A Bochner like theorem about infinitesimal contact automorphisms*

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Abstract

We prove that on a compact contact manifold there are no infinitesimal contact automorphisms, apart from the scalar multiples of the Reeb vector field, provided the manifold carries an admissible Riemannian metric whose Jacobi operator is negative definite on the contact subbundle.

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1 Introduction

The classical Bochner theorem states that on a compact Riemiannan manifold with negative definite Ricci tensor there are no non trivial infinitesimal isometries (Killing vector fields). In this note we treat a similar result concerning compact contact manifolds, showing that the presence of a suitable Riemannian metric with a sufficient amount of negative curvature forces the absence of non trivial *infinitesimal contact automorphisms*. Recall that if (M, η) is a contact manifold, with contact form η, an infinitesimal contact automorphism is a vector field X preserving η, i.e. a vector field such that

$$\mathcal{L}_X \eta = 0,$$

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where $\mathcal{L}$ is the Lie derivative (for more information on the subject, see [1], [7] and also [4]).

In order to formulate a more precise statement of our result, we recall that on (M, η) there is a distinguished vector field ξ transverse to the contact subbundle $D := \mathrm{Ker}(\eta)$, called the *Reeb vector field* of the contact form η; it is the unique globally defined vector field such that:

$$i_\xi d\eta = 0, \qquad \eta(\xi) = 1,$$

where i_ξ denotes the interior product. The Reeb vector field of a contact form η will be also denoted by ξ_η. From these conditions, it follows that ξ is an infinitesimal contact automorphism of (M, η). Our interest is to investigate a geometric setting forcing the infinitesimal automorphisms to reduce to the scalar multiples of the Reeb vector field.

We shall consider a natural class of Riemannian metrics on (M, η), i.e. those metrics such that ξ is of unit lenght and orthogonal to D. More explicitly a metric g is said to be *admissible to the contact form* provided:

$$g(\xi, \xi) = 1, \ g(\xi, X) = 0$$

for every smooth section X of the contact subbundle D.

This condition can be reformulated as the requirement that the contact form and its Reeb vector field should be dual with respect to g, that is

$$\xi_\eta = \eta^\sharp.$$

Here $\sharp : \Lambda^1(M) \to \mathfrak{X}(M)$ is the natural isomorphism between 1-forms and vector fields induced by the Riemannian metric by "raising the index", namely for every 1-form $\theta \in \Lambda^1(M)$, $\theta^\sharp \in \mathfrak{X}(M)$ denotes the unique vector field such that:

$$g(\theta^\sharp, X) = \theta(X)$$

for every $X \in \mathfrak{X}(M)$.

Notice that, because of the global splitting

$$TM = D \oplus [\xi], \tag{1.1}$$

$[\xi]$ being the 1-dimensional distribution spanned by ξ, these metrics abound and arise naturally as extensions of sub-Riemannan metrics defined on the contact subbundle. The class of admissible metrics includes the *associated metrics*, fully studied in the literature, cf. the textbook [2] by D. Blair. Here we recall that a metric g is *associated to a contact form* η if there exists a $(1,1)$ tensor field φ such that

$$\varphi^2 = -Id + \eta \otimes \xi, \quad \eta(X) = g(X, \xi), \quad d\eta(X,Y) = g(X, \varphi Y)$$

for every X, Y vector fields on M. The tensors (φ, ξ, η, g) make up a *contact metric structure* on M and (M, η, g) is called a *contact metric manifold.* Every contact form carries infinitely many associated metrics, which can be constructed by means of a polarization process, starting from the symplectic form induced by $d\eta$ on the contact distribution (see [2, Chapter 4]).

In our discussion, the role of the Ricci tensor will be played by the *Jacobi operator* $l = l(g, \eta)$ determined by the pair (g, η), which is the bundle endomorphism

$$l : TM \to TM$$

defined by

$$lX := R(X, \xi)\xi,$$

where R is the curvature tensor of an admissible metric g; it is a symmetric operator. We remark that, since by definition $l(\xi) = 0$, and the splitting (1.1) is orthogonal with respect to the metric, l restricts canonically to a bundle endomorphism $l : D \to D$.

With these notations and terminology, our result is the following.

Theorem 1.1. *Let (M, η) be a compact contact manifold. Assume that M admits an admissible Riemannian metric g for which the Jacobi operator*

$$l : D \to D$$

of (g, η), restricted to the contact distribution, is negative definite. Then the infinitesimal contact automorphisms of (M, η) are exactly the vector fields $\alpha\xi_\eta$, with $\alpha \in \mathbb{R}$.

