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Regular Article

Weighted Sobolev spaces and Morse estimates for quasilinear elliptic equations

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A B S T R A C T

We establish critical groups estimates for the weak solutions of $-\Delta_p u = f(x,u)$ in Ω and $u = 0$ on $\partial\Omega$ via Morse index, where Ω is a bounded domain, $f \in C^1(\overline{\Omega} \times \mathbb{R})$ and $f(x,s) > 0$ for all $x \in \overline{\Omega}$, $s > 0$ and $f(x,s) = 0$ for all $x \in \overline{\Omega}$, $s \leq 0$. The proof relies on new uniform Sobolev inequalities for approximating problems. We also prove critical groups estimates when Ω is the ball or the annulus and f is a sign changing function.

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Contents

1. Introduction

Let us consider the model problem

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{P}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with $C^{1,\alpha}$ boundary, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p \geq 2$, and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$. From now on, we assume that f satisfies:

(f) $f \in C^1(\overline{\Omega} \times \mathbb{R})$; if $N \geq p$ *we also assume the estimate*

$$|D_s f(x,s)| \leq c_1 |s|^q + c_2$$

with c_1, c_2 *positive constants and* $0 \leq q \leq p^* - 2$ *with* $p^* = Np/(N-p)$ *if* $N > p$, *while* q *is any positive number, if* $N = p$.

Solutions u of (P) have to be understood in the weak distributional meaning and correspond to critical points of the C^2-functional $J : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by setting

$$J(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \int_\Omega F(x,u)\, dx\,, \tag{1.1}$$

where $F(x,t) = \int_0^t f(x,s)\, ds$. It is well known (see [22,32,38,39]) that each critical point u of J belongs to $C^{1,\beta}(\overline{\Omega})$ for some $\beta \in]0,1]$. Moreover, we have

$$\langle J''(u)v, v\rangle = \int_\Omega \left[|\nabla u|^{p-2} \, |\nabla v|^2 \, + (p-2)|\nabla u|^{p-4} \, (\nabla u|\nabla v)^2 \right] \, dx$$
$$- \int_\Omega D_s f(x,u)\, v^2 \, dx \qquad \text{for all } u, v \in W_0^{1,p}(\Omega)\,. \tag{1.2}$$

Such quasilinear elliptic equations, involving the p-Laplace operator or the p-area operator, arise in the study of non-Newtonian fluids, in Plasma Physics and Differential Geometry.

An open problem related to (P) is the evaluation of the critical groups of the functional (1.1) at its critical points via differential notions.

We recall some basic concepts (see [7,8,21,34]).

Definition 1.1. Let $u \in W_0^{1,p}(\Omega)$ be a critical point of J at level $c = J(u)$. The *m-th critical group* of J at u (with respect to a given field $\mathbb{K}$) is defined by

$$C_m(J, u) = H^m\left(J^c, J^c \setminus \{u\}; \mathbb{K}\right),$$

where H^* stands for Alexander-Spanier cohomology [37] and $J^c = \left\{v \in W_0^{1,p}(\Omega) : J(v) \le c\right\}$.

We also introduce the formal series of powers with coefficients in $\mathbb{N} \cup \{\infty\}$

$$P_t(J, u) = \sum_{m=0}^{\infty} \left(\dim_{\mathbb{K}} C_m(J, u)\right) t^m .$$

Definition 1.2. Let $u \in W_0^{1,p}(\Omega)$ be a critical point of J. For every integer $m \ge 1$, first define

$$\lambda_m(J''(u)) = \inf_{V \in \mathcal{V}_m} \max_{v \in V \setminus \{0\}} \frac{\langle J''(u)v, v\rangle}{\|v\|_2^2},$$

where $\mathcal{V}_m$ is the family of the m-dimensional linear subspaces of $W_0^{1,p}(\Omega)$ and $\|\cdot\|_2$ denotes the usual $L^2(\Omega)$-norm. For convenience, set also $\lambda_0(J''(u)) = -\infty$.

Then the *Morse index* of J at u, denoted by $m(J, u)$, and the *large Morse index* of J at u, denoted by $m^*(J, u)$, are defined as

$$\begin{aligned} m(J, u) &= \sup\{m : \lambda_m(J''(u)) < 0\}, \\ m^*(J, u) &= \sup\{m : \lambda_m(J''(u)) \le 0\}. \end{aligned}$$

It is easily seen that $m(J, u)$ is also the supremum of the dimensions of the linear subspaces of $W_0^{1,p}(\Omega)$ where the quadratic form $\{v \mapsto \langle J''(u)v, v\rangle\}$ is negative definite.

For quasilinear elliptic equations, containing the p-laplacian ($p > 2$), the attempt of extending Morse theory to Banach spaces presents some conceptual difficulties (cf. [42,41,6,7]). In particular the classical definition of *nondegenerate* critical point in Hilbert spaces does not seem reasonable in a Banach (not Hilbert) one and several obstructions arise due to the lack of Fredholm properties of the linearized operators at the critical points. Therefore the classical Morse Lemma, its generalized versions, due to Gromoll-Meyer [26], and perturbation results, due to Marino and Prodi [33], cannot be successfully applied.

In order to introduce a form of regularization, let us embed the functional J into a family of C^2-functionals $J_\varepsilon : W_0^{1,p}(\Omega) \to \mathbb{R}$, $\varepsilon \geq 0$, defined by setting

$$J_\varepsilon(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx + \frac{\varepsilon}{2} \int_\Omega |\nabla u|^2 \, dx - \int_\Omega F(x,u) \, dx \tag{1.3}$$

and associated to the quasilinear elliptic equation

$$\begin{cases} -\Delta_p u - \varepsilon \, \Delta u = f(x,u) & \text{in } \Omega \,, \\ u = 0 & \text{on } \partial\Omega \,. \end{cases} \tag{P_ε}$$

First of all, in [15,17,18], some ideas are introduced for studying the critical groups of the Euler functional J_1. We stress that each critical point u of J_1 is degenerate in the classical sense given in a Hilbert setting, namely $J_1''(u) : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is not bijective and, more precisely, is not a Fredholm operator. However the presence of the semilinear term gives rise to a linearized operator that is not degenerate in the sense of elliptic partial differential equations, and thus one can perform a suitable finite dimensional reduction. In this way it is possible to give a good extension of local Morse theory in the framework of functional J_1 and the right notion of nondegeneracy seems to be the injectivity of $J_1''(u)$. Indeed one can show that if $J_1''(u)$ is injective, then u is isolated and the critical groups at u can be computed by means of its Morse index.

In [16], such critical groups estimates are also extended to a class of admissible functionals of the form $J_1(u) + h(P_V(u))$ where P_V is a linear continuous projection on a finite dimensional subspace V of $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $h \in C^2(V, \mathbb{R})$.

Successively in [18] the authors proved that the above notion of nondegeneracy is generally verified in the sense of Marino-Prodi perturbation type results, showing that each degenerate critical point u of J_1 (namely $J_1''(u)$ is not injective) can be solved in a finite number of nondegenerate critical points of a locally approximating functional, belonging to the above admissible class.

In the present work we are concerned with problem (P), with $p > 2$, which gives rise, according to (1.2), to a linearized operator which is degenerate (in the sense of partial differential equations) at any point where $|\nabla u|$ vanishes; thus the linearized equation at u fails to be uniformly elliptic.

We recall the classical result in the Hilbert setting (see [7, Theorem I.5.1] or [34, Theorem 8.3]).

Theorem 1.3. *Let $p = 2$ and let $u_0 \in W_0^{1,2}(\Omega)$ be an isolated critical point of J. Then $m(J,u_0)$ and $m^*(J,u_0)$ are finite and it results*

$$C_m(J,u_0) = \{0\} \qquad \textit{for all } m < m(J,u_0) \textit{ and all } m > m^*(J,u_0) \,.$$

Nevertheless, the estimate concerning the case $m < m(J,u_0)$ has been derived by Lancelotti in [31, Theorem 3.1] in a general Banach space.

Theorem 1.4. *Let $p > 2$ and let $u_0 \in W_0^{1,p}(\Omega)$ be a critical point of J. Then we have*

$$C_m(J, u_0) = \{0\} \qquad \text{for all } m < m(J, u_0)\,.$$

On the contrary, the case $m > m^*(J, u_0)$ is unsolved and the evaluation of the high critical groups remains a challenging open problem. In [11], the first two authors proved that each isolated critical point of J has critical groups of finite type and that the Poincaré-Hopf formula holds.

For special geometries, high critical groups computations have been obtained by Aftalion and Pacella in [1]. Precisely they considered problem (P) when Ω is the unit ball of $\mathbb{R}^N$, $f(x, s) = f(s)$ is a nonlinear term and u_0 is a radial, positive solution to (P), such that $|\nabla u_0(x)| \neq 0$ for $x \neq 0$. We remark that this last condition is satisfied by every positive solution u of (P) if $F(s) \leq 0$ at all values $s \leq u(0)$ for which $f(s) = 0$ (see Proposition 1.2.6 of [24]). From L'Hôpital's rule, the solution u_0 belongs to the space of radial functions

$$\begin{aligned} X \quad &= \{v \in C^2(\overline{B} \setminus \{0\} \cap C^1(\overline{B}) \,|\, v(x) = w(|x|), \forall x \in B,\ v = 0\ \partial B, \\ &\quad |w'(|x|)| \leq c|x|^{1/(p-1)},\ |w''(|x|)| \leq c|x|^{(2-p)/(p-1)}\ \forall x \in B \setminus \{0\}\}. \end{aligned} \tag{1.4}$$

Then, the authors performed a finite-dimensional reduction in the regular space X and derived critical groups estimates for the functional J restricted to X.

In this present paper we consider a general bounded domain Ω and a critical point u_0 of J, without any symmetry property. Thus the understanding of the properties of the linearized operator $J''(u_0)$ and in particular its spectrum can be very difficult. Differently from [15,17,18], it does not seem possible, in general, to perform a finite dimensional reduction in the variational setting. Therefore we need to adopt a different strategy. We derive critical groups estimates for the functional J by means of a perturbation argument. Under an abstract condition at the critical point u_0, called (PC) condition, which guarantees the uniform compact embedding of some weighted Sobolev spaces, we are able to estimate the high critical groups of J at u_0 (see Theorem 1.7).

In particular, we recognize that the abstract condition (PC) holds at u_0 if we have a sign condition on the nonlinearity f for general bounded domains, using a refined version of some techniques introduced in [20].

Our analysis requires to exploit the geometric and qualitative properties of the solutions to quasilinear elliptic equations and their critical sets.

In order to state our main results, we stress that actually, since the principal part of $J''(u_0)$ degenerates where $|\nabla u_0| = 0$, at least two different definitions of $\lambda_m(J''(u_0))$ can be considered.

Let us also point out that our regularity assumptions on $\partial\Omega$ imply that $W_0^{1,2}(\Omega) = W_0^{1,1}(\Omega) \cap W^{1,2}(\Omega)$.

Let $p \geq 2$. For every $u_0 \in C^1(\overline{\Omega})$, denote by $\widetilde{W}_{u_0}$ the set of w's in $W_0^{1,1}(\Omega) \cap L^2(\Omega)$ such that

$$\int_\Omega |\nabla u_0|^{p-2}\,|\nabla w|^2\,dx < +\infty\,.$$

It is easily seen that $\widetilde{W}_{u_0}$ is a linear subspace of $W^{1,1}_0(\Omega)\cap L^2(\Omega)$ and that

$$\|w\|_{u_0} := \left\{\int_\Omega \left[|\nabla u_0|^{p-2}\,|\nabla w|^2 + (p-2)|\nabla u_0|^{p-4}\,(\nabla u_0|\nabla w)^2 + w^2\right]\,dx\right\}^{\frac{1}{2}}$$

is a norm (coming from a scalar product) in $\widetilde{W}_{u_0}$. If we denote by W_{u_0} its completion, then W_{u_0} is a Hilbert space such that

$$W^{1,2}_0(\Omega)\subseteq W_{u_0}\subseteq L^2(\Omega)$$

with continuous embeddings. We still denote by $\|\cdot\|_{u_0}$ the associated norm in W_{u_0}. Finally, we denote by H_{u_0} the closure of $C^\infty_c(\Omega)$ in W_{u_0}, so that

$$W^{1,p}_0(\Omega)\subseteq W^{1,2}_0(\Omega)\subseteq H_{u_0}\subseteq W_{u_0}\subseteq L^2(\Omega)\,.$$

Of course, we have $H_{u_0}=W_{u_0}=W^{1,2}_0(\Omega)$ in the case $p=2$.

Now we come back to Definition 1.2, taking into account the presence of the two spaces H_{u_0} and W_{u_0}.

Definition 1.5. Let $u_0\in W^{1,p}_0(\Omega)$ be a critical point of J. For every integer $m\geq 1$, first define

$$\lambda^W_m(J''(u_0)) = \inf_{V\in\mathcal{W}_m}\ \max_{w\in V\setminus\{0\}} \frac{\|w\|^2_{u_0} - \displaystyle\int_\Omega \left[1+D_sf(x,u_0)\right]w^2\,dx}{\|w\|_2^2}\,,$$

where $\mathcal{W}_m$ is the family of the m-dimensional linear subspaces of W_{u_0}. For convenience, set also $\lambda^W_0(J''(u_0))=-\infty$.

Define also

$$\lambda^H_m(J''(u_0)) = \inf_{V\in\mathcal{V}_m}\ \max_{v\in V\setminus\{0\}} \frac{\|v\|^2_{u_0} - \displaystyle\int_\Omega \left[1+D_sf(x,u_0)\right]v^2\,dx}{\|v\|_2^2}\,,$$

where $\mathcal{V}_m$ is the family of the m-dimensional linear subspaces of H_{u_0}. For convenience, set also $\lambda^H_0(J''(u_0))=-\infty$.

Finally, define

$$\begin{aligned} m_H(J,u_0) &= \sup\left\{m:\ \lambda^H_m(J''(u_0))<0\right\}\,,\\ m^*_W(J,u_0) &= \sup\left\{m:\ \lambda^W_m(J''(u_0))\leq 0\right\}\,. \end{aligned}$$

Since $W_0^{1,p}(\Omega)$ is dense in H_{u_0}, it is easily seen that

$$\lambda_m^H(J''(u_0)) = \lambda_m(J''(u_0))\,, \qquad m_H(J,u_0) = m(J,u_0)\,,$$

while the inclusion $H_{u_0} \subseteq W_{u_0}$ implies that

$$\lambda_m^W(J''(u_0)) \leq \lambda_m(J''(u_0))\,, \qquad m_W^*(J,u_0) \geq m^*(J,u_0)\,.$$

In the end, it is

$$m_H(J,u_0) = m(J,u_0) \leq m^*(J,u_0) \leq m_W^*(J,u_0)\,.$$

Of course, if $H_{u_0} = W_{u_0}$ we have

$$m_W^*(J,u_0) = m^*(J,u_0)\,.$$

Definition 1.6. Let u_0 be a critical point of J. The condition (PC) is said to hold at u_0 if, for every sequence (ε_n) converging to 0 with $\varepsilon_n \geq 0$, every sequence (u_n) converging to u_0 in $C^1(\overline{\Omega})$ and every sequence (w_n) in $W_0^{1,1}(\Omega) \cap L^2(\Omega)$ such that each u_n is a critical point of J_{ε_n} and

$$\sup_n \int_\Omega \left[\left(\varepsilon_n + |\nabla u_n|^{p-2}\right)|\nabla w_n|^2 + w_n^2\right]\,dx < +\infty\,,$$

we have that (w_n) is weakly precompact in $W_0^{1,1}(\Omega)$ and strongly precompact in $L^2(\Omega)$.

Now we can state our first result.

Theorem 1.7. *Let $u_0 \in W_0^{1,p}(\Omega)$ be an isolated critical point of J and assume that condition (PC) holds at u_0.*

Then $m_W^*(J,u_0) < \infty$ and we have

$$C_m(J,u_0) = \{0\} \qquad \text{for all } m < m(J,u_0) \text{ and all } m > m_W^*(J,u_0)\,.$$

In particular, if $m(J,u_0) = m_W^*(J,u_0)$ and $C_m(J,u_0) \neq \{0\}$ for some $m \in \mathbb{N}$, then we have $m(J,u_0) = m_H(J,u_0) = m_W^*(J,u_0) = m^*(J,u_0) = m$.

Taking advantage of the regularity results of the next sections, we can infer the next statement.

Theorem 1.8. *Let $p > 2$ and let $u_0 \in W_0^{1,p}(\Omega) \setminus \{0\}$ be an isolated critical point of J. Assume that*

$$
\begin{aligned}
&f(x,s) > 0 \qquad \text{for all } x \in \overline{\Omega} \text{ and } s > 0\,,\\
&f(x,s) = 0 \qquad \text{for all } x \in \overline{\Omega} \text{ and } s \leq 0\,.
\end{aligned}
$$

Then $m(J,u_0)$ and $m_W^*(J,u_0)$ are finite and it results

$$C_m(J,u_0) = \{0\} \qquad \text{for all } m < m(J,u_0) \text{ and all } m > m_W^*(J,u_0)\,.$$

In particular, if $m(J,u_0) = m_W^*(J,u_0)$ and $C_m(J,u_0) \neq \{0\}$ for some $m \in \mathbb{N}$, then we have $m(J,u_0) = m_H(J,u_0) = m_W^*(J,u_0) = m^*(J,u_0) = m$.

Remark 1.9. Under the assumptions of Theorem 1.8, we also have that

$$\{x \in \Omega : \ \nabla u_0(x) = 0\}$$

is a compact subset of Ω. Therefore, the sense of "zero at $\partial\Omega$" in W_{u_0} is the same of $W_0^{1,2}(\Omega)$.

Whether or not it is $H_{u_0} = W_{u_0}$ is a delicate question (see also [9,10,29,30,40,44]). In particular, let us mention a consequence of [44, Theorem 4.7].

