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Hilbert series of T -spaces

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ABSTRACT. We show that the Hilbert series of a T -space in a free associative algebra A is either a rational function, or differs from the Hilbert series of the commutator $[A, A]$ by a rational function.

Keywords: Hilbert series, PI -algebra, regular word, T -space.

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1 Introduction

We consider a field F and an infinite countable set of indeterminates $X = \{x_1, x_2, \dots, x_n, \dots\}$; we denote by $F\langle X \rangle$ the free (unital) associative algebra freely generated by X over F and we shall call *polynomials* its elements. An ideal of $F\langle X \rangle$ that is generated by the values of a system of polynomials closed under substitutions, is called a T -ideal. It is remarkable any T -ideal corresponds to the set of *polynomial identities* of a certain algebra that is, of course, called *algebra satisfying polynomial identities* or PI -algebra. By analogy with the notion of T -ideal, we have the notion of a T -space that is simply a vector subspace of $F\langle X \rangle$ generated by the values of a system of polynomials closed under substitutions. Indeed, any T -ideal is a T -space

but the converse is not true. For instance, we can consider the commutator subalgebra $[F\langle X \rangle, F\langle X \rangle]$ of the free associative algebra. As well as for T -ideals, to any T -space corresponds the set of *central polynomials* of a certain algebra, i.e., polynomials getting values in the center of the algebra.

In this paper we will handle *Hilbert series*, too. Hence, let us recall the key role of the study of Hilbert series in Mathematics, in the last century.

If A is a finitely generated commutative algebra, then the *Hilbert-Serre Theorem* says that the Hilbert series of A is rational (see [6] for the proof). Nevertheless, Hilbert-Serre theorem is not true in the case the algebra is noncommutative and, in this regard, we cite the work [5] by Anick where the author showed a famous counterexample. On the other hand, it is easy to show classes of noncommutative finitely generated algebras whose Hilbert series is rational. For instance, we have the class of finitely generated relatively free algebras, i.e., algebras isomorphic to the quotient of a finitely generated free algebra by a T -ideal. The rationality of Hilbert series for relatively free algebras carried a lot of results in PI-theory and we would like to cite among them the paper [15] by Berele and Regev in which the authors showed the exact asymptotic behaviour of the codimension sequence of a PI-algebra satisfying the Capelli identity. The analog of Hilbert-Serre Theorem also holds for classes of free algebras with additional structure, such as the class of relatively free G -graded algebras, where G is a finite group, satisfying an ordinary polynomial identity (Aljadeff and Kanel-Belov in [3]) as well as for the class of relatively free algebras of superalgebras with superinvolutions and of relatively free algebras of H -module algebras, where H is a finite dimensional semisimple Hopf algebra [18].

In this paper we aim to study Hilbert series of T -spaces. In particular, we have the following result (see Theorem 6 in the text).

Theorem 1 (Main Result). *Let S be a T -space in a finitely generated (relatively) free algebra A over a field of characteristic 0; we denote by $[A, A]$ the commutator space of A . Then one of the following two statements holds:*

- i) either $S \setminus [A, A] \neq \emptyset$, and the Hilbert series $H(S)$ of S is rational;*
- ii) $S \subseteq [A, A]$ and the difference between $H(S)$ and*

$$H([A, A]) = \sum_{n=1}^{\infty} (s^n - \sum_{d|n} \frac{1}{d} \sum_{r|d} \mu(r) s^{d/r}) t^n$$

is a rational function, where s is the number of generators of A .

Our result has, as a natural consequence, the following fact:

The Hilbert series of the center of a generic algebra is rational.

This can be compared with the last sentence of the following paper by Berele [13].

Let us spend a few words about the assumption on the field in the hypotheses of our Main Result. If the characteristic of the field is positive, then, according to Shchigolev's result (see [34]), the *locally finite basis property* (even for the case of 2-generated algebras) of T -spaces does not occur (see also [16] and [17]), although if the characteristic of the field is 0, then the global finite basis property holds. Hence one can find out that there exist continuum many pairwise different Hilbert series for T -spaces, differing two by two by a Hilbert series of a transcendental function. This is why we will only work over a field of characteristic 0.

