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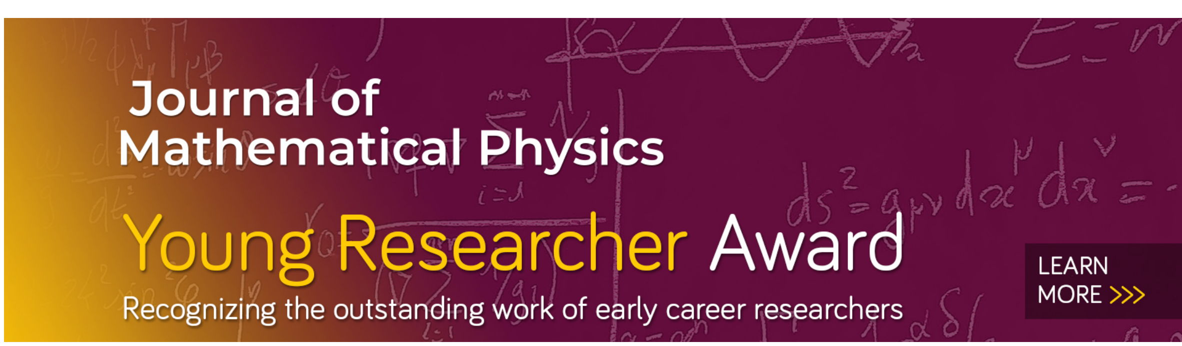

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ABSTRACT

In this paper, the *local* iterative Lie–Schwinger block-diagonalization method, introduced and developed in our previous work for quantum chains, is extended to higher-dimensional quantum lattice systems with Hamiltonians that can be written as the sum of an unperturbed gapped operator, consisting of a sum of on-site terms, and a perturbation, consisting of bounded interaction potentials of short range multiplied by a real coupling constant t. Our goal is to prove that the spectral gap above the ground-state energy of such Hamiltonians persists for sufficiently small values of $|t|$, *independently* of the size of the lattice. New ideas and concepts are necessary to extend our method to systems in dimension $d > 1$: As in our earlier work, a sequence of *local* block-diagonalization steps based on judiciously chosen unitary conjugations of the original Hamiltonian is introduced. The supports of effective interaction potentials generated in the course of these block-diagonalization steps can be identified with what we call *minimal rectangles* contained in the lattice, a concept that serves to tackle combinatorial problems that arise in the course of iterating the block-diagonalization steps. For a given minimal rectangle, control of the effective interaction potentials generated in each block-diagonalization step with support in the given rectangle is achieved by exploiting a variety of rather subtle mechanisms, which include, for example, the use of *weighted sums of paths* consisting of overlapping rectangles and of *large denominators*, expressed in terms of sums of orthogonal projections, which serve to control analogous sums of projections in the numerators resulting from the unitary conjugations of the interaction potential terms involved in the local block-diagonalization step.

I. MODELS OF GAPPED QUANTUM LATTICE SYSTEMS AND SURVEY OF RESULTS

In this paper, we introduce and study a family of quantum lattice systems describing insulating materials in two or more dimensions. We are interested in analyzing the low-energy spectrum of the Hamiltonians of these systems and, in particular, in showing that the ground-state energies of these Hamiltonians are separated from the rest of their energy spectrum by a strictly positive gap.[1] Our analysis is based on a novel method consisting in iteratively block-diagonalizing the Hamiltonians with respect to the ground-state subspace. The block-diagonalization is accomplished by a sequence of unitary conjugations of the Hamiltonians. Our analysis is motivated, in part, by recent interest in characterizing "*topological phases*" (see, e.g., Refs. 2–4) and, more specifically, by studying Hamiltonians of a class of "*topological insulators*" whose ground-state energy is separated from the higher-lying spectrum by a strictly positive energy gap. However, the scope of our techniques is actually more general.

To be concrete, we consider tight-binding models of electrons hopping on a lattice $\mathbb{Z}^d, d \geq 2$, with Hamiltonians that are given as the sum of an *unperturbed operator*, K_0, and a *perturbation*, K_I, consisting of a sum of bounded *interaction potentials*. The operator K_0 can be written as a sum of terms, $H_{\mathbf{i}}$, only depending on the degrees of freedom located at single sites $\mathbf{i} \in \mathbb{Z}^d$, while the interaction potentials contributing to K_I only couple degrees of freedom located on subsets of the lattice of strictly bounded diameter. We focus our attention on unperturbed operators K_0 with a unique ground-state, Ω, and a positive energy gap above their ground-state energy. However, our methods can be extended to families of operators with degenerate ground-state energies. Indeed, in Ref. 6, our scheme has been employed to deal with small perturbations of the Hamiltonian of the Kitaev chain, which has a degenerate ground-state. The extension to unbounded operators for quantum chains (e.g., the relativistic, massive ϕ^4 model on a one-dimensional lattice) was discussed in Refs. 7 and 8, where we specify the general structure of the degenerate ground-state subspace that allows us to implement our block-diagonalization scheme (see Remark 1.1 of Ref. 7). Under the same assumption on the ground-state subspace, our method works in any dimensions.

Our aim is to iteratively construct an anti-self-adjoint operator $S \equiv S(t) = -S(t)^*, t \in \mathbb{R}$, such that the ground state of the operator $e^S(K_0 + t \cdot K_I)e^{-S}$ is given again by Ω, and the spectrum of the restriction of this operator to the subspace orthogonal to Ω lies strictly above the ground-state energy, provided that the absolute value of the coupling constant t is small enough. Our method to construct the operator $S = S(t)$ is inspired by a novel technique introduced in Ref. 6, which, in its original form, has been limited to chains, i.e., to one-dimensional systems. This technique represents an interesting example of multi-scale, iterative perturbation theory: it consists in successively block-diagonalizing the Hamiltonians associated with sequences of bounded, connected subsets of the lattice. In one dimension, such subsets are intervals. However, for $d > 1$, the number of connected subsets of a given cardinality, R, containing a fixed point of the lattice grows exponentially in R, and this causes certain difficulties that make it necessary to refine the methods in Ref. 6 in a rather subtle way; see Sec. II.

We remark that the procedure described here is amenable to be extended to analogous lattice systems but with unbounded interactions.[9]

It is appropriate to comment on earlier work addressing problems closely related to the ones treated in our paper. Methods somewhat similar to those developed in this paper, but much simpler, have been applied to the analysis of phase diagrams of quantum lattice systems, see Refs. 5, 11, and 17. Our main results are similar to some that can be found in the literature. In Ref. 10, fermionic path integral methods have been used for the same purpose, and in Refs. 12–14, quasi-adiabatic flows have been constructed to establish results related to ours. In Refs. 15 and 16, similar results have been obtained by using cluster expansions based on operator methods. (Reference 15 is the only one among the quoted papers where unbounded interactions are considered.) More recent results concern developments of the technique introduced in Refs. 3 and 4 in order to prove gap stability for the so-called lattice fermions (see Refs. 18 and 19) and for insulators in the presence of edge states (see Refs. 20 and 21). In Refs. 22 and 23, criteria for SPT (Symmetry-Protected-Topological) phases in two dimensions have been introduced.

