



Freedman's Theorem for Unitarily Invariant States on the CCR Algebra

Vitonofrio Crismale, Simone Del Vecchio, Tommaso Monni, Stefano Rossi 

Dipartimento di Matematica, Università degli studi di Bari, Via E. Orabona, 4, 70125 Bari, Italy.
E-mail: vitonofrio.crismale@uniba.it; simone.delvecchio@uniba.it;
tommaso.monni@uniba.it; stefano.rossi@uniba.it

Received: 7 November 2023 / Accepted: 3 January 2024
© The Author(s) 2024

Abstract: The set of states on $\text{CCR}(\mathcal{H})$, the CCR algebra of a separable Hilbert space $\mathcal{H}$, is here looked at as a natural object to obtain a non-commutative version of Freedman's theorem for unitarily invariant stochastic processes. In this regard, we provide a complete description of the compact convex set of states of $\text{CCR}(\mathcal{H})$ that are invariant under the action of all automorphisms induced in second quantization by unitaries of $\mathcal{H}$. We prove that this set is a Bauer simplex, whose extreme states are either the canonical trace of the CCR algebra or Gaussian states with variance at least 1.

1. Introduction

A sequence of real-valued random variables is said to be orthogonally invariant (or rotatable, as in [13]) if the joint distribution of any finite subset of variables taken from the sequence has spherical symmetry. Rotatability is among the strongest distributional symmetries that a stochastic process may enjoy. Indeed, it is a classical result due to Freedman [12] that the joint distribution of an orthogonally invariant sequence is a mixture (i.e. convex combination) of measures each of which is an infinite product of a single centered Gaussian distribution. Furthermore, extreme measures correspond to sequences of independent, identically distributed variables with centered Gaussian distribution with arbitrary variance. A weaker distributional symmetry, exchangeability only requires invariance of the finite-dimensional distributions under permutations. Exchangeable sequences are ruled by de Finetti's theorem, which says that their joint distribution is still a mixture, and that extreme measures are again infinite products. The common distribution, however, remains undetermined, see e.g. [13]. Unlike the classical framework, in most of the commonly used non-commutative models already the structure of the convex set of exchangeable (also known as symmetric) states is too poor to allow for a strict inclusion of the convex set of orthogonally invariant states in the set of symmetric states. For instance, the only symmetric state on the C^* -algebra

generated by the q -deformed canonical commutation relations¹ with $|q| < 1$ ($q = 0$ corresponds to the free case) is the Fock vacuum, see e.g. [7], whereas symmetric states on both Boolean and monotone C^* -algebra make up a mere segment, see [5, 6]. The case of the CAR algebra is the odd one out: although it does feature a rich convex set of symmetric states, thoroughly analyzed in [4], its rotatable states have nonetheless been proved to be the same as symmetric states in [3]. Far-reaching general results can be stated all the same, at the cost, though, of working with symmetries implemented by the natural coaction of the quantum orthogonal/unitary groups on the $*$ -algebra of non-commutative polynomials. Doing so, the Gaussian distribution must be replaced by free operator-valued circular distribution, see [9].

The motivation of the present paper, however, is to exhibit a natural model where rotatable states are fewer than symmetric states and may be described in a meaningful way. The CCR algebra associated with a separable complex Hilbert space $\mathcal{H}$, $\text{CCR}(\mathcal{H})$, turns out to provide a model particularly suited to this end for three main reasons. First, it is naturally acted upon by unitary transformations and permutations, which puts one in a position to consider unitarily invariant and symmetric states. Second, it contains a subalgebra $\mathfrak{B}_0$, which is still a CCR algebra, and, more importantly, is an infinite tensor product to which classical results of Størmer in [19] apply, hence making the structure of exchangeable states completely known. Third, it also has the merit of providing a setting where a genuine generalization of Freedman's classical theorem may be stated, see Remark 3.9.

It is worth remarking, however, that the actual quantum random variables associated with the CCR algebra are in fact given by a sequence of pairs $\{(Q_i, P_i) : i \in \mathbb{Z}\}$ of non-commuting self-adjoint operators, each of which should be conceived as a complex variable. Therefore, unitary invariance appears the natural symmetry to take into account as opposed to orthogonal invariance.

With this in mind, we start our analysis by considering our subalgebra $\mathfrak{B}_0 \subset \text{CCR}(\mathcal{H})$ generated by Weyl operators $W(x)$, with x running through the dense subspace $\mathcal{H}_0$ of the Hilbert space $\mathcal{H}$ of those vectors with only finitely many Fourier coefficients different from 0 with respect to a given basis. Working with $\mathfrak{B}_0$ is particularly convenient because of its factoring into an infinite tensor product of $\text{CCR}(\mathbb{C})$ with itself, as well as being invariant under the action of both $\mathbb{O}_{\mathbb{Z}}$, the group of localized orthogonal transformations, and $\mathbb{U}_{\mathbb{Z}}$, the group of localized unitary transformations. As a benefit of the tensor structure of $\mathfrak{B}_0$, the restricted dynamics is asymptotically abelian, from which we derive a number of consequences. In particular, both the sets of $\mathbb{O}_{\mathbb{Z}}$ -invariant states and of $\mathbb{U}_{\mathbb{Z}}$ -invariant states on $\mathfrak{B}_0$ are Choquet simplices, and their extreme states are infinite products of a single state φ on $\text{CCR}(\mathbb{C})$, Proposition 3.4, in analogy with what happens in the classical setting. Moreover, knowing what the extreme states of our simplices look like proves key to showing that each of our simplices of invariant states is contained in the larger ones as a face, Proposition 3.5.

The characteristic function of the state φ above, that is $\mathbb{C} \ni z \mapsto \varphi(W(z)) \in \mathbb{C}$, must satisfy a certain functional equation, the solutions of which are Gaussians of the type $\varphi(W(z)) = e^{-\sigma^2(\arg z)|z|^2}$, where the variance σ^2 only depends on the phase of z , and is also allowed to take possibly infinite values, Lemma 3.10. As long as rotatable states are focused on, the variance may and will vary with the phase, as shown in Example 3.11. As soon as one requires unitary invariance, though, the dependence of σ^2 on the phase disappears. More precisely, there are only two possibilities. In the first, the characteristic

¹ Namely, for $-1 < q < 1$, the universal unital C^* -algebra generated by a countable family $\{a_i : i \in \mathbb{Z}\}$ satisfying $a_i^* a_j - q a_j a_i^* = \delta_{i,j}$, for all $i, j \in \mathbb{Z}$, where $\delta_{i,j}$ is the Kronecker symbol.

function is $\varphi(W(z)) = \mathbb{1}_{\{0\}}(z)$, the indicator function of the singleton $\{0\}$. In the second, the characteristic function is Gaussian, namely $\varphi(W(z)) = e^{-\frac{1}{2}\sigma^2|z|^2}$, where σ^2 cannot be chosen as small as one wishes. In fact, Heisenberg's uncertainty principle drives σ^2 to be at least 1, Proposition 3.12. Phrased differently, all regular extreme $\mathbb{U}_{\mathbb{Z}}$ -invariant states on $\mathfrak{B}_0$ are quasi-free states of the type $\omega_{\sigma^2}(W(x)) = e^{-\frac{\sigma^2}{2}\|x\|^2}$, $x \in \mathcal{H}_0$, with $\sigma^2 \geq 1$. In addition, taking σ^2 as small as possible, *i.e.* $\sigma^2 = 1$, yields the only pure $\mathbb{U}_{\mathbb{Z}}$ -invariant state, which is nothing but the Fock vacuum. Along the way, we also show that our techniques lead us to a new proof of Freedman's theorem based on non-commutative ergodic theory for C^* -algebras, Remark 3.9, by restricting the dynamics to a suitable commutative subalgebra, which is the infinite tensor product of $\text{CCR}(\mathbb{R})$ with itself. In addition, states on the whole $\mathfrak{B}_0$ lend themselves to being interpreted as quantum stochastic processes with sample algebra given by $\text{CCR}(\mathbb{C})$, the CCR algebra of a single particle.

Once $\mathbb{U}_{\mathbb{Z}}$ -invariant states on $\mathfrak{B}_0$ have been dealt with, we draw our attention to the whole $\text{CCR}(\mathcal{H})$. Now $\mathbb{U}_{\mathbb{Z}}$ -invariance is too weak a symmetry to determine the structure of the resulting states, not least because the map that restricts $\mathbb{U}_{\mathbb{Z}}$ -invariant states of $\text{CCR}(\mathcal{H})$ to $\mathfrak{B}_0$ fails to be injective, as a drawback of the existence of the so-called flat extensions, Remark 3.19. By considering the larger symmetry group $\mathcal{U}(\mathcal{H})$ of all unitaries on $\mathcal{H}$, though, we manage to come to a complete description of the compact convex set of the corresponding invariant states, Theorem 3.18. This turns out to be a Bauer simplex (a Choquet simplex whose extreme points make up a compact set) whose extreme points are either quasi-free Gaussian states of the type $\omega_{\sigma^2}(W(x)) = e^{-\frac{\sigma^2\|x\|^2}{2}}$, $x \in \mathcal{H}$, with $\sigma^2 \geq 1$, or the canonical trace τ of $\text{CCR}(\mathcal{H})$ defined by $\tau(W(x)) = 0$ for all non-zero x in $\mathcal{H}$. In particular, the canonical trace is thus seen to be the only extreme $\mathcal{U}(\mathcal{H})$ -invariant state on $\text{CCR}(\mathcal{H})$ which is not regular.