Proposition 1.10. *Let $p > 2$ and let $u_0 \in W_0^{1,p}(\Omega)$ be a critical point of J such that*

$$Z = \{x \in \Omega : \ \nabla u_0(x) = 0\}$$

is negligible and such that

$$\inf\left\{\int_\Omega |\nabla v|^2\, |\nabla u_0|^{p-2}\, dx : \ v \in C_c^\infty(\Omega),\ v = 1 \text{ in a neighborhood of } Z\right\} = 0\,.$$

Then $H_{u_0} = W_{u_0}$.

Remark 1.11. From Theorem 1.8 it follows that, if u_0 is an isolated Mountain Pass critical point of J in the sense of [8, Definition 5.1.19], with $m(J,u_0) = m_W^*(J,u_0)$, then we have $m(J,u_0) = m_W^*(J,u_0) = 1$.

Corollary 1.12. *Under the assumptions of Theorem 1.8, set*

$$Z = \{x \in \Omega : \ \nabla u_0(x) = 0\}$$

and suppose that

$$\inf\left\{\int_\Omega |\nabla v|^2\, |\nabla u_0|^{p-2}\, dx : \ v \in C_c^\infty(\Omega),\ v = 1 \text{ in a neighborhood of } Z\right\} = 0\,.$$

Then $H_{u_0} = W_{u_0}$, $m(J, u_0)$ and $m^*(J, u_0)$ are finite and we have

$$C_m(J, u_0) = \{0\} \qquad \textit{for all } m < m(J, u_0) \textit{ and all } m > m^*(J, u_0)\,.$$

In particular, if $m(J, u_0) = m^*(J, u_0)$ and $C_m(J, u_0) \neq \{0\}$ for some $m \in \mathbb{N}$, then we have $m(J, u_0) = m^*(J, u_0) = m$.

When Ω is a ball, $f = f(s)$ satisfies the assumptions of Theorem 1.8 and u_0 is a positive radial critical point of J, then all the assumptions of Corollary 1.12 are satisfied.

Furthermore, in the special case of the ball, a complete description of the critical groups can be given in the nondegenerate case, by means of a stability result in [17], improving Theorem 2.2 in [1]. See also [23] for Liouville theorems for p-Laplace equations.

In the special cases of the ball and the annulus, we can also consider sign changing nonlinearities and establish the following critical groups computations of J at u_0.

Theorem 1.13. *Let $p > 2$. Assume that B is the unit ball of $\mathbb{R}^N$ and $u_0 \in C^1(\overline{B})$ is a positive* radial *solution to*

$$(R) \quad \begin{cases} -\Delta_p u = f(u) & \textit{in } B \\ u = 0 & \textit{on } \partial B \end{cases}$$

where $f(s) > 0$ for any $s > 0$ and $f(s) = 0$ for any $s \leq 0$. If $J''(u_0)$ is injective and $C_{\overline{m}}(J, u_0) \neq 0$ for some $\overline{m} \in \mathbb{N}$, then $\overline{m} = m(J, u_0)$ and

$$C_m(J, u_0) \cong \delta_{m, m(J,u_0)} \mathbb{K} \qquad \textit{for all } m\,.$$

Moreover, if $f(s) = s^q - \Lambda s^{p-1}$ with $2 \leq p < q + 1 \leq p^*$, $\Lambda > 0$, then $H_{u_0} = W_{u_0}$, the indices $m(J, u_0)$, $m^*(J, u_0)$ are finite and

$$C_m(J, u_0) = \{0\} \qquad \textit{for all } m < m(J, u_0) \textit{ and all } m > m^*(J, u_0)\,.$$

When the domain is an annulus, we can also consider changing sign nonlinearities and establish the following critical groups computations of J at u_0.

Theorem 1.14. *Let $p > 2$. Assume that A is an annulus of $\mathbb{R}^N$ and $u_0 \in C^1(\overline{A})$ is a positive* radial *solution to*

$$(R) \quad \begin{cases} -\Delta_p u = f(u) & \textit{in } A \\ u = 0 & \textit{on } \partial A \end{cases}$$

where $f(s) = s^q - \Lambda s^{p-1}$, with $2 \leq p < q + 1 \leq p^*$ and $\Lambda > 0$.

Then the indices $m(J, u_0)$ and $m^*_W(J, u_0)$ are finite and

$$C_m(J, u_0) = \{0\} \qquad \textit{for all } m < m(J, u_0) \textit{ and all } m > m^*_W(J, u_0)\,.$$

The proof of Theorem 1.7 will be given in next section, while Theorems 1.8, 1.13, 1.14 and Corollary 1.12 will be proved in Section 8.

2. A general result

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with $C^{1,\alpha}$ boundary, let $p \geq 2$ and assume condition (f). We denote by $\|\cdot\|_q$ the usual norm in L^q, $1 \leq q \leq \infty$, we set

$$B_r(u) = \left\{ v \in W_0^{1,p}(\Omega) : \ \|\nabla v - \nabla u\|_p < r \right\}$$

and consider, for every $\varepsilon \geq 0$, the C^2-functional J_ε given by (1.3) and the associated quasilinear elliptic problem (P_ε). It is easily seen that

$$\langle J_\varepsilon''(u)v, v\rangle = \int_\Omega \left[\left(\varepsilon + |\nabla u|^{p-2}\right)|\nabla v|^2 + (p-2)|\nabla u|^{p-4}|(\nabla u|\nabla v)|^2 \right] dx \\ - \int_\Omega D_s f(x,u) v^2 \, dx \qquad \text{for all } u, v \in W_0^{1,p}(\Omega)$$

and the indices $m(J_\varepsilon, u)$ and $m^*(J_\varepsilon, u)$ can be introduced as in Definition 1.2.

Moreover, it is well known (see [22,32,38,39]) that each critical point u of J_ε belongs to $C^{1,\beta}(\overline{\Omega})$ for some $\beta \in]0,1]$. Consequently, when $\varepsilon > 0$ the quadratic form

$$v \mapsto \int_\Omega \left[\left(\varepsilon + |\nabla u|^{p-2}\right)|\nabla v|^2 + (p-2)|\nabla u|^{p-4}|(\nabla u|\nabla v)|^2 \right] dx - \int_\Omega D_s f(x,u) v^2 \, dx$$

is defined and well behaved in $W_0^{1,2}(\Omega)$ and then other equivalent definitions of $m(J_\varepsilon, u)$ and $m^*(J_\varepsilon, u)$ are possible.

Theorem 2.1. *Let $u_0 \in W_0^{1,p}(\Omega)$ be an isolated critical point of J and let $m^* \in \mathbb{N}$. Suppose that there exist $\overline{\varepsilon} > 0$ and $r > 0$ such that*

$$m^*(J_\varepsilon, u) \leq m^*$$

whenever $0 < \varepsilon \leq \overline{\varepsilon}$ and $u \in B_r(u_0)$ is a critical point of J_ε.

Then we have

$$C_m(J, u_0) = \{0\} \qquad \text{for all } m > m^* .$$

Proof. By [2, Theorem 3.4] there exists $r > 0$ such that the map

$$\begin{array}{ccc} \overline{B_r(u_0)} \times [0,1] & \to & W^{-1,p'}(\Omega) \\ (u, \varepsilon) & \mapsto & J_\varepsilon{}'(u) \end{array}$$

is of class $(S)_+$, hence proper. In particular, every J_ε is weakly lower semicontinuous and satisfies the Palais-Smale condition on $\overline{B_r(u_0)}$ (see also [17, Lemma 3.2]). We may also assume that u_0 is the unique critical point of J in $\overline{B_r(u_0)}$, so that there exists $\sigma > 0$ such that

$$\|J'(u)\| \geq \sigma \qquad \text{whenever } u \in \overline{B_r(u_0)} \setminus B_{r/2}(u_0)\,,$$

and that

$$m^*(J_\varepsilon, u) \leq m^* \qquad \text{whenever } 0 < \varepsilon \leq \overline{\varepsilon} \text{ and } u \in \overline{B_r(u_0)} \text{ is a critical point of } J_\varepsilon, \tag{2.5}$$

$$\|J'_\varepsilon(u)\| \geq \sigma/2 \qquad \text{whenever } 0 < \varepsilon \leq \overline{\varepsilon} \text{ and } u \in \overline{B_r(u_0)} \setminus B_{r/2}(u_0). \tag{2.6}$$

We first apply [14, Theorem 7.1] with $A = B_{2r}(u_0)$, $U = B_r(u_0)$ and $f = J$. Let $\overline{\mu} > 0$ be as in the assertion of that theorem. Then choose $\varepsilon \in]0, \overline{\varepsilon}]$ small enough to guarantee that

$$\|J_\varepsilon - J\|_{C^1(B_{2r}(u_0))} < \frac{1}{2}\,\overline{\mu}\,.$$

The set of critical points of J_ε in $\overline{B_r(u_0)}$ is a compact subset of $B_{r/2}(u_0)$. We aim to apply [18, Corollary 4.4] with $A = B_{2r}(u_0)$, $U'' = B_r(u_0)$, $U' = B_{r/2}(u_0)$ and $f = J_\varepsilon$. In that result it is assumed that $q < p^* - 2$, but the same proof works with $q = p^* - 2$, as J_ε is weakly lower semicontinuous and satisfies the Palais-Smale condition on $\overline{B_r(u_0)}$. Moreover, in [18, Corollary 4.4] the nonlinearity f is independent of x, but under assumption (f) the adaptation of the proofs is straightforward. According to the assertion of that result, there exists a sequence of C^2-functions $\tilde{g}_n : B_{2r}(u_0) \to \mathbb{R}$ such that $\tilde{g}_n$ has no critical point in $\overline{B_r(u_0)} \setminus B_{r/2}(u_0)$, $\tilde{g}_n$ satisfies the Palais-Smale condition on $\overline{B_r(u_0)}$ and $(\tilde{g}_n)$ is convergent to J_ε in $C^2(B_{2r}(u_0))$. Moreover, if we denote by K_n the set of critical points of $\tilde{g}_n$ in $B_{r/2}(u_0)$, then K_n is a finite set and $\tilde{g}''_n(u)$ is injective for all $u \in K_n$. Then, without loss of generality, we may assume that

$$\|\tilde{g}_n - J_\varepsilon\|_{C^1(B_{2r}(u_0))} < \frac{1}{2}\,\overline{\mu} \qquad \text{for all } n\,,$$

whence

$$\|\tilde{g}_n - J\|_{C^1(B_{2r}(u_0))} < \overline{\mu} \qquad \text{for all } n\,.$$

Then, from [14, Theorem 7.1] we infer that, for every n, there exists a formal series of powers Q_n with coefficients in $\mathbb{N} \cup \{\infty\}$ such that

$$\sum_{u \in K_n} P_t(\tilde{g}_n, u) = P_t(J, u_0) + (1 + t)Q_n(t)\,.$$

To conclude the proof, it is enough to show that there exists n such that

$$C_m(\tilde{g}_n, u) = \{0\} \qquad \text{for all } u \in K_n \text{ and } m > m^* \,.$$

Combining [18, Corollary 4.4] with [16, Theorem 1.2], it is enough to prove that there exists n such that

$$m(\tilde{g}_n, u) \le m^* \qquad \text{for all } u \in K_n \,.$$

Assume, for a contradiction, that for every n there exists $u_n \in K_n$ with $m(\tilde{g}_n, u) \ge m^*+1$. Since

$$\lim_n \|\tilde{g}_n - J_\varepsilon\|_{C^1(B_{2r}(u_0))} = 0 \,,$$

we have $J_\varepsilon'(u_n) \to 0$. Therefore, up to a subsequence (u_n) is convergent to some u in $\overline{B_r(u_0)}$ with ${J_\varepsilon}'(u) = 0$, as J_ε satisfies the Palais-Smale condition on $\overline{B_r(u_0)}$.

From [18, Corollary 4.4] we conclude that

$$m^* + 1 \le \limsup_n m(\tilde{g}_n, u_n) \le m^*(J_\varepsilon, u)$$

and a contradiction follows by (2.5). □

Let us point out a simple fact.

Remark 2.2. Let $u_0 \in W_0^{1,p}(\Omega)$ be a critical point of J. Then there exist $\overline{\varepsilon} > 0$ and $r > 0$ such that

$$m(J_\varepsilon, u) \ge m(J, u_0) = m_H(J, u_0)$$

whenever $0 < \varepsilon \le \overline{\varepsilon}$ and $u \in B_r(u_0)$ is a critical point of J_ε.

Assume also that $p = 2$. Then there exist $\overline{\varepsilon} > 0$ and $r > 0$ such that

$$m^*(J_\varepsilon, u) \le m^*(J, u_0) = m_W^*(J, u_0)$$

whenever $0 < \varepsilon \le \overline{\varepsilon}$ and $u \in B_r(u_0)$ is a critical point of J_ε.

The first fact is related to the general estimate of critical groups provided in [31, Theorem 3.1], while the second one is easily proved, as

$$J''(u_0) : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$$

is a Fredholm operator (of index 0).

On the contrary, when $p > 2$ an upper semicontinuity property of m^* does not hold, in general. However, we will prove a suitable estimate for m_W^* under the precompactness assumption (PC) we have introduced.

Theorem 2.3. *Let u_0 be a critical point of J. If condition (PC) holds at u_0, then $m^*_W(J,u_0) < \infty$ and there exist $\overline{\varepsilon} > 0$ and $r > 0$ such that*

$$m^*(J_\varepsilon, u) \leq m^*_W(J, u_0)$$

whenever $0 < \varepsilon \leq \overline{\varepsilon}$ and $u \in B_r(u_0)$ is a critical point of J_ε.

Proof. Let (w_n) be a bounded sequence in $\widetilde{W}_{u_0}$. From condition (PC), with $\varepsilon_n = 0$ and $u_n = u_0$, we infer that (w_n) is precompact in $L^2(\Omega)$, so that W_{u_0} is compactly embedded in $L^2(\Omega)$. Then

$$\left\{ w \mapsto \int_\Omega \left[1 + D_s f(x, u_0)\right] w^2 \, dx \right\}$$

is a smooth quadratic form in W_{u_0} with compact gradient. It follows that $m^*_W(J,u_0) < \infty$.

Now set $m^* = m^*_W(J, u_0)$ and

$$\delta = \lambda^W_{m^*+1}(J''(u_0)) \,.$$

According to Definition 1.5, we have $\delta > 0$ and

$$\max_{w \in V \setminus \{0\}} \frac{\|w\|^2_{u_0} - \displaystyle\int_\Omega \left[1 + D_s f(x, u_0)\right] w^2 \, dx}{\|w\|_2^2} \geq \delta \tag{2.7}$$

whenever V is a linear subspace of W_{u_0} with $m^*_W(J, u_0) < \dim(V) < \infty$.

Let $m \geq 1$, let (ε_n) be strictly decreasing to 0 and let (u_n) be converging to u_0 in $W^{1,p}_0(\Omega)$, where each u_n is a critical point of J_{ε_n} and

$$m^*(J_{\varepsilon_n}, u_n) \geq m \qquad \text{for all } n \geq 1 \,.$$

We need to show that $m^*_W(J, u_0) \geq m$. From the regularity results of [22,32,38,39] it follows that (u_n) is bounded in $C^{1,\beta}(\overline{\Omega})$ for some $\beta \in]0,1]$, hence convergent to u in $C^1(\overline{\Omega})$.

Since $\lambda_m(J''_{\varepsilon_n}(u_n)) \leq 0$, there exists an m-dimensional subspace V_n of $W^{1,p}_0(\Omega)$ such that

$$\langle J''_{\varepsilon_n}(u_n) w, w\rangle \leq \frac{\delta}{2} \, \|w\|_2^2 \qquad \text{for all } w \in V_n \,.$$

Let $\left\{ e_n^{(1)}, \ldots, e_n^{(m)} \right\}$ be an $L^2(\Omega)$-orthonormal basis in V_n. Since

$$\langle J''_{\varepsilon_n}(u_n)e_n^{(j)}, e_n^{(j)}\rangle \le \frac{\delta}{2} \qquad \text{for all } n \ge 1 \text{ and } j = 1, \ldots, m\,,$$

$$\int\limits_\Omega \big[1 + D_s f(x, u_n)\big](e_n^{(j)})^2\,dx \le 1 + \|D_s f(x, u_n)\|_\infty \qquad \text{for all } n \ge 1 \text{ and } j = 1, \ldots, m\,,$$

we have

$$\sup_{n,j} \int\limits_\Omega \Big[\big(\varepsilon_n + |\nabla u_n|^{p-2}\big)|\nabla e_n^{(j)}|^2 + (e_n^{(j)})^2\Big]\,dx < +\infty\,.$$

By assumption (PC), up to a subsequence each $(e_n^{(j)})$ with $1 \le j \le m$ is convergent to some $e^{(j)}$ weakly in $W_0^{1,1}(\Omega)$ and strongly in $L^2(\Omega)$. Moreover, we have that $\{e^{(1)}, \ldots, e^{(m)}\}$ is orthonormal in $L^2(\Omega)$. Let V be the linear span of $\{e^{(1)}, \ldots, e^{(m)}\}$, so that V is a linear subspace of $W_0^{1,1}(\Omega) \cap L^2(\Omega)$ with $\dim(V) = m$.

For every $w \in V$, there exists a sequence (w_n) converging to w weakly in $W_0^{1,1}(\Omega)$ and strongly in $L^2(\Omega)$ such that $w_n \in V_n$ for all $n \ge 1$. Since

$$\int\limits_\Omega \big[\big(\varepsilon_n + |\nabla u_n|^{p-2}\big)|\nabla w_n|^2 + (p-2)|\nabla u_n|^{p-4}(\nabla u_n|\nabla w_n)^2\big]\,dx$$
$$- \int\limits_\Omega D_s f(x, u_n) w_n^2\,dx \le \frac{\delta}{2} \int\limits_\Omega w_n^2\,dx\,,$$

from the semicontinuity theorem of [28] we infer that

$$\int\limits_\Omega \big[|\nabla u_0|^{p-2}|\nabla w|^2 + (p-2)|\nabla u_0|^{p-4}(\nabla u_0|\nabla w)^2\big]\,dx - \int\limits_\Omega D_s f(x, u_0) w^2\,dx \le \frac{\delta}{2} \int\limits_\Omega w^2\,dx\,,$$

whence $V \subseteq \widetilde{W}_{u_0} \subseteq W_{u_0}$ and

$$\|w\|_{u_0}^2 - \int\limits_\Omega \big[1 + D_s f(x, u_0)\big] w^2\,dx \le \frac{\delta}{2} \int\limits_\Omega w^2\,dx \qquad \text{for all } w \in V\,.$$

From (2.7) we infer that

$$m = \dim(V) \le m_W^*(J, u_0)$$

and the assertion follows. □

Proof of Theorem 1.7. The assertion concerning $m(J, u_0)$ follows from [31, Theorem 3.1], while that concerning $m_W^*(J, u_0)$ is a consequence of Theorems 2.1 and 2.3. □

In the following sections we will show that condition (PC) is satisfied for positive solutions to (P) under suitable assumptions on the nonlinearities or the domain.

