaystyle\int_{x_0}^x\frac{y-x_0}{a(y)}dy \geq 0$ for every $x\in[0,1]$. Hence, it remains to bound the term $\int_0^T\int_0^1\Theta^{3/2}w^2dxdt$. Using the Young inequality, we find

$$\begin{aligned}
&s\frac{c_1c_2}{2}c\int_0^1\Theta^{3/2}w^2dx\\
&= s\frac{c_1c_2}{2}c\int_0^1\left(\Theta\frac{a^{1/3}}{|x-x_0|^{2/3}}w^2\right)^{3/4}\left(\Theta^3\frac{|x-x_0|^2}{a}w^2\right)^{1/4}\\
&\leq s\varepsilon\frac{3c_1c_2}{8}c\int_0^1\Theta\frac{a^{1/3}}{|x-x_0|^{2/3}}w^2dx + s\frac{c_1c_2}{8\varepsilon^3}c\int_0^1\Theta^3\frac{|x-x_0|^2}{a}w^2dx.
\end{aligned} \tag{4.11}$$

Now, we concentrate on the integral $\displaystyle\int_0^1\frac{a^{1/3}}{|x-x_0|^{2/3}}w^2dx$.

If $K\leq\dfrac{4}{3}$ (where K is the constant appearing in Hypothesis 1.1 or 1.2), we introduce the function $p(x)=|x-x_0|^{4/3}$. Obviously, there exists $q\in\left(1,\dfrac{4}{3}\right)$ such that the function $x\mapsto\dfrac{p(x)}{|x-x_0|^q}$ is nonincreasing on the left of $x=x_0$ and nondecreasing on the right of $x=x_0$. Thus, using the Hardy-Poincaré inequality (see Proposition 1.1) and Lemma 2.1, one has

$$\begin{aligned}
\int_0^1\frac{a^{1/3}}{|x-x_0|^{2/3}}w^2dx &\leq \max_{[0,1]}a^{1/3}\int_0^1\frac{1}{|x-x_0|^{2/3}}w^2dx\\
&= \max_{[0,1]}a^{1/3}\int_0^1\frac{p}{(x-x_0)^2}w^2dx\\
&\leq \max_{[0,1]}a^{1/3}C_{HP}\int_0^1 p(w_x)^2dx\\
&= \max_{[0,1]}a^{1/3}C_{HP}\int_0^1 a\frac{|x-x_0|^{4/3}}{a}(w_x)^2dx\\
&\leq \max_{[0,1]}a^{1/3}C_{HP}C_2\int_0^1 a(w_x)^2dx,
\end{aligned} \tag{4.12}$$

where C_{HP} is the Hardy-Poincaré constant and $C_2:=\max\left\{\dfrac{x_0^{4/3}}{a(0)},\dfrac{(1-x_0)^{4/3}}{a(1)}\right\}$. In this way, we find

$$\begin{aligned}
\frac{s}{2}\int_{Q_T}\ddot{\Theta}\psi w^2dxdt \geq &-s\varepsilon\frac{3c_1c_2}{8}c\max_{[0,1]}a^{1/3}C_{HP}C_2\int_0^1 a(w_x)^2dx\\
&-s\frac{c_1c_2}{8\varepsilon^{1/3}}c\int_0^1\Theta^3\frac{|x-x_0|^2}{a}w^2dx.
\end{aligned} \tag{4.13}$$

If $K > 4/3$ consider the function $p(x) = (a(x)|x-x_0|^4)^{1/3}$. It is clear that, setting

$$C_1 := \max\left\{ \left(\frac{x_0^2}{a(0)}\right)^{2/3}, \left(\frac{(1-x_0)^2}{a(1)}\right)^{2/3} \right\},$$

by Lemma 2.1 we have

$$p(x) = a(x)\left(\frac{(x-x_0)^2}{a(x)}\right)^{2/3} \le C_1 a(x)$$

and $\dfrac{a^{1/3}}{|x-x_0|^{2/3}} = \dfrac{p(x)}{(x-x_0)^2}$. Moreover, using (4.1), one has that the function $x \mapsto \dfrac{p(x)}{|x-x_0|^q}$ is nonincreasing on the left of $x = x_0$ and nondecreasing on the right of $x = x_0$ for $q = \dfrac{4+\vartheta}{3} \in (1,2)$. Thus, the Hardy-Poincaré inequality (see Proposition 1.1) implies

$$\begin{aligned} \int_0^1 \frac{a^{1/3}}{|x-x_0|^{2/3}} w^2 dx &= \int_0^1 \frac{p}{(x-x_0)^2} w^2 dx \le C_{HP} \int_0^1 p(w_x)^2 dx \\ &\le C_{HP} C_1 \int_0^1 a(w_x)^2 dx, \end{aligned} \tag{4.14}$$

where C_{HP} and C_1 are the Hardy-Poincaré constant and the constant introduced before, respectively.

Using the estimates above, from (4.10) we finally obtain

$$\begin{aligned} \frac{s}{2}\int_{Q_T} \ddot{\Theta}\psi w^2 dxdt \ge &-s\varepsilon \frac{3c_1c_2}{8} c C_{HP} C_1 \int_{Q_T} \Theta a(w_x)^2 dxdt \\ &- s\frac{c_1c_2}{8\varepsilon^3} c \int_0^T \int_0^1 \Theta^3 \frac{(x-x_0)^2}{a} w^2 dxdt. \end{aligned} \tag{4.15}$$

Thus, in every case, we can choose ε so small and s so large that, by (4.8), (4.9), (4.10), (4.13) and (4.15), we can estimate the distributed terms from below with

$$s\frac{\bar{C}}{2}\int_{Q_T} \Theta a(w_x)^2 dxdt + \frac{c_1^3}{2} s^3 \int_0^1 \Theta^3 \frac{(x-x_0)^2}{a} w^2 dxdt.$$

□

As for the boundary terms, similarly to Lemma 3.4, we have the following calculation, whose proof parallels the one of Lemma 3.4 and is thus omitted.

LEMMA 4.2. *The boundary terms reduce to*

$$-sc_1 \int_0^T \left[\Theta a(x-x_0)(w_x)^2\right]_{x=0}^{x=1} dt.$$

From Lemma 4.1, and Lemma 4.2, we deduce immediately that there exist two positive constants C and s_0, such that all solutions w of (3.1) satisfy, for all $s \geq s_0$,

$$\begin{aligned}\int_0^T \int_0^1 L_s^+ w L_s^- w dxdt \geq Cs &\int_{Q_T} \Theta a (w_x)^2 dxdt \\ &+ Cs^3 \int_{Q_T} \Theta^3 \frac{(x-x_0)^2}{a} w^2 dxdt \\ &- sc_1 \int_0^T \left[\Theta a (x-x_0)(w_x)^2\right]_{x=0}^{x=1} dt.\end{aligned} \tag{4.16}$$

Again, we immediately find

Proposition 4.1. *Assume Hypothesis* 4.1. *Then, there exist two positive constants C and s_0, such that all solutions w of* (3.1) *in $\mathcal{S}_1$ satisfy, for all $s \geq s_0$,*

$$\begin{aligned}&s\int_{Q_T} \Theta a (w_x)^2 dxdt + s^3 \int_{Q_T} \Theta^3 \frac{(x-x_0)^2}{a} w^2 dxdt \\ &\leq C\left(\int_{Q_T} h^2 e^{2s\varphi} dxdt + sc_1 \int_0^T \left[a\Theta (x-x_0)(w_x)^2\right]_{x=0}^{x=1} dt\right).\end{aligned}$$

Recalling the definition of w and starting as in the end of the proof of Theorem 3.1, from Proposition 4.1 we immediately obtain Theorem 4.1.

2. Carleman estimate for the problem in non divergence form

Now, we consider the parabolic problem in non divergence form

$$\begin{cases} v_t + a v_{xx} = h & (t,x) \in Q_T, \\ v(t,0) = v(t,1) = 0 & t \in (0,T), \end{cases} \tag{4.17}$$

where a satisfies one of the assumptions describing the (WD) or the (SD) case, plus an additional condition, which we briefly recollect in the following

Hypothesis 4.2. The function a satisfies Hypothesis 1.1 or Hypothesis 1.2. Moreover,

$$\frac{(x-x_0)a'(x)}{a(x)} \in W^{1,\infty}(0,1),$$

and if $K \geq \frac{1}{2}$ (4.1) holds.

Remark 8. We underline the fact that in this subsection (4.2) is not necessary since all integrals and integrations by parts are justified by the definition of $D(\mathcal{A}_2)$.

Now, we introduce the function $\varphi(t,x) := \Theta(t)\psi(x)$, where Θ is defined as in (3.2) and

$$\psi(x) := d_1 \left(\int_{x_0}^x \frac{y-x_0}{a(y)} e^{R(y-x_0)^2} dy - d_2\right), \tag{4.18}$$

with $R > 0$, $d_2 > \max\left\{\frac{(1-x_0)^2 e^{R(1-x_0)^2}}{(2-K)a(1)}, \frac{x_0^2 e^{Rx_0^2}}{(2-K)a(0)}\right\}$ and $d_1 > 0$. A more precise restriction on d_1 will be given in Chapter 5, while the reason for the choice of d_2 will be immediately clear: observe that, by Lemma 2.1 and operating as in (4.4), we have that

$$-d_1 d_2 \le \psi(x) < 0 \quad \text{for every } x \in [0,1].$$

The basic result concerning Carleman estimates is the following inequality, which is the counterpart of [**33**, Theorem 3.1] or of Theorem 4.1 for the non divergence case:

THEOREM 4.2. *Assume Hypothesis* 4.2. *Then, there exist two positive constants* C *and* s_0 *such that every solution* v *of* (4.17) *in*

$$\mathcal{S}_2 := H^1\big(0,T;\mathcal{H}^1_{\frac{1}{a}}(0,1)\big) \cap L^2\big(0,T;\mathcal{H}^2_{\frac{1}{a}}(0,1)\big) \tag{4.19}$$

satisfies

$$\begin{aligned}
&\int_{Q_T} \left(s\Theta(v_x)^2 + s^3\Theta^3 \left(\frac{x-x_0}{a}\right)^2 v^2 \right) e^{2s\varphi} dxdt \\
&\le C\left(\int_{Q_T} h^2 \frac{e^{2s\varphi}}{a} dxdt + sd_1 \int_0^T \left[\Theta e^{2s\varphi}(x-x_0)(v_x)^2 dt\right]_{x=0}^{x=1} \right)
\end{aligned} \tag{4.20}$$

for all $s \ge s_0$, *where* d_1 *is the constant introduced in* (4.18).

2.1. Proof of Theorem 4.2**.** The proof of Theorem 4.2 follows the ideas of the proof of [**33**, Theorem 3.1] or Theorem 4.1, but the non divergence structure introduces several technicalities which were absent before. We start as in the proof of Theorem 3.2: for every $s > 0$ consider the function

$$w(t,x) := e^{s\varphi(t,x)} v(t,x),$$

where v is any solution of (4.17) in $\mathcal{S}_2$, so that also $w \in \mathcal{S}_2$, since $v \in \mathcal{S}_2$ and $\varphi < 0$. Moreover, w satisfies (3.22), which we re–write as (3.23). Moreover, Lemma 3.5 still holds also in this case. We underline the fact that also in the degenerate case all integrals and integrations by parts are justified by the definition of $D(\mathcal{A}_2)$ and the choice of φ, while in [**16**] they were guaranteed by the choice of Dirichlet boundary conditions at $x = 0$, i.e. where their operator degenerates. Thus we start with the analogue of Lemma 3.6 in the weakly and in the strongly degenerate case, which now gives the following estimate:

LEMMA 4.3. *Assume Hypothesis* 4.2. *Then there exists a positive constant* s_0 *such that for all* $s \ge s_0$ *the distributed terms of* (3.25) *satisfy the estimate*

$$\begin{aligned}
&s\int_{Q_T} (a\varphi_{xx} + (a\varphi_x)_x)(w_x)^2 dxdt + s^3 \int_{Q_T} (\varphi_x)^2 (a\varphi_{xx} + (a\varphi_x)_x) w^2 dxdt \\
&- 2s^2 \int_{Q_T} \varphi_x \varphi_{xt} w^2 dxdt + \frac{s}{2}\int_{Q_T} \frac{\varphi_{tt}}{a} w^2 dxdt + s\int_{Q_T} (a\varphi_{xx})_x w w_x dxdt \\
&\ge \frac{C}{2} s \int_{Q_T} \Theta (w_x)^2 dxdt + \frac{C^3}{2} s^3 \int_{Q_T} \Theta^3 \left(\frac{x-x_0}{a}\right)^2 w^2 dxdt,
\end{aligned}$$

for a universal positive constant C.

PROOF. Using the definition of φ, the distributed terms of $\displaystyle\int_{Q_T} \frac{1}{a} L_s^+ w L_s^- w dxdt$ take the form

$$\frac{s}{2}\int_{Q_T} \ddot{\Theta}\frac{\psi}{a} w^2 dxdt - 2s^2 \int_{Q_T} \Theta\dot{\Theta}(\psi')^2 w^2 dxd + s\int_{Q_T} \Theta(2a\psi'' + a'\psi')(w_x)^2 dxdt$$
$$+ s^3 \int_{Q_T} \Theta^3(2a\psi'' + a'\psi')(\psi')^2 w^2 dxdt + s\int_{Q_T} \Theta(a\psi'')' w w_x dxdt.$$

Because of the choice of $\psi(x)$, one has

$$2a(x)\psi''(x) + a'(x)\psi'(x) = d_1 e^{R(x-x_0)^2} \frac{2a(x) - a'(x)(x-x_0) + 4R(x-x_0)^2 a(x)}{a(x)}.$$

By Hypothesis (1.1) or (1.2), we immediately find

$$2 - \frac{(x-x_0)a'}{a} \geq 2 - K > 0 \quad \text{a.e. } \; x \in [0,1];$$

hence, for every $R > 0$ we get

$$2 - \frac{(x-x_0)a'}{a} + 4R(x-x_0)^2 \geq 2 - K \quad \text{a.e. } \; x \in [0,1].$$

Thus, since $e^{R(x-x_0)^2}$ is bounded and bounded away from 0 in $[0,1]$, the distributed terms satisfy the estimate

$$\begin{aligned} \{D.T.\} \geq{}& \frac{s}{2}\int_{Q_T} \ddot{\Theta}\frac{\psi}{a} w^2 dxdt - s^2 C\int_{Q_T} |\Theta\dot{\Theta}| \left(\frac{x-x_0}{a}\right)^2 w^2 dxdt \\ &+ sC\int_{Q_T} \Theta(w_x)^2 dxdt + s^3 C\int_{Q_T} \Theta^3 \left(\frac{x-x_0}{a}\right)^2 w^2 dxdt \\ &+ s\int_{Q_T} \Theta(a\psi'')' w w_x dxdt, \end{aligned} \tag{4.21}$$

where $C > 0$ denotes some universal positive constant which may vary from line to line.