2. Preliminaries

Let $(\mathcal{H}, \sigma)$ be a symplectic real space, that is a pair where $\mathcal{H}$ is a real vector space and $\sigma : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ an antisymmetric bilinear form. We recall that associated with any such pair there is a CCR algebra, denoted by $\text{CCR}(\mathcal{H}, \sigma)$, which is by definition the universal C^* -algebra generated by the family of elements $\{W(x) \mid x \in \mathcal{H}\}$ satisfying the so-called canonical commutation rules in the Weyl form

$$\begin{aligned} W(x)^* &= W(-x) \\ W(x)W(y) &= e^{i\sigma(x,y)}W(x+y) \end{aligned} \tag{2.1}$$

for all $x, y \in \mathcal{H}$. Details on how this C^* -algebra may be shown to exist are found in *e.g.* [15]. In particular, $\text{CCR}(\mathcal{H}, \sigma)$ is a unital C^* -algebra with unit $W(0)$, and $W(x)$ is a unitary element for every $x \in \mathcal{H}$.

If in addition suppose that σ is non-degenerate ($x \in \mathcal{H}$ with $\sigma(x, y) = 0$ for all $y \in \mathcal{H}$ implies $x = 0$), then $\text{CCR}(\mathcal{H}, \sigma)$ turns out to be a simple C^* -algebra, see *e.g.* [15].

Throughout the present paper we will be working with $\mathcal{H}$ being given by the real vector space associated with a (separable) complex Hilbert space $\mathcal{H}$, thought of as a symplectic space with canonical symplectic form

$$\sigma(x, y) := \Im \langle x, y \rangle \text{ for all } x, y \in \mathcal{H},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of $\mathcal{H}$ (assumed linear in the first variable and anti-linear in the second). As is well known, $\text{CCR}(\mathcal{H}, \sigma)$ admits a (faithful) remarkable representation, the Fock representation, whose construction we briefly sketch for the reader's convenience. For each integer $n \geq 0$, we denote by $\mathcal{H}^{\otimes n}$ the tensor product of $\mathcal{H}$ with itself n times, with the convention that $\mathcal{H}^{\otimes 0}$ is a one-dimensional space, which we denote by $\mathbb{C}\Omega$. The full Fock space is then defined as the infinite direct sum below

$$\mathfrak{F}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n},$$

and Ω is referred to as the vacuum vector in the literature.

We next consider the orthogonal projection P_+ of $\mathfrak{F}(\mathcal{H})$ uniquely determined by

$$P_+ x_1 \otimes \cdots \otimes x_n := \frac{1}{n!} \sum_{s \in S_n} x_{s(1)} \otimes \cdots \otimes x_{s(n)},$$

with S_n the group of all permutations of $\{1, \dots, n\}$. The symmetric (or Bose) Fock space is then defined as the range of the projection P_+ , that is

$$\mathfrak{F}_+(\mathcal{H}) := P_+(\mathfrak{F}(\mathcal{H})).$$

With any $x \in \mathcal{H}$ it is possible to associate a vector $e(x)$ in $\mathfrak{F}_+(\mathcal{H})$, known as the exponential vector of x , defined as

$$e(x) := \bigoplus_{n=0}^{\infty} \frac{1}{n!} x^{\otimes n}.$$

We continue our quick overview of the Bose Fock space by recalling that, for any given $A \in \mathcal{B}(\mathcal{H})$, it is possible to define a (possibly unbounded) operator $\Gamma(A)$ on $\mathfrak{F}_+(\mathcal{H})$, the second quantization of A , whose action on simple tensors is

$$\Gamma(A)(x_1 \otimes \cdots \otimes x_n) := Ax_1 \otimes \cdots \otimes Ax_n, \quad n \in \mathbb{N}, \quad x_1, \dots, x_n \in \mathcal{H}.$$

Accordingly, $\Gamma(A)$ sends exponential vectors to exponential vectors, for

$$\Gamma(A)e(x) = e(Ax), \quad x \in \mathcal{H}.$$

Furthermore, one has $\Gamma(A^*) = \Gamma(A)^*$ and $\Gamma(AB) = \Gamma(A)\Gamma(B)$ for all $A, B \in \mathcal{B}(\mathcal{H})$, hence $\Gamma(U)$ is unitary if U is so.

For every $x \in \mathcal{H}$, define on $\mathfrak{F}(\mathcal{H})$ a pair of unbounded densely-defined operators $a(x)$ and $a^\dagger(x)$, known as full annihilator and creator, respectively. For all $n \in \mathbb{N}$ and $x_1, \dots, x_n$ in $\mathcal{H}$, set

$$a(x)\Omega = 0$$

$$a(x)(x_1 \otimes \cdots \otimes x_n) := n^{\frac{1}{2}} \langle x_1, x \rangle x_2 \otimes \cdots \otimes x_n$$

and

$$a^\dagger(x)\Omega = x$$

$$a^\dagger(x)(x_1 \otimes \cdots \otimes x_n) = (n+1)^{\frac{1}{2}} x \otimes x_1 \otimes \cdots \otimes x_n.$$

At this point, for every $x \in \mathcal{H}$ Bose annihilators and creators can be simply defined as

$$\begin{aligned} a_+(x) &:= P_+ a(x) P_+ \\ a_+^\dagger(x) &:= P_+ a^\dagger(x) P_+. \end{aligned}$$

Details on the domains of Bose annihilators and creators are to be found in e.g. [2]. For every $x \in \mathcal{H}$, the sum $a_+(x) + a_+^\dagger(x)$ turns out to be an essentially self-adjoint operator, whose unique self-adjoint extension is denoted by $\Phi(x)$, which is called the field operator. By exponentiating the field operators, we finally arrive at the following family of unitary operators

$$W'(x) := e^{i\Phi(x)}, \quad x \in \mathcal{H}$$

which can be seen to satisfy the CCR relations (2.1).

Therefore, by universality of the CCR algebra there exists a unique $*$ -representation of $\text{CCR}(\mathcal{H}, \sigma)$ that sends $W(x)$ to $W'(x)$, for all $x \in \mathcal{H}$. This is precisely the Fock representation of $\text{CCR}(\mathcal{H}, \sigma)$ and is known to be irreducible. The vector state associated with Ω , the vacuum vector, is thus a pure state, known as the Fock vacuum, which is still denoted with Ω by a slight abuse of notation. Note that $\Omega(W(x)) = e^{-\frac{1}{2}\|x\|^2}$.

The Fock representation is the most important example of a regular representation of $\text{CCR}(\mathcal{H}, \sigma)$, by which we mean a representation π such that for every $x \in \mathcal{H}$ the one-parameter unitary group $\{\pi(W(tx)) : t \in \mathbb{R}\}$ is strongly continuous. A state ω on $\text{CCR}(\mathcal{H}, \sigma)$ is accordingly said to be regular if its GNS representation is regular.

Our next goal is to define quite a vast class of endomorphisms (automorphisms) on CCR algebras. To this end, let O be any real operator on $\mathcal{H}$ preserving σ , that is $\sigma(Ox, Oy) = \sigma(x, y)$ for all x, y in $\mathcal{H}$. Then it is immediate to see that the set of unitaries $\{W(Ox) \mid x \in \mathcal{H}\}$ continues to satisfy the canonical commutation rules. Therefore, by universality of CCR algebras there must exist a $*$ -monomorphism ρ_O of $\text{CCR}(\mathcal{H}, \sigma)$ uniquely determined by

$$\rho_O(W(x)) = W(Ox), \quad x \in \mathcal{H}. \quad (2.2)$$

Since the symplectic form we are working with is the imaginary part of the canonical Hermitian product of $\mathcal{H}$, for any isometry S of $\mathcal{H}$ there exists a $*$ -monomorphism ρ_S of $\text{CCR}(\mathcal{H}, \sigma)$ such that

$$\rho_S(W(x)) = W(Sx), \quad x \in \mathcal{H}.$$

Furthermore, $\rho_{S_1 S_2} = \rho_{S_1} \circ \rho_{S_2}$ for all isometries S_1, S_2 acting on $\mathcal{H}$. We denote by $\mathcal{U}(\mathcal{H})$ the group of all unitary operators acting on $\mathcal{H}$. Clearly, if U is in $\mathcal{U}(\mathcal{H})$, then ρ_U is an automorphism.

In this paper, however, we will be mostly concerned with isometries and unitaries of a particular form. To define them, we first need to fix an orthonormal basis, say $\{e_k : k \in \mathbb{Z}\}$, once and for all. For any injective map Ψ of $\mathbb{Z}$ to itself, consider the isometry S_Ψ acting on the basis as

$$S_\Psi e_i := e_{\Psi(i)}, \quad i \in \mathbb{Z}.$$

By a slight abuse of notation we simply denote by ρ_Ψ the endomorphism of $\text{CCR}(\mathcal{H}, \sigma)$ corresponding to S_Ψ . Note that

$$\rho_\Psi(W(e_i)) = W(e_{\Psi(i)}), \quad i \in \mathbb{Z}.$$

In particular, corresponding to $\Psi(i) = i + 1, i \in \mathbb{Z}$, we get the so-called shift automorphism.

