https://doi.org/10.1007/s00025-022-01785-3

# INVARIANT CONDITIONAL EXPECTATIONS AND UNIQUE ERGODICITY FOR ANZAI SKEW-PRODUCTS

SIMONE DEL VECCHIO, FRANCESCO FIDALEO, AND STEFANO ROSSI

Abstract. Anzai skew-products are shown to be uniquely ergodic with respect to the fixed-point subalgebra if and only if there is a unique conditional expectation onto such a subalgebra which is invariant under the dynamics. For the particular case of skew-products, this solves a question raised by B. Abadie and K. Dykema in the wider context of $C^*$-dynamical systems.

## 1. introduction

A $C^*$-dynamical system, *i.e.* a pair $(\mathfrak{A}, \Phi)$ given by a unital $C^*$-algebra $\mathfrak{A}$ with unit $\mathbf{I}_{\mathfrak{A}}$ and a unital $*$-automorphism $\Phi$ of $\mathfrak{A}$, is said to be uniquely ergodic when there exists exactly one $\Phi$-invariant state $\omega$ on $\mathfrak{A}$. This condition turns out to be equivalent to the seemingly stronger condition that, for every $a \in \mathfrak{A}$, the Cesàro averages $\frac{1}{N}\sum_{k=0}^{N-1}\Phi^k(a)$ converge to $\omega(a)\mathbf{I}_{\mathfrak{A}}$ in norm. As is known, either condition implies that the fixed-point subalgebra $\mathfrak{A}^\Phi := \{a \in \mathfrak{A} \mid \Phi(a) = a\}$ is trivial, *i.e.* $\mathfrak{A}^\Phi = \mathbb{C}\mathbf{I}_{\mathfrak{A}}$.

It is then natural to turn one's attention to dynamical systems $(\mathfrak{A}, \Phi)$ for which the fixed-point subalgebra is allowed to be nontrivial but nevertheless any state on $\mathfrak{A}^\Phi$ only admits a unique $\Phi$-invariant extension to the whole $\mathfrak{A}$. To our knowledge, systems of this type were first introduced by Abadie and Dykema in [1], where they are referred to as dynamical systems uniquely ergodic with respect to the fixed-point subalgebra, in that they are a broad generalization of uniquely ergodic systems. Among other things, in that paper a number of equivalent conditions are given for a system to be uniquely ergodic with respect to the fixed point subalgebra. For instance, one is that, for any $a \in \mathfrak{A}$, the Cesàro averages $\frac{1}{N}\sum_{k=0}^{N-1}\Phi^k(a)$ still converge to some necessarily $\Phi$-invariant element $E(a)$. Note that $E$ defines a conditional expectation of $\mathfrak{A}$ onto $\mathfrak{A}^\Phi$ that is $\Phi$-invariant, namely $E \circ \Phi = E$. Moreover, $E$ is actually the only such conditional expectation.

2000 *Mathematics Subject Classification.* 37A05, 37A55, 37B05, 46L30.

*Key words and phrases.* Conditional Expectations, Skew-Products, Ergodic Dynamical Systems, Fixed-point Subalgebras, Unique Ergodicity.

It is not known, however, whether the existence of a unique $\Phi$-invariant conditional expectation as above is enough to obtain unique ergodicity with respect to the fixed-point subalgebra, and this was indeed formulated as a question in [1]. The present work settles the problem for so-called skew products, which are a remarkable family of classical dynamical systems, namely the $C^*$-algebra is commutative with the underlying topological space being a product of the form $X_o \times \mathbb{T}$, where $X_o$ is any compact Hausdorff space, and the dynamic is assigned through a homeomorphism $\Phi_{\theta_o,f}$ acting on $X_o \times \mathbb{T}$ as $\Phi_{\theta_o,f}(x,z) := (\theta_o(x), f(x)z)$, where $\theta_o$ is a uniquely ergodic homeomorphism of $X_o$ and $f : X_o \to \mathbb{T}$ is a continuous function.

We prove that any such system is uniquely ergodic with respect to the fixed point-subalgebra if and only if there exists a unique conditional expectation onto the fixed point-subalgebra. To accomplish this goal, we make use of another characterization of unique ergodicity with respect to the fixed-point algebra for skew products which has been proved in our previous work [4]. Indeed, a skew product $(X_o \times \mathbb{T}, \Phi_{\theta_o,f})$ is there seen to be uniquely ergodic with respect to the fixed point algebra if and only if, for any $n \in \mathbb{Z}$, a function $g : X_o \to \mathbb{C}$ satisfying $g(\theta_o(x))f^n(x) = g(x)$ is provided by a continuous (possibly zero) function. Our strategy, therefore, will be to show that when non-continuous solutions do exist, it is always possible to exhibit uncountably many $\Phi_{\theta_o,f}$-invariant conditional expectations.

Quite interestingly, the analysis of the invariant conditional expectations can be pushed further, for we also show that all invariant conditional expectations are dominated by a distinguished expectation as long as the fixed-point subalgebra is not trivial. Furthermore, this conditional expectation is exactly the only invariant conditional expectation when the skew product is uniquely ergodic with respect to the fixed-point subalgebra. Ultimately these facts allow us to spell out an extension of Fustenberg's characterization of uniquely ergodic skew-product dynamical systems ([7], Lemma 2.1) to the case of uniquely ergodic systems with respect to the fixed point subalgebra in terms of invariant conditional expectations (Theorem 4.2).

## 2. PRELIMINARIES

A (discrete) $C^*$-dynamical system is a pair $(\mathfrak{A}, \Phi)$ made of a $C^*$-algebra and positive map $\Phi : \mathfrak{A} \to \mathfrak{A}$. Suppose that $\mathfrak{A}$ is unital with identity $\mathbf{1} := \mathbf{1}_{\mathfrak{A}}$, and $\Phi$ completely positive and unital, that is $\Phi(\mathbf{1}) = \mathbf{1}$. It is said to be *topologically ergodic* if $\mathfrak{A}^\Phi = \mathbb{C}\mathbf{1}$ for the fixed-point subspace $\mathfrak{A}^\Phi := \{a \in \mathfrak{A} : \Phi(a) = a\}$.

The set $\mathcal{S}(\mathfrak{A})^\Phi := \{\varphi \in \mathcal{S}(\mathfrak{A}) \mid \varphi \circ \Phi = \varphi\}$ of the invariant states is convex and weak-* compact. The extremal invariant states are said to be *ergodic*. If the set of the invariant states is a singleton, that is $\mathcal{S}(\mathfrak{A})^\Phi = \{\varphi\}$, the $C^*$-dynamical system $(\mathfrak{A}, \Phi)$ is said to be *uniquely ergodic*. If in addition $\varphi$ is faithful, it is said to be *strictly ergodic*. In the uniquely ergodic case, we have $\mathfrak{A}^\Phi = \mathbb{C}\mathbf{I}$, $E(\,\cdot\,) := \varphi(\,\cdot\,)\mathbf{I}$ is an invariant completely positive projection onto the fixed-point subspace, and for the Cesáro averages,

$$\lim_n \frac{1}{n}\sum_{k=0}^{n-1}\Phi^k(a) = E(a)\,, \quad a \in \mathfrak{A}\,,$$

in norm.

The notion of unique ergodicity was recently generalised to the case when the fixed-point subspace is non trivial. The reader is referred to [1], Definition 3.3, for $*$-automorphisms where $\mathfrak{A}^\Phi$ is a $C^*$-subalgebra, and [6], Definition 2.2, for the more general case of completely positive maps.[1] For the purpose of the present paper, we adopt the following definition of unique ergodicity w.r.t. the fixed-point subalgebra.

**Definition 2.1.** *A $C^*$-dynamical system $(\mathfrak{A}, \alpha)$, with $\mathfrak{A}$ unital and $\alpha \in \mathrm{Aut}(\mathfrak{A})$ a $*$-automorphism, is said to be uniquely ergodic w.r.t. the fixed point subalgebra if the sequence $\big(\frac{1}{n}\sum_{k=0}^{n-1}\alpha^k(a)\big)_n$ converges in norm for each $a \in \mathfrak{A}$.*

With an abuse of notation, we say that the automorphism itself $\alpha$ is uniquely ergodic if it causes no confusion.

For uniquely ergodic systems as in Definition 2.1, the limit of the Cesáro averages defines a $\alpha$-invariant conditional expectation $E : \mathfrak{A} \to \mathfrak{A}^\alpha$ onto the fixed-point subalgebra given by

$$E(a) := \lim_n \frac{1}{n}\sum_{k=0}^{n-1}\alpha^k(a)\,, \quad a \in \mathfrak{A}\,,$$

which is necessarily unique.

Therefore, if $\mathfrak{A}^\alpha = \mathbb{C}\mathbf{I}$, the unique ergodicity (*i.e.* the convergence in norm of all averages $\big(\frac{1}{n}\sum_{k=0}^{n-1}\alpha^k(a)\big)_n$) is equivalent to the existence of a unique invariant conditional expectation $E : \mathfrak{A} \to \mathfrak{A}^\alpha$ which, due to the triviality of the fixed-point subalgebra, leads to $E = \varphi(\,\cdot\,)\mathbf{I}$, $\varphi \in \mathcal{S}(\mathfrak{A})$ being the unique invariant state.