2 Preliminaries

2.1 PI-algebras

From now on every field is supposed to be of characteristic zero and any algebra is supposed to be associative with unit 1.

Let $X = \{x_1, x_2, \dots, x_n, \dots\}$ be a countable set of indeterminates and consider $F\langle X \rangle$ the free algebra freely generated by X over F ; let us call *polynomials* its elements. Moreover we say a polynomial $f = f(x_1, \dots, x_n)$ is *multilinear* if every x_i appears only one time in each summand of f for every $i = 1, \dots, n$.

Definition 1. *Let $f = f(x_1, \dots, x_n)$ be a polynomial and A be an associative algebra. We say f is a polynomial identity for A if $f(a_1, \dots, a_n) = 0$ for any $a_1, \dots, a_n \in A$. If an algebra A satisfies a nontrivial polynomial identity, then A is said to be an algebra with polynomial identities (PI-algebra). We will denote the set of polynomial identities of a given algebra A by $T(A)$ (the T -ideal of A).*

Given a subset S of $F\langle X \rangle$, we define the T -ideal generated by S , and denoted by $\langle S \rangle^T$, as the smallest T -ideal containing S . Moreover, a T -ideal is an ideal of the free associative algebra that is invariant under endomorphisms of the free associative algebra. A vector space that is invariant under endomorphisms of the free associative algebra is called a T -space. Let $k \geq 1$ be an integer and consider the finite set of variables $X_k = \{x_1, \dots, x_k\}$. The quotient algebra $F_k(A) = F\langle X_k \rangle / (F\langle X_k \rangle \cap T(A))$ is the *relatively free algebras of A in k variables*. If one considers all the variables, the quotient algebra $F(A) := F\langle X \rangle / (F\langle X \rangle \cap T(A))$ is called the *relatively free algebra* of A .

The class of PI-algebras contains the class of commutative algebras and finite dimensional algebras, for instance. In particular, we have the following crucial result due to Amitsur and Levitzki.

Theorem 2. *Let F be any field. Then $M_n(F)$ satisfies a multilinear identity of degree $2n$.*

The definition of a PI-algebra carries the definition of a *variety* that is simply a class of algebras satisfying the identities of a given PI-algebra.

The theory of algebras with polynomial identities is now a big research area and it has been investigated since the first half of the 20-th century. For an exhaustive account toward algebras with polynomial identities we refer to the books by Giambruno and Zaicev [21], Drensky [20], Aljadeff, Giambruno, Procesi and Regev [1].

Being F a field of characteristic 0, the ideal of identities of a given F -algebra is determined by its multilinear part. Moreover, it is finitely generated as a T -ideal in the light of the precious work by Kemer (see [25] and [27]) and generalized to other polynomial identities settings by many authors (see for instance [2], [24], [18]). Let P_n denote the set of all multilinear polynomials in the variables x_1, \dots, x_n . For any $n \in \mathbb{N}$, let us consider the factor space

$$P_n(A) = \frac{P_n}{P_n \cap T(A)}.$$

The n -th *codimension* of A , denoted by $c_n(A)$, is the dimension of $P_n(A)$ as a vector space over F . Moreover, since $P_n(A)$ is a left S_n -module we shall denote its character by $\chi_n(A)$ and call it the n -th *cocharacter* of A . As a matter of fact, is more efficient to study $P_n(A)$ instead of $P_n \cap T(A)$ as showed by Regev in [33] where he proved if A is a *PI*-algebra, then its codimension sequence is exponentially bounded whereas $T(A) \cap P_n$ grows factorially.

