Interest Rates Term Structure Models Driven by Hawkes Processes

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This paper includes a marked Hawkes process in the original HJM set-up, and investigates the impact of this assumption on the pricing of the popular vanilla fixed-income derivatives. Our model exhibits a smile that can fit the implied volatility of swaptions for a given key rate (tenor). We harness on the log-normality of the model, conditionally with respect to jumps, and derive formulae to evaluate both caplets/floorlets and swaptions. Our model exhibits negative jumps on the zero-coupon (hence positive on the rates). Therefore, its behaviour is compatible with the situation where globally low interest rates can suddenly show cluster of positive jumps in case of tensions on the market. One of the main difficulties when dealing with the HJM model is to keep a framework that is Markovian. In this paper we show how to preserve the relevant features of the Hull and White version, especially the reconstruction formula that provides the zero-coupon bonds in terms of the underlying model factors.

Keywords: Heath-Jarrow-Morton Model; Forward Rates; Hawkes processes; Jumps Clustering; Swaptions; Caplets/Floorlets.

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1 Introduction

The cornerstone paper by Heath, Jarrow and Morton [27] (hereafter HJM model) proposes a general approach which takes the whole yield curve as an input and provides a dynamics of all forward rates. One particular specification of this class of models, proposed by Hull and White [32], offers a Gaussian, Markov version which offers tractable formulae to deal with vanilla interest rates derivatives. The simplicity of this model allows fruitful applications for the valuation and the hedging of some complex derivatives (bermuda swaptions, callable floaters etc.). In particular, efficient numerical methods such as trinomial trees has been developed to deal with these products. See, for instance, Brigo and Mercurio [13]. However, the HJM model does not provide a smile that can replicate adequately the implied volatilities of a given key rate.

The HJM model, in its original formulation, exploits a stochastic framework generated by a Brownian motion. Even if rates exhibit low volatility compared to equities, large fluctuations were highlighted in econometric literature, see for instance [15, 16, 19, 31, 36]. A large amount of literature then proposed to integrate jumps in HJM approach mainly by exploiting Lévy processes, as, for instance, in [20, 21]. The general law of Lévy processes is useful to fit the data but can give rise also to drawbacks. In fact, the market convention is to express the prices of fixed-income derivatives (caps, floors, swaps, etc.) in term of Black/Bachelier volatilities, that is by using a log-normal or normal law. It is not surprising that a large part of market models are derived from Black Scholes model, as highlighted in Brigo and Mercurio [13, chapter 10]. Moreover the volatility surfaces obtained by models based on Lévy processes do not look very realistic for all ranges of strike/maturities. When stochastic volatility is added, it is often assumed a zero correlation between rate and its volatility in order to have a conditional log-normal setup. See Renault and Touzi [42] for the theoretical setting and [13, pages 495-496] for the applications in interest rate modelling.

It is easy to remark that the interest in including Lévy processes in fixed income modelling reached his peak in the first decade of the 21-th century and faded away during the second decade due to a new phenomenon: the persistence of unusual low interest rates, see [5, 23]. That is, after, or because of, the global financial crisis of 2008 and the European debt crisis of 2010-13 the yield curves have been drowned downward by Central Banks' quantitative easing. For some countries, the curve can be partially (or even totally) negative. The situation has changed after the corona-virus crisis with a strong recovery and the increase of inflation.

We summarize the previous empirical results by noticing that jumps cannot be neglected in interest rate modeling, but that their distribution in time cannot be assumed as homogeneous, as pointed out by a large and increasing literature on term structure and credit derivatives, see for instance [6, 22, 25, 33]. Jumps are in fact clustered, that is concentrated in relatively small time windows as highlighted by many empirical studies also before the financial crisis of 2008 [2, 3, 18, 45]. Hawkes processes [26] can reproduce this behaviour and are now often exploited to model self-exciting effects in different setup as equity [1, 8, 35, 41], limit order book [39], credit risk [6, 22], water management [46, 47] commodities [12, 14, 24, 34] and more recently cyber risk [9, 28, 29]. Hawkes and branching processes have been already proposed in order to model the short rate dynamics, see [25, 33], but to our knowledge, no attempt was made insofar to introduce Hawkes type models in order to describe the dynamics of forward rates. The main goal of the present paper is to include in the original HJM set-up a marked Hawkes process and to investigate the impact of this assumption on the pricing of some popular financial instruments of fixed-income markets. Conditionally on the jumps and the intensity process, the bond evolution is log-normal. This feature is particular important for numerical efficiency of swaption pricing and is consistent with market convention. In contrast with previous conditional log-normal models, see [13, Chapter 11], the implied volatility is not only smiled but also skewed due to the presence of jumps.

A further argument justifying the choice of Marked Hawkes processes to describe forward rates is that central banks, when changes are required by the general economic conditions, change interest rates in a step-wise way, and this makes the probability of a new change more likely to happen when a change has just occurred, this implying in a natural way a clustering of the jumps observed on the market. Although the interest rates modifications planned by the central banks usually affect mainly short rates, this clustering effect is in fact inherited by the forward curves, and this can be explained by the *rational expectation hypothesis*, according to which forward rates can be interpreted as suitable expectations by market operators of future spot rates.

Our model exhibits a smile that can reproduce the implied volatility of swaptions for a given key rate (tenor). We harness on the log-normality of the model, conditionally with respect to jumps, and derive formulae to evaluate both caplets/floorlets and swaptions. Our model exhibits negative jumps on the zero-coupon (hence positive on the rates). Therefore, its behaviour is compatible with the situation where globally low interest rates can suddenly show cluster of positive jumps in case of tensions on the market. One of the main difficulties when dealing with Hawkes jumps in the HJM model is to keep a framework that is Markovian. In particular, it is important to preserve the important features of the Hull and White version [32], especially the reconstruction formula that provides the zero-coupons in terms of the underlying model factors. In our case, this formula is based on two factors: a classical Gaussian one and a pure jump martingale based on Hawkes process.