We next establish some notation to introduce the group actions we are interested in. Let $\mathbb{P}_{\mathbb{Z}}$ denote the group of all finite permutations of $\mathbb{Z}$, that is of all bijective maps σ of $\mathbb{Z}$ into itself such that $\sigma(i) \neq i$ for finitely many i in $\mathbb{Z}$. By $\mathbb{U}_{\mathbb{Z}}$ we denote the subgroup of $\mathcal{U}(\mathcal{H})$ made up of all unitaries U for which there exists $N > 0$ (depending on U) such that $Ue_i = e_i$ for all $|i| > N$. If we denote by $U_{i,j}$ the entries of a given U in $\mathbb{U}_{\mathbb{Z}}$, i.e. $U_{i,j} := \langle Ue_i, e_j \rangle$, for all $i, j \in \mathbb{Z}$, we have $U_{i,j} = \delta_{i,j}$ for all i, j with $|i|, |j| > N$. In particular, it follows that $\mathbb{U}_{\mathbb{Z}}$ is an inductive limit of unitary matrix groups. Thus, it is convenient to think of an element U of $\mathbb{U}_{\mathbb{Z}}$ as an infinite matrix $(U_{i,j})_{i,j \in \mathbb{Z}}$.

Lastly, $\mathbb{O}_{\mathbb{Z}}$ is the subgroup of $\mathbb{U}_{\mathbb{Z}}$ obtained by considering real unitary maps, namely O in $\mathbb{U}_{\mathbb{Z}}$ belongs to $\mathbb{O}_{\mathbb{Z}}$ if and only if $\langle Oe_i, e_j \rangle$ are real numbers for all $i, j \in \mathbb{Z}$. The three groups introduced all act through automorphisms on $\text{CCR}(\mathcal{H}, \sigma)$ by (2.2). This allows us to consider the invariant states of the corresponding C^* -dynamical systems.

By C^* -dynamical system we mean here a triplet $(\mathfrak{A}, G, \alpha)$, where $\mathfrak{A}$ is a unital C^* -algebra, G a group, and $\alpha : G \rightarrow \text{Aut}(\mathfrak{A})$ a group homomorphism, i.e. $\alpha_{gh} = \alpha_g \circ \alpha_h$, $g, h \in G$.

A state ω on $\mathfrak{A}$ is G -invariant (or invariant under the action α of G) if $\omega \circ \alpha_g = \omega$ for every $g \in G$. The set of all G -invariant states is denoted by $S^G(\mathfrak{A})$.

Going back to $\text{CCR}(\mathcal{H}, \sigma)$, its states invariant under $\mathbb{P}_{\mathbb{Z}}$ are known as exchangeable (or symmetric) states. Borrowing the terminology from Classical Probability, we shall henceforth refer to $\mathbb{O}_{\mathbb{Z}}$ -invariant states as orthogonally invariant or rotatable states. We shall refer to $\mathbb{U}_{\mathbb{Z}}$ -invariant states as unitarily invariant states.

For any C^* -dynamical system $(\mathfrak{A}, G, \alpha)$, $S^G(\mathfrak{A})$ is a weakly* compact convex subset of the set of all states of $\mathfrak{A}$. The set of extreme points of $S^G(\mathfrak{A})$, which is non-empty by virtue of the Krein-Milman theorem, is denoted by $\mathcal{E}(S^G(\mathfrak{A}))$. The elements of $\mathcal{E}(S^G(\mathfrak{A}))$ are often referred to as the ergodic states of the given C^* -dynamical system.

On the GNS space $\mathcal{H}_{\omega}$ of any G -invariant state ω , it is possible to define a family of unitaries $\{U_g^{\omega} : g \in G\}$ that implement the action of the group itself, namely for each $g \in G$

$$U_g^{\omega} \pi_{\omega}(a) (U_g^{\omega})^* = \pi_{\omega}(\alpha_g(a)), \quad a \in \mathfrak{A}.$$

The closed subspace of invariant vectors is then defined as

$$\mathcal{H}_{\omega}^G := \bigcap_{g \in G} \{\xi \in \mathcal{H}_{\omega} : U_g^{\omega} \xi = \xi\}.$$

Note that $\mathcal{H}_{\omega}^G$ is never zero as it always contains $\mathbb{C}\xi_{\omega}$, where $\xi_{\omega} \in \mathcal{H}_{\omega}$ is the GNS vector of ω . The orthogonal projection onto $\mathcal{H}_{\omega}^G$ is denoted by E_{ω} . We recall that $\mathcal{H}_{\omega}^G = \mathbb{C}\xi_{\omega}$ implies that ω is ergodic. Unlike what happens in the classical theory, the converse implication may fail to hold; it does hold, however, if we assume that the system is G -abelian, that is if for any $\omega \in S^G(\mathfrak{A})$, the family of operators $\{E_{\omega} \pi_{\omega}(a) E_{\omega} : a \in \mathfrak{A}\}$ is commutative, see Theorem 3.1.12 in [18]. Moreover, G -abelianness is often obtained as a consequence of a stronger property known as asymptotical abelianness: a C^* -dynamical system $(\mathfrak{A}, G, \alpha)$ is asymptotically abelian if there exists a sequence $\{g_n : n \in \mathbb{N}\} \subset G$ such that, for every $a, b \in \mathfrak{A}$, $\lim_{n \rightarrow \infty} \|\alpha_{g_n}(a)b - b\alpha_{g_n}(a)\| = 0$, see Proposition 3.1.16 in [18].

3. Invariant States

We denote by $\mathfrak{A}$ the CCR algebra associated with $\mathbb{C}$ endowed with its canonical symplectic form $\sigma(z, w) := \Im z \bar{w}$, $z, w \in \mathbb{C}$. As is well known, $\mathfrak{A}$ is a (non-separable) simple nuclear C^* -algebra, see e.g. [11]. To ease the notation, we will from now on denote by $\mathfrak{B}$ the CCR algebra associated with $(\mathcal{H}, \sigma)$. Let $\mathcal{H}_n \subset \mathcal{H}$ denote the finite-dimensional subspace generated by $\{e_k : |k| \leq n\}$. We will think of $\mathcal{H}_n \subset \mathcal{H}$ as a symplectic real space whose symplectic form is the restriction of σ to $\mathcal{H}_n$. The corresponding CCR algebra, that is $\text{CCR}(\mathcal{H}_n, \sigma)$, will be denoted by $\mathfrak{B}_n$. Since $\mathfrak{B}_n \subset \mathfrak{B}_{n+1}$, we can consider the inductive limit $\mathfrak{B}_0 \subset \mathfrak{B}$ of the sequence of C^* -algebras $\{\mathfrak{B}_n : n \in \mathbb{N}\}$. More concretely, $\mathfrak{B}_0$ is the norm completion in $\mathfrak{B}$ of $\bigcup_n \mathfrak{B}_n$. Note that $\mathfrak{B}_0$ is itself a CCR algebra, in that $\mathfrak{B}_0 = \text{CCR}(\mathcal{H}_0, \sigma)$ with $\mathcal{H}_0 = \bigcup_n \mathcal{H}_n$. The inclusion $\mathfrak{B}_0 \subset \mathfrak{B}$ is proper, as follows from e.g. [2, Proposition 5.2.9.]. What is more relevant to our analysis is that the subalgebra $\mathfrak{B}_0$ decomposes as an infinite tensor product of $\mathfrak{A}$ with itself. Let us quickly recall what the isomorphism looks like. For each $k \in \mathbb{Z}$, denote by $i_k : \mathfrak{A} \rightarrow \bigotimes_{\mathbb{Z}} \mathfrak{A}$ the k -th embedding, that is

$$i_k(a) = \cdots \otimes 1 \otimes 1 \otimes \underbrace{a}_{k^{\text{th}}} \otimes 1 \otimes 1 \otimes \cdots.$$

That said, the isomorphism alluded to above is nothing but the map $\Psi : \mathfrak{B}_0 \rightarrow \bigotimes_{\mathbb{Z}} \mathfrak{A}$ uniquely determined by

$$\Psi\left(W\left(\sum_{|k| \leq n} z_k e_k\right)\right) = \prod_{|k| \leq n} i_k(W(z_k)) \quad (3.1)$$

for all $n \in \mathbb{N}$ and for all $z_k \in \mathbb{C}$ with $k = -n, \dots, n$. Elements of $\bigotimes_{\mathbb{Z}} \mathfrak{A}$ of the form $i_{k_1}(a_1) \cdots i_{k_n}(a_n)$, for $n \in \mathbb{N}$, $k_1, \dots, k_n \in \mathbb{Z}$ and $a_1, \dots, a_n \in \mathfrak{A}$, will be sometimes referred to as localized tensors.