By (3.14), we conclude that, for s large enough,

$$\begin{aligned} s^2 C\int_{Q_T} |\Theta\dot{\Theta}| \left(\frac{x-x_0}{a}\right)^2 w^2 dxdt &\leq cCs^2 \int_{Q_T} \Theta^3 \left(\frac{x-x_0}{a}\right)^2 w^2 dxdt \\ &\leq \frac{C^3}{8} s^3 \int_{Q_T} \Theta^3 \left(\frac{x-x_0}{a}\right)^2 w^2 dxdt. \end{aligned}$$

Moreover, by (3.14) we get

$$\begin{aligned} \left|\frac{s}{2}\int_{Q_T} \ddot{\Theta}\frac{\psi}{a} w^2 dxdt\right| &\leq \frac{s}{2} c\int_0^T \int_0^1 \Theta^{3/2} \frac{-\psi}{a} w^2 dxdt \\ &\leq s\frac{d_1 d_2}{2} c\int_0^T \int_0^1 \Theta^{3/2} \frac{w^2}{a} dxdt, \end{aligned} \tag{4.22}$$

by the very definition of ψ. In order to estimate the last integral, we distinguish the cases $K < \frac{1}{2}$ and $K \geq \frac{1}{2}$. In the former case, using the Young inequality, we get

$$
\begin{aligned}
&s\frac{d_1d_2}{2}c\int_0^1 \Theta^{3/2}\frac{w^2}{a}dx\\
&= s\frac{d_1d_2}{2}c\left|\int_0^1\left(\Theta\frac{w^2}{a^{2/3}|x-x_0|^{2/3}}\right)^{3/4}\left(\Theta^3\left(\frac{x-x_0}{a}\right)^2w^2\right)^{1/4}\right| \qquad (4.23)\\
&\le 3s\frac{d_1d_2}{8}c\int_0^1\Theta\frac{w^2}{a^{2/3}|x-x_0|^{2/3}}dx + s\frac{d_1d_2}{8}c\int_0^1\Theta^3\left(\frac{x-x_0}{a}\right)^2w^2dx.
\end{aligned}
$$

Now, we introduce the function

$$
p(x)=\frac{|x-x_0|^{4/3}}{a^{2/3}}=\left(\frac{|x-x_0|^2}{a(x)}\right)^{2/3}\to 0 \text{ as } x\to 0 \text{ by Lemma 2.1,}
$$

and we take $q=\frac{4}{3}-\frac{2}{3}K$. Then the function

$$
x\mapsto\frac{p(x)}{|x-x_0|^q}=\left(\frac{|x-x_0|^K}{a}\right)^{2/3}
$$

is nonincreasing on the left of $x=x_0$ and nondecreasing on the right of $x=x_0$ by Lemma 2.1. Since $K<1/2$, we have that $q\in(1,2)$. Thus, using the Hardy-Poincaré inequality (see Proposition 1.1), one has

$$
\begin{aligned}
\int_0^1\frac{w^2}{a^{2/3}|x-x_0|^{2/3}}dx &= \int_0^1\frac{p(x)}{|x-x_0|^2}w^2dx\\
&\le C_{HP}\int_0^1 p(w_x)^2dx \qquad (4.24)\\
&\le C_{HP}\max\left\{\frac{x_0^{4/3}}{a(0)^{2/3}},\frac{|1-x_0|^{4/3}}{a(1)^{2/3}}\right\}\int_0^1(w_x)^2dx,
\end{aligned}
$$

by Lemma 2.1. Thus, by (4.23) and (4.24), we have that for s large enough

$$
s\frac{d_1d_2}{2}c\int_0^1\Theta^{3/2}\frac{w^2}{a}dx\le\frac{C}{4}s\int_0^1\Theta(w_x)^2dx+\frac{C^3}{8}s^3\int_0^1\Theta^3\left(\frac{x-x_0}{a}\right)^2w^2dx, \qquad (4.25)
$$

for a positive constant C. Using (4.25), from (4.22) we finally obtain

$$
\begin{aligned}
\left|\frac{s}{2}\int_{Q_T}\ddot{\Theta}\frac{\psi}{a}w^2dxdt\right| &\le \frac{C}{4}s\int_{Q_T}\Theta(w_x)^2dxdt \qquad (4.26)\\
&\quad+\frac{C^3}{4}s^3\int_{Q_T}\Theta^3\left(\frac{x-x_0}{a}\right)^2w^2dxdt.
\end{aligned}
$$

If $K\ge\dfrac{1}{2}$ we proceed as follows. We take $r>2$, $\gamma<2$ and $\alpha,\beta>0$ to be chosen later, and, by (3.14) and the Young inequality, we get

$$
\begin{aligned}
&\int_0^1\Theta^{3/2}\frac{w^2}{a}dx\le c\int_0^1\left(\Theta\frac{w^2}{x^2}\right)^{1/r}\left(\Theta^\alpha\frac{|x-x_0|^\beta}{a^\gamma}w^2\right)^{1-1/r}dx \qquad (4.27)\\
&\le\frac{c}{r}\int_0^1\Theta\frac{w^2}{x^2}dx+\frac{r}{r-1}\int_0^1\Theta^\alpha\frac{|x-x_0|^\beta}{a^\gamma}w^2dx,
\end{aligned}
$$

which holds true provided that:

$$\alpha = \frac{3r-2}{2r-2} (< 3 \text{ since } r > 2),$$

and

$$\frac{1}{a} \le \bar{c}\frac{1}{x^{2/r}}\frac{|x-x_0|^{\beta\frac{r-1}{r}}}{a^{\gamma\frac{r-1}{r}}},$$

which, by (4.1), is true if

$$\gamma > \frac{r}{r-1} \tag{4.28}$$

and

$$\frac{\beta(r-1)}{\gamma(r-1)-r} \le \vartheta.$$

Notice that (4.28) is consistent with the requirement $\gamma < 2$ since $r > 2$.

Moreover, we shall clearly use the inequality

$$\frac{|x-x_0|^\beta}{a^\gamma} \le \tilde{c}\left(\frac{|x-x_0|}{a}\right)^2,$$

which is true if

$$\frac{2-\beta}{2-\gamma} \le \vartheta.$$

Hence, choosing β and γ satisfying the conditions above, from (4.27) and the classical Hardy inequality (recall that $\mathcal{H}_{\frac{1}{a}}(0,1) \subset H^1_0(0,1)$) we get

$$\int_0^1 \Theta^{3/2}\frac{w^2}{a}dx \le c_1\int_0^1 \Theta(w_x)^2dx + c_2\int_0^1 \Theta^3\left(\frac{x-x_0}{a}\right)^2 w^2dx \tag{4.29}$$

for some universal constant $c_1, c_2 > 0$. Hence, as before, (4.26) also holds in this case, if s is large enough.

Now, we consider the last term in (4.21), i.e. $s\int_{Q_T}\Theta(a\psi'')'ww_xdxdt$. Observe that, using the definition of ψ and Hypothesis 4.2, we have

$$\begin{aligned}\|(a\psi'')'\|_{L^\infty(0,1)} \le d_1e^R\Bigg(&4R^2+6R+2R\left\|\frac{(x-x_0)a'}{a}\right\|_{L^\infty(0,1)}\\ &+\left\|\left(\frac{(x-x_0)a'}{a}\right)'\right\|_{L^\infty(0,1)}\Bigg) := C_R.\end{aligned}$$

Hence, proceeding as for (4.25), one has

$$\begin{aligned}&\left|s\int_{Q_T}\Theta(a\psi'')'ww_xdxdt\right| \le \frac{1}{2}s\int_{Q_T}\Theta|(a\psi'')'|^2w^2dxdt\\ &+\frac{1}{2}s\int_{Q_T}\Theta(w_x)^2dxdt\\ &\le \frac{1}{2}sc\|(a\psi'')'\|^2_{L^\infty(0,1)}\int_{Q_T}\Theta^{3/2}w^2dxdt + \frac{1}{2}s\int_{Q_T}\Theta(w_x)^2dxdt\\ &\le \frac{C}{4}s\int_{Q_T}\Theta(w_x)^2dxdt + s^3\frac{C^3}{8}\int_{Q_T}\Theta^3\left(\frac{x-x_0}{a}\right)^2w^2dxdt.\end{aligned}$$

Summing up, we obtain

$$\begin{aligned}
\{D.T.\} \geq & -\frac{C}{4}s\int_{Q_T}\Theta(w_x)^2dxdt - \frac{C^3}{4}s^3\int_{Q_T}\Theta^3\left(\frac{x-x_0}{a}\right)^2w^2dxdt\\
&-\frac{C^3}{8}s^3\int_{Q_T}\Theta^3\left(\frac{x-x_0}{a}\right)^2w^2dxdt\\
&+sC\int_{Q_T}\Theta(w_x)^2dxdt + s^3C\int_{Q_T}\Theta^3\left(\frac{x-x_0}{a}\right)^2w^2dxdt\\
&-\frac{C}{4}s\int_{Q_T}\Theta(w_x)^2dxdt - \frac{C^3}{8}s^3\int_{Q_T}\Theta^3(w_x)^2dxdt\\
&=\frac{C}{2}s\int_{Q_T}\Theta(w_x)^2dxdt + \frac{C^3}{2}s^3\int_{Q_T}\Theta^3\left(\frac{x-x_0}{a}\right)^2w^2dxdt.
\end{aligned}$$

□

As for the boundary terms, similarly to Lemma 3.7, we have the following result, whose proof parallels the one of Lemma 3.7 and is thus omitted.

Lemma 4.4. *Assume Hypothesis* 4.2. *Then the boundary terms in* (3.25) *reduce to*

$$-sd_1\int_0^T\Theta(t)\left[(x-x_0)e^{R(x-x_0)^2}(w_x)^2\right]_{x=0}^{x=1}dt.$$

By Lemmas 4.3 and 4.4, there exist $C>0$ and $s_0>0$ such that all solutions w of (3.22) satisfy, for all $s\geq s_0$,

$$\begin{aligned}
\int_{Q_T}\frac{1}{a}L_s^+wL_s^-wdxdt \geq & \; Cs\int_{Q_T}\Theta(w_x)^2dxdt\\
&+Cs^3\int_{Q_T}\Theta^3\left(\frac{x-x_0}{a}\right)^2w^2dxdt\\
&-sd_1\int_0^T\Theta(t)\left[(x-x_0)e^{R(x-x_0)^2}(w_x)^2\right]_{x=0}^{x=1}dt.
\end{aligned} \tag{4.30}$$

Thus, by (3.24) and (4.30), we obtain the next Carleman inequality for w:

$$\begin{aligned}
&s\int_{Q_T}\Theta(w_x)^2dxdt + s^3\int_{Q_T}\Theta^3\left(\frac{x-x_0}{a}\right)^2w^2dxdt\\
&\leq C\left(\int_{Q_T}h^2\frac{e^{2s\varphi}}{a}dxdt + sd_1\int_0^T\left[\Theta(x-x_0)e^{R(x-x_0)^2}(w_x)^2\right]_{x=0}^{x=1}dt\right)
\end{aligned}$$

for all $s\geq s_0$.

Theorem 4.2 follows recalling the definition of w and starting as in the end of the proof of Theorem 3.1.

CHAPTER 5

Observability inequalities and application to null controllability

In this chapter we assume that the control set ω satisfies the following assumption:

HYPOTHESIS 5.1. The subset ω is such that

- it is an interval which contains the degeneracy point, more precisely:

$$\omega = (\alpha, \beta) \subset (0,1) \text{ is such that } x_0 \in \omega. \tag{5.1}$$

or

- it is an interval lying on one side of the degeneracy point, more precisely:

$$\omega = (\alpha, \beta) \subset (0,1) \text{ is such that } x_0 \notin \bar{\omega}. \tag{5.2}$$

1. The divergence case

Now, we consider the problem in divergence form and we make the following assumptions on the function a:

HYPOTHESIS 5.2. Hypothesis 4.1 is satisfied. Moreover, if Hypothesis 1.1 holds, we assume that there exist two functions $\mathfrak{g} \in L^\infty_{\text{loc}}([0,1] \setminus \{x_0\})$, $\mathfrak{h} \in W^{1,\infty}_{\text{loc}}([0,1] \setminus \{x_0\}, L^\infty(0,1))$ and two strictly positive constants $\mathfrak{g}_0$, $\mathfrak{h}_0$ such that $\mathfrak{g}(x) \geq \mathfrak{g}_0$ for a.e. x in $[0,1]$ and

$$-\frac{a'(x)}{2\sqrt{a(x)}} \left(\int_x^B \mathfrak{g}(t)dt + \mathfrak{h}_0 \right) + \sqrt{a(x)}\mathfrak{g}(x) = \mathfrak{h}(x,B) \quad \text{for a.e. } x, B \in [0,1] \tag{5.3}$$

with $x < B < x_0$ or $x_0 < x < B$.

REMARK 9. Contrary to the non degenerate case, the identity in (5.3) is assumed to hold with functions which are bounded only *far from* x_0. Indeed, (5.3) will be applied in sets where a is non degenerate and the corresponding identity given in Hypothesis 3.1 will be applied. For this reason, functions $\mathfrak{g}$ and $\mathfrak{h}$ can be easily found, once a is given.

To the linear problem (1.2) we associate the homogeneous adjoint problem

$$\begin{cases} v_t + (av_x)_x = 0, & (t,x) \in Q_T, \\ v(t,0) = v(t,1) = 0, & t \in (0,T), \\ v(T,x) = v_T(x) \in L^2(0,1), \end{cases} \tag{5.4}$$

where $T > 0$ is given. By the Carleman estimate in Theorem 4.1, we will deduce the following observability inequality for both the weakly and the strongly degenerate cases:

PROPOSITION 5.1. *Assume Hypotheses* 5.1 *and* 5.2. *Then there exists a positive constant C_T such that every solution $v \in C([0,T]; L^2(0,1)) \cap L^2(0,T; \mathcal{H}^1_a(0,1))$ of* (5.4) *satisfies*

$$\int_0^1 v^2(0,x)dx \le C_T \int_0^T \int_\omega v^2(t,x)dxdt. \tag{5.5}$$

Using the observability inequality (5.5) and a standard technique (e.g., see [**40**, Section 7.4]), one can prove the null controllability result for the linear degenerate problem (1.2), another fundamental result of this paper.

THEOREM 5.1. *Assume Hypotheses* 5.1 *and* 5.2. *Then, for every $u_0 \in L^2(0,1)$ there exists $h \in L^2(Q_T)$ such that the solution u of* (1.2) *satisfies*

$$u(T,x) = 0 \quad \textit{for every} \quad x \in [0,1].$$

Moreover

$$\int_0^T \int_0^1 h^2 dxdt \le C \int_0^1 u_0^2 dx,$$

for some universal positive constant C.

We remark that Proposition 5.1 has an immediate application also in the case in which the control set ω is the union of two intervals ω_i, $i = 1,2$ each of them lying on one side of the degeneracy point. More precisely, we have the following observability inequality, whose proof is straightforward:

COROLLARY 5.1. *Assume Hypothesis* 5.2 *and $\omega = \omega_1 \cup \omega_2$, where ω_i, $i = 1,2$ are intervals each of them lying on one side of the degeneracy point, more precisely:*

$$\omega_i = (\lambda_i, \beta_i) \subset (0,1),\ i = 1,2, \ \textit{and}\ \beta_1 < x_0 < \lambda_2. \tag{5.6}$$

Then there exists a positive constant C_T such that every solution v of (5.4) *satisfies*

$$\int_0^1 v^2(0,x)dx \le C_T \int_0^T \int_\omega v^2(t,x)dxdt.$$

As a consequence, one has the next null controllability result:

THEOREM 5.2. *Assume* (5.6) *and Hypothesis* 5.2. *Then, for every $u_0 \in L^2(0,1)$, there exists $h \in L^2(Q_T)$ such that the solution u of* (1.2) *satisfies*

$$u(T,x) = 0 \quad \textit{for every} \quad x \in [0,1].$$

Moreover

$$\int_0^T \int_0^1 h^2 dxdt \le C \int_0^1 u_0^2(x) dx,$$

for some positive constant C.

1.1. Proof of Proposition 5.1. In this subsection we will prove, as a consequence of the Carleman estimate established in Theorem 4.1, the observability inequality (5.5). For this purpose, we will give some preliminary results. As a first step, we consider the adjoint problem with more regular final–time datum

$$\begin{cases} v_t + (av_x)_x = 0, & (t,x) \in Q_T, \\ v(t,0) = v(t,1) = 0, & t \in (0,T), \\ v(T,x) = v_T(x) \in D(\mathcal{A}_1^2), \end{cases} \tag{5.7}$$

where

$$D(\mathcal{A}_1^2) = \Big\{ u \in D(\mathcal{A}_1) \;\big|\; \mathcal{A}_1 u \in D(\mathcal{A}_1) \Big\}$$

and, we recall, $\mathcal{A}_1 u := (au_x)_x$. Observe that $D(\mathcal{A}_1^2)$ is densely defined in $D(\mathcal{A}_1)$ (see, for example, [**12**, Lemma 7.2]) and hence in $L^2(0,1)$. As in [**16**], [**17**] or [**31**], letting v_T vary in $D(\mathcal{A}_1^2)$, we define the following class of functions:

$$\mathcal{W}_1 := \Big\{ v \in C^1\big([0,T]; L^2(0,1)\big) \cap C\big([0,T]; D(\mathcal{A}_1)\big) \,|\, v \text{ is a solution of (5.7)} \Big\}.$$

Obviously (see, for example, [**12**, Theorem 7.5])

$$\mathcal{W}_1 \subset C^1\big([0,T];\, \mathcal{H}_a^2(0,1)\big) \subset \mathcal{S}_1 \subset \mathcal{U}_1,$$

where, $\mathcal{S}_1$ is defined in (4.5) and

$$\mathcal{U}_1 := C([0,T]; L^2(0,1)) \cap L^2(0,T; \mathcal{H}_a^1(0,1)).$$

We start with the following Proposition, for whose proof we refer to [**33**, Proposition 4.2], since also in this weaker setting that proof is still valid. We underline the fact that the degeneracy point is allowed to belong even to the control set.

Proposition 5.2 (Caccioppoli's inequality). *Assume Hypothesis* 1.1 *or Hypothesis* 1.2. *Let* ω' *and* ω *two open subintervals of* $(0,1)$ *such that* $\omega' \subset\subset \omega \subset (0,1)$ *and* $x_0 \notin \bar{\omega}'$. *Let* $\varphi(t,x) = \Theta(t)\Upsilon(x)$, *where* Θ *is defined in* (3.2) *and*

$$\Upsilon \in C([0,1], (-\infty,0)) \cap C^1([0,1] \setminus \{x_0\}, (-\infty,0))$$

is such that

$$|\Upsilon_x| \le \frac{c}{\sqrt{a}} \text{ in } [0,1] \setminus \{x_0\} \tag{5.8}$$

for some $c > 0$. *Then, there exist two positive constants* C *and* s_0 *such that every solution* $v \in \mathcal{W}_1$ *of the adjoint problem* (5.7) *satisfies*

$$\int_0^T \int_{\omega'} (v_x)^2 e^{2s\varphi} dxdt \;\le\; C \int_0^T \int_\omega v^2 dxdt,$$

for all $s \ge s_0$.