We are going to see now the interplay between polynomial identities and representation theory of the symmetric group. Let $n \geq 1$ be a non negative integer. Remind a partition of n is simply an r -tuple $\lambda = (\lambda_1, \dots, \lambda_r)$ of integers such that

$$\lambda_1 \geq \dots \geq \lambda_r \text{ and } \lambda_1 + \dots + \lambda_r = n.$$

We shall use the notation $\lambda \vdash n$ to say λ is a partition of n and $|\lambda| = r$ to say that λ is a partition of height r .

We say that two partitions $\lambda = (\lambda_1, \dots, \lambda_r)$ and $\mu = (\mu_1, \dots, \mu_s)$ are equal if $r = s$ and $\lambda_1 = \mu_1, \dots, \lambda_r = \mu_s$.

Consider a partition $\lambda = (\lambda_1, \dots, \lambda_r)$ of a certain non negative integer. We associate to λ its *Young Diagram* having r rows such that the i -th row contains λ_i square.

Recall from the classical representation theory of groups that if G is a finite group and F is a field of characteristic 0 or $p \nmid |G|$, then *Maschke's Theorem* says that every finite dimensional representation is completely reducible, i.e., the group algebra $F[G]$ is semisimple and isomorphic to the direct sum of matrix algebras with entries from a division ring. Moreover, every finite dimensional left G -module is a direct sum of irreducible G -modules that are isomorphic to a minimal left ideals of $F[G]$. Putting $G = S_n$, the left irreducible S_n -modules, and therefore their related irreducible characters, may be described in terms of partitions and Young Diagrams.

Let M be an irreducible $F[S_n]$ -module with a Young diagram D , and let M' be a restriction of M , i.e., a module M considered as an $F[S_{n-1}]$ -module. The *Branching rule* states that M' is a direct sum of modules whose Young diagrams can be obtained from D by deleting one of its corner cells. Let f be a multilinear polynomial of degree n ; then it generates an $F[S_n]$ -module whose irreducible components are in one-to-one correspondence with the Young diagrams D_i . These components correspond to polynomials that are contained in the T -ideal generated by f and generate it. Let us call the diagrams D_i *Young diagrams corresponding to f* .

There is also a right action of S_n on P_n which permutes the position of variables in any monomial (see Section 3.1 of [22], page 124 and following); this action commutes with the left action. As already remarked, the left action preserves T -spaces and T -ideals but right action. For instance, let f, g be central polynomials of the 3×3 matrix algebra $M_3(F)$ that are, of course, identities for the 2×2 matrix algebra $M_2(F)$. Consider now the polynomials $h_r := f[x, g]$ and $h_l := [g, x]f$: the first one is an identity of the semidirect product of $M_2(F)$ with $M_3(F)$ while it is not an identity of the opposite semidirect product of $M_3(F)$ with $M_2(F)$; vice versa for the second one. Anyway, via the right action we can obtain h_l from h_r .

2.2 Hilbert series and PI-algebras

Another consequence of the knowledge of the cocharacter sequence of a PI-algebra relies to the knowledge of the Hilbert series of its relatively free algebra. But let us introduce first the key character of this paper, the *Hilbert series* of a module. We recall that given any \mathbb{Z} -graded F -algebra A and a finitely generated \mathbb{Z} -graded A -module $M = \bigoplus_{n \in \mathbb{N}} M_n$ such that for any $n \geq 0$ we have $\dim_F(M_n) < \infty$, if we denote by a_n the dimension of this subspace of A , the Hilbert series of M is defined to be the following formal power series:

$$H(M, t) = \sum_{n=0}^{\infty} a_n t^n,$$

where t is a variable. Indeed, the previous definition includes the definition of the Hilbert series of an algebra and of a T -space. It is a very big deal trying to build up a complete list of papers about Hilbert series of algebras. Nevertheless we would like to mention the papers by La Scala [28, 29] for the reduction of the computation of the Hilbert series of algebras to the case of a monomial algebras, i.e., an algebra generated by monomials in one of its generating sets.

In the following theorem we state a classical result by Serre (see, for instance, Section 11 of [6]).