In next section we set-out the stochastic framework and define a general HJM model driven by a Hawkes process exhibiting the evolution of the term structure, i.e. all zero coupon bonds, in term of a log-Brownian term and the marked Hawkes process. Then, Section 3 considers a special case of a two-factor non-Markovian linear model. In this setting, we derive pricing formula for caplets/floorlets and swaptions. The fourth section provides some numerical applications. More specifically, we fit the model on the implied volatility of swaptions for a given tenor and investigate the influence of the model parameters on the shape of the smile, by performing a sensitivity analysis. Finally, in Section 5 we provide a summary and a discussion of the results obtained.

2 General HJM framework with Hawkes processes

Let us set-out the following stochastic framework. Let $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F}, \mathbb{Q})$ be a filtered probability space, satisfying the usual conditions [37, page 10], equipped with a standard Brownian motion W and a marked Hawkes process, represented by its counting measure $\nu(dt, dz)$, independent from W. The intensity of the Hawkes process reads

$$\lambda_t = \lambda_0 + \beta \int_0^t (\lambda_0 - \lambda_s) ds + \int_0^{t^-} \int_{\mathbb{R}^+} z \nu(ds, dz). \tag{1}$$

According to the definition of Marked Hawkes process see [6, Definition 2] and [10], the compensator of ν is $\Theta(dz)\lambda_t dt$, where $\Theta(dz)$ is a measure on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$. Throughout the paper, we assume that $\Theta(dz)$ admits moments of every order. In particular, in the numerical setup, we will set out that Θ follows an exponential law, i.e $\Theta(dz) = \alpha e^{-\alpha z} dz$. We will denote by

$\widetilde{\nu}(dt,dz)$ the compensated measure of $\nu(dt,dz)$. The role of clustering in interest rates has been highlighted in a large literature before [2, 18, 45] and after the financial crisis of 2008 [25, 33, 48].

The choice of a Marked Hawkes process rather than usual Hawkes process is justified by the fact that the impact of news arrival and/or market shocks could not be generally assumed constant as in the pioneer model of Errais et al. [22]. At the opposite, the perfect proportional effect of the model by Jiao et al. [33], where the intensity of the jumps of the short interest rate is proportional to the same interest rate, is also probably too binding. The choice of a Marked Hawkes process could be then seen as a intermediate setup since the marks will of course impact the evolution of the bond prices but not in a multiplicative way. Of course, the choice of a Marked Hawkes increases the number of the parameters. Therefore, we assume the particular choice of an initial intensity λ_0 equal to the long run λ_0 in order to reduce the number of parameters to be estimated. The intensity is a hidden process and then its estimation is an issue, as pointed out in the literature. See for instance [6, 12, 24, 22].

We introduce the sequence of jumps $\{T_i\}_{i\leq 1}$ and marks $\{Z_i\}_{i\leq 1}$ of the Hawkes process. From now on, we assume the following:

Assumption 1 (Mean reverting condition) $\beta > \displaystyle\int_{\mathbb{R}^+} z\Theta(dz)$.

Under this assumption, the Hawkes process is well defined, mean reverting admits moments of every order and its Laplace Transform is known (see [7], Proposition 7.3, p. 176). For the particular setup chosen of an exponential law, this assumption is equivalent to assume $\beta\alpha > 1$.

A zero-coupon is a fixed-income instrument which pays 1 at a given maturity $T \geq 0$. Its price at any time $0 \leq t \leq T$ is denoted by $B_t(T)$, and, of course satisfies $B_T(T) = 1$. We now turn our attention to the dynamics of zero-coupons bonds. Let $T_\star < \infty$ be the finite maximal horizon of the set-up. We introduce the spot risk-free interest rate process $\{r(t)\}_{0\leq t\leq T_\star}$.

Assumption 2 *[Zero-coupons Dynamics]. For any* $0 \leq t \leq T \leq T_\star$*,*

$$\frac{dB_t(T)}{B_{t^-}(T)} = r(t)dt + \Gamma(t,T)dW_t + \int_{\mathbb{R}^+} J(t,T,z)\widetilde{\nu}(dt,dz) \tag{2}$$

under the initial condition $B_0(t)$ *where* Γ *and* J *are deterministic functions and satisfy the following conditions, for any* $0 \leq t \leq T \leq T_\star$*:*

Well-posedness and integrability for Brownian part: *For all* $t < T$ *we have* $\Gamma(t,T) > 0$*, that is the Brownian contribution never vanishes during the bond life, and* $\int_0^T \Gamma(t,T)^2 dt < +\infty$*, that is the Brownian contribution is square integrable.*

Well-posedness and integrability for jump part: $J(t,T,z) > -1$ *meaning that the jumps never pushes the value of the bond below zero.*

Closure: $\Gamma(T,T) = 0$ *and* $J(T,T,z) = 0$*, that is the usual condition to insure that the randomness has no impact on the bond price at the maturity.*

These assumptions are quite classical at this stage. In particular, following Hull and White paradigm [32], we assume that the integrands of the stochastic drivers W and ν are deterministic functions. In this sense, conditional on the path of the marked Hawkes process, the evolution of the zero-coupons is log-normal in the spirit of Romano and Touzi [43]. We make this choice for sake of parsimony. The goal of the numerical section is to show that this setup is not only manageable but also able to capture the behaviour of implied volatility of swaptions, in particular its skew and not only its smile.

We define also the money market account dynamics as follows:

$$Q(t) := \exp\left(-\int_0^t r(u)du\right). \tag{3}$$

In this framework, the probability $\mathbb{Q}$ represents the *spot risk-neutral* probability, i.e. the probability with the cash as *numéraire*.

A direct integration of (2) yields

$$\begin{aligned} B_t(T) = B_0(T)\mathcal{E}\Bigg(\int_0^{\cdot} r(s)ds &+ \int_0^{\cdot} \Gamma(s,T)dW_s \\ &+ \int_0^{\cdot}\int_{\mathbb{R}^+} J(s,T,z)\widetilde{\nu}(ds,dz)\Bigg)_t \end{aligned} \tag{4}$$

where $\mathcal{E}\left(\right)$ denotes the Doléans-Dade exponential.