It is primarily the *mathematical methods* used in our analysis that are novel. Compared to previous work, our algorithm is extremely robust: it applies with both self-adjoint and non-self-adjoint Hamiltonians; it can be used to control interactions of both fermions and bosons; and it can cope with some general degeneracy structure of the ground-state subspace. In contrast to some other work on related problems, our paper is self-contained. Indeed, when applied to fermion systems, our methods require no other preliminaries than elementary linear algebra.

Ideas sharing some similarities with the ones presented in our paper were used in Ref. 24 for purposes analogous to ours and in Refs. 25 and 26 for a partial analysis of many-body localization in one dimension.

A. A family of quantum lattice systems

We consider a finite, d-dimensional lattice, $\Lambda_N^d \subset \mathbb{Z}^d$, with sides consisting of N vertices, where $N < \infty$ is arbitrary (but fixed). Each vertex in Λ_N^d is labeled by a multi-index $\mathbf{i} := (i_1, \ldots, i_d)$ with $i_j \in (1, \ldots, N), j = 1, \ldots, d$. The Hilbert space of pure state vectors of the quantum lattice systems studied in this paper is given by

$$\mathcal{H}^{(N)} := \bigotimes_{\mathbf{i} \in \Lambda_N^d} \mathcal{H}_{\mathbf{i}} \quad \text{with} \quad \mathcal{H}_{\mathbf{i}} \simeq \mathbb{C}^M, \; \forall\, \mathbf{i} \in \Lambda_N^d, \tag{1.1}$$

where M is an arbitrary, but finite, N-independent integer. Let H be a non-negative $M \times M$ matrix with the properties that 0 is an eigenvalue of H corresponding to an eigenvector $\Omega \in \mathbb{C}^M$, and

$$H \upharpoonright_{\{\mathbb{C}\Omega\}^\perp} \geq \mathbb{1},$$

where $\mathbb{1}$ is the identity matrix.

We define

$$H_{\mathbf{i}} := \left(\bigotimes_{\Lambda_N^d \ni \mathbf{j} \neq \mathbf{i}} \mathbb{1}_{\mathbf{j}} \right) \bigotimes \underset{\mathbf{i}^{th}\text{slot}}{\underset{\uparrow}{H}}, \tag{1.2}$$

where $\mathbb{1}_{\mathbf{j}}$ is the identity matrix on $\mathcal{H}_{\mathbf{j}}$. By $P_{\Omega_{\mathbf{i}}}$, we denote the orthogonal projection onto the subspace,

$$\left(\bigotimes_{\Lambda_N^d \ni \mathbf{j} \neq \mathbf{i}} \mathcal{H}_{\mathbf{j}} \right) \bigotimes \underset{\mathbf{i}^{th}\text{slot}}{\underset{\uparrow}{\{\mathbb{C}\Omega\}}} \subset \mathcal{H}^{(N)} \quad \text{and} \quad P_{\Omega_{\mathbf{i}}}^\perp := \mathbb{1} - P_{\Omega_{\mathbf{i}}}. \tag{1.3}$$

Then,

$$H_{\mathbf{i}} = P^{\perp}_{\Omega_{\mathbf{i}}}\, H_{\mathbf{i}}\, P^{\perp}_{\Omega_{\mathbf{i}}} + P_{\Omega_{\mathbf{i}}}\, H_{\mathbf{i}}\, P_{\Omega_{\mathbf{i}}}$$

with

$$P_{\Omega_i}\, H_{\mathbf{i}}\, P_{\Omega_{\mathbf{i}}} = 0 \quad \text{and} \quad P^{\perp}_{\Omega_{\mathbf{i}}}\, H_{\mathbf{i}}\, P^{\perp}_{\Omega_{\mathbf{i}}} \geq P^{\perp}_{\Omega_{\mathbf{i}}}. \tag{1.4}$$

We study quantum systems on the lattice Λ^d_N with Hamiltonians of the form

$$K_N \equiv K_N(t) := \underbrace{\sum_{\mathbf{i}\in\Lambda^d_N} H_{\mathbf{i}}}_{K_0} + t \cdot \underbrace{\sum_{J_{\mathbf{k},\mathbf{q}}\subset\Lambda^d_N,\, k\leq\bar{k}} V_{J_{\mathbf{k},\mathbf{q}}}}_{K_I}, \tag{1.5}$$

where we have the following:

(i) $J_{\mathbf{k},\mathbf{q}} \equiv J_{k_1,\ldots,k_d\,;\,q_1,\ldots,q_d}$ denotes the rectangle in Λ^d_N with sides of lengths $k_1, k_2, \ldots, k_d$, respectively, whose 2^d corners are the sites given by $(q_1 + \varepsilon_1 k_1, \ldots, q_d + \varepsilon_d k_d)$, $\varepsilon_j = 0$ or 1, for $j = 1, \ldots, d$. [Note that $\Lambda^d_N \equiv J_{\mathbf{N-1},\mathbf{1}}$, where $\mathbf{N} - \mathbf{1} = (N-1, \ldots, N-1)$ and $\mathbf{1} = (1, \ldots, 1)$].

(ii) $k \equiv |\mathbf{k}|$ denotes the circumference (= sum of the side lengths) of a rectangle $J_{\mathbf{k},\mathbf{q}}$, i.e.,

$$k \equiv |\mathbf{k}| := \sum_{i=1}^{d} k_i. \tag{1.6}$$

(iii) The range of the interaction potentials, namely, the integer $\bar{k} < \infty$ with the property that $|\mathbf{k}| \leq \bar{k}$, $\forall$ rectangles $J_{\mathbf{k},\mathbf{q}}$ appearing in (1.5), is arbitrary, but fixed and *N−independent.*

(iv) $V_{J_{\mathbf{k},\mathbf{q}}}$ is a symmetric matrix on $\mathcal{H}^{(N)}$ with the property that

$$V_{J_{\mathbf{k},\mathbf{q}}} \ \text{acts as the identity on} \bigotimes_{\mathbf{j}\in\Lambda^d_N,\, \mathbf{j}\notin J_{\mathbf{k},\mathbf{q}}} \mathcal{H}_{\mathbf{j}} \quad \text{and} \quad \|V_{J_{\mathbf{k},\mathbf{q}}}\| \leq 1 \tag{1.7}$$

for all $\mathbf{k}, \mathbf{q}$, with $|\mathbf{k}| \leq \bar{k} < \infty$, as in (iii) (and $V_{J_{\mathbf{k},\mathbf{q}}} = 0$ whenever $|\mathbf{k}| > \bar{k}$). The rectangle $J_{\mathbf{k},\mathbf{q}}$ is called the "support" of $V_{J_{\mathbf{k},\mathbf{q}}}$.