3. Weighted Sobolev inequalities

In order to derive uniform inequalities in weighted Sobolev spaces, we consider the following approximating problem:

$$\begin{cases} -\Delta_p u - \varepsilon \Delta u = f(x,u) + r(x) & \text{in } \Omega \\ \qquad u > 0 & \text{in } \Omega \\ \qquad u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.8}$$

where $N \geq 2$, Ω is a bounded smooth domain in $\mathbb{R}^N$, $\varepsilon \geq 0$, $f \in C^1(\overline{\Omega} \times \mathbb{R})$ and $r \in C^1(\overline{\Omega})$.

In view of further developments, it is convenient to consider also the term $r(x)$ in the equation, as it appears in some approximation results (see e.g. [19]).

Under very general growth assumptions on the nonlinearity, by standard L^∞-estimates, it follows that each solution u of problem (3.8) is actually locally bounded. Consequently by [22,39,32,27] we deduce that $u \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$.

Fix $\varepsilon \geq 0$, let u_ε be a solution to problem (3.8) and set

$$\varrho_{u_\varepsilon} := \left((\varepsilon + |\nabla u_\varepsilon|^{p-2})|\nabla u_\varepsilon|\right)^{\frac{p-2}{p-1}}.$$

We consider $W_\varepsilon \equiv W_0^{1,1}(\Omega, \varrho_\varepsilon)$ the completion of the subspace $W_0^{1,1}(\Omega) \cap L^2(\Omega)$ with respect the

$$\|v\|_{\varrho_\varepsilon} = \|\nabla v\|_{L^2(\Omega,\varrho_\varepsilon)} := \left(\int_\Omega |\nabla v|^2 \varrho_\varepsilon\right)^{\frac{1}{2}}. \tag{3.9}$$

We stress that for $\varepsilon = 0$ the space $W_0^{1,1}(\Omega, \varrho_{u_0})$ depends on the solution u_0 to problem (P), considered via the weight $|\nabla u_0|^{p-2}$. This causes some technical difficulties and obstructions when dealing for example with existence multiplicity and uniqueness of the solutions, or more generally when it is necessary to consider more than one fixed solution. Nevertheless, one might think that if a solution u_0 is in some norm near a solution v_0, then the space $W_0^{1,1}(\Omega, \varrho_{u_0})$ is in some reasonable meaning similar to $W_0^{1,1}(\Omega, \varrho_{v_0})$. This is related to the weights ϱ_{u_0} and ϱ_{v_0} and their behaviors. We take into account this task and prove actually that, if u_0 is in a C^1-neighborhood of v_0, then the spaces $W_0^{1,1}(\Omega, \varrho_{u_0})$, $W_0^{1,1}(\Omega, \varrho_{v_0})$ have a compact-uniform embedding.

Recalling that, for $w \in C^1(\overline{\Omega})$, we set

$$\mathcal{B}_R(w) := \{v \in C^1(\overline{\Omega}) \mid \|v - w\|_{C^1(\overline{\Omega})} \leq R\}.$$

We also denote by $\mathcal{K}$ the family of the compact sets contained in some finite dimensional subspace of $C_c^\infty(\Omega)$. We will establish the following regularity result:

Theorem 3.1. *Let $p > 2$ and $u_0 \in C^1(\overline{\Omega})$ be a fixed nontrivial solution of problem (P) with $f \in C^1(\overline{\Omega} \times \mathbb{R})$ with $f(x,s) > 0$ for any $x \in \Omega$, $s > 0$, $f(x,s) = 0$ for any $x \in \Omega$, $s \leq 0$.*
Then given any $K \in \mathcal{K}$, there exist $R = R(u_0, K) > 0$ and $\varepsilon_0 = \varepsilon_0(u_0, K) > 0$ such that, for

- $r \in K$
- $0 \leq \varepsilon \leq \varepsilon_0$ *and* $u_\varepsilon \in \mathcal{B}_R(u_0)$ *solution of* $-\Delta_p u - \varepsilon \Delta u = f(x,u) + r(x)$
- $1 \leq q < 2^*(p)$ *with* $2^*(p)$ *defined by* $\frac{1}{2^*(p)} = \frac{1}{2} - \frac{1}{N} + \frac{p-2}{p-1} \cdot \frac{1}{N}$,

it results that, setting

$$\varrho_{u_\varepsilon} := \left((\varepsilon + |\nabla u_\varepsilon|^{p-2})|\nabla u_\varepsilon|\right)^{\frac{p-2}{p-1}},$$

the space $W_0^{1,1}(\Omega, \varrho_{u_\varepsilon})$ is uniformly *continuously embedded in $L^q(\Omega)$, where* uniformly *means that there exists a constant $\mathcal{C}_S(K)$, which does not depend on ε and u_ε, such that*

$$\|w\|_{L^q(\Omega)} \leq \mathcal{C}_S(K) \left(\int_\Omega |\nabla w|^2 \varrho_\varepsilon \right)^{\frac{1}{2}} \tag{3.10}$$

and

$$\|\nabla w\|_{L^t(\Omega)} \leq \mathcal{C}_S(K) \left(\int_\Omega |\nabla w|^2 \varrho_\varepsilon \right)^{\frac{1}{2}} \tag{3.11}$$

for some $t > 1$.

The same result holds if we assume that Ω is a ball or an annulus and $u_0 \in C^1(\overline{\Omega})$ is a positive radial *solution of (3.8) with $f(x,u) + r(x) = u^q - \Lambda u^{p-1}$, with $\Lambda > 0$ a fixed parameter, $2 \leq p < q+1 \leq p^*$.*

The same result follows if we replace the weight ϱ_{u_ε} by the weight

$$\tilde{\varrho}_{u_\varepsilon} := \left(\varepsilon + |\nabla u_\varepsilon|^{p-2}\right).$$

From Theorem 3.1, we can deduce the uniform compact embeddings of a family of weighted Sobolev spaces. For any $K \in \mathcal{K}$ and $\varepsilon > 0$, we set

$$\mathcal{M}_K^\varepsilon = \{u \in W_0^{1,p}(\Omega) \,|\, J'_\varepsilon(u) \in K\}.$$

Precisely we derive the following result.

Corollary 3.2. *Let $p > 2$. Suppose also that $f(x,s) > 0$ for every $x \in \Omega$, $s > 0$ and $f(x,s) = 0$ for every $x \in \Omega$, $s \leq 0$. Finally, let $u_0 \in C^1(\overline{\Omega})$ be a fixed nontrivial solution of problem (P).*

Then, for every $K \in \mathcal{K}$, there exist $R = R(u_0, K) > 0$ and $\varepsilon_0 = \varepsilon_0(u_0, K) > 0$ such that the set

$$\mathcal{D}_0(K) := \bigcup_{0 \leq \varepsilon \leq \varepsilon_0} \bigcup_{u \in \mathcal{M}_K^\varepsilon \cap \mathcal{B}_{R_0}(u_0)} \left\{ v \in W_0^{1,1}(\Omega) : \int_\Omega \left[\left(\varepsilon + |\nabla u|^{p-2} \right) |\nabla v|^2 + v^2 \right] dx \leq 1 \right\}$$

has weakly compact closure in $W_0^{1,q}(\Omega)$ for some $q > 1$ and compact closure in $L^2(\Omega)$.

The same result holds if we assume that Ω is a ball or an annulus and $u_0 \in C^1(\overline{\Omega})$ is a radial *solution of (3.8) with $f(x,u) + r(x) = u^q - \Lambda u^{p-1}$, with $\Lambda > 0$ a fixed parameter, $2 \leq p < q + 1 \leq p^*$.*

Finally we establish the following result which furnishes some sufficient conditions to derive (PC).

Theorem 3.3. *Let $p > 2$ and $u_0 \in C^1(\overline{\Omega}) \setminus \{0\}$ be a solution to problem (P) with f satisfying $f(x,s) > 0$ for any $x \in \overline{\Omega}$, $s > 0$, $f(x,s) = 0$ for any $x \in \overline{\Omega}$, $s \leq 0$.*
Then $|Z| = 0$ and Condition (PC) is fulfilled at u_0.
The same result follows ($|Z| = 0$ and Condition (PC) is fulfilled at u_0) if we assume that Ω is a ball or an annulus and $u_0 \in C^1(\overline{\Omega})$ is a positive radial *solution to problem (P) with $f(x,s) = s^q - \Lambda s^{p-1}$ where $2 \leq p < q + 1 \leq p^*$ and $\Lambda > 0$ is a fixed parameter.*

Theorem 3.1, Corollary 3.2 and Theorem 3.3 will be proved at the end of Section 6.

We point out that Theorem 3.1 is strongly related to the case $p > 2$. If $1 < p \leq 2$ in fact, it follows that the weighted Sobolev inequality reduces to the classic (not weighted) one, which is standard. This is clear in view of the fact that, if $u \in C^1(\overline{\Omega})$ and $1 < p \leq 2$, we have $k < |\nabla u|^{p-2}$ (see [12,13]). We address in fact the case $p > 2$ and prove (3.10) also following some techniques from [20].

4. Global summability of the weight

Let us start fixing some notations.
For $\xi \geq 0$ we consider $G_\xi(t) = (2t - 2\xi)\chi_{[\xi\,,\,2\xi]}(t) + t\chi_{]2\xi\,,\,\infty)}(t)$ for $t \geq 0$, while $G_\xi(t) = -G_\xi(-t)$ for $t \leq 0$ ($\chi_A(\cdot)$ denoting the characteristic function of the set A).
In what follows we will assume that the ball $B_{2\varrho}(x_0)$ is contained in Ω, and we will consider a cut-off function φ_ϱ, in such a way that $\varphi_\varrho \in C_c^\infty(B_{2\varrho}(x_0))$, $\varphi_\varrho = 1$ in $B_\varrho(x_0)$ and $|\nabla \varphi_\varrho| \leq \frac{1}{\varrho}$. For shortness we will write φ instead of φ_ϱ.
Also, for $0 \leq \beta < 1$ and $\gamma < (N-2)$ ($\gamma = 0$ for $N = 2$) fixed, we set

$$T_\xi(t) = \frac{G_\xi(t)}{|t|^\beta} \qquad \text{and} \qquad H_\delta(t) = \frac{G_\delta(t)}{|t|^{\gamma+1}} \tag{4.12}$$

and we set

$$\begin{aligned}
&M(\varrho, x_0, \gamma, \Omega) \\
&= \max \{\sup_{y\in\Omega} \int\limits_{B_{2\varrho}(x_0)} \frac{1}{|x-y|^\gamma}\, dx\,;\, \sup_{y\in\Omega} \int\limits_{B_{2\varrho}(x_0)} \frac{1}{|x-y|^{\gamma+1}}\, dx\,;\, \sup_{y\in\Omega} \int\limits_{B_{2\varrho}(x_0)} \frac{1}{|x-y|^{\gamma+2}}\, dx\ \} \\
&K(\varrho, x_0) = \sup_{x\in B_{2\varrho}(x_0)} |\nabla u| \\
&V(\varrho, x_0, f) = \sup_{x\in B_{2\varrho}(x_0)} \sum_{i=1}^{N} |f_i(x, u(x))| \\
&W(\varrho, x_0, f) = \sup_{x\in B_{2\varrho}(x_0)} |f_u(x, u)| \\
&Z(\varrho, x_0, r) = \sup_{x\in B_{2\varrho}(x_0)} |r_i(x)|
\end{aligned} \tag{4.13}$$

We will generally use the short notation $M(\varrho, x_0, \gamma, \Omega) = M$, $K(\varrho, x_0) = K$, $V(\varrho, x_0, f) = V$, $W(\varrho, x_0, f) = W$, and $Z(\varrho, x_0, r) = Z$. Also the definition of $M(\varrho, x_0, \gamma, \Omega)$ makes sense for $N \geq 3$ while it is not needed in the case $N = 2$.

Let u be a solution of the problem (3.8). We begin to notice that the derivatives of the solution are solutions of the linearized equation, that is

$$\begin{aligned}
\mathcal{L}_u(u_i, \phi) = &\int\limits_\Omega |\nabla u|^{p-2}(\nabla u_i, \nabla\phi) + \varepsilon \int\limits_\Omega (\nabla u_i, \nabla\phi) \\
&+ (p-2) \int\limits_\Omega |\nabla u|^{p-4}(\nabla u, \nabla u_i)(\nabla u, \nabla\phi) \\
&- \int\limits_\Omega f_i(x, u)\phi + f_u(x, u)u_i\phi + r_i(x)\phi = 0
\end{aligned} \tag{4.14}$$

for every $\phi \in W_\varepsilon$, with $u_i(x) = \frac{\partial u}{\partial x_i}(x)$.

In what follows we will consider the second derivative of the solution u of the problem (3.8) and get summability results. We point out that for $\varepsilon = 0$ the second distributional derivative of the solution coincides with the classic one in $\Omega \setminus Z = \Omega \setminus \{\nabla u = 0\}$. In fact the solution is smooth in $\Omega \setminus Z$ since the operator in strictly-uniformly elliptic there and standard regularity results apply. Furthermore the second derivatives are intended to be zero a.e. in Z, accordingly to Stampacchia's Theorem.

We state the following local regularity results, which can be proved using arguments strictly related to the previous papers [20,35]. The proof will be provided in the appendix.

Theorem 4.1. *Let $1 < p < \infty$ and $u \in C^1_{loc}(\Omega) \cap W^{1,p}_0(\Omega)$ be a solution of*

$$-\Delta_p u - \varepsilon \Delta u = f(x,u) + r(x) \tag{4.15}$$

where $f \in C^1_{loc}(\Omega \times \mathbb{R})$ and $r(x) \in C^1_{loc}(\Omega)$. Assume that $B_{2\varrho}(x_0) \subset \Omega$. Then we have

$$\int\limits_{B_\varrho(x_0)} \frac{|\nabla u|^{p-2-\beta} |\nabla u_i|^2}{|x-y|^\gamma} + \varepsilon \int\limits_{B_\varrho(x_0)} \frac{|\nabla u_i|^2}{|\nabla u|^\beta |x-y|^\gamma} \leq \mathcal{C} \qquad \forall\, i = 1, \ldots, N$$

with

$$\begin{aligned}
&\mathcal{C}(p\,,\gamma\,,\beta\,,f\,,r\,,\|u\|_\infty\,,\|\nabla u\|_\infty\,,\varrho\,,x_0\,,\varepsilon) = \\
&= \frac{2\,M \cdot \max\{(p-1)\,,1\}^2}{1-\beta} \Big(\frac{\gamma^2 \cdot K^{p-\beta}}{1-\beta} + \frac{4K^{p-\beta}}{(1-\beta)\varrho^2} + \frac{K^{1-\beta}}{p-1} (V + W \cdot K + Z) + \\
&\frac{\varepsilon 4 K^{2-\beta}}{(1-\beta)} \Big(\frac{\gamma^2}{4} + \frac{1}{\varrho^2} \Big) \Big)
\end{aligned}$$

for $0 \leq \beta < 1$, and $\gamma < (N-2)$ (with $\gamma = 0$ if $N = 2$).

We assume here that Ω is a bounded smooth domain of $\mathbb{R}^N$, and that $u \in C^1(\overline{\Omega})$. We further assume that there exists $\delta > 0$ such that, given the neighborhood

$$\mathcal{N}(3\delta, \partial\Omega) = \{x \in \Omega \,|\, d(x, \partial\overline{\Omega}) \leq 3\delta\}$$

it occurs that $|\nabla u| > 0$ in $\overline{\mathcal{N}(3\delta, \partial\Omega)}$ and $u \in C^2(\overline{\mathcal{N}(3\delta, \partial\Omega)})$.
We set

$$\begin{aligned}
&\overline{M} = \max\,\Big\{ \sup_{y \in \Omega} \int\limits_\Omega \frac{1}{|x-y|^\gamma}\,dx\,;\, \sup_{y \in \Omega} \int\limits_\Omega \frac{1}{|x-y|^{\gamma+1}}\,dx\,;\, \sup_{y \in \Omega} \int\limits_\Omega \frac{1}{|x-y|^{\gamma+2}}\,dx\ \Big\} \\
&\overline{K} = \sup_{x \in \Omega} |\nabla u| < \infty \\
&\overline{L} = \sup_{x \in \mathcal{N}(3\delta, \partial\Omega)} \|D^2 u\| \\
&\overline{\lambda} = \inf_{x \in \mathcal{N}(3\delta, \partial\Omega)} |\nabla u|\ > 0 \\
&\overline{V} = \sup_{x \in \Omega} \sum_{i=1}^N |f_i(x, u(x))| \\
&\overline{W} = \sup_{x \in \Omega} |f_u(x,u)| \\
&\overline{Z} = \sup_{x \in \Omega} |r_i(x)|
\end{aligned} \tag{4.16}$$

where we mean $\|D^2 u\| \equiv \sum\limits_{i,j} |u_{ij}|$.

We state a global regularity results, which will be proved in the appendix.