Set $T \geq 0$ and define

$$C_t(T) = Q(t)\frac{B_t(T)}{B_0(T)}\,. \tag{5}$$

From Equation (4), we have that the process C can be decomposed into two terms,

$$\begin{aligned} C_t(T) &= M_t(T)\, N_t(T) \text{ with} && (6)\\ M_t(T) &= \mathcal{E}\left(\int_0^{\cdot} \Gamma(s,T)dW_s\right)_t && (7)\\ N_t(T) &= \mathcal{E}\left(\int_0^{\cdot}\int_{\mathbb{R}^+} J(s,T,z)\widetilde{\nu}(ds,dz)\right)_t && (8)\end{aligned}$$

The process $\{M_t(T)\}_{0\leq t\leq T}$ has continuous paths whereas $\{N_t(T)\}_{0\leq t\leq T}$ has finite variation and captures the jumps.

Remark 1 *We have the following formulation of the Doléans-Dade martingale, $\{N_t(T)\}_{0\leq t\leq T}$:*

$$N_t(T) = \prod_{T_i\leq t} (1 + J(T_i,T,Z_i)) \exp\left(-\int_0^t \int_{\mathbb{R}^+} J(s,T,z)\Theta(dz)\lambda_s ds\right)$$

see [40, Theorem 36, p.77].

From now on, set $\overline{J}(t,T) = \int_{\mathbb{R}^+} J(t,T,z)\Theta(dz)$, the expected value with respect to the mark distribution of the jump part of the zero-coupon term structure.

Proposition 1 *Assume that, for any $z \geq 0$, $0 \leq S \leq t \leq T$,*

$$\ln(1 + J(S,T,z)) - z\int_S^t e^{-\beta(s-S)}\overline{J}(s,t)ds \leq 0 \tag{9}$$

Then, $\{C_t(T)\}_{0\leq t\leq T}$, defined by equation (6), is a martingale with moments of every order.

Proof. The process C is the product of two local martingales, as Doléans-Dade exponentials of martingales. According to the square integrability of Γ, see Definition 2, M is a true martingale with log-normal law and then it admits moment of every order.

We will, now, focus on N. First, by recalling that the dynamics of λ is given by $\lambda_t = \lambda_0 + \sum_{T_i<t} Z_i e^{-\beta(t-T_i)}$, thanks to the fact that the initial condition

λ_0 coincides with the long run level in (1), the explicit form can be obtained by following the arguments in [8, Section 4.4] or the simulation scheme in [41, Appendix B]. Using this expression together with the form of $N_t(T)$ given in Remark 1, we have the following upper-bound:

$$N_t(T) \leq K \prod_{T_i \leq t} \exp\left(\ln\left(1 + J(T_i, T, Z_i)\right) - Z_i \int_{T_i}^{t} \overline{J}(s,T) e^{-\beta(s-T_i)} ds \right) \quad (10)$$

where

$$K := \sup_{0 \leq t \leq T} \exp\left(-\lambda_0 \int_0^t \overline{J}(s,T) ds \right). \quad (11)$$

Under assumption 9 that prevails, the local martingale $\{N_t(T)\}_{0 \leq t \leq T}$, defined by equation (10), is a true martingale because $\sup\limits_{0 \leq t \leq T} N_t(T)$ is integrable, and it also has moments of every orders. Then, the process $\{C_t(T)\}_{0 \leq t \leq T}$, defined by (6), is a martingale with moments of every order as the product of two independent martingales with moments of every order. □

One of the key elements when dealing with interest rates derivatives is the expression of forward zero-coupons.

Proposition 2 *For any* $0 \leq s \leq t \leq T$, *we obtain*

$$\frac{B_s(T)}{B_s(t)} = \frac{B_0(T)}{B_0(t)} M_s(t,T) N_s(t,T)$$

with

$$\begin{aligned} M_s(t,T) &= \mathcal{E}\left(\int_0^{\cdot} \left[\Gamma(u,T) - \Gamma(u,t)\right]\left[dW_u - \Gamma(u,t)du\right] \right)_s \\ N_s(t,T) &= \mathcal{E}\left(\int_0^{\cdot} \int_{\mathbb{R}+} \frac{J(u,T,z) - J(u,t,z)}{1 + J(u,t,z)} \widehat{\nu}^t(du,dz) \right)_s \\ \widehat{\nu}_t(du,dz) &= \nu(du,dz) - (1 + J(u,t,z))\Theta(dz)\lambda_u du \end{aligned} \quad (12)$$

Let $C_t(T)$ *as defined in equations (6). Then,* $\frac{d\mathbb{Q}^t}{d\mathbb{Q}}(s) = C_s(t)$ *defines a probability, known as the* t-forward probability. *Under* $\mathbb{Q}^t$, $\left\{ \frac{B_s(T)}{B_s(t)} \right\}_{0 \leq s \leq t}$ *is a martingale , moreover,* $dW_u - \Gamma(u,t)du$ *is a standard Brownian motion and* ν *admits for compensator* $(1 + J(u,t,z))\Theta(dz)\lambda_u du$.

Proof. The expressions of the forward zero-coupon is direct from (4) and the reorganisation of both Brownian and jump parts. The existence of probability $\mathbb{Q}^t$ is a consequence of Proposition 1. We have, for any $0 \leq u \leq s \leq t$,

$$C_u(t)\mathbb{E}^{\mathbb{Q}^t}\left[\frac{B_s(T)}{B_s(t)}\;\middle|\;\mathcal{F}_u\right] = \mathbb{E}^{\mathbb{Q}}\left[C_s(t)\frac{B_s(T)}{B_s(t)}\;\middle|\;\mathcal{F}_u\right] \tag{13}$$

By inserting the expression of $C_s(t)$, see equation (7), in the right-hand member of (13), we obtain, after simplification,

$$B_u(t)\mathbb{E}^{\mathbb{Q}^t}\left[\frac{B_s(T)}{B_s(t)}\;\middle|\;\mathcal{F}_u\right] = \mathbb{E}^{\mathbb{Q}}\left[exp\left(-\int_u^s r(v)dv\right)B_s(T)\;\middle|\;\mathcal{F}_u\right] \tag{14}$$

By Equation (4), and Proposition 1, we can simplify the right-hand member of (14) into

$$\mathbb{E}^{\mathbb{Q}^t}\left[\frac{B_s(T)}{B_s(t)}\;\middle|\;\mathcal{F}_u\right] = \frac{B_u(T)}{B_u(t)}.$$

It shows that $\frac{B_s(T)}{B_s(t)}$ is a t-forward martingale.