Remark 10. Of course, our prototype for Υ are the functions ψ defined in (3.3) or in (4.3). Indeed, if ψ is as in (4.3), then

$$|\psi'(x)| = c_1 \frac{|x-x_0|}{a(x)} = c_1 \sqrt{\frac{|x-x_0|^2}{a(x)}} \frac{1}{\sqrt{a(x)}} \le c \frac{1}{\sqrt{a(x)}}$$

by Lemma 2.1. In the case of (3.3), inequality (5.8) is obvious.

Remark 11. Actually, in the proof of Proposition 5.2, only the regularity on a required in Hypothesis 1.1 or Hypothesis 1.2 is used, and not the inequality $(x-x_0)a' \le Ka$.

We shall also need the two following lemmas, that deal with the different situations in which x_0 is inside or outside the control region ω. The statements of the conclusions are in fact the same, however, we state the results in two separate lemmas, since their applications are related to different situations and the proofs, though inspired by the same ideas, are different. First, we state both the results and then we will prove them.

LEMMA 5.1. *Assume* (5.1) *and Hypothesis* 5.2. *Then there exist two positive constants* C *and* s_0 *such that every solution* $v \in \mathcal{W}_1$ *of* (5.7) *satisfies, for all* $s \geq s_0$,

$$\int_{Q_T} \left(s\Theta a(v_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a} v^2 \right) e^{2s\varphi} dxdt \leq C \int_0^T \int_\omega v^2 dxdt.$$

Here Θ *and* φ *are as in* (3.2) *and* (4.3), *respectively, with* c_1 *sufficiently large.*

LEMMA 5.2. *Assume* (5.2) *and Hypothesis* 5.2. *Then there exist two positive constants* C *and* s_0 *such that every solution* $v \in \mathcal{W}_1$ *of* (5.7) *satisfies, for all* $s \geq s_0$,

$$\int_{Q_T} \left(s\Theta a(v_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a} v^2 \right) e^{2s\varphi} dxdt \leq C \int_0^T \int_\omega v^2 dxdt.$$

Here Θ *and* φ *are as in* (3.2) *and* (4.3), *respectively, with* c_1 *sufficiently large.*

We underline the fact that for the proof of the previous lemmas a crucial rôle will be played also by the Carleman estimate for nondegenerate equations with nonsmooth coefficient proved in Theorem 3.1.

PROOF OF LEMMA 5.1. By assumption, we can find two subintervals $\omega_1 = (\lambda_1, \beta_1) \subset (0, x_0), \omega_2 = (\lambda_2, \beta_2) \subset (x_0, 1)$ such that $(\omega_1 \cup \omega_2) \subset\subset \omega \setminus \{x_0\}$. Now, consider a smooth function $\xi : [0,1] \to [0,1]$ such that

$$\xi(x) = \begin{cases} 0 & x \in [0, \alpha], \\ 1 & x \in [\lambda_1, \beta_2] \\ 0 & x \in [\beta, 1], \end{cases}$$

and define $w := \xi v$, where v solves (5.7). Hence, w satisfies

$$\begin{cases} w_t + (aw_x)_x = (a\xi_x v)_x + \xi_x a v_x =: f, & (t,x) \in (0,T) \times (0,1), \\ w(t,0) = w(t,1) = 0, & t \in (0,T). \end{cases} \tag{5.9}$$

Applying Theorem 4.1 and using the fact that $w = 0$ in a neighborhood of $x = 0$ and $x = 1$, we have

$$\int_{Q_T} \Big(s\Theta a(w_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a} w^2 \Big) e^{2s\varphi}\, dxdt \leq C \int_{Q_T} e^{2s\varphi} f^2 dxdt \tag{5.10}$$

for all $s \geq s_0$. Then, using the definition of ξ and in particular the fact that ξ_x and ξ_{xx} are supported inside $\tilde{\omega} := [\alpha, \lambda_1] \cup [\beta_2, \beta]$, from Hypothesis 5.2 we can write

$$f^2 = ((a\xi_x v)_x + a\xi_x v_x)^2 \leq C(v^2 + (v_x)^2)\chi_{\tilde{\omega}}.$$

Hence, applying Proposition 5.2 and inequality (5.10), we get

$$\begin{aligned} &\int_0^T \int_{\lambda_1}^{\beta_2} \left(s\Theta a(v_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a} v^2 \right) e^{2s\varphi} dxdt \\ &= \int_0^T \int_{\lambda_1}^{\beta_2} \Big(s\Theta a(w_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a} w^2 \Big) e^{2s\varphi}\, dxdt \\ &\leq \int_0^T \int_0^1 \Big(s\Theta a(w_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a} w^2 \Big) e^{2s\varphi}\, dxdt \\ &\leq C \int_0^T \int_{\tilde{\omega}} e^{2s\varphi}(v^2 + (v_x)^2) dxdt \leq C \int_0^T \int_\omega v^2 dxdt, \end{aligned} \tag{5.11}$$

for a positive constant C.

Now, we consider a smooth function $\tau : [0,1] \to [0,1]$ such that

$$\tau(x) = \begin{cases} 0 & x \in \left[0, \frac{\lambda_2+\beta_2}{2}\right], \\ 1 & x \in [\beta_2, 1]. \end{cases}$$

Define $z := \tau v$, where v is the solution of (5.7). Then z satisfies (3.1), with $h := (a\tau_x v)_x + a\tau_x v_x$, $A = \lambda_2$ and $B = 1$. Since h is supported in $\left[\frac{\lambda_2+\beta_2}{2}, \beta_2\right]$, by Proposition 5.2, Theorem 3.1 applied with $A = \lambda_2$, $B = 1$ and Remark 4, we get

$$\begin{aligned} &\int_0^T \int_{\lambda_2}^1 s\Theta(z_x)^2 e^{2s\Phi} dxdt + \int_0^T \int_{\lambda_2}^1 s^3\Theta^3 z^2 e^{2s\Phi} dxdt \\ &\le c\int_0^T \int_{\lambda_2}^1 e^{2s\Phi} h^2 dxdt \le C\int_0^T \int_{\tilde\omega_1} v^2 dxdt + C\int_0^T \int_{\tilde\omega_1} e^{2s\Phi}(v_x)^2 dxdt \\ &\le C\int_0^T \int_\omega v^2 dxdt, \end{aligned} \tag{5.12}$$

where $\tilde\omega_1 = (\lambda_2, \beta_2)$. Let us remark that the boundary term in $x = 1$ is nonpositive, while the one in $x = \lambda_2$ is 0, so that they can be neglected in the classical Carleman estimate.

Now, choose the constant c_1 in (4.3) so that

$$c_1 \ge \begin{cases} \dfrac{r\left[\displaystyle\int_{\lambda_2}^1 \frac{1}{\sqrt{a(t)}}\int_t^1 \mathfrak{g}(s)dsdt + \int_{\lambda_2}^1 \frac{\mathfrak{h}_0}{\sqrt{a(t)}}dt\right] + \mathfrak{c}}{c_2 - \dfrac{(1-x_0)^2}{a(1)(2-K)}} =: \Pi & \text{in the (WD) case,} \\ \dfrac{\mathfrak{c}-1}{c_2 - \dfrac{(1-x_0)^2}{a(1)(2-K)}} & \text{in the (SD) case,} \end{cases} \tag{5.13}$$

where $\mathfrak{c}$ is the constant appearing in (3.3). Then, by definition of φ, the choice of c_1 and by Lemma 2.1, one can prove that there exists a positive constant k, for example

$$k = \max\left\{\max_{[\lambda_2,1]} a, \frac{(1-x_0)^2}{a(1)}\right\},$$

such that

$$a(x)e^{2s\varphi(t,x)} \le ke^{2s\Phi(t,x)} \tag{5.14}$$

and

$$\frac{(x-x_0)^2}{a(x)}e^{2s\varphi(t,x)} \le ke^{2s\Phi(t,x)} \tag{5.15}$$

for every $(t,x) \in [0,T] \times [\lambda_2, 1]$. Note that the value of k can be immediately found by estimating the coefficients of $e^{2s\varphi(t,x)}$ in (5.14) and (5.15), once known that $e^{2s\varphi(t,x)} \le e^{2s\Phi(t,x)}$, using Lemma 2.1. Finally, condition (5.13) is a sufficient one to get $e^{2s\varphi(t,x)} \le e^{2s\Phi(t,x)}$, and it can be found by using Lemma 2.1 and rough estimates.

Thus, by (5.12), one has

$$\begin{aligned}&\int_0^T\int_{\lambda_2}^1 \Big(s\Theta a(z_x)^2 + s^3\Theta^3\frac{(x-x_0)^2}{a}z^2\Big)e^{2s\varphi}dxdt\\&\le k\int_0^T\int_{\lambda_2}^1 s\Theta(z_x)^2e^{2s\Phi}dxdt + k\int_0^T\int_{\lambda_2}^1 s^3\Theta^3z^2e^{2s\Phi}dxdt\\&\le kC\int_0^T\int_\omega v^2dxdt,\end{aligned}$$

for a positive constant C. As a trivial consequence,

$$\begin{aligned}&\int_0^T\int_{\beta_2}^1 \Big(s\Theta a(v_x)^2 + s^3\Theta^3\frac{(x-x_0)^2}{a}v^2\Big)e^{2s\varphi}dxdt\\&=\int_0^T\int_{\beta_2}^1 \Big(s\Theta a(z_x)^2 + s^3\Theta^3\frac{(x-x_0)^2}{a}z^2\Big)e^{2s\varphi}dxdt\\&\le\int_0^T\int_{\lambda_2}^1 \Big(s\Theta a(z_x)^2 + s^3\Theta^3\frac{(x-x_0)^2}{a}z^2\Big)e^{2s\varphi}dxdt\\&\le\int_0^T\int_\omega v^2dxdt,\end{aligned}\tag{5.16}$$

for a positive constant C.

Thus (5.11) and (5.16) imply

$$\int_0^T\int_{\lambda_1}^1 \Big(s\Theta a(v_x)^2 + s^3\Theta^3\frac{(x-x_0)^2}{a}v^2\Big)e^{2s\varphi}\,dxdt \le C\int_0^T\int_\omega v^2dxdt,\tag{5.17}$$

for some positive constant C.

To complete the proof it is sufficient to prove a similar inequality on the interval $[0,\lambda_1]$. To this aim, we perform a reflection procedure introducing the functions

$$W(t,x) := \begin{cases} v(t,x), & x\in[0,1],\\ -v(t,-x), & x\in[-1,0],\end{cases}\tag{5.18}$$

where v solves (5.7), and

$$\tilde a(x) := \begin{cases} a(x), & x\in[0,1],\\ a(-x), & x\in[-1,0].\end{cases}\tag{5.19}$$

Then W satisfies the problem

$$\begin{cases} W_t + (\tilde a W_x)_x = 0, & (t,x)\in(0,T)\times(-1,1),\\ W(t,-1)=W(t,1)=0, & t\in(0,T).\end{cases}\tag{5.20}$$

As above, we introduce a smooth function $\rho:[-1,1]\to[0,1]$ such that

$$\rho(x) = \begin{cases} 0 & x\in[-1,-\frac{\lambda_1+\beta_1}{2}],\\ 1 & x\in[-\lambda_1,\lambda_1],\\ 0 & \in[\frac{\lambda_1+\beta_1}{2},1].\end{cases}$$

Finally, set $Z:=\rho W$, where W is the solution of (5.20). Then Z satisfies (3.1) with $h := (\tilde a\rho_x W)_x + \tilde a\rho_x W_x$. Observe that $Z_x(t,-\beta_1) = Z_x(t,\beta_1) = 0$. Using

Proposition 5.2, Theorem 3.1 with $A = -\beta_1$ and $B = \beta_1$, Remark 4, the definition of W and the fact that h is supported in $\left[-\frac{\lambda_1+\beta_1}{2}, -\lambda_1\right] \cup \left[\lambda_1, \frac{\lambda_1+\beta_1}{2}\right]$ give

(5.21)

$$\begin{aligned}
&\int_0^T \int_{-\beta_1}^{\beta_1} s\Theta(Z_x)^2 e^{2s\Phi} dxdt + \int_0^T \int_{-\beta_1}^{\beta_1} s^3\Theta^3 Z^2 e^{2s\Phi} dxdt \\
&\le C \int_0^T \int_{-\beta_1}^{\beta_1} e^{2s\Phi} h^2 dxdt \\
&\le C \int_0^T \int_{-\frac{\lambda_1+\beta_1}{2}}^{-\lambda_1} e^{2s\Phi}(W^2 + (W_x)^2) dxdt + C \int_0^T \int_{\lambda_1}^{\frac{\lambda_1+\beta_1}{2}} e^{2s\Phi}(W^2 + (W_x)^2) dxdt \\
&\le C \int_0^T \int_{\lambda_1}^{\frac{\lambda_1+\beta_1}{2}} e^{2s\Phi}(W^2 + (W_x)^2) dxdt \\
&\le C \int_0^T \int_{\lambda_1}^{\frac{\lambda_1+\beta_1}{2}} v^2 dxdt + C \int_0^T \int_{\lambda_1}^{\frac{\lambda_1+\beta_1}{2}} e^{2s\Phi}(v_x)^2 dxdt \le C \int_0^T \int_\omega v^2 dxdt,
\end{aligned}$$

for some positive constants C, which we allow to vary from line to line. Now, define

$$\tilde{\varphi}(t,x) := \Theta(t)\tilde{\psi}(x),$$

where

$$\tilde{\psi}(x) := \begin{cases} \psi(x), & x \ge 0, \\ \psi(-x) = c_1 \left[\int_{-x_0}^{x} \dfrac{t + x_0}{\tilde{a}(t)} dt - c_2 \right], & x < 0. \end{cases} \tag{5.22}$$

and choose the constant c_1 so that

$$c_1 \ge \begin{cases} \max \left\{ \Pi, \dfrac{r \left[\int_{-\beta_1}^{\beta_1} \dfrac{1}{\sqrt{a(t)}} \int_t^1 \mathfrak{g}(s) ds dt + \int_{-\beta_1}^{\beta_1} \dfrac{\mathfrak{h}_0}{\sqrt{a(t)}} dt \right] + \mathfrak{c}}{c_2 - \dfrac{x_0^2}{a(0)(2-K)}} \right\} & \text{in the (WD) case,} \\ \max \left\{ \dfrac{\mathfrak{c} - 1}{c_2 - \dfrac{(1-x_0)^2}{a(1)(2-K)}}, \dfrac{\mathfrak{c} - 1}{c_2 - \dfrac{x_0^2}{a(0)(2-K)}} \right\} & \text{in the (SD) case.} \end{cases}$$

Thus, by definition of $\tilde{\varphi}$, one can prove as before that there exists a positive constant k, for example

$$k = \max \left\{ \max_{[-\beta_1, \beta_1]} \tilde{a}, \frac{(x_0)^2}{a(0)} \right\},$$

such that

$$\tilde{a}(x) e^{2s\tilde{\varphi}(t,x)} \le k e^{2s\Phi(t,x)}$$

and

$$\frac{(x - x_0)^2}{\tilde{a}(x)} e^{2s\tilde{\varphi}(t,x)} \le k e^{2s\Phi(t,x)}$$

for every $(t, x) \in [0, T] \times [-\beta_1, \beta_1]$. Thus, by (5.21), one has

$$\begin{aligned}
&\int_0^T \int_{-\beta_1}^{\beta_1} \Big(s\Theta \tilde{a}(Z_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{\tilde{a}} Z^2 \Big) e^{2s\tilde{\varphi}} dxdt \\
&\le k \int_0^T \int_{-\beta_1}^{\beta_1} s\Theta (Z_x)^2 e^{2s\Phi} dxdt + k \int_0^T \int_{-\beta_1}^{\beta_1} s^3\Theta^3 Z^2 e^{2s\Phi} dxdt \\
&\le kC \int_0^T \int_\omega v^2 dxdt.
\end{aligned} \tag{5.23}$$

Hence, by (5.23) and the definition of W and Z, we get

$$\begin{aligned}
&\int_0^T \int_0^{\lambda_1} \Big(s^3\Theta^3 \frac{(x-x_0)^2}{a} v^2 + s\Theta a (v_x)^2 \Big) e^{2s\varphi} dxdt \\
&= \int_0^T \int_0^{\lambda_1} \Big(s^3\Theta^3 \frac{(x-x_0)^2}{a} W^2 + s\Theta a (W_x)^2 \Big) e^{2s\varphi} dxdt \\
&\le \int_0^T \int_{-\lambda_1}^{\lambda_1} \Big(s^3\Theta^3 \frac{(x-x_0)^2}{\tilde{a}} W^2 + s\Theta \tilde{a} (W_x)^2 \Big) e^{2s\tilde{\varphi}} dxdt \\
&= \int_0^T \int_{-\lambda_1}^{\lambda_1} \Big(s^3\Theta^3 \frac{(x-x_0)^2}{\tilde{a}} Z^2 + s\Theta \tilde{a} (Z_x)^2 \Big) e^{2s\tilde{\varphi}} dxdt \\
&\le \int_0^T \int_{-\beta_1}^{\beta_1} \Big(s^3\Theta^3 \frac{(x-x_0)^2}{\tilde{a}} Z^2 + s\Theta \tilde{a} (Z_x)^2 \Big) e^{2s\tilde{\varphi}} dxdt \\
&\le C \int_0^T \int_\omega v^2 dxdt,
\end{aligned} \tag{5.24}$$

for a positive constant C.