Rather than work in $\mathfrak{B}_0$ only, we will sometimes use its isomorphic copy. To do so, we first need to rewrite the action of $\mathbb{U}_{\mathbb{Z}}$ on $\bigotimes_{\mathbb{Z}} \mathfrak{A}$. This is done by combining (2.2) and (3.1), which for any $U = (U_{i,j})_{i,j \in \mathbb{Z}}$ in $\mathbb{U}_{\mathbb{Z}}$ gives

$$\rho_U(i_k(W(z))) = \prod_j i_j(W(U_{j,k}z)), \quad k \in \mathbb{Z}, \quad z \in \mathbb{C}. \quad (3.2)$$

As an immediate consequence of (3.2), for every $U \in \mathbb{U}_{\mathbb{Z}}$, one has $\rho_U(\mathfrak{B}_0) \subseteq \mathfrak{B}_0$.

We start our analysis of invariant states on the CCR algebra by considering the smaller dynamical system obtained by restricting the dynamics to $\mathfrak{B}_0$. To this aim, we will need to consider the following sequence $\{g_n : n \in \mathbb{N}\}$ of bijections of $\mathbb{Z}$:

$$g_n(k) = \begin{cases} k + 2^{n-1}, & 0 \leq k < 2^{n-1} \\ k - 2^{n-1}, & 2^{n-1} \leq k < 2^n \\ k, & k \geq 2^n \\ k - 2^{n-1}, & -2^{n-1} \leq k \leq -1 \\ k + 2^{n-1}, & -2^n \leq k < -2^{n-1} \\ k, & k < -2^n \end{cases}$$

For each n , g_n is an element of the group $\mathbb{P}_{\mathbb{Z}}$. In particular, the sequence $\{g_n : n \in \mathbb{N}\}$ is contained in $\mathbb{O}_{\mathbb{Z}}$. By a slight abuse of notation, we continue to denote by g_n the corresponding matrix. Note that for each line of this matrix there is exactly one entry that is equal to 1, while the others are all 0.

Lemma 3.1. *The restricted C^* -dynamical system $(\mathfrak{B}_0, \mathbb{O}_{\mathbb{Z}}, \rho)$ is asymptotically abelian.*

Proof. We will prove that, for any $a, b \in \bigotimes_{\mathbb{Z}} \mathfrak{A}$, one has

$$\lim_{n \rightarrow \infty} \|\rho_{g_n}(a)b - b\rho_{g_n}(a)\| = 0.$$

A standard density argument shows it is enough to deal with a, b lying in a dense subalgebra of $\bigotimes_{\mathbb{Z}} \mathfrak{A}$. With this in mind, we can consider localized simple tensors a, b . In other words, there is no lack of generality if we take $a = \prod_{|k| \leq m} i_k(W(\lambda_k))$ and $b = \prod_{|l| \leq m} i_l(W(\mu_l))$ for some $m > 0$ and $\lambda_k, \mu_l \in \mathbb{C}$ for all $k, l \in \{-m, \dots, 0, \dots, m\}$. Applying (3.2), one finds

$$\rho_{g_n}(a) = \prod_j \prod_{|k| \leq m} i_j(W(\lambda_k(g_n)_{j,k})).$$

Now if n satisfies $2^{n-1} > m$, the definition of g_n implies that the entries $(g_n)_{j,k}$ are 0 for all j with $|j| > m$. Phrased differently, for n such that $2^{n-1} > m$ one has that $\rho_{g_n}(a)$ is a simple tensor whose j -th factors are different from the identity I only when $|j| > m$, hence $\rho_{g_n}(a)$ commutes with b . $\square$

As a consequence of the lemma above, we find that $(\mathfrak{B}_0, \mathbb{O}_{\mathbb{Z}}, \rho)$ is an $\mathbb{O}_{\mathbb{Z}}$ -abelian dynamical system. Moreover, $\mathbb{O}_{\mathbb{Z}}$ -abelianness in turn implies that $\mathcal{S}^{\mathbb{O}_{\mathbb{Z}}}(\mathfrak{B}_0)$ is a Choquet simplex, see Theorem 3.1.14 in [18].

The next goal we want to achieve is to show that any extreme state of $\mathcal{S}^{\mathbb{O}_{\mathbb{Z}}}(\mathfrak{B}_0)$ factors into an infinite product of a single state ρ of the sample algebra $\mathfrak{A}$. To do so, we first need a couple of lemmas.

Lemma 3.2. *For every localized simple tensor a in $\mathfrak{B}_0 \cong \bigotimes_{\mathbb{Z}} \mathfrak{A}$ and every O in $\mathbb{O}_{\mathbb{Z}}$, one has*

$$\rho_{Og_n}(a) = \rho_{g_n}(a)$$

eventually for all n .

Proof. By hypothesis a can be taken as $\prod_{|k| \leq m} i_k(W(\lambda_k))$ for some $m > 0$ and $\lambda_k \in \mathbb{C}$ for all k 's in $\{-m, \dots, m\}$. In addition, there exists $d > 0$ such that $O_{i,j} = \delta_{i,j}$ for all i, j with $|i|, |j| > d$. The statement holds if n satisfies $2^{n-1} \geq \max\{m, d\}$. Indeed, ρ_O acts as the identity on simple tensors whose factors different from I lie outside the discrete interval $\{-d, \dots, d\}$, and $\rho_{g_n}(a)$ is a tensor of this form. $\square$

Lemma 3.3. *If ω is an extreme state in $\mathcal{S}^{\mathbb{O}_{\mathbb{Z}}}(\mathfrak{B}_0)$, then for every a in $\mathfrak{B}_0$ one has*

$$\lim_{n \rightarrow \infty} \pi_{\omega}(\rho_{g_n}(a))\xi_{\omega} = \omega(a)\xi_{\omega}$$

in the weak topology of $\mathcal{H}_{\omega}$.

Proof. As usual, we will identify $\mathfrak{B}_0$ with the infinite tensor product $\bigotimes_{\mathbb{Z}} \mathfrak{A}$. We do not harm the generality of the proof if we suppose that a is localized. By weak compactness of the unit ball of the Hilbert space $\mathcal{H}_{\omega}$, the bounded sequence $\{\pi_{\omega}(\rho_{g_n}(a))\xi_{\omega} : n \in \mathbb{N}\} \subset \mathcal{H}_{\omega}$ must have at least one accumulation point, say ξ . Now it is easy to verify that any such vector is $\mathbb{O}_{\mathbb{Z}}$ -invariant. Indeed, by definition there exists a subnet $\{\pi_{\omega}(\rho_{g_{n_{\alpha}}}(a))\xi_{\omega} :$

$\alpha \in I$ such that $\xi = \lim_{\alpha} \pi_{\omega}(\rho_{g_{n(\alpha)}}(a))\xi_{\omega}$ (in the weak topology of $\mathcal{H}_{\omega}$). But then for any $O \in \mathbb{O}_{\mathbb{Z}}$ we have

$$\begin{aligned} U_O^{\omega}\xi &= \lim_{\alpha} U_O^{\omega}\pi_{\omega}(\rho_{g_{n(\alpha)}}(a))\xi_{\omega} = \lim_{\alpha} U_O^{\omega}\pi_{\omega}(\rho_{g_{n(\alpha)}}(a))(U_O^{\omega})^*\xi_{\omega} \\ &= \lim_{\alpha} \pi_{\omega}(\rho_{Og_{n(\alpha)}}(a))\xi_{\omega} = \lim_{\alpha} \pi_{\omega}(\rho_{g_{n(\alpha)}}(a))\xi_{\omega} = \xi \end{aligned}$$

where the second-last equality is a consequence of Lemma 3.2. Since ω is extreme, we must have $\xi = \lambda\xi_{\omega}$ for some λ in $\mathbb{C}$ due to Lemma 3.1 and Proposition 3.1.12 in [18]. Now λ is easily seen to equal $\omega(a)$ irrespective of the chosen subsequence. In other words, $\omega(a)\xi_{\omega}$ is the only accumulation point of the set $\{\pi_{\omega}(\rho_{g_n}(a))\xi_{\omega} : n \in \mathbb{N}\}$. Therefore, by weak compactness of the unit ball of $\mathcal{H}_{\omega}$ the whole sequence $\{\pi_{\omega}(\rho_{g_n}(a))\xi_{\omega} : n \in \mathbb{N}\}$ must converge to $\omega(a)\xi_{\omega}$, which ends the proof. $\square$

Proposition 3.4. *If ω is an extreme state in $\mathcal{S}^{\mathbb{O}_{\mathbb{Z}}}(\mathfrak{B}_0)$, then there exists a state φ on $\mathfrak{A}$ such that $\omega = \otimes_{\mathbb{Z}}\varphi$.*

Proof. We start by showing that any extreme state is strongly clustering, that is

$$\lim_n \omega(\rho_{g_n}(a)b) = \omega(a)\omega(b)$$

for all $a, b \in \mathfrak{B}_0$. This is a consequence of Lemma 3.3, for

$$\begin{aligned} \lim_n \omega(\rho_{g_n}(a)b) &= \lim_n \langle \pi_{\omega}(\rho_{g_n}(a))\pi_{\omega}(b)\xi_{\omega}, \xi_{\omega} \rangle \\ &= \lim_n \langle \pi_{\omega}(b)\xi_{\omega}, \pi_{\omega}(\rho_{g_n}(a^*))\xi_{\omega} \rangle = \omega(a)\omega(b). \end{aligned}$$

We next define φ as a state on $\mathfrak{A}$ by setting $\varphi(a) := \omega(i_j(a))$, $a \in \mathfrak{A}$, $j \in \mathbb{Z}$. The definition is well posed since it does not depend on j , for ω is exchangeable being even rotatable. In order to prove the thesis, we need to show that

$$\omega(i_{j_1}(a_1) \cdots i_{j_n}(a_n)) = \varphi(a_1) \cdots \varphi(a_n)$$

for every $n \in \mathbb{N}$, for every $j_1, \dots, j_n$ in $\mathbb{Z}$, and $a_1, \dots, a_n$ in $\mathfrak{A}$. This can be done by induction on n exactly as in the proof of [19, Theorem 2.7.], to which the reader is referred for the missing details. $\square$

We denote by $\mathcal{S}^{\mathbb{Z}}(\mathfrak{B}_0)$ the set of stationary states on $\mathfrak{B}_0$, that is of those states that are invariant under the shift automorphism. In order to state the following result, we recall that a convex subset $\Delta \subset C$ of a convex set C is a face if given $x, y \in C$ such that $tx + (1-t)y$ lies in Δ for some t with $0 < t < 1$, then x, y are in Δ , in which case we write $\Delta \subset_F C$.

