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Generalized Quasilinear Elliptic Equations in $\mathbb{R}^N$

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Abstract. In this paper, we aim at establishing a new existence result for the quasilinear elliptic equation

$$-\mathrm{div}(a(x,u,\nabla u)) + A_t(x,u,\nabla u) + V(x)|u|^{p-2}u = g(x,u) \qquad \text{in } \mathbb{R}^N$$

with $p > 1$, $N \geq 2$ and $V : \mathbb{R}^N \to \mathbb{R}$ suitable measurable positive function. Here, we suppose $A : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a given C^1-Carathéodory function which grows as $|\xi|^p$, with $A_t(x,t,\xi) = \frac{\partial A}{\partial t}(x,t,\xi)$, $a(x,t,\xi) = \nabla_\xi A(x,t,\xi)$, $V : \mathbb{R}^N \to \mathbb{R}$ is a suitable measurable function and $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a given Carathéodory function which grows as $|\xi|^q$ with $1 < q < p$. Since the coefficient of the principal part depends on the solution itself, under suitable assumptions on $A(x,t,\xi), V(x)$ and $g(x,t)$, we study the interaction of two different norms in a suitable Banach space with the aim of obtaining a good variational approach. Thus, a minimization argument on bounded sets can be used to state the existence of a nontrivial weak bounded solution on an arbitrary bounded domain. Then, one nontrivial bounded solution of the given equation can be found by passing to the limit on a sequence of solutions on bounded domains. Finally, under slightly stronger hypotheses, we can able to find a positive solution of the problem.

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1. Introduction

In this paper, we aim at investigating the existence of a weak bounded solution for the generalized quasilinear Schrödinger equation

$$-\mathrm{div}(a(x,u,\nabla u)) + A_t(x,u,\nabla u) + V(x)|u|^{p-2}u = g(x,u) \quad \text{in}\mathbb{R}^N \tag{1.1}$$

where $p > 1$ and $N \geq 2$, $A : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is C^1-Carathéodory function with partial derivatives

$$A_t(x,t,\xi) = \frac{\partial A}{\partial t}(x,t,\xi), \quad a(x,t,\xi) = \left(\frac{\partial A}{\partial \xi_1}(x,t,\xi), ..., \frac{\partial A}{\partial \xi_N}(x,t,\xi)\right) \tag{1.2}$$

and $V : \mathbb{R}^N \to \mathbb{R}$ and $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are suitable measurable functions.

We note that in the very special case $A(x,t,\xi) = \mathcal{A}|\xi|^p$ with $\mathcal{A}$ real constant, equation (1.1) turns out to be the p-Laplacian equation

$$-\Delta_p u + V(x)|u|^{p-2}u = g(x,u) \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

In the particular case $p = 2$, equation (1.3) reduces to the following Schrödinger equation

$$-\Delta u + V(x)u = g(x,u) \quad \text{in } \mathbb{R}^N$$

which is a central topic in Nonlinear Analysis (see, e.g., [4, 6, 20, 21, 24, 31, 32]).

Equation (1.3) has been widely studied also in the general case $p > 1$ (see, e.g., [3, 5, 26, 29]).

It is worth of knowing that equation (1.3) has a variational structure, but there is a lack of compactness as such a problem is settled in the whole Euclidean space $\mathbb{R}^N$ and classical variational tools do not work easily; thus, suitable assumptions on potential $V(x)$ and on the function g are required.

More recently, many authors investigated equation (1.1) when the function $A(x,t,\xi)$ has the particular form $\frac{1}{p}\mathcal{A}(x,t)|\xi|^p$ with the coefficient $\mathcal{A}$ not constant. In this case, besides the difficulties arising for the problem (1.3) due to the lack of compactness, the presence of a coefficient which depends on the solution itself (and in the general case $A(x,t,\xi)$ depends also on the derivatives of the solution) makes the variational approach more complicated.

In fact, in this case, under suitable assumption on A, V and g, the "natural" functional associated to (1.3) is

$$\mathcal{I}(u) = \frac{1}{p}\int_{\mathbb{R}^N} \mathcal{A}(x,u)|\nabla u|^p dx + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p dx - \int_{\mathbb{R}^N} G(x,u)dx$$

with $G(x,t) = \displaystyle\int_0^t g(x,s)ds$, but, even if $\mathcal{A}(x,t)$ is a smooth strictly positive bounded function, if $\mathcal{A}_t(x,t) \not\equiv 0$ functional $\mathcal{I}$ is well defined in $W^{1,p}(\mathbb{R}^N)$, while it is Gâteaux differentiable only along directions in $X = W^{1,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

More in general, if $A(x,t,\xi)$ grows as $|\xi|^p$ with respect to ξ, where $p > 1$, suitable additional growth assumptions on the functions involved guarantee that to the equation (1.1) can be associated the functional

$$\mathcal{J}(u) = \int_{\mathbb{R}^N} A(x,u,\nabla u)dx + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p dx - \int_{\mathbb{R}^N} G(x,u)dx$$

defined on a subset $\mathcal{D}$ of $W^{1,p}(\mathbb{R}^N)$ which contains X.

The loss of regularity for $\mathcal{J}$ has been overcome by introducing suitable definitions of critical point (see, e.g., [1,2,9,19,23]), or by using a suitable change of variable but only if the coefficient $A(x,t,\xi)$ has a very particular form (see, e.g., [22,28,33,34]).

More recently, following a different approach developed in [11–13], it has been proved that, if $g(x,\cdot)$ has a super-p-linear but subcritical growth, functional J is C^1 in the Banach space X equipped with the intersection norm, then abstract results can be applied since a "weaker compactness" condition holds (see, Definition 2.1). Hence, the existence of at least one bounded solution of (1.1) in the case $A(x,t,\xi) = \frac{1}{p}\mathcal{A}(x,t)|\xi|^p$ has been stated if V verifies suitable assumptions in [18] or if $V \equiv 1$ and all the involved functions are radially symmetric in [17], while in [14] an existence result holds for a general term $A(x,t,\xi)$ when the involved functions are $1-$periodic in x.

Using the same variational approach, in [30], we state the existence of a weak bounded solution for (1.1) when $A(x,t,\xi) = \frac{1}{p}\mathcal{A}(x,t)|\xi|^p$ and V are as in [18] and $g(x,t) = \eta(x)|t|^{q-2}t$ with $\eta : \mathbb{R}^N \to \mathbb{R}$ a suitable measurable function and $1 < q < p$. In this paper, we want to generalize this result considering the term $A(x,t,\xi)$ and a Carathéodory function g having a sub-p-linear growth of the type

$$|g(x,t)| \leq \eta(x)|t|^{q-1}$$

with η and q as above.

The paper is organized as follows. In Sect. 2, we introduce the abstract framework and we recall a weaker version of the Cerami's variant of the Palais-Smale condition and the related Minimum Principle (see, Proposition 2.2). In Section 3 we introduce some preliminary assumptions on the functions $A(x,t,\xi)$, $V(x)$ and $\eta(x)$ which allow to give a variational principle for equation (1.1). In Sect. 4, we consider some further hypotheses, we state our main result (see, Theorem 4.4) and we prove some properties for the action functional associated to the equation (1.1). Then, in Sect. 5, we prove Theorem 4.4 by means of a method of approximation on bounded domains, while in Sect. 6, a slightly strong assumption on the function $A(x,t,\xi)$ allows us to state the existence of a nontrivial weak positive solution.

2. Abstract setting

In this section, we assume that

- $(X, \|\cdot\|_X)$ is a Banach space with dual $(X', \|\cdot\|_{X'})$;
- $(W, \|\cdot\|_W)$ is a Banach space such that $X \hookrightarrow W$ continuously, i.e., $X \subset W$ and a constant $\sigma_0 > 0$ exists such that

$$\|u\|_W \ \leq \ \sigma_0 \ \|u\|_X \qquad \text{for all } u \in X;$$

- $J : \mathcal{D} \subset W \to \mathbb{R}$ and $J \in C^1(X,\mathbb{R})$ with $X \subset \mathcal{D}$.

Nevertheless, to avoid any ambiguity, we will henceforth denote by X the space equipped with its norm $\|\cdot\|_X$, while, if the norm $\|\cdot\|_W$ is involved, we will write it explicitly.

Taking $\beta \in \mathbb{R}$, we say that a sequence $(u_n)_n \subset X$ is a *Cerami–Palais–Smale sequence at level* β, briefly $(CPS)_\beta$*–sequence*, if

$$\lim_{n\to+\infty} J(u_n) = \beta \quad \text{and} \quad \lim_{n\to+\infty} \|dJ(u_n)\|_{X'}(1+\|u_n\|_X) = 0.$$

Moreover, β is a Cerami–Palais–Smale level, briefly (CPS)*–level*, if there exists a $(CPS)_\beta$–sequence. The functional J satisfies the classical Cerami–Palais–Smale condition in X at the level β if every sequence $(CPS)_\beta$–sequence converges in X up to subsequences. However, considering the setting of our problem, in general a $(CPS)_\beta$–sequence may also exist which is unbounded in $\|\cdot\|_X$ but which converges with respect to $\|\cdot\|_W$. Then, we are able to weaken the classical Cerami–Palais–Smale condition in an appropriate way according to ideas that have already been developed in previous papers (see, for example, [11–13]).

Definition 2.1. The functional J satisfies the weak Cerami–Palais–Smale condition at level β ($\beta \in \mathbb{R}$), briefly $(wCPS)_\beta$ condition, if for every $(CPS)_\beta$–sequence $(u_n)_n$, a point $u \in X$ exists such that

(i) $\lim_{n\to+\infty} \|u_n - u\|_W = 0$ (up to subsequences),

(ii) $J(u) = \beta,\ dJ(u) = 0.$

If J satisfies the $(wCPS)_\beta$ condition at each level $\beta \in I$, I real interval, we say that J satisfies the $(wCPS)$ condition in I.

Let us point out that, due to the convergence only in the norm of W, the $(wCPS)_\beta$ condition implies that the set of critical points of J at the β level is compact with respect to $\|\cdot\|_W$, so that we can state a Deformation Lemma and some abstract theorems about critical points (see [13]). In particular, the following Minimum Principle applies (for the proof, (see, [13, Theorem 1.6])).