The last assertion is twofold. For the Brownian part, the result is well known: see [37, Theorem 5.1, p 191]. The form of M given by (7) gives the form of the density of Girsanov Theorem. For the pure jump part, we can refer to [38, Theorem 10.2.6, p. 339]. In this case, Equation (8) is the expression of the density of the change of probability from compensator $\Theta(dz)\lambda_u du$ to $(1 + J(u,t,z))\Theta(dz)\lambda_u du$. □

3 A Two-Factor Non-Markovian Linear Model

In order to carry on a formal computation of zero-coupons bonds and short-term rate, we shall need to specify the structure of volatility and jumps. This is the purpose of the following assumption.

Assumption 3 *For any* $0 \leq s \leq t$, $\Gamma(s,t) = \sigma(s)\frac{1-e^{-\gamma(t-s)}}{\gamma}$ *and, for a.s any* $z \in \mathbb{R}^+$, $J(s,t,z) = exp\left(z\, j(s)\ \frac{1-e^{-\gamma(t-s)}}{\gamma}\right) - 1$, *where* $\sigma > 0$ *and* $j \leq 0$.

Assumption 3 implies that the amortizing factor of both volatility and jump factor is the same. It will play a crucial part in order to obtain a tractable form for the short-rate. Besides, $j \leq 0$ implies that $J(s,t,Z) \leq 0$, hence, the jumps of the zero-coupons are negative. It implies that the jumps of the short rate are positive. It is clear that both Γ and J satisfy the requirements of Definition 2.

The volatility chosen takes the form of the Hull and White model and a similar feature is taken for the jumps. The Hull and White form of the volatility is very close to the general form required to have a Markovian model, as shown in [44]. So it is not a strong requirement on the Brownian part if we want to keep tractable pricing formulae. Even with this assumption, the model with jumps is not Markovian yet, since the expression of the zero-coupon will depend on the whole integrated path of the intensity λ multiplied by a deterministic function.

When the jumps are set to 0, i.e. $J = 0$, we find again the diffusion version of the Hull and White model which is linear, Gaussian and Markovian.

Proposition 3 *Assume that the jumps are exponentially distributed, i.e.* $\Theta(dz) = \alpha e^{-\alpha z}\mathbb{I}_{z\geq 0}dz$. *Assume that* $\beta \leq \gamma$ *and* $j < -1$, *the process* $\{N_t(T)\}_{0\leq t\leq T}$ *is a martingale with moments of every order.*

Proof. The idea of the proof is to apply Proposition 1. In this case, direct calculation shows that

$$N_t(T) \leq K \prod_{T_i \leq t} e^{Z_i f(t,T_i,T)}$$

where $K > 0$ is given by (11) and

$$\begin{aligned} f(t,T_i,T) :=& j(T_i)A_\gamma(T-T_i) - \int_{T_i}^t J(s,T)e^{-\beta(s-T_i)}ds \\ =& j(T_i)A_\gamma(T-T_i) + A_\beta(t-T_i) \\ & - \int_{T_i}^t \frac{\alpha}{\alpha - j(s)A_\gamma(T-s)} e^{-\beta(z-T_i)}ds \end{aligned} \tag{15}$$

with

$$A_\gamma(u) := \frac{1-e^{-\gamma u}}{h} \quad \forall u \geq 0 \text{ and } h \in \mathbb{R}_+ \,. \tag{16}$$

It is clear, by the definition (15), that

$$\sup_{0\leq t\leq T} f(t,T_i,T) = f(T,T_i,T).$$

If $\gamma_1 \geq \gamma_2$, we have $A_{\gamma_1}(s) \geq A_{\gamma_2}(s)$ for any $s \geq 0$. Since

$$f(T,u,T) \leq j(u)A_\gamma(T-u) + A_\beta(u) \quad \forall\, 0 \leq u \leq T,$$

we can conclude that $f(T,T_i,T) \leq 0$.

Hence, the end of proof. □

3.1 Short Rate Dynamics

First, let us define the initial (deterministic) forward rate curve $f(0,\cdot)$ by $B_0(t) := \exp(-\int_0^t f(0,u)du)$,

Let us set $\hat{r}(\cdot) := r(\cdot) - f(0,\cdot)$. Take $t = T$ in (4). We obtain:

$$\begin{aligned} -\int_0^t \hat{r}(u)du \quad = \quad & \int_0^t \sigma(s)\frac{1-e^{-\gamma(t-s)}}{\gamma}dW_s \\ & -\frac{1}{2}\int_0^t \sigma^2(u)\left(\frac{1-e^{-\gamma(t-u)}}{\gamma}\right)^2 du \\ & +\int_0^t\int_{\mathbb{R}^+} zj(u)\frac{1-e^{-\gamma(t-u)}}{\gamma}\nu(du,dz) \\ & -\int_0^t\int_{\mathbb{R}^+} J(u,t,z)\Theta(dz)\lambda_u(du) \end{aligned}$$

The next step consists in inverting the stochastic integration and the integration with respect to time, in order to obtain, on both left and right members, an integral with respect to time. Of course we assume the integrability conditions required to perform this step. It yields

$$\begin{aligned} -\int_0^t \hat{r}(u)du = & \int_0^t\int_0^u \sigma(s)e^{-\gamma(u-s)}dW_s du \\ & -\frac{1}{2}\int_0^t \sigma^2(u)\left(\frac{1-e^{-\gamma(t-u)}}{\gamma}\right)^2 du \\ & +\int_0^t\int_0^u j(s)e^{-\gamma(u-s)}\int_{\mathbb{R}^+} z\nu(ds,dz)du \\ & -\int_0^t\int_{\mathbb{R}^+} J(u,t,z)\Theta(dz)\lambda_u(du) \end{aligned} \qquad (17)$$

In order to extract $\hat{r}$, we need to differentiate with respect to t. We obtain the following result:

Proposition 4 *Under Assumption 3, the dynamics of the short rate is given by*

$$\begin{aligned} r(t) = f(0,t) - \int_0^t \sigma(s) e^{-\gamma(t-s)} dW_s + \int_0^t e^{-\gamma(t-s)} \Phi(s) ds \\ - \int_0^t j(s) e^{-\gamma(t-s)} \int_{\mathbb{R}^+} z\nu(ds,dz) + \int_0^t h(s,t)\lambda_s ds \end{aligned} \tag{18}$$

where

$$\Phi(t) := \int_0^t \sigma^2(s) e^{-\gamma(t-s)} ds \tag{19}$$

$$h(s,t) := j(s) e^{-\gamma(t-s)} \int_{\mathbb{R}^+} z e^{zj(s)\frac{1-e^{-\gamma(t-s)}}{\gamma}} \Theta(dz) \tag{20}$$

The proof is straightforward remarking that the partial derivative with respect to t of the last term in (17) reads

$$\int_{\mathbb{R}^+} \frac{\partial J(s,t,z)}{\partial t} \Theta(dz) = j(s) e^{-\gamma(t-s)} \int_{\mathbb{R}^+} z e^{zj(s)\frac{1-e^{-\gamma(t-s)}}{\gamma}} \Theta(dz) \, .$$

Now, let us provide the *reconstruction formula* which enables to express the forward zero-coupons value at time t, in terms of $r(t)$ and of $\lambda(\cdot)$.

Proposition 5 (Reconstruction Formula) *For any* $0 \le t \le T$,

$$B(t,T) = \frac{B(0,T)}{B(0,t)} \, e^{A_\gamma(T-t) r(t) + \int_0^t b(u,t,T)\lambda_u du + C(t,T)} \tag{21}$$

where A_γ *is given by relation (16) and with the following terms*

$$\begin{aligned} b(s,t,T) &:= A_\gamma(T-t) h(s,t) - \int_t^T h(s,v) dv \\ C(t,T) &:= A_\gamma(T-t) \int_0^t \frac{\sigma^2(s)}{\gamma^2} \left[e^{-\gamma(t-u)} - e^{-2\gamma(t-u)} \right] du \\ &+ A_\gamma(T-t) \int_0^t e^{-\gamma(t-s)} \Phi(s) ds \\ &+ A_\gamma(T-t) f(0,t) \end{aligned}$$

where $\Phi(s)$ *and* $h(s,v)$ *are given by equations (19) and (19) respectively.*

The proof of equation (21) is a standard but tedious computation exploiting the explicit form for $r(t)$ given in equation (18).

3.2 Caplets/floorlets pricing

A caplet (respectively, a floorlet) is a call option on a so-called "Ibor" rate (Euribor, Libor USD, Libor GBP...), paid at the "end date" of the interest period which defines the rate. Indeed, this Ibor rate is caracterized by 3 dates $T^f \leq T^s < T^e$:

- The date T^f is the fixing date, where the rate is known
- The date T^s is the start date of the interest period of the rate
- The date T^e is the end date of the interest period of the rate.

The interest period $[T^s, T^e]$ defines the so-called "frequency" of the rate (3, 6, 12 months for the most common frequencies). It also defined, with a specific day-count fraction, the coverage δ, which is the year fraction between T^s and T^e. The link between the Ibor rate and the zero-coupons is given by:

$$L(T^f, T^s, T^e) = \frac{1}{\delta}\left(\frac{B_{T^f}(T^s)}{B_{T^f}(T^e)} - 1\right)$$

The caplet price of strike K at time $t = 0$ writes

$$C(K) = \delta \, \mathbb{E}\left[Q(T^e)\left(L(T^f, T^z, T^e) - K\right)_+\right]$$

By inserting the expression of the Ibor rate in terms of zero-coupons, and then, switching to the T^s-forward probability, we obtain

$$C(K) = B(0, T^s)\, \mathbb{E}^{T^s}\left[\left(1 - (1 + \delta K)\frac{B_{T^f}(T^e)}{B_{T^f}(T^s)}\right)_+\right]$$

i.e the caplet is a put on the forward zero-coupon.

According to Proposition 2, $\left\{\frac{B_t(T^e)}{B_t(T^s)}\right\}_{0 \leq t \leq T}$ is a $\mathbb{Q}^{T^s}$-martingale. This leads us to the following result:

Proposition 6 *The price at time $t = 0$ of the caplet of strike K written on the Ibor rate $L(T^f, T^s, T^e)$ is given by*

$$C(K) = B_0(T^s)\mathbb{E}^{T^s}\Big[BS_p\Big\{(1+\delta K)\tfrac{B_0(T^e)}{B_0(T^s)}N_{T^f}(T^s,T^e);\; 1;\; T^f;\; \sigma^{LGM}(T^f,T^s,T^e)\Big\}\Big] \tag{22}$$

where $BS_p(f;k;t;v)$ denotes the price in the Black model of a put option with forward f, strike k, time-to expiry t and annualized volatility v. Besides, $\sigma^{LGM}(T^f,T^s,T^e)$ is the volatility of the caplet under the Linear, Gaussian Markov version (i.e. the model with no jumps). It is simply given by

$$\sigma^{LGM}(T^f,T^s,T^e) = \sqrt{\frac{A_\gamma(T^e-T^s)^2\int_0^{T^f}\sigma^2(u)e^{-2\gamma(T^s-u)}}{T^f}du}$$

Proof: The Brownian motion and the Hawkes process being independent, let us consider the price of the instrument conditional to the jumps. The formula is similar to the one in the Hull and White model (cf. [13]). The results follows. □