[1] In [6], Theorem 2.1, it was also shown that the a-priori weaker condition (v), also characterises the unique ergodicity w.r.t. the fixed-point subspace.

It is of certain interest to decide when the question raised by B. Abadie and K. Dykema (*cf.* Question 3.4) of whether the unique ergodicity is equivalent to the existence of a unique invariant conditional expectations holds true. However, for a wide class of Anzai skew-products, called also *processes on the torus* in [7], for which $\mathfrak{A}^\alpha$ is always either trivial or infinite dimensional, we will show that the existence of a unique invariant conditional expectation onto the fixed-point subalgebra is indeed equivalent to unique ergodicity w.r.t. the fixed-point subalgebra.

**Remark 2.2.** *For the case $\mathfrak{A}^\alpha = \mathbb{C}\mathbf{1}$, Definition 2.1 is equivalent to the usual one: $(\mathfrak{A}, \alpha)$ is uniquely ergodic if, by definition, $\mathcal{S}(\mathfrak{A})^\alpha$ is a singleton, see e.g.* [13], *Theorem 4.1.8.*

From now on, we suppose that $\mathfrak{A}$ is a unital abelian $C^*$-algebra, and $\alpha$ is a $*$-automorphism. It is well known that any such a $C^*$-dynamical system arises as follows, $\mathfrak{A} \sim C(X)$, $X \sim \sigma(\mathfrak{A})$ being a compact Hausdorff space uniquely determined up to topological isomorphisms, and $\alpha(f) := f \circ \theta_o$ for some $\theta_o \in \mathrm{Homeo}(X)$, the space of all homeomorphisms of $X$. With a slight abuse of notation, we denote any such a $C^*$-dynamical system as above with $(X, \theta_o)$, and call these simply "a dynamical system".

One of such dynamical systems is thus uniquely ergodic if, by definition, there is only one regular Borel probability measure $\mu$ (*i.e.* a positive normalised Radon measure) on $X$ which is invariant under the transposed action $\nu \to \nu\circ\theta_o^{-1}$ of $\theta_o$ induced on measures $\nu$, $\mu = \mu\circ\theta_o^{-1}$. It is strictly ergodic if, in addition, $\mathrm{supp}(\mu) = X$.

A triplet $(X, \theta_o, \mu)$ denotes also a dynamical system, as soon as we want to point out any invariant measure $\mu$ as above, in particular when $(X, \theta_o)$ is uniquely ergodic and $\mu$ is its unique invariant measure.

The dynamical systems with which we deal with, called in [7] *processes on the torus*, are those on the cartesian product $(X_o \times \mathbb{T})$, where $X_o$ is a compact Hausdorff space and $\mathbb{T}$ is the one dimensional torus.

On $X_o \times \mathbb{T}$, for each $n > 1$ we consider the periodic homeomorphism $\mathrm{id}_{X_o} \times R_{2\pi\imath/n}$, together with the fixed-point subalgebra

$$C(X_o \times \mathbb{T})^{\beta_n} = \overline{\left\{ \sum_{l\in F} g_l(x) z^{l n_o} \mid g_l(x) \in C(X_o), F \,\text{finite subset of}\, \mathbb{Z} \right\}}$$

w.r.t. the canonical (dual) action $\beta_n$ on $C(X_o \times \mathbb{T})$ associated to such an homeomorphism.

A canonical $\beta_n$-invariant conditional expectation onto $C(X_o\times\mathbb{T})^{\beta_n}$ is uniquely defined by its action on generators

$$\mathcal{E}_n\big(h(x)z^k\big) := h(x)z^{ln}\delta_{k,ln}\,,\quad k,l\in\mathbb{Z}\,. \tag{2.1}$$

Indeed, since

$$\mathcal{E}_n(f)=\frac{1}{n}\sum_{l=0}^{n-1}\beta_n^l(f)\,,$$

we deduce that $\mathcal{E}_n$ is a faithful conditional expectation of $C(X_o\times\mathbb{T})$ onto $C(X_o\times\mathbb{T})^{\beta_n}$ which is invariant under the action of $\beta_n$.[2]

Our starting point will be a uniquely ergodic dynamical system $(X_o,\theta_o,\mu_o)$. Corresponding to a given continuous function $f\in C(X_o;\mathbb{T})$, we consider the *Anzai skew-product* (*cf.* [2]) $\Phi_{\theta_o,f}\in\mathrm{Homeo}(X_o\times\mathbb{T})$ defined as

$$\Phi_{\theta_o,f}(x,z) := \big(\theta_o(x),f(x)z\big)\,,\quad (x,z)\in X_o\times\mathbb{T}\,. \tag{2.2}$$

It is seen in [7] that the product measure $\mu:=\mu_o\times m$, where

$$m=\frac{\mathrm{d}\theta}{2\pi}=\frac{\mathrm{d}z}{2\pi\imath z}\,,\quad z=e^{\imath\theta}\in\mathbb{T}\,,$$

is the Haar-Lebesgue measure of the unit circle $\mathbb{T}$, is invariant for the dynamics induced by $\Phi_{\theta_o,f}$ on $(X_o\times\mathbb{T})$.

Most of the ergodic properties of $(X_o\times\mathbb{T},\Phi_{\theta_o,f})$ can be read through the kind of the solutions of the so-called cohomological equations, one for each $n\in\mathbb{Z}$.[3]

More precisely, for $g\in L^\infty(X_o,\mu_o)$ consider the multiplication operator $M_g\in\mathcal{B}\big(L^2(X_o,\mu_o)\big)$ given by

$$(M_g\xi)(x):=g(x)\xi(x)\,,\quad \xi\in L^2(X_o,\mu_o)\,.$$

We also have a (cyclic, with cyclic vector $\xi(x):=1$, $\mu_o$ a.e.) representation $\pi_{\mu_o}$ of $C(X_o)$ by multiplication operators, given for $G\in C(X_o)$ as

$$(\pi_{\mu_o}(G)\xi)(x):=G(x)\xi(x)\,,\quad \xi\in L^2(X_o,\mu_o)\,.$$

Corresponding to a skew-product $\Phi_{\theta_o,f}$, for each $n\in\mathbb{Z}$ we consider the *cohomological equations*

$$g(\theta_o(x))f(x)^n=g(x)\,,\ \mu_o\text{- a.e.}\,, \tag{2.3}$$

[2] The case $n=1$ corresponds to the trivial homeomorphism $\mathrm{id}_{(X_o\times\mathbb{T})}$ leading to the trivial fixed-point subalgebra $C(X_o\times\mathbb{T})$ and trivial conditional expectation $\mathcal{E}_1=\mathrm{id}_{C(X_o\times\mathbb{T})}$.

[3] The case $n=0$ corresponds always to the trivial solution $f(x)=1$, $\mu_o$ a.e., up to a multiplicative constant.

in the unknown complex function $g \in L^\infty(X_o, \mu_o)$. We also consider the twin equation

$$g(\theta_o(x))f(x)^n = g(x)\,, \tag{2.4}$$

where the unknown is now a function $g \in C(X_o)$. Obviously if $G$ satisfies (2.4), $\pi_{\mu_o}(G)$ satisfies (2.3).

For a fixed $n$, it is therefore natural to say that solutions $g = \pi_{\mu_o}(G)$ with $G$ satisfying (2.4) are said to be *continuous*, whereas the remaining one are named *measurable non-continuous*.

By unique ergodicity of $\theta_o$, the equation only has constant solutions for $n = 0$. Also note that, for every $n \in \mathbb{Z}$, the function that is zero is is a solution of (2.4), whereas that which is zero $\mu_o$-a.e. is a solution of (2.3). We we shall refer to those as the trivial solutions of the cohomological equation.

Throughout the paper, if a nontrivial solution of (2.3) is continuous (up to being re-defined on a $\mu_o$-negligible set) and satisfies (2.4), we will simply say that the cohomological equations have non-trivial continuous solutions. Note also that, if $g$ is a solution of (2.3) at the level $n$, then the two-variable function $h(x, z) := g(x)z^n$ is a continuous $\Phi_{\theta_o,f}$-invariant function if and only if $g$ is continuous and satisfies (2.4).[4]

We remark that, due to ergodicity of $(X_o, \theta_o, \mu_0)$, the solution of (2.3) for a fixed $n \in \mathbb{Z}$ is unique up to a multiplicative scalar. This was seen in [4], Proposition 8.2, by adapting the proof of [5], Proposition 2.2, to the present situation. Moreover, there is no loss of generality if those solutions are multiple of a single function with absolute value 1, almost everywhere w.r.t. $\mu_o$, see [3], Remark 4.2.