Proposition 2.2. *(Minimum Principle) If $J \in C^1(X,\mathbb{R})$ is bounded from below in X and $(wCPS)_\beta$ holds at level $\beta = \inf_X J \in \mathbb{R}$, then J attains its infimum, i.e., $u_0 \in X$ exists such that $J(u_0) = \beta$.*

3. Variational framework and regularity result

Let $\mathbb{N} = \{1, 2, ...\}$ be the set of the strictly positive integers and, taking any Ω open subset of $\mathbb{R}^N$, $N \ge 2$, we denote by:

- $B_R(x) = \{y \in \mathbb{R}^N : |x-y| < R\}$ the open ball in $\mathbb{R}^N$ with center in $x \in \mathbb{R}^N$ and radius $R > 0$;
- $|D|$ the usual Lebesgue measure of a measurable set D in $\mathbb{R}^N$;
- $L^r(\Omega)$ the Lebesgue space endowed with norm $|u|_{\Omega,r} = \left(\int_\Omega |u|^r dx\right)^{\frac{1}{r}}$ if $1 \le r < +\infty$;
- $L^\infty(\Omega)$ the space of Lebesgue–measurable and essentially bounded functions $u : \Omega \to \mathbb{R}$ with norm $|u|_{\Omega,\infty} = \operatorname{ess\,sup}_\Omega |u|$;

- $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ the classical Sobolev spaces both equipped with the standard norm $\|u\|_\Omega = (|\nabla u|_{\Omega,p}^p + |u|_{\Omega,p}^p)^{\frac{1}{p}}$ if $1 \le p < +\infty$.

Moreover, if $V : \mathbb{R}^N \to \mathbb{R}$ is a measurable function such that

$$(V_1) \qquad V_0 = \text{ess}\inf_{\mathbb{R}^N} V(x) > 0,$$

we denote by

- $L_V^r(\Omega)$ the weighted Lebesgue space

$$L_V^r(\Omega) = \left\{u \in L^r(\Omega) : \int_\Omega V(x)|u|^r dx < +\infty\right\} \quad \text{for} \quad 1 \le r < +\infty,$$

equipped with the norm

$$|u|_{\Omega,V,r} = \left(\int_\Omega V(x)|u|^r dx\right)^{\frac{1}{r}};$$

- $W_V^{1,p}(\Omega)$ and $W_{0,V}^{1,p}(\Omega)$, if $1 \le r < +\infty$ the weighted Sobolev spaces

$$W_V^{1,p}(\Omega) = \left\{u \in W^{1,p}(\Omega) : \int_\Omega V(x)|u|^p dx < +\infty\right\},$$
$$W_{0,V}^{1,p}(\Omega) = \left\{u \in W_0^{1,p}(\Omega) : \int_\Omega V(x)|u|^p dx < +\infty\right\},$$

both equipped with the norm

$$\|u\|_{\Omega,V} = \left(\int_\Omega |\nabla u|^p + V(x)|u|^p dx\right)^{\frac{1}{p}} = (|\nabla u|_{\Omega,p}^p + |u|_{\Omega,V,p}^p)^{\frac{1}{p}}. \tag{3.1}$$

For simplicity, we put $B_R = B_R(0)$ and, if $\Omega = \mathbb{R}^N$, we omit the subscript in the previous notations, i.e., we put

- $|\cdot|_r = |\cdot|_{\mathbb{R}^N,r}$ for the norm in $L^r(\mathbb{R}^N)$, for all $1 \le r < +\infty$;
- $|\cdot|_{V,r} = |\cdot|_{\mathbb{R}^N,V,r}$ for the norm in $L_V^r(\mathbb{R}^N)$, for all $1 \le r < +\infty$;
- $\|\cdot\| = \|\cdot\|_{\mathbb{R}^N}$ for the norm in $W^{1,p}(\mathbb{R}^N)$ and in $W_0^{1,p}(\mathbb{R}^N)$;
- $\|\cdot\|_V = \|\cdot\|_{\mathbb{R}^N,V}$ for the norm in $W_V^{1,p}(\mathbb{R}^N) = W_{0,V}^{1,p}(\mathbb{R}^N)$.

Remark 3.1. By assumption (V_1), the following continuous embeddings hold:

$$L_V^r(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N) \quad \text{for all } 1 \le r < +\infty \tag{3.2}$$

and

$$W_V^{1,p}(\mathbb{R}^N) \hookrightarrow W^{1,p}(\mathbb{R}^N) \quad \text{for all } 1 \le p < +\infty. \tag{3.3}$$

From now on, we suppose that V satisfies also the following additional hypothesis:

$$(V_2) \qquad \int_{B_1(x)} \frac{1}{V(y)} dy \longrightarrow 0 \quad \text{as} \quad |x| \to +\infty.$$

From Remark 3.1 and Sobolev Embedding Theorems, we deduce the following result (for the compact embeddings, we refer to [7, Theorem 3.1]).

Proposition 3.2. *Let $V : \mathbb{R}^N \to \mathbb{R}$ be a Lebesgue measurable function satisfying assumption (V_1).*

Then, the following continuous embeddings hold:

- *if* $p < N$, *then*
$$W_V^{1,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N) \text{ for any } p \le r \le \frac{Np}{N-p}; \tag{3.4}$$
- *if* $p = N$, *then*
$$W_V^{1,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N) \text{ for any } p \le r < +\infty; \tag{3.5}$$
- *if* $p > N$, *then*
$$W_V^{1,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N) \text{ for any } p \le r \le +\infty. \tag{3.6}$$

Furthermore, if assumption (V_2) *also occurs, the compact embedding*
$$W_V^{1,p}(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^r(\mathbb{R}^N) \text{ for any } p \le r < p^* \tag{3.7}$$
holds, with
$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \ge N. \end{cases}$$

From Proposition 3.2, it follows that for any $r \ge p$ as in (3.4), respectively (3.5) or (3.6), a constant $\tau_r > 0$ exists such that
$$|u|_r \le \tau_r \|u\|_V \text{ for all } u \in W_V^{1,p}(\mathbb{R}^N). \tag{3.8}$$
On the other hand, if we consider the case in which Ω is an open bounded domain of class C^1 in $\mathbb{R}^N$ with $\partial\Omega$ bounded and $p < N$, from a classical embedding Theorem (see, e.g., [10, Corollary 9.14]) a constant $\sigma^* > 0$, independent of Ω and depending only on p and N, exists such that
$$|u|_{\Omega,p^*} \le \sigma^* \|u\|_\Omega \text{ for all } u \in W_0^{1,p}(\Omega).$$
From now on, we assume that V is a measurable function verifying (V_1) and we set
$$X = W_V^{1,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \text{ with } \|u\|_X = \|u\|_V + |u|_\infty \text{ for any } u \in X.$$
In the following, we assume $p \le N$ as, otherwise, from (3.6), it follows that $X = W_V^{1,p}(\mathbb{R}^N)$ and all the arguments and the proofs can be simplified.

The following lemmas hold.

Lemma 3.3. *Let* $V : \mathbb{R}^N \to \mathbb{R}$ *be a Lebesgue measurable function satisfying* (V_1). *For any* $r \ge p$, *the Banach space* X *is continuously embedded in* $L_V^r(\mathbb{R}^N)$, *i.e.,* $X \hookrightarrow L_V^r(\mathbb{R}^N)$. *More precisely, it results*
$$|u|_{V,r} \le \|u\|_X \text{ for all } u \in X. \tag{3.9}$$

Proof. For the proof, (see, [18, Lemma 3.3]). □

Remark 3.4. Observe that, thanks to (3.2), Lemma 3.3 ensures that
$$X \hookrightarrow L^r(\mathbb{R}^N) \text{ for any } p \le r \le +\infty.$$

Lemma 3.5. *If* $(u_n)_n \subset X, u \in X$ *and* $M > 0$ *are such that*

$$\|u_n - u\|_V \to 0 \qquad \text{as } n \to +\infty \tag{3.10}$$

and

$$|u_n|_\infty \leq M \qquad \text{for all } n \in \mathbb{N}, \tag{3.11}$$

then,

$$u_n \to u \qquad \text{in } L^r_V(\mathbb{R}^N) \qquad \text{for any } p \leq r < +\infty.$$

Proof. For the proof, (see, [18, Lemma 3.5]). □

Here and in the following, let $A : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be such that, using the notation in (1.2), the following hypotheses hold:

(h_0) A is a C^1–Carathéodory function, i.e., $A(\cdot, t, \xi)$ is measurable for all $(t, \xi) \in \mathbb{R} \times \mathbb{R}^N$, and $A(x, \cdot, \cdot)$ is C^1 for a.e. $x \in \mathbb{R}^N$;

(h_1) some positive continuous functions $\Phi_i, \phi_i : \mathbb{R} \to \mathbb{R}$, $i \in \{0, 1, 2\}$, exist such that:

$$\begin{aligned}
|A(x,t,\xi)| &\leq \Phi_0(t)|t|^p + \phi_0(t)|\xi|^p && \text{a.e. in } \mathbb{R}^N, \text{ for all } (t,\xi) \in \mathbb{R} \times \mathbb{R}^N, \\
|A_t(x,t,\xi)| &\leq \Phi_1(t)|t|^{p-1} + \phi_1(t)|\xi|^p && \text{a.e. in } \mathbb{R}^N, \text{ for all } (t,\xi) \in \mathbb{R} \times \mathbb{R}^N, \\
|a(x,t,\xi)| &\leq \Phi_2(t)|t|^{p-1} + \phi_2(t)|\xi|^{p-1} && \text{a.e. in } \mathbb{R}^N, \text{ for all } (t,\xi) \in \mathbb{R} \times \mathbb{R}^N;
\end{aligned}$$

(g_0) $g(x, t)$ is a Carathéodory function;

(g_1) a function $\eta \in L^{\frac{p}{p-q}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ exists, with $1 < q < p$, such that

$$0 \leq g(x,t)t \leq \eta(x)|t|^q \qquad \text{a.e. in } \mathbb{R}^N, \text{for all } t \in \mathbb{R}.$$

Remark 3.6. From (g_1), it results

$$|g(x,t)| \leq \eta(x)|t|^{q-1} \qquad \text{a.e. in } \mathbb{R}^N, \text{ for all } t \in \mathbb{R}.$$

Moreover, from (g_0)–(g_1), it follows that $G(x,t) = \displaystyle\int_0^t g(x,s)ds$ is a well defined C^1–Carathéodory function in $\mathbb{R}^N \times \mathbb{R}$ and

$$0 \leq G(x,t) \leq \frac{1}{q}\eta(x)|t|^q \qquad \text{a.e. in } \mathbb{R}^N, \text{ for all } t \in \mathbb{R}. \tag{3.12}$$

Remark 3.7. From (h_0)–(h_1), it follows that

$$A(x,0,0) = A_t(x,0,0) = 0 \quad \text{and} \quad a(x,0,0) = 0 \qquad \text{for a.e. } x \in \mathbb{R}^N.$$

Moreover, from (g_0)–(g_1) and Remark 3.6, we have that

$$G(x,0) = g(x,0) = 0 \qquad \text{for a.e. } x \in \mathbb{R}^N.$$

Clearly, it follows that $u = 0$ is a trivial solution of (1.1).