Theorem 3. *If A is a commutative algebra, then $H(A, t)$ is a rational function.*

In 1997, Belov gave the following analog of Serre's result for relatively free algebras (see [8]).

Theorem 4 (Rationality of Hilbert series of relatively free algebras (ordinary case)). *The Hilbert series $H(F_k(A), t)$ of the relatively free algebra of a PI-algebra A in k variables is a rational function.*

Here we have the result obtained independently by Berele in [12] and Drensky in [19] linking the Hilbert series of the relatively free algebra and the cocharacter sequence of a PI-algebra.

Theorem 5. *Let A be a PI-algebra and suppose its cocharacter is*

$$\chi_n(A) = \sum m_\lambda \chi_\lambda,$$

then the Hilbert series of the relatively free of A is

$$H(A, T) = \sum_{n=0}^{\infty} \sum_{\lambda \vdash n} m_\lambda S_\lambda(t),$$

where $S_\lambda(T)$ denotes the Schur polynomial of shape λ in the variables t .

When the variables t are unambiguous, we write $H(M)$ instead of $H(M, t)$. We close this section with the following definition.

Definition 2. *Let S be a T -space and consider a PI-algebra A jointly with its relatively free algebra $F(A)$. Then we consider a projection $\pi(S)$ of S inside $F(A)$, then we define the Hilbert series of S in $F(A)$ as $H_{F(A)}(S, t) := H(\pi(S), t)$.*

3 Hilbert series of T -spaces

We alert the reader for the sake of simplicity we will slightly change the classical notations through this section.

The content of this chapter is devoted to the proof of the main result of this paper, as announced in the introduction. Here we recall the complete statement keeping in mind we use the symbol $[A, A]$ for the commutator space generated by A .

Theorem 6. *Let S be a T -space in a finitely generated (relatively) free algebra A over a field of characteristic 0. Then one of the following two issues holds:*

- a) $S \setminus [A, A] \neq \emptyset$, and the Hilbert series $H(S)$ is rational;
- b) $S \subseteq [A, A]$ and the difference between $H(S)$ and

$$H([A, A]) = \sum_{n=1}^{\infty} (s^n - \sum_{d|n} \frac{1}{d} \sum_{r|d} \mu(r) s^{d/r}) t^n$$

is a rational function, where s is the number of generators of A .

We start off by recalling the next result obtained by one of the authors in [9].

Theorem 7. *Let \mathcal{M} be a variety of algebras generated by a finite dimensional algebra over a field of characteristic zero such that all its subvarieties are representable. Then the Hilbert series $H_{F(\mathcal{M})}(S)$ of an arbitrary T -space S in a relatively free algebra $F(\mathcal{M})$ of \mathcal{M} is rational.*

Before carrying on, it is worth saying we will make use of the well known fact the Hilbert series of the quotient A/I , where I is an ideal of the algebra A , equals the difference $H(A, t) - H(I, t)$.

We will also make use of the following result by Schigolev [35].

Theorem 8. *Let S be a T -space in an absolutely free algebra A over a field of characteristic zero. Then a) If $S \setminus [A, A] \neq \emptyset$, then S contains a nonzero T -ideal Γ . b) If $S \subseteq [A, A]$, then there exists a T -ideal Γ such that $[\Gamma, A] \subseteq S$.*

If the condition a) occurs, then $H(S) = H(\Gamma) + H_{A/\Gamma}(S)$. By Theorem 4 and by Theorem 7 we easily the first assertion of Theorem 6. Thus from now on we assume $S \subseteq [A, A]$.

Hence we will focus on the following.