(v) $t \in \mathbb{R}$ is a coupling constant independent of N.

B. Main result

Our main result is the following theorem proven in Sec. V (see Theorem 5.3).

Theorem. *Under the assumption that (1.4) and (1.7) hold for an arbitrary, but fixed finite range $\bar{k} < \infty$, the Hamiltonian $K_N(t)$ defined in (1.5) has the following properties.*

There exists some $t_d > 0$ independent of N such that for any coupling constant $t \in \mathbb{R}$ with $|t| < t_d$ and for all $N < \infty$, we have the following.

(i) *$K_N(t)$ has a unique ground-state and*

(ii) *the energy spectrum of $K_N(t)$ has a strictly positive gap, $\Delta_N(t) \geq \frac{1}{2}$, above the ground-state energy.*

Results similar to this theorem have appeared in the literature; see, e.g., Ref. 10. The main novelty of our paper is the method of proof. We define

$$P_{vac} := \bigotimes_{\mathbf{i}\in\Lambda^d_N} P_{\Omega_{\mathbf{i}}}, \tag{1.8}$$

which is the orthogonal projection onto the ground-state subspace of the unperturbed operator $K_{0,N} \equiv K_N(t=0) = \sum_{\mathbf{i}\in\Lambda^d_N} H_{\mathbf{i}}$. We will construct an anti-symmetric matrix $S_N(t) = -S_N(t)^*$ acting on $\mathcal{H}^{(N)}$ (so that $\exp[\pm S_N(t)]$ are unitary matrices) with the property that, after conjugation, the operator

$$e^{S_N(t)} K_N(t) e^{-S_N(t)} =: \widetilde{K}_N(t) \tag{1.9}$$

is "*block-diagonal*" with respect to the pair $\left(P_{vac}, P_{vac}^{\perp} := \mathbb{1} - P_{vac}\right)$ of projections in the sense that P_{vac} projects onto the ground-state of $\widetilde{K}_N(t)$,

$$\widetilde{K}_N(t) = P_{vac}\widetilde{K}_N(t)P_{vac} + P_{vac}^{\perp}\widetilde{K}_N(t)P_{vac}^{\perp} \tag{1.10}$$

and

$$\text{infspec}\left(P_{vac}^{\perp}\widetilde{K}_N(t)P_{vac}^{\perp}\restriction_{P_{vac}^{\perp}\mathcal{H}^{(N)}}\right) \geq \text{infspec}\left(P_{vac}\widetilde{K}_N(t)P_{vac}\restriction_{P_{vac}\mathcal{H}^{(N)}}\right) + \Delta_N(t) \tag{1.11}$$

with $\Delta_N(t) \geq \frac{1}{2}$ for $|t| < t_d$, *uniformly* in N.

The Hamiltonian we will study in the following has the special form,

$$K_N(t) := \sum_{\mathbf{i} \in \Lambda_N^{(d)}} H_{\mathbf{i}} + t\sum_{j=1}^{d} \sum_{q_1=1}^{N} \cdots \sum_{q_j=1}^{N-1} \cdots \sum_{q_d=1}^{N} V_{J_{\mathbf{1}_j,\mathbf{q}}}, \tag{1.12}$$

where

$$(\mathbf{1}_j, \mathbf{q}) := (0, \ldots, k_j = 1, \ldots, 0\,;\, q_1, \ldots, q_d), \tag{1.13}$$

i.e., the range of the interaction potentials is $\bar{k} = 1$. We could study potentials with an arbitrary finite range. However, in order to keep our exposition as transparent as possible, we restrict our attention to the nearest neighbor "hopping terms." For simplicity, we also assume that the coupling constant is positive, i.e., $t > 0$.

Organization of this paper. In Sec. II, we explain the formal aspects of our construction. In Sec. II A, we introduce the notion of "*minimal rectangles*" that will play an important role in our analysis. In Sec. II B, we describe the *local* (so-called Lie–Schwinger) conjugations of the Hamiltonian associated with minimal rectangles. Next, in Sec. II C, we introduce an algorithm that describes the flow of effective interactions determined by the iterative conjugations of the Hamiltonian used to block-diagonalize it. Moreover, we outline the new features and the complications of our strategy arising in dimensions $d \geq 2$, as compared to the one used in Ref. 6 for chains.

In Sec. III, we describe a scheme of re-expansions of collections of effective interaction potentials and a method to derive estimates on the norms of these operators that involve keeping track of paths of connected rectangles.

In Sec. IV, we recall how to provide a lower bound on the spectral gap $\Delta_N(t)$ for sufficiently small values of the coupling constant t following the same procedure as in Ref. 6.

In Sec. V, the proof of convergence of our construction of the operator $S_N(t)$ is presented with a few technicalities deferred to the Appendix. Theorem 5.1 is the core result in our proof of convergence, enabling us to control the norms of the effective interactions by using a composite strategy combining different mechanisms, depending on the regime of the growth processes of rectangles; see Sec. II C. From Theorem 5.1, the final result of this paper, Theorem 5.3, follows.

Notation.

(1) For chains, i.e., $d = 1$, the rectangles $J_{\mathbf{k},\mathbf{q}}$ coincide with the connected one-dimensional graphs, $I_{k,q}$, $k \in \mathbb{N}$, used in Ref. 6, with k edges connecting the $k+1$ vertices $q, 1+q, \ldots, k+q$, which can also be seen as "intervals" of length k whose left end-point coincides with q.
(2) We use the same symbol for the operator $O_{\mathbf{j}}$ acting on $\mathcal{H}_{\mathbf{j}}$ and the corresponding operator

$$O_{\mathbf{j}} \bigotimes \mathbb{1}_{J_{\mathbf{k},\mathbf{q}}\setminus\{\mathbf{j}\}}$$

acting on $\otimes_{\mathbf{i}\in J_{\mathbf{k},\mathbf{q}}}\mathcal{H}_{\mathbf{i}}$ for any $\mathbf{j} \in J_{\mathbf{k},\mathbf{q}}$. Similarly, with a slight abuse of notation, we do not make a distinction between an operator $O_{J_{\mathbf{l},\mathbf{i}}}$ acting on $\mathcal{H}_{J_{\mathbf{l},\mathbf{i}}} := \otimes_{\mathbf{j}\in J_{\mathbf{l},\mathbf{i}}}\mathcal{H}_{\mathbf{j}}$ and the corresponding operator acting on the whole Hilbert space $\mathcal{H}^{(N)}$, which is obtained out of $O_{J_{\mathbf{l},\mathbf{i}}}$ by tensoring by the identity matrix operator on all the remaining sites.
(3) With the symbol "$\subset$," we denote strict inclusion; otherwise, we use the symbol "$\subseteq$."
(4) The multiplicative constant implicit in the symbol $\mathcal{O}(\cdot)$ can depend on the spatial dimension d.