3.3 Swaption Pricing

A swaption is an option to enter into an interest rate swap at a given date, t^e, called the expiry (at least the physically settled version that we are dealing with). When the swap pays (respectively, receives) the fixed rate, the swaption is called a payer (respectively, receiver) swaption and relates to a call (respectively, a put) on the swap rate. The swap is characterized mainly by the dates of the fixed leg. Let t_0 be the start date of the swap (typically, 2 business days after the expiry t^e for Euro swaps). The payment dates of the fixed leg are given by the schedule $\mathcal{T} := \{t_1,\ldots,t_M\}$, with $t_1 < t_2 < \cdots < t_M$. The distance $t_M - t_0$, expressed as a number of years, is called the tenor of the swap. The associated payment coverages (representing the year fractions of each interest periods are given by the $(\delta_i)_{1\le i\le M}$. We define the associated level (or annuity) by

$$\mathrm{LVL}(t,[T_0,\mathcal{T}]) := \sum_{i=1}^{M}\delta_i B_t(t_i)$$

The floating leg represents the sequence of consecutive Ibor rates. When neglecting the difference between the start date of a rate and the end date of the previous one (typically, when we do not use business days), the value of the floating leg at time t^e writes (with the convention that $s_0 = t_0$ and $s_N = t_M$)

$$\sum_{j=1}^{N} \delta \; B_{T^e}(s_j) \; \frac{1}{\delta} \; \left(\frac{B_{t^e}(s_{j-1})}{B_{t^e}(s_j)} - 1 \right) = B_{t^e}(t_0) - B_{t^e}(t_M)$$

The price of a payer swaption at time $t = 0$, with strike κ, writes

$$\begin{aligned} \mathrm{Swpn}_p(0, [t_0, \mathcal{T}], \kappa) &= \mathbb{E}\left[Q(t^e) \left\{ B_{t^e}(t_0) - B_{t^e}(t_M) - \kappa \, \mathrm{LVL}(t^e, [t_0, \mathcal{T}]) \right\}_+ \right] \\ &= B_0(t_0) \mathbb{E}^{t_0} \left[\left(1 - \sum_{i=1}^{M} c_i \frac{B_{t^e}(t_i)}{B_{t^e}(t_0)} \right)_+ \right] \end{aligned}$$

where

$$c_i := \begin{cases} 1 + \kappa \; \delta_M, & \text{if } i = M \\ \kappa \; \delta_i, & \text{elsewhere} \end{cases}$$

As in the context of Proposition 6, let us work conditionally to the jumps and marks. In this case, according to Proposition 2, under the T_0-forward probability any of the $\frac{B_{t^e}(t_i)}{B_{t^e}(t_0)}$, $1 \le i \le M$, is a log-normal random variable with volatility ν_i, defined by

$$\nu_i^2 := (A_\gamma(t_i - t_0))^2 \int_0^{t^e} \sigma^2(u) e^{-2\gamma(t_0 - u)} du$$

Morevover, with the separability of Γ, their underlying normal laws are correlated to 1. Thus, we can write

$$\begin{aligned} &\mathrm{Swpn}_p(0, [t_0, \mathcal{T}], \kappa) \\ =& B_0(t_0) \mathbb{E}^{t_0} \left\{ \int_{-\infty}^{x^*} \left(1 - \sum_{i=1}^{M} c_i \frac{B_0(t_i)}{B_0(t_0)} N_{t^e}(t_0, t_i) e^{\nu_i x - \frac{\nu_i^2}{2}} \right) \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx \right\} \end{aligned} \tag{23}$$

where x^* (depending on the jumps) is the solution of

$$\sum_{i=1}^{M} c_i \frac{B_0(t_i)}{B_0(t_0)} N_{t^e}(t_0, t_i) e^{\nu_i x^* - \frac{\nu_i^2}{2}} = 1 \tag{24}$$

This solution always exists as the ν_i are positive as well as the zero-coupons and the $N_{t^e}(t_0, t_i)$. With Equations (23) and (24) at hand, a simple integration with respect to the Gaussian density yields

$$\begin{aligned} &\mathrm{Swpn}_p(0, [t_0, \mathcal{T}], \kappa) \\ =&\mathbb{E}^{t_0}\left[B_0(t_0)\mathcal{N}(x^*) - \sum_{i=1}^{M} c_i B_0(t_i) N_{t^e}(t_0, t_i)\mathcal{N}(x^* - \nu_i)\right] \\ =&\mathbb{E}\left[N_{t^e}(t_0)\left(B_0(t_0)\mathcal{N}(x^*) - \sum_{i=1}^{M} c_i B_0(t_i) N_{t^e}(t_0, t_i)\mathcal{N}(x^* - \nu_i)\right)\right] \end{aligned} \tag{25}$$

In Equation (25), the only random parts of the terms in the expectation operator are $N_{t^e}(t_0)$ and x^*. They both depend of the jumps occurring on $]0, t^e]$.

4 Numerical Illustration and Discussion on the Results

In this section we provide a numerical illustration of the results obtained in order to show the relevance of the model introduced and to price swaptions. More precisely, we show that the model, with a well-chosen set of parameters, is able to reproduce stylized facts of this asset class. We work with a set of swaption contracts of start date t_0 in 5 years, tenor $t_M - t_0$ equal to 10 years, and annual payments. The strike of these swaptions is in the interval $[\kappa^{ATM} - 4\%, \kappa^{ATM} + 4\%]$, where κ^{ATM} is the strike at the money, defined by the forward swap rate

$$\kappa^{ATM} = \frac{B_0(t_0) - B_0(t_M)}{\mathrm{LVL}(0, [t_0, \mathcal{T}])}.$$

In our dataset, which corresponds to the observation of the Euro zero-coupon rate curve as of 14^{th} June 2021, we have κ^{ATM} equal to 0.678%. We also observe the prices of these swaptions at the same date, from which we determine the annual implied volatility under the Bachelier model (i.e. normal volalitily), which provides us with the call price of forward f, strike k, maturity t, and volatility v,

$$C_B(f, k, t, v) = (f - k)\,\mathcal{N}\left(\frac{f - k}{v\sqrt{t}}\right) + v\sqrt{t}\,g\left(\frac{f - k}{v\sqrt{t}}\right),$$

where $\mathcal{N}$ and g are respectively the Gaussian cdf and the Gaussian pdf. In the case of a swaption of strike κ and market price $\text{Swpn}^{\text{market}}$, the forward rate is κ^{ATM} and the implied volatility $\sigma^{\text{Bachelier}}$ is solution of the equation

$$C_B(\kappa^{ATM}, \kappa, T_0, \sigma^{\text{Bachelier}}) = \text{Swpn}^{\text{market}}.$$

Solving this equation numerically with the Newton-Raphson algorithm, we represent the smile of implied volatilities in Figure 2.