Proposition 3.5. *There holds the inclusion of faces:*

$$\mathcal{S}^{\mathbb{U}_{\mathbb{Z}}}(\mathfrak{B}_0) \subset_F \mathcal{S}^{\mathbb{O}_{\mathbb{Z}}}(\mathfrak{B}_0) \subset_F \mathcal{S}^{\mathbb{P}_{\mathbb{Z}}}(\mathfrak{B}_0) \subset_F \mathcal{S}^{\mathbb{Z}}(\mathfrak{B}_0).$$

Proof. We start by pointing out that the four sets in the statement are all Choquet simplices by asymptotical abelianness of the corresponding dynamical systems. Therefore, we can argue as in the proof of Theorem 3.5 in [3] and reduce the problem to showing that any extreme state in one of the first three sets (say $\mathcal{S}^{\mathbb{U}_{\mathbb{Z}}}(\mathfrak{B}_0)$) remains extreme in the next set to the right ($\mathcal{S}^{\mathbb{O}_{\mathbb{Z}}}(\mathfrak{B}_0)$). But this is trivial for the first two inclusions because extreme $\mathbb{U}_{\mathbb{Z}}$ -invariant/ $\mathbb{O}_{\mathbb{Z}}$ -invariant/symmetric states are product states, and product states are extreme in $\mathcal{S}^{\mathbb{P}_{\mathbb{Z}}}(\mathfrak{B}_0)$ as follows from Theorem 2.7 in [19]. As for $\mathcal{S}^{\mathbb{P}_{\mathbb{Z}}}(\mathfrak{B}_0)$ being a face of $\mathcal{S}^{\mathbb{Z}}(\mathfrak{B}_0)$, one can proceed in the same way as done in the proof of Theorem 3.5 in [3]. $\square$

Now the state $\varphi \in \mathcal{S}(\mathfrak{A})$ whose existence is guaranteed by Proposition 3.4 cannot be chosen arbitrarily. In fact, rotatability on ω imposes strict constraints on the distribution of φ on position (and momentum) operators. More precisely, the distributions turn out to be Gaussian, as we are going to show.

To any state $\varphi \in \mathcal{S}(\mathfrak{A})$ we associate the function $G_\varphi : \mathbb{C} \rightarrow \mathbb{C}$ defined as

$$G_\varphi(z) := \varphi(W(z)), \quad z \in \mathbb{C}.$$

Note that G is bounded, $|G_\varphi(z)| \leq 1$ for all $z \in \mathbb{C}$, and $G_\varphi(0) = 1$. It is worth mentioning that in the literature G_φ is sometimes referred to as the characteristic function of the state φ .

Lemma 3.6. *Let φ be a state on $\mathfrak{A}$. Then the product $\omega = \otimes_{\mathbb{Z}} \varphi$ is a rotatable state on $\mathfrak{B}_0$ if and only if G_φ satisfies the equations*

$$G_\varphi(z) = \prod_j G_\varphi(O_{j,k}z), \quad z \in \mathbb{C}, \quad k \in \mathbb{Z},$$

for any given matrix $O = (O_{k,j})_{k,j \in \mathbb{Z}}$ in $\mathbb{O}_{\mathbb{Z}}$.

Proof. It is a matter of direct computation. If ω is rotatable, then thanks to Equation (3.2) we have

$$\begin{aligned} G_\varphi(z) &= \varphi(W(z)) = \omega(i_k(W(z))) = \omega(\rho_O(i_k(W(z)))) \\ &= \omega\left(\prod_j i_j(W(O_{j,k}z))\right) = \prod_j \varphi(W(O_{j,k}z)) \\ &= \prod_j G_\varphi(O_{j,k}z) \end{aligned}$$

for all z in $\mathbb{C}$. On the other hand, if the equation is satisfied, then similar calculations show that ω is rotatable. $\square$

Remark 3.7. Clearly, both Proposition 3.4 and Lemma 3.6 hold for $\mathbb{U}_{\mathbb{Z}}$ -invariant states as well since Lemma 3.3 applies to extreme states in $\mathcal{S}^{\mathbb{U}_{\mathbb{Z}}}(\mathfrak{B}_0)$.

The next result spells out in what sense the distribution of position (or momentum) operators in a rotatable state is Gaussian, as announced.

Proposition 3.8. *Let φ be a state on $\mathfrak{A}$. If $\omega = \otimes_{\mathbb{Z}} \varphi$ is a rotatable state on $\mathfrak{B}_0$, then either $G_\varphi = \mathbb{1}_{\{0\}}$, the indicator function of $\{0\}$, or there exists $\sigma^2 \geq 0$ such that*

$$G_\varphi(t) = e^{-\frac{\sigma^2 t^2}{2}}, \quad t \in \mathbb{R}.$$

Proof. Since G_φ must satisfy the equations in the statement of Lemma 3.6, in particular we find $G_\varphi(z) = G_\varphi(-z)$ for all $z \in \mathbb{C}$. As a consequence, the restriction of G_φ to the real line is completely determined by its restriction to the positive half-line. Furthermore, by Proposition 3.1 in [15], the restriction of G_φ to the real line must be a definite-positive function.

Let now $z = x + iy$ be any given complex number. By considering a 2×2 orthogonal matrix given by a rotation by an angle α the functional equation yields

$$G_\varphi(|z|) = G_\varphi(\cos \alpha |z|) G_\varphi(\sin \alpha |z|).$$

If we now choose α any argument of z , we find $G_\varphi(\sqrt{x^2 + y^2}) = G_\varphi(x)G_\varphi(y)$. Henceforth we will denote by g the restriction of G_φ to the real line. From what we have seen, g is even and

$$g(\sqrt{x^2 + y^2}) = g(x)g(y) \quad (3.3)$$

for all $x, y \geq 0$. Since g is even, we may as well work with the restriction of g to the positive half line. Let us define $h : [0, \infty) \rightarrow \mathbb{C}$ as $h(t) = g(\sqrt{t})$, $t \in \mathbb{R}$. From (3.3) one has that h satisfies $h(t+s) = h(t)h(s)$, for $t, s \geq 0$. Let ρ be the absolute value of h , i.e. $\rho(t) = |h(t)|$, $t \geq 0$. We clearly have $\rho(t+s) = \rho(t)\rho(s)$, for all $t, s \geq 0$. There are two cases to deal with depending on whether ρ may vanish at a point or not. If it does, that is if there exists $t_0 > 0$ with $\rho(t_0) = 0$, then $\rho = \mathbb{1}_{\{0\}}$. Indeed, if $t \geq t_0$, then $\rho(t) = \rho(t - t_0)\rho(t_0) = 0$. Furthermore, $\rho(\frac{t_0}{n}) = \rho(t_0)^{\frac{1}{n}} = 0$ for every $n \in \mathbb{N}$ with $n \geq 1$. In conclusion $\rho(t) = 0$ for all $t > 0$, hence $\rho = \mathbb{1}_{\{0\}}$.

If ρ never vanishes, then there exists $\Psi : [0, \infty) \rightarrow \mathbb{R}$ such that $\rho(t) = e^{\Psi(t)}$ and $\Psi(t+s) = \Psi(t) + \Psi(s)$ for all $t, s \geq 0$. Clearly, any such Ψ is the restriction of a function $\tilde{\Psi}$ defined on the whole real line that satisfies Cauchy's functional equation, namely $\tilde{\Psi}(t+s) = \tilde{\Psi}(t) + \tilde{\Psi}(s)$, t, s in $\mathbb{R}$. If $\tilde{\Psi}$ is continuous, then $\tilde{\Psi}(t) = ct$, $t \in \mathbb{R}$, for some real constant c . In addition, for G_φ to be bounded, it is necessary to take only non-positive values, say $c = -\frac{\sigma^2}{2}$ for some σ in $\mathbb{R}$. Discontinuous (non-measurable) solutions of Cauchy's equation are ruled out because their graph is well known to be dense in $\mathbb{R}^2$, thus giving rise to an unbounded ρ .