Therefore, by (5.17) and (5.24), Lemma 5.1 follows. □

PROOF OF LEMMA 5.2. The idea is quite similar to that of the proof of Lemma 5.1, so we will be faster in the calculations. Suppose that $x_0 < \alpha$ (the proof is analogous if we assume that $\beta < x_0$ with obvious adaptations); moreover, set $\lambda := \frac{2\alpha+\beta}{3}$ and $\gamma := \frac{\alpha+2\beta}{3}$, so that $\alpha < \lambda < \gamma < \beta$. Now, fix $\tilde{\alpha} \in (\alpha, \lambda)$, $\tilde{\beta} \in (\gamma, \beta)$ and consider a smooth function $\xi : [0, 1] \to [0, 1]$ such that

$$\xi(x) = \begin{cases} 0 & x \in [0, \tilde{\alpha}], \\ 1 & x \in [\lambda, \gamma], \\ 0 & x \in [\tilde{\beta}, 1]. \end{cases}$$

Then, define $w := \xi v$, where v is any fixed solution of (5.7), so that w satisfies (5.9) with

$$f^2 = ((a\xi_x v)_x + a\xi_x v_x)^2 \le C(v^2 + (v_x)^2)\chi_{\hat{\omega}}, \tag{5.25}$$

where $\hat{\omega} = (\tilde{\alpha}, \lambda) \cup (\gamma, \tilde{\beta})$, by Hypothesis 5.2.

Applying Theorem 4.1 and using the fact that $w \equiv 0$ in a neighborhood of $x = 0$ and $x = 1$, we have

$$\int_0^T \int_0^1 \Big(s\Theta a (w_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a} w^2 \Big) e^{2s\varphi} dxdt \le C \int_0^T \int_0^1 e^{2s\varphi} f^2 dxdt, \tag{5.26}$$

for all $s \geq s_0$. Hence, we find

$$
\begin{aligned}
&\int_0^T \int_\lambda^\gamma \left(s\Theta a(v_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a} v^2 \right) e^{2s\varphi} dxdt \\
&= \int_0^T \int_\lambda^\gamma \left(s\Theta a(w_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a} w^2 \right) e^{2s\varphi}\, dxdt \\
&\leq \int_0^T \int_0^1 \left(s\Theta a(w_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a} w^2 \right) e^{2s\varphi}\, dxdt \\
&\text{(by (5.10) and (5.25))} \qquad (5.27)\\
&\leq C \int_0^T \int_{\hat\omega} e^{2s\varphi}(v^2 + (v_x)^2) dxdt \\
&\text{(by Proposition 5.2 with } \varphi = \Theta\psi\text{, since } \hat\omega \subset\subset \omega\text{, and using the fact that } e^{2s\varphi} \text{ is bounded)} \\
&\leq C \int_0^T \int_\omega v^2 dxdt.
\end{aligned}
$$

Analogously, we define a smooth function $\tau : [0,1] \to [0,1]$ such that

$$
\tau(x) = \begin{cases} 0 & x \in [0, \lambda], \\ 1 & x \in [\gamma, 1]. \end{cases}
$$

Defining $z := \tau v$, then z satisfies

$$
\begin{cases} z_t + (az_x)_x = h, & (t,x) \in (0,T) \times (\alpha, 1) \\ z(t,\alpha) = z(t,1) = 0, & t \in (0,T), \end{cases} \tag{5.28}
$$

with $h := (a\tau_x v)_x + a\tau_x v_x$, which is supported in $\tilde\omega = (\lambda, \gamma)$.

Observe that (5.28) is a nondegenerate problem, hence, thanks to Remark 4, we can apply the classical Carleman estimate (3.4) with $A = \alpha$ and $B = 1$, obtaining

$$
\begin{aligned}
&\int_0^T \int_\alpha^1 s\Theta(z_x)^2 e^{2s\Phi} dxdt + \int_0^T \int_\alpha^1 s^3\Theta^3 z^2 e^{2s\Phi} dxdt \\
&\leq c \int_0^T \int_\alpha^1 e^{2s\Phi} h^2 dxdt,
\end{aligned}
$$

where $r > 0$, $s \geq s_0$ and $c > 0$. Let us note that the boundary term which appears in the original estimate is nonpositive and thus is neglected.

Now, we use Proposition 5.2, getting as above

$$
\begin{aligned}
&\int_0^T \int_\alpha^1 s\Theta(z_x)^2 e^{2s\Phi} dxdt + \int_0^T \int_\alpha^1 s^3\Theta^3 z^2 e^{2s\Phi} dxdt \\
&\leq C \int_0^T \int_{\tilde\omega} e^{2s\Phi}(v^2 + (v_x)^2) dxdt \leq C \int_0^T \int_{\tilde\omega} v^2 dxdt + C \int_0^T \int_{\tilde\omega} e^{2s\Phi}(v_x)^2 dxdt \\
&\leq C \int_0^T \int_\omega v^2 dxdt.
\end{aligned} \tag{5.29}
$$

Now, choose the constant c_1 in (4.3) so that

$$c_1 \geq \begin{cases} \dfrac{r\left[\displaystyle\int_\alpha^1 \frac{1}{\sqrt{a(t)}}\int_t^1 \mathfrak{g}(s)dsdt + \int_\alpha^1 \frac{\mathfrak{h}_0}{\sqrt{a(t)}}dt\right] + \mathfrak{c}}{c_2 - \dfrac{(1-x_0)^2}{a(1)(2-K)}} & \text{in the (WD) case,} \\ \dfrac{\mathfrak{c}-1}{c_2 - \dfrac{(1-x_0)^2}{a(1)(2-K)}} & \text{in the (SD) case.} \end{cases}$$

Then, by definition of φ and the choice of c_1, one can prove that there exists a positive constant k, for example

$$k = \max\left\{\max_{[\alpha,1]} a(x), \frac{(1-x_0)^2}{a(1)}\right\},$$

such that

$$a(x)e^{2s\varphi(t,x)} \leq ke^{2s\Phi(t,x)}$$

and

$$\frac{(x-x_0)^2}{a(x)}e^{2s\varphi(t,x)} \leq ke^{2s\Phi(t,x)}$$

$\forall\,(t,x) \in [0,T] \times [\alpha, 1]$. Thus, by (5.14) and (5.15), via (5.29), we find

$$\begin{aligned} &\int_0^T\int_\alpha^1 \left(s\Theta a(z_x)^2 + s^3\Theta^3\frac{(x-x_0)^2}{a}z^2\right)e^{2s\varphi}dxdt \\ &\leq k\int_0^T\int_\alpha^1 s\Theta(z_x)^2e^{2s\Phi}dxdt + k\int_0^T\int_\alpha^1 s^3\Theta^3z^2e^{2s\Phi}dxdt \\ &\leq C\int_0^T\int_\omega v^2dxdt, \end{aligned}$$

for a positive constant C and s large enough. Hence, by definition of z and by the inequality above, we get

$$\begin{aligned} &\int_0^T\int_\gamma^1 \left(s\Theta a(v_x)^2 + s^3\Theta^3\frac{(x-x_0)^2}{a}v^2\right)e^{2s\varphi}dxdt \\ &= \int_0^T\int_\gamma^1 \left(s\Theta a(z_x)^2 + s^3\Theta^3\frac{(x-x_0)^2}{a}z^2\right)e^{2s\varphi}dxdt \\ &\leq \int_0^T\int_\alpha^1 \left(s\Theta a(z_x)^2 + s^3\Theta^3\frac{(x-x_0)^2}{a}z^2\right)e^{2s\varphi}dxdt \\ &\leq C\int_0^T\int_\omega v^2dxdt, \end{aligned} \tag{5.30}$$

for a positive constant C and for s large enough.

Thus (5.27) and (5.30) imply

$$\int_0^T\int_\lambda^1 \left(s\Theta a(v_x)^2 + s^3\Theta^3\frac{(x-x_0)^2}{a}v^2\right)e^{2s\varphi}\,dxdt \leq C\int_0^T\int_\omega v^2dxdt, \tag{5.31}$$

for some positive constant C and $s \geq s_0$.

To complete the proof it is sufficient to prove a similar inequality for $x \in [0,\lambda]$. To this aim, we follow the reflection procedure introduced in the proof of Lemma 5.1:

consider the functions W and $\tilde a$ introduced in (5.18) and (5.19), so that W satisfies (5.20).

Now, consider a smooth function $\rho : [-1,1] \to [0,1]$ such that

$$\rho(x) = \begin{cases} 0 & x \in [-1,-\gamma], \\ 1 & x \in [-\lambda, \lambda], \\ 0 & x \in [\gamma, 1], \end{cases}$$

and define $Z := \rho W$; thus Z satisfies

$$\begin{cases} Z_t + (\tilde a Z_x)_x = \tilde h, & (t,x) \in (0,T) \times (-\beta,\beta), \\ Z(t,-\beta) = Z(t,\beta) = 0, & t \in (0,T), \end{cases} \tag{5.32}$$

with $\tilde h = (\tilde a \rho_x W)_x + \tilde a \rho_x W_x$, which is supported in $[-\gamma,-\lambda] \cup [\lambda,\gamma]$.

Now, define $\tilde\varphi(t,x) := \Theta(t)\tilde\psi(x)$, where $\tilde\psi(x)$ is defined as in (5.22). Using the analogue of Theorem 4.1 on $(-\beta,\beta)$ in place of $(0,1)$ and with φ replaced by $\tilde\varphi$, the equalities $Z_x(t,-\beta) = Z_x(t,\beta) = 0$, and the definition of W, we get

$$\begin{aligned}
&\int_0^T \int_{-\beta}^{\beta} \left(s\Theta \tilde a (Z_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{\tilde a} Z^2 \right) e^{2s\tilde\varphi} dxdt \\
&\le c \int_0^T \int_{-\beta}^{\beta} e^{2s\tilde\varphi} \tilde h^2 dxdt \\
&\le C \int_0^T \int_{-\gamma}^{-\lambda} (W^2 + (W_x)^2) e^{2s\tilde\varphi} dxdt + C \int_0^T \int_{\lambda}^{\gamma} (W^2 + (W_x)^2) e^{2s\varphi} dxdt \\
&\text{(since } \tilde\psi(x) = \psi(-x) \text{ for } x < 0) \\
&= 2C \int_0^T \int_{\lambda}^{\gamma} (W^2 + (W_x)^2) e^{2s\varphi} dxdt = 2C \int_0^T \int_{\lambda}^{\gamma} (v^2 + (v_x)^2) e^{2s\varphi} dxdt \\
&\text{(by Propositions 5.2)} \\
&\le C \int_0^T \int_{\omega} v^2 dxdt,
\end{aligned}$$

for some positive constants c and C and s large enough.

Hence, by the definitions of Z, W and ρ, and using the previous inequality one has

$$\begin{aligned}
&\int_0^T \int_0^{\lambda} \left(s\Theta a (v_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a} v^2 \right) e^{2s\varphi} dxdt \\
&= \int_0^T \int_0^{\lambda} \left(s\Theta a (W_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a} W^2 \right) e^{2s\varphi} dxdt \\
&= \int_0^T \int_0^{\lambda} \left(s\Theta a (Z_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a} Z^2 \right) e^{2s\varphi} dxdt \\
&\le \int_0^T \int_0^{\beta} \left(s\Theta a (Z_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a} Z^2 \right) e^{2s\varphi} dxdt \\
&\le \int_0^T \int_{-\beta}^{\beta} \left(s\Theta \tilde a (Z_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{\tilde a} Z^2 \right) e^{2s\varphi} dxdt \\
&\le C \int_0^T \int_{\omega} v^2 dxdt,
\end{aligned} \tag{5.33}$$

for a positive constant C and s large enough. Therefore, by (5.31) and (5.33), the conclusion follows. $\square$

We shall also use the following

LEMMA 5.3. *Assume Hypotheses* 5.1 *and* 5.2. *Then there exists a positive constant* C_T *such that every solution* $v \in \mathcal{W}_1$ *of* (5.7) *satisfies*

$$\int_0^1 v^2(0,x)dx \le C_T \int_0^T \int_\omega v^2(t,x)dxdt.$$

PROOF. Multiplying the equation of (5.7) by v_t and integrating by parts over $(0,1)$, one has

$$0 = \int_0^1 (v_t + (av_x)_x)v_t dx = \int_0^1 (v_t^2 + (av_x)_x v_t)dx = \int_0^1 v_t^2 dx + [av_x v_t]_{x=0}^{x=1}$$
$$- \int_0^1 av_x v_{tx} dx = \int_0^1 v_t^2 dx - \frac{1}{2}\frac{d}{dt}\int_0^1 a(v_x)^2 \ge -\frac{1}{2}\frac{d}{dt}\int_0^1 a(v_x)^2 dx.$$

Thus, the function $t \mapsto \int_0^1 a(v_x)^2 dx$ is increasing for all $t \in [0,T]$. In particular,

$$\int_0^1 av_x(0,x)^2 dx \le \int_0^1 av_x(t,x)^2 dx \text{ for every } t \in [0,T].$$

Integrating the last inequality over $\left[\frac{T}{4}, \frac{3T}{4}\right]$, Θ being bounded therein, we find

$$\begin{aligned}\int_0^1 a(v_x)^2(0,x)dx &\le \frac{2}{T}\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^1 a(v_x)^2(t,x)dxdt \\ &\le C_T \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^1 s\Theta a(v_x)^2(t,x)e^{2s\varphi}dxdt.\end{aligned}$$

Hence, by Lemma 5.1 or by Lemma 5.2 and the previous inequality, there exists a positive constant C such that

$$\int_0^1 a(v_x)^2(0,x)dx \le C\int_0^T \int_\omega v^2 dxdt. \tag{5.34}$$

Proceeding again as in the proof of Lemma 4.1 and applying the Hardy-Poincaré inequality, by (5.34), one has

$$\begin{aligned}\int_0^1 \left(\frac{a}{(x-x_0)^2}\right)^{1/3} v^2(0,x)dx &\le \int_0^1 \frac{p}{(x-x_0)^2} v^2(0,x)dx \\ &\le C_{HP}\int_0^1 p(v_x)^2(0,x)dx \\ &\le \max\{C_1, C_2\}C_{HP}\int_0^1 a(v_x)^2(0,x)dx \\ &\le C\int_0^T \int_\omega v^2 dxdt,\end{aligned}$$

for a positive constant C. Here $p(x) = (a(x)|x-x_0|^4)^{1/3}$ if $K > \dfrac{4}{3}$ or $p(x) = \max_{[0,1]} a|x-x_0|^{4/3}$ otherwise,

$$C_1 := \max\left\{\left(\frac{x_0^2}{a(0)}\right)^{2/3}, \left(\frac{(1-x_0)^2}{a(1)}\right)^{2/3}\right\},$$

$C_2 := \max\left\{\dfrac{x_0^{4/3}}{a(0)}, \dfrac{(1-x_0)^{4/3}}{a(1)}\right\}$ and C_{HP} is the Hardy-Poincaré constant, as before.

By Lemma 2.1, the function $x \mapsto \dfrac{a(x)}{(x-x_0)^2}$ is nondecreasing on $[0, x_0)$ and nonincreasing on $(x_0, 1]$; then

$$\left(\frac{a(x)}{(x-x_0)^2}\right)^{1/3} \geq C_3 := \min\left\{\left(\frac{a(1)}{(1-x_0)^2}\right)^{1/3}, \left(\frac{a(0)}{x_0^2}\right)^{1/3}\right\} > 0.$$

Hence

$$C_3 \int_0^1 v(0,x)^2 dx \leq C \int_0^T \int_\omega v^2 dx dt$$

and the thesis follows. □

Proof of Proposition 5.1. The proof is now standard, but we give it with some precise references: let $v_T \in L^2(0,1)$ and let v be the solution of (5.4) associated to v_T. Since $D(\mathcal{A}_1^2)$ is densely defined in $L^2(0,1)$, there exists a sequence $(v_T^n)_n \subset D(\mathcal{A}_1^2)$ which converges to v_T in $L^2(0,1)$. Now, consider the solution v_n associated to v_T^n.