II. OUTLINE OF THE PROOF STRATEGY

The conjugations used to block-diagonalize the Hamiltonian in (1.5) determine a flow of effective Hamiltonians. These operators are expressed in terms of effective interaction potentials with supports that can be represented as connected unions of the rectangles $J_{\mathbf{k},\mathbf{i}}$ labeling interaction terms in formula (1.5). Whereas for chains, $d = 1$, when starting from a family of intervals (i.e., $I_{k,q} \equiv J_{\mathbf{k},\mathbf{q}}$ with $\mathbf{k} = k$ and $1 \le q \le N - k$), the connected sets associated with the new interaction potentials are again intervals, the situation is much more complicated in higher dimensions, $d > 1$, because connected sets of arbitrary shape arise in the flow. The control of growth processes giving rise to each fixed shape that can appear in our construction is crucial in order to accomplish the block-diagonalization of the Hamiltonian. For an arbitrary connected set of a fixed shape, the number of growth processes scales factorially in the number of edges of the set. This crude estimate is, however, not good enough to control the norms of the interaction potentials associated with a given shape since the expected prefactor, t^n, in the norm of the interaction potential labeled by a connected set of cardinality n with a fixed shape arising from all possible growth processes terminating in the given shape cannot compensate the number, $\mathcal{O}(n!)$, of such growth processes when n tends to ∞ (here, t is the coupling constant). Hence, in our estimates, we cannot simply count all growth processes giving rise to each fixed shape since some of them are, in fact, forbidden by the ordering encoded in the block-diagonalization procedure. In this paper, we circumvent this problem with a strategy outlined in Sec. II C, which involves the notion of "*minimal rectangles*" introduced in Subsection II A.

A. Minimal rectangles

We recall that the symbol $J_{\mathbf{k},\mathbf{q}} \equiv J_{k_1,\dots,k_d\,;\,q_1,\dots,q_d}$ denotes a rectangle in Λ_N^d whose sides have lengths $k_1, k_2, \dots, k_d$ and that $|\mathbf{k}|$ denotes the sum of these lengths, i.e., $|\mathbf{k}| := \sum_{i=1}^d k_i$. The coordinates of the 2^d corners of $J_{\mathbf{k},\mathbf{q}}$ are d-tuples of integers given by either q_j or $q_j + k_j$ at the jth position for all $1 \le j \le d$ with $q_j \le N - k_j$.

The rectangles $J_{\mathbf{k},\mathbf{q}}$ play the role of the intervals $I_{k,q}$ in the one-dimensional case considered in Ref. 6. Similarly to the one-dimensional case, the pairs $(\mathbf{k},\mathbf{q})$ label the block-diagonalization steps, which are ordered according to the ordering relation "$>$" defined as follows:[27]

$$(\mathbf{k}',\mathbf{q}') > (\mathbf{k},\mathbf{q}) \qquad \text{iff} \tag{2.14}$$

- $\sum_{j=1}^d k'_j > \sum_{j=1}^d k_j$;
- or, if $\sum_{j=1}^d k'_j = \sum_{j=1}^d k_j$, $k'_j > k_j$, for some $1 \le j \le d$, with $k'_l = k_l$, $\forall l > j$;
- or, if $k'_l = k_l$, for all l, $q'_j > q_j$, for some $1 \le j \le d$, with $q'_l = q_l$, $\forall l > j$.

As will become clear from our description of the block-diagonalization flow in Sec. II B, the ordering among rectangles must ensure that rectangles with larger circumference $|\mathbf{k}|$ succeed those of smaller circumference. With this requirement fulfilled, the ordering chosen here is convenient, but it is definitely not the only possible ordering.

With the symbols $(\mathbf{k},\mathbf{q})_{+j}$ and $(\mathbf{k},\mathbf{q})_{-j}$, we denote the jth successor and the jth predecessor of $(\mathbf{k},\mathbf{q})$, respectively, in the ordering introduced above. The initial step is $(\mathbf{0},\mathbf{N})$ because the "potentials" associated with the degenerate rectangles consisting of a single point are the on-site terms, $H_{\mathbf{i}}$, which are already block-diagonal with respect to the pair of projections defined in (2.20) and (2.21). The final step is $(\mathbf{N}-\mathbf{1},\mathbf{1})$, where $\mathbf{N}-\mathbf{1} = (N-1,\dots,N-1)$ and $\mathbf{1} = (1,\dots,1)$.

Definition 2.1. Given an arbitrary rectangle $J_{\mathbf{k},\mathbf{q}}$ of sites in Λ_N^d, we define

$$\mathcal{H}_{J_{\mathbf{k},\mathbf{q}}} := \bigotimes_{\mathbf{i}\in J_{\mathbf{k},\mathbf{q}}} \mathcal{H}_{\mathbf{i}}. \tag{2.15}$$

Definition 2.2. Consider two rectangles, $J_{\mathbf{k},\mathbf{q}}$ and $J_{\mathbf{k}',\mathbf{q}'}$, with nonempty intersection. The *minimal rectangle* associated with $J_{\mathbf{k},\mathbf{q}} \cup J_{\mathbf{k}',\mathbf{q}'}$ is defined to be the *smallest* rectangle containing $J_{\mathbf{k},\mathbf{q}}$ and $J_{\mathbf{k}',\mathbf{q}'}$. Note that its corners are the 2^d numbers with either

$$\min\{q_j, q'_j\} \quad \text{or} \quad \max\{q_j + k_j, q'_j + k'_j\} \tag{2.16}$$

at the jth position. The minimal rectangle associated with $J_{\mathbf{k},\mathbf{q}}$ and $J_{\mathbf{k}',\mathbf{q}'}$ is denoted by

$$[J_{\mathbf{k},\mathbf{q}} \cup J_{\mathbf{k}',\mathbf{q}'}]. \tag{2.17}$$

Definition 2.3. Let $J_{\mathbf{k},\mathbf{q}} \subset J_{\mathbf{l},\mathbf{i}}$. We define a family, $\mathcal{G}_{J_{\mathbf{l},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})}$, of rectangles by

$$\mathcal{G}_{J_{\mathbf{l},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})} := \left\{ J_{\mathbf{k}',\mathbf{q}'} \,\middle|\, J_{\mathbf{k}',\mathbf{q}'} \neq J_{\mathbf{l},\mathbf{i}} \quad \text{and} \quad [J_{\mathbf{k},\mathbf{q}} \cup J_{\mathbf{k}',\mathbf{q}'}] = J_{\mathbf{l},\mathbf{i}} \right\}. \tag{2.18}$$