Remark 2 *The use of Bachelier model (normal framework) to quote swaptions has been extended to Euro rates, which have been negative for many years. Before this situation, the quotation was made according to Black and Scholes model (lognormal framework). The occurrence of negative rates (and therefore strikes) required to switch to a normal framework as a market standard.*

Following the Hawkes-HJM approach, we can generate swaption prices with Equation (25), which defines the price as an expectation of a given transformation of the Hawkes process. We evaluate the expectation via a Monte Carlo method. We thus have to simulate Hawkes processes in the time interval $[0, t_M]$. To this end, we use the exact simulation method of Dassios and Zhao [17]. This method provides us with the quadruplet

$$(n_{t_M}, \{T_i\}_{i=1}^{n_{t_M}}, \{\lambda_{T_i}\}_{i=1}^{n_{t_M}}, \{Z_i\}_{i=1}^{n_{t_M}}),$$

where n_{t_M} is the number of jumps simulated in the interval $[0, t_M]$, T_i is the i^{th} jump time, λ_{T_i} is the intensity at this time, and Z_i is the mark of the i-th jump. Figure 1 represents a simulated trajectory of the processes used in the model.

In the general framework of the Hawkes-HJM model, we focus on a particular specification. Indeed, we assume that the distribution Θ is of exponential type: $\Theta(dz) = \alpha \exp(-\alpha z)dz$. This choice is justified for parsimony reasons and is coherent with the previous literature, see for instance [4, 8, 25]. We also assume that the functions σ and j, introduced in Assumption 3, are constant. Finally, our model has globally six parameters: σ, j, γ, β, λ_0, and α.

For a given set of these six parameters, we calculate the price of eleven swaptions of various strikes with the Hawkes-HJM model. Then, we translate these model prices in implied volatilities under the Bachelier model.

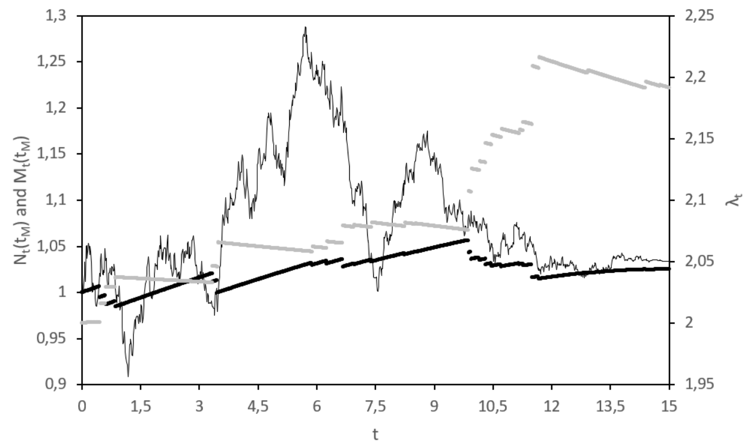

Figure 1: Simulated intensity λ_t (grey line) and simulated processes $N_t(t_M)$ (thick black line) and $M_t(t_M)$ (thin black line), for parameters $\gamma = 0.1$, $\beta = 0.05$, $\lambda_0 = 2$, $\alpha = 100$, $\sigma = 0.011$, $j = -0.1$, and $t \in [0, t_M]$.

The results is displayed in Figure 2, in which each curve represents a particular set of parameters, namely $\gamma = 0.1$, $\beta = 0.05$, $\lambda_0 = 2$, and $\alpha = 100$, with the parameters σ and j fixed so as to get an implied volatility equal to the true implied volatility for the at-the-money swaption. We also point out that these parameters fulfill Assumption 1 and the fit gives $\sigma = 0.0057$, $j = -1$ such that Assumption 3 is also verified.

We observe that the Hawkes-HJM model reproduces properly the smile of the true implied volatilities of swaptions. This smile is even more pronounced for larger jumps, that is for a larger $|j|$. On the contrary, values of j close to 0 lead to an almost flat curve, indicating the relevance of the jumps in the rate dynamics.

For each set of parameters, all the prices are computed with the same series of pseudo-random numbers, in order to avoid that the differences of prices for swaptions of different strikes come from an inaccuracy of the Monte Carlo method. In this numerical application, we have considered 1,000 simulations in the Monte Carlo method, which leads to a satisfying accuracy, as reported in Figure 3, in which we display the standard deviation of the output of our method as a function of the number of Monte Carlo simulations.

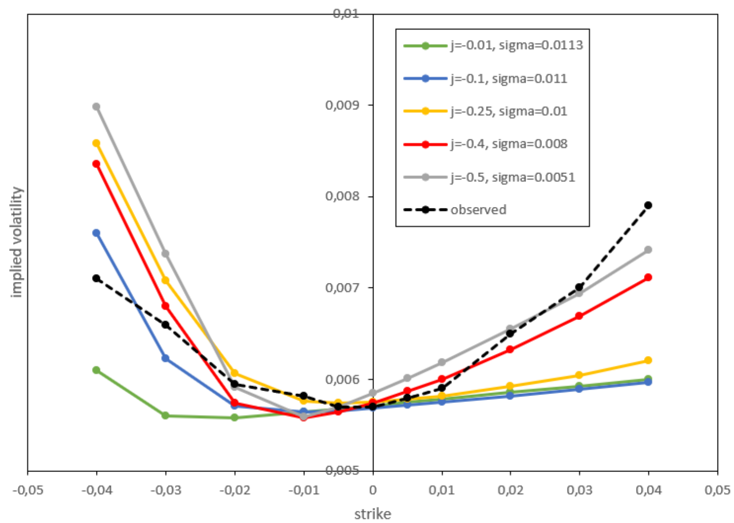

Figure 2: Implied volatility of the swaption contracts under the Bachelier model for the market prices (dotted line) and for prices obtained by simulations with the Hawkes-HJM model (continuous lines), as a function of $\kappa - \kappa^{ATM}$.