It should be remarked that, due to a deep result of A. Zeghib on geodesic line fields, no *negatively curved* associated metric can exist on a compact contact manifold: see [2, Corollary 7.2]. His result is applicable in this context because the integral curves of the Reeb vector field of every contact metric manifold are always geodesics. The same facts are true also for the larger class of admissible metrics in our sense (concerning the geodesics of ξ_η, see Proposition 2.2 in Section 2). This motivates trying to get a better understanding of compact contact metric manifolds with some amount of negative curvature; Theorem 1.1 is a first step in this direction, yielding an obstruction for negative sectional curvature for plane sections containing ξ (negative ξ-sectional curvature).

Familiar examples of compact contact metric manifolds with negative ξ-sectional curvature are the compact quotients $\widetilde{SL(2,\mathbb{R})}/\Gamma$, Γ being a discrete subgroup of the universal covering group of $SL(2,\mathbb{R})$, and the tangent sphere bundle $T_1M(c)$ of any compact Riemannian manifold $M(c)$ with negative constant curvature $-1 < c < 0$.

More precisely, in the first case, one can consider any contact metric structure induced by a left-invariant contact metric structure on $\widetilde{SL(2,\mathbb{R})}$, while in the second case there exists a suitable D-homothetic deformation of the standard contact metric structure on $T_1M(c)$, chosen in such a way that $l : D \to D$ coincides with the operator κId_D, where $\kappa < 0$ is a constant. In the terminology used in Blair's book, this example is a contact metric $(\kappa, 0)$-manifold. For more details and information about the terms not explained here, see secton 7.3 of [2]. For a discussion of geometric properties of contact metric manifolds with negative definite Jacobi operator $l : D \to D$, related to the notion of Anosow flow, we also refer the reader to Chapter 11 in [2].

As an application of Theorem 1.1, we consider the case of regular contact manifolds. Recall that a regular contact form is one that determines a regular 1-dimensional foliation on M, i.e. the space $B = M/\xi$ of maximal integral curves of the Reeb vector field is a manifold. It is known that the class of regular contact manifolds includes the homogeneous ones, i.e. those admitting a smooth transitive Lie group action preserving the contact form (for more details, see [3]).

Corollary 1.2. *Let (M, η) be a compact contact manifold. If M admits an admissible Riemannian metric such that the Jabobi operator $l : D \to D$ of (η, g) is negative definite, then the contact form η is not regular. In particular, (M, η) is not a homogeneous contact manifold.*

Corollary 1.3. *There exist no homogeneous compact contact metric manifolds with negative ξ-sectional curvature.*

These corollaries follows immediately from Theorem 1.1, since every regular contact manifold always admits infinitesimal contact automorphisms which are not scalar multiples of the Reeb vector field. Indeed, it is known that, in the general case, the mapping

$$X \mapsto \eta(X) \tag{1.2}$$

is a bijection between the set of contact vector fields X on a contact manifold (M, η), i.e. smooth vector fields satisfying

$$\mathcal{L}_X \eta = f\eta, \qquad f \in C^\infty(M) \tag{1.3}$$

and the set $C^\infty(M)$ of smooth functions on M. See for instance [1, Proposition II.3] or [7, §3]. If X is a contact vector field, then the function f in (1.3) is $\xi\eta(X)$.

In particular, the map (1.2) induces a bijection between infinitesimal contact automorphisms and first integrals of the Reeb vector field, i.e. smooth functions $h : M \to \mathbb{R}$ such that $\xi h = 0$. In particular, the scalar multiples $\alpha\xi$, $\alpha \in \mathbb{R}$ of ξ correspond to the constant functions.

If (M, η) is regular, denoting by $\pi : M \to B$ the canonical projection, every non constant smooth function $g : B \to \mathbb{R}$ lifts to a non constant function $h = g \circ \pi$ determining an automorphism which is not a scalar multiple of the Reeb vector field.

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2 Proof of the result

We start by discussing the following general fact. Given a vector field ξ on a Riemannian manifold (M, g) whose Levi-Civita connection is ∇, we shall denote by A the $(1, 1)$ tensor field defined by

$$A(X) := -\nabla_X \xi$$

for every smooth vector field X. This definition is coherent with the one adopted in [5] and [6], where A is denoted by A_ξ. The covariant derivative $\nabla_Z A = [\nabla_Z, A]$ of A with respect to any vector field Z is still a (1,1) tensor field. Denoting by R the curvature tensor of ∇, it is known that, if X, Y are arbitrary vector fields, then

$$R(X, Y)\xi = (\nabla_Y A)X - (\nabla_X A)Y. \tag{2.1}$$

For a proof, which is a straightforward computation based on the definition of the curvature tensor, namely $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, see e.g. [6], p. 533.