In [7], it was proved that the system $(X_o \times \mathbb{T}, \Phi_{\theta_o,f}, \mu)$ is ergodic if and only if, for every $n \neq 0$, (2.3) only have the trivial solution. Remarkably, the system $(X_o \times \mathbb{T}, \Phi_{\theta_o,f})$ is uniquely ergodic if and only if $(X_o \times \mathbb{T}, \Phi_{\theta_o,f}, \mu)$ is ergodic, provided that $(X_o, \theta_o)$ is uniquely ergodic with $\mu_o$ the unique invariant measure. In addition, topological ergodicity of $(X_o \times \mathbb{T}, \Phi_{\theta_o,f})$, that is $h \in C(X_o \times \mathbb{T})$ with $h \circ \Phi_{\theta_o,f} = h$ implies that $f$ is constant, is equivalent to the weaker request that continuous solutions of the cohomological equations are null for each $n \neq 0$.

The analysis in [7] for processes on the torus leaves open the case when the fixed-point subalgebra is non trivial, that is unique ergodicity w.r.t. the fixed-point subalgebra, which has been recently addressed

[4]To be more precise, whenever $g$ is a solution of (2.3) at the level $n$, $h(x, z) := g(x)z^n$ is $\mu_o \times m$-equivalent to a $\Phi_{\theta_o,f}$-invariant continuous function if and only if $g = \pi_{\mu_o}(G)$, and $G$ satisfies the twin equation (2.4).

in [4]. Indeed, in Theorem 10.7 of this paper it has been proved to amount to the condition that, for any $n \in \mathbb{Z}$, any solution of (2.3) is automatically continuous, that is its class of $\mu_o$-equivalence contains a continuous function satisfying (2.4).

The following simple result helps to further clarify the relation between solutions of cohomological equations (2.3) and (2.4).

**Proposition 2.3.** *If $g \in C(X)$ satisfies* (2.3) *for $n \in \mathbb{Z}$, then $g(\theta_o(x))f(x)^n = g(x)$ for $x \in \mathrm{supp}(\mu_o)$, and therefore $g$ satisfies automatically* (2.4) *if $(X, \theta_o)$ is strictly ergodic.*

*Proof.* Suppose that If $g \in C(X)$ satisfies (2.3) and choose any Borel set $A \subset X$ of full measure and, necessarily, $A \bigcap \mathrm{supp}(\mu_o)$ is dense in $\mathrm{supp}(\mu_o)$. For each $x \in \mathrm{supp}(\mu_o)$, choose a net $(x_\iota)_\iota \subset A \bigcap \mathrm{supp}(\mu_o)$ converging to $x$. We get

$$\begin{aligned} g(x) =& g\big(\lim_\iota x_\iota\big) = \lim_\iota g(x_\iota) = \lim_\iota \big(g(\theta_o(x_\iota))f(x_\iota)^n\big) \\ =& g\big(\theta_o(\lim_\iota x_\iota)\big)f(\lim_\iota x_\iota)^n = g(\theta_o(x))f(x)^n \,. \end{aligned}$$

□

**Remark 2.4.** *If $g \in C(X)$ satisfies* (2.3) *for $n \in \mathbb{Z}$, then $g(\theta_o(x))f(x)^n = g(x)$ for $x \in \mathrm{supp}(\mu_o)$, and therefore $g$ satisfies automatically* (2.4) *if $(X, \theta_o)$ is strictly ergodic.*

*Proof.* Indeed, suppose that If $g \in C(X)$ satisfies (2.3) and choose any Borel set $A \subset X$ of full measure and, necessarily, $A \bigcap \mathrm{supp}(\mu_o)$ is dense in $\mathrm{supp}(\mu_o)$. For each $x \in \mathrm{supp}(\mu_o)$, choose a net $(x_\iota)_\iota \subset A \bigcap \mathrm{supp}(\mu_o)$ converging to $x$. We get

$$\begin{aligned} g(x) =& g\big(\lim_\iota x_\iota\big) = \lim_\iota g(x_\iota) = \lim_\iota \big(g(\theta_o(x_\iota))f(x_\iota)^n\big) \\ =& g\big(\theta_o(\lim_\iota x_\iota)\big)f(\lim_\iota x_\iota)^n = g(\theta_o(x))f(x)^n \,. \end{aligned}$$

□

We end the section by remarking that all sums arising from the Fourier analysis on the unit circle $\mathbb{T}$ are understood to converge in the sense of Cesáro w.r.t. a fixed topology, usually that generated by the norm if is not differently specified, see *e.g.* [4],

## 3. ON INVARIANT CONDITIONAL EXPECTATIONS

We start with some results which are useful in the sequel. The first one provides a parametric generalisation of the Fejér-Riesz Theorem which has a self-containing interest.

**Proposition 3.1.** *Let $X_o$ be a topological space together with a strictly positive trigonometric polynomial*

$$p_K(x,z) := \sum_{|k|\leqslant K} b_k(x)z^k, \quad (x,z) \in X_o \times \mathbb{T}$$

*(i.e. $p_K(x,z) > 0$ for every $(x,z) \in X_o \times \mathbb{T}$), where the coefficients $b_k$, $k = -K, \dots, K$, are complex-valued (bounded) Borel functions on $X_o$.*

*Then there exists a trigonometric polynomial $g_K(x,z) = \sum_{k=0}^{K} a_k(x)z^k$, where the coefficients $a_k$, $k = 0, \dots, K$ are (bounded) Borel functions, such that $p_K = \overline{g_K} g_K$.*

*Proof.* We start by recalling that the Fejér-Riesz Theorem (see *e.g.* [10], Lemma 2.5) gives an explicit formula for the square root of a strictly positive trigonometric polynomial $q_K(z) = \sum_{|k|\leqslant K} b_k z^k$ with complex coefficients in terms of the roots of the polynomial $z^K q_K(z)$.

More precisely, let $q_K(z) = \sum_{|k|\leqslant K} b_k z^k$ with $b_K \neq 0$, be positive for $z \in \mathbb{T}$, and consider the $K$-roots $z_1, \cdots, z_K$ (counted with their multiplicity) of the polynomial $z^K q_K(z)$ which lie in the complement of the unit disk $\{z \in \mathbb{C} : |z| > 1\}$. Then the trigonometric polynomial

$$g_K(z) = \left|\frac{b_K}{z_1 \cdots z_K}\right|^{1/2} \prod_{i=1}^{K}(z - z_i) \tag{3.1}$$

satisfies the required property: $|g_K(z)|^2 = q_K(z)$, $z \in \mathbb{T}$. We now start by handling the case where $b_K(x) \neq 0$ for every $x \in X_o$.

Define $C := \{(w_0, w_1, \dots, w_{2K}) \in \mathbb{C}^{2K+1} \mid w_{2K} = 0\} \subset \mathbb{C}^{2K+1}$. Consider the map

$$\Sigma : \mathbb{C}^{2K+1} \backslash C \to \mathbb{C}^{2K}/S^{2K}$$

that, to the $2K+1$-tuple $(w_0, w_1, \dots, w_{2K})$, associates the set of the $2K$ roots of the polynomial $p(z) = \sum_{j=0}^{2K} w_j z^j$ considered as an element of the quotient of $\mathbb{C}^{2K}$ by the natural action of the symmetric group $S^{2K}$. By [8], Theorem A, the map $\sigma$ is continuous.

Given $q_K(z) = \sum_{|k|\leqslant K} b_k z^k$, we denote by $w_j$, $j = 0, 1, \dots, 2K$ the coefficients of $z^K q_K(z)$, that is $\sum_{j=0}^{2K} w_j z^j := z^K q_K(z)$. We note that

$$D := \left\{(w_0, w_1, \dots, w_{2K}) \in \mathbb{C}^{2K+1} \backslash C \,\middle|\, \sum_{j=0}^{2K} w_j z^{j-K} > 0, \;\; z \in \mathbb{T}\right\}.$$

is a Borel subset of $\mathbb{C}^{2K+1}$. Indeed, $D = (C^{2K+1} \backslash C) \cap D_1 \cap D_2$, where

$$D_1 := \left\{(w_0, w_1, \dots, w_{2K}) \in \mathbb{C}^{2K+1} \,\middle|\, \operatorname{Re} \sum_{j=0}^{2K} w_j z^{j-K} > 0, \;\; z \in \mathbb{T}\right\}$$

is open, and

$$D_2 := \left\{ (w_0, w_1, \dots, w_{2K}) \in \mathbb{C}^{2K+1} \,\middle|\, \operatorname{Im} \sum_{j=0}^{2K} w_j z^{j-K} = 0, \;\; z \in \mathbb{T} \right\}$$

is closed.

Consider the restriction of $\Sigma$ to $D$, and note that $\Sigma(D)$ is contained in the set of those non ordered $2K$-tuples such that $K$ entries have absolute value strictly greater than 1 and $K$ entries have absolute value strictly less than 1.

Define $\pi := P \circ \Sigma$ from $D$ to $\mathbb{C}^K/S^K$, where $P : \Sigma(D) \to \mathbb{C}^K/S^K$ is the map that selects the $K$ entries whose absolute value is greater than 1. The map $\pi$ is continuous on $D$ as it is the composition of continuous maps.