Theorem 4.2. *Let $1 < p < \infty$ and $u \in C^1(\overline{\Omega})$ be a solution of (3.8) with $f \in C^1(\overline{\Omega} \times \mathbb{R})$, $r \in C^1(\overline{\Omega})$. Let us assume that*

$$\Omega \setminus \mathcal{N}(3\delta, \partial\Omega) \subset \bigcup_{i=1}^{S} B_\varrho(x_i)$$

and assume without loose of generality that $x_i \in \overline{\Omega \setminus \mathcal{N}(3\delta, \partial\Omega)}$ and $\varrho < \delta$. Then for every $i = 1, \ldots, N$, $0 \leq \beta < 1$ and $\gamma < N - 2$ ($\gamma = 0$ if $N = 2$), we have

$$\begin{aligned}
&\int_\Omega \frac{|\nabla u|^{p-2-\beta}|\nabla u_i|^2}{|x-y|^\gamma} + \varepsilon \int_\Omega \frac{|\nabla u_i|^2}{|\nabla u|^\beta |x-y|^\gamma} \leq \overline{\mathcal{C}} = \\
&= \frac{2\, S\, \overline{M} \cdot \max\{(p-1)\,,\, 1\}^2}{1-\beta} \Big(\frac{\gamma^2 \cdot \overline{K}^{p-\beta}}{1-\beta} + \frac{4\overline{K}^{p-\beta}}{(1-\beta)\varrho^2} + \frac{\overline{K}^{1-\beta}}{p-1}(\overline{V} + \overline{W} \cdot \overline{K} + \overline{Z}) + \\
&+ \frac{\varepsilon 4 \overline{K}^{2-\beta}}{(1-\beta)} \Big(\frac{\gamma^2}{4} + \frac{1}{\varrho^2} \Big) \Big) + \\
&+ \max\{\overline{K}^{p-2-\beta}\,;\, \overline{\lambda}^{p-2-\beta}\,;\, \overline{K}^{-\beta}\,;\, \overline{\lambda}^{-\beta}\} \cdot \overline{L}^2 \cdot \overline{M}\,.
\end{aligned} \tag{4.17}$$

We will prove a local summability result for $\frac{1}{|\nabla u|}$.

Theorem 4.3. *Let $1 < p < \infty$ and $u \in C^1_{loc}(\Omega) \cap W^{1,p}_0(\Omega)$ be a solution of*

$$-\Delta_p u \,-\, \varepsilon \Delta u = f(x, u) + r(x) \tag{4.18}$$

where $f \in C^1_{loc}(\Omega \times \mathbb{R})$ and $r(x) \in C^1_{loc}(\Omega)$.

Assume that $B_{2\varrho}(x_0) \subset \Omega$ and that:

$$0 < \mu = \mu(\varrho\,,\, x_0) = \inf_{x \in B_{2\varrho}(x_0)} f(x, u) + r(x)\,.$$

Then we have

$$\int_{B_\varrho(x_0)} \frac{1}{(\varepsilon + |\nabla u|^{p-2})|\nabla u|^t} \frac{1}{|x-y|^\gamma} \leq \mathcal{C}^*\,,$$

with $t < 1$, and

$$
\begin{aligned}
&\mathcal{C}^* = \mathcal{C}^*(\mu\,,\,p\,,\,\gamma\,,\,t\,,\,f\,,\,r\,,\,\|u\|_\infty\,,\,\|\nabla u\|_\infty\,,\,\varrho\,,\,x_0) =\\
&\frac{2}{\mu}\Big[\frac{N^2(|p-2|+t)^2}{2\mu}\Big[\frac{2\,M\cdot\max\{(p-1)\,,\,1\}^2}{1-t}\cdot\\
&\cdot\Big(\frac{\gamma^2\cdot K^{p-t}}{1-t}+\frac{4K^{p-t}}{(1-t)\varrho^2}+\frac{K^{1-t}}{p-1}(V+W\cdot K+Z)+\frac{\varepsilon 4K^{2-t}}{(1-t)}\big(\frac{\gamma^2}{4}+\frac{1}{\varrho^2}\big)\Big)\Big]+\\
&+\gamma K^{1-t}\cdot M+\frac{2}{\varrho}\cdot K^{1-t}\cdot M\Big]\,.
\end{aligned}
$$

The same result follows if we assume that $B_{2\varrho}(x_0)\subset\Omega$ and that

$$
\sup_{x\in B_{2\varrho}(x_0)} f(x,u)+r(x)\leq -\mu=-\mu(\varrho\,,\,x_0)<0
$$

Proof. We prove the theorem in the case when $f(x,u)+r(x)$ is positive in $B_{2\varrho}(x_0)$. If else $f(x,u)+r(x)$ is negative in $B_{2\varrho}(x_0)$ the proof is exactly the same, only changing sign in the first inequality (4.19) here below. Consider

$$
\phi=\frac{1}{(\varepsilon+|\nabla u|^{p-2})|\nabla u|^t+\xi}\cdot H_\delta\cdot\varphi^2
$$

as test function in (4.18), with $H_\delta=H_\delta(|x-y|)$, and get

$$
\begin{aligned}
&\mu\int_{B_{2\varrho}(x_0)}\phi\leq\int_{B_{2\varrho}(x_0)}(f(x,u)+r(x))\phi=\int_{B_{2\varrho}(x_0)}(\varepsilon+|\nabla u|^{p-2})(\nabla u\,,\,\nabla\phi)=\\
&=-\int_{B_{2\varrho}(x_0)}(\nabla u\,,\,\nabla|\nabla u|)\frac{t|\nabla u|^{t-1}}{(|\nabla u|^t+\xi)^2}\cdot H_\delta\varphi^2\\
&-(p-2)\int_{B_{2\varrho}(x_0)}\frac{|\nabla u|^{p-3}}{(\varepsilon+|\nabla u|^{p-2})}\frac{1}{(|\nabla u|^t+\xi)}(\nabla u\,,\,\nabla|\nabla u|)\cdot H_\delta\varphi^2\\
&+\int_{B_{2\varrho}(x_0)}(\nabla u\,,\,\nabla H_\delta)\frac{1}{|\nabla u|^t+\xi}\cdot\varphi^2+\\
&+2\int_{B_{2\varrho}(x_0)}(\nabla u\,,\,\nabla\varphi)\frac{1}{|\nabla u|^t+\xi}\cdot H_\delta\varphi,
\end{aligned}
\tag{4.19}
$$

that gives

$$
\begin{aligned}
&\mu\int_{B_{2\varrho}(x_0)}\phi\leq(|p-2|+t)\int_{B_{2\varrho}(x_0)}\frac{1}{(|\nabla u|^t+\xi)}\sum_{i=1}^N|\nabla u_i|\cdot H_\delta\varphi^2+\\
&+\int_{B_{2\varrho}(x_0)}|\nabla u|\frac{|\nabla H_\delta|}{|\nabla u|^t+\xi}\cdot\varphi^2+2\int_{B_{2\varrho}(x_0)}|\nabla u|\nabla\varphi|\frac{1}{|\nabla u|^t+\xi}\cdot H_\delta\varphi\,.
\end{aligned}
$$

We now let as above δ goes to zero, and get

$$\mu \int\limits_{B_{2\varrho}(x_0)} \frac{1}{(|\nabla u|^t + \xi)} \frac{1}{|x-y|^\gamma} \cdot \varphi^2 \le \tilde{A} + \tilde{B} + \tilde{C} \tag{4.20}$$

where

$$\tilde{A} = (|p-2| + t) \int\limits_{B_{2\varrho}(x_0)} \frac{1}{(|\nabla u|^t + \xi)} \sum_{i=1}^N |\nabla u_i| \frac{1}{|x-y|^\gamma} \cdot \varphi^2$$

$$\tilde{B} = \gamma \int\limits_{B_{2\varrho}(x_0)} \frac{|\nabla u|}{|\nabla u|^t + \xi} \frac{1}{|x-y|^{\gamma+1}} \cdot \varphi^2$$

$$\tilde{C} = 2 \int\limits_{B_{2\varrho}(x_0)} \frac{|\nabla u|^{p-1}}{|\nabla u|^t + \xi} |\nabla \varphi| \frac{1}{|x-y|^\gamma} \cdot \varphi\,.$$

We now estimate the terms $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$. We have

$$\begin{aligned}
\tilde{A} &= (|p-2| + t) \int\limits_{B_{2\varrho}(x_0)} \frac{1}{(|\nabla u|^t + \xi)} \sum_{i=1}^N |\nabla u_i| \frac{1}{|x-y|^\gamma} \cdot \varphi^2 \le \\
&\le \int\limits_{B_{2\varrho}(x_0)} \frac{1}{(|\nabla u|^t + \xi)^{\frac{1}{2}}} \frac{1}{|x-y|^{\frac{\gamma}{2}}} \cdot \varphi \cdot \frac{(|p-2|+t)}{(|\nabla u|^t + \xi)^{\frac{1}{2}}} \frac{\sum_{i=1}^N |\nabla u_i|}{|x-y|^{\frac{\gamma}{2}}} \cdot \varphi \le \\
&\le \vartheta \int\limits_{B_{2\varrho}(x_0)} \frac{1}{(\varepsilon + |\nabla u|^{p-2})(|\nabla u|^t + \xi)} \frac{1}{|x-y|^\gamma} \cdot \varphi^2 + \\
&\frac{(|p-2|+t)^2}{4\vartheta} \int\limits_{B_{2\varrho}(x_0)} \frac{(\varepsilon + |\nabla u|^{p-2})}{|\nabla u|^t |x-y|^\gamma} \Big(\sum_{i=1}^N |\nabla u_i| \Big)^2 \cdot \varphi^2 \le \\
&\le \vartheta \int\limits_{B_{2\varrho}(x_0)} \frac{1}{(\varepsilon + |\nabla u|^{p-2})(|\nabla u|^t + \xi)} \frac{1}{|x-y|^\gamma} \cdot \varphi^2 + \\
&\frac{(|p-2|+t)^2}{4\vartheta} \int\limits_{B_{2\varrho}(x_0)} \frac{(\varepsilon + |\nabla u|^{(p-2)})}{|\nabla u|^t\ |x-y|^\gamma} N^2 \Big(\sum_{i=1}^N |\nabla u_i|^2 \Big) \cdot \varphi^2 \le \\
&\le \vartheta \int\limits_{B_{2\varrho}(x_0)} \frac{1}{(\varepsilon + |\nabla u|^{p-2})(|\nabla u|^t + \xi)} \frac{1}{|x-y|^\gamma} \cdot \varphi^2 + \frac{N^2(|p-2|+t)^2}{4\vartheta} \mathcal{C}
\end{aligned}$$

where $\mathcal{C}$ is the constant given by Theorem 4.1, recalling that $t < 1$. We also have

$$\tilde{B} = \gamma \int_{B_{2\varrho}(x_0)} \frac{|\nabla u|}{|\nabla u|^t + \xi} \frac{1}{|x-y|^{\gamma+1}} \cdot \varphi^2 \leq$$
$$\leq \gamma \cdot K^{1-t} \int_{B_{2\varrho}(x_0)} \frac{1}{|x-y|^{\gamma+1}} \leq \gamma \cdot K^{1-t} \cdot M$$

and

$$\tilde{C} = 2 \int_{B_{2\varrho}(x_0)} \frac{|\nabla u|}{|\nabla u|^t + \xi} |\nabla \varphi| \frac{1}{|x-y|^{\gamma}} \cdot \varphi \leq$$
$$\leq \frac{2}{\varrho} \cdot K^{1-t} \int_{B_{2\varrho}(x_0)} \frac{1}{|x-y|^{\gamma}} \cdot \varphi \leq \frac{2}{\varrho} \cdot K^{1-t} \cdot M\,.$$

We now take $\vartheta = \frac{\mu}{2}$ and by (4.20), exploiting the above estimates, it follows

$$\begin{aligned}\int_{B_{2\varrho}(x_0)} \frac{1}{(\varepsilon + |\nabla u|^{p-2})(|\nabla u|^t + \xi)} \frac{1}{|x-y|^{\gamma}} \cdot \varphi^2 \leq \frac{2}{\mu} \Big(\frac{N^2(|p-2|+t)^2}{2\mu} \mathcal{C} \\ + \gamma K^{1-t} \cdot M + \frac{2}{\varrho} \cdot K^{1-t} \cdot M \Big)\end{aligned} \tag{4.21}$$

where $\mathcal{C}$ is the constant given by Theorem 4.1.
Letting ξ goes to zero we get the thesis. □

We prove here a global summability result for $\frac{1}{|\nabla u|}$, basing on Theorem 4.3.

Theorem 4.4. *Let $1 < p < \infty$ and $u \in C^1(\overline{\Omega})$ be a solution of*

$$-\Delta_p u \, - \, \varepsilon \Delta u = f(x,u) + r(x) \tag{4.22}$$

with $f \in C^1(\overline{\Omega} \times \mathbb{R})$, $r \in C^1(\overline{\Omega})$. Let $\overline{M}$, $\overline{K}$, $\overline{V}$, $\overline{W}$ and $\overline{W}$ as in (4.16). Assume that there exist $\Omega^+ \subset\subset \Omega$ and $\Omega^- \subset\subset \Omega$ and $\mathcal{N} \subset \Omega$ such that:

$$\Omega \subseteq \overline{\mathcal{N}} \cup \overline{\Omega^+} \cup \overline{\Omega^-}$$

and that there exists $\bar{\varrho}$ such that $f(x,u) + r(x)$ is (strictly) positive in $\overline{\Omega^+_{\bar{\varrho}}}$ where

$$\Omega^+_{\bar{\varrho}} = \{x \in \Omega \,|\, d(x, \Omega^+) < \bar{\varrho}\},$$

and that $f(x,u) + r(x)$ is (strictly) negative in $\overline{\Omega^-_{\bar{\varrho}}}$ where

$$\Omega^-_{\bar{\varrho}} = \{x \in \Omega \,|\, d(x, \Omega^-) < \bar{\varrho}\}.$$

Assume also with no loose of generality that $\Omega^+_{\bar{\varrho}} \subset\subset \Omega$ *and* $\Omega^-_{\bar{\varrho}} \subset\subset \Omega$. *Consider the finite covering* $\Omega^+ \subseteq \bigcup_{i=1}^{S_1} B_\varrho(x_i)$ *and* $\Omega^- \subseteq \bigcup_{i=1}^{S_2} B_\varrho(y_i)$ *with* $x_i \in \Omega^+$ *and* $y_i \in \Omega^-$. *With no loose of generality, assume that* $2\varrho < \bar{\varrho}$. *Also assume that* $|\nabla u| > 0$ *in* $\overline{\mathcal{N}}$ *and let*

$$\begin{aligned} &\tilde{L} = \sup_{x\in\mathcal{N}} \sum_{i=1}^{N} |\nabla u_i|^2 < \infty \\ &0 < \overline{\lambda} = \inf_{x\in\mathcal{N}} |\nabla u| \\ &0 < \overline{\mu} \le \inf_{x\in\Omega^+_{\bar{\varrho}}} (f(x,u) + r(x)) \\ &0 > -\overline{\mu} \ge \sup_{x\in\Omega^-_{\bar{\varrho}}} (f(x,u) + r(x)), \end{aligned} \tag{4.23}$$

then we get:

$$\int_\Omega \frac{1}{(\varepsilon + |\nabla u|^{p-2})|\nabla u|^t} \frac{1}{|x-y|^\gamma} \le \overline{\mathcal{C}}^* \tag{4.24}$$

with $t < 1$, *and*

$$\begin{aligned} \overline{\mathcal{C}}^* &= \overline{\mathcal{C}}^*(\mu\,,\, p\,,\, \gamma\,,\, t\,,\, f\,,\, r\,,\, \|u\|_\infty\,,\, \|\nabla u\|_\infty\,,\, \varrho\,,\, x_0\,,\, \varepsilon) = \frac{1}{\overline{\lambda}^{t+p-2}} \cdot \overline{M} + \\ &+ \frac{2\,(S_1+S_2)}{\overline{\mu}} \Big[\frac{N^2(|p-2|+t)^2}{2\overline{\mu}} \Big[\frac{2\,\overline{M} \cdot \max\{(p-1)\,,\,1\}^2}{1-t} \cdot \\ &\cdot \left(\frac{\gamma^2 \cdot \overline{K}^{2-t}}{1-t} + \frac{4\overline{K}^{p-t}}{(1-t)\varrho^2} + \frac{\overline{K}^{1-t}}{p-1}(\overline{V} + \overline{W}\cdot\overline{K} + \overline{Z}) + \frac{\varepsilon 4\overline{K}^{2-t}}{(1-t)}(\frac{\gamma^2}{4} + \frac{1}{\varrho^2}) \right) \Big] + \\ &+ \gamma \overline{K}^{1-t} \cdot \overline{M} + \frac{2}{\varrho} \cdot \overline{K}^{1-t} \cdot \overline{M} \Big]. \end{aligned} \tag{4.25}$$

Proof. By Theorem 4.3, since $\overline{\mu} \le \mu$ (μ as in Theorem 4.3), we get that:

$$\begin{aligned} &\int_{B_\varrho(x_i)} \frac{1}{(\varepsilon + |\nabla u|^{p-2})|\nabla u|^t} \frac{1}{|x-y|^\gamma} \le \overline{\mathcal{C}} = \\ &\le \frac{2}{\overline{\mu}} \Big[\frac{N^2(|p-2|+t)^2}{2\overline{\mu}} \Big[\frac{2\,\overline{M} \cdot \max\{(p-1)\,,\,1\}^2}{1-t} \cdot \\ &\cdot \left(\frac{\gamma^2 \cdot \overline{K}^{p-t}}{1-t} + \frac{4\overline{K}^{p-t}}{(1-t)\varrho^2} + \frac{\overline{K}^{1-t}}{p-1}(\overline{V} + \overline{W}\cdot\overline{K} + \overline{Z}) + \frac{\varepsilon 4\overline{K}^{2-t}}{(1-t)}(\frac{\gamma^2}{4} + \frac{1}{\varrho^2}) \right) \Big] + \\ &+ \gamma \overline{K}^{1-t} \cdot \overline{M} + \frac{2}{\varrho} \cdot \overline{K}^{1-t} \cdot \overline{M} \Big]\,. \end{aligned}$$

In the same way $\int_{B_\varrho(y_i)} \frac{1}{(\varepsilon+|\nabla u|^{p-2})|\nabla u|^t} \frac{1}{|x-y|^\gamma} \leq \overline{\mathcal{C}}$. Also

$$\int\limits_{\mathcal{N}} \frac{1}{(\varepsilon+|\nabla u|^{p-2})|\nabla u|^t} \frac{1}{|x-y|^\gamma} \leq \frac{1}{\overline{\lambda}^{t+p-2}} \cdot \overline{M}\,.$$

Consequently the thesis follows now via a standard covering argument since we have:

$$\begin{aligned}
&\int\limits_{\Omega} \frac{1}{(\varepsilon+|\nabla u|^{p-2})|\nabla u|^t} \frac{1}{|x-y|^\gamma} \leq \\
&\int\limits_{\mathcal{N}} \frac{1}{(\varepsilon+|\nabla u|^{p-2})|\nabla u|^t} \frac{1}{|x-y|^\gamma} + \int\limits_{\Omega^+} \frac{1}{(\varepsilon+|\nabla u|^{p-2})|\nabla u|^t} \frac{1}{|x-y|^\gamma} + \int\limits_{\Omega^-} \frac{1}{|\nabla u|^t} \frac{1}{|x-y|^\gamma} \leq \\
&\leq \frac{1}{\overline{\lambda}^{t+p-2}} \cdot \overline{M} + \sum_{i=1}^{S_1} \int\limits_{B_\varrho(x_i)} \frac{1}{(\varepsilon+|\nabla u|^{p-2})|\nabla u|^t} \frac{1}{|x-y|^\gamma} \\
&+ \sum_{i=1}^{S_2} \int\limits_{B_\varrho(y_i)} \frac{1}{(\varepsilon+|\nabla u|^{p-2})|\nabla u|^t} \frac{1}{|x-y|^\gamma}\,. \quad \square
\end{aligned}$$

5. Summability of the weight in a C^1-neighborhood of the solution

In this section we furnish some results concerning the uniform summability of the weight $\frac{1}{|\nabla v|}$ where v belongs to a C^1 neighborhood of a fixed solution u to (3.8) and it solves a perturbing quasilinear problem. We will show that the constants involved are controlled in terms of the C^1 distance of v from u.