B. Effective Hamiltonians

Each conjugation step in the block-diagonalization of the original Hamiltonian is labeled by a rectangle $J_{\mathbf{k},\mathbf{q}}$ and, consequently, by a pair $(\mathbf{k},\mathbf{q})$. In the effective Hamiltonian arising from a conjugation step, a potential term, $V_{J_{\mathbf{l},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})}$, is associated with each rectangle $J_{\mathbf{l},\mathbf{i}}$. More precisely, after the conjugation step $(\mathbf{k},\mathbf{q})$, the effective Hamiltonian reads

$$
\begin{aligned}
K^{(\mathbf{k},\mathbf{q})}_{\Lambda^d_N} = \sum_{\mathbf{i}\in\Lambda^{(d)}_N} H_{\mathbf{i}} + t\sum_{\mathbf{k}'_{(1)},\mathbf{q}'} V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{k}'_{(1)},\mathbf{q}'}} + t\sum_{\mathbf{k}'_{(2)},\mathbf{q}'} V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{k}'_{(2)},\mathbf{q}'}} + \cdots + t\sum_{\mathbf{k}'_{(|\mathbf{k}|)},\mathbf{q}'} V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{k}'_{(|\mathbf{k}|)},\mathbf{q}'}} \\
+t\sum_{\mathbf{k}'_{(|\mathbf{k}|+1)},\mathbf{q}'} V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{k}'_{(|\mathbf{k}|+1)},\mathbf{q}'}} + \cdots + tV^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{N-1},\mathbf{1}}},
\end{aligned} \tag{2.19}
$$

where we have the following.

1. The pairs $(\mathbf{k}'_{(j)},\mathbf{q}')$ are used to index all rectangles $J_{\mathbf{k}',\mathbf{q}'}$ with $|\mathbf{k}'| = j$.
2. For a fixed rectangle $J_{\mathbf{l},\mathbf{i}}$, the corresponding potential term may change in each conjugation step of the block-diagonalization procedure until the step $(\mathbf{k},\mathbf{q}) = (\mathbf{l},\mathbf{i})$ is reached; hence, $V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{l},\mathbf{i}}}$ is the potential term associated with $J_{\mathbf{l},\mathbf{i}}$ arising in step $(\mathbf{k},\mathbf{q})$ of the block-diagonalization, the superscript $(\mathbf{k},\mathbf{q})$ keeping track of the changes in the potential term arising in step $(\mathbf{k},\mathbf{q})$. The operator $V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{l},\mathbf{i}}}$ depends on the coupling constant t, but this is not made explicit in our notation; it acts as the identity on the spaces $\mathcal{H}_{\mathbf{j}}$ for $\mathbf{j}\notin J_{\mathbf{l},\mathbf{i}}$. A more precise description of how these operators arise in our procedure and an outline of the strategy to control their norms are deferred to Sec. II C.
3. For all rectangles $J_{\mathbf{l},\mathbf{i}}$ with $(\mathbf{k},\mathbf{q}) \succ (\mathbf{l},\mathbf{i})$ and for the rectangle $J_{\mathbf{l},\mathbf{i}} = J_{\mathbf{k},\mathbf{q}}$, the associated effective potential $V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{l},\mathbf{i}}}$ is block-diagonal with respect to the decomposition of the identity acting on $\mathcal{H}^{(N)}$ into the sum of projections,

$$
P^{(-)}_{J_{\mathbf{l},\mathbf{i}}} := \bigotimes_{\mathbf{j}\in J_{\mathbf{l},\mathbf{i}}} P_{\Omega_{\mathbf{j}}}, \tag{2.20}
$$

$$
P^{(+)}_{J_{\mathbf{l},\mathbf{i}}} := \left(\bigotimes_{\mathbf{j}\in J_{\mathbf{l},\mathbf{i}}} P_{\Omega_{\mathbf{j}}}\right)^{\perp}. \tag{2.21}
$$

The effective Hamiltonian $K^{(\mathbf{k},\mathbf{q})}_{\Lambda^d_N}$ of (2.19) is obtained after the conjugation step labeled by $(\mathbf{k},\mathbf{q})$. Starting from

$$
K^{(\mathbf{k},\mathbf{q})-1}_{\Lambda^d_N} = \sum_{\mathbf{i}\in\Lambda^{(d)}_N} H_{\mathbf{i}} + t\sum_{\mathbf{k}'_{(1)},\mathbf{q}'} V^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{k}'_{(1)},\mathbf{q}'}} + t\sum_{\mathbf{k}'_{(2)},\mathbf{q}'} V^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{k}'_{(2)},\mathbf{q}'}} + \cdots + t\sum_{\mathbf{k}'_{(|\mathbf{k}|)},\mathbf{q}'} V^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{k}'_{(|\mathbf{k}|)},\mathbf{q}'}} \tag{2.22}
$$

$$
+t\sum_{\mathbf{k}'_{(|\mathbf{k}|+1)},\mathbf{q}'} V^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{k}'_{(|\mathbf{k}|+1)},\mathbf{q}'}} + \cdots + tV^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{N-1},\mathbf{1}}}, \tag{2.23}
$$

the conjugation step labeled by $(\mathbf{k},\mathbf{q})$ is given by

$$
e^{S_{J_{\mathbf{k},\mathbf{q}}}}\, K^{(\mathbf{k},\mathbf{q})-1}_{\Lambda^d_N}\, e^{-S_{J_{\mathbf{k},\mathbf{q}}}} =: K^{(\mathbf{k},\mathbf{q})}_{\Lambda^d_N}, \tag{2.24}
$$

where the anti-symmetric matrix $S_{J_{\mathbf{k},\mathbf{q}}}$ is chosen in such a way that the interaction potential $V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{k},\mathbf{q}}}$ is block-diagonal; see Sec. IV. More precisely, following the Lie–Schwinger procedure, $S_{J_{\mathbf{k},\mathbf{q}}}$ is built so as to block-diagonalize the *local* operator given by the sum of all terms in $K^{(\mathbf{k},\mathbf{q})-1}_{\Lambda^d_N}$ whose support is contained in $J_{\mathbf{k},\mathbf{q}}$. In other words, $S_{J_{\mathbf{k},\mathbf{q}}}$ is chosen in such a way that the conjugation in (2.24) renders the operator