Now, from (3.1) one sees that, under our hypotheses, the coefficients $a_k$, $k = 0, \dots, K$, are Borel functions on $X_o$. Indeed, these are obtained as continuous symmetric functions of the roots $z_1, z_2, \dots, z_K$, which are in turn measurable functions on $X_o$ since they are the composition of the Borel measurable map

$$X_o \ni x \to (b_{-K}(x), \dots, b_K(x)) \in D^{2K+1}$$

with the continuous map $\pi$.

The general case can be dealt with by defining recursively for $l = 0, 1, \dots, K-1$,

$$M_{K-l} := \{ x \in X_o \mid b_j(x) = 0,\, j = K, \dots, K-l+1, \text{ and } c_{K-l}(x) \neq 0 \} ,$$

which are Borel subsets of $X_o$.

The proof ends by employing the same technique as above on the subsets $M_{k-l}$, $l = 0, 1, \dots, K-1$ to obtain the coefficients of $g_K$, by gluing finitely many Borel functions.

Concerning the boundedness of the coefficients $a_l$, we easily have

$$\left| g_K(x,z) \right|^2 = p_K(x,z) \leqslant \sup_{(x,z) \in X_o \times \mathbb{T}} p_K(x,z) \, , \quad (x,z) \in X_o \times \mathbb{T} \, .$$

Therefore, for each $x \in X_o$,

$$|a_l(x)| = \left| \oint g_K(x,z) z^{-l} \frac{\mathrm{d}z}{2\pi \imath z} \right| \leqslant \sqrt{\sup_{(x,z) \in X_o \times \mathbb{T}} p_K(x,z)} \, ,$$

which leads to $\left| g_K(x,z) \right| \leqslant (K+1) \sqrt{\sup_{(x,z) \in X_o \times \mathbb{T}} p_K(x,z)}$ . □

**Proposition 3.2.** *Suppose $U \in \mathfrak{A}$ is a unitary in the unital $C^*$-algebra $\mathfrak{A}$ such that, if $(\lambda_k)_{k\in\mathbb{Z}} \subset \ell^1(\mathbb{Z})$ and*

$$\sum_{k\in\mathbb{Z}} \lambda_k U^k = 0 \Rightarrow \lambda_k = 0\,, \quad k \in \mathbb{Z}\,.$$

*Then $C^*(U, \mathbf{I}_{\mathfrak{A}}) \sim C(\mathbb{T})$ through the $*$-isomorphism that sends $U^n$ to the character $\chi_n(z) := z^n$, $n \in \mathbb{Z}$.*

*Proof.* The same argument employed in the proof of Proposition 10.8 in [4], which we report for the convenience of the reader.

Put

$$\mathfrak{A}_o := \left\{ \sum_{m\in\mathbb{Z}} c_l U^l \mid \sum_{m\in\mathbb{Z}} |c_l| < \infty \right\},$$

endowed with the $\ell_1$-type norm

$$\left[\!\left| \sum_{l\in\mathbb{Z}} c_l U^l \right|\!\right] := \sum_{z\in\mathbb{Z}} |c_l|\,,$$

and observe that any element in $\mathfrak{A}_o$ provides a well-defined element of $\mathfrak{A}$ because the above sums defining the elements of $\mathfrak{A}_o$ are absolutely convergent in $\mathfrak{A}$.

In addition, $\mathfrak{A}_o$ is seen at once to be isometrically isomorphic with the Banach algebra $\ell_1(\mathbb{Z})$ understood as the convolution algebra of $\mathbb{Z}$.

Since the latter has only one $C^*$-completion, that is $C(\mathbb{T})$, we end the proof. ☐

**Example 3.1.** For the integer $k \geqslant 1$, consider the unitary in $M_k(C(\mathbb{T}))$ given by

$$U_k = \begin{pmatrix} 0 & 0 & \cdots & z \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Then $U_k^k$ is the diagonal matrix $zI_k = z\mathbf{I}_{\mathbb{M}_k(\mathbb{C})}$.

It easy to check that Proposition 3.5 tells us that

$$C(\mathbb{T}) \ni z^l \mapsto U_k^l \in \mathbb{M}_k\big(C(\mathbb{T})\big)$$

realises a $*$-monomorphism denoted by $\pi_k$.[5]

For the skew-product in (2.2), we recall some properties of the solutions of the cohomological equations introduced above.

[5]The case $k = 1$ corresponds to the trivial case when $M_k(C(\mathbb{T})) = C(\mathbb{T})$ and $\pi_k = \mathrm{id}$.

**Proposition 3.3.** *The elements of* $\mathbb{Z}$ *for which* (2.3) *(resp.* (2.4)*) admits nontrivial solutions provide a subgroup, and therefore there is an integer* $n_o \geqslant 0$ *(resp.* $m_0 \geqslant 0$*) for which such a subgroup is given by* $\{ln_o \mid l \in \mathbb{Z}\}$ *(resp.* $\{lm_o \mid l \in \mathbb{Z}\}$*). Since for a fixed* $n \in \mathbb{Z}$*, if* $g \in C(X)$ *satisfies* (2.4)*, it satisfies* (2.3)*, there is an integer* $k_o \geqslant 0$ *such that* $m_o = k_o n_o$*.*

*In addition, choosing a unitary* $u_{n_o}$ *satisfying* (2.3) *for* $n = n_o$ *(resp. satisfying* (2.4) *for* $n = m_o$*), all solutions of* (2.3) *for* $n = ln_o$ *(resp.* (2.4)*)* $n = lm_o$*) are a multiple of the powers of* $u^l_{n_o}$ *(resp.* $u^l_{m_o}$*).*

*Proof.* The proof is analogous to that of [4], Proposition 10.2. □

**Remark 3.4.** *According to the results of Furstenberg,* $n_o = 0$ *corresponds to unique ergodicity. The case* $k_o = 0$ *corresponds to topological ergodicity, and finally* $k_o = 1$ *to unique ergodicity w.r.t. the fixed-point subalgebra thanks to Proposition 10.7 in* [4]*.*

From now on, we suppose that $n_o > 0$ if is not otherwise specified, where $n_o$ is defined in Proposition 3.3.

For $k \geqslant 1$, denote by $\mathcal{A}_k \subset L^\infty(X_o \times \mathbb{T})$ the $C^*$-algebra generated by the functions $a_{lk}$ with

$$a_n(x,z) := \big(u_{n_0}(x) z^{n_o}\big)^n\,, \quad n \in \mathbb{Z}\,. \tag{3.2}$$

Obviously, all functions of $\mathcal{A}_k$ are $\Phi_{\theta_o,f}$-invariant, $\mu$-a.e. .

We report the following facts which are direct consequences of Proposition 3.2 and the Fejér-Riesz Theorem, respectively.

**Corollary 3.5.** *For each integer* $k \geqslant 1$*,* $\mathcal{A}_k \sim C(\mathbb{T})$ *in the* $*$*-isomorphism that sends* $\big(u_{n_0}\chi_{n_o}\big)^{kl}$ *to the character* $\chi_l(z) := z^l$*,* $l \in \mathbb{Z}$*.*

*Proof.* It will follow from Proposition 3.2, once we have verified its hypothesis is satisfied.

To this aim, let $(\lambda_l)_{l\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$, such that $\sum_{l\in\mathbb{Z}} \lambda_l \big(u_{n_o}\chi_{n_o}\big)^{kl} = 0$. Note that the above series converges totally.

For every $l' \in \mathbb{Z}$, multiplying both members of the above series by $\big(u_{n_0}\chi_{n_o}\big)^{-kl}$ one has

$$0 = \sum_{l\in\mathbb{Z}} \lambda_l \big(u_{n_o}\chi_{n_o}\big)^{kl} \big(u_{n_o}\chi_{n_o}\big)^{-kl'} = \sum_{l\in\mathbb{Z}} \lambda_l u_{n_o}^{k(l-l')} \chi_{n_o}^{k(l-l')}\,.$$

Integrating both members against the product measure $\mathrm{d}\mu_o \times \mathrm{d}m$, exchanging the integral with the sum and finally applying Fubini Theorem, we get

$$\begin{aligned} 0 &= \int_{X_o\times\mathbb{T}}\bigg(\sum_{l\in\mathbb{Z}}\lambda_l u_{n_o}(x)^{k(l-l')}\chi_{n_o}(z)^{k(l-l')}\bigg)\mathrm{d}\mu_o(x)\times\mathrm{d}m(z)\\ &= \sum_{l\in\mathbb{Z}}\lambda_l\int_{X_o}u_{n_o}^{k(l-l')}(x)\mathrm{d}\mu_o(x)\oint z^{kn_o(l-l')}\frac{\mathrm{d}z}{2\pi\imath z}\\ &= \sum_{l\in\mathbb{Z}}\lambda_l\delta_{l,l'} = \lambda_{l'}\,, \end{aligned}$$

which concludes the proof. □

**Remark 3.6.** *Notice that, if $\mathcal{A}_k \ni b = \sum_{|l|\leqslant L} b_l\big(u_{n_0}(x)z^{n_o}\big)^{kl}$ is positive, that is $b = c^*c$ for some $c \in \mathcal{A}_k$, then there exists $a = \sum_{l=0}^{L} a_l\big(u_{n_0}(x)z^{n_o}\big)^{kl}$ such that $b = a^*a$.*

*Proof.* It easily follows by Corollary 3.5 and Fejér-Riesz Theorem. □

We denote by $\rho_k : \mathcal{A}_k \to C(\mathbb{T})$ the isomorphisms described by Corollary 3.5. We also note that $C(X_o \times \mathbb{T})^{\Phi_{\theta_o,f}}$ is isomorphic with $C(\mathbb{T})$ whenever $k_o > 0$, see also [4], Proposition 10.8. For $x \in C(X_o)$, $z \in \mathbb{T}$ and $l \in \mathbb{Z}$, we indeed denote by $\sigma$ the map given by

$$C(\mathbb{T}) \ni \chi_l \mapsto \sigma(\chi_l)(x,z) := \big(v_{n_o}(x)z^{n_o}\big)^{k_o l} \in C(X_o \times \mathbb{T})^{\Phi_{\theta_o,f}}\,, \tag{3.3}$$

where $v_{n_o k_o} \in C(X_o)$ is a unitary satisfying (2.4) for $n = n_o k_o$.