Let us start with the following:

Lemma 5.1. *Let $1 < p < \infty$ and $u \in C^1(\overline{\Omega})$ be solution of (3.8) with $f(x, u(x)) + r(x) > 0$ in Ω. Then the assumption in Theorem 4.4 is fulfilled and with the notation above we have*

$$\int\limits_{\Omega} \frac{1}{(\varepsilon+|\nabla u|^{p-2})|\nabla u|^t} \frac{1}{|x-y|^\gamma} \leq \overline{\mathcal{C}}^* \tag{5.26}$$

with $t < 1$, $\gamma = 0$ if $N = 2$ while $0 \leq \gamma < N-2$ if $N \geq 3$. Moreover $\overline{\mathcal{C}}^*$ is the constant given by Theorem 4.4.

Proof. It follows in this case by Hopf boundary Lemma [43] that there exists a neighborhood, say $\mathcal{N}(3\delta, \partial\Omega)$, with $|\nabla u| > 0$ in $\overline{\mathcal{N}(3\delta, \partial\Omega)}$. Consequently the solution is C^2 in $\mathcal{N}(3\delta, \partial\Omega)$ by standard regularity theory, and the thesis follows by Theorem 4.4 with $\Omega \subseteq \overline{\mathcal{N}} \cup \overline{\Omega}^+ \cup \overline{\Omega^-}$ for $\mathcal{N} = \mathcal{N}(3\delta, \partial\Omega)$, $\Omega^+ = \Omega \setminus \overline{\mathcal{N}(3\delta, \partial\Omega)}$ and $\Omega^- = \emptyset$. $\square$

Theorem 5.2. *Let $1 < p < +\infty$ and $u_0 \in C^1(\overline{\Omega})$ be a nontrivial solution of problem (P), with $f \in C^1(\overline{\Omega} \times \mathbb{R})$ with $f(x,s) > 0$ for any $x \in \Omega$, $s > 0$, $f(x,s) = 0$ for any $x \in \Omega$, $s \leq 0$.*
For any $K \in \mathcal{K}$ and for any for every $\nu > 0$, there exists $\sigma = \sigma(\nu, K) > 0$ such that for any $r \in K$, for any $v \in C^1(\overline{\Omega})$ solution to $-\Delta_p v - \varepsilon\Delta v = f(x,v) + r(x)$ with

$$\|v - u_0\|_{C^1(\overline{\Omega})} \leq \sigma \qquad \text{and} \qquad 0 \leq \varepsilon \leq \sigma,$$

it results that the assumption in Theorem 4.4 is fulfilled for v and with the notation above we have

$$\int_\Omega \frac{1}{(\varepsilon + |\nabla v|^{p-2})|\nabla v|^t} \frac{1}{|x-y|^\gamma} \leq \overline{\mathcal{C}}^* + \nu$$

with $t < 1$ and $\gamma = 0$ if $N = 2$ while $0 \leq \gamma < N - 2$ if $N \geq 3$. Moreover $\overline{\mathcal{C}}^* = \overline{\mathcal{C}}^*(u)$ is the constant given by Theorem 4.4.

Proof. Let K be a compact subset of a finite dimensional subspace V of $C_c^\infty(\Omega)$ and let $\nu > 0$. Since u_0 is a solution to problem (P), by Hopf boundary Lemma [43] as above, it follows that $|\nabla u_0| > 0$ in $\overline{\mathcal{N}(3\delta, \partial\Omega)}$ and also $u \in C^2(\mathcal{N}(3\delta, \partial\Omega))$.
Then there exists $\sigma_0 > 0$ such that for any $r \in K$ and for any $v \in C^1(\overline{\Omega})$ solution of $-\Delta_p v - \varepsilon\Delta v = f(x,v) + r(x)$ with $\|u_0 - v\|_{C^1(\overline{\Omega})} \leq \sigma_0$, it results $|\nabla v| > 0$ in $\overline{\mathcal{N}(3\delta, \partial\Omega)}$ too and consequently $v \in C^2(\mathcal{N}(3\delta, \partial\Omega))$ by standard regularity theory.

Now fix $\bar{\varrho} > 0$ small such that $f(x, u_0(x)) \geq c > 0$ on the closed neighborhood of radius $\bar{\varrho}$ of $\overline{\Omega \setminus \mathcal{N}(3\delta, \partial\Omega)}$.

Then we can eventually reduce σ_0 so that for any $0 < \sigma \leq \sigma_0$, for any $r \in K$ and for any $v \in C^1(\overline{\Omega})$ solution of $-\Delta_p v - \varepsilon\Delta v = f(x,v) + r(x)$ with $\|u_0 - v\|_{C^1(\overline{\Omega})} \leq \sigma$, we have $f(x,v) + r(x) \geq c/2$ in the closed neighborhood of radius $\bar{\varrho}$ of $\overline{\Omega \setminus \mathcal{N}(3\delta, \partial\Omega)}$.

Indeed arguing by contradiction there exist $\{r_n\} \subset K$, $\{v_n\} \subset C^1(\overline{\Omega})$, $\{x_n\} \subset \Omega$ such that $-\Delta_p v_n - \varepsilon_n \Delta v_n = f(x, v_n) + r_n(x)$, v_n tends to u_0 in $C^1(\overline{\Omega})$ and x_n belongs to the closed neighborhood of radius $\bar{\varrho}$ of $\overline{\Omega \setminus \mathcal{N}(3\delta, \partial\Omega)}$, $\varepsilon_n \to 0$, such that

$$f(x_n, v_n(x_n)) + r_n(x_n) < c/2.$$

Then there exists $\bar{x}$ belonging to the closed neighborhood of radius $\bar{\varrho}$ of $\overline{\Omega \setminus \mathcal{N}(3\delta, \partial\Omega)}$, such, up to subsequences, $x_n \to \bar{x}$ as $n \to +\infty$.

Since there exists $M > 0$ such that $\|r_n\|_{C^1(\overline{\Omega})} \leq M\|v_n - u_0\|_{C^1(\overline{\Omega})}$ for any $n \in \mathbb{N}$, it follows that $f(x_n, v_n(x_n)) + r_n(x_n) \to f(\bar{x}, u_0(\bar{x}))$ as $n \to \infty$, and thus $c \leq f(\bar{x}, u_0(\bar{x})) \leq c/2$ which is a contradiction.

We can consequently exploit Theorem 4.4 as in Lemma 5.1 and get

$$\int_\Omega \frac{1}{(\varepsilon + |\nabla v|^{p-2})|\nabla v|^t} \frac{1}{|x-y|^\gamma} \leq \overline{\mathcal{C}}^*(v)$$

where $\overline{\mathcal{C}}^*(v)$ is the constant given in Theorem 4.4 via Lemma 5.1, where the dependence on u of the quantities involved, is replaced by the dependence on v.
By the expression for $\overline{\mathcal{C}}^*(\cdot)$, it is now clear that for $\nu > 0$, we can take σ sufficiently small such that

$$\overline{\mathcal{C}}^*(v) \leq \overline{\mathcal{C}}^*(u) + \nu$$

and the thesis. $\square$

We consider here below an application to a problem involving a sign changing nonlinearity.

Theorem 5.3. *Let $1 < p < \infty$ and assume that Ω is a ball or an annulus and let $u_0 \in C^1(\overline{\Omega})$ be a positive* radial *solution to problem (P) with $f(x, u_0) + r(x) = u_0^q - \Lambda u_0^{p-1}$ where $\Lambda > 0$ is a fixed parameter.*

For any $K \in \mathcal{K}$ and for every $\nu > 0$, there exists $\sigma = \sigma(\nu, K) > 0$ such that for any $r \in K$ and for any $v \in C^1(\overline{\Omega})$ solution to $-\Delta_p v - \varepsilon\Delta v = v^q - \Lambda v^{p-1} + r(x)$ with

$$\|v - u_0\|_{C^1(\overline{\Omega})} \leq \sigma \qquad \text{and} \qquad 0 \leq \varepsilon \leq \sigma,$$

it results that the assumption in Theorem 4.4 is fulfilled for v and with the notation above we have

$$\int_\Omega \frac{1}{(\varepsilon + |\nabla v|^{p-2})|\nabla v|^t} \frac{1}{|x - y|^\gamma} \leq \overline{\mathcal{C}}^* + \nu$$

with $t < 1$ and $\gamma = 0$ if $N = 2$ while $0 \leq \gamma < N - 2$ if $N \geq 3$. Moreover $\overline{\mathcal{C}}^* = \overline{\mathcal{C}}^*(u)$ is the constant given by Theorem 4.4.

Proof. Let us start considering the case $\Omega = B(0, R)$. It is standard to see that, since u is positive and radial, we have that $u'(r) < 0$ for $r \in (0, R]$ and $u'(0) = 0$, so that the only critical point of u (and the maximum point) is at the origin for $r = 0$. Also necessary $u(0)^q - \Lambda u(0)^{p-1} > 0$, in fact if this is not the case, u could not be positive. It follows in this case that we can take $\mathcal{N}$, with $|\nabla u| > 0$ in $\overline{\mathcal{N}}$ in such a way that, setting $\Omega^+ = \Omega \backslash \overline{\mathcal{N}}$, we may and do assume that $u^q - \Lambda u^{p-1}$ is strictly positive in a neighborhood of radius $\bar{\varrho}$ of $\overline{\Omega^+}$. It is sufficient to do this that $\overline{\Omega^+}$ is a small neighborhood of the origin.
The solution is C^2 in $\mathcal{N}$ by standard regularity theory, and we can now set $\Omega \subseteq \overline{\mathcal{N}} \cup \overline{\Omega}^+ \cup \overline{\Omega^-}$ for $\mathcal{N}$, Ω^+ as above and $\Omega^- = \emptyset$.
For σ sufficiently small, we can now exploit Theorem 4.4 exactly as in Theorem 5.2 and get the thesis.
Consider now the case $\Omega = B(0, R_2) \setminus \overline{B(0, R_1)}$. Since u is positive and radial, we have that $u(r)$ is a solution of

$$(r^{N-1}|u'|^{p-2}u')' = -r^{N-1}(u^q - \Lambda u^{p-1}),$$

for $r \in (R_1\,,\,R_2)$, with $u(R_1) = u(R_2) = 0$, $u'(R_1) > 0$ and $u'(R_2) < 0$ by Hopf boundary Lemma. It is easy to see in this case that there exists only one value, say $\bar{r}$, such that $u'(\bar{r}) = 0$. That is $Z = \{\nabla u = 0\} = \{u = u(\bar{r})\}$. Also it follows that necessary $u(\bar{r}) > 0$. It follows now that we can take $\mathcal{N}$, with $|\nabla u| > 0$ in $\overline{\mathcal{N}}$ in such a way that, setting $\Omega^+ = \Omega \setminus \overline{\mathcal{N}}$ as above, $u^q - \Lambda u^{p-1}$ is strictly positive in a neighborhood of radius $\bar{\varrho}$ of $\overline{\Omega^+}$ $\overline{\Omega^+}$. To do this only let Ω^+ be a small neighborhood of the level set $\{u = u(\bar{r})\}$. Setting now $\Omega \subseteq \overline{\mathcal{N}} \cup \overline{\Omega}^+ \cup \overline{\Omega^-}$ for $\mathcal{N}$, Ω^+ as above and $\Omega^- = \emptyset$, the thesis follows by Theorem 4.4 exploited exactly as in Theorem 5.2. $\square$

6. Uniform Sobolev inequality

Let us recall here some known facts from potential estimates. Given a function $f \in L^a(\Omega)$, with $a \geq 1$, considering some $0 < \alpha < N$, we set

$$\mathcal{U}_\alpha[f](x) = \int\limits_\Omega \frac{f(y)}{|x-y|^{N-\alpha}}\, dy\,.$$

Defining b by

$$\frac{1}{b} = \frac{1}{a} - \frac{\alpha}{N}$$

and assuming $b \geq 1$, by Marcinkiewicz Interpolation Theorem (see e.g. [25] section 7.8 pg. 159), it follows that the $\mathcal{U}_\alpha$ defines a *continuous* map

$$\mathcal{U}_\alpha[\cdot]\ :\ L^a(\Omega)\ \to\ L^b(\Omega).$$

That is, there exists a constant $c_M = c_M(N\,,\,\alpha\,,\,a)$ such that

$$\|\mathcal{U}_\alpha[f]\|_{L^r(\Omega)} \leq c_M|\Omega|^{\frac{1}{r}-\frac{1}{b}}\|f\|_{L^a(\Omega)} \tag{6.27}$$

for every $1 \leq r \leq b$.

Theorem 6.1. *Let $p > 2$ and $u \in C^1(\overline{\Omega})$ be a solution of (3.8). Let ϱ such that*

$$\int\limits_\Omega \frac{1}{\varrho^t \cdot |x-y|^\gamma} \leq \tilde{C}^*$$

with $t = \frac{p-1}{p-2}r$, $\frac{p-2}{p-1} < r < 1$, $\gamma < N-2$ ($\gamma = 0$ if $N = 2$).
Assume, in the case $N \geq 3$, with no loose of generality that:

$$\gamma > N\,-\,2t$$

which also implies $Nt - 2N + 2t + \gamma > 0$.[1] *Then the space* $W_0^{1,1}(\Omega, \varrho)$ *is continuously embedded in* $L^{\hat{2}^*(t)}$ *where*

$$\frac{1}{\hat{2}^*(t)} = \frac{1}{2} - \frac{1}{N} + \frac{1}{t}\big(\frac{1}{2} - \frac{\gamma}{2\,N}\big)\,.$$

More precisely there exists a constant $\mathcal{C}_S$ *such that*

$$\|w\|_{L^q(\Omega)} \leq \mathcal{C}_S \|\nabla w\|_{L^2(\Omega,\varrho)} = \mathcal{C}_S \left(\int\limits_\Omega |\nabla w|^2 \varrho \right)^{\frac{1}{2}} \tag{6.28}$$

for any $1 \leq q \leq 2^*(t)$. *Explicitly* $\mathcal{C}_S = C_N \cdot (\tilde{\mathcal{C}}^*)^{\frac{1}{2t}} \cdot (c_M)^{\frac{1}{(2t)'}}$, *with* $\tilde{\mathcal{C}}^*$ *as in the statement of the theorem,* C_N *given by standard potential estimates (see* (6.29) *and [25]) and* c_M *as in* (6.27).

Note that the largest value of $2^*(t)$ *is obtained at the limiting case* $t \approx \frac{p-1}{p-2}$, *and* $\gamma \approx (N-2)$, $\gamma = 0$ *for* $N = 2$. *We have therefore that* $H_0 = H_0^{1,2}(\Omega, \varrho)$ *is continuously embedded in* $L^{2^*(t)}$ *where*

$$\frac{1}{\hat{2}^*(t)} = \frac{1}{2} - \frac{1}{N} + \frac{p-2}{p-1} \cdot \frac{1}{N}\,,$$

for $1 \leq q < 2^*(t)$.