As shown in Theorem 2.1, the semigroup generated by $\mathcal{A}_1$ is analytic, hence $\mathcal{A}_1$ is closed (for example, see [**24**, Theorem I.1.4]; thus, by [**24**, Theorem II.6.7], we get that $(v_n)_n$ converges to a certain v in $C([0,T]; L^2(0,1))$, so that

$$\lim_{n\to+\infty} \int_0^1 v_n^2(0,x)dx = \int_0^1 v^2(0,x)dx,$$

and also

$$\lim_{n\to+\infty} \int_0^T \int_\omega v_n^2 dx dt = \int_0^T \int_\omega v^2 dx dt.$$

But, by Lemma 5.3 we know that

$$\int_0^1 v_n^2(0,x)dx \leq C_T \int_0^T \int_\omega v_n^2 dx dt.$$

Thus Proposition 5.1 is now proved. □

2. The non divergence case

In this section we make the following assumptions on the degenerate function a (see also Remark 9 concerning the divergence case):

HYPOTHESIS 5.3. Hypothesis 4.2 is satisfied. Moreover, if Hypothesis 1.1 holds, then there exist two functions $\mathfrak{g} \in L^\infty_{\text{loc}}([0,1]\setminus\{x_0\})$, $\mathfrak{h} \in W^{1,\infty}_{\text{loc}}([0,1]\setminus\{x_0\}, L^\infty(0,1))$ and two strictly positive constants $\mathfrak{g}_0$, $\mathfrak{h}_0$ such that $\mathfrak{g}(x) \geq \mathfrak{g}_0$ for a.e. x in $[0,1]$ and

$$\frac{a'(x)}{2\sqrt{a(x)}}\left(\int_x^B \mathfrak{g}(t)dt + \mathfrak{h}_0\right) + \sqrt{a(x)}\mathfrak{g}(x) = \mathfrak{h}(x,B) \quad \text{for a.e. } x, B \in [0,1] \tag{5.35}$$

with $x < B < x_0$ or $x_0 < x < B$.

As for the case in divergence form, by the Carleman estimates given in Theorems 3.2 and 4.2, we will deduce a fundamental observability inequality for the homogeneous adjoint problem to (1.2), i.e.

$$\begin{cases} v_t + av_{xx} = 0, & (t,x) \in Q_T, \\ v(t,0) = v(t,1) = 0, & t \in (0,T), \\ v(T,x) = v_T(x) \in L^2_{\frac{1}{a}}(0,1), \end{cases} \tag{5.36}$$

where $T > 0$ is given. Such an observability inequality will hold true both in the weakly and in the strongly degenerate cases, as the next proposition shows.

PROPOSITION 5.3. *Assume* (5.2) *and Hypothesis* 5.3. *Then there exists a positive constant* C_T *such that the solution* $v \in C([0,T]; L^2_{\frac{1}{a}}(0,1)) \cap L^2(0,T; \mathcal{H}^1_{\frac{1}{a}}(0,1))$ *of* (5.36) *satisfies*

$$\int_0^1 v^2(0,x)\frac{1}{a}dx \leq C_T \int_0^T \int_\omega v^2 \frac{1}{a} dxdt. \tag{5.37}$$

Using inequality (5.37) the following null controllability result for (1.2) in non divergence form holds:

THEOREM 5.3. *Assume* (5.2) *and Hypothesis* 5.3. *Then, given* $u_0 \in L^2_{\frac{1}{a}}(0,1)$, *there exists* $h \in L^2_{\frac{1}{a}}(Q_T)$ *such that the solution* u *of* (1.2) *satisfies*

$$u(T,x) = 0 \quad \text{for every} \quad x \in [0,1].$$

Moreover

$$\int_0^T \int_0^1 h^2 \frac{1}{a} dxdt \leq C \int_0^1 u_0^2 \frac{1}{a} dx,$$

for some positive constant C *independent of* u_0.

We refer to Comment 2 in Chapter 7 to explain why in Proposition 5.3 and Theorem 5.1 we consider only the case in which the degeneracy point is *outside* the control region.

As for the divergence case, a straightforward consequence of Proposition 5.3 and of Theorem 5.3 are the following results, which are of interest when the control region lies on both the two sides of the degeneracy point.

COROLLARY 5.2. *Assume Hypothesis* 5.3 *and* (5.6). *Then there exists a positive constant* C_T *such that every solution* $v \in \mathcal{W}_2$ *of* (5.38) *satisfies*

$$\int_0^1 v^2(0,x)\frac{1}{a}dx \leq C_T \int_0^T \int_\omega v^2(t,x)\frac{1}{a}dxdt.$$

As a standard consequence one has the next null controllability result:

THEOREM 5.4. *Assume Hypothesis* 5.3 *and* (5.6)*. Then, given* $u_0 \in L^2_{\frac{1}{a}}(0,1)$*, there exists* $h \in L^2_{\frac{1}{a}}(Q_T)$ *such that the solution* u *of* (1.2) *satisfies*

$$u(T,x) = 0 \quad \text{for every} \quad x \in [0,1].$$

Moreover

$$\int_0^T \int_0^1 h^2 \frac{1}{a} dxdt \le C \int_0^1 u_0^2(x) \frac{1}{a} dx$$

for some positive constant C.

2.1. Proof of Proposition 5.3**.** As for the problem in divergence form, we will start giving some preliminary results for the following homogeneous adjoint final–time problems having *more regular final–time datum*:

$$\begin{cases} v_t + av_{xx} = 0, & (t,x) \in Q_T, \\ v(t,0) = v(t,1) = 0, & t \in (0,T), \\ v(T,x) = v_T(x) \in D(\mathcal{A}_2^2), \end{cases} \tag{5.38}$$

where, we recall, $\mathcal{A}_2 u := au_{xx}$ with $D(\mathcal{A}_2) = \mathcal{H}^2_{\frac{1}{a}}(0,1)$, and

$$D(\mathcal{A}_2^2) = \Big\{ u \in D(\mathcal{A}_2) \;\big|\; \mathcal{A}_2 u \in D(\mathcal{A}_2) \Big\}.$$

Observe that $D(\mathcal{A}_2^2)$ is densely defined in $D(\mathcal{A}_2)$ (see, for example, [**12**, Lemma VII.2]) and hence in $L^2_{\frac{1}{a}}(0,1)$. As in [**16**], [**17**] or [**31**], letting v_T vary in $D(\mathcal{A}_2^2)$, we introduce the class of solutions to (5.38), i.e.

$$\mathcal{W}_2 := \Big\{ v \in C^1\big([0,T]; L^2_{\frac{1}{a}}(0,1)\big) \cap C\big([0,T]; D(\mathcal{A}_2)\big) \mid v \text{ solves } (5.38) \Big\},$$

with the obvious meaning that it is a class and not a set of one function, since v_T vary.

Obviously (see, for example, [**12**, Theorem VII.5])

$$\mathcal{W}_2 \subset C^1\big([0,T];\, \mathcal{H}^2_{\frac{1}{a}}(0,1)\big) \subset \mathcal{S}_2 \subset \mathcal{U}_2,$$

where $\mathcal{S}_2$ is defined in (4.19) and

$$\mathcal{U}_2 := C([0,T]; L^2_{\frac{1}{a}}(0,1)) \cap L^2(0,T; \mathcal{H}^1_{\frac{1}{a}}(0,1)).$$

Also in this case the Caccioppoli's inequality is crucial. In the non divergence case it reads as follows:

PROPOSITION 5.4 (Caccioppoli's inequality). *Assume that either Hypothesis* 1.1 *and* (5.35) *or Hypothesis* 1.2 *hold. Let* ω' *and* ω *two open subintervals of* $(0,1)$ *such that* $\omega' \subset\subset \omega \subset (0,1)$ *and* $x_0 \notin \overline{\omega}$*. Let* $\varphi(t,x) = \Theta(t)\Upsilon(x)$*, where* Θ *is defined in* (4.3) *and*

$$\Upsilon \in C([0,1], (-\infty,0)) \cap C^1([0,1] \setminus \{x_0\}, (-\infty,0))$$

satisfies (5.8)*. Then, there exist two positive constants* C *and* s_0 *such that every solution* $v \in \mathcal{W}_2$ *of the adjoint problem* (5.38) *satisfies*

$$\int_0^T \int_{\omega'} (v_x)^2 e^{2s\varphi} dxdt \;\le\; C \int_0^T \int_{\omega} v^2 dxdt \;\le\; C \int_0^T \int_{\omega} v^2 \frac{1}{a} dxdt,$$

for all $s \ge s_0$.

Observe that we require $x_0 \notin \bar{\omega}$, since in the applications below the control region ω is assumed to satisfy (5.2). Moreover, as in Remark 10, one can prove that our prototype for Υ is the function ψ defined in (4.18).

PROOF OF PROPOSITION 5.4. The proof is an adaptation of the one of [**33**, Proposition 4.2], so we will skip some details. Let us consider a smooth function $\xi : [0,1] \to \mathbb{R}$ such that

$$\begin{cases} 0 \le \xi(x) \le 1 & \text{for all } x \in [0,1], \\ \xi(x) = 1 & x \in \omega', \\ \xi(x) = 0 & x \in [0,1] \setminus \omega. \end{cases}$$

Hence, by the very definition of φ, we have

$$\begin{aligned} 0 &= \int_0^T \frac{d}{dt}\left(\int_0^1 \xi^2 e^{2s\varphi} v^2 dx\right) dt = \int_{Q_T} \left(2s\xi^2\varphi_t e^{2s\varphi} v^2 + 2\xi^2 e^{2s\varphi} v v_t\right) dxdt \\ &\text{(since } v \text{ solves (5.38) and has homogeneous boundary conditions)} \\ &= 2\int_{Q_T} s\xi^2\varphi_t e^{2s\varphi} v^2 dxdt + 2\int_{Q_T} (\xi^2 e^{2s\varphi} a)_x v v_x dxdt \\ &+ 2\int_{Q_T} \xi^2 e^{2s\varphi} a (v_x)^2 dxdt. \end{aligned}$$

Therefore, by definition of ξ, the previous identity gives

$$\begin{aligned} &2\int_0^T \int_\omega \xi^2 e^{2s\varphi} a (v_x)^2 dxdt = -2\int_0^T \int_\omega s\xi^2 \varphi_t e^{2s\varphi} v^2 dxdt \\ &- 2\int_0^T \int_\omega \left(\xi^2 e^{2s\varphi} a\right)_x \frac{\xi e^{s\varphi}\sqrt{a}}{\xi e^{s\varphi}\sqrt{a}} v v_x \, dxdt \\ &\text{(by the Cauchy–Schwarz inequality)} \\ &\le -2\int_0^T \int_\omega s\xi^2\varphi_t e^{2s\varphi} v^2 dxdt + \int_0^T \int_\omega \left(\xi e^{s\varphi}\sqrt{a} v_x\right)^2 dxdt \\ &+ \int_0^T \int_\omega \left(\frac{(\xi^2 e^{2s\varphi} a)_x}{\xi e^{s\varphi}\sqrt{a}} v\right)^2 dxdt \\ &= -2\int_0^T \int_\omega s\xi^2\varphi_t e^{2s\varphi} v^2 dxdt + \int_0^T \int_\omega \xi^2 e^{2s\varphi} a (v_x)^2 dxdt \\ &+ 4\int_0^T \int_\omega [\left(\xi e^{s\varphi}\sqrt{a}\right)_x]^2 v^2 dxdt. \end{aligned}$$

Thus

$$\begin{aligned} \int_0^T \int_\omega \xi^2 e^{2s\varphi} a (v_x)^2 dxdt &\le -2\int_0^T \int_\omega s\xi^2\varphi_t e^{2s\varphi} v^2 dxdt \\ &\quad + 4\int_0^T \int_\omega [\left(\xi e^{s\psi}\sqrt{a}\right)_x]^2 v^2 dxdt. \end{aligned}$$

Since $x_0 \notin \overline{\omega'}$, then

$$\begin{aligned}\inf_{\omega'} a(x)\int_0^T\int_{\omega'} e^{2s\varphi}(v_x)^2dxdt &\le \int_0^T\int_{\overline{\omega'}} \xi^2 e^{2s\varphi}a(v_x)^2dxdt\\ &\le \int_0^T\int_{\omega} \xi^2 e^{2s\varphi}a(v_x)^2dxdt\\ &\le -2\int_0^T\int_{\omega} s\xi^2\varphi_t e^{2s\varphi}v^2dxdt + 4\int_0^T\int_{\omega}[(\xi e^{s\varphi}\sqrt{a})_x]^2v^2dxdt.\end{aligned}$$

As in [**33**, Proposition 4.2], one can show that $s\varphi_t e^{2s\varphi}$ is uniformly bounded if $s \ge s_0 > 0$, since Υ is strictly negative, a rough estimate being

$$|s\varphi_t e^{2s\varphi}| \le cs|\Upsilon|\Theta^{5/4}e^{2s\Upsilon\Theta} \le c\frac{1}{s_0^{5/4}(-\max\Upsilon)^{5/4}}.$$

On the other hand, $(\xi e^{s\varphi}\sqrt{a})_x$ can be estimated by

$$C\left(e^{2s\varphi} + s^2(\varphi_x)^2e^{2s\varphi} + e^{2s\varphi}\frac{(a')^2}{a}\right).$$

Of course, $e^{2s\varphi} < 1$, $\dfrac{(a')^2}{a}$ exists and is bounded in ω since $x_0 \notin \bar\omega$ and (5.35) holds with Hypothesis 1.1 or Hypothesis 1.2 is in force, while $s^2(\varphi_x)^2e^{2s\varphi}$ can be estimated with

$$\frac{c}{(-\max\Upsilon)^2}(\Upsilon_x)^2 \le \frac{c}{a}$$

by (5.8), for some constants $c > 0$ (see [**33**, Proposition 4.2]). Hence, there exists a positive constant C such that

$$\begin{aligned}-2\int_0^T\int_{\omega} s\xi^2\varphi_t e^{2s\varphi}v^2dxdt + 4\int_0^T\int_{\omega}[(\xi e^{s\varphi}\sqrt{a})_x]^2v^2dxdt\\ \le C\int_0^T\int_{\omega} v^2dxdt \le C\int_0^T\int_{\omega} v^2\frac{1}{a}dxdt,\end{aligned}$$

and the claim follows. □

LEMMA 5.4. *Assume* (5.2) *and Hypothesis* 5.3. *Then there exist two positive constants C and s_0 such that every solution $v \in \mathcal{W}_2$ of* (5.38) *satisfies*

$$\int_{Q_T}\left(s\Theta(v_x)^2 + s^3\Theta^3\left(\frac{x-x_0}{a}\right)^2v^2\right)e^{2s\varphi}dxdt \le C\int_0^T\int_{\omega} v^2\frac{1}{a}dxdt$$

for all $s \ge s_0$ and d_1 sufficiently large.

PROOF. Suppose that $x_0 < \alpha$ (the proof is similar if we assume that $\beta < x_0$ with simple adaptations); moreover, set $\lambda := \frac{2\alpha+\beta}{3}$ and $\gamma := \frac{\alpha+2\beta}{3}$, so that $\alpha < \lambda < \gamma < \beta$. Now, fix $\tilde\alpha \in (\alpha, \lambda)$ and $\tilde\beta \in (\gamma, \beta)$ and consider a smooth function $\xi : [0, 1] \to \mathbb{R}$ such that

$$\begin{cases} 0 \le \xi(x) \le 1, & \text{for all } x \in [0,1],\\ \xi(x) = 1, & x \in [\lambda, \gamma],\\ \xi(x) = 0, & x \in [0,1]\setminus(\tilde\alpha, \tilde\beta).\end{cases}$$

Define $w := \xi v$, where v is any fixed solution of (5.38). Hence, neglecting the final–time datum (of no interest in this context), w satisfies

$$\begin{cases} w_t + a w_{xx} = a(\xi_{xx} v + 2\xi_x v_x) =: f, & (t,x) \in (0,T) \times (0,1), \\ w(t,0) = w(t,1) = 0, & t \in (0,T). \end{cases}$$

Applying Theorem 4.2 and using the fact that $w \equiv 0$ in a neighborhood of $x = 0$ and $x = 1$, we have

$$\int_0^T \int_0^1 \left(s\Theta (w_x)^2 + s^3 \Theta^3 \left(\frac{x - x_0}{a} \right)^2 w^2 \right) e^{2s\varphi} \, dxdt \le C \int_{Q_T} \frac{e^{2s\varphi}}{a} f^2 dxdt, \tag{5.39}$$

for all $s \ge s_0$. Then, using the definition of ξ and in particular the fact that ξ_x and ξ_{xx} are supported in $\hat{\omega}$, where $\hat{\omega} := (\tilde{\alpha}, \lambda) \cup (\gamma, \tilde{\beta})$, we can write

$$\frac{f^2}{a} = a(\xi_{xx} v + 2\xi_x v_x)^2 \le C(v^2 + (v_x)^2)\chi_{\hat{\omega}}. \tag{5.40}$$

Hence, we find

$$\begin{aligned} &\int_0^T \int_\lambda^\gamma \left(s\Theta (v_x)^2 + s^3 \Theta^3 \left(\frac{x - x_0}{a} \right)^2 v^2 \right) e^{2s\varphi} dxdt \\ &= \int_0^T \int_\lambda^\gamma \left(s\Theta (w_x)^2 + s^3 \Theta^3 \left(\frac{x - x_0}{a} \right)^2 w^2 \right) e^{2s\varphi} \, dxdt \\ &\le \int_{Q_T} \left(s\Theta (w_x)^2 + s^3 \Theta^3 \left(\frac{x - x_0}{a} \right)^2 w^2 \right) e^{2s\varphi} \, dxdt \\ &\text{(by (5.39) and (5.40))} \\ &\le C \int_0^T \int_{\hat{\omega}} e^{2s\varphi} (v^2 + (v_x)^2) dxdt \\ &\text{(by Proposition 5.4 with } \varphi = \Theta\psi, \text{ since } \hat{\omega} \subset\subset \omega, \text{ and using the fact that} \\ &ae^{2s\varphi} \text{ is bounded)} \\ &\le C \int_0^T \int_\omega \frac{v^2}{a} x dt, \end{aligned} \tag{5.41}$$

Now, consider a smooth function $\eta : [0,1] \to \mathbb{R}$ such that

$$\begin{cases} 0 \le \eta(x) \le 1, & \text{for all } x \in [0,1], \\ \eta(x) = 1, & x \in [\gamma, 1], \\ \eta(x) = 0, & x \in [0, \lambda], \end{cases}$$

and define $z := \eta v$. Then z satisfies

$$\begin{cases} z_t + a z_{xx} = h, & (t,x) \in (0,T) \times (\alpha, 1) \\ z(t,\alpha) = z(t,1) = 0, & t \in (0,T), \end{cases} \tag{5.42}$$

with $h := a(\eta_{xx} v + 2\eta_x v_x) \in L^2\big((0,T) \times (\alpha,1)\big)$. Observe that (5.42) is non degenerate, since $x \in (\alpha, 1)$.