Finally, we want to get some insight on the role of each parameter of the Hawkes-HJM model on the shape of the volatility smile. We thus study the sensitivity of the smile to a change in only one parameter. Results are gathered in Figure 4. A higher γ, that is a smaller impact of jumps and volatility on rates, tends to move globally the smile downwards but may also reinforce the skew, with still a strong volatility for in-the-money swaptions. A higher value of λ_0, that is the more frequent occurrence of jumps, tends to translate the smile upwards without any obvious impact on the skew. A higher value of α, that is the occurrence of smaller jumps, moves the smile downwards and tends to flatten the right-hand side of the curve. Regarding β, the strength of the mean reversion of the intensity process, the sensitivity study has not led to any obvious interpretation, since the smile does not change significantly when increasing or decreasing, even to a large extent, the value of this parameter.

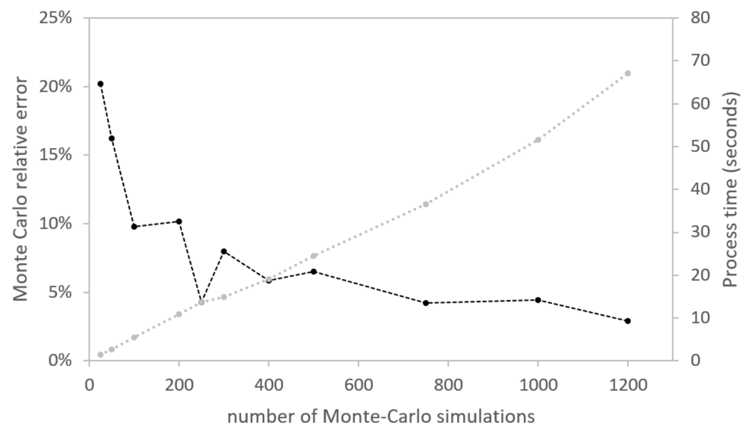

Figure 3: Monte Carlo error (black line), defined as the standard deviation of the implied volatility (obtained from the model prices evaluated with a given number of simulations in the Monte Carlo method), relatively to the average implied volatility (obtained with the same method). Parameters of the model are $\sigma = 0.0057$, $j = -1$, and the same values as in Figure 2 for the other parameters. The swaption considered is at the money. The grey line provides the computation time with an Intel Core i5-7300U CPU @ 2.60GHz.

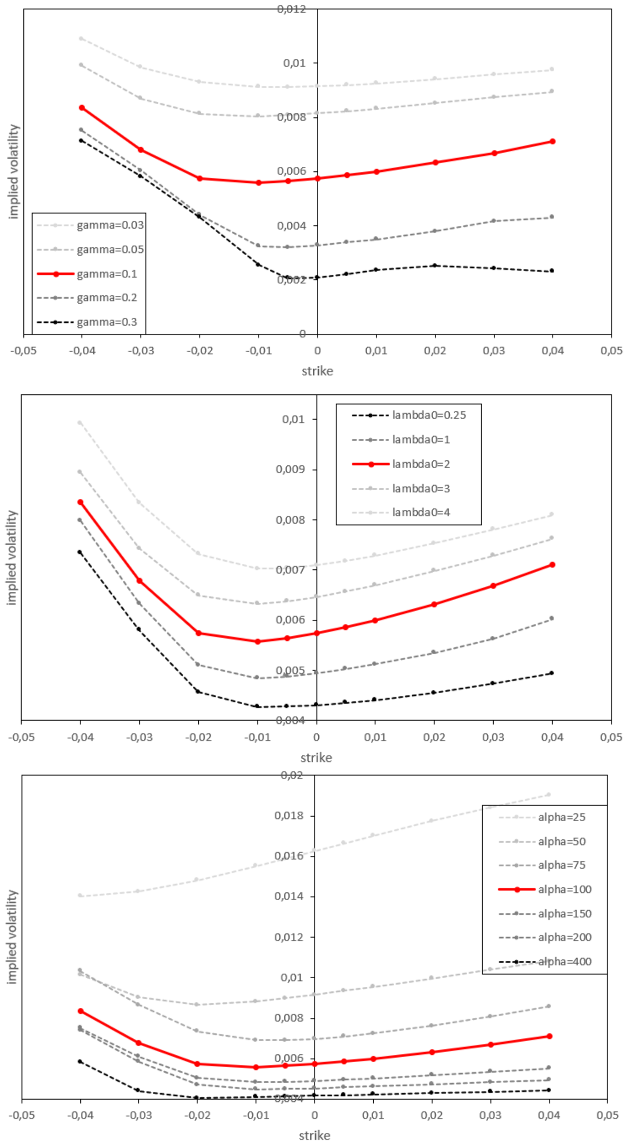

Figure 4: Volatility smiles when changing only one parameter: γ, λ_0, and α. The reference set of parameters (red curve) corresponds to $(\sigma, j, \gamma, \beta, \lambda_0, \alpha) = (0.008,\ -0.4,\ 0.1,\ 0.05,\ 2,\ 100)$.

5 Concluding Remarks and Discussion on the Results

In the present paper, we provide a new modeling framework for forward interest rates based on Hawkes processes, coherent with a large empirical literature [2, 3, 16, 18, 45, 48]. These processes have been proved to be an adequate instrument in order to capture jump clustering features and to describe the price dynamics in many asset classes, like credit risk derivatives [22], equity [1], volatility [35], and short interest rates [25, 33]. Jump clustering features have been detected in forward curves dynamics in power markets [14], and this suggests that Hawkes-type models could arise as a natural modeling framework for forward interest rates as well, in particular in periods of financial turmoil. The same clustering features is highlighted in insurance applications in particular to model and manage cyber risks, see [9, 28], epidemics [29] and water management [46, 47].

After presenting the model setting and the conditions under which the corresponding dynamics for short rates becomes Markovian, we provided some explicit valuation formulae for some of the most popular derivatives in fixed-income markets. Finally we illustrated a numerical example in which a calibration is performed on real data in order to check the validity of this new modeling framework and we have shown that the smile shapes arising are realistic enough to support a model based on Hawkes processes.
A systematic investigation of econometric type on historical forward rates series would be the most natural next step in order to check the validity of this modeling approach, but this, together with new derivatives valuation with proper numerical methods, will be the subject of future research.

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