From now on we can work under the hypothesis that $\rho(t) > 0$ for all $t \geq 0$, which allows us to decompose h as $h(t) = \rho(t)\chi(t)$, for all $t \geq 0$, with $\chi(t) \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ for all $t \geq 0$. Now χ satisfies the same equation as h , that is $\chi(t+s) = \chi(t)\chi(s)$. The conclusion will be fully reached once we have shown that χ must be identically 1 (when $\rho > 0$). To this end, note that χ cannot take non-real values. Suppose on the contrary there exists $t_0 > 0$ such that $\chi(t_0)$ is not real. The function $G_\varphi(t) = \chi(t^2)\rho(t^2)$, $t \in \mathbb{R}$, then turns out not to be positive definite. Indeed, the 2×2 matrix $(G_\varphi(t_i - t_j))_{i,j=1,2}$ with $t_1 = 0$ and $t_2 = t_0$ is

$$\begin{pmatrix} 1 & \chi(t_0^2)\rho(t_0^2) \\ \chi(t_0^2)\rho(t_0^2) & 1 \end{pmatrix},$$

which even fails to be Hermitian.

In other terms, the only possible values for χ are 1 and -1 . But, again, the values -1 is ruled out. Indeed, if there existed t_0 with $\chi(t_0) = -1$, then from the very equation satisfied by χ we would have $\chi(\frac{t_0}{2}) = \sqrt{\chi(t_0)}$, and both square roots of -1 are not real, being $\pm i$. $\square$

Remark 3.9. The arguments employed in the proposition above along with Lemma 3.1 and Proposition 3.4 provide a novel proof of Freedman's theorem for commutative random variables. Precisely, the classical theorem is obtained by restricting our dynamics to the commutative subalgebra $C^*(W(te_i) : i \in \mathbb{Z}, t \in \mathbb{R}) \cap \mathfrak{B}_0$, which still factors into a tensor product in that there holds the $*$ -isomorphism

$$C^*(W(te_i) : i \in \mathbb{Z}, t \in \mathbb{R}) \cap \mathfrak{B}_0 \cong \bigotimes_{\mathbb{Z}} \text{CCR}(\mathbb{R}),$$

where $\text{CCR}(\mathbb{R})$ is the CCR of $\mathbb{R}$ as a symplectic space with null symplectic form, which can be concretely thought of the C^* -subalgebra generated by the set $\{W(te_1) : t \in \mathbb{R}\}$.

It is well known that $\text{CCR}(\mathbb{R})$ is isomorphic with $\text{AP}(\mathbb{R})$, the C^* -algebra of almost periodic functions on $\mathbb{R}$. Moreover, $\text{AP}(\mathbb{R})$ can be identified with $C(\beta\mathbb{R})$, the C^* -algebra of continuous functions on the Bohr compactification of the real line, see e.g. Chapter 16 in [10]. Working with $\text{AP}(\mathbb{R})$ rather than $C_0(\mathbb{R})$ causes no loss of generality, for the set of states of the latter (Borel probability measures on the real line) naturally injects into those of the former.

Our next aim is to address the CCR algebra so as to say something more precise about the structure of its rotatable states. To this end, the first thing to do is to exhibit all solutions of the functional equation considered in Lemma 3.6.

Lemma 3.10. *If $G : \mathbb{C} \rightarrow \mathbb{C}$ is a bounded non-constant function satisfying*

$$G(z) = \prod_j G(O_{j,k}z), \quad z \in \mathbb{C}, k \in \mathbb{Z},$$

for all orthogonal matrices $O = (O_{j,k})_{k,j \in \mathbb{Z}}$ in $\mathbb{O}_{\mathbb{Z}}$, then

$$G(z) = e^{-\sigma^2(\arg z)|z|^2}, \quad z \in \mathbb{C},$$

where σ^2 is a positive (possibly infinite) function depending only on the argument of z .

Proof. It is an application of Proposition 3.8, thanks to which the solutions of our equations can be determined by considering their restrictions to lines through the origin. More precisely, let $z_0 = x_0 + iy_0$ be a fixed complex number. Let us consider the line in the complex plane $\{tz_0 = tx_0 + ity_0 : t \in \mathbb{R}\}$ and denote by $g : \mathbb{R} \rightarrow \mathbb{C}$ the restriction of G to this line, that is $g(t) := G(tz_0)$, $t \in \mathbb{R}$. Now the function g can be determined in the same way as in the proof of Proposition 3.8, which means we must

have $g(t) = e^{-\sigma^2(z_0)t^2} = e^{-\frac{\sigma^2(z_0)t^2|z_0|^2}{|z_0|^2}}$ for all real t . Setting $z = tz_0$, one finally sees that $G(z) = e^{-\sigma^2(z)|z|^2}$, where $\sigma^2(z) = \frac{\sigma^2(z_0)}{|z_0|^2}$ depends only on the argument of $z = x + iy$, say on the ratio $\frac{y}{x}$. $\square$

We next exhibit a simple class of rotatable states.

Example 3.11. For all reals $\lambda, \mu \geq 0$

$$\varphi(W(z)) := e^{-\frac{1}{2}(\lambda x^2 + \mu y^2)}, \quad z = x + iy,$$

defines a state on $\mathfrak{A}$ if and only if $\lambda\mu \geq 1$. This is a direct application of [14, Proposition 10], which says that for $\varphi(W(z)) = e^{-\frac{1}{2}\alpha(z,z)}$, z in $\mathbb{C}$, to define a state, it is necessary and sufficient that

$$\alpha(z, z)\alpha(w, w) \geq \sigma(z, w)^2$$

for all z, w in $\mathbb{C}$, where $\alpha(z, z) = \lambda x^2 + \mu y^2$ with $z = x + iy$.

Take $z = x_1 + iy_1$ and $w = x_2 + iy_2$. If the inequality holds, then $(\lambda x_1^2 + \mu y_1^2)(\lambda x_2^2 + \mu y_2^2) \geq (y_1 x_2 - x_1 y_2)^2$. Choosing $x_2 = 0$, we find $(\lambda\mu - 1)x_1^2 + \mu^2 y_1^2 \geq 0$ for all $x_1, y_1 \in \mathbb{R}$, hence $\lambda\mu - 1 \geq 0$.

Conversely, if $\lambda\mu \geq 1$, then

$$\begin{aligned}
 \alpha(z, z)\alpha(w, w) &= (\lambda x_1^2 + \mu y_1^2)(\lambda x_2^2 + \mu y_2^2) \\
 &= \lambda^2 x_1^2 x_2^2 + \lambda\mu x_1^2 y_2^2 + \lambda\mu y_1^2 x_2^2 + \mu^2 y_1^2 y_2^2 \\
 &\geq -2\lambda\mu x_1 x_2 y_1 y_2 + \lambda\mu x_1^2 y_2^2 + \lambda\mu y_1^2 x_2^2 \\
 &= \lambda\mu (x_1 y_2 - y_1 x_2)^2 \\
 &\geq (x_1 y_2 - y_1 x_2)^2 = \sigma(z, w)^2.
 \end{aligned}$$

Accordingly, for any choice of λ, μ with $\lambda\mu \geq 1$, the infinite tensor product $\otimes_{\mathbb{Z}} \varphi$ provides a rotatable state on $\mathfrak{B}_0$.

Lemma 3.10 allows us to completely determine what G_φ looks like when $\omega = \otimes_{\mathbb{Z}} \varphi$ is a $\mathbb{U}_{\mathbb{Z}}$ -invariant state. In particular, the rotatable states discussed in Example 3.11 are ruled out unless $\lambda = \mu$.

Proposition 3.12. *If $\omega = \otimes_{\mathbb{Z}} \varphi$ is a $\mathbb{U}_{\mathbb{Z}}$ -invariant state on $\mathfrak{B}_0$, then either $G_\varphi = \mathbb{1}_{\{0\}}$ or*

$$G_\varphi(z) = e^{-\frac{\sigma^2 |z|^2}{2}}, \quad z \in \mathbb{C},$$

where σ^2 is a constant with $\sigma^2 \geq 1$.

Proof. Under the current hypotheses, the function G_φ now satisfies the stronger conditions $G(z) = \prod_j G(U_{j,k}z)$, $z \in \mathbb{C}, k \in \mathbb{Z}$, for all $U = (U_{k,j})_{j,k \in \mathbb{Z}} \in \mathbb{U}_{\mathbb{Z}}$. In particular, $\mathbb{U}_{\mathbb{Z}}$ -invariance also imposes the extra condition $G_\varphi(z) = G_\varphi(e^{i\theta}z)$, $z \in \mathbb{C}$, which holds for all $\theta \in \mathbb{R}$. But then $G_\varphi(z)$ only depends on $|z|$, which means it is uniquely determined by its restriction to the real line. Therefore, Proposition 3.8 gives $G_\varphi = \mathbb{1}_{\{0\}}$ or $G_\varphi(z) = e^{-\frac{\sigma^2 |z|^2}{2}}$, $z \in \mathbb{C}$, for some $\sigma^2 > 0$. Finally, the condition $\sigma^2 \geq 1$ comes from Example 3.11 with $\lambda = \mu = \sigma^2$. $\square$

Remark 3.13. Conversely, for any $\sigma^2 \geq 1$, there exists a unique (quasi-free) state φ_{σ^2} on $\mathfrak{A} = \text{CCR}(\mathbb{C}, \sigma)$ such that $\varphi_{\sigma^2}(W(z)) = e^{-\frac{\sigma^2 |z|^2}{2}}$, $z \in \mathbb{C}$, as follows from Theorem 3.4 in [15]. If we now let φ_∞ denote the state on $\mathfrak{A}$ defined by $\varphi_\infty(W(z)) = \mathbb{1}_{\{0\}}(z)$, we clearly have that

$$\lim_{\sigma^2 \rightarrow \infty} \varphi_{\sigma^2} = \varphi_\infty$$

in the weak* topology.