Lemma 1 (Main Lemma). *Let $S \subseteq [A, A]$. Then there exists a T -ideal Δ in A such that $\Delta \cap [A, A] \subseteq S$.*

We will show now how the proof of item b) of Theorem 6 will follow easily from Lemma 1. Notice that $H(S) = H_A(S) = H_{A/\Delta}(S) + H(\Delta \cap S) = H_{A/\Delta}(S) + H(\Delta \cap [A, A]) = H_{A/\Delta}(S) + H([A, A]) - H_{A/\Delta}([A, A])$. Because of Theorem 7, it follows $H(S)$ differs from the Hilbert series of the commutator $[A, A]$ by a rational function. Let us check for any $n \geq 0$, the dimension g_n of the n -th homogeneous component of $[A, A]$ satisfies the next equality:

$$g_n = s^n - \sum_{d|n} \frac{1}{d} \sum_{r|d} \mu(r) s^{d/r}. \quad (1)$$

For, two words u and v are said to be *cyclically conjugated* if for some words s, t we get $u = st, v = ts$; we say operation of *cyclic conjugation* any transformation of the word into one of its cyclic conjugates. A word is called *correct* if it is larger than any of its cyclic conjugate; a *semicorrect* word is a power of a correct word. Each word, which is not a power, uniquely corresponds to a cyclically conjugated correct word, and the degree of any word equals the degree of any of its cyclically conjugated word. Therefore the algebra A modulo $[A, A]$ has a basis of semicorrect words with dimension $g_n = s^n - h_n$, where h_n is the number of semicorrect words of length n . The number of correct words of the length n equals the dimension L_n of the space of Lie monomials of degree n and $L_n = \frac{1}{n} \sum_{r|n} \mu(r) s^{n/r}$, where $\mu(n)$ is the

Möbius function (see Theorem 1 on page 74 of [7]); hence the equality (1) follows from the next

$$h_n = \sum_{d|n} L_d = \sum_{d|n} \frac{1}{d} \sum_{r|d} \mu(r) s^{d/r}$$

and now the proof is complete.

Now we are in position to prove the Main Lemma. We are first and foremost interested in the case, when a multilinear polynomial f of degree n corresponds to one diagram. We denote by \mathcal{I}_m the T -ideal generated by multilinear polynomials corresponding to the Young diagrams containing an $m \times m$ square, where $m = n^4$. According to Chapter 5 of [23], $\mathcal{I}_m \subseteq \langle f \rangle^T$.

Lemma 2. *We have the following:*

a) *let a diagram D , corresponding to a polynomial f , contain an $(m+1) \times (m+1)$ square. Let $f = \sum f_i$, where each polynomial f_i is a linear combination of words containing a variable x_0 on the i -th position. Then all Young diagrams corresponding to f_i contain a square of size $m \times m$;*

b) *for each monomial of f_i , let us implement a cyclic conjugation in such a way that the variable x_0 lies in the first position; we obtain a polynomial f'_i . Then the Young diagrams corresponding to f'_i contain a square of size $m \times m$;*

c) *the polynomial of item b) can be taken so that $f_i - f'_i$ belongs to $[A, \mathcal{I}_{m-1}]$;*

d) *if $f \in [A, A]$, then $\sum f'_i = 0$.*

Proof. Let n be the degree of f , where $n \geq (m+1)^2$. We consider the action of the symmetric group S_n on words in the alphabet $\{x_0, x_1, \dots, x_{n-1}\}$. Then part a) follows from the Branching rule applied to the diagram D which eventually turns the $(m+1) \times (m+1)$ square into an $m \times m$ square (see also Theorem 3.2.14 of [22]). In fact we are inducing from S_n , considered as the permutations of $\{x_0, \dots, x_{n-1}\}$, to S_{n-1} , considered as the permutations of $\{x_1, \dots, x_{n-1}\}$.

Part b) follows from a). The polynomial $f = \sum f_i$ is a linear combination of its components according to the the position of x_0 . We consider the left action of S_n of the stabilizer of x_0 that is isomorphic to the action of S_{n-1} . This action of S_{n-1} preserves all f_i 's and the Young diagrams of this action contain an $m \times m$ square because of the Branching rule. For each monomial of f_i we have the same cyclicly conjugation according to the right action sending x_0 to the first position, obtaining a polynomial f'_i . Because the right action commutes with the left one, f'_i also corresponds to the Young diagram of our left S_{n-1} action containing an $m \times m$ square.