Proposition 3.8. *The assumptions* (g_0)*–*(g_1) *imply that*

$$\int_{\mathbb{R}^N} G(x,u)dx \in \mathbb{R} \qquad \text{for all } u \in X \quad (\text{ or better for all } u \in W^{1,p}(\mathbb{R}^N))$$

and

$$\int_{\mathbb{R}^N} g(x,u)vdx \in \mathbb{R} \qquad \text{for all } u, v \in X \quad (\text{ or better for all } u, v \in W^{1,p}(\mathbb{R}^N)).$$

Proof. Let $u \in W^{1,p}(\mathbb{R}^N)$. As $\eta \in L^{\frac{p}{p-q}}(\mathbb{R}^N)$ and $|u|^q \in L^{\frac{p}{q}}(\mathbb{R}^N)$, by Hölder's inequality with $\frac{p}{p-q}$ and $\frac{p}{q}$ conjugate exponents and by (3.12) it follows that

$$0 \le \int_{\mathbb{R}^N} G(x,u)dx \le \frac{1}{q}\int_{\mathbb{R}^N} \eta(x)|u|^q dx \le \frac{1}{q}|\eta|_{\frac{p}{p-q}}|u|_p^q. \tag{3.13}$$

Moreover, for all $u, v \in W^{1,p}(\mathbb{R}^N)$, it is

$$\left|\int_{\mathbb{R}^N} g(x,u)v dx\right| \le \int_{\mathbb{R}^N} \left|\eta(x)|u|^{q-1}v\right| dx \le |\eta|_{\frac{p}{p-q}}|u|_p^{q-1}|v|_p \tag{3.14}$$

by applying again Hölder's inequality with $\frac{p}{p-q}$, $\frac{p}{q-1}$ and p conjugate exponents. □

Remark 3.9. From (g_0)-(g_1), we have that

$$g(x,u) \in L^{\frac{p}{p-1}}\left(\mathbb{R}^N\right) \qquad \text{for all } u \in W^{1,p}(\mathbb{R}^N).$$

Indeed, Hölder's inequality with $\frac{p-1}{p-q}$ and $\frac{p-1}{q-1}$ conjugate exponents implies that

$$\int_{\mathbb{R}^N} |g(x,u)|^{\frac{p}{p-1}} dx \le |\eta|^{\frac{p}{p-1}}_{\frac{p}{p-q}}|u|_p^{\frac{p(q-1)}{p-1}}.$$

Let us point out that (h_0)–(h_1) imply that $A(x,u,\nabla u) \in L^1(\mathbb{R}^N)$ for any $u \in X$.

Therefore, from (3.9) and (3.13) it follows that the functional

$$\mathcal{J}(u) = \int_{\mathbb{R}^N} A(x,u,\nabla u)dx + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p dx - \int_{\mathbb{R}^N} G(x,u)dx \tag{3.15}$$

is well defined for all $u \in X$. Moreover, taking $v \in X$, from (3.14), the Gâteaux differential of functional $\mathcal{J}$ in u along the direction v is given by

$$\begin{aligned}\langle d\mathcal{J}(u), v\rangle = &\int_{\mathbb{R}^N} a(x,u,\nabla u)\nabla v dx + \int_{\mathbb{R}^N} A_t(x,u,\nabla u)v dx \\ &+ \int_{\mathbb{R}^N} V(x)|u|^{p-2}uv dx - \int_{\mathbb{R}^N} g(x,u)v dx.\end{aligned} \tag{3.16}$$

Now, we are ready to prove the following regularity result.

Proposition 3.10. *Let $p > 1$ and assume that conditions (V_1), (h_0)–(h_1) and (g_0)-(g_1) hold.*

If $(u_n)_n \subset X$ and $u \in X$ are such that

$$u_n \to u \text{ a.e. in } \mathbb{R}^N \tag{3.17}$$

and (3.10), (3.11) hold for a constant $M > 0$, then

$$\mathcal{J}(u_n) \to \mathcal{J}(u) \quad \text{and} \quad \|d\mathcal{J}(u_n) - d\mathcal{J}(u)\|_{X'} \to 0 \quad \text{as } n \to +\infty.$$

Hence, $\mathcal{J}$ is a C^1 functional on X with Fréchet differential as in (3.16).

Proof. For simplicity, we set $\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2 - \mathcal{J}_3$, where

$$\mathcal{J}_1 : u \in X \;\mapsto\; \mathcal{J}_1(u) = \int_{\mathbb{R}^N} A(x, u, \nabla u) dx \in \mathbb{R},$$
$$\mathcal{J}_2 : u \in X \;\mapsto\; \mathcal{J}_2(u) = \frac{1}{p} \int_{\mathbb{R}^N} V(x)|u|^p dx \in \mathbb{R},$$
$$\mathcal{J}_3 : u \in X \mapsto \mathcal{J}_3(u) = \int_{\mathbb{R}^N} G(x, u) dx \in \mathbb{R}$$

with related Gâteaux differentials

$$\begin{aligned}
\langle d\mathcal{J}_1(u), v\rangle &= \int_{\mathbb{R}^N} a(x, u, \nabla u) \cdot \nabla v \; dx \;+\; \int_{\mathbb{R}^N} A_t(x, u, \nabla u) v \; dx,\\
\langle d\mathcal{J}_2(u), v\rangle &= \int_{\mathbb{R}^N} V(x)|u|^{p-2} uv \; dx,\\
\langle d\mathcal{J}_3(u), v\rangle &= \int_{\mathbb{R}^N} g(x, u) v \; dx.
\end{aligned}$$

Let $(u_n)_n \subset X$, $u \in X$ and $M > 0$ such that (3.10), (3.11) and (3.17) hold.

Reasoning as in [14], we obtain that

$$\mathcal{J}_1(u_n) \to \mathcal{J}_1(u) \text{ and } \|d\mathcal{J}_1(u_n) - d\mathcal{J}_1(u)\|_{X'} \to 0 \quad \text{as } n \to +\infty.$$

For completeness, we report the various computations.

As a consequence of (3.11) and the growth assumption (h_1) a constant $b > 0$, $b = b(M)$ exists such that for all $n \in \mathbb{N}$ and a.e. $x \in \mathbb{R}^N$, we have

$$|A(x, u, \nabla u)| \le b(|u|^p + |\nabla u|^p), \; |A(x, u_n, \nabla u_n)| \le b(|u_n|^p + |\nabla u_n|^p), \tag{3.18}$$
$$|A_t(x, u, \nabla u)| \le b(|u|^{p-1} + |\nabla u|^p), \; |A_t(x, u_n, \nabla u_n)| \le b(|u_n|^{p-1} + |\nabla u_n|^p), \tag{3.19}$$
$$|a(x, u, \nabla u)| \le b(|u|^{p-1} + |\nabla u|^{p-1}), \; |a(x, u_n, \nabla u_n)| \le b(|u_n|^{p-1} + |\nabla u_n|^{p-1}). \tag{3.20}$$

From (3.10) and (3.18), it follows that $(\mathcal{J}_1(u_n))_n$ is bounded. Arguing by contradiction, assume that the limit

$$\mathcal{J}_1(u_n) \to \mathcal{J}_1(u)$$

does not hold; then $\beta \in \mathbb{R}$ and a subsequence $(\mathcal{J}_1(u_{k_n}))_n$ exist such that

$$\mathcal{J}_1(u_{k_n}) \to \beta \quad \text{and} \quad \beta \ne \mathcal{J}_1(u). \tag{3.21}$$

On the other hand, (3.10) implies that both $u_{k_n} \to u$ and $\nabla u_{k_n} \to \nabla u$ in $L^p(\mathbb{R}^N)$ and also two functions ψ_1, $\psi_2 \in L^1(\mathbb{R}^N)$ exist such that, up to subsequences, for all $n \in \mathbb{N}$

$$\begin{aligned}
&\nabla u_{k_n} \to \nabla u \;\text{ a.e. in } \mathbb{R}^N, |u_{k_n}(x)|^p \le \psi_1(x),\\
&|\nabla u_{k_n}(x)|^p \le \psi_2(x) \text{ a.e. in } \mathbb{R}^N
\end{aligned} \tag{3.22}$$

(see, [10, Theorem 4.9]). Thus, from (h_0) and (3.17), we have

$$A(x, u_{k_n}, \nabla u_{k_n}) \to A(x, u, \nabla u) \quad \text{a.e. in } \mathbb{R}^N,$$

where (3.18) and (3.22) imply

$$|A(x, u_{k_n}, \nabla u_{k_n})| \le b(\psi_1(x) + \psi_2(x)) \quad \text{a.e. in } \mathbb{R}^N, \text{ with } b(\psi_1 + \psi_2) \in L^1(\mathbb{R}^N).$$

Then, by Lebesgue's Dominated Convergence Theorem, we have

$$\int_{\mathbb{R}^N} |A(x, u_{k_n}, \nabla u_{k_n}) - A(x, u, \nabla u)| dx \to 0$$

hence,

$$\mathcal{J}_1(u_{k_n}) \to \mathcal{J}_1(u)$$

in contradiction with (3.21).