Moreover, since the problem is *non degenerate*, we can apply Theorem 3.2 with $A = \alpha$, $B = 1$ and Remark 5, obtaining

$$\int_0^T \int_\alpha^1 \left(s\Theta (z_x)^2 + s^3 \Theta^3 z^2 \right) e^{2s\Phi} dxdt \le C \int_0^T \int_\alpha^1 h^2 e^{2s\Phi} dxdt,$$

for $s \geq s_0$. Let us note that the boundary term which appears in the original estimate is nonpositive and thus is neglected.

Now, we use the analogue of (5.40) for h, Proposition 5.4 and, recalling what the support of η is, we get

(5.43)
$$\begin{aligned}&\int_0^T \int_\alpha^1 \left(s\Theta(z_x)^2 + s^3\Theta^3 z^2\right) e^{2s\Phi} dxdt \\ &\leq C\int_0^T \int_{\tilde\omega} e^{2s\Phi}(v^2 + (v_x)^2) dxdt \leq C\int_0^T \int_{\tilde\omega} v^2 dxdt + C\int_0^T \int_{\tilde\omega} e^{2s\Phi}(v_x)^2 dxdt \\ &\leq C\int_0^T \int_\omega \frac{v^2}{a} dxdt\end{aligned}$$

where $\tilde\omega = (\lambda, \gamma)$.

Now, choose the constant d_1 in (4.18) so that

(5.44)
$$d_1 \geq \begin{cases} \dfrac{r\left[\displaystyle\int_\alpha^1 \frac{1}{\sqrt{a(t)}} \int_t^1 \mathfrak{g}(\tau) d\tau dt + \int_\alpha^1 \frac{\mathfrak{h}_0}{\sqrt{a(t)}} dt\right] + \mathfrak{c}}{d_2 - \dfrac{(1-x_0)^2 e^{R(1-x_0)^2}}{a(1)(2-K)}} & \text{in the (WD) case,} \\ \dfrac{\mathfrak{c} - 1}{d_2 - \dfrac{(1-x_0)^2 e^{R(1-x_0)^2}}{a(1)(2-K)}} & \text{in the (SD) case.} \end{cases}$$

Then, by definition of φ and the choice of d_1, one can prove that there exists a positive constant k, for example

$$k = \max\left\{1, \left(\frac{1-x_0}{a(1)}\right)^2\right\},$$

such that

$$e^{2s\varphi(t,x)} \leq k e^{2s\Phi(t,x)} \tag{5.45}$$

and

$$\left(\frac{x-x_0}{a(x)}\right)^2 e^{2s\varphi(t,x)} \leq k e^{2s\Phi(t,x)} \tag{5.46}$$

$\forall\,(t,x) \in [0,T] \times [\alpha, 1]$.

Thus, by (5.45) and (5.46), via (5.43), we find

$$\begin{aligned}&\int_0^T \int_\alpha^1 \left(s\Theta(z_x)^2 + s^3\Theta^3 \left(\frac{x-x_0}{a}\right)^2 z^2\right) e^{2s\varphi} dxdt \\ &\leq k\int_0^T \int_\alpha^1 s\Theta(z_x)^2 e^{2s\Phi} dxdt + k\int_0^T \int_\alpha^1 s^3\Theta^3 z^2 e^{2s\Phi} dxdt \\ &\leq C\int_0^T \int_\omega \frac{v^2}{a} dxdt,\end{aligned}$$

for a positive constant C and s large enough. Hence, by definition of z and by the inequality above, we get

$$\begin{aligned}&\int_0^T\int_\gamma^1 \left(s\Theta(v_x)^2 + s^3\Theta^3\left(\frac{x-x_0}{a}\right)^2 v^2\right)e^{2s\varphi}dxdt\\ &= \int_0^T\int_\gamma^1 \left(s\Theta(z_x)^2 + s^3\Theta^3\left(\frac{x-x_0}{a}\right)^2 z^2\right)e^{2s\varphi}dxdt\\ &\le \int_0^T\int_\alpha^1 \left(s\Theta(z_x)^2 + s^3\Theta^3\left(\frac{x-x_0}{a}\right)^2 z^2\right)e^{2s\varphi}dxdt\\ &\le C\int_0^T\int_\omega \frac{v^2}{a}dxdt,\end{aligned} \tag{5.47}$$

for a positive constant C and for s large enough.

Thus (5.41) and (5.47) imply

$$\int_0^T\int_\lambda^1 \left(s\Theta(v_x)^2 + s^3\Theta^3\left(\frac{x-x_0}{a}\right)^2 v^2\right)e^{2s\varphi}\,dxdt \le C\int_0^T\int_\omega \frac{v^2}{a}dxdt, \tag{5.48}$$

for some positive constant C and $s \ge s_0$. To complete the proof it is sufficient to prove a similar inequality for $x \in [0, \lambda]$. To this aim, we follow a reflection procedure already introduced in [**33**], considering the functions

$$W(t,x) := \begin{cases} v(t,x), & x \in [0,1],\\ -v(t,-x), & x \in [-1,0]\end{cases}$$

and

$$\tilde{a}(x) := \begin{cases} a(x), & x \in [0,1],\\ a(-x), & x \in [-1,0],\end{cases}$$

so that W satisfies the problem

$$\begin{cases} W_t + \tilde{a}W_{xx} = 0, & (t,x) \in (0,T)\times(-1,1),\\ W(t,-1) = W(t,1) = 0, & t \in (0,T).\end{cases}$$

Now, consider a cut off function $\rho : [-1,1] \to \mathbb{R}$ such that

$$\begin{cases} 0 \le \rho(x) \le 1, & \text{for all } x \in [-1,1],\\ \rho(x) = 1, & x \in [-\lambda,\lambda],\\ \rho(x) = 0, & x \in [-1,-\gamma]\cup[\gamma,1],\end{cases}$$

and define $Z := \rho W$. Then Z satisfies

$$\begin{cases} Z_t + \tilde{a}Z_{xx} = \tilde{h}, & (t,x) \in (0,T)\times(-\beta,\beta),\\ Z(t,-\beta) = Z(t,\beta) = 0, & t \in (0,T),\end{cases} \tag{5.49}$$

where $\tilde{h} = \tilde{a}\rho_{xx}W + 2\tilde{a}\rho_x W_x$. Now, defining $\tilde{\varphi}(t,x) := \Theta(t)\tilde{\psi}(x)$, with

$$\tilde{\psi}(x) := \begin{cases} \psi(x), & x \ge 0,\\ \psi(-x) = d_1\left[\int_{-x_0}^{x} \frac{t+x_0}{\tilde{a}(t)}e^{R(t+x_0)^2}dt - d_2\right], & x < 0,\end{cases}$$

we use the analogue of Theorem 4.2 on $(-\beta, \beta)$ in place of $(0,1)$ and with φ replaced by $\tilde\varphi$. Moreover, using the fact that $Z_x(t,-\beta) = Z_x(t,\beta) = 0$, the definition of W and the fact that ρ is supported in $[-\gamma,-\lambda]\cup[\lambda,\gamma]$, we get

$$\begin{aligned}
&\int_0^T\int_{-\beta}^{\beta}\left(s\Theta(Z_x)^2 + s^3\Theta^3\left(\frac{x-x_0}{\tilde a}\right)^2 Z^2\right)e^{2s\tilde\varphi}dxdt\\
&\le c\int_0^T\int_{-\beta}^{\beta}\tilde h^2\frac{e^{2s\tilde\varphi}}{\tilde a}dxdt\\
&\le C\int_0^T\int_{-\gamma}^{-\lambda}(W^2+(W_x)^2)e^{2s\tilde\varphi}dxdt + C\int_0^T\int_{\lambda}^{\gamma}(W^2+(W_x)^2)e^{2s\varphi}dxdt\\
&\text{(since } \tilde\psi(x)=\psi(-x), \text{ for } x<0)\\
&= 2C\int_0^T\int_{\lambda}^{\gamma}(W^2+(W_x)^2)e^{2s\varphi}dxdt = 2C\int_0^T\int_{\lambda}^{\gamma}(v^2+(v_x)^2)e^{2s\varphi}dxdt\\
&\text{(by Propositions 5.4)}\\
&\le C\int_0^T\int_{\omega}\frac{v^2}{a}dxdt,
\end{aligned}$$

for some positive constants c and C and s large enough.

Hence, by the definitions of Z, W and ρ, and using the previous inequality one has

$$\begin{aligned}
&\int_0^T\int_0^{\lambda}\left(s\Theta(v_x)^2 + s^3\Theta^3\left(\frac{x-x_0}{a}\right)^2 v^2\right)e^{2s\varphi}dxdt\\
&= \int_0^T\int_0^{\lambda}\left(s\Theta(W_x)^2 + s^3\Theta^3\left(\frac{x-x_0}{a}\right)^2 W^2\right)e^{2s\varphi}dxdt\\
&= \int_0^T\int_0^{\lambda}\left(s\Theta(Z_x)^2 + s^3\Theta^3\left(\frac{x-x_0}{a}\right)^2 Z^2\right)e^{2s\varphi}dxdt\\
&\le \int_0^T\int_0^{\beta}\left(s\Theta(Z_x)^2 + s^3\Theta^3\left(\frac{x-x_0}{a}\right)^2 Z^2\right)e^{2s\varphi}dxdt\\
&\le \int_0^T\int_{-\beta}^{\beta}\left(s\Theta(Z_x)^2 + s^3\Theta^3\left(\frac{x-x_0}{a}\right)^2 Z^2\right)e^{2s\tilde\varphi}dxdt\\
&\le C\int_0^T\int_{\omega}\frac{v^2}{a}dxdt,
\end{aligned}\tag{5.50}$$

for a positive constant C and s large enough. Therefore, by (5.48) and (5.50), the conclusion follows. □

We are now ready to prove the observability inequality in the case of a regular final–time datum:

Lemma 5.5. *Assume* (5.2) *and Hypothesis* 5.3. *Then there exists a positive constant* C_T *such that every solution* $v \in \mathcal{W}_2$ *of* (5.38) *satisfies*

$$\int_0^1 v^2(0,x)\frac{1}{a}dx \le C_T\int_0^T\int_{\omega}v^2\frac{1}{a}dxdt.$$

PROOF. Multiplying the equation of (5.38) by $\dfrac{v_t}{a}$ and integrating by parts over $(0,1)$, one has

$$\begin{aligned}0 &= \int_0^1 (v_t + av_{xx})\frac{v_t}{a}dx = \int_0^1 \frac{(v_t)^2}{a}dx + [v_xv_t]_{x=0}^{x=1} - \int_0^1 v_xv_{tx}dx \\ &= \int_0^1 \frac{(v_t)^2}{a}dx - \frac{1}{2}\frac{d}{dt}\int_0^1 (v_x)^2 \geq -\frac{1}{2}\frac{d}{dt}\int_0^1 (v_x)^2dx.\end{aligned}$$

Thus, the function $t \mapsto \int_0^1 (v_x)^2dx$ is nondecreasing for all $t \in [0,T]$. In particular, $\int_0^1 (v_x)^2(0,x)dx \leq \int_0^1 (v_x)^2(t,x)dx$. Integrating the last inequality over $\left[\frac{T}{4}, \frac{3T}{4}\right]$, we find

$$\begin{aligned}\int_0^1 (v_x)^2(0,x)dx &\leq \frac{2}{T}\int_{\frac{T}{4}}^{\frac{3T}{4}}\int_0^1 (v_x)^2dxdt \\ &\leq C_T\int_{\frac{T}{4}}^{\frac{3T}{4}}\int_0^1 \Theta(v_x)^2e^{2s_0\varphi}dxdt.\end{aligned}$$

Hence, by Lemma 5.4, there exists a positive constant C such that

$$\int_0^1 (v_x)^2(0,x)dx \leq C\int_0^T\int_\omega \frac{v^2}{a}dxdt.$$

First, in the strongly degenerate case, by Lemma 2.3, there exists a positive constant $C > 0$ such that

$$\int_0^1 v^2(t,x)\frac{1}{a}dx \leq C\int_0^1 (v_x)^2(t,x)dx,$$

for all $t \in [0,T]$. Thus, from the previous two inequalities, we get

$$\int_0^1 v^2(0,x)\frac{1}{a}dx \leq C\int_0^T\int_\omega v^2\frac{1}{a}dxdt$$

for a positive constant C, and the conclusion follows.

In the weakly degenerate case, proceeding as in the proof of Lemma 4.3 and applying the Hardy–Poincaré inequality of Proposition 1.1, one has

$$\begin{aligned}\int_0^1 v^2(0,x)\frac{1}{a}dx &= \int_0^1 \frac{p(x)}{(x-x_0)^2}v^2(0,x)dx \\ &\leq C_{HP}\int_0^1 p(x)(v_x)^2(0,x)dx \\ &\leq C_1C_{HP}\int_0^1 (v_x)^2(0,x)dx \leq C\int_0^T\int_\omega \frac{v^2}{a}dxdt,\end{aligned}$$

for a positive constat C. Here $p(x) = \dfrac{(x-x_0)^2}{a(x)}$, C_{HP} is the Hardy-Poincaré constant and $C_1 := \max\left\{\dfrac{x_0^2}{a(0)}, \dfrac{(1-x_0)^2}{a(1)}\right\}$. Observe that the function p satisfies the assumptions of Proposition 1.1 (with $q = 2-K$) thanks to Lemma 2.1. Hence, also in this case, the conclusion follows. □

Using Lemma 5.5 and proceeding as in the proof of Proposition 5.1, one can prove Proposition 5.3.

CHAPTER 6

Linear and Semilinear Extensions

In this chapter we will extend the global null controllability result proved in the previous chapter to the linear problem

$$(6.1)\qquad \begin{cases} u_t - \mathcal{A}u + c(t,x)u = h(t,x)\chi_\omega(x), & (t,x) \in (0,T) \times (0,1), \\ u(t,1) = u(t,0) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,1), \end{cases}$$

where $u_0 \in X$, $h \in L^2(0,T;X)$, $c \in L^\infty(Q_T)$, ω is as in (5.1) or in (5.2). We recall that X is $L^2(0,1)$ in the divergence case and $L^2_{\frac{1}{a}}(0,1)$ in the non divergence one. Concerning a, we assume that it satisfies Hypothesis 4.1 or Hypothesis 4.2 in order to prove the Carleman estimates in Corollary 6.1, and Hypothesis 5.2 or Hypothesis 5.3 to prove the observability inequalities in Propositions 6.1 and 6.2. Observe that the well-posedness of (6.1) follows by [**32**, Theorems 4.1, 4.3]. As for the previous case, the global null controllability of (6.1) follows in a standard way from an observability inequality for the solution of the associated adjoint problem

$$(6.2)\qquad \begin{cases} v_t + \mathcal{A}v - cv = 0, & (t,x) \in (0,T) \times (0,1), \\ v(t,1) = v(t,0) = 0, & t \in (0,T), \\ v(T) = v_T \in L^2(0,1). \end{cases}$$

To obtain an observability inequality like the one in Proposition 5.1 or in Proposition 5.3, the following Carleman estimate, corollary of Theorem 4.1 and Theorem 4.2, is crucial. For this, consider the problem

$$(6.3)\qquad \begin{cases} v_t + \mathcal{A}v - cv = h, & (t,x) \in (0,T) \times (0,1), \\ v(t,1) = v(t,0) = 0, & t \in (0,T) \end{cases}$$

and denote with $\mathcal{S}$ the space $\mathcal{S}_1$ if we consider the divergence case and $\mathcal{S}_2$ if we consider the non divergence one.