Denote by ω_{σ^2} the state on $\mathfrak{B}_0$ which is the infinite product $\otimes_{\mathbb{Z}} \varphi_{\sigma^2}$, where φ_{σ^2} is the state on the sample algebra $\mathfrak{A}$ defined in the remark above. Note that the infinite product of φ_∞ with itself is the canonical trace τ of $\mathfrak{B}_0$, that is $\tau(W(x)) = 0$ for all $x \in \mathcal{H}_0$ with $x \neq 0$. For convenience, the canonical trace will be here denoted by ω_∞ . In the following corollary, $[1, +\infty]$ denotes the one-point compactification of the half-line $[1, +\infty)$ thought of as a topological space with respect to the usual Euclidean topology, whereas $\mathcal{E}(\mathcal{S}^{\mathbb{U}_{\mathbb{Z}}}(\mathfrak{B}_0))$ is endowed with the weak* topology inherited from $\mathcal{S}(\mathfrak{B}_0)$.

Corollary 3.14. *The map $\Psi : [1, +\infty] \rightarrow \mathcal{E}(\mathcal{S}^{\mathbb{U}_{\mathbb{Z}}}(\mathfrak{B}_0))$ given by*

$$\Psi(\sigma^2) = \begin{cases} \omega_{\sigma^2}, & \text{if } \sigma^2 < \infty \\ \omega_{\infty}, & \text{if } \sigma^2 = \infty \end{cases}$$

is a homeomorphism of topological spaces. In particular, $\mathcal{E}(\mathcal{S}^{\mathbb{U}_{\mathbb{Z}}}(\mathfrak{B}_0))$ is a compact subset of $\mathcal{S}^{\mathbb{U}_{\mathbb{Z}}}(\mathfrak{B}_0)$

Proof. The map is well defined on $[1, +\infty]$ thanks to Remark 3.13 and Theorem 2.7 in [19]. Injectivity is obvious and surjectivity is a consequence of Remark 3.7.

The continuity of the map and of its inverse is a matter of straightforward computations, which we can safely leave out. $\square$

Proposition 3.15. *If ω is a $\mathbb{U}_{\mathbb{Z}}$ -invariant state on $\mathfrak{B}_0$, then there exists a unique Borel probability measure μ on $[1, +\infty]$ such that*

$$\omega = \int_{[1, +\infty]} \omega_{\sigma^2} d\mu(\sigma^2).$$

More explicitly, for every x in $\mathcal{H}_0$ one has

$$\omega(W(x)) = \int_1^{+\infty} e^{-\frac{\sigma^2 \|x\|^2}{2}} d\mu(\sigma^2) + \mu(\{\infty\})\omega_{\infty}(W(x)).$$

Proof. The first formula is a direct consequence of $\mathcal{S}^{\mathbb{U}_{\mathbb{Z}}}(\mathfrak{B}_0)$ being a Choquet simplex along with Corollary 3.14. The second follows from $\omega_{\sigma^2}(W(x)) = e^{-\frac{\sigma^2 \|x\|^2}{2}}$ for x in $\mathcal{H}_0$, which can be seen by direct computation. Indeed, for $x = \sum_{i \in F} \lambda_i e_i$, where $F \subset \mathbb{Z}$ is a finite subset and λ_i is a complex number for every $i \in F$, we have

$$\begin{aligned} \omega_{\sigma^2}(W(x)) &= \omega_{\sigma^2} \left(\prod_{i \in F} W(\lambda_i e_i) \right) = \prod_{i \in F} \varphi_{\sigma^2}(W(\lambda_i)) \\ &= \prod_{i \in F} e^{-\frac{\sigma^2 |\lambda_i|^2}{2}} = e^{-\frac{\sigma^2 \|x\|^2}{2}}. \end{aligned}$$

$\square$

The next result shows that most of $\mathbb{U}_{\mathbb{Z}}$ -invariant states on $\mathfrak{B}_0$ are regular.

Proposition 3.16. *If μ is a Borel probability measure on $[1, +\infty]$ with $\mu(\{\infty\}) = 0$, then the corresponding $\mathbb{U}_{\mathbb{Z}}$ -invariant state $\omega := \int_1^{+\infty} \omega_{\sigma^2} d\mu(\sigma^2)$ is regular.*

Proof. In light of Proposition 3.15, for any x in $\mathcal{H}_0$ we have

$$\omega(W(tx)) = \int_1^{+\infty} e^{-\frac{\sigma^2 t^2 \|x\|^2}{2}} d\mu(\sigma^2)$$

The continuity in t of the above expression is a consequence of Lebesgue's dominated convergence theorem since $e^{-\frac{\sigma^2 t^2 \|x\|^2}{2}} \leq 1$ for all $t \in \mathbb{R}$ and $\sigma^2 \in [1, +\infty)$. $\square$

The proof of the above result actually shows more. Indeed, the function $\mathbb{R} \ni t \mapsto \omega(W(tx)) \in \mathbb{C}$ is even analytic under the assumption $\mu(\{\infty\}) = 0$.

Remark 3.17. If ρ is any non-regular state on $\mathfrak{A} = \text{CCR}(\mathbb{C}, \sigma)$, then $\otimes_{\mathbb{Z}} \rho$ is certainly an extreme symmetric state on $\mathfrak{B}_0$, which by construction fails to be regular. In this way it is possible to exhibit a good many non-regular extreme symmetric states, in stark contrast with $\mathbb{U}_{\mathbb{Z}}$ -invariant states, whose only non-regular extreme state is the canonical trace.

By a slight abuse of notation, for any $\sigma^2 \geq 1$ (including ∞) we continue to denote by ω_{σ^2} the quasi-free state on the whole $\mathfrak{B}$ uniquely determined by $\omega_{\sigma^2}(W(x)) = e^{-\frac{\sigma^2 \|x\|^2}{2}}$, $x \in \mathcal{H}$.

We are now ready to state our main result.

Theorem 3.18. *The compact convex $\mathcal{S}^{\mathcal{U}(\mathcal{H})}(\mathfrak{B})$ is a Bauer simplex whose extreme point set is $\{\omega_{\sigma^2} : \sigma^2 \in [1, +\infty]\}$. Explicitly, if ω is a $\mathcal{U}(\mathcal{H})$ -invariant state on $\mathfrak{B}$, then there exists a unique Borel probability measure μ on $[1, +\infty]$ such that*

$$\omega = \int_{[1, +\infty]} \omega_{\sigma^2} d\mu(\sigma^2).$$

Proof. We start by observing that if ω is a $\mathcal{U}(\mathcal{H})$ -invariant state on $\mathfrak{B}$, its restriction $\omega|_{\mathfrak{B}_0}$ to $\mathfrak{B}_0$ is certainly a $\mathbb{U}_{\mathbb{Z}}$ -invariant state on $\mathfrak{B}_0$. Let us then consider the restriction map $\Psi : \mathcal{S}^{\mathcal{U}(\mathcal{H})}(\mathfrak{B}) \rightarrow \mathcal{S}^{\mathbb{U}_{\mathbb{Z}}}(\mathfrak{B}_0)$ defined as $\Psi(\omega) := \omega|_{\mathfrak{B}_0}$, ω in $\mathcal{S}^{\mathcal{U}(\mathcal{H})}(\mathfrak{B})$. We aim to show that Ψ is an affine homeomorphism between compact convex sets. Injectivity of Ψ follows from transitivity of $\mathcal{U}(\mathcal{H})$ on $\mathcal{H}$: if x lies in $\mathcal{H} \setminus \mathcal{H}_0$, it is possible to find $y \in \mathcal{H}_0$ and $U \in \mathcal{U}(\mathcal{H})$ such that $\|x\| = \|y\|$ and $y = Ux$. But then we must have $\omega(W(x)) = \omega(W(Ux)) = \omega(W(y))$. In other words, ω is completely determined by its restriction to $\mathfrak{B}_0$. The map Ψ is clearly surjective thanks to Proposition 3.15. The continuity of Ψ is trivial. By compactness of $\mathcal{S}^{\mathcal{U}(\mathcal{H})}(\mathfrak{B})$ it follows that Ψ is a closed map, which means its inverse is continuous as well. $\square$

Remark 3.19. The invariance under $\mathbb{U}_{\mathbb{Z}}$ for states on the whole CCR algebra $\mathfrak{B}$ is not enough to obtain the thesis of Theorem 3.18. This can be seen by looking at the so-called flat extensions of a state defined on $\mathfrak{B}_0$. To this end, pick any $\mathbb{U}_{\mathbb{Z}}$ -invariant state ω on $\mathfrak{B}_0$. Then one can define a state Ω on the whole $\mathfrak{B}$ by setting

$$\Omega(W(x)) = \begin{cases} \omega(W(x)), & x \in \mathcal{H}_0 \\ 0, & x \in \mathcal{H} \setminus \mathcal{H}_0. \end{cases}$$

The proof that the definition above yields a state on $\mathfrak{B}$ is to be found in [1, Proposition 2.1]. However, it remains to show that Ω is invariant under the action of $\mathbb{U}_{\mathbb{Z}}$ on $\mathfrak{B}$. But this easily follows from the inclusions $U\mathcal{H}_0 \subset \mathcal{H}_0$ and $U(\mathcal{H} \setminus \mathcal{H}_0) \subset \mathcal{H} \setminus \mathcal{H}_0$, which hold true for any unitary U in $\mathbb{U}_{\mathbb{Z}}$. Phrased differently, the map that restricts $\mathbb{U}_{\mathbb{Z}}$ -invariant states on $\mathfrak{B}$ to $\mathfrak{B}_0$ fails to be injective.