Now, from (3.10) and (3.19), (3.20) it follows that $(d\mathcal{J}_1(u_n))_n$ is bounded in X'. So, if by contradiction we assume that $\|dJ_1(u_n) - dJ_1(u)\|_{X'} \to 0$ does not hold, then $\bar{\varepsilon} > 0$, a subsequence $(u_{k_n})_n$ and a sequence $(v_n)_n \subset X$ exist such that $\|v_n\|_X \leq 1$ and

$$|\langle d\mathcal{J}_1(u_{k_n}) - d\mathcal{J}_1(u), v_n\rangle| \geq \bar{\varepsilon} \quad \text{for all } n \in \mathbb{N}. \tag{3.23}$$

But

$$\begin{aligned} |\langle d\mathcal{J}_1(u_{k_n}) - d\mathcal{J}_1(u), v_n\rangle| \leq & \int_{\mathbb{R}^N} |a(x, u_{k_n}, \nabla u_{k_n}) - a(x, u, \nabla u)|\ |\nabla v_n|\ dx \\ & + \int_{\mathbb{R}^N} |A_t(x, u_{k_n}, \nabla u_{k_n}) - A_t(x, u, \nabla u)|\ |v_n|\ dx \end{aligned}$$

and from (3.10), (3.11), (3.19), (3.20), Hölder inequality, (3.22) and the same arguments pointed out previously, together with the Lebesgue's Dominated Convergence Theorem, it follows that, up to subsequences,

$$\int_{\mathbb{R}^N} |a(x, u_{k_n}, \nabla u_{k_n}) - a(x, u, \nabla u)|\ |\nabla v_n|\ dx \to 0$$

and

$$\int_{\mathbb{R}^N} |A_t(x, u_{k_n}, \nabla u_{k_n}) - A_t(x, u, \nabla u)|\ |v_n|\ dx \to 0$$

which is in contradiction with (3.23).

Arguing as in [18, Proposition 3.11], we have that

$$\mathcal{J}_2(u_n) \to \mathcal{J}_2(u) \text{ and } \|d\mathcal{J}_2(u_n) - d\mathcal{J}_2(u)\|_{X'} \to 0 \quad \text{as } n \to +\infty.$$

Finally, in order to state the regularity of $\mathcal{J}_3(u)$, we have to prove that $d\mathcal{J}_3(u)$ is continuous or better

$$\|d\mathcal{J}_3(u_n) - d\mathcal{J}_3(u)\|_{X'} \to 0 \quad \text{as } n \to +\infty$$

if $(u_n)_n$ satisfies the assumptions (3.10), (3.11) and (3.17).

From Remark 3.9, we have that

$$\begin{aligned} |\langle d\mathcal{J}_3(u_n) - d\mathcal{J}_3(u), v\rangle| \leq & \int_{\mathbb{R}^N} |g(x, u_n) - g(x, u)||v| dx \\ \leq & |g(x, u_n) - g(x, u)|_{\frac{p}{p-1}} |v|_p. \end{aligned} \tag{3.24}$$

where

$$\begin{aligned}
&|g(x,u_n)-g(x,u)|^{\frac{p}{p-1}}\\
&\quad\le c\left(|g(x,u_n)|^{\frac{p}{p-1}}+|g(x,u)|^{\frac{p}{p-1}}\right)\\
&\quad\le c\left((\eta(x))^{\frac{p}{p-1}}\,|u_n|^{\frac{p}{p-1}(q-1)}+(\eta(x))^{\frac{p}{p-1}}\,|u|^{\frac{p}{p-1}(q-1)}\right)\\
&\quad\le c\left((\eta(x))^{\frac{p}{p-1}}\,|u_n-u|^{\frac{p}{p-1}(q-1)}+(\eta(x))^{\frac{p}{p-1}}\,|u|^{\frac{p}{p-1}(q-1)}\right)
\end{aligned}$$

as $|u_n|\le|u_n-u|+|u|$ and (g_1) holds. From Fatou's Lemma, we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^N}\liminf_n\left[c\left((\eta(x))^{\frac{p}{p-1}}\,|u_n-u|^{\frac{p}{p-1}(q-1)}+(\eta(x))^{\frac{p}{p-1}}\,|u|^{\frac{p}{p-1}(q-1)}\right)\right.\\
&\qquad\left.-|g(x,u_n)-g(x,u)|^{\frac{p}{p-1}}\right]dx\\
&\quad\le\liminf_n\int_{\mathbb{R}^N}\left[c\left((\eta(x))^{\frac{p}{p-1}}\,|u_n-u|^{\frac{p}{p-1}(q-1)}\right.\right.\\
&\qquad\left.\left.+\,(\eta(x))^{\frac{p}{p-1}}\,|u|^{\frac{p}{p-1}(q-1)}\right)-|g(x,u_n)-g(x,u)|^{\frac{p}{p-1}}\right]dx.
\end{aligned}\tag{3.25}$$

From (3.17) and (g_0), it follows that

$$|g(x,u_n)-g(x,u)|^{\frac{p}{p-1}}\to 0\qquad\text{a.e. in }\mathbb{R}^N.$$

Moreover, from (g_1), (3.10) and (3.4), we have that

$$\int_{\mathbb{R}^N}\left((\eta(x))^{\frac{p}{p-1}}\,|u_n-u|^{\frac{p}{p-1}(q-1)}\right)dx\le|\eta|^{\frac{p}{p-1}}_{\frac{p}{p-q}}|u_n-u|_p^{p\frac{(q-1)}{p-1}}\to 0.$$

Taking into account condition (3.25), we obtain that

$$\begin{aligned}
c\int_{\mathbb{R}^N}\left((\eta(x))^{\frac{p}{p-1}}\,|u|^{\frac{p}{p-1}(q-1)}\right)dx&\le c\int_{\mathbb{R}^N}\left((\eta(x))^{\frac{p}{p-1}}\,|u|^{\frac{p}{p-1}(q-1)}\right)dx\\
&\quad+\liminf_n\left(-\int_{\mathbb{R}^N}|g(x,u_n)-g(x,u)|^{\frac{p}{p-1}}dx\right).
\end{aligned}$$

Therefore, we have

$$0\le-\limsup_n\int_{\mathbb{R}^N}|g(x,u_n)-g(x,u)|^{\frac{p}{p-1}}dx,$$

whence it follows that

$$\begin{aligned}
0&\le\liminf_n\int_{\mathbb{R}^N}|g(x,u_n)-g(x,u)|^{\frac{p}{p-1}}dx\\
&\le\limsup_n\int_{\mathbb{R}^N}|g(x,u_n)-g(x,u)|^{\frac{p}{p-1}}dx\le 0.
\end{aligned}$$

Then, it is

$$|g(x,u_n)-g(x,u)|_{\frac{p}{p-1}}\to 0\qquad\text{for } n\to\infty.$$

and from (3.24) we can conclude that

$$\|d\mathcal{J}_3(u_n)-d\mathcal{J}_3(u)\|_{X'}\to 0\qquad\text{as } n\to+\infty.$$

□

4. Set of hypotheses and the main result

From now on, besides $(h_0)-(h_1)$, $(g_0)-(g_1)$, $(V_1)-(V_2)$ for the functions $A(x,t,\xi)$, $V(x)$ and $g(x,t)$ we introduce the following set of hypotheses:

(h_2) a constant $\alpha_0 > 0$ exists such that

$$A(x,t,\xi) \geq \alpha_0|\xi|^p \quad \text{a.e. in } \mathbb{R}^N, \text{ for all}(t,\xi) \in \mathbb{R}\times\mathbb{R}^N;$$

(h_3) a constant $\alpha_1 > 0$ exists such that

$$a(x,t,\xi)\cdot\xi \geq \alpha_1|\xi|^p \quad \text{a.e. in } \mathbb{R}^N, \text{ for all } (t,\xi) \in \mathbb{R}\times\mathbb{R}^N;$$

(h_4) a constant $\alpha_2 > 0$ exists such that

$$a(x,t,\xi)\cdot\xi + A_t(x,t,\xi)t \geq \alpha_2 a(x,t,\xi)\cdot\xi \quad \text{a.e. in } \mathbb{R}^N, \text{ for all } (t,\xi) \in \mathbb{R}\times\mathbb{R}^N;$$

(h_5) some constants $\mu > p$ and $\alpha_3 > 0$ exist such that

$$\mu A(x,t,\xi) - a(x,t,\xi)\cdot\xi - A_t(x,t,\xi)t \geq \alpha_3 A(x,t,\xi) \quad \text{a.e. in } \mathbb{R}^N, \text{ for all } (t,\xi) \in \mathbb{R}\times\mathbb{R}^N;$$

(h_6) for all ξ, $\xi^* \in \mathbb{R}^N$, $\xi \neq \xi^*$, it is

$$[a(x,t,\xi) - a(x,t,\xi^*)]\cdot[\xi-\xi^*] > 0 \quad \text{a.e. in } \mathbb{R}^N, \text{ for all } t \in \mathbb{R};$$

(V_3) for any $\varrho > 0$ a constant $C_\varrho > 0$ exists such that

$$\operatorname{ess\,sup}_{|x|\leq\varrho} V(x) \leq C_\varrho;$$

(g_2) it holds

$$\lim_{t\to 0^+} \frac{g(x,t)}{t^{p-1}} = +\infty \quad \text{uniformly for a.e. } x \in \mathbb{R}^N.$$

We point out some direct consequences of the previous hypotheses.

Remark 4.1. In the assumptions (h_2)-(h_3), we may always suppose $\alpha_0 \leq 1$ and $\alpha_1 \leq 1$.

Remark 4.2. From (h_3) and (h_4), it follows that $\alpha_2 \leq 1$.

Remark 4.3. From (h_4)–(h_5), we have that, a.e. in $\mathbb{R}^N$, for all $(t,\xi) \in \mathbb{R}\times\mathbb{R}^N$

$$(\mu-\alpha_3)A(x,t,\xi) \;\geq\; \alpha_2\; a(x,t,\xi)\cdot\xi;$$

hence, if also (h_2)–(h_3) hold, it is $\alpha_3 < \mu$. So,

$$A(x,t,\xi) \;\geq\; \alpha_4\; a(x,t,\xi)\cdot\xi \tag{4.1}$$

with $\alpha_4 = \frac{\alpha_2}{\mu-\alpha_3} > 0$. Moreover, from (4.1) and (h_5) we have that

$$\mu A(x,t,\xi) - a(x,t,\xi)\cdot\xi - A_t(x,t,\xi)t \;\geq\; \alpha_3\alpha_4\; a(x,t,\xi)\cdot\xi. \tag{4.2}$$

We note that (4.2) implies hypothesis (H_5) in [12].