COROLLARY 6.1. *Assume Hypothesis* 4.1 *or Hypothesis* 4.2*. Then, there exist two positive constants C and s_0, such that every solution v in $\mathcal{S}$ of* (6.3) *satisfies, for all $s \geq s_0$,*

$$\begin{aligned} &\int_{Q_T} \left(s\Theta a (v_x)^2 + s^3\Theta^3 \frac{(x-x_0)^2}{a} v^2 \right) e^{2s\varphi} dxdt \\ &\leq C \left(\int_{Q_T} h^2 e^{2s\varphi} dxdt + sc_1 \int_0^T \left[a\Theta e^{2s\varphi}(x-x_0)(v_x)^2 dt \right]_{x=0}^{x=1} \right), \end{aligned}$$

if Hypothesis 4.1 *holds and*

$$\int_{Q_T}\left(s\Theta(v_x)^2+s^3\Theta^3\left(\frac{x-x_0}{a}\right)^2v^2\right)e^{2s\varphi}dxdt$$
$$\leq C\left(\int_{Q_T}h^2\frac{e^{2s\varphi}}{a}dxdt+sd_1\int_0^T\left[\Theta e^{2s\varphi}(x-x_0)(v_x)^2dt\right]_{x=0}^{x=1}\right),$$

if Hypothesis 4.2 *is in force. Here* c_1 *and* d_1 *are the constants introduced in* (4.3) *and* (4.18)*, respectively.*

PROOF. Rewrite the equation of (6.3) as $v_t+(av_x)_x=\bar{h}$, where $\bar{h}:=h+cv$. Hence $\bar{h}^2\leq 2h^2+2\|c\|_{L^\infty(Q_T)}v^2$. Now, we will distinguish between the divergence and the non divergence case.

Divergence case. If Hypothesis 4.1 holds, then, applying Theorem 4.1, there exists two positive constants C and $s_0>0$, such that

(6.4)
$$\int_{Q_T}\left(s\Theta a(v_x)^2+s^3\Theta^3\frac{(x-x_0)^2}{a}v^2\right)e^{2s\varphi}dxdt$$
$$\leq C\left(\int_{Q_T}\bar{h}^2e^{2s\varphi}dxdt+sc_1\int_0^T\left[a\Theta e^{2s\varphi}(x-x_0)(v_x)^2dt\right]_{x=0}^{x=1}\right)$$
$$\leq C\left(\int_{Q_T}h^2e^{2s\varphi}dxdt+\int_{Q_T}e^{2s\varphi}v^2dxdt+sc_1\int_0^T\left[a\Theta e^{2s\varphi}(x-x_0)(v_x)^2dt\right]_{x=0}^{x=1}\right)$$

for all $s\geq s_0$. Applying the Hardy-Poincaré inequality (see Proposition 1.1) to $w(t,x):=e^{s\varphi(t,x)}v(t,x)$ and proceeding as in (4.11), recalling that $0<\inf\Theta\leq\Theta\leq c\Theta^2$, one has

$$\int_0^1e^{2s\varphi}v^2dx=\int_0^1w^2dx\leq C\int_0^1a(w_x)^2dx+\frac{s}{2}\int_0^1\frac{(x-x_0)^2}{a}w^2dx$$
$$\leq C\Theta\int_0^1ae^{2s\varphi}(v_x)^2dx+C\Theta^3s^2\int_0^1e^{2s\varphi}v^2\frac{(x-x_0)^2}{a}dx.$$

Using this last inequality in (6.4), we have

$$\int_{Q_T}\left(s\Theta a(v_x)^2+s^3\Theta^3\frac{(x-x_0)^2}{a}v^2\right)e^{2s\varphi}dxdt$$
$$\leq C\left(\int_{Q_T}h^2e^{2s\varphi}dxdt+\int_{Q_T}\Theta ae^{2s\varphi}(v_x)^2dxdt\right.\tag{6.5}$$
$$\left.+s^2\int_{Q_T}e^{2s\varphi}\Theta^3\frac{(x-x_0)^2}{a}v^2dxdt+sc_1\int_0^T\left[a\Theta e^{2s\varphi}(x-x_0)(v_x)^2dt\right]_{x=0}^{x=1}\right)$$

for a positive constant C. Hence, for all $s\geq s_0$, where s_0 is assumed sufficiently large, the claim follows.

Non divergence case. If Hypothesis 4.2 holds, then, applying Theorem 4.2, there exist two positive constants C and s_0 such that

(6.6)
$$\begin{aligned}
&\int_{Q_T}\left(s\Theta(v_x)^2+s^3\Theta^3\left(\frac{x-x_0}{a}\right)^2v^2\right)e^{2s\varphi}dxdt\\
&\le C\left(\int_{Q_T}\bar h^2\frac{e^{2s\varphi}}{a}dxdt+sd_1\int_0^T\left[\Theta e^{2s\varphi}(x-x_0)(v_x)^2dt\right]_{x=0}^{x=1}\right)\\
&\le C\left(\int_{Q_T}h^2\frac{e^{2s\varphi}}{a}dxdt+\int_{Q_T}v^2\frac{e^{2s\varphi}}{a}dxdt+sd_1\int_0^T\left[\Theta e^{2s\varphi}(x-x_0)(v_x)^2dt\right]_{x=0}^{x=1}\right)
\end{aligned}$$

for all $s\ge s_0$. Applying again the Hardy-Poincaré inequality to $w:=e^{s\varphi}v$, setting $p(x)=\dfrac{(x-x_0)^2}{a(x)}$, using Lemma 2.1 and proceeding as in Lemma 5.5, one has

$$\begin{aligned}
\int_0^1 w^2\frac{1}{a}dx&=\int_0^1\frac{p(x)}{(x-x_0)^2}w^2dx\le C_{HP}\int_0^1p(x)(w_x)^2dx\le C\int_0^1(w_x)^2dx\\
&\le C\left(\int_{Q_T}\Theta e^{2s\varphi}(v_x)^2dxdt+s^2\int_{Q_T}\Theta^3e^{2s\varphi}\left(\frac{x-x_0}{a}\right)^2v^2dxdt\right),
\end{aligned}$$

for a positive constat C (we recall that C_{HP} is the Hardy-Poincaré constant). Finally, using the previous inequality in (6.6), one has

$$\begin{aligned}
&\int_{Q_T}\left(s\Theta(v_x)^2+s^3\Theta^3\left(\frac{x-x_0}{a}\right)^2v^2\right)e^{2s\varphi}dxdt\\
&\le C\left(\int_{Q_T}h^2\frac{e^{2s\varphi}}{a}dxdt+\int_{Q_T}\Theta e^{2s\varphi}(v_x)^2dxdt\right.\\
&\left.+s^2\int_{Q_T}\Theta^3e^{2s\varphi}\left(\frac{x-x_0}{a}\right)^2v^2dxdt+sd_1\int_0^T\left[\Theta e^{2s\varphi}(x-x_0)(v_x)^2dt\right]_{x=0}^{x=1}\right)
\end{aligned}$$

As for the divergence case, choosing s_0 sufficiently large, the claim follows.

□

As a consequence of the previous corollary, one can deduce an observability inequality for the homogeneous adjoint problem (6.2). In fact, without loss of generality we can assume that $c\ge 0$ (otherwise one can reduce the problem to this case introducing $\tilde v:=e^{-\lambda t}v$ for a suitable λ). Using this assumption we can prove that the analogous of Lemma 5.1, of Lemma 5.2 and of Lemma 5.4 still hold true. Thus, as before, one can prove the following observability inequalities:

PROPOSITION 6.1. *Assume Hypotheses* 5.1 *and* 5.2. *Then there exists a positive constant* C_T *such that every solution* $v\in C([0,T];L^2(0,1))\cap L^2(0,T;\mathcal{H}^1_a(0,1))$ *of* (6.2) *satisfies*

$$\int_0^1 v^2(0,x)dx\le C_T\int_0^T\int_\omega v^2(t,x)dxdt. \tag{6.7}$$

PROPOSITION 6.2. *Assume Hypothesis* 5.3 *and* (5.2). *Then there exists a positive constant* C_T *such that the solution* $v\in C([0,T];L^2_{\frac{1}{a}}(0,1))\cap L^2(0,T;\mathcal{H}^1_{\frac{1}{a}}(0,1))$

of (6.2) *satisfies*

$$\int_0^1 v^2(0,x)\frac{1}{a}dx \le C_T \int_0^T \int_\omega v^2 \frac{1}{a} dxdt. \tag{6.8}$$

Using (6.7) and (6.8) one can prove that the analogous of Theorems 5.1, 5.3 still hold for (6.1). We underline the fact that also Corollaries 5.1, 5.2 and Theorems 5.2, 5.4 still hold for (6.2) and (6.1), respectively.

Finally, the controllability result can be extended to a semilinear degenerate parabolic equation of the type

$$\begin{cases} u_t - \mathcal{A}u + f(t,x,u) = h(t,x)\chi_\omega(x), & (t,x) \in (0,T) \times (0,1), \\ u(t,1) = u(t,0) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,1), \end{cases} \tag{6.9}$$

where $u_0 \in X$, $h \in L^2(0,T;X)$ and ω satisfies Hypothesis 5.1. However, we can treat this semilinear problem only in the (WD) case, since we need to use the following compactness result:

Theorem 6.1 ([**32**], Theorems 5.4, 5.5). *Suppose that $a \in C[0,1]$ and $a^{-1} \in L^1(0,1)$. Then the spaces*

$$H^1(0,T;L^2(0,1)) \cap L^2(0,T;\mathcal{H}^2_a(0,1)) \quad \textit{and} \quad H^1(0,T;L^2_{\frac{1}{a}}(0,1)) \cap L^2(0,T;H^2_{\frac{1}{a}}(0,1))$$

are compactly imbedded in

$$L^2(0,T;\mathcal{H}^1_a(0,1)) \cap C(0,T;L^2(0,1)) \quad \textit{and} \quad L^2(0,T;H^1_{\frac{1}{a}}(0,1)) \cap C(0,T;L^2_{\frac{1}{a}}(0,1)),$$

respectively.

Also in this case, we assume that a satisfies the degeneracy conditions stated in Hypothesis 1.1. Moreover, in order to prove observability inequalities analogous to those proved in Chapter 5, we assume that the conditions assumed therein are satisfied. More precisely, we suppose that:

Hypothesis 5.2 holds in the divergence case
Hypothesis 5.3 holds in the non divergence case.

Concerning $f : Q_T \times \mathbb{R} \to \mathbb{R}$ we make the following assumptions:

- f is a Carathéodory function, i.e.

  the map $(t,x) \mapsto f(t,x,q)$ is measurable for all $q \in \mathbb{R}$ and
  the map $q \mapsto f(t,x,q)$ is continuous for a.e. $(t,x) \in Q_T$;

- $f(t,x,0) = 0$ for a.e. $(t,x) \in Q_T$;
- $f_q(t,x,q)$ exists for a.e. $(t,x) \in Q_T$;
- f_q is a Carathéodory function;
- there exists $C > 0$ such that

  $$|f_q(t,x,q)| \le C$$

  for a.e. $(t,x) \in Q_T$ and for every $q \in \mathbb{R}$.

To prove a null controllability result for (6.9) one can use, as in [**3**] or in [**15**], a fixed point method, considering a suitable sequence of linear problem associated to the semilinear one, apply a related approximate null controllability property for the linear case, and pass to the limit. We omit the details, which are now standard, just stating the following

THEOREM 6.2. *Under the assumptions above, problem* (6.9) *is globally null controllable.*

CHAPTER 7

Final Comments

Comment 1. If $\omega = \omega_1 \cup \omega_2$, ω_i intervals, with $\omega_1 \subset\subset (0, x_0)$, $\omega_2 \subset\subset (x_0, 1)$, and $x_0 \notin \overline{\omega}$, the global null controllability for (1.2) follows by [**3**, Theorem 4.1] when $\mathcal{A} = \mathcal{A}_1$ and by [**16**, Theorem 4.5] when $\mathcal{A} = \mathcal{A}_2$, at least in the strongly degenerate case and if the initial datum is more regular. Indeed, in this case, given $u_0 \in \mathcal{H}^1_a(0,1)$ or $u_0 \in \mathcal{H}^1_{\frac{1}{a}}(0,1)$, u is a solution of (1.2) if and only if the restrictions of u to $[0, x_0)$ and to $(x_0, 1]$, $u_{|_{[0,x_0)}}$ and $u_{|_{(x_0,1]}}$, are solutions to

$$\begin{cases} u_t - \mathcal{A}u = h(t,x)\chi_{\omega_1}(x), & (t,x) \in (0,T) \times (0,x_0), \\ u(t,0) = 0, & t \in (0,T), \\ \begin{cases} (au_x)(t,x_0) = 0, & \text{in the divergence case} \\ u(t,x_0) = 0, & \text{in the non divergence case,} \end{cases} & t \in (0,T), \\ u(0,x) = u_0(x)_{|_{[0,x_0)}}, \end{cases} \tag{7.1}$$

and

$$\begin{cases} u_t - \mathcal{A}u = h(t,x)\chi_{\omega_2}(x), & (t,x) \in (0,T) \times (x_0,1), \\ u(t,1) = 0, & t \in (0,T), \\ \begin{cases} (au_x)(t,x_0) = 0, & \text{in the divergence case} \\ u(t,x_0) = 0, & \text{in the non divergence case,} \end{cases} & t \in (0,T), \\ u(0,x) = u_0(x)_{|_{(x_0,1]}}, \end{cases} \tag{7.2}$$

respectively. This fact is implied by the characterization of the domains of $\mathcal{A}_1$ and $\mathcal{A}_2$ given in Propositions 2.2, 2.5 and by the Regularity Theorems 2.1, 2.2 when the initial datum is more regular. On the other hand if u_0 is only of class $L^2(0,1)$ or $L^2_{\frac{1}{a}}(0,1)$, the solution is not sufficiently regular to verify the additional condition at (t, x_0) and this procedure cannot be pursued.

Moreover, in the weakly degenerate case, the lack of characterization of the domains of $\mathcal{A}_1$ and $\mathcal{A}_2$ doesn't let us consider a decomposition of the system in two disjoint systems like (7.1) and (7.2), in order to apply the results of [**3**] and [**16**], not even in the case of a regular initial datum.

For this reason, using observability inequalities and Carleman estimates, in Chapter 5 we have proved a null controllability result both in the (WD) and (SD) cases, also in the case of a control region of the form $\omega = \omega_1 \cup \omega_2$ as above.

Comment 2. It is well known that observability inequalities for the adjoint homogeneous problem imply the validity of null controllability results for the original parabolic problem. In fact, as a corollary of the observability inequalities, we give the associated null controllability results for (1.2) providing an estimate of the control function h, see Theorems 5.1, 5.2, 5.3 and 5.4.

Of course, null controllability results can be obtained also in other ways, but the approach with observability inequalities is very general and permits to cover all possible situations. For example, if $x_0 \in \omega$, one could think to obtain the null controllability result directly by a localization argument based on cut-off functions, as in the non degenerate case. But we now show that, at least in the divergence case, this is *not always possible* in presence of a weakly degenerate a, but only in the (SD) case. Indeed, assume that the degenerate point x_0 belongs to the control region ω, consider $0 < r' < r$ with $(x_0 - r, x_0 + r) \subset \omega$, the cut-off functions $\phi_i \in C^\infty([0,1])$, $i = 0,1,2$, defined as

$$\phi_1(x) := \begin{cases} 0, & x \in [x_0 - r', 1], \\ 1, & x \in [0, x_0 - r], \end{cases} \qquad \phi_2(x) := \begin{cases} 0, & x \in [0, x_0 + r'], \\ 1, & x \in [x_0 + r, 1], \end{cases}$$

and $\phi_0 = 1 - \phi_1 - \phi_2$. Then, given an initial condition $u_0 \in L^2(0,1)$, by classical controllability results in the nondegenerate case, there exist two control functions $h_1 \in L^2((0,T) \times \big[\omega \cap (0, x_0 - r')\big)\big]$ and $h_2 \in L^2((0,T) \times \big[\omega \cap (x_0 + r', 1)\big]\big)$, such that the corresponding solutions v_1 and v_2 of the parabolic problems analogous to (1.2) in the domains $(0,T) \times (0, x_0 - r')$ and $(0,T) \times (x_0 + r', 1)$, respectively, satisfy $v_1(T,x) = 0$ for all $x \in (0, x_0 - r')$ and $v_2(T,x) = 0$ for all $x \in (x_0 + r', 1)$ with

$$\int_0^T \int_0^{x_0 - r'} h_1^2 dxdt \le C \int_0^T \int_0^{x_0 - r'} u_0^2 dxdt$$

and

$$\int_0^T \int_{x_0 + r'}^1 h_2^2 dxdt \le C \int_0^T \int_{x_0 + r'}^1 u_0^2 dxdt$$

for some constant C.