$$
G_{J_{\mathbf{k},\mathbf{q}}} + tV^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{k},\mathbf{q}}}, \tag{2.25}
$$

block-diagonal, where

$$
G_{J_{\mathbf{k},\mathbf{q}}} := \sum_{\mathbf{i}\subset J_{\mathbf{k},\mathbf{q}}} H_{\mathbf{i}} + t\sum_{J_{\mathbf{k}'_{(1)},\mathbf{q}'}\subset J_{\mathbf{k},\mathbf{q}}} V^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{k}'_{(1)},\mathbf{q}'}} + \cdots + t\sum_{J_{\mathbf{k}'_{(|\mathbf{k}|-1)},\mathbf{q}'}\subset J_{\mathbf{k},\mathbf{q}}} V^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{k}'_{(|\mathbf{k}|-1)},\mathbf{q}'}}. \tag{2.26}
$$

Here, "block-diagonalization" refers to the projections $P^{(-)}_{J_{\mathbf{k},\mathbf{q}}}$ and $P^{(+)}_{J_{\mathbf{k},\mathbf{q}}}$ corresponding to the decomposition of the Hilbert space $\otimes_{\mathbf{i}\in J_{\mathbf{k},\mathbf{q}}}\mathcal{H}_{\mathbf{i}}$ into vacuum subspace and its orthogonal complement, respectively. The operator $G_{J_{\mathbf{k},\mathbf{q}}}$ plays the role of the "unperturbed" operator since it is already block-diagonal with respect to the decomposition of the identity,

$$
\mathbb{1} = P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} + P^{(-)}_{J_{\mathbf{k},\mathbf{q}}},
$$

i.e.,

$$G_{J_{\mathbf{k},\mathbf{q}}} = P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} G_{J_{\mathbf{k},\mathbf{q}}} P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} + P^{(-)}_{J_{\mathbf{k},\mathbf{q}}} G_{J_{\mathbf{k},\mathbf{q}}} P^{(-)}_{J_{\mathbf{k},\mathbf{q}}}. \tag{2.27}$$

The construction outlined here works because one can show inductively that the energy gap in the spectrum of the Hamiltonian $G_{J_{\mathbf{k},\mathbf{q}}}$ above its ground-state eigenvalue is bounded away from zero, *uniformly* in the size of the rectangle $J_{\mathbf{k},\mathbf{q}}$, when a suitable upper bound on the operators norms of the interaction potentials is imposed. The control of this gap (see Sec. IV) relies on the fact that all the effective potentials appearing in $G_{J_{\mathbf{k},\mathbf{q}}}$ have been block-diagonalized already in the previous steps.

These properties of the operator $G_{J_{\mathbf{k},\mathbf{q}}}$, combined with bounds on the norms of the effective potentials obtained at the previous conjugation step, enable us to construct the anti-symmetric matrix $S_{J_{\mathbf{k},\mathbf{q}}}$ used at the next conjugation step and control the norms of the effective potentials obtained after conjugation with $\exp[S_{J_{\mathbf{k},\mathbf{q}}}]$. This is described in more detail in Sec. II C.

C. The algorithm and the different regimes in the growth processes of rectangles

Our strategy to control the norms of the effective potentials $V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}$ is based on the following key ideas, which will give rise to a concrete algorithm.

(I) The number of shapes of connected sets of lattice sites arising in our construction is limited by making use of "minimal rectangles" in such a way that, instead of two connected sets, only the minimal rectangle containing them will be recorded (i.e., the rectangle with the property that any rectangle of smaller size cannot contain the union of those sets). Only keeping track of minimal rectangles reduces the combinatorial divergence because the number of rectangles with a given circumference $k(:= \sum_{i=1}^{d} k_i)$ containing a specified site of the lattice grows polynomially in k, namely, like $\mathcal{O}(k^{d-1})$. We, then, lump together all effective potential terms whose support is contained in a given rectangle in such a way that no rectangle of smaller size can contain it. The sum of the norms of these terms is expected to be bounded above by $\mathcal{O}(t^{c\cdot k})$, where c is a universal constant.

(II) We will exploit some subtle mechanisms to identify and control the growth processes allowed by the algorithm introduced below. Depending on the relation between the size, k, of $J_{\mathbf{k},\mathbf{q}}$ and the size, r, of $J_{\mathbf{r},\mathbf{i}}$, we will distinguish three different regimes for the growth processes that may give rise to the term $V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}$ in (2.31).

As implicitly indicated in (2.22) and (2.23) for the effective Hamiltonian $K^{(\mathbf{k},\mathbf{q})_{-1}}_{\Lambda^d_N}$, the potentials must be re-combined properly after each conjugation step $(\mathbf{k},\mathbf{q})$ so as to determine a well-defined flow of operators, $V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}$, for every fixed support $J_{\mathbf{r},\mathbf{i}}$. This flow is obtained with the help of a specific algorithm described in Definition 2.4. In Theorem 4.1, we check that our algorithm is consistent with the conjugation in (2.24). This amounts to showing that the right-hand side of (2.24) has the form given in (2.22) and (2.23), with $(\mathbf{k},\mathbf{q})_{-1}$ replaced by $(\mathbf{k},\mathbf{q})$ and effective potentials $V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{l},\mathbf{i}}}$ as defined in Definition 2.4, formulated next.

The algorithm is supposed to enable us to iteratively determine effective potentials $V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}$ in terms of the potentials obtained at the previous step $(\mathbf{k},\mathbf{q})_{-1}$, starting from

$$V^{(\mathbf{0},\mathbf{N})}_{J_{\mathbf{0},\mathbf{i}}} := H_{\mathbf{i}}, \qquad V^{(\mathbf{0},\mathbf{N})}_{J_{\mathbf{1}_j,\mathbf{q}}} := V_{J_{\mathbf{1}_j,\mathbf{q}}}, \quad \text{and} \quad V^{(\mathbf{0},\mathbf{N})}_{J_{\mathbf{k},\mathbf{i}}} = 0 \quad \text{for } |\mathbf{k}| \ge 2. \tag{2.28}$$

Definition 2.4. Assuming that, at fixed $(\mathbf{k},\mathbf{q})_{-1}$ with $(\mathbf{k},\mathbf{q})_{-1} > (\mathbf{0},\mathbf{N})$, for any $\mathbf{r},\mathbf{i}$, the operators $V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{r},\mathbf{i}}}$ and $S_{J_{\mathbf{k},\mathbf{q}}}$ [defined as in (4.51) and (4.52)] are well defined or assuming that $(\mathbf{k},\mathbf{q}) = (\mathbf{1}_1,\mathbf{1})$ [where $\mathbf{1}_1 = (1,0,\ldots,0)$ and $\mathbf{1} = (1,\ldots,1)$, respectively] and $S_{J_{\mathbf{1}_1,\mathbf{1}}}$ are well defined, then we define the following:

(a) if $J_{\mathbf{k},\mathbf{q}} \not\subseteq J_{\mathbf{r},\mathbf{i}}$,

$$V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}} := V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{r},\mathbf{i}}}; \tag{2.29}$$

(b) if $J_{\mathbf{r},\mathbf{i}} = J_{\mathbf{k},\mathbf{q}}$,

$$V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}} := \sum_{j=1}^{\infty} t^{j-1} \big(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{r},\mathbf{i}}}\big)^{diag}_{j}, \tag{2.30}$$

where $\big(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{r},\mathbf{i}}}\big)^{diag}_{j}$ is defined such as in (4.53) and diag means the diagonal part with respect to the projections $P^{(-)}_{J_{\mathbf{r},\mathbf{i}}}$ and $P^{(+)}_{J_{\mathbf{r},\mathbf{i}}}$;

(c) if $J_{\mathbf{k},\mathbf{q}} \subset J_{\mathbf{r},\mathbf{i}}$,

$$V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}} := e^{S_{J_{\mathbf{k},\mathbf{q}}}} V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{r},\mathbf{i}}} e^{-S_{J_{\mathbf{k},\mathbf{q}}}} + \sum_{J_{\mathbf{k}',\mathbf{q}'} \in \mathcal{G}^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}} \sum_{n=1}^{\infty} \frac{1}{n!} ad^{n} S_{J_{\mathbf{k},\mathbf{q}}} \big(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}}\big), \tag{2.31}$$

where *ad* is defined in (4.49) and (4.50). We observe that the set $\mathcal{G}_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})}$ [see (2.18)] is not empty only if the rectangle $J_{\mathbf{k},\mathbf{q}}$ has a nonempty intersection with the boundary of the rectangle $J_{\mathbf{r},\mathbf{i}}$.

The rationale motivating the recombination of terms described in Definition 2.4 is explained in Sec. IV. Here, a remark on item (c) of Definition 2.4 may be helpful in order to understand the key ideas used to control the operator norms of the effective potentials.

Remark 2.5. The sum on the right-hand side of (2.31) accounts for all contributions to the term $V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})}$ with support $J_{\mathbf{r},\mathbf{i}}$ that correspond to "growth processes" of rectangles, i.e., to processes where the union of a rectangle $J_{\mathbf{k}',\mathbf{q}'} \neq J_{\mathbf{r},\mathbf{i}}$ and of the fixed rectangle $J_{\mathbf{k},\mathbf{q}}$ labeling the conjugation step in the block-diagonalization is a set with the property that $J_{\mathbf{r},\mathbf{i}}$ is the minimal rectangle associated with it, i.e., such that $[J_{\mathbf{k}',\mathbf{q}'} \cup J_{\mathbf{k},\mathbf{q}}] \equiv J_{\mathbf{r},\mathbf{i}}$.

To control the operator norms of the effective potentials, we begin by observing that, by construction, the potential $V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})}$ does not change anymore whenever $(\mathbf{k},\mathbf{q}) \succ (\mathbf{r},\mathbf{i})$. Using this observation, we will prove by induction that, for every pair $(\mathbf{r},\mathbf{i})$, an upper bound of the form

$$\| V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})} \| \le C_j \frac{t^{\frac{r-1}{3}}}{r^{\rho_j}}, \quad j = 1,2,3, \tag{2.32}$$

holds true at all steps $(\mathbf{k},\mathbf{q})$ up to step $(\mathbf{r},\mathbf{i})$ (included), where C_j and the exponent $\rho_j \equiv \rho_j(d) > 0$ (d being the space dimension) depend on the regime $\mathfrak{R}j$ introduced below, and the different regimes, $\mathfrak{R}1, \mathfrak{R}2$, and $\mathfrak{R}3$, depend on the relative magnitude of the circumferences $k = |\mathbf{k}|$ and $r = |\mathbf{r}|$.

We recall that, for quantum chains, control of the norms relies on a feature of formula (2.31) that holds only in dimension $d = 1$: An interval can only grow at the two end-points and, hence, at a number of vertices independent of the size of the interval. However, in higher dimensions, $d > 1$, the number of terms in the sum in formula (2.31) labeled by rectangles, $J_{\mathbf{k}',\mathbf{q}'}$, that intersect the rectangle $J_{\mathbf{k},\mathbf{q}}$ only at the boundary grows like a positive power of r (depending on the dimension d). This motivates the introduction of three different regimes, $\mathfrak{R}1, \mathfrak{R}2$, and $\mathfrak{R}3$, enabling us to exploit a different mechanism to estimate the number of terms in each of the regimes, as outlined below; see also Fig. 1.

($\mathfrak{R}1$) The first regime deals with rectangles labeled by $(\mathbf{k},\mathbf{q})$ that are "small" as compared to the rectangle labeled by $(\mathbf{r},\mathbf{i})$, namely, with pairs $(\mathbf{k},\mathbf{q})$ such that $k \le \lfloor r^{\frac{1}{4}} \rfloor$. In order to establish the desired estimate (2.32), we iterate the re-expansion of the potential $V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})}$ by applying formulas (2.29) and (2.31). As a consequence, each potential term resulting from the re-expansion can, then, be associated with a connected sequence of rectangles $J_{\mathbf{k}'',\mathbf{q}''}$ labeling the operators $S_{\mathbf{k}'',\mathbf{q}''}$, plus one labeling one of the potentials appearing in the Hamiltonian of definition (1.5) or a potential of the type $V_{J_{\mathbf{k}',\mathbf{q}'}}^{(\mathbf{k}',\mathbf{q}')}$ (where $k' \le \lfloor r^{\frac{1}{4}} \rfloor$), with the property that $J_{\mathbf{r},\mathbf{i}}$ is the minimal rectangle associated with this sequence. Roughly speaking, the result, then, holds for the following reasons:

(1) At least $\mathcal{O}(r/\lfloor r^{\frac{1}{4}} \rfloor)$ rectangles $J_{\mathbf{k}'',\mathbf{q}''}$ are present in each connected set, and all the corresponding operators $S_{\mathbf{k}'',\mathbf{q}''}$ have norms of order $|t| \cdot \| V_{\mathbf{k}'',\mathbf{q}''}^{(\mathbf{k}'',\mathbf{q}'')_{-1}} \|$; apart from the resulting product of norms $\| V_{\mathbf{k}'',\mathbf{q}''}^{(\mathbf{k}'',\mathbf{q}'')_{-1}} \|$, which is also crucial in the argument, it is important that a total factor $|t|^{\mathcal{O}(r/\lfloor r^{\frac{1}{4}} \rfloor)}$ or smaller is gained from the re-expansion (due to the constraint $k \le \lfloor r^{\frac{1}{4}} \rfloor$ that holds in this regime).