Part c) follows from part a), the Branching rule and the following fact: an operation of cyclic conjugation after which a variable from the i -th position moves to the first position, can be realized by $i - 1$ steps. At each step only

one first letter moves to the end and this procedure cannot corrupt an action of the group algebra of the symmetric group on other variables.

Towards item *d*), it is enough to consider a commutator of two words u and v and suppose uv contains the variable x_0 just once; the words uv and vu are cyclically conjugated. If we provide cyclically conjugation of uv and vu such that x_0 lies in the first position, then we get the same words s , defined uniquely, i.e., $s = (uv)' = (vu)'$; hence $([u, v])' = 0$.

□

Remark 1. *There is also a "trace argument" for item *d*). One can naturally define a free algebra A with trace Tr . The algebra A can be approximated by its factors which are $n \times n$ generic matrix algebras with traces when n goes to infinity; This shows Tr over A is non degenerate. Then part *d*) follows from the fact in the free algebra with a non degenerate trace an equality $h = 0$ follows from $\text{Tr}(xh) = 0$, where a polynomial h does not contain x . More precisely, here we have $0 = \text{Tr}(f) = \text{Tr}(x_0 \sum f'_i)$ implying $\sum f'_i = 0$ and we are done.*

Now we are ready to complete the proof of the Main Theorem.

Completion of the proof of the Main Lemma. Assume that a T -ideal Γ contains \mathcal{I}_m (such that $[\Gamma, A] \subseteq S$ as in Theorem 8). Let us show that for Δ one can take \mathcal{I}_{m+2} . In order to prove this fact, it is enough to show that, if $f \in \Delta \cap [A, A]$, then $\bar{f} = 0$ because in this case we will have $f \in \mathcal{I}_m \subseteq \Gamma$ which means $f \in [\Gamma, A] \subseteq S$ and the proof will follow. Of course, here \bar{f} stands for the polynomial obtained from f by Lemma 2. Hence it remains to show $\bar{f} = 0$. For, $f - \bar{f} \in [A, A]$, then $\bar{f} \in [A, A]$. Now, according to part *d*) of Lemma 2, a linear combination of terms, beginning with the letter x_0 , which does not appear anywhere else, cannot belong to the commutator. This shows $\bar{f} = 0$ and we are done.

□

4 Final remarks, positive characteristic

Let us consider the infinite dimensional Grassmann algebra over a field of characteristic $p > 2$ in the variety generated by the identity $[[x, y], z] = 0$ and the following polynomials

$$Q_k(x, y) := [x, y]x^{p^k-1}y^{p^k-1},$$

where $k \geq 1$ is an integer. The set $M_Q := \{Q_k(x, y) | k \geq 1\}$ generates an infinitely generated T -space as in [34] (see also its \mathbb{Z}_2 -graded version in [17]). Moreover, the polynomials $Q_k(x, y)$'s are independent and different subsets of M_Q do generate different T -spaces. Hence we have the following result.

Theorem 9. *There is a continuum of different Hilbert series of T -spaces for any $p > 0$ in two generated algebras such that all but countably many are transcendental functions.*

Remark 2. *As already said before, the Specht property for T -ideals holds for finitely generated associative algebras unlikely for T -spaces, as can be shown in [11, 26, 10]. It is opportune mentioning for a variety in characteristic 0 satisfying the Specht property implies the rationality of the Hilbert series of its relatively free algebra. This shows handling T -spaces is much more delicate than handling T -ideals.*

About future directions, from the top of our knowledge, it could be of great interest generalizing all the results of this work and those of Shchigolev for arbitrary G -gradings and for H -module algebras, where G stands for a finite group whereas H stands for a finite dimensional semisimple Hopf algebra.

Moreover, we think the manipulations made in order to get the main lemma of the present paper can be used in order to generalize the results in [14] to T -spaces, too. Supported by the recent paper [31], we would also address the reader to the study of codimensions of T -spaces with additional structures in the context of Belov's theorem.

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