Now, we are ready to state our main result.

Theorem 4.4. *Assume that (h_0)–(h_6), (V_1)–(V_3) and (g_0)–(g_2) hold, then problem (1.1) admits at least one weak nontrivial bounded solution.*

Remark 4.5. The existence of solutions for quasilinear elliptic problems like (1.1), but in a bounded domain, has been established in [16]. On the other hand, equation (1.1) in the whole Euclidean space has been studied in the particular case $A(x,t,\xi) = \frac{1}{p}\mathcal{A}(x,t)|\xi|^p$ and $g(x,t) = \eta(x)|t|^{p-1}$ (see [30]).

Proposition 4.6. *Assume that conditions* $(h_0)-(h_2)$, (V_1) *and* (g_0)-(g_1) *hold. Then, some positive constants* c_1 *and* c_2 *exist such that*

$$\mathcal{J}(u) \geq c_1\|u\|_V^p - c_2\|u\|_V^q \quad \textit{for any } u \in X.$$

Hence, functional $\mathcal{J}$ *is bounded from below, i.e., a constant* $\alpha \in \mathbb{R}$ *exists such that*

$$\mathcal{J}(u) \geq \alpha \quad \textit{for any } u \in X, \ \textit{with } \alpha = \min_{s\geq 0}(c_1 s^p - c_2 s^q).$$

Proof. From (h_2) and (3.13), we have

$$\begin{aligned}\mathcal{J}(u) &= \int_{\mathbb{R}^N} A(x,u,\nabla u)dx + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p dx - \int_{\mathbb{R}^N} G(x,u)dx \\ &\geq \alpha_0 \int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p dx - \frac{1}{q}|\eta|_{\frac{p}{p-q}}|u|_p^q.\end{aligned}$$

Hence, the conclusion follows from (3.8) with $c_1 = \min\left\{\alpha_0, \frac{1}{p}\right\}$ and $c_2 = \frac{1}{q}\tau_p^q|\eta|_{\frac{p}{p-q}}$.

□

From Proposition 4.6, the following convergence result can be stated.

Proposition 4.7. *Suppose that hypotheses* (h_0)–(h_2), (V_1) *and* (g_0)-(g_1) *hold. Then, taking any* $\beta \in \mathbb{R}$ *and a* $(CPS)_\beta$*–sequence* $(u_n)_n \subset X$*, it follows that* $(u_n)_n$ *is bounded in* $W_V^{1,p}(\mathbb{R}^N)$*. Furthermore,* $u \in W_V^{1,p}(\mathbb{R}^N)$ *exists such that, up to subsequence,*

$$u_n \rightharpoonup u \ \textit{weakly in } W_V^{1,p}(\mathbb{R}^N), \tag{4.3}$$

$$u_n \to u \ \textit{strongly in } L^r(\mathbb{R}^N) \ \textit{for each } r \in [p, p^*[, \tag{4.4}$$

$$u_n \to u \ \textit{a.e. in } \mathbb{R}^N \tag{4.5}$$

as $n \to +\infty$.

Proof. The conditions (4.3) - (4.5) follow from Proposition 4.6, the reflexivity of the space $W_V^{1,p}(\mathbb{R}^N)$ and (3.7). □

Remark 4.8. We note that the thesis of Proposition 4.7 holds even if we replace $\|\cdot\|_V$ with the usual norm $\|\cdot\|$ in $W^{1,p}(\mathbb{R}^N)$.

Remark 4.9. We are not able to prove that $\mathcal{J}$ satisfies the condition $(wCPS)$ in the whole space X. In particular, we are not able to prove that the weak limit $u \in L^\infty(\mathbb{R}^N)$ because the result of Ladyzhenskaya-Ural'tseva (see, [25, Theorem II 5.1]) holds for bounded domains. In order to overcome this problem, we use an approximation method on bounded domains.

From now on, fixed Ω an open bounded domain in $\mathbb{R}^N$, we define

$$X_\Omega = W_{0,V}^{1,p}(\Omega) \cap L^\infty(\Omega)$$

endowed with the norm

$$\|u\|_{X_\Omega} = \|u\|_{\Omega,V} + |u|_{\Omega,\infty} \quad \text{for any } u \in X_\Omega \tag{4.6}$$

and we denote by X'_Ω its dual.

Remark 4.10. Since Ω is a bounded domain, it follows that $\|u\|_\Omega$ and $|\nabla u|_{p,\Omega}$ are equivalent norms. Moreover, as (V_3) holds, a constant $c_\Omega \geq 1$ exists such that

$$\|u\|_{\Omega,V}^p = \int_\Omega (|\nabla u|^p + V(x)|u|^p)dx \leq \int_\Omega |\nabla u|^p dx + c_\Omega \int_\Omega |u|^p dx \leq c_\Omega \|u\|_\Omega^p,$$

which, together with (3.3), implies that the norms $\|\cdot\|_{\Omega,V}$ and $\|\cdot\|_\Omega$ are equivalent, too. We have, in particular, that the norm in (4.6) can be replaced with the equivalent one, still denoted by $\|\cdot\|_{X_\Omega}$, given by

$$\|u\|_{X_\Omega} = \|u\|_\Omega + |u|_{\Omega,\infty} \quad \text{for any } u \in X_\Omega,$$

and it results

$$X_\Omega = W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \tag{4.7}$$

Actually, since any function $u \in X_\Omega$ can be trivially extended to a function $\tilde{u} \in X$ just assuming $\tilde{u}(x) = 0$ for all $x \in \mathbb{R}^N \setminus \Omega$, then

$$\|\tilde{u}\| = \|u\|_\Omega, \quad \|\tilde{u}\|_V = \|u\|_{\Omega,V}, \quad |\tilde{u}|_\infty = |u|_{\Omega,\infty}, \quad \|\tilde{u}\|_X = \|u\|_{X_\Omega}.$$

Thus, we can consider the restriction $\mathcal{J}_\Omega$ of the functional $\mathcal{J}$ to X_Ω, i.e., $\mathcal{J}_\Omega = \mathcal{J}|_{X_\Omega}$.

Clearly, from (3.15) for all $u \in X_\Omega$ it results

$$\begin{aligned}\mathcal{J}_\Omega(u) &= \mathcal{J}|_{X_\Omega}(u) \\ &= \int_\Omega A(x,u,\nabla u)dx + \frac{1}{p}\int_\Omega V(x)|u|^p dx - \int_\Omega G(x,u)dx,\end{aligned} \tag{4.8}$$

Hence, by simplifying the arguments in the proof in [12, Proposition 3.1], it follows that the functional $\mathcal{J}_\Omega : X_\Omega \to \mathbb{R}$ is C^1 and, for any $u, v \in X_\Omega$, its Fréchet differential in u evaluated in v is given by

$$\begin{aligned}\langle d\mathcal{J}_\Omega(u), v\rangle = &\int_\Omega a(x,u,\nabla u)\nabla v dx + \int_\Omega A_t(x,u,\nabla u)v dx \\ &+ \int_\Omega V(x)|u|^{p-2}uv dx - \int_\Omega g(x,u)v dx.\end{aligned} \tag{4.9}$$

We need to state the following result.

Proposition 4.11. *Let $1 < q < p$ and suppose that hypotheses (h_0)–(h_3), (h_5)-(h_6), (V_1)–(V_3) and (g_0)-(g_1) hold. Then, $\mathcal{J}_\Omega$ satisfies the $(wCPS)$ condition in $\mathbb{R}$.*

Proof. Taking $\beta \in \mathbb{R}$, let $(u_n)_n \subset X_\Omega$ be a $(CPS)_\beta$–sequence, i.e.,

$$\mathcal{J}_\Omega(u_n) \to \beta \quad \text{and} \quad \|d\mathcal{J}_\Omega(u_n)\|_{X'_\Omega}(1 + \|u_n\|_{X_\Omega}) \to 0 \qquad \text{if } n \to +\infty.$$

We want to prove that $u \in X_\Omega$ exists such that

(i) $\|u_n - u\|_\Omega \to 0$ (up to subsequences),
(ii) $\mathcal{J}_\Omega(u) = \beta$, $d\mathcal{J}_\Omega(u) = 0$.

Reasoning as in Proposition 4.7 with $\mathcal{J}_\Omega$ instead of $\mathcal{J}$ and using the Sobolev Embedding Theorem for bounded domains, we have that $(u_n)_n$ is bounded in $W_0^{1,p}(\Omega)$. Hence, there exists $u \in W_0^{1,p}(\Omega)$ such that, up to subsequences, if $n \to +\infty$, then

$$\begin{aligned} &u_n \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega),\\ &u_n \to u \text{ strongly in } L^r(\Omega) \text{ for each } r \in [1, p^*[,\\ &u_n \to u \text{ a.e. in } \Omega. \end{aligned}$$

The remainder of this proof follows reasoning as in [12, Proposition 4.6] since (4.2) holds and the term $\bar{g}(x,t) = g(x,t) - V(x)|t|^{p-2}t$ satisfies the hypothesis (G_1) required in [12]. □

Proposition 4.12. *Assume conditions $(h_0)-(h_3)$, $(h_5)-(h_6)$, $(V_1)-(V_3)$ and $(g_0)-(g_1)$. Then functional $\mathcal{J}_\Omega$ has at least a critical point $u_\Omega \in X_\Omega$. Moreover, if (g_2) holds, then u_Ω is not trivial.*

Proof. The functional $\mathcal{J}_\Omega$ is bounded from below in X_Ω (see, Proposition 4.6) and satisfies condition $(wCPS)$ in $\mathbb{R}$ (see, Proposition 4.11), thus, from Proposition 2.2, $\mathcal{J}_\Omega$ admits a minimum point u_Ω in X_Ω. Clearly, it is

$$\mathcal{J}_\Omega(u_\Omega) = \min_{u \in X_\Omega} \mathcal{J}_\Omega(u) \le \mathcal{J}_\Omega(0) = 0.$$

In order to prove that u_Ω is non trivial since $\mathcal{J}_\Omega(u_\Omega) < 0$, we consider $\varphi_1 \in X_\Omega$ the unique eigenfunction associated to the first eigenvalue λ_1 of $-\Delta_p$ in Ω (see [27]) such that

$$\varphi_1 > 0 \text{ a.e. in } \Omega,\ \varphi_1 \in L^\infty(\Omega),\ |\varphi_1|_{\Omega,p} = 1, \text{ and so } |\nabla\varphi_1|^p_{\Omega,p} = \lambda_1.$$

Then, taking $\tau \in (0,1)$, from (h_1) and (V_3) we have

$$\begin{aligned} \mathcal{J}_\Omega(\tau\varphi_1) &= \int_\Omega A(x, \tau\varphi_1, \nabla(\tau\varphi_1))dx + \int_\Omega V(x)|\tau\varphi_1|^p dx - \int_\Omega G(x, \tau\varphi_1)dx\\ &\le \int_\Omega \left(\Phi_0(\tau\varphi_1(x))|\tau\varphi_1(x)|^p + \phi_0(\tau\varphi_1(x))|\nabla(\tau\varphi_1(x))|^p\right) dx\\ &+ \tau^p \int_\Omega V(x)|\varphi_1|^p dx - \int_\Omega G(x, \tau\varphi_1)dx\\ &\le K_\Omega \tau^p - \int_\Omega G(x, \tau\varphi_1)dx, \end{aligned}$$

where $K_\Omega = \max_{0 \le t \le |\varphi_1|_\infty} \Phi_0(t) + \lambda_1 \max_{0 \le t \le |\varphi_1|_\infty} \phi_0(t) + \sup_\Omega V(x)$.