Therefore, the fixed-point subalgebra $C(X_o \times \mathbb{T})^{\Phi_{\theta_o,f}}$ turns out to be isomorphic with $\mathcal{A}_{k_o}$ under $\rho_{k_o}^{-1} \circ \sigma^{-1}$, which is nothing but the map that sends each function in $C(X_o \times \mathbb{T})^{\Phi_{\theta_o,f}}$ to its equivalence class in $L^\infty(X_o \times \mathbb{T}, \mu)$.

We are going to construct $\Phi_{\theta_o,f}$-invariant conditional expectations from $C(X_o \times \mathbb{T})$ onto the fixed-point subalgebra $C(X_o \times \mathbb{T})^{\Phi_{\theta_o,f}}$. For such a purpose, we next single out a canonical contractive linear map $T$ from $C(X_o \times \mathbb{T})$ to $\mathcal{A}_1$ which is also $\Phi_{\theta_o,f}$-invariant. To do this, we start by recalling that $h \in C(X_o \times \mathbb{T})$ can be expressed as a series $h(x,z) = \sum_{n\in\mathbb{Z}} h_n(x)z^n$, with $h_n(x) = \oint h(x,z)z^{-n}\frac{\mathrm{d}z}{2\pi\imath z}$. Thanks to Fejér's theorem, the convergence of the series holds in norm in the Cesàro sense.

**Proposition 3.7.** *For $h(x,z) = \sum_{n\in\mathbb{Z}} h_n(x)z^n$,*

$$T(h) := \sum_{l\in\mathbb{Z}} a_l \int_{X_o} h_{ln_0}(x)u_{n_0}^{-l}(x)\mathrm{d}\mu_o(x)$$

*with the convergence being understood in norm in the Cesàro sense, defines a linear contractive map of* $C(X_o \times \mathbb{T})$ *to* $\mathcal{A}_1$.

*Moreover, for* $h \in C(X_o \times \mathbb{T})$ *and* $g \in C(X_o \times \mathbb{T})^{\Phi_{\theta_o,f}}$, $T(h \circ \Phi_{\theta_o,f}) = T(h)$, *and*

$$T(gh) = [g]_\mu T(h)\,, \quad T(hg) = T(h)[g]_\mu\,,$$

*where* $[g]_\mu$ *denotes the equivalence class of* $g$ *in* $L^\infty(X_o \times \mathbb{T}, \mu)$.

*Proof.* On $X_o \times \mathbb{T}$, we consider the periodic homeomorphism $\mathrm{id} \times R_{2\pi\imath/n_o}$, together with the fixed-point subalgebra

$$C(X_o \times \mathbb{T})^{\beta_{n_o}} = \left\{ \sum_{l \in \mathbb{Z}} g_l(x) z^{l n_o} \mid g_l(x) \in C(X_o) \right\}$$

w.r.t. the dual action associated to such an homeomorphism. A canonical $\beta_{n_o}$-invariant faithful conditional expectation $\mathcal{E}_{n_o}$ onto $C(X_o \times \mathbb{T})^{\beta_{n_o}}$ is given (2.1).

We now claim that $C(X_o \times \mathbb{T})^{\beta_{n_o}}$ embeds into the tensor product $\mathcal{A}_1 \otimes L^\infty(X_o, \mu_o)$ through the map $\iota$, completely determined on generators by

$$\iota(h(x) z^{l n_o}) = a_l(x_1, z) \otimes h(x_2) \overline{u_{n_0}(x_2)^l}\,.$$

Indeed, on the involutive subalgebra

$$\mathcal{C}_o := \left\{ \sum_{l \in F} g_l(x) z^{l n_o} \mid g_l(x) \in C(X_o),\ F \subset \mathbb{Z} \text{ finite} \right\},$$

$\iota$ is a well-defined $*$-homomorphism.

We want to show that $\iota$ is a positive map between the operator system $\mathcal{C}_o$ and the $C^*$-algebra $\mathcal{A}_1 \otimes L^\infty(X_o, \mu_o)$. To this aim, we first note that $\iota$ extends to $\mathcal{BB}_o$, the $*$-algebra made of elements of the form $\sum_{l \in F} g_l(x) z^{l n_o}$, $F \subset \mathbb{Z}$ finite and $g_l$ bounded Borel functions on $X_o$.

We now fix an element $h \in \mathcal{C}_o$ such that $h = c^* c$ for some $c \in \mathcal{C}$, and define $h_n := h + 1/n$. The $h_n$ belong to $\mathcal{BB}_o$ and are strictly positive, and thus by Proposition 3.1, there exists $(b_n)_n \subset \mathcal{BB}_o$ such that $h = b_n^* b_n$. Therefore,

$$\iota(h) + \frac{1}{n} = \iota(h_n) = \iota(b_n^* b_n) = \iota(b_n)^* \iota(b_n) \in V\,,$$

the convex cone of the positive elements of the $C^*$-algebra $\mathcal{A}_1 \otimes L^\infty(X_o, \mu_o)$, which is closed by [12], Theorem I.6.1.

By taking the limit on $n$, we easily deduce that $\iota : \mathcal{C}_o \to \mathcal{A}_1 \otimes L^\infty(X_o, \mu_o)$ is a positive map, and thus we can apply [10], Proposition 2.1, to conclude that $\iota$ is bounded on $\mathcal{C}_o$. Therefore, it extends to a bounded map on the whole $C(X_o \times \mathbb{T})^{\beta_{n_o}} = \overline{\mathcal{C}_o}$ which will be also a $*$-homomorphism denoted again by $\iota$ with an abuse of notation. The claim is now proved by noticing that $T$ can be expressed as the composition of three bounded maps as $T=\left(\mathrm{id}\otimes\int\cdot\,\mathrm{d}\mu_o\right)\circ\iota\circ\mathcal{E}_{n_o}$.

To end the proof, it is enough to verify the $\Phi_{\theta_o,f}$ invariance on the total set of generators of the form $h(x,z)=g(x)z^n$ of $C(X_o\times\mathbb{T})$, the proof of the module-map properties being similar.

Indeed, if $n\neq 0$ is not a multiple of $n_o$, then $T(h)=0=T(h\circ\Phi_{\theta_o,f})$. If instead $n=ln_o$, then

$$\begin{aligned}T(h\circ\Phi_{\theta_o,f})=&a_l\int_{X_o}g(\theta_o(x))f(x)^{ln_0}u_{n_0}(x)^{-l}\mathrm{d}\mu_o\\=&a_l\int_{X_o}g(\theta_o(x))u_{ln_0}(\theta_o(x))^{-l}\mathrm{d}\mu_o\\=&a_l\int_{X_o}g(x)u_{ln_0}(x)^{-l}\mathrm{d}\mu_o=T(h).\end{aligned}$$

□

**Remark 3.8.** *Note that, when there are only continuous solutions of the cohomological equation (i.e. $k_o=1$), the map $\sigma\circ\rho_1\circ T$, $\sigma$ given in* (3.3)*, yields an invariant conditional expectation onto the fixed-point subalgebra. By* [4]*, Theorem 10.7, and* [1]*, Theorem 3.2, it is in fact the unique invariant conditional expectation on the fixed-point subalgebra.*

*When instead there are no nontrivial solutions of the cohomological equation (i.e. $n_o=0$), it is easily checked that the map $T$ yields the state on $C(X_o\times\mathbb{T})$ corresponding to $\int_{X_o\times\mathbb{T}}\cdot\,\mathrm{d}\mu_o\times\mathrm{d}\theta/2\pi$ which is the unique invariant measure under the action of $\Phi_{\theta_o,f}$.*

The following ought to be known. Nevertheless, we include a sketched proof for convenience and establish some notation. Given a square matrix $C=(c_{i,j})_{i,j=1}^k\in\mathbb{M}_k(\mathbb{C})$, for $0\leqslant l<k-1$ its $l$-diagonal is the set of the entries $\{c_{1,l+1},c_{2,l+2},\dots,c_{k-l,k}\}$, and its $l$-trace is the number $\mathrm{tr}_l(C):=\sum_{j=1}^{k-l}c_{j,l+j}$. Note that $\mathrm{tr}_0$ is the usual trace Tr of $\mathbb{M}_k(\mathbb{C})$.