Proof. Let us first note that by density arguments we may and do assume that $w \in W_0^{1,1}(\Omega)$ and consequently by potential estimates

$$|w(x)| \leq C_N \int\limits_\Omega \frac{|\nabla w|(y)}{|x-y|^{N-1}}\, dy \tag{6.29}$$

that gives

$$\begin{aligned} |w(x)| &\leq C_N \int\limits_\Omega \frac{1}{\varrho^{\frac{1}{2}} |x-y|^{\frac{\gamma}{2t}}} \cdot \frac{|\nabla w|(y) \cdot \varrho^{\frac{1}{2}}}{|x-y|^{N-1-\frac{\gamma}{2t}}}\, dy \leq \\ &\leq C_N \cdot \left(\int\limits_\Omega \frac{1}{\varrho^t \cdot |x-y|^\gamma} \right)^{\frac{1}{2t}} \cdot \left(\int\limits_\Omega \frac{(|\nabla w| \varrho^{\frac{1}{2}})^{(2t)'}}{|x-y|^{(N-1-\frac{\gamma}{2t})(2t)'}} \right)^{\frac{1}{(2t)'}} \leq \\ &\leq C_N \cdot (\tilde{\mathcal{C}}^*)^{\frac{1}{2t}} \cdot \Big[\mathcal{U}_\alpha [(|\nabla w| \varrho^{\frac{1}{2}})^{(2t)'}] \Big]^{\frac{1}{(2t)'}} \end{aligned} \tag{6.30}$$

[1] Note that the condition $\gamma > N - 2t$ holds true for $r \approx 1$ and $\gamma \approx N - 2$ that we may assume with no loose of generality.

by setting $N-\alpha=(N-1-\frac{\gamma}{2t})(2t)'$ that is $\alpha=N-(N-1-\frac{\gamma}{2t})(2t)'$. We note now that $(|\nabla w|\varrho^{\frac{1}{2}})^{(2t)'}\in L^{\frac{2}{(2t)'}}(\Omega)$ and we set $a=\frac{2}{(2t)'}$ in order to apply Marcinkiewicz interpolation theorem. We therefore set

$$\frac{1}{b}=\frac{(2t)'}{2}-\frac{\alpha}{N}.$$

It follows now by the assumption $\gamma>N-2t$ that $\alpha>0$, and by the assumption $Nt-2N+2t+\gamma>0$ that $b>1$, and

$$\begin{aligned}\|w(x)\|_{L^{b(2t)'}(\Omega)}&\le C_N\cdot(\tilde{\mathcal{C}}^*)^{\frac{1}{2t}}\left[\left(\int_\Omega\left[\mathcal{U}_\alpha[(|\nabla w|\varrho^{\frac{1}{2}})^{(2t)'}]\right]^b\right)^{\frac{1}{b}}\right]^{\frac{1}{(2t)'}}\le\\&\le C_N\cdot(\tilde{\mathcal{C}}^*)^{\frac{1}{2t}}\cdot c_M^{\frac{1}{(2t)'}}\cdot\left(\int_\Omega|\nabla w|^2\varrho\right)^{\frac{1}{2}}\end{aligned}\tag{6.31}$$

that gives the thesis since $b(2t)'=\hat{2}^*(t)$. □

Proof of Theorem 3.1. Let $p>2$ and $u_0\in C^1(\overline{\Omega})$ be a solution of (P). Given $K\in\mathcal{K}$ and $\nu>0$, we can fix $\sigma(\nu,K)>0$ so that Theorem 5.2 holds. Let $u_\varepsilon\in C^1(\overline{\Omega})$ be a solution to

$$-\Delta_p u-\varepsilon\Delta u=f(x,u)+r(x)\tag{6.32}$$

with $r\in K$, $0\le\varepsilon\le\sigma$ and $\|v-u_0\|_{C^1(\overline{\Omega})}\le\sigma$. Taking into account Theorem 5.2, we derive that

$$\int_\Omega\frac{1}{(\varepsilon+|\nabla u_\varepsilon|^{p-2})|\nabla u_\varepsilon|^t}\frac{1}{|x-y|^\gamma}\le\overline{\mathcal{C}}^*+\nu$$

with $t<1$ and $\gamma=0$ if $N=2$ while $0\le\gamma<N-2$ if $N\ge3$. Moreover $\overline{\mathcal{C}}^*=\overline{\mathcal{C}}^*(u_0)$ is the constant given by Theorem 4.4 and ν can be fixed (arbitrary small).
Since $|\nabla u_\varepsilon|^{p-2}$ is bounded, this can be rewritten as follows

$$\int_\Omega\frac{1}{\varrho_\varepsilon^\tau}\frac{1}{|x-y|^\gamma}\le\overline{\mathcal{C}}^*+\nu$$

with $\tau=\frac{(p-1)s}{p-2}$, for some $\frac{p-2}{p-1}\le s<1$. This allows to exploit Theorem 6.1 and get (3.10).
Finally, taking into account Theorem 5.2, for

$$1 < r < \frac{2p-2}{2p-3},$$

we have that for any $w \in W_0^{1,1}(\Omega, \varrho_\epsilon)$

$$\left(\int_\Omega |\nabla w|^r\right)^{\frac{1}{r}} \leq \left(\int_\Omega (\varepsilon + |\nabla u_\varepsilon|^{p-2})|\nabla w|^2\right)^{\frac{1}{2}} \cdot \left(\int_\Omega \frac{1}{(\varepsilon + |\nabla u_\varepsilon|^{p-2})^{\frac{r}{2-r}}}\right)^{\frac{2-r}{2r}}$$

and (3.11) follows.

If else we assume that Ω is a ball or an annulus and that $u_0 \in C^1(\overline{\Omega})$ is a positive *radial* solution of (3.8) with $f(x,u) + r(x) = u^q - \Lambda u^{p-1}$, the same result follows exploiting in this case Theorem 5.3 instead of Theorem 5.2, and again (3.10) follows.
The embedding is now compact as showed in Lemma 5.1 of [4] (see also [5]).

If we replace the weight ϱ_{u_ε} by

$$\tilde{\varrho}_{u_\varepsilon} := \left(\varepsilon + |\nabla u_\varepsilon|^{p-2}\right),$$

it is enough to observe that $\varrho_{u_\varepsilon} \leq \tilde{\varrho}_{u_\varepsilon}$, so that

$$\frac{1}{\tilde{\varrho}_{u_\varepsilon}} \leq \frac{1}{\varrho_{u_\varepsilon}},$$

and the above arguments can be therefore repeated. □

Proof of Corollary 3.2. Let $u_0 \in C^1(\overline{\Omega}) \setminus \{0\}$ be solution to problem (P).
Given $K \in \mathcal{K}$, let us fix $R_0 > 0$ and $\varepsilon_0 > 0$ such Theorem 3.1 applies. It follows immediately that

$$\mathcal{D}_0(K) := \bigcup_{0 \leq \varepsilon \leq \varepsilon_0} \bigcup_{u \in \mathcal{M}_K^\varepsilon \cap \mathcal{B}_{R_0}(u_0)} \left\{ v \in W_0^{1,1}(\Omega) : \int_\Omega \left[\left(\varepsilon + |\nabla u|^{p-2}\right)|\nabla v|^2 + v^2 \right] \, dx \leq 1 \right\},$$

is compactly embedded in $L^q(\Omega)$, for any $1 \leq q < 2^*(p)$.

Fix $\delta > 0$ so that $q + \delta < 2^*(p)$, and set

$$\frac{1}{q} = \alpha + \frac{(1-\alpha)}{q+\delta}, \qquad \text{for } 0 < \alpha \leq 1.$$

Now, let $\omega \subset\subset \Omega$ and consider h such that

$$|h| < \operatorname{dist}(\omega, \Omega^c).$$

Given $\phi \in \mathcal{D}_0(K)$, by interpolation we get:

$$\|\phi(x+h) - \phi(x)\|_{L^q(\omega)} = \|\tau_h(\phi) - \phi\|_{L^q(\omega)} \leq \|\tau_h(\phi) - \phi\|_{L^1(\omega)}^\alpha \, \|\tau_h(\phi) - \phi\|_{L^{q+\delta}(\omega)}^{1-\alpha}.$$

Taking into account Theorem 5.2, we have also that $\phi \in W_0^{1,1}(\Omega)$, with

$$\int_\Omega |\nabla \phi| \le \left(\int_\Omega (\varepsilon + |\nabla u_\varepsilon|^{p-2})|\nabla \phi|^2 \right)^{\frac{1}{2}} \cdot \left(\int_\Omega \frac{1}{(\varepsilon + |\nabla u_\varepsilon|^{p-2})} \right)^{\frac{1}{2}}$$

for some $u_\varepsilon \in \mathcal{B}_{R_0}(u_0)$, given by the definition of $\mathcal{D}_0(K)$, with

$$\|\phi\|^2_{\varepsilon,u_\varepsilon} := \int_\Omega \left[\left(\varepsilon + |\nabla u|^{p-2}\right)|\nabla v|^2 + v^2 \right] \le 1.$$

Therefore, for every $\phi \in \mathcal{D}_0(K)$, we have

$$\|\tau_h(\phi) - \phi\|_{L^1(\omega)} \le |h| \|\nabla \phi\|_{L^1(\Omega)} \le C_0|h|$$

so that, exploiting (3.10) we get

$$\|\tau_h(\phi) - \phi\|_{L^q(\omega)} \le C_0^\alpha \, |h|^\alpha \, \left(2\|\phi\|_{L^{q+\delta}(\Omega)}\right)^{1-\alpha} \le C|h|^\alpha.$$

Since

$$\|\phi\|_{L^q(\Omega\backslash\omega)} \le \|\phi\|_{L^{q+\delta}(\Omega\backslash\omega)} |\Omega\backslash\omega|^{\frac{1}{q}-\frac{1}{q+\delta}} \le C \, |\Omega\backslash\omega|^{\frac{1}{q}-\frac{1}{q+\delta}} \le \epsilon$$

for $\Omega\backslash\omega$ sufficiently small, by Corollary 4.27 in [3] we deduce that the closure of $\mathcal{D}_0(K)$ is a compact set in $L^q(\Omega)$.

Finally, for

$$1 < q < \frac{2p-2}{2p-3}$$

and $\phi \in W_0^{1,q}(\Omega)$, we have that

$$\left(\int_\Omega |\nabla \phi|^q \right)^{\frac{1}{q}} \le \left(\int_\Omega (\varepsilon + |\nabla u_\varepsilon|^{p-2})|\nabla \phi|^2 \right)^{\frac{1}{2}} \cdot \left(\int_\Omega \frac{1}{(\varepsilon + |\nabla u_\varepsilon|^{p-2})^{\frac{q}{2-q}}} \right)^{\frac{2-q}{2q}} \le C\|\phi\|_{\varepsilon,u_\varepsilon}$$

for some $u_\varepsilon \in \mathcal{B}_{R_0}(u_0)$, given by the definition of $\mathcal{D}_0(K)$, with $\|\phi\|_{\varepsilon,u_\varepsilon} \le 1$.
If we now assume that Ω is a ball or an annulus and $u_0 \in C^1(\overline{\Omega})$ is a positive *radial* solution to problem (P) with $f(x, u_0) = u_0^q - \Lambda u_0^{p-1}$, the same proof works exploiting Theorem 5.3 instead of Theorem 5.2. □

Proof of Theorem 3.3. Let $u_0 \in C^1(\overline{\Omega}) \setminus \{0\}$ be solution to problem (P) and $Z = \{\nabla u_0 = 0\}$ be the critical set of u_0. Since $f(x, u_0(x)) > 0$ for any $x \in \Omega$, we can deduce that $|Z| = 0$. This is a consequence of [20, Theorem 1.1]. More precisely in our case we can exploit the local results in [35, Proposition 3.1] (carried out with general positive source terms) and the fact that u_0 has not critical points near the boundary (see [43]). Nevertheless the fact that $|Z| = 0$ also follows by (4.24) in Theorem 4.4, exploited in the case $\varepsilon = 0$ and $r = 0$ recalling that u_0 has not critical points near the boundary. Now we show that condition (PC) holds at u_0. Firstly let (ε_n) be a sequence in $[0, \infty[$ converging to 0 and (u_n) be a sequence converging to u_0 in $C^1(\overline{\Omega})$. Let (w_n) be a sequence in $W_0^{1,1}(\Omega)$ such that each u_n is a critical point of J_{ε_n} and

$$\sup_n \int_\Omega \left[\left(\varepsilon_n + |\nabla u_n|^{p-2}\right)|\nabla w_n|^2 + w_n^2\right] dx < +\infty\,.$$

From Corollary 3.2 applied for $K = \{0\}$, we deduce that there exist (w_{n_k}), and $w \in L^2(\Omega) \cap W_0^{1,q}(\Omega)$ for some $q > 1$ such that (w_{n_k}) weakly converges to w in $W_0^{1,q}(\Omega)$ and strongly converges to w in $L^2(\Omega)$. Therefore we have that (w_n) is weakly precompact in $W_0^{1,1}(\Omega)$ and strongly precompact in $L^2(\Omega)$.
If we now assume that Ω is a ball or an annulus and $u_0 \in C^1(\overline{\Omega})$ is a positive *radial* solution to problem (P) with $f(x, u_0) = u_0^q - \Lambda u_0^{p-1}$, the same proof works exploiting Theorem 5.3 instead of Theorem 5.2. □

7. The nondegenerate case for radial solutions

In this section we restrict to consider the case $\Omega = B$ is the unit ball of $\mathbb{R}^N$. We fix $u_0 \in C^1(\overline{B})$ a positive *radial* solution to

$$(R) \quad \begin{cases} -\Delta_p u = f(u) & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases}$$

where f satisfies assumption (f) with $f(s) > 0$ for any $s > 0$ and $f(s) = 0$ for any $s \le 0$ or $f(s) = s^q - \Lambda s^{p-1}$, with $2 \le p < q + 1 \le p^*$ and $\Lambda > 0$.

By Proposition 1.2.6 in [24], it is known that $u_0'(r) < 0$ for $r > 0$ and $f(u_0(0)) > 0$. Moreover one can derive the precise behavior of $|\nabla u_0|$ near the origin using L'Hôpital's rule, which is $|x|^{1/(p-1)}$ (see [36]).

Indeed, let u_0 be a solution of the initial value problem in radial coordinates

$$(I) \quad \begin{cases} (|u'|^{p-2}u')' + \frac{N-1}{r}|u'|^{p-2}u' + f(u) = 0, & r \in]0,1[\\ u(r) > 0, \ r \in]0,1[, & u'(0) = u(1) = 0. \end{cases}$$

By Hopf's Lemma applied to u_0, we deduce $u_0'(1) < 0$. Equation (I) can be rewritten in the form

$$(r^{N-1}u' \, |u'|^{p-2})' = -r^{N-1} f(u),$$

and thus integrating from 0 to r we have

$$r^{-1}u'(r)|u'(r)|^{p-2} = -r^{-N}\int_0^r s^{N-1}f(u(s))\,ds.$$

Using L'Hôpital's rule we infer

$$\lim_{r\to 0} r^{-1/(p-1)}u_0' = -\big(\frac{f(u_0(0))}{N}\big)^{1/(p-1)}. \tag{7.33}$$

Now we consider the family of approximating functionals

$$J_\varepsilon(u) = \frac{1}{p}\int_B |\nabla u|^p\,dx + \frac{\varepsilon}{2}\int_B |\nabla u|^2\,dx - \int_B F(u)\,dx \tag{7.34}$$

associated to the quasilinear elliptic equation

$$(P)_\epsilon \quad \begin{cases} -\Delta_p u - \varepsilon\Delta u = f(u) & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases}$$

In the following lemma, we show an uniform convexity of the second derivative of J_ε in a neighborhood of the solution u_0 along the direction of W. For general domains, it seems very difficult to establish a similar result since the precise behavior of the gradient of the approximating solution near its zeros is not known.

Lemma 7.1. *There exist $0 < \bar r$, $0 < \bar\varepsilon$, $\gamma_1 > 0$ such that for any $\varepsilon \in]0, \bar\varepsilon[$, if u_ε and z_ε are positive, radial critical points of J_ε with $\|u_\varepsilon - u_0\| < \bar r$ and $\|z_\varepsilon - u_0\| < \bar r$, then*

$$\langle J_\varepsilon''(\vartheta u_\varepsilon + (1-\vartheta)z_\varepsilon)w, w\rangle \geq \gamma_1 \|w\|^2_{\varepsilon,\psi_\varepsilon} \quad \forall w \in W,\ \forall \vartheta \in]0,1[\tag{7.35}$$

where $\psi_\varepsilon := \vartheta u_\varepsilon + (1-\vartheta)z_\varepsilon$.