Now, from assumption (g_2), there exists a constant $\delta = \delta_{K_\Omega} > 0$ such that for any $s \in [0, \delta]$ and for a.e. $x \in \Omega$ it is $G(x,s) > 2K_\Omega s^p$.

Then, for any $\tau > 0$ small enough, in particular $0 < \tau < \frac{\delta}{|\varphi_1|_\infty}$, it results

$$\mathcal{J}_\Omega(\tau\varphi_1) \le K_\Omega\tau^p - 2K_\Omega\tau^p < 0.$$

□

5. Proof of the Theorem 4.4

For the proof of our main result, we follow an approach similar to those in [14] and [18]. Anyway, the different assumptions for our problem requires to rehash the proofs and provide them with all the details.

Throughout this section, we suppose that all the hypotheses in Theorem 4.4 are satisfied. Thus, for any $k \in \mathbb{N}$, we can consider the spaces X_{B_k} as in (4.7) and the related functionals

$$\mathcal{J}_k(u) = \mathcal{J}_{B_k}(u) = \mathcal{J}|_{X_{B_k}}(u)$$

as in (4.8). For the sake of convenience, since any $u \in X_{B_k}$ can be trivially extended setting $u = 0$ a.e. in $\mathbb{R}^N \setminus B_k$, we still denote by u such an extension.

Remark 5.1. From the expression of $\mathcal{J}_k$, it follows that if $u \in X_{B_k}$ then

$$\langle d\mathcal{J}_k(u), v\rangle = \langle d\mathcal{J}(u), v\rangle \text{ for all } v \in X_{B_k}.$$

Thus, from Propositions 4.6 and 4.12, a sequence $(u_k)_k \subset X$ exists such that for every $k \in \mathbb{N}$ it results:

(i) $u_k|_{B_k} \in X_{B_k}$ with $u_k = 0$ a.e. in $\mathbb{R}^N \setminus B_k$,
(ii) $\alpha \le \mathcal{J}(u_k) \le \mathcal{J}(u_1) < 0$,
(iii) $\langle d\mathcal{J}(u_k), v\rangle = 0$ for all $v \in X_{B_k}$,

where, from Proposition 4.6, in (ii) we can choose α independent of k. Now, our aim is proving that sequence $(u_k)_k$ is bounded in X. First of all, we recall the following result which is a refinement of [25, Theorem II.51].

Lemma 5.2. *Let Ω be an open bounded domain in $\mathbb{R}^N$ and consider p, s so that $1 < p \le N$ and $p \le s < p^*$ (if $N = p$ we just require that p^* is any number larger than s) and take $u \in W_0^{1,p}(\Omega)$. If $a^* > 0$ and $m_0 \in \mathbb{N}$ exist such that*

$$\int_{\Omega_m^+} |\nabla u|^p dx \le a^* \left(m^s \ |\Omega_m^+| + \int_{\Omega_m^+} |u|^s dx \right) \quad \textit{for all } m \ge m_0,$$

with $\Omega_m^+ = \{x \in \Omega : \ u(x) > m\}$, then $\operatorname{ess\,sup}_\Omega u$ is bounded from above by a positive constant which can be chosen so that it depends only on $|\Omega_{m_0}^+|$, N, p, s, a^*, m_0, $\|u\|_\Omega$, or better by a positive constant which can be chosen so that it depends only on N, p, s, a^*, m_0 and a_0^* for any a_0^* such that $\max\{|\Omega_{m_0}^+|, \|u\|_\Omega\} \le a_0^*$.

Vice versa, if inequality

$$\int_{\Omega_m^-} |\nabla u|^p dx \le a^* \left(m^s \ |\Omega_m^-| + \int_{\Omega_m^-} |u|^s dx \right) \quad \textit{for all } m \ge m_0,$$

holds, with $\Omega_m^- = \{x \in \Omega : \ u(x) < -m\}$, then $\operatorname{ess\,sup}_\Omega(-u)$ is bounded from above by a positive constant which can be chosen so that it depends only on N, p, s, a^*, m_0 and any constant which is greater than both $|\Omega_{m_0}^-|$ and $\|u\|_\Omega$.

Proof. For the proof, (see, [18, Lemma 5.6]). □

We proceed stating a boundedness result in X.

Proposition 5.3. *A positive constant M_0 exists such that*

$$\|u_k\|_X \le M_0 \quad \text{for all } k \in \mathbb{N}. \tag{5.1}$$

Proof. From Proposition 4.6 and (ii), since $X_{B_1} \subset X_{B_k}$ for any $k \in \mathbb{N}$, it is

$$c_1\|u_k\|_V^p - c_2\|u_k\|_V^q \le \mathcal{J}(u_k) \le \mathcal{J}(u_1),$$

so the condition $q < p$ implies that $(\|u_k\|_V)_k$ is bounded. Furthermore, by (3.3), the sequence $(\|u_k\|)_k$ is bounded, too.

Now, we need to prove that $(|u_k|_\infty)_k$ is bounded.

To reach our aim, we suppose that $|u_k|_\infty > 1$, i.e., ess $\sup_{\mathbb{R}^N} u_k > 1$ or ess $\sup_{\mathbb{R}^N}(-u_k) > 1$.

Assume that ess $\sup_{\mathbb{R}^N} u_k > 1$ and consider the set

$$B_{1,k}^+ = \{x \in \mathbb{R}^N : \ u_k(x) > 1\}.$$

Since condition (i) holds, clearly we have

$$B_{1,k}^+ \subset \ B_k.$$

In particular, it follows that $B_{1,k}^+$ is an open bounded domain such that $|B_{1,k}^+| > 0$ and also

$$|B_{1,k}^+| \le \int_{B_{1,k}^+} |u_k|^p dx \le \int_{\mathbb{R}^N} |u_k|^p dx \le \|u_k\|^p \le C,$$

since $(\|u_k\|)_k$ is bounded.

Now, for any $m \in \mathbb{N}$, we define the function $R_m^+ : \mathbb{R} \to \mathbb{R}$ such that

$$R_m^+ t = \begin{cases} 0 & \text{if } t \le m \\ t - m & \text{if } t > m. \end{cases}$$

It follows that $R_m^+ u_k \in X_{B_k}$. Hence, from condition (iii), (4.9), (h_4), Remark 4.2, (h_3), (V_1) and from direct computations we obtain

$$\begin{aligned} 0 &= \langle d\mathcal{J}(u_k), R_m^+ u_k\rangle \\ &= \int_{B_{m,k}^+} \left(1 - \frac{m}{u_k}\right) (a(x, u_k, \nabla u_k) \cdot \nabla u_k + A_t(x, u_k, \nabla u_k)u_k)\, dx \\ &+ \int_{B_{m,k}^+} \frac{m}{u_k} a(x, u_k, \nabla u_k) \cdot \nabla u_k dx + \int_{B_{m,k}^+} \left(1 - \frac{m}{u_k}\right) V(x)|u_k|^p dx \\ &- \int_{B_{m,k}^+} g(x, u_k) R_m^+ u_k dx \\ &\ge \alpha_2 \int_{B_{m,k}^+} a(x, u_k, \nabla u_k) \cdot \nabla u_k dx - \int_{B_{m,k}^+} g(x, u_k) R_m^+ u_k dx \\ &\ge \alpha_1\alpha_2 \int_{B_{m,k}^+} |\nabla u_k|^p dx - \int_{B_{m,k}^+} g(x, u_k) R_m^+ u_k dx \end{aligned}$$

which implies

$$\alpha_1\alpha_2 \int_{B_{m,k}^+} |\nabla u_k|^p dx \le \int_{B_{m,k}^+} g(x, u_k) R_m^+ u_k dx, \tag{5.2}$$

where $B^+_{m,k} = \left\{x \in \mathbb{R}^N : u_k(x) > m\right\}$ and obviously it is $B^+_{m,k} \subset B^+_{1,k}$.

Since (g_1) implies that $g(x,t) \geq 0$ if $t > 0$ for a.e. $x \in \mathbb{R}^N$, then $g(x,u_k(x))m \geq 0$ for a.e. $x \in B^+_{m,k}$. So, applying again (g_1) it results

$$\int_{B^+_{m,k}} g(x,u_k)R^+_m u_k dx \leq \int_{B^+_{m,k}} g(x,u_k)u_k dx$$
$$\leq \int_{B^+_{m,k}} \eta(x)|u_k|^q dx \leq |\eta|_\infty \int_{B^+_{m,k}} |u_k|^q dx.$$

Thus, from (5.2), it follows that

$$\int_{B^+_{m,k}} |\nabla u_k|^p dx \leq \frac{1}{\alpha_1\alpha_2} |\eta|_\infty \int_{B^+_{m,k}} |u_k|^q dx.$$

Since $q < p$ and $u_k(x) > 1$ for any $x \in B^+_{m,k}$, it follows that

$$\int_{B^+_{m,k}} |\nabla u_k|^p dx \leq c^* \left(|B^+_{m,k}| + \int_{B^+_{m,k}} |u_k|^p dx\right) \quad \text{for all } m \geq 1,$$

where $c^* > 0$ is a constant independent of m and k.