In the next proposition we show that the states focused on in the above result belong to the folium of all normal states in the Fock representation as long as they come from a measure μ that assigns 0 to the point at infinity.

Proposition 3.20. *Any regular $\mathcal{U}(\mathcal{H})$ -invariant state on $\mathfrak{B}$ is normal in the Fock representation.*

Proof. Thanks to Theorem 3.18 any such state is of the form

$$\omega = \int_{[1, +\infty]} \omega_{\sigma^2} d\mu(\sigma^2),$$

for a suitable probability measure on $[1, +\infty]$ with $\mu(\{\infty\}) = 0$.

We recall that for each $\sigma^2 \geq 1$, the state ω_{σ^2} is normal in the Fock representation $\pi_{\mathfrak{F}}$, which means there exists a trace-class operator T_{σ^2} in $\mathfrak{F}_+(\mathcal{H})$ such that $\omega_{\sigma^2}(a) = \text{Tr}(\pi_{\mathfrak{F}}(a)T_{\sigma^2})$, $a \in \mathfrak{B}$. Moreover, the operator T_{σ^2} can be seen to equal $\frac{\Gamma(A(I+A)^{-1})}{\text{Tr}(\Gamma(A(I+A)^{-1}))}$ with $A = \frac{\sigma^2-1}{2}I$, cf. [16] and Lemma 6.4.1 in [17]. That said, we can easily come to the conclusion. Note that

$$\tilde{\omega}(S) := \int_1^{+\infty} \text{Tr}(ST_{\sigma^2})d\mu(\sigma^2), \quad S \in \mathcal{B}(\mathfrak{F}_+(\mathcal{H}))$$

defines a state on $\mathcal{B}(\mathfrak{F}_+(\mathcal{H}))$ such that $\omega(a) = \tilde{\omega}(\pi_{\mathfrak{F}}(a))$ for every $a \in \mathfrak{B}$. The thesis will be achieved once we prove that $\tilde{\omega}$ is normal. Let $\{S_n : n \in \mathbb{N}\} \subset \mathcal{B}(\mathfrak{F}_+(\mathcal{H}))$ be a bounded increasing sequence of positive operator, and let S be its sup. As is known, S is the strong limit of S_n , hence for every $\sigma^2 \geq 1$ we have $\lim_n \text{Tr}(S_n T_{\sigma^2}) = \text{Tr}(S T_{\sigma^2})$. Finally, by the monotone convergence theorem we find

$$\lim_n \tilde{\omega}(S_n) = \lim_n \int_1^{+\infty} \text{Tr}(S_n T_{\sigma^2})d\mu(\sigma^2) = \int_1^{+\infty} \text{Tr}(S T_{\sigma^2})d\mu(\sigma^2) = \tilde{\omega}(S),$$

and we are done. $\square$

We would like to remark that a $\mathcal{U}(\mathcal{H})$ -invariant state ω on $\mathfrak{B}$ can be thought of as a quantum stochastic process of a particular kind. Indeed, any such state is uniquely determined by its restriction ω_0 to $\mathfrak{B}_0$, and the tensor product structure of $\mathfrak{B}_0$ allows us to recover a stochastic process with sample algebra $\mathfrak{A} = \text{CCR}(\mathbb{C}, \sigma)$ from ω_0 , see e.g. Section 3 of [8]. In particular, one may consider the tail algebra of the process, as defined as in e.g. [5, 7], and ask what it looks like. The question is very easily answered for extreme states. Precisely, if ω_0 is an extreme state in any of the sets $\mathcal{S}^{\mathbb{U}_{\mathbb{Z}}}(\mathfrak{B}_0)$, $\mathcal{S}^{\mathbb{O}_{\mathbb{Z}}}(\mathfrak{B}_0)$ or $\mathcal{S}^{\mathbb{P}_{\mathbb{Z}}}(\mathfrak{B}_0)$, then its tail algebra is trivial. This is a consequence of Proposition 2.9 in [8], which addresses extreme symmetric states (in the wider context of local action on quasi-local C^* -algebras). In fact, extreme states of both $\mathcal{S}^{\mathbb{U}_{\mathbb{Z}}}(\mathfrak{B}_0)$ and $\mathcal{S}^{\mathbb{O}_{\mathbb{Z}}}(\mathfrak{B}_0)$ are all instances of extreme symmetric states, being product states. The tail algebra of non-extreme states can of course fail to be trivial. Nevertheless, it will always be commutative, cf. Proposition 2.10 in [8] and the references therein. Furthermore, there is only one conditional expectation from $\pi_{\omega_0}(\mathfrak{B}_0)''$ onto the tail algebra, and, more importantly, the process itself is conditionally independent and identically distributed (with centered Gaussian distribution for states in $\mathcal{S}^{\mathbb{U}_{\mathbb{Z}}}(\mathfrak{B}_0)$ or $\mathcal{S}^{\mathbb{O}_{\mathbb{Z}}}(\mathfrak{B}_0)$) with respect to the conditional expectation.

Acknowledgement We acknowledge the support of Italian INDAM-GNAMP. The authors are partially supported by Progetto GNAMPA 2023 CUP E55F22000270001 “Metodi di Algebre di Operatori in Probabilità non Commutativa” and by Italian PNRR Partenariato Esteso PE4, NQSTI.

Funding Open access funding provided by Università degli Studi di Bari Aldo Moro within the CRUI-CARE Agreement.

Data Availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study. The authors are not aware of a conflict of interest on their side related to this article.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

1. Acerbi, F.: Nonregular Representations of CCR Algebras. Ph.D. thesis, SISSA (1993)
2. Bratteli, O., Robinson, D.W.: Operator Algebras and Quantum Statistical Mechanics. II. Texts and Monographs in Physics, 2nd edn. Springer, Berlin (1997)
3. Crismale, V., Del Vecchio, S., Rossi, S.: On distributional symmetries on the CAR algebra. arXiv preprint [arXiv:2209.05431](https://arxiv.org/abs/2209.05431)
4. Crismale, V., Fidaleo, F.: De Finetti theorem on the CAR algebra. *Commun. Math. Phys.* **315**, 135–152 (2012)
5. Crismale, V., Fidaleo, F.: Exchangeable stochastic processes and symmetric states in quantum probability. *Ann. Mat. Pura Appl.* **194**, 969–993 (2015)
6. Crismale, V., Fidaleo, F., Griseta, M.E.: Wick order, spreadability and exchangeability for monotone commutation relations. *Ann. Henri Poincaré* **19**, 3179–3196 (2018)
7. Crismale, V., Rossi, S.: Tail algebras for monotone and q -deformed exchangeable stochastic processes. *Ann. Mat. Pura Appl.* (4) **202**, 497–518 (2023)
8. Crismale, V., Rossi, S., Zurlo, P.: de Finetti-type theorems on quasi-local algebras and infinite Fermi tensor products. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **26**, Paper No. 2250028 (2023)
9. Curran, S.: Quantum rotatability. *Trans. Am. Math. Soc.* **362**, 4831–4851 (2010)
10. Dixmier, J.: C^* -algebras. North-Holland Publishing Co., Amsterdam (1977)
11. Evans, D.E., Lewis, J.T.: Dilations of irreversible evolutions in algebraic quantum theory. *Communications Dublin Inst. Advanced Studies. Ser. A* (1977)
12. Freedman, D.A.: Invariants under mixing which generalize de Finetti's theorem. *Ann. Math. Stat.* **33**, 916–923 (1962)
13. Kallenberg, O.: Probabilistic Symmetries and Invariance Principles. Springer, Berlin (2005)
14. Manuceau, J., Verbeure, A.: Quasi-free states of the C.C.R.-algebra and Bogoliubov transformations. *Commun. Math. Phys.* **9**, 293–302 (1968)
15. Petz, D.: An Invitation to the Algebra of Canonical Commutation Relations, Leuven Notes in Mathematical and Theoretical Physics. Series A: Mathematical Physics, vol. 2. Leuven University Press, Leuven (1990)
16. Petz, D., Pitrik, J.: Markov property of Gaussian states of canonical commutation relation algebras. *J. Math. Phys.* **50**(11), 113517 (2009)
17. Pitrik, J.: Markov property in non-commutative probability. Ph.D. thesis, Mathematical Institute of Budapest University of Technology and Economics (2009)
18. Sakai, S.: C^* -algebras and W^* -algebras. Springer, Berlin (1971)
19. Størmer, E.: Symmetric states of infinite tensor products of C -algebras. *J. Funct. Anal.* **3**, 48–68 (1969)

Communicated by E. Smith.