Proof. We begin to prove that there exist $0 < C^*$, $0 < R^*$, $0 < \varepsilon^*$ such that for any $\varepsilon \in]0, \varepsilon^*[$ if u_ε and z_ε are critical points of J_ε with $\|u_\varepsilon - u_0\| < R^*$ and $\|z_\varepsilon - u_0\| < R^*$, then

$$(|\vartheta u_\varepsilon'(r) + (1-\vartheta)z_\varepsilon'(r)|^{p-2} + \varepsilon) \geq C^*|u_0'(r)|^{p-2} \quad \forall r \in]0,1], \forall \vartheta \in [0,1]. \tag{7.36}$$

By contradiction we assume that there exist $\beta_n \to 0$, $\varepsilon_n \to 0$, two sequences u_n, z_n of positive, radial critical points of J_{ε_n}, converging to u_0 as $n \to +\infty$, a sequence $r_n \in]0,1]$ and a sequence $\vartheta_n \in [0,1]$ such that for n large

$$(|\vartheta_n u_n'(r_n) + (1-\vartheta_n)z_n'(r_n)|^{p-2} + \varepsilon_n) < \beta_n\,|u_0'(r_n)|^{p-2} \tag{7.37}$$

and thus

$$r_n^{-(p-2)/(p-1)}(|\vartheta_n u_n'(r_n)+(1-\vartheta_n)z_n'(r_n)|^{p-2}+\varepsilon_n) < \beta_n\, r_n^{-(p-2)/(p-1)}|u_0'(r_n)|^{p-2}. \quad (7.38)$$

From standard regularity theory, we can infer that u_n and z_n converge to u_0 in $C^1(\overline{B})$. As ϑ_n is bounded, we have $\vartheta_n \to \vartheta \in [0,1]$, as $n \to +\infty$, up to subsequences. If $r_n \to \bar{r} \neq 0$, we immediately derive a contradiction by (7.37), since $u_0'(\bar{r}) \neq 0$. We suppose that $\bar{r} = 0$. Firstly by (7.33) we derive that

$$\lim_{n\to+\infty} r_n^{-(p-2)/(p-1)}|u_0'(r_n)|^{p-2} = \alpha^{p-2} \quad (7.39)$$

where $\alpha = \left(\frac{f(u_0(0))}{N}\right)^{1/((p-1)}$. Moreover since $J_n'(u_n) = 0$ and $J_n'(v_n) = 0$, we have

$$\lim_{n\to+\infty} r_n^{-1}(|u_n'(r_n)|^{p-1} + \varepsilon_n|u_n'(r_n)|) = \frac{f(u_0(0))}{N} = \alpha^{p-1}, \quad (7.40)$$

and

$$\lim_{n\to+\infty} r_n^{-1}(|v_n'(r_n)|^{p-1} + \varepsilon_n|v_n'(r_n)|) = \frac{f(u_0(0))}{N} = \alpha^{p-1}. \quad (7.41)$$

From (7.40) (7.41) we deduce that

$$r_n^{-1}|u_n'(r_n)|^{p-1} \to \alpha^{p-1} \quad (7.42)$$

and

$$r_n^{-1}|v_n'(r_n)|^{p-1} \to \alpha^{p-1}. \quad (7.43)$$

Indeed, we firstly notice that by (7.38) and (7.39) we have that $\varepsilon_n = o(r_n^{(p-2)/(p-1)})$ as $n \to +\infty$. Moreover by (7.40) we derive

$$\frac{\varepsilon_n|u_n'(r_n)|}{r_n} = \frac{\varepsilon_n|u_n'(r_n)|}{r_n^{1/(p-1)}}\frac{r_n^{1/(p-1)}}{r_n} = \frac{\varepsilon_n}{r_n^{(p-2)/(p-1)}}\ \frac{|u_n'(r_n)|}{r_n^{1/(p-1)}} \to 0$$

which implies (7.42). Analogously we can infer (7.43). Now we observe that

$$\begin{aligned} r_n^{-1/(p-1)}|\vartheta_n u_n'(r_n) + (1-\vartheta_n)v_n'(r_n)| &= -r_n^{-1/(p-1)}(\vartheta_n u_n'(r_n) + (1-\vartheta_n)v_n'(r_n)) \\ &= r_n^{-1/(p-1)}(\vartheta_n|u_n'(r_n)| + (1-\vartheta_n)|v_n'(r_n))|. \quad (7.44) \end{aligned}$$

Since $\vartheta_n r_n^{-1/(p-1)}|u_n'(r_n)| + (1-\vartheta_n)r_n^{-1/(p-1)}|v_n'(r_n)| \to \vartheta\alpha + (1-\vartheta)\alpha = \alpha \neq 0$ as $n \to +\infty$, we derive a contradiction from (7.44) and the fact that

$$\beta_n^{1/(p-2)}\, r_n^{-1/(p-1)}|u_0'(r_n)| \to 0.$$

Now we prove the statement of the lemma. Arguing by contradiction, we assume that there exist a sequence $\varepsilon_n \to 0$, and two sequence u_n, z_n of positive critical points of $J_{\varepsilon_n} := J_n$ and two sequence $\vartheta_n \in]0,1[$ and $w_n \in W$, $w_n \neq 0$ such that

$$\langle J_n''(\vartheta_n u_n + (1-\vartheta_n)z_n)w_n, w_n\rangle < \frac{1}{n}\|w_n\|^2_{\varepsilon,\psi_n} \tag{7.45}$$

where $\psi_n = \vartheta_n u_n + (1-\vartheta_n)z_n$. Set $\varphi_n = \frac{w_n}{\|w_n\|_{\varepsilon,\psi_n}}$, we divide (7.45) by $\|w_n\|^2_{\varepsilon,\psi_n}$ and we get

$$\langle J_n''(\psi_n)\varphi_n, \varphi_n\rangle < \frac{1}{n}. \tag{7.46}$$

By (7.36), we have that

$$\left(\int\limits_B |\nabla u_0|^{p-2}|\nabla \varphi_n|^2\right)^{\frac{1}{2}} \leq C_1\left(\int\limits_B (\varepsilon_n + |\nabla \psi_n|^{p-2})|\nabla \varphi_n|^2\right)^{\frac{1}{2}} \leq C\|\varphi_n\|_{\varepsilon_n,\psi_n} \tag{7.47}$$

for suitable constants $C, C_1 > 0$. Since $\|\varphi_n\|_{\varepsilon,\psi_n} = 1$, there exists $\varphi \in W$, such that $\varphi_n \to \varphi \in W$ weakly in $W^{1,q}_0(B)$, $q > 1$ and strongly in $L^2(B)$. Since

$$1/n \geq \langle J_n''(\psi_n)\varphi_n, \varphi_n\rangle \geq 1 + \int\limits_B f'(\psi_n)\varphi_n^2 dx, \tag{7.48}$$

we derive that $\varphi \neq 0$. We infer that

$$\|\varphi\|^2_{u_0} \leq \langle J''(u_0)\varphi, \varphi\rangle \leq \liminf_{n\to+\infty}\langle J_n''(\psi_{\varepsilon_n})\varphi_n, \varphi_n\rangle \leq 0 \tag{7.49}$$

which is absurd. $\square$

Lemma 7.2. *Suppose that u_0 is a isolated positive, radial critical point of J, such that $J''(u_0)$ is injective. Then there exists $0 < \varepsilon_1 \leq \bar{\varepsilon}$ such that for any $\varepsilon \in (0, \varepsilon_1]$ the functional J_ε has at most one critical point in $B_{\bar{r}}(u_0)$.*

Proof. Assume that there exist two sequences u_n and w_n of critical points of $J_n \equiv J_{\varepsilon_n}$ in $B_{\bar{r}}(u_0)$, with $\varepsilon_n \to 0$, as $n \to +\infty$. By Lieberman's regularity results [32], we have $\|u_n\|_{C^{1,\beta}(\overline{B})} \leq C$, $\|w_n\|_{C^{1,\beta}(\overline{B})} \leq C$, independently of n. We can assume that u_n and w_n are positive radial functions as n is large enough. So as u_0 is an isolated critical point of J, u_n and w_n converges to u_0 in $C^{1,\alpha}(\overline{B})$, with $\alpha \in]0,\beta[$, as $n \to +\infty$. We aim to prove that $u_n = w_n$ for n large. Let us define $u_n - w_n = v_n^1 + v_n^2$, with $v_n^1 \in V$, $v_n^2 \in W$. Since $J_0(u_0)$ is injective, we have

$$\langle J''(u_0)v, v\rangle \leq -\mu\|v\|_2^2 \quad \forall v \in V, \tag{7.50}$$

and thus we can deduce that

$$\langle J_n''(z)v, v\rangle \le -\mu/3\|v\|_2^2 \quad \forall v \in V, \tag{7.51}$$

for any $\|z - u_0\| < r_0$, r_0 small enough. Moreover for any $n \in \mathbb{N}$, there exists $\vartheta_n \in]0, 1[$ such that

$$\begin{aligned} 0 &= \langle J_n'(u_n) - J_n'(w_n), v_n^1 - v_n^2\rangle = \langle J_n''(\xi_n)(v_n^1 + v_n^2), v_n^1 - v_n^2\rangle \\ &= \langle J_n''(\xi_n)v_n^1, v_n^1\rangle - \langle J_n''(\xi_n)v_n^2, v_n^2\rangle \end{aligned} \tag{7.52}$$

where $\xi_n = \vartheta_n u_n + (1 - \vartheta_n)w_n$. It follows that $\xi_n \to u_0$ in $C^{1,\alpha}(\overline{B})$ as $n \to +\infty$, from Lemma 7.1 we derive for n large

$$-(\mu/3 - \varepsilon_n)\|v_n^1\|_2^2 \ge \langle J_n''(\xi_n)v_n^1, v_n^1\rangle = \langle J_n''(\xi_n)v_n^2, v_n^2\rangle \ge \tilde{\gamma}\|v_n^2\|_2^2 \tag{7.53}$$

which implies $v_n^1 = v_n^2 = 0$ for n large and the thesis follows. □

8. Proofs of the main results

In this section we provide the proofs of the results stated in the Introduction.

Proofs of Theorem 1.8 and Corollary 1.12. From Theorems 3.3 and 1.7, we can infer Theorem 1.8. Corollary 1.12 follows from Theorems 3.3, 1.8 and Proposition 1.10. □

Proof of Theorem 1.13. We begin to suppose that $f(s) > 0$ for all $s > 0$ and $f(s) = 0$ for all $s \le 0$. From Theorem 1.12, we have that $m(J, u_0)$ and $m^*(J, u_0)$ are finite and $C_k(J, u_0) = \{0\}$ for $k > m(J, u_0)$ and $q > m^*(J, u_0)$. Assume that $m(J, u_0) = m^*(J, u_0)$ and $C_{\overline{m}}(J, u_0) \ne \{0\}$ for some $\overline{m} \in \mathbb{N}$. By Lemma 7.2, there exist $\varepsilon_n \to 0^+$ and $u_{\varepsilon_n} \to u_0$ as $n \to +\infty$ such that $J_{\varepsilon_n}'(u_{\varepsilon_n}) = 0$ and $J_{\varepsilon_n}''(u_{\varepsilon_n})$ is injective, namely $m(J, u_0) = m(J_{\varepsilon_n}, u_{\varepsilon_n})$. Applying Theorem 1.3 in [17] we deduce $C_q(J, u_0) \simeq C_q(J_{\varepsilon_n}, u_{\varepsilon_n})$ and then by Theorem 1.1 in [15], we conclude $\overline{m} = m(J, u_0)$ and $C_q(J, u_0) \simeq \delta_{q,m(J,u_0)}\mathbb{K}$. Conversely, if f changes sign, the thesis follows from Theorems 3.3 and 1.7. □

Proofs of Theorem 1.14. From Theorems 3.3, 1.7 and 1.8, we can infer Theorem 1.14. □

9. Appendix

Proofs of Theorem 4.1. Let us consider the test function

$$\phi = T_\xi(u_i) \cdot H_\delta(|x - y|) \cdot \varphi^2 = T_\xi(u_i) \cdot H_\delta \cdot \varphi^2$$

with the notation $H_\delta(|x - y|) = H_\delta$. If $N = 2$ the presence of H_δ is omitted. Note that ϕ can be used as test-function in linearized equation (4.14), since it vanishes near the singularity and in a neighborhood of the critical set Z. Consequently by (4.14) we get:

$$\int_\Omega \left(\varepsilon+|\nabla u|^{p-2}\right)|\nabla u_i|^2\cdot T'_\xi(u_i)\cdot H_\delta\cdot\varphi^2+(p-2)|\nabla u|^{p-4}(\nabla u\,,\,\nabla u_i)^2\cdot T'_\xi(u_i)\cdot H_\delta\cdot\varphi^2+$$
$$+\int_\Omega \left(\varepsilon+|\nabla u|^{p-2}\right)(\nabla u_i\,,\,\nabla H_\delta)\cdot T_\varepsilon(u_i)\cdot\varphi^2+(p-2)|\nabla u|^{p-4}(\nabla u\,,\,\nabla u_i)(\nabla u\,,\,\nabla H_\delta)\cdot$$
$$\cdot T_\xi(u_i)\cdot\varphi^2+2\int_\Omega \left(\varepsilon+|\nabla u|^{p-2}\right)(\nabla u_i\,,\,\nabla\varphi)\cdot T_\xi(u_i)\cdot H_\delta\cdot\varphi$$
$$+(p-2)|\nabla u|^{p-4}(\nabla u\,,\,\nabla u_i)(\nabla u\,,\,\nabla\varphi)\cdot T_\xi(u_i)\cdot H_\delta\cdot\varphi=$$
$$=\int_\Omega f_i(x,u)\cdot T_\xi(u_i)\cdot H_\delta\cdot\varphi^2+\int_\Omega f_u(x,u)u_i\cdot T_\xi(u_i)\cdot H_\delta\cdot\varphi^2+\int_\Omega r_i(x)\cdot T_\xi(u_i)\cdot H_\delta\cdot\varphi^2. \tag{9.54}$$

We now set

$$\begin{aligned}
A(\xi\,,\,\delta)&\equiv\int_\Omega |\nabla u|^{p-2}|\nabla u_i||\nabla H_\delta|\cdot|T_\xi(u_i)|\cdot\varphi^2\\
B(\xi\,,\,\delta)&\equiv 2\int_\Omega |\nabla u|^{p-2}|\nabla u_i||\nabla\varphi|\cdot|T_\xi(u_i)|\cdot H_\delta\cdot\varphi\\
C(\xi\,,\,\delta)&\equiv\int_\Omega |f_i(x,u)|\cdot|T_\xi(u_i)|\cdot H_\delta\cdot\varphi^2\\
D(\xi\,,\,\delta)&\equiv\int_\Omega |f_u(x,u)||u_i|\cdot|T_\xi(u_i)|\cdot H_\delta\cdot\varphi^2\\
E(\xi\,,\,\delta)&\equiv\int_\Omega |r_i(x)|\cdot|T_\xi(u_i)|\cdot H_\delta\cdot\varphi^2\,.\\
F(\xi\,,\,\delta)&\equiv\varepsilon\int_\Omega |\nabla u_i||\nabla H_\delta|\cdot|T_\xi(u_i)|\cdot\varphi^2\\
G(\xi\,,\,\delta)&\equiv 2\varepsilon\int_\Omega |\nabla u_i||\nabla\varphi|\cdot|T_\xi(u_i)|\cdot H_\delta\cdot\varphi\,.
\end{aligned} \tag{9.55}$$

We now observe that

$$\min\{(p-1)\,,\,1\}\int_\Omega |\nabla u|^{p-2}|\nabla u_i|^2\cdot T'_\xi(u_i)\cdot H_\delta\cdot\varphi^2\le$$
$$\le\int_\Omega |\nabla u|^{p-2}|\nabla u_i|^2\cdot T'_\xi(u_i)\cdot H_\delta\cdot\varphi^2+(p-2)|\nabla u|^{p-4}(\nabla u\,,\,\nabla u_i)^2\cdot T'_\xi(u_i)\cdot H_\delta\cdot\varphi^2$$

so that, from (9.54), estimating the terms involved, we get

$$\begin{aligned}
&\frac{\min\{(p-1)\,,\,1\}}{p-1}\int\limits_{\Omega}|\nabla u|^{p-2}|\nabla u_i|^2\cdot T'_\xi(u_i)\cdot H_\delta\cdot\varphi^2\\
&+\frac{\varepsilon}{p-1}\int\limits_{\Omega}|\nabla u_i|^2\cdot T'_\xi(u_i)\cdot H_\delta\cdot\varphi^2\le \qquad (9.56)\\
&A(\xi\,,\,\delta)+B(\xi\,,\,\delta)+\frac{C(\xi\,,\,\delta)+D(\xi\,,\,\delta)+E(\xi\,,\,\delta)+F(\xi\,,\,\delta)+G(\xi\,,\,\delta)}{p-1}\,.
\end{aligned}$$

By the Dominated Convergence Theorem, letting δ goes to zero, we now have that

$$\begin{aligned}
&A(\xi\,,\,\delta)\underset{\delta\to0}{\longrightarrow}A(\xi,0)\equiv\gamma\int\limits_{\Omega}|\nabla u|^{p-2}|\nabla u_i|\frac{1}{|x-y|^{\gamma+1}}\cdot|T_\xi(u_i)|\cdot\varphi^2\\
&B(\xi\,,\,\delta)\underset{\delta\to0}{\longrightarrow}B(\xi,0)\equiv2\int\limits_{\Omega}|\nabla u|^{p-2}|\nabla u_i||\nabla\varphi|\cdot|T_\xi(u_i)|\cdot\frac{1}{|x-y|^{\gamma}}\cdot\varphi\\
&C(\xi\,,\,\delta)\underset{\delta\to0}{\longrightarrow}C(\xi,0)\equiv\int\limits_{\Omega}|f_i(x,u)|\cdot|T_\xi(u_i)|\cdot\frac{1}{|x-y|^{\gamma}}\cdot\varphi^2\\
&D(\xi\,,\,\delta)\underset{\delta\to0}{\longrightarrow}D(\xi,0)\equiv\int\limits_{\Omega}|f_u(x,u)||u_i|\cdot|T_\xi(u_i)|\cdot\frac{1}{|x-y|^{\gamma}}\cdot\varphi^2 \qquad (9.57)\\
&E(\xi\,,\,\delta)\underset{\delta\to0}{\longrightarrow}E(\xi,0)\equiv\int\limits_{\Omega}|r_i(x)|\cdot|T_\xi(u_i)|\cdot\frac{1}{|x-y|^{\gamma}}\cdot\varphi^2\\
&F(\xi\,,\,\delta)\underset{\delta\to0}{\longrightarrow}A(\xi,0)\equiv\varepsilon\gamma\int\limits_{\Omega}|\nabla u_i|\frac{1}{|x-y|^{\gamma+1}}\cdot|T_\xi(u_i)|\cdot\varphi^2\\
&G(\xi\,,\,\delta)\underset{\delta\to0}{\longrightarrow}B(\xi,0)\equiv2\varepsilon\int\limits_{\Omega}|\nabla u_i||\nabla\varphi|\cdot|T_\xi(u_i)|\cdot\frac{1}{|x-y|^{\gamma}}\cdot\varphi\,.
\end{aligned}$$

So that we have from (9.56)

$$\begin{aligned}
&\frac{\min\{(p-1)\,,\,1\}}{p-1}\int\limits_{\Omega}|\nabla u|^{p-2}|\nabla u_i|^2\cdot T'_\xi(u_i)\cdot\frac{1}{|x-y|^{\gamma}}\cdot\varphi^2\\
&+\frac{\varepsilon}{p-1}\int\limits_{\Omega}|\nabla u_i|^2\cdot T'_\xi(u_i)\cdot\frac{1}{|x-y|^{\gamma}}\cdot\varphi^2\le \qquad (9.58)\\
&A(\xi\,,\,0)+B(\xi\,,\,0)+\frac{C(\xi\,,\,0)+D(\xi\,,\,0)+E(\xi\,,\,0)}{p-1}\,.
\end{aligned}$$