From Lemma 5.2 with $\Omega = B_k$ and from the boundedness of $(\|u_k\|)_k$ and $\left(|B^+_{m,k}|\right)_k$, it follows that a constant $M > 0$, independent of $k \in \mathbb{N}$, exists such that

$$\text{ess sup}_{B_k} u_k \leq M.$$

Similar arguments apply if $\text{ess sup}_{\mathbb{R}^N}(-u_k) > 1$.

Thus, it results that $(|u_k|_\infty)_k$ is bounded and the proof of (5.1) is complete.

At last, (5.3) follows from estimate (3.9). □

Remark 5.4. From (5.1) and (3.9), it follows that for all $\tau \geq p$

$$|u_k|_{V,\tau} \leq M_0 \quad \text{for all } k \in \mathbb{N}. \tag{5.3}$$

We observe that the boundedness of $(\|u_k\|_V)_k$ stated in the proof of Proposition 5.3 and the reflexivity of $W^{1,p}_V(\mathbb{R}^N)$ ensure the existence of $u^* \in W^{1,p}_V(\mathbb{R}^N)$ such that, up to subsequences,

$$u_k \rightharpoonup u^* \quad \text{in } W^{1,p}_V(\mathbb{R}^N). \tag{5.4}$$

From (3.7), it results

$$u_k \to u^* \quad \text{strongly in } L^r(\mathbb{R}^N) \text{ for any } r \in [p, p^*), \tag{5.5}$$

and

$$u_k \to u^* \quad \text{a.e. in } \mathbb{R}^N. \tag{5.6}$$

Proposition 5.5. *$u^* \in L^\infty(\mathbb{R}^N)$.*

Proof. For the proof, see, [18, Proposition 6.4]. □

Corollary 5.6. *For all $1 \leq r < +\infty$ it results that*

$$u_k \to u^* \quad \text{in } L^r_V(B_R) \text{ for all } R \geq 1. \tag{5.7}$$

Proof. For the proof (see, [18, Corollary 6.6]). $\square$

Proposition 5.7. *It results*

$$u_k \to u^* \quad \textit{strongly in } W_V^{1,p}(B_R) \textit{ for all } R \geq 1.$$

Proof. Fixing any $R \geq 1$, from (5.7), it is enough to prove that

$$|\nabla u_k - \nabla u^*|_{B_R,p} \to 0.$$

To achieve this aim, following an idea introduced in [8], let us consider the real map $\psi(t) = te^{\bar{\eta}t^2}$, where $\bar{\eta} > \left(\frac{\beta_2}{2\beta_1}\right)^2$ will be fixed once β_1, $\beta_2 > 0$ are chosen in a suitable way later on.

By definition,

$$\beta_1\psi'(t) - \beta_2|\psi(t)| > \frac{\beta_1}{2} \qquad \text{for all } t \in \mathbb{R}.$$

Defining $v_k = u_k - u^*$, we have that (5.4) implies

$$v_k \rightharpoonup 0 \text{ weakly in } W_V^{1,p}(\mathbb{R}^N),$$

while from (5.5), respectively (5.6), it follows that

$$v_k \to 0 \quad \text{strongly in } L^p(\mathbb{R}^N), \tag{5.8}$$

respectively

$$v_k \to 0 \quad \text{a.e. in } \mathbb{R}^N. \tag{5.9}$$

Moreover, from (5.1) and Proposition 5.5, it is

$$v_k \in X \quad \text{and} \quad |v_k|_\infty \leq \overline{M}_0 \quad \text{for all } k \in \mathbb{N} \tag{5.10}$$

with $\overline{M}_0 = M_0 + |u^*|_\infty$. Now, let $\chi_R \in C^\infty(\mathbb{R}^N)$, $R \geq 1$, be a cut-off function such that

$$\chi_R(x) = \begin{cases} 1 \text{ if } |x| \leq R \\ 0 \text{ if } |x| \geq R+1 \end{cases}, \qquad \text{with} \quad 0 \leq \chi_R(x) \leq 1 \quad \text{for all } x \in \mathbb{R}^N, \tag{5.11}$$

and

$$|\nabla \chi_R(x)| \leq 2 \quad \text{for all } x \in \mathbb{R}^N. \tag{5.12}$$

Thus, for every $k \in \mathbb{N}$, we consider the new function

$$w_{R,k} : x \in \mathbb{R}^N \mapsto w_{R,k}(x) = \chi_R(x)\psi(v_k(x)) \in \mathbb{R}.$$

We note that (5.10) gives

$$|\psi(v_k)| \leq \psi(\overline{M}_0), \quad 0 < \psi'(v_k) \leq \psi'(\overline{M}_0) \qquad \text{a.e. in } \mathbb{R}^N, \tag{5.13}$$

while (5.9) implies

$$\psi(v_k) \to 0, \quad \psi'(v_k) \to 1 \qquad \text{a.e. in } \mathbb{R}^N. \tag{5.14}$$

Then, from (5.11) and (5.13), we have that $\operatorname{supp} w_{R,k} \subset \operatorname{supp} \chi_R \subset B_{R+1}$ and also

$$|w_{R,k}(x)| \leq \psi(\overline{M}_0) \qquad \text{a.e. in } \mathbb{R}^N. \tag{5.15}$$

Moreover, (5.14) implies that

$$w_{R,k} \to 0 \qquad \text{a.e. in } \mathbb{R}^N,$$

while from

$$\nabla w_{R,k} = \psi(v_k)\nabla\chi_R + \chi_R\psi'(v_k)\nabla v_k \qquad \text{a.e. in } \mathbb{R}^N \tag{5.16}$$

and (5.10)–(5.13) and (5.15) it follows that $w_{R,k} \in X_{B_{R+1}}$. Hence, for all $k \geq R+1$, it results that

$$w_{R,k} \in X_{B_k}$$

so (iii), (3.16) and (5.16) imply that

$$\begin{aligned}
0 &= \langle d\mathcal{J}(u_k), w_{R,k}\rangle \\
&= \int_{B_{R+1}} \psi(v_k)a(x,u_k,\nabla u_k)\cdot\nabla\chi_R dx \\
&+ \int_{B_{R+1}} \chi_R\psi'(v_k)a(x,u_k,\nabla u_k)\cdot\nabla v_k dx \\
&+ \int_{B_{R+1}} A_t(x,u_k,\nabla u_k)w_{R,k}dx + \int_{B_{R+1}} V(x)|u_k|^{p-2}u_k w_{R,k}dx \\
&- \int_{B_{R+1}} g(x,u_k)w_{R,k}dx.
\end{aligned}$$

Now, we note that (g_0)–(g_1) together with (5.11), (5.10), (3.14), (5.1) and (5.8) imply

$$\begin{aligned}
\left|\int_{B_{R+1}} g(x,u_k)w_{R,k}dx\right| &\leq \int_{B_{R+1}} |g(x,u_k)||v_k|\mathrm{e}^{\bar\eta v_k^2}dx \\
&\leq \mathrm{e}^{\bar\eta\overline{M}_0^2}|\eta|_{\frac{p}{p-q}}|u_k|_p^{q-1}|v_k|_p \to 0.
\end{aligned}$$

Moreover, from (V_3), (5.11), (5.10), Hölder inequality, (5.1) and (5.8), it follows that

$$\begin{aligned}
\left|\int_{B_{R+1}} V(x)|u_k|^{p-2}u_k w_{R,k}dx\right| &\leq C_{R+1}\int_{B_{R+1}} |u_k|^{p-1}|v_k|\mathrm{e}^{\bar\eta v_k^2}dx \\
&\leq C_{R+1}\mathrm{e}^{\bar\eta\overline{M}_0^2}|u_k|_p^{p-1}|v_k|_p \to 0.
\end{aligned}$$

The remainder of the proof follows as in [14] (see also, [18, Proposition 6.7]) since function A with its derivatives satisfy similar hypotheses in [14]. $\square$

Proposition 5.8. *It results*

$$\langle d\mathcal{J}(u^*), \varphi\rangle = 0 \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^N) \tag{5.17}$$

with $C_c^\infty(\mathbb{R}^N) = \{\varphi \in C^\infty(\mathbb{R}^N) : \ \operatorname{supp}\varphi \subset\subset \mathbb{R}^N\}$. *Hence,* $d\mathcal{J}(u^*) = 0$ *in* X.

Proof. Taking $\varphi \in C_c^\infty(\Omega)$, a radius $R \geq 1$ exists such that $\operatorname{supp}\varphi \subset B_R$. Thus, for all $k \geq R$, it results $\varphi \in X_k$, so (iii) applies and $\langle d\mathcal{J}(u_k), \varphi\rangle = 0$;

direct computations give

$$
\begin{aligned}
0 \ &\le \ |\langle d\mathcal{J}(u^*),\varphi\rangle| = |\langle d\mathcal{J}(u_k),\varphi\rangle - \langle d\mathcal{J}(u^*),\varphi\rangle| \\
&\le \int_{B_R} |a(x,u_k,\nabla u_k) - a(x,u^*,\nabla u^*)||\nabla\varphi|dx \\
&+ |\varphi|_\infty \int_{B_R} |A_t(x,u_k,\nabla u_k) - A_t(x,u^*,\nabla u^*)|dx \\
&+ |\varphi|_\infty \int_{B_R} V(x)\big||u_k|^{p-2}u_k - |u^*|^{p-2}u^*\big|dx \\
&+ \int_{B_R} \big|\big[g(x,u_k) - g(x,u^*)\big]\varphi\big|dx.
\end{aligned}
\tag{5.18}
$$

Exploiting assumptions (h_0)-(h_1), (V_3) and (g_0)-(g_1) it is easy to prove that (5.1), (5.6) and Proposition 5.5 allows us to apply repeatedly the Dominated Convergence Theorem, hence all the integrals in (5.18) go to 0 and (5.17) follows. The density of $C_c^\infty(\mathbb{R}^N)$ in $W_V^{1,p}(\mathbb{R}^N)$ completes the proof. □