Denote by $\mathcal{B}_k\subset C(\mathbb{T})$ the $C^*$-algebra generated by all powers of the function $z^k$. For each $A\in M_k(\mathbb{C})_+$ with $\mathrm{Tr}(A)=1$ and $x\in\pi_k(C(\mathbb{T}))\subset M_k(C(\mathbb{T}))$, where $\pi_k:C(\mathbb{T})\to M_k(C(\mathbb{T}))$ is the $^*$-monomorphism considered in Example 3.1, set

$$\widetilde{F}_A(x):=\begin{pmatrix}\mathrm{Tr}(Ax)&0&\cdots&0\\0&\mathrm{Tr}(Ax)&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&\cdots&0&\mathrm{Tr}(Ax)\end{pmatrix}\tag{3.4}$$

By easy computations, one can verify that $\widetilde{F}_A$ is a conditional expectation from $\pi_k(C(\mathbb{T}))$ onto $\pi_k(\mathcal{B}_k)$.

**Lemma 3.9.** *For any $A \in M_k(\mathbb{C})_+$ with $\mathrm{Tr}(A) = 1$, the map $F_A := \pi_k^{-1} \circ \widetilde{F}_A \circ \pi_k$ provides a conditional expectation of $C(\mathbb{T})$ onto $\mathcal{B}_k$. Moreover, $F_A = F_B$ if and only if $\mathrm{tr}_l(A) = \mathrm{tr}_l(B)$ for every $l = 1, \ldots, k-1$.*

*Proof.* That $F_A$ is a conditional expectation is a straightforward consequence of its definition, for $\widetilde{F}_A$ is a conditional expectation and $F_A$ is obtained out of the latter by conjugation by the $*$-isomorphism $\pi_k$.

As for the second part of the statement, given two positive matrices $A = (a_{i,j}), B = (b_{i,j}) \in \mathbb{M}_k(\mathbb{C})$ with $\mathrm{tr}_0(A) = \mathrm{tr}_0(B) = 1$, one has $\widetilde{F}_A = \widetilde{F}_B$ if and only if $\mathrm{tr}_0(Ax) = \mathrm{tr}_0(Bx)$ for any $x \in C^*(U)$, which is the same as $\mathrm{tr}_0(AU^l) = \mathrm{tr}_0(BU^l)$ for every $l \in \mathbb{Z}$. It is actually enough to consider only positive values of the integer $l$ thanks to the equality

$$\mathrm{tr}_0(AU^{-l}) = \mathrm{tr}_0((U^l A)^*) = \overline{\mathrm{tr}_0(U^l A)} = \overline{\mathrm{tr}_0(AU^l)}\,,$$

which holds for every $l \in \mathbb{N}$. For $l = 1, 2, \ldots, k-1$, the conditions $\mathrm{tr}_0(AU^l) = \mathrm{tr}_0(BU^l)$ can be written explicitly as

$$z\mathrm{tr}_{k-l}(A) + \mathrm{tr}_l(A) = z\mathrm{tr}_{k-l}(B) + \mathrm{tr}_l(B), \text{ for any } z \in \mathbb{T}. \tag{3.5}$$

If $\mathrm{tr}_l(A) = \mathrm{tr}_l(B)$ for every $l = 1, 2, \ldots, k-1$ then the equalities (3.5) are certainly satisfied. Conversely, rewrite (3.5) as $z(\mathrm{tr}_{k-l}(A) - \mathrm{tr}_{k-l}(B)) = \mathrm{tr}_l(B) - \mathrm{tr}_l(A)$, for any $z \in \mathbb{T}$. Since the r.h.s. of the last equality does not depend on $z$, the only possibility is $\mathrm{tr}_{k-l}(A) - \mathrm{tr}_{k-l}(B) = 0$, hence $\mathrm{tr}_l(A) = \mathrm{tr}_l(B)$ for any $l = 1, 2, ..., k-1$.

Finally if $l \geqslant k$, we can rewrite $l$ as $l = mk + l'$, for some $m \in \mathbb{N}$ and $l' = 0, 1, \ldots, k-1$. But then $\mathrm{tr}_0(AU^l) = \mathrm{tr}_0(AU^{mk}U^{l'}) = \mathrm{tr}_0(U^{mk}AU^{l'}) = z^m \mathrm{tr}_0(AU^{l'}) = z^m(z\mathrm{tr}_{k-l'}(A) + \mathrm{tr}_{l'}(A))$ (or simply $z^m \mathrm{tr}_0(A)$ when $l' = 0$), which means $\mathrm{tr}_0(AU^l) = \mathrm{tr}_0(BU^l)$ is still satisfied. This ends the proof. □

**Proposition 3.10.** *Let $m_o = n_o k_o$ be different from 0. Then, for $A \in M_{k_o}(\mathbb{C})_+$ with $\mathrm{Tr}(A) = 1$, the map*

$$E_A := \sigma \circ F_A \circ \rho_1 \circ T \tag{3.6}$$

*provides an invariant conditional expectation of $C(X_o \times \mathbb{T})$ onto the fixed-point subalgebra $C(X_o \times \mathbb{T})^{\Phi_{\theta_o,f}}$.*

*Moreover, $E_A = E_B$ if and only if $\mathrm{tr}_l(A) = \mathrm{tr}_l(B)$ for every $l = 1, \ldots, k-1$, where $\mathrm{tr}_l$ is the $l$-trace.*

*Proof.* The fact that $E_A$ is an invariant conditional expectation follows immediately from the properties of $T$ stated in Propostion 3.7, by taking into account the identification $C(\mathbb{T}) \sim C(X_o \times \mathbb{T})^{\Phi_{\theta_o,f}}$ given in (3.3).

Since the range of $T$ is dense in $\mathcal{A}_1$, the equality $E_A = E_B$ holds if and only if $F_A = F_B$, and the second statement follows directly from Lemma 3.9. □

As noticed above (*cf.* Remark 3.8), in the case $k_o = 1$ that is when there are only continuous solutions of the cohomological equations (2.3), (3.6) provides the unique $\Phi_{\theta_o,f}$-invariant conditional expectation of $C(X_o \times \mathbb{T})$ onto the fixed-point subalgebra.

## 4. Invariant conditional expectations and ergodicity w.r.t. the fixed-point subalgebra

We start the present section to prove the main result of the paper, that is to answer in positive Question 3.4 in [1] for the (classical) Anzai skew-product. For such a purpose, we start with the following

**Remark 4.1.** *If there are no non trivial continuous solutions to the cohomological equations, then the fixed-point subalgebra $C(X_o \times \mathbb{T})^{\Phi_{\theta_o,f}}$ is trivial. Therefore, a $\Phi_{\theta_o,f}$-invariant conditional expectation is simply described by a $\Phi_{\theta_o,f}$-invariant state.*

*Since there are non trivial measurable non-continuous solutions, the system cannot be uniquely ergodic and thus there exist infinitely many $\Phi_{\theta_o,f}$-invariant states. For instance, a family of such states is obtained by considering the composition $\varphi \circ T$, where $\varphi$ is any state on $\mathcal{A}_1$ and $T$ is described in Proposition 3.7.*

We recall that, for the definition of ergodicity w.r.t the fixed-point subalgebra, we are adopting Definition 2.1 which turns out to be equivalent to the conditions (i)-(v) listed in [1], Theorem 3.2 (see also [6], Theorem 2.1).

**Theorem 4.2.** *Let $(X_o, \theta_o, \mu_o)$ be a uniquely ergodic dynamical system, together with the Anzai skew product $\Phi_{\theta_o,f}$ associated to $f \in C(X_o; \mathbb{T})$. Then the following are equivalent:*

(i) *$\Phi_{\theta_o,f}$ is uniquely ergodic w.r.t. the fixed-point subalgebra,*
(ii) *there exists only one $\Phi_{\theta_o,f}$-invariant conditional expectation onto the fixed-point subalgebra.*

*Proof.* We start by recalling Proposition 3.3 describing the structure of the solutions of the solutions of the cohomological equations (2.3) and (2.4). In the sequel we adopt the notations used therein.

The case when there exists no nontrivial solutions (but the constant ones) corresponds to $n_o = 0$ for which the result holds true (*e.g.* Remark 2.2). Therefore, we assume that (2.3) admits nontrivial solutions for some $n \in \mathbb{Z} \setminus \{0\}$, which corresponds $n_o > 0$.