Now, let v_0 be the solution of the analogous of problem (1.2) in divergence form in the domain $(0,T) \times (x_0 - r, x_0 + r)$ without control, and with the same initial condition u_0. Finally, define the function

$$u(t,x) = \phi_1(x) v_1(t,x) + \phi_2(x) v_2(t,x) + \frac{T-t}{T} \phi_0(x) v_0(t,x). \tag{7.3}$$

Then, $u(T,x) = 0$ for all $x \in (0,1)$ and u satisfies problem (1.2) in the domain Q_T with

$$\begin{aligned} h = {} & \phi_1 h_1 \chi_{\omega \cap (0, x_0 - r')} + \phi_2 h_2 \chi_{\omega \cap (x_0 + r', 1)} - \frac{1}{T} \phi_0 v_0 - \phi_1' a v_{1,x} - \phi_2' a v_{2,x} \\ & - \phi_0' \frac{T-t}{T} a v_{0,x} - \left(\phi_1' a v_1 + \phi_2' a v_2 + \phi_0' \frac{T-t}{T} a v_0 \right)_x . \end{aligned}$$

We strongly remark that in the (WD) case this function *is not* in $L^2\big((0,T) \times \omega\big)$ since the degenerate function in this case is only $W^{1,1}(0,1)$, so that the problem fails to be controllable in the Hilbert space L^2. For this reason, we think that our approach via Carleman estimates can be extremely interesting also to prove null controllability results, which could not be obtained in other ways.

On the other hand, using the previous technique, one can prove that (1.2) in *non divergence* form is global null controllable if x_0 belongs to the control region ω. Being the observability inequality equivalent to the null controllability, it is superfluous to obtain the first inequality as a consequence of Carleman estimate.

For this reason we have proved Proposition 5.3 only when x_0 does not belong to the control region ω.

Comment 3. Finally, let us conclude with a remark on the fact that in the definition of degeneracy (both weak and strong) we admit only that $K \in (0,2)$. This technical assumption, which is essential to prove Lemmas 4.1 and 4.3, was already introduced, for example, in [**16**] or in [**17**], with the following motivation: if $K \geq 2$ and the degeneracy occurs at the boundary of the domain, the problem fails to be null controllable on the whole interval $[0,1]$ (see, e.g., [**16**], [**18**]), and in this case the only property that can be expected is the *regional null* controllability (see also [**15**], [**19**] and [**33**]). Let us briefly show that the same phenomenon appears, for example, in the non divergence case, inspired by [**16**, Remark 4.6]. Indeed, let us introduce the following variant of a classical change of variables:

$$X = \begin{cases} x_0 - \displaystyle\int_x^{x_0} \frac{1}{y^{K/2}} dy & \text{if } 0 < x \leq x_0, \\ x_0 + \displaystyle\int_{x_0}^{x} \frac{1}{\sqrt{a(1+x_0-y)}} dy & \text{if } x_0 < x < 1, \end{cases}$$

so that $(0,1)$ is stretched to $(-\infty,\infty)$, and $U(t,X) = a^{-1/4}(x)u(t,x)$, where u solves (1.2). Now, take the reference function $a(x) = |x-x_0|^K$ with $K > 2$, so that we find that U solves a nondegenerate heat equation of the form $U_t - U_{xx} + b(X)U = \tilde{h}\chi_{\tilde{\omega}}$, where $\tilde{\omega}$ is a bounded domain compactly contained both in $(-\infty,0)$ and in $(0,\infty)$. Adapting a result of [**45**], the new equation is not controllable, see [**16**, Remark 4.6].

In particular, proceeding as in [**18**], [**19**], one can prove that if $a \in W^{1,\infty}(0,1)$, $a^{-1} \notin L^1(0,1)$ and the control set ω is an interval $\omega = (\alpha,\beta)$ lying on one side of x_0, for every $\lambda, \gamma \in (0,1)$ such that

$$\begin{aligned} & 0 \leq \alpha < \lambda < \beta < x_0 < 1 \quad (\text{if } x_0 > \beta) \\ \text{or } \ & 0 < x_0 < \alpha < \gamma < \beta \leq 1 \quad (\text{if } x_0 < \alpha), \end{aligned}$$

there exists $h \in L^2(0,T;H)$ so that the solution u of (1.2) (or (6.1)) satisfies

$$u(T,x) = 0 \ \text{ for every } x \in [0,\lambda] \ (\text{if } x_0 > \beta) \text{ or for every } x \in [\gamma,1] \ (\text{if } x_0 < \alpha). \tag{7.4}$$

Moreover, there exists a positive constant C_T such that

$$\int_{Q_T} h^2 dxdt \leq C_T \int_0^1 u_0^2 dx,$$

in the divergence case and

$$\int_{Q_T} h^2 \frac{1}{a} dxdt \leq C_T \int_0^1 u_0^2 \frac{1}{a} dx,$$

in the non divergence one.

As pointed out in [**18**], [**19**], we note that the global null controllability is a stronger property than (7.4), in the sense that the former is automatically preserved with time. More precisely, if $u(T,x) = 0$ for all $x \in [0,1]$ and if we stop controlling the system at time T, then for all $t \geq T$, $u(t,x) = 0$ for all $x \in [0,1]$. On the contrary, regional null controllability is a weaker property: in general, (7.4) is no more preserved with time if we stop controlling at time T. Thus, it is important to improve the previous result, as shown in [**18**] or in [**19**], proving that the solution can be forced to vanish identically on $[0, \lambda]$ (if $x_0 > \beta$) or in $[\gamma, 1]$ (if $x_0 < \alpha$) during a given time interval (T, T'), i.e. that the solution is *persistent regional null controllable*.

These results can be extended also in our situation, i.e. with an interior degeneracy, for the problem

$$
\begin{cases} u_t - \mathcal{A}u + c(t,x)u + b(t,x)u_x = h(t,x)\chi_\omega(x), & (t,x) \in Q_T, \\ u(t,1) = u(t,0) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,1), \end{cases} \tag{7.5}
$$

where u_0, h, ω, a and c are as before, while $b \in L^\infty(Q_T)$ and $|b(t,x)| \le C\sqrt{a(x)}$ for a positive constant C. Observe that, in this case, the well-posedness of (7.5) follows by [**32**, Theorems 4.1 and 4.2], and the persistent regional null controllability follows by using cut-off functions, adapting the technique developed in [**29**] or in [**19**].

APPENDIX A

Rigorous derivation of Lemma 3.5

Here we show that all integrations by parts used in the proof of Lemma 3.5 are well justified, both in the non degenerate and in the degenerate case. We will not prove all integration by parts, which can be treated in similar ways, so we just consider the (probably more involved) term

$$2\int_{Q_T} a^2(\varphi_x)^3 w w_x dxdt = \int_0^T \int_0^1 a^2(\varphi_x)^3 (w^2)_x dxdt,$$

where, we recall $w \in \mathcal{V}_1$, and φ stands for Φ in the non degenerate case. First, let us note that such an integral is always well defined. Indeed, in the non degenerate case, $w(\varphi_x)^3 = w\Theta^3(\psi_x)^3 \in L^\infty(Q_T)$, since $w = e^{s\Theta\psi}v$ with $v \in \mathcal{V}_1 \subset L^\infty(Q_T)$, while $\int_A^B a(w_x)^2 \in L^\infty(0,T)$ and $a^{3/2} \in L^\infty(A,B)$.

On the other hand, in the degenerate cases we have that

$$a^2(\varphi_x)^3 w w_x = \left(\frac{(x-x_0)^2}{a}\right)^{3/2} \Theta^3 w \sqrt{a} w_x.$$

First, by Lemma 2.1.1, we have that the map $x \mapsto \frac{(x-x_0)^2}{a}$ is bounded. Then, as for the (WD) case, by definition of $\mathcal{S}_1$, we immediately find that $\sqrt{a}w_x \in L^2(Q_T)$ and $\sqrt{a}(x-x_0)^3, \Theta^3 w^3 \in L^\infty(Q_T)$, so that the integral is well defined. Finally, in the (SD) case we have that $\sqrt{a}w_x, \Theta^3 w \in L^2(Q_T)$ and $\sqrt{a}(x-x_0)^3 \in L^\infty(Q_T)$.

Since the considerations in the degenerate case are more general, from now on we shall confine to this case; hence we shall prove the version of Lemma 3.5 just for the degenerate case.

Thus, for any sufficiently small $\delta > 0$, we get

$$
\begin{aligned}
\int_{Q_T} a^2(\varphi_x)^3(w^2)_x dxdt &= \int_0^T \int_0^{x_0-\delta} a^2(\varphi_x)^3(w^2)_x dxdt \\
&+ \int_0^T \int_{x_0-\delta}^{x_0+\delta} a^2(\varphi_x)^3(w^2)_x dxdt + \int_0^T \int_{x_0+\delta}^{1} a^2(\varphi_x)^3(w^2)_x dxdt \\
&= \int_0^T [(a^2(\varphi_x)^3 w^2)(x_0-\delta) - (a^2(\varphi_x)^3 w^2)(0)]dt \\
&- \int_0^T \int_0^{x_0-\delta} (a^2(\varphi_x)^3)_x w^2 dxdt + \int_0^T \int_{x_0-\delta}^{x_0+\delta} a^2(\varphi_x)^3(w^2)_x dxdt \\
&+ \int_0^T [(a^2(\varphi_x)^3 w^2)(1) - (a^2(\varphi_x)^3 w^2)(x_0+\delta)]dt \\
&- \int_0^T \int_{x_0+\delta}^{1} (a^2(\varphi_x)^3)_x w^2 dxdt \\
&= \int_0^T (a^2(\varphi_x)^3 w^2)(x_0-\delta)dt - \int_0^T (a^2(\varphi_x)^3 w^2)(x_0+\delta)dt \\
&- \int_0^T \int_0^{x_0-\delta} (a^2(\varphi_x)^3)_x w^2 dxdt + \int_0^T \int_{x_0-\delta}^{x_0+\delta} a^2(\varphi_x)^3(w^2)_x dxdt \\
&- \int_0^T \int_{x_0+\delta}^{1} (a^2(\varphi_x)^3)_x w^2 dxdt,
\end{aligned} \tag{A.1}
$$

since the functions $a^2(\cdot)(\varphi_x)^3(t,\cdot)$, $w^2(t,\cdot)$ belong to $H^1(0,x_0-\delta) \cap H^1(x_0+\delta,1)$ for a.e. $t \in (0,T)$ and $w(t,0) = w(t,1) = 0$. Now, we prove that

$$
\begin{aligned}
\lim_{\delta\to 0} \int_0^T \int_0^{x_0-\delta} (a^2(\varphi_x)^3)_x w^2 dx &= \int_0^T \int_0^{x_0} (a^2(\varphi_x)^3)_x w^2 dx, \\
\lim_{\delta\to 0} \int_0^T \int_{x_0+\delta}^{1} (a^2(\varphi_x)^3)_x w^2 dx &= \int_0^T \int_{x_0}^{1} (a^2(\varphi_x)^3)_x w^2 dx
\end{aligned}
$$

and

$$
\lim_{\delta\to 0} \int_0^T \int_{x_0-\delta}^{x_0+\delta} a^2(\varphi_x)^3(w^2)_x dx = 0. \tag{A.2}
$$

Toward this end, observe that

$$
\int_0^T \int_0^{x_0-\delta} (a^2(\varphi_x)^3)_x w^2 dx = \int_0^T \int_0^{x_0} (a^2(\varphi_x)^3)_x w^2 dx - \int_0^T \int_{x_0-\delta}^{x_0} (a^2(\varphi_x)^3)_x w^2 dx \tag{A.3}
$$

and

$$
\int_0^T \int_{x_0+\delta}^{1} (a^2(\varphi_x)^3)_x w^2 dx = \int_0^T \int_{x_0}^{1} (a^2(\varphi_x)^3)_x w^2 dx - \int_0^T \int_{x_0}^{x_0+\delta} (a^2(\varphi_x)^3)_x w^2 dx. \tag{A.4}
$$

We notice that the identities above are justified by the fact that $(a^2(\varphi_x)^3)_x w^2 \in L^1(Q_T)$. Indeed, by Lemma 2.1.1 applied with $\gamma = 2$, we immediately have

$$\begin{aligned}
|(a^2(\varphi_x)^3)_x w^2| &= |c_1^3 \Theta^3 \left(\frac{(x-x_0)^3}{a}\right)_x w^2| \\
&= \left| c_1^3 \Theta^3 \frac{3(x-x_0)^2}{a} w^2 - c_1^3 \Theta^3 \frac{a'(x)(x-x_0)^3}{a^2} w^2 \right| \\
&\le 3c_1^3 \max\left\{ \frac{x_0^2}{a(0)}, \frac{(1-x_0)^2}{a(1)} \right\} \Theta^3 w^2 + c_1^3 \Theta^3 \left| \frac{(x-x_0)^3 a'}{a^2} \right| w^2.
\end{aligned}$$

Now, in the (WD) case we have that $\Theta^3 w^2 = \Theta^3 e^{2s\varphi} v^2 \in L^\infty(Q_T)$, while

$$\Theta^3 \left| \frac{(x-x_0)^3 a'}{a^2} \right| w^2 = \Theta^3 |a'| \left(\frac{|x-x_0|^{3/2}}{a} \right)^2 w^2 \le c\Theta^3 |a'| w^2$$

by Lemma 2.1.1 applied with $\gamma = 3/2 > K$, and where c is a positive constant. At this point, $a' \in L^1(Q_T)$, while $\Theta^3 w^2 \in L^\infty(Q_T)$, and the claim follows. In the (SD) case for $K \le 3/2$, from the previous inequality we have that $\Theta^3 w^2 = \Theta^3 e^{2s\varphi} v^2 \in L^2(Q_T)$, while $a' \in L^\infty(Q_T)$, and again this is enough. For the (SD) case when $K \in (3/2, 2)$, we observe that

$$\Theta^3 \left| \frac{(x-x_0)^3 a'}{a^2} \right| w^2 = \Theta^3 e^{2s\varphi} \left(\frac{|x-x_0|^{\vartheta}}{a} \right)^2 |x-x_0|^{3-2\vartheta} |a'| v^2.$$

By the last requirement in condition (4.1) and from (4.2), since $\Theta^3 e^{2s\varphi} \in L^\infty(Q_T)$ and $v^2 \in L^1(Q_T)$, also this case is finished (recall Remark 7), and (A.3) and (A.4) are justified.

Thus, for any $\epsilon > 0$, by the absolute continuity of the integral, there exists $\delta := \delta(\epsilon) > 0$ such that

$$\left| \int_0^T \int_{x_0-\delta}^{x_0} (a^2(\varphi_x)^3)_x w^2 dxdt \right| < \epsilon,$$

$$\left| \int_0^T \int_{x_0-\delta}^{x_0+\delta} a^2(\varphi_x)^3 (w^2)_x dxdt \right| < \epsilon,$$

$$\left| \int_0^T \int_{x_0}^{x_0+\delta} (a^2(\varphi_x)^3)_x w^2 dxdt \right| < \epsilon.$$

Now, take such a δ in (A.1). Thus, ϵ being arbitrary,

$$\begin{aligned}
&\lim_{\delta\to 0} \int_0^T \int_{x_0-\delta}^{x_0} (a^2(\varphi_x)^3)_x w^2 dxdt = \lim_{\delta\to 0} \int_0^T \int_{x_0-\delta}^{x_0+\delta} a^2(\varphi_x)^3 (w^2)_x dxdt \\
&= \lim_{\delta\to 0} \int_0^T \int_{x_0}^{x_0+\delta} (a^2(\varphi_x)^3)_x w^2 dxdt = 0.
\end{aligned}$$

The previous limits, (A.3), (A.4), together with the integrability conditions proved above, imply

$$\lim_{\delta\to 0} \int_0^T \int_0^{x_0-\delta} (a^2(\varphi_x)^3)_x w^2 dxdt = \int_0^T \int_0^{x_0} (a^2(\varphi_x)^3)_x w^2 dxdt$$

and

$$\lim_{\delta\to 0}\int_0^T\int_{x_0+\delta}^1 au'v'dxdt = \int_0^T\int_{x_0}^1 (a^2(\varphi_x)^3)_x w^2 dxdt.$$

In order to conclude the proof of the desired result, it is sufficient to prove that

$$\lim_{\delta\to 0}\int_0^T (a^2(\varphi_x)^3w^2)(x_0-\delta)dt = \lim_{\delta\to 0}\int_0^T (a^2(\varphi_x)^3w^2)(x_0+\delta)dt, \tag{A.5}$$

and in particular they are 0, as it follows from the identity

$$a^2(\varphi_x)^3w^2 = c_1^3\Theta^3e^{2s\varphi}(x-x_0)^3av^2.$$

Indeed, in the (WD) case (A.5) follows from the fact that v is absolutely continuous in Q_T and from Lebesgue's Theorem, since the map $(t,x)\mapsto\Theta^3(t)e^{2s\varphi(t,x)}$ is bounded; in the (SD) case we use the characterization of the domain given by Proposition 2.1.

The other integrations by parts in Lemma 3.5 are easier and can be proved proceeding as above.

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