With this notation, we can state:

Proposition 2.1. *Let ξ be a vector field on a Riemannian manifold (M, g), whose integral curves are geodesics. Let ∇ be the Levi-Civita connection of g. Then the Jacobi operator $l := R(\cdot, \xi)\xi$ of g determined by ξ and the (1,1) tensor field A are related as follows:*

$$l = \nabla_\xi A - A^2. \tag{2.2}$$

Proof. Choosing $Y = \xi$ in (2.1) we get:

$$lX = (\nabla_\xi A)X - \nabla_X A\xi + A(\nabla_X \xi) = (\nabla_\xi A)X - A^2 X,$$

because $A\xi = -\nabla_\xi \xi = 0$ by the assumption that the integral curves of ξ are geodesics.

$\square$

Proposition 2.2. *Let (M, η) be a contact manifold and let g be an admissible metric on it. Then the integral curves of the Reeb vector field ξ are geodesics with respect to g.*

Proof. We denote again by ∇ the Levi-Civita connection with respect to g. Since, by the definition of an admissible metric, ξ is a unit vector field, we have $\xi g(\xi, \xi) = 0$, so that $g(\nabla_\xi \xi, \xi) = 0$, thus it suffices to prove that $g(\nabla_\xi \xi, X) = 0$ for every section X of the contact subbundle. This can be readily checked by means of the Koszul formula, using the assumption that ξ is othogonal to $D = Ker(\eta)$ with respect to g and the fact that $[\xi, X] \in D$, since ξ is an infinitesimal contact automorphism. $\square$

Now we are ready to prove Theorem 1.1. Let X be an infinitesimal contact automorphism of (M, η). It suffices to prove that for each $p \in M$, we have $X_p \in \mathbb{R}\xi_p$. Indeed, this implies $X = \alpha\xi$ where $\alpha \in C^\infty(M)$ is a smooth function; but α must actually be a constant, because of $\mathcal{L}_X \eta = d\alpha = 0$.

Arguing by contradiction, assume that there exists a point $p \in M$ such that $X_p \notin \mathbb{R}\xi_p$ and let

$$\gamma : \mathbb{R} \to M$$

be the maximal integral curve of ξ passing through p. Consider the function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(t) = \frac{1}{2} g_{\gamma(t)}(X_{\gamma(t)}, X_{\gamma(t)}),$$

that is

$$F = \frac{1}{2} g(X, X) \circ \gamma.$$

Observe that, since M is compact, F is bounded. Now, we remark that, since X preserves η, its flow must also preserve ξ, i.e. $[X, \xi] = 0$; using this, computing F' we get, taking into account that γ is an integral curve of ξ:

$$F' = \frac{1}{2}\xi g(X, X) \circ \gamma = g(\nabla_\xi X, X) \circ \gamma = g(\nabla_X \xi, X) \circ \gamma = -g(AX, X) \circ \gamma$$

where A denotes the (1,1) tensor field considered in Proposition 2.1.

Next we compute F'':

$$\begin{aligned} F'' &= -\xi g(AX, X) \circ \gamma = -(g(\nabla_\xi AX, X) + g(AX, \nabla_\xi X)) \circ \gamma \\ &= -(g((\nabla_\xi A)X, X) + g(A\nabla_\xi X, X) + g(AX, \nabla_\xi X)) \circ \gamma. \end{aligned}$$

Using again $[X, \xi] = 0$ and (2.2), which is applicable according to Proposition 2.2, we can rewrite this as follows:

$$F'' = (-g((l + A^2)X, X) + g(A^2X, X) + g(AX, AX)) \circ \gamma,$$

and we can conclude that:

$$F''(t) = -g_{\gamma(t)}(lX_{\gamma(t)}, X_{\gamma(t)}) + ||AX_{\gamma(t)}||^2. \tag{2.3}$$

According to the assumption, for each $x \in M$, the operator $l_x : D_x \to D_x$ restricted to D_x is negative definite; since $l_x : T_xM \to T_xM$ is symmetric and $l_x(\xi) = 0$, this implies that $l_x : T_xM \to T_xM$ is negative semidefinite; thus we obtain that $F'' \geq 0$. Hence, since F is bounded, it must be constant. But (2.3) yieds:

$$F''(0) = -g(lX_p, X_p) + ||AX_p||^2.$$

This implies $F''(0) > 0$; indeed, according to the splitting (1.1) we can write

$$X_p = Z_p + \eta(X_p)\xi_p$$

where $Z_p \in D_p$ and $Z_p \neq 0$, because $X_p \notin \mathbb{R}\xi_p$; hence:

$$g(lX_p, X_p) = g(lZ_p, Z_p) < 0,$$

taking into account that $g(Z_p, \xi_p) = 0$.

We have arrived at a contradiction.

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