Since the implication (i)⇒(ii) holds in general, it remains to prove the implication (ii)⇒(i) for a nontrivial fixed-point subalgebra $C(X_o \times \mathbb{T})^{\Phi_{\theta_o,f}}$. If (i) does not hold, by Theorem 10.7 in [4] there must exist measurable non-continuous solutions of the cohomological equations, which means Proposition 3.10 applies providing plenty of conditional expectations, and (ii) does not hold either. □

**Remark 4.3.** *As noticed before (cf. [4], Theorem 10.7), the (equivalent) conditions (i) and (ii) are also equivalent to*

(iii) *the cohomological equations* (2.3) *admit only continuous solutions, that is if $g$ satisfies* (2.3) *for some $n \in \mathbb{Z}$, then $g = \pi_{\mu_o}(G)$ where $G$ satisfies* (2.4) *for the same $n$ or, in other words, $G \in C(\mathbb{T})$ and $g = G$, $\mu_o$-a.e. .*

By Proposition (3.3) (and with the notations used therein), (iii) corresponds either to $n_o = 0$, the uniquely ergodic case, or $n_o > 0$ and $k_o = 1$, the uniquely ergodic cases with nontrivial fixed-point subalgebra.

We now pass to study some properties of the set of all invariant conditional expectations which allow us to characterise unique ergodicity w.r.t. the fixed-point subalgebra. With $F_A$ and $\mathcal{E}_n$, we refer to the maps in Lemma 3.9 and (2.1), respectively.

**Proposition 4.4.** *Suppose $m_o > 0$. Then the $\Phi_{\theta_o,f}$-invariant conditional expectation $E_{\frac{1}{k_o}\mathbf{1}} = \sigma \circ F_{\frac{1}{k_o}\mathbf{1}} \circ \rho_1 \circ T$ associated with $\frac{1}{k_o}\mathbf{1}$ satisfies*

$$E_{\frac{1}{k_o}\mathbf{1}} \circ \mathcal{E}_{m_0} = E_{\frac{1}{k_o}\mathbf{1}}. \tag{4.1}$$

*In addition, every $\Phi_{\theta_o,f}$-invariant conditional expectation $E$ from $C(X_o \times \mathbb{T})$ onto $C(X_o \times \mathbb{T})^{\Phi_{\theta_o,f}}$ satisfies*

$$E(a) \leqslant m_o E_{\frac{1}{k_o}\mathbf{1}}(a)\,, \quad a \in C(X_o \times \mathbb{T})_+\,. \tag{4.2}$$

*Proof.* For the first part of the statement it is enough to show

$$E_{\frac{1}{k_o}\mathbf{1}} \circ \mathcal{E}_{m_o}(h \cdot \chi_l) = E_{\frac{1}{k_o}\mathbf{1}}(h \cdot \chi_l)$$

for $l \in \mathbb{Z}$ and $h \in C(X_o)$, where $\chi_l(z) := z^l$, $z \in \mathbb{T}$.

Note that, if $l$ is not a multiple of $m_o$, then by the definition of $\mathcal{E}_{m_o}$, the l.h.s. is zero. The same can be said for the r.h.s. since for $l$ not a multiple of $n_o$, by the definition of the map $T$, one has $T(h \cdot \chi_l) = 0$ and thus $E_{\frac{1}{k_o}\mathbf{1}}(h \cdot \chi_l) = 0$.

We now handle the case when $l$ is a multiple of $n_o$ but not of $m_o$. From the equality $T(h \cdot \chi_l) = \omega_o(h \cdot \bar{u}_l)[u_l\chi_l]_\mu$, we see that

$$E_{\frac{1}{k_o}\mathbf{1}}(h \cdot \chi_l) = \sigma \circ F_{\frac{1}{k_o}\mathbf{1}} \circ \rho_1 \circ T(h \cdot \chi_l) = \omega_o(h \cdot \bar{u}_l)\, \sigma \circ F_{\frac{1}{k_o}\mathbf{1}} \circ \rho_1([u_l\chi_l]_\mu)\,.$$

But then, $E_{\frac{1}{k_o}\mathbf{1}}(h\cdot\chi_l)=0$ because $F_{\frac{1}{k_o}\mathbf{1}}\circ\rho_1([u_l\chi_l]_\mu)=F_{\frac{1}{k_o}\mathbf{1}}(\chi_{\frac{l}{n_o}})=0$.

Finally, if $l$ is a multiple of $m_0$, then $\mathcal{E}_{m_0}(h\chi_l)=h\chi_l$ for any $h\in C(X_o)$, and the sought equality follows.

For the second statement, first note that for every invariant conditional expectation $E$ from $C(X_o\times\mathbb{T})$ onto the fixed-point subalgebra $C(X_o\times\mathbb{T})^{\Phi_f}$, one has

$$E_{\frac{1}{k_o}\mathbf{1}}=E_{\frac{1}{k_o}\mathbf{1}}\circ\mathcal{E}_{m_o}=E\circ\mathcal{E}_{m_o}\,.$$

Indeed, by density, linearity and what we saw above, it is enough to verify the equality only on functions $h$ of the form $h(x,z)=\ell(x)z^{lm_o}$, where $\ell\in C(X_o)$ and $l\in\mathbb{Z}$.

Recalling the definition of the functions $a_n$ as $a_n(x,z):=\big(u_{n_0}(x)z^{n_o}\big)^n$, one has

$$\begin{aligned}(E\circ\mathcal{E}_{m_o}(h))\,(x,z)=&(E(h))(x,z)=\left(E(\ell\overline{u_{lk_o}}a_{lk_o})\right)(x,z)\\ =&\left(\frac{1}{n}\sum_{j=0}^{n-1}E(h\circ\Phi^j_{\theta_o,f})\right)(x,z)\\ =&a_{lk_o}(x,z)E\left(\frac{1}{n}\sum_{j=0}^{n-1}\ell(\theta^j(x))\overline{u_{lm_o}(\theta^j(x))}\right)\\ =&a_{lk_o}(x,z)\omega_o(\ell\overline{u_{lm_o}}),\end{aligned}$$

where the last equality has been obtained by exploting the unique ergodicity of $\theta_o$, whose unique invariant state is $\omega_o$.

Now, from the equality $\mathcal{E}_{m_o}=\frac{1}{m_o}\sum_{l=0}^{m_o-1}\beta^l_{m_o}$, we see that

$$E_{\frac{1}{k_o}\mathbf{1}}=\frac{1}{m_o}E+\frac{1}{m_o}\sum_{l=1}^{m_o-1}E\circ\beta^l_{m_o}\,,$$

hence $E_{\frac{1}{k_o}\mathbf{1}}(a)\geqslant\frac{1}{m_o}E(a)$ for all positive functions $a$ in $C(X_o\times\mathbb{T})$, as stated. □

Consider the convex set $\mathcal{K}:=\big\{E\mid E$ is a $\Phi_{\theta_o,f}$-invariant conditional expectation of $C(X_o\times\mathbb{T})$ onto $C(X_o\times\mathbb{T})^{\Phi_{\theta_o,f}}\big\}$. In the topologically ergodic situation (*i.e.* $C(X_o\times\mathbb{T})^{\Phi_{\theta_o,f}}=\mathbb{C}\mathbf{1}$), any such conditional expectation is associated with a $\Phi_{\theta_o,f}$-invariant state $\varphi$: $E\equiv E_\varphi:=\varphi(\,\cdot\,)\mathbf{1}$.

It was proved in [7] (see also [3, 4] for the noncommutative cases) that

$\big(X_o\times\mathbb{T},\Phi_{\theta_o,f}\big)$ topologically ergodic & $E_{\varphi_\mu}$ extremal $\Rightarrow$ $\big(X_o\times\mathbb{T},\Phi_{\theta_o,f}\big)$ uniquely ergodic, or equivalently $\mathcal{K}$ is a singleton.

We want to extend this result to the non-topologically ergodic situation corresponding to $m_o > 0$, where $E_{\varphi_\mu}$ is replaced by $E_{\frac{1}{k_o}\mathbf{1}}$. We set

$$E_{\text{can}} := \begin{cases} E_{\varphi_\mu} & \text{if } m_o = 0\,, \\ E_{\frac{1}{k_o}\mathbf{1}} & \text{if } m_o > 0\,. \end{cases}$$

**Theorem 4.5.** *For the Anzai skew-product $C^*$-dynamical system $(X_o \times \mathbb{T}, \Phi_{\theta_o,f})$, the equivalent conditions* (i), (ii) *in Theorem 4.2 and* (iii) *in Remark 4.3 are equivalent to*

(iv) *$E_{can}$ is extreme among all $\Phi_{\theta_o,f}$-invariant conditional expectations of $C(X_o \times \mathbb{T})$ onto $C(X_o \times \mathbb{T})^{\Phi_{\theta_o,f}}$.*

*Proof.* We only need to focus on the case $m_o > 1$, and thus $E_{\text{can}} = E_{\frac{1}{k_o}\mathbf{1}}$. Indeed, the assertion is certainly true when $m_o = 0$, *i.e.* in the topologically ergodic case, whereas the case $m_o = 1$ corresponds to a uniquely ergodic system w.r.t. the fixed-point subalgebra, see *e.g.* Remark 3.4.