Let us now estimate the term $A(\xi,0)$. We use Young's inequality $ab\le\vartheta a^2+\frac{b^2}{4\vartheta}$ and the fact that $|T_\xi(t)|\le t^{1-\beta}$

$$
\begin{aligned}
A(\xi,0) &\le \int_\Omega \frac{|\nabla u|^{\frac{p-2}{2}}|\nabla u_i|}{|x-y|^{\frac{\gamma}{2}}|u_i|^{\frac{\beta}{2}}}\chi_{\{|u_i|\ge\xi\}}\varphi\cdot\gamma\frac{|\nabla u|^{\frac{p-2}{2}}|u_i|^{\frac{2-\beta}{2}}}{|x-y|^{\frac{\gamma+2}{2}}}\varphi \le \\
&\le \vartheta\int_\Omega \frac{|\nabla u|^{p-2}|\nabla u_i|^2}{|x-y|^{\gamma}|u_i|^{\beta}}\chi_{\{|u_i|\ge\xi\}}\varphi^2+\frac{\gamma^2}{4\vartheta}\int_\Omega\frac{|\nabla u|^{p-\beta}}{|x-y|^{\gamma+2}}\varphi^2 \le \\
&\le \vartheta\int_\Omega \frac{|\nabla u|^{p-2}|\nabla u_i|^2}{|x-y|^{\gamma}|u_i|^{\beta}}\chi_{\{|u_i|\ge\xi\}}\varphi^2+\frac{\gamma^2\cdot K^{p-\beta}\cdot M}{4\vartheta}\,.
\end{aligned}
$$

To estimate the term $B(\xi,0)$ we use the fact that $|\nabla\varphi|\le\frac{1}{\varrho}$ and get:

$$
\begin{aligned}
B(\xi,0) &= \frac{2}{\varrho}\int_\Omega \frac{|\nabla u|^{\frac{p-2}{2}}|\nabla u_i|}{|x-y|^{\frac{\gamma}{2}}|u_i|^{\frac{\beta}{2}}}\chi_{\{|u_i|\ge\xi\}}\varphi\cdot\frac{|\nabla u|^{\frac{p-2}{2}}|u_i|^{\frac{2-\beta}{2}}}{|x-y|^{\frac{\gamma}{2}}} \le \\
&\le \vartheta\int_\Omega \frac{|\nabla u|^{p-2}|\nabla u_i|^2}{|x-y|^{\gamma}|u_i|^{\beta}}\chi_{\{|u_i|\ge\xi\}}\varphi^2+\frac{1}{\vartheta\varrho^2}\int_{B_{2\varrho}(x_0)}\frac{|\nabla u|^{p-\beta}}{|x-y|^{\gamma}} \le \\
&\le \vartheta\int_\Omega \frac{|\nabla u|^{p-2}|\nabla u_i|^2}{|x-y|^{\gamma}|u_i|^{\beta}}\chi_{\{|u_i|\ge\xi\}}\varphi^2+\frac{K^{p-\beta}\cdot M}{\vartheta\varrho^2}\,.
\end{aligned}
$$

The term $C(\xi,0)$ is simply estimated as follows:

$$
C(\xi,0)\le V\cdot\int_\Omega\frac{|u_i|^{1-\beta}}{|x-y|^{\gamma}}\varphi^2\le V\cdot K^{1-\beta}\int_\Omega\frac{1}{|x-y|^{\gamma}}\le M\cdot V\cdot K^{1-\beta}\,.
$$

In the same way:

$$
D(\xi,0)\le W\cdot\int_\Omega\frac{|u_i|^{2-\beta}}{|x-y|^{\gamma}}\varphi^2\le W\cdot K^{2-\beta}\int_\Omega\frac{1}{|x-y|^{\gamma}}\le M\cdot W\cdot K^{2-\beta}\,,
$$

and

$$
E(\xi,0)\le Z\cdot\int_\Omega\frac{|u_i|^{1-\beta}}{|x-y|^{\gamma}}\varphi^2\le Z\cdot K^{1-\beta}\int_\Omega\frac{1}{|x-y|^{\gamma}}\le M\cdot Z\cdot K^{1-\beta}\,.
$$

Also we have

$$
\begin{aligned}
F(\xi,0) &\le \int\limits_{\Omega} \varepsilon \frac{|\nabla u_i|}{|x-y|^{\frac{\gamma}{2}}|u_i|^{\frac{\beta}{2}}}\chi_{\{|u_i|\ge\xi\}}\varphi\cdot\gamma\frac{|u_i|^{\frac{2-\beta}{2}}}{|x-y|^{\frac{\gamma+2}{2}}}\varphi \le \\
&\le \tilde{\vartheta}\varepsilon\int\limits_{\Omega}\frac{|\nabla u_i|^2}{|x-y|^{\gamma}|u_i|^{\beta}}\chi_{\{|u_i|\ge\xi\}}\varphi^2+\frac{\gamma^2}{4\tilde{\vartheta}}\varepsilon\int\limits_{\Omega}\frac{|\nabla u|^{2-\beta}}{|x-y|^{\gamma+2}}\varphi^2 \le \\
&\le \tilde{\vartheta}\varepsilon\int\limits_{\Omega}\frac{|\nabla u_i|^2}{|x-y|^{\gamma}|u_i|^{\beta}}\chi_{\{|u_i|\ge\xi\}}\varphi^2+\frac{\varepsilon\gamma^2\cdot K^{2-\beta}\cdot M}{4\tilde{\vartheta}}\,.
\end{aligned}
$$

Finally

$$
\begin{aligned}
G(\xi,0) &= \frac{2\varepsilon}{\varrho}\int\limits_{\Omega}\frac{|\nabla u_i|}{|x-y|^{\frac{\gamma}{2}}|u_i|^{\frac{\beta}{2}}}\chi_{\{|u_i|\ge\xi\}}\varphi\cdot\frac{|u_i|^{\frac{2-\beta}{2}}}{|x-y|^{\frac{\gamma}{2}}} \le \\
&\le \tilde{\vartheta}\varepsilon\int\limits_{\Omega}\frac{|\nabla u_i|^2}{|x-y|^{\gamma}|u_i|^{\beta}}\chi_{\{|u_i|\ge\xi\}}\varphi^2+\frac{\varepsilon}{\tilde{\vartheta}\varrho^2}\int\limits_{B_{2\varrho}(x_0)}\frac{|\nabla u|^{2-\beta}}{|x-y|^{\gamma}} \le \\
&\le \tilde{\vartheta}\varepsilon\int\limits_{\Omega}\frac{|\nabla u_i|^2}{|x-y|^{\gamma}|u_i|^{\beta}}\chi_{\{|u_i|\ge\xi\}}\varphi^2+\frac{\varepsilon K^{2-\beta}\cdot M}{\tilde{\vartheta}\varrho^2}\,.
\end{aligned}
$$

Taking into account (9.58), exploiting the above estimates and evaluating T'_ξ, we get

$$
\begin{aligned}
&\frac{\min\{(p-1)\,,1\}}{p-1}\int\limits_{\Omega}\frac{|\nabla u|^{p-2}|\nabla u_i|^2}{|u_i|^{\beta}|x-y|^{\gamma}}\big(G'_\xi(u_i)- \\
&\beta\frac{G_\xi(u_i)}{|u_i|}-2\vartheta\max\{(p-1)\,,1\}\chi_{\{|u_i|\ge\xi\}}\big)\cdot\varphi^2+ \\
&\frac{\varepsilon}{p-1}\int\limits_{\Omega}\frac{|\nabla u_i|^2}{|u_i|^{\beta}|x-y|^{\gamma}}\big(G'_\xi(u_i)-\beta\frac{G_\xi(u_i)}{|u_i|}-2\tilde{\vartheta}\chi_{\{|u_i|\ge\xi\}}\big)\cdot\varphi^2+ \\
&\le\frac{\gamma^2\cdot K^{p-\beta}\cdot M}{4\vartheta}+\frac{K^{p-\beta}\cdot M}{\vartheta\varrho^2}+\frac{M}{p-1}(V\cdot K^{1-\beta}+W\cdot K^{2-\beta}+Z\cdot K^{1-\beta})+ \\
&\frac{\varepsilon\gamma^2K^{2-\beta}M}{4\tilde{\vartheta}}+\frac{\varepsilon K^{2-\beta}M}{\tilde{\vartheta}\varrho^2}\,.
\end{aligned}
$$

We now chose $\tilde{\vartheta}$ sufficiently small such that $\big(G'_\xi(u_i)-\beta\frac{G_\xi(u_i)}{|u_i|}-2\tilde{\vartheta}\chi_{\{|u_i|\ge\xi\}}\big)\ge 0$, say

$$
\tilde{\vartheta}=\frac{1-\beta}{4\,,}\qquad\text{and}\qquad\vartheta=\frac{1-\beta}{4\max\{(p-1)\,,1\}},
$$

which gives $\big(G'_\xi(u_i)-\beta\frac{G_\xi(u_i)}{|u_i|}-2\vartheta\max\{(p-1)\,,1\}\chi_{\{|u_i|\ge\xi\}}\big)\ge 0$. We then pass to the limit and let $\xi\to 0$. By Fatou's Lemma, since $\big(G'_\xi(u_i)-\beta\frac{G_\xi(u_i)}{|u_i|}-2\vartheta\max\{(p-$

$1), 1\}\chi_{\{|u_i|\geq\xi\}}\big) \to (1-\beta-2\vartheta) = \frac{1-\beta}{2}$ and $\big(G'_\xi(u_i) - \beta\frac{G_\xi(u_i)}{|u_i|} - 2\tilde{\vartheta}\chi_{\{|u_i|\geq\xi\}}\big) \to (1-\beta-2\tilde{\vartheta}) = \frac{1-\beta}{2}$, we get

$$\frac{1-\beta}{2\cdot\max\{(p-1)\,,\,1\}}\int\limits_\Omega \frac{|\nabla u|^{p-2}|\nabla u_i|^2}{|u_i|^\beta|x-y|^\gamma}\cdot\varphi^2 + \frac{\varepsilon\,(1-\beta)}{2\,\max\{(p-1)\,,\,1\}}\int\limits_\Omega \frac{|\nabla u_i|^2}{|u_i|^\beta|x-y|^\gamma}\cdot\varphi^2 \leq$$
$$\frac{1-\beta}{2\cdot\max\{(p-1)\,,\,1\}}\int\limits_\Omega \frac{|\nabla u|^{p-2}|\nabla u_i|^2}{|u_i|^\beta|x-y|^\gamma}\cdot\varphi^2 + \frac{\varepsilon\,(1-\beta)}{2\,(p-1)}\int\limits_\Omega \frac{|\nabla u_i|^2}{|u_i|^\beta|x-y|^\gamma}\cdot\varphi^2 \leq$$
$$\leq \frac{\gamma^2\cdot K^{p-\beta}\cdot M}{4\vartheta} + \frac{K^{p-\beta}\cdot M}{\vartheta\varrho^2} + \frac{M}{p-1}(V\cdot K^{1-\beta} + \cdot W\cdot K^{2-\beta} + \cdot Z\cdot K^{1-\beta})$$
$$+ \frac{\varepsilon\gamma^2 K^{2-\beta}\,M}{4\tilde{\vartheta}} + \frac{\varepsilon K^{2-\beta}\,M}{\tilde{\vartheta}\varrho^2} \tag{9.59}$$

and consequently

$$\int\limits_{B_\varrho(x_0)} \frac{(\varepsilon+|\nabla u|^{p-2})|\nabla u_i|^2}{|u_i|^\beta|x-y|^\gamma} \leq \int\limits_{B_{2\varrho}(x_0)} \frac{(\varepsilon+|\nabla u|^{p-2})|\nabla u_i|^2}{|u_i|^\beta|x-y|^\gamma}\varphi^2 \leq$$
$$\mathcal{C}(p\,,\,\gamma\,,\,\beta\,,\,f\,,\,r\,,\,\|u\|_\infty\,,\,\|\nabla u\|_\infty\,,\,\varrho,\,x_0\,,\,\varepsilon)\,, \tag{9.60}$$

where

$$\mathcal{C}(p\,,\,\gamma\,,\,\beta\,,\,f\,,\,r\,,\,\|u\|_\infty\,,\,\|\nabla u\|_\infty\,,\,\varrho,\,x_0\,,\,\varepsilon) = \frac{2\,M\cdot\max\{(p-1)\,,\,1\}}{1-\beta}\times$$
$$\left(\frac{\gamma^2\cdot K^{p-\beta}}{4\vartheta} + \frac{K^{p-\beta}}{\vartheta\varrho^2} + \frac{K^{1-\beta}}{p-1}(V+W\cdot K+Z) + \frac{\varepsilon K^{2-\beta}}{\tilde{\vartheta}}(\frac{\gamma^2}{4}+\frac{1}{\varrho^2})\right)$$

which implies the thesis recalling the choice $\vartheta = \frac{1-\beta}{4\max\{(p-1)\,,1\}}$ and $\tilde{\vartheta} = \frac{1-\beta}{4}$. □

Proof of Theorem 4.2. We get from Theorem 4.1 that

$$\int\limits_{B_\varrho(x_i)} \frac{|\nabla u|^{p-2-\beta}|\nabla u_i|^2}{|x-y|^\gamma} + \varepsilon\int\limits_{B_\varrho(x_i)} \frac{|\nabla u_i|^2}{|\nabla u|^\beta|x-y|^\gamma}$$
$$\leq \frac{2\,M\cdot\max\{(p-1)\,,\,1\}^2}{1-\beta}$$
$$\times\left(\frac{\gamma^2\cdot K^{p-\beta}}{1-\beta} + \frac{4K^{p-\beta}}{(1-\beta)\varrho^2} + \frac{K^{1-\beta}}{p-1}(V+W\cdot K+Z) + \frac{\varepsilon 4K^{2-\beta}}{(1-\beta)}(\frac{\gamma^2}{4}+\frac{1}{\varrho^2})\right)$$
$$\leq \frac{2\,\overline{M}\cdot\max\{(p-1)\,,\,1\}^2}{1-\beta}$$
$$\times\left(\frac{\gamma^2\cdot \overline{K}^{p-\beta}}{1-\beta} + \frac{4\overline{K}^{p-\beta}}{(1-\beta)\varrho^2} + \frac{\overline{K}^{1-\beta}}{p-1}(\overline{V}+\overline{W}\cdot\overline{K}+\overline{Z}) + \frac{\varepsilon 4\overline{K}^{2-\beta}}{(1-\beta)}(\frac{\gamma^2}{4}+\frac{1}{\varrho^2})\right)$$

which gives

$$
\begin{aligned}
&\int\limits_{\Omega\setminus\mathcal{N}(3\delta,\partial\Omega)} \frac{|\nabla u|^{p-2-\beta}|\nabla u_i|^2}{|x-y|^{\gamma}} + \varepsilon \int\limits_{\Omega\setminus\mathcal{N}(3\delta,\partial\Omega)} \frac{|\nabla u_i|^2}{|\nabla u|^{\beta}|x-y|^{\gamma}} \\
\leq &\sum_{i=1}^{S} \int\limits_{B_\varrho(x_i)} \frac{|\nabla u|^{p-2-\beta}|\nabla u_i|^2}{|x-y|^{\gamma}} + \varepsilon \int\limits_{B_\varrho(x_i)} \frac{|\nabla u_i|^2}{|\nabla u|^{\beta}|x-y|^{\gamma}} \\
\leq &\frac{2\,S\,\overline{M}\cdot\max\{(p-1)\,,\,1\}^2}{1-\beta} \\
&\times\left(\frac{\gamma^2\cdot\overline{K}^{p-\beta}}{1-\beta} + \frac{4\overline{K}^{p-\beta}}{(1-\beta)\varrho^2} + \frac{\overline{K}^{1-\beta}}{p-1}(\overline{V}+\overline{W}\cdot\overline{K}+\overline{Z}) + \frac{\varepsilon 4\overline{K}^{2-\beta}}{(1-\beta)}(\frac{\gamma^2}{4}+\frac{1}{\varrho^2})\right).
\end{aligned}
\tag{9.61}
$$

Also it is easy to see that

$$
\begin{aligned}
&\int\limits_{\mathcal{N}(3\delta,\partial\Omega)} \frac{|\nabla u|^{p-2-\beta}|\nabla u_i|^2}{|x-y|^{\gamma}} + \varepsilon \int\limits_{\mathcal{N}(3\delta,\partial\Omega)} \frac{|\nabla u_i|^2}{|\nabla u|^{\beta}|x-y|^{\gamma}} \leq \\
&\leq \max\{\overline{K}^{p-2-\beta}\,;\,\overline{\lambda}^{p-2-\beta}\,;\,\overline{K}^{-\beta}\,;\,\overline{\lambda}^{-\beta}\}\cdot\overline{L}^2\cdot\overline{M}\,.
\end{aligned}
\tag{9.62}
$$

By (9.61) and (9.62) and the fact that

$$
\begin{aligned}
&\int\limits_{\Omega} \frac{|\nabla u|^{p-2-\beta}|\nabla u_i|^2}{|x-y|^{\gamma}} + \varepsilon \int\limits_{\Omega} \frac{|\nabla u_i|^2}{|\nabla u|^{\beta}|x-y|^{\gamma}} \leq \\
&\int\limits_{\Omega\setminus\mathcal{N}(3\delta,\partial\Omega)} \frac{|\nabla u|^{p-2-\beta}|\nabla u_i|^2}{|x-y|^{\gamma}} + \varepsilon \int\limits_{\Omega\setminus\mathcal{N}(3\delta,\partial\Omega)} \frac{|\nabla u_i|^2}{|\nabla u|^{\beta}|x-y|^{\gamma}} + \\
&\int\limits_{\mathcal{N}(3\delta,\partial\Omega)} \frac{|\nabla u|^{p-2-\beta}|\nabla u_i|^2}{|x-y|^{\gamma}} + \varepsilon \int\limits_{\mathcal{N}(3\delta,\partial\Omega)} \frac{|\nabla u_i|^2}{|\nabla u|^{\beta}|x-y|^{\gamma}}\,,
\end{aligned}
\tag{9.63}
$$

it follows (4.17) and the thesis. □

Data availability

No data was used for the research described in the article.

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