Proof of Theorem 4.4. From Proposition 5.8, we have that u^* is a critical point of $\mathcal{J}$, i.e., a weak bounded solution of (1.1). Suppose by contradiction that $u^* = 0$. From (5.5), it follows that $u_k \to 0$ in $L^p(\mathbb{R}^N)$. Hence, (3.13) and (3.14) imply that

$$
\int_{\mathbb{R}^N} G(x,u_k)dx \to 0 \tag{5.19}
$$

and

$$
\int_{\mathbb{R}^N} g(x,u_k)u_k dx \to 0. \tag{5.20}
$$

Furthermore, from (iii), (3.16), (h_4) and (h_3) it follows that

$$
\begin{aligned}
0 &= \langle d\mathcal{J}(u_k),u_k\rangle \\
&= \int_{\mathbb{R}^N} a(x,u_k,\nabla u_k)\cdot\nabla u_k dx + \int_{\mathbb{R}^N} A_t(x,u_k,\nabla u_k)u_k dx \\
&+ \int_{\mathbb{R}^N} V(x)|u_k|^p dx - \int_{\mathbb{R}^N} g(x,u_k)u_k dx \\
&\ge \alpha_2 \int_{\mathbb{R}^N} a(x,u_k,\nabla u_k)\cdot\nabla u_k dx + \int_{\mathbb{R}^N} V(x)|u_k|^p dx \\
&- \int_{\mathbb{R}^N} g(x,u_k)u_k dx \\
&\ge \alpha_1\alpha_2 \int_{\mathbb{R}^N} |\nabla u_k|^p dx + \int_{\mathbb{R}^N} V(x)|u_k|^p dx - \int_{\mathbb{R}^N} g(x,u_k)u_k dx,
\end{aligned}
$$

whence, by means of (3.1) and (5.20) it results that

$$
\|u_k\|_V^p \to 0 \quad \text{ as } k \to +\infty. \tag{5.21}
$$

Hence, from (3.15), (5.1), (h_1), (5.19) and (5.21) a positive constant $c > 0$ exists such that

$$
\mathcal{J}(u_k) \le c\|u_k\|_V^p - \int_{\mathbb{R}^N} G(x,u_k)dx \to 0
$$

which contradicts (ii). Then, it must be $u^* \not\equiv 0$. □

6. Existence of a positive solution

In this section, we prove the existence of a weak positive bounded solution of equation (1.1), i.e., a nontrivial weak bounded solution of the problem

$$\begin{cases} -\text{div}(a(x,u,\nabla u)) + A_t(x,u,\nabla u) + V(x)|u|^{p-2}u = g(x,u) & \text{in } \mathbb{R}^N, \\ \quad u \ \geq \ 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{6.1}$$

To this aim, we replace (h_1) with the stronger following assumption

(h_1') some positive continuous functions $\Phi_i, \phi_j : \mathbb{R} \to \mathbb{R}$, $i \in \{0,2\}, j \in \{0,1,2\}$ exist such that:

$$|A(x,t,\xi)| \le \Phi_0(t)|t|^p + \phi_0(t)|\xi|^p \qquad \text{a.e. in } \mathbb{R}^N, \text{ for all } (t,\xi) \in \mathbb{R} \times \mathbb{R}^N,$$
$$|A_t(x,t,\xi)| \le \phi_1(t)|\xi|^p \qquad \text{a.e. in } \mathbb{R}^N, \text{ for all } (t,\xi) \in \mathbb{R} \times \mathbb{R}^N,$$
$$|a(x,t,\xi)| \le \Phi_2(t)|t|^{p-1} + \phi_2(t)|\xi|^{p-1} \text{ a.e. in } \mathbb{R}^N, \text{ for all } (t,\xi) \in \mathbb{R} \times \mathbb{R}^N.$$

Moreover, we assume that $g(x,t)$ satisfies (g_0),(g_2) and

(g_1') a function $\eta \in L^{\frac{p}{p-q}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ exists, with $1 < q < p$, such that

$$0 \le g(x,t) \le \eta(x)t^{q-1} \qquad \text{a.e. in } \mathbb{R}^N, \text{ for all } t \ge 0.$$

Then, Theorem 4.4 can be improved as follows.

Theorem 6.1. *Assume that (h_0), (h_1'), (h_2)–(h_6), (V_1)–(V_3) and (g_0), (g_1') and (g_2) hold, then problem (6.1) admits at least a nontrivial weak bounded solution.*

From now on, we assume that all the hypotheses in Theorem 6.1 are verified. Clearly, (g_1') implies that $g(x,0) = 0$ for a.e. $x \in \mathbb{R}^N$.

Then, we introduce the new function $g_+ : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ defined as

$$g_+(x,t) = \begin{cases} g(x,u) & \text{a.e. in } \mathbb{R}^N, \text{ for all } t \ge 0 \\ 0 & \text{a.e. in } \mathbb{R}^N, \text{ for all } t < 0 \end{cases}$$

and the related primitive

$$G_+(x,t) = \begin{cases} G(x,u) & \text{a.e. in } \mathbb{R}^N, \text{ for all } t \ge 0 \\ 0 & \text{a.e. in } \mathbb{R}^N, \text{ for all } t < 0 \end{cases}$$

Remark 6.2. We note that $g_+(x,t)$ satisfies conditions (g_0)-(g_2).

From the previous remark and Proposition 3.10, it follows that the functional

$$\mathcal{J}_+(u) = \int_{\mathbb{R}^N} A(x,u,\nabla u)dx + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p dx - \int_{\mathbb{R}^N} G_+(x,u)dx$$

is of class C^1 on X and for any $u, v \in X$ it is

$$\begin{aligned} \langle d\mathcal{J}_+(u), v\rangle = &\int_{\mathbb{R}^N} a(x,u,\nabla u)\nabla v dx + \int_{\mathbb{R}^N} A_t(x,u,\nabla u)v dx \\ &+ \int_{\mathbb{R}^N} V(x)|u|^{p-2}uv dx - \int_{\mathbb{R}^N} g_+(x,u)v dx. \end{aligned} \tag{6.2}$$

Now, we prove that each critical point of $\mathcal{J}_+$ in X is nonnegative. The result has been stated in [15, Proposition 4.5] (see also, [1, Lemma 1.3]) in the case of bounded domains, here we give the details of the proof in $\mathbb{R}^N$.

Proposition 6.3. *If $u \in X$ is a critical point of $\mathcal{J}_+$, then $u \geq 0$ a.e. in $\mathbb{R}^N$. Hence, $\mathcal{J}(u) = \mathcal{J}_+(u)$ and $d\mathcal{J}(u) = 0$.*

Proof. Let u be a critical point of $\mathcal{J}_+$. Then, as in the proof of Proposition 5.7, we consider the real map $\psi(t) = te^{\bar{\eta}}t^2$ where, taking

$$K_u = \max_{|t|\leq|u|_\infty} \phi_1(t) \qquad \text{with } \phi_1(t) \text{ as in } (h_1'),$$

we choose $\bar{\eta} > \left(\dfrac{K_u}{2\alpha_1}\right)^2$. By definition, ψ is odd in $\mathbb{R}$ and

$$\alpha_1\psi'(t) - K_u|\psi(t)| > \frac{\alpha_1}{2} \qquad \text{for all } t \in \mathbb{R} \tag{6.3}$$

and

$$\psi(t) \geq t \qquad \text{for all } t \geq 0. \tag{6.4}$$

Since $d\mathcal{J}_+(u) = 0$, taking $v = \psi(-u_-)$ with $u_- = \max\{0, -u\}$, from (6.2) and $g_+(x,u)\psi(-u_-) = 0$ a.e. in $\mathbb{R}^N$ it follows that

$$\int_{\mathbb{R}^N} \psi'(-u_-)a(x,u,\nabla u)\cdot\nabla(-u_-)dx + \int_{\mathbb{R}^N} A_t(x,u,\nabla u)\psi(-u_-)dx + \int_{\mathbb{R}^N} V(x)|u|^{p-2}u\psi(-u_-)dx = 0.$$

Hence, as $u = -u_-$ in $\left(\mathbb{R}^N\right)_- = \{x \in \mathbb{R}^N : u(x) \leq 0\}$ and $u_- = 0$ a.e. in $\mathbb{R}^N \setminus \left(\mathbb{R}^N\right)_-$, from (h_1'), (h_3), (6.3), (6.4) and Remark 4.3 we have

$$\begin{aligned}
0 &= \int_{(\mathbb{R}^N)_-} \psi'(-u_-)a(x,-u_-,\nabla(-u_-))\cdot\nabla(-u_-)dx \\
&\quad + \int_{(\mathbb{R}^N)_-} A_t(x,-u_-,\nabla(-u_-))\psi(-u_-)dx \\
&\quad + \int_{(\mathbb{R}^N)_-} V(x)|-u_-|^{p-2}(-u_-)\psi(-u_-)dx \\
&\geq \int_{(\mathbb{R}^N)_-} \left(\alpha_1\psi'(u_-) - K_u|\psi(u_-)|\right)|\nabla u_-|^p dx \\
&\quad + \int_{(\mathbb{R}^N)_-} V(x)|u_-|^{p-2}(u_-)\psi(u_-)dx \\
&\geq \frac{\alpha_1}{2}\int_{(\mathbb{R}^N)_-} |\nabla(u_-)|^p dx + \int_{(\mathbb{R}^N)_-} V(x)|u_-|^p dx \\
&\geq \frac{\alpha_1}{2}\|u_-\|^p
\end{aligned}$$

which implies $u_- = 0$ a.e. in $\mathbb{R}^N$. □

Proof of Theorem 6.1. From Remark 6.2, all the assumptions of Theorem 4.4 hold, hence $\mathcal{J}_+$ has at least a nontrivial critical point $u^* \in X$. Then, Proposition 6.3 implies $u^* \geq 0$ a.e. in $\mathbb{R}^N$, so it is a nontrivial weak bounded solution of problem (6.1). □

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Declarations

Conflict of Interest *Always applicable and includes interests of a financial or personal nature:* This declaration is "not applicable", because we declare that the authors have no competing interests as defined by Springer, or other interests that might be perceived to influence the results and/or discussion reported in our paper.

Ethical Approval Applicable for both human and/ or animal studies. Ethical committees, Internal Review Boards and guidelines followed must be named. When applicable, additional headings with statements on consent to participate and consent to publish are also required: This declaration is not relevant to the content of our submission and for this reason we state that this declaration is "not applicable".

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