The implication (ii)⇒(iv) is entirely obvious. We limit ourselves to the reverse implication. Suppose that $E_{\frac{1}{k_o}\mathbf{1}}$ is extreme among all invariant conditional expectations. If there existed an invariant conditional expectation $E$ different from $E_{\frac{1}{k_o}\mathbf{1}}$ then, thanks to Proposition 4.4, we would have $E(a^*a) \leqslant m_o E_{\frac{1}{k_o}\mathbf{1}}(a^*a)$, for every $a \in C(X_o \times \mathbb{T})$. But then $F := \frac{1}{m_o-1}(m_o E_{\frac{1}{k_o}\mathbf{1}} - E)$ would be an invariant conditional expectation as well, and $E_{\frac{1}{k_o}\mathbf{1}}$ could be written as a proper convex combination $E_{\frac{1}{k_o}\mathbf{1}} = \frac{1}{m_o}E + \frac{m_o-1}{m_o}F$, which is a contradiction. □

## 5. A SIMPLE EXAMPLE

For the convenience of the reader, we briefly revisit Example 11.3 of [4] and compute the conditional expectations described in Proposition 3.10.

Indeed, let $\mathbb{Z}_\infty$ be the one-point compactification of the integers $\mathbb{Z}$, together with the homeomorphism $\theta_o : \mathbb{Z}_\infty \to \mathbb{Z}_\infty$ given by

$$\theta_o(l) := \begin{cases} l+1 & \text{if } l \in \mathbb{Z}\,, \\ \infty & \text{if } l = \infty\,. \end{cases}$$

The dynamical system $(\mathbb{Z}_\infty, \theta)$ is uniquely ergodic (but neither minimal, nor strictly ergodic) with the unique invariant measure

$$\mu_o(f) := f(\infty)\,, \quad f \in C(\mathbb{Z}_\infty)\,.$$

We also note that

$$C(\mathbb{Z}_\infty)^{\theta_o} := \{f \in C(\mathbb{Z}_\infty) \mid f \circ \theta = f\} = \mathbb{C}1 \sim L^\infty(\mathbb{Z}_\infty, \mu_o)\,,$$

with 1 being the function identically equal to one.

For $f\in C(\mathbb{Z}_\infty;\mathbb{T})$, we can associate the process on the torus $\Phi_{\theta_o,f}:\mathbb{Z}_\infty\times\mathbb{T}\to\mathbb{Z}_\infty\times\mathbb{T}$ given by $(\Phi_{\theta_o,f})(l,z)=(\theta_o(l),f(l)z)$. We now particularise the situation for

$$f(l):=\begin{cases}-1 & \text{if } l=0\,,\\ 1 & \text{otherwise}\,.\end{cases}$$

It is immediate to show that for each $n\in\mathbb{Z}$,

$$\mathrm{const}\,f(l)^n=\mathrm{const}\,,\ \mu_o\text{-a.e.}\,,$$

and therefore the (2.3) admit nontrivial solutions for each $n\in\mathbb{Z}$.

On the other hand, if the two-sided sequence $(g(l))_{l\in\mathbb{Z}}$ satisfies (2.4) for some $n$, then

$$g(l):=\begin{cases}g(0)(-1)^{-n} & \text{if } l>0\,,\\ g(0) & \text{if } l\leqslant 0\,,\end{cases}$$

Imposing continuity, we are led to

$$g(0)=\lim_{l\to-\infty}g(l)=\lim_{l\to+\infty}g(l)=(-1)^{-n}g(0)\,.$$

Therefore, the (2.4) admit nontrivial solutions if $n$ is even, and correspondingly $n_o=1$, $m_o=2=k_o$. The fixed-point subalgebra is linearly generated by elements of the form $g(l,z)=az^{2n}$, $n\in\mathbb{Z}$.

For $h\in C(\mathbb{Z}_\infty,\mathbb{T})$, with

$$h(l,z)=\sum_{n\in\mathbb{Z}}\big(h_{2n}(l)z^{2n}+h_{2n+1}(l)z^{2n+1}\big)\,,$$

we easily obtain

$$T(h)(z)=h(\infty,z)=\sum_{n\in\mathbb{Z}}\big(h_{2n}(\infty)z^{2n}+h_{2n+1}(\infty)z^{2n+1}\big)\,.$$

After some computations, for $A\in\mathbb{M}_2(\mathbb{C})$, positive and normalised, we deduce

$$E_A(h)(l,z)=\sum_{n\in\mathbb{Z}}\big(h_{2n}(\infty)+h_{2n+1}(\infty)(a_{12}+a_{21}z^2)\big)z^{2n}\,,$$

and thus $E_{\frac{1}{2}\mathbf{1}}(h)(l,z)=\sum_{n\in\mathbb{Z}}h_{2n}(\infty)z^{2n}$.

We now sketch some computations in order to verify (4.2) for $\widetilde{E}=E_A$, and for the simplest nontrivial situation $h=\overline{g}g$ where $g(l,z)=g_0(l)+g_1(l)z$ (see also Proposition 3.1). For such a purpose, we first note that, if $A=\begin{pmatrix}\lambda & a_{12}\\ \overline{a_{12}} & 1-\lambda\end{pmatrix}$ and $0\leqslant\lambda\leqslant 1$, $|a_{12}|\leqslant a(\lambda)\leqslant 1/2$, where $a(\lambda)$ is the greatest value that $|a_{12}|$ can assume. On the other hand, $E_A(|g|^2)\leqslant E_{\frac{1}{2}\mathbf{1}}(|g|^2)$ leads to $4|g_0(\infty)||g_1(\infty)||a_{12}|\leqslant|g_0(\infty)|^2+|g_1(\infty)|^2$, which is automatically satisfied if $|g_0(\infty)|$ or $|g_1(\infty)|$ is 0. If indeed $|g_0(\infty)||g_1(\infty)| \neq 0$, $|a_{12}| \leqslant \frac{|g_0(\infty)|^2+|g_1(\infty)|^2}{4|g_0(\infty)||g_1(\infty)|}$ with the r.h.s. always greater than 1/2, and the assertion follows.

## Acknowledgements

S.D.V. and F.F. acknowledge the "MIUR Excellence Department Project" awarded to the Department of Mathematics, University of Rome Tor Vergata, CUP E83C18000100006.

## References

[1] Abadie B., Dykema K. *Unique ergodicity of free shifts and some other automorphisms of $C^*$-algebras,*, J. Operator Theory **61** (2009), 279-294.
[2] Anzai H. *Ergodic Anzai skew-product transformations on the torus*, Osaka Math. J. **3** (1951), 83-99
[3] Del Vecchio S., Fidaleo F., Giorgetti L., Rossi S. *Ergodic properties of the Anzai skew-product for the noncommutative torus*, Ergod. Theory Dyn. Syst. **41** (2021), 1064-1085.
[4] Del Vecchio S., Fidaleo F., Rossi S. *Skew-product dynamical systems for crossed product $C^*$-algebras and their ergodic properties*, preprint.
[5] Fidaleo F. *An ergodic theorem for quantum diagonal measures*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **12** (2009), 307-320.
[6] Fidaleo F., Mukhamedov F. *Strict weak mixing of some $C^*$-dynamical systems based on free shifts*, J. Math. Anal. Appl. **336** (2007), 180-187.
[7] Furstenberg H. *Strict ergodicity and transformation on the torus*, Amer. J. math. **83** (1961), 573-601.
[8] Harris G., Martin C. *The roots of a polynomial vary continuously as a function of the coefficients*, Proc. Amer. Math. Soc. **100** (1987), 390–392.
[9] Oxtoby J.C.. *Ergodic sets*, Bull. Amer. Math. Soc. **58** (1952), 116-136.
[10] Paulsen V. *Completely bounded maps and operator algebras*, Cambridge University Press, Cambridge-New York-Melbourne, 2002.
[11] Rosenblum M., Rovnyak J. *Hardy classes and operator theory*, Dover, 1997. *Completely bounded maps and operator algebras*, Cambridge University Press, Cambridge-New York-Melbourne, 2002.
[12] Takesaki M. *Theory of operator algebras I*, Springer, Berlin-Heidelberg-New York, 1979.
[13] van Wyk D. W. *Unique ergodicity in $C^*$-dynamical systems*, Magister Scientiæ, University of Pretoria 2013.

Simone Del Vecchio, Dipartimento di Matematica, Università di Roma Tor Vergata, Via della Ricerca Scientifica 1, Roma 00133, Italy
*E-mail address*: delvecchio@mat.uniroma2.it

Francesco Fidaleo, Dipartimento di Matematica, Università di Roma Tor Vergata, Via della Ricerca Scientifica 1, Roma 00133, Italy
*E-mail address*: fidaleo@mat.uniroma2.it

Stefano Rossi, Dipartimento di Matematica, Università degli Studi di Bari Aldo Moro, Via Edoardo Orabona 4, Bari 70125, Italy
*E-mail address*: stefano.rossi@uniba.it