(2) Note that the rectangles contained in the considered connected set are ordered according to $\succ$, and consequently, only one growth process can yield each such a set. Due to this observation, the number of connected sets of rectangles resulting from the re-expansion, when each connected set is properly weighted in accordance with the inductive hypothesis on the norms of the potentials $V_{J_{\mathbf{k}'',\mathbf{q}''}}^{(\mathbf{k}'',\mathbf{q}'')_{-1}}$, provides an upper bound to $\| V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})} \|$. In fact, for $|t|$ small enough but independent of N, this weighted number yields the sought bound (2.32) for $\| V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})} \|$.

($\mathfrak{R}2$) The second regime is associated with pairs $(\mathbf{k},\mathbf{q})$ with the property that $\lfloor r^{\frac{1}{4}} \rfloor \le k \le r - \lfloor r^{\frac{1}{4}} \rfloor$. In this regime, thanks to the upper bound on k, the size of the rectangles $J_{\mathbf{k}',\mathbf{q}'}$ in formula (2.31) is so large that it is enough to carry out only one re-expansion step and to then use the inductive hypotheses, similarly to the treatment of chains in Ref. 6. In this regime, we use a basic mechanism involving the use of the denominator r^{ρ_2} in the inductive estimate [see (2.32)] of the potential. If k^{ρ_2} and $(r-k)^{\rho_2}$ are both large as it happens in this regime, we can still control the polynomially growing number of terms in the sum of formula (2.31).

($\mathfrak{R}3$) The third regime is associated with "large" rectangles $(\mathbf{k},\mathbf{q})$ since the $r - \lfloor r^{\frac{1}{4}} \rfloor \le k \le r$. In this regime, we exploit a mechanism based on *large denominators*. This means that we shall collect the contributions in (2.31) corresponding to potentials $V_{J_{\mathbf{k}',\mathbf{q}'}}^{(\mathbf{k},\mathbf{q})_{-1}}$ that are already block-diagonal and, then, estimate them in terms of a sum of projections $P_{J_{\mathbf{k}',\mathbf{q}'}}^{(+)}$ controlled, through an induction, by the denominator appearing in the expression of $(S_{J_{\mathbf{r},\mathbf{i}}})_1$ [see formula (4.52)]; in the proof by induction for this regime, we make use of the auxiliary quantities displayed in (5.91).

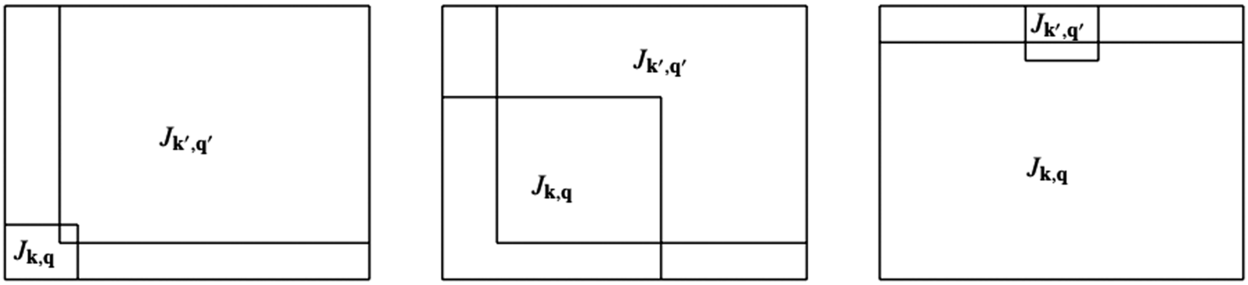

FIG. 1. Examples of configurations of $\mathfrak{R}1$, $\mathfrak{R}2$, $\mathfrak{R}3$, respectively.

III. TREE STRUCTURE AND PATHS OF RECTANGLES

In order to study regime $\mathfrak{R}1$, we shall re-expand the potentials $V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})}$ using the recursive definition (Definition 2.4) repeatedly. The method we develop to single out the terms in the re-expansion contributing to a certain effective potential and to, then, count and *weight* them is of some independent interest, irrespective of the crucial role it will play in our analysis of regime $\mathfrak{R}1$. We, therefore, describe it carefully in this section.

For the purpose of re-expanding $V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})}$ using Definition 2.4, we observe that, for $r \gg 1$, case (b) of Definition 2.4 can occur only after many steps of the re-expansion because $k \le \lfloor r^{\frac{1}{4}} \rfloor$ in regime $\mathfrak{R}1$. In order to streamline our formulas, we introduce the notation

$$\sum_{n=1}^{\infty} \frac{1}{n!}\, ad^n S_{J_{\mathbf{k},\mathbf{q}}}(\cdots) =: \mathcal{A}_{J_{\mathbf{k},\mathbf{q}}}(\cdots). \tag{3.33}$$

Depending on the relative position between $J_{\mathbf{k},\mathbf{q}}$ and $J_{\mathbf{r},\mathbf{i}}$, we are instructed to use either

$$V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})} = V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})_{-1}} \tag{3.34}$$

$$+\mathcal{A}_{J_{\mathbf{k},\mathbf{q}}}\big(V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})_{-1}}\big) \tag{3.35}$$

$$+\sum_{J_{\mathbf{k}',\mathbf{q}'} \in \mathcal{G}_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})}} \mathcal{A}_{J_{\mathbf{k},\mathbf{q}}}\big(V_{J_{\mathbf{k}',\mathbf{q}'}}^{(\mathbf{k},\mathbf{q})_{-1}}\big) \tag{3.36}$$

or

$$V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})} = V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})_{-1}}, \tag{3.37}$$

corresponding to cases (a) and (c) in Definition 2.4, respectively. We will use formulas (a) and (c) of Definition 2.4 iteratively for the potentials on the right-hand side of (3.34)–(3.37) when they apply, if it is the case all the way down to step $(\mathbf{0},\mathbf{N})$, but do not re-expand potentials of the type $V_{J_{\mathbf{k}'',\mathbf{q}''}}^{(\mathbf{k}'',\mathbf{q}'')}$ when they appear (i.e., we stop the re-expansion), which corresponds to case (b) of Definition 2.4.

The strategy can be summarized as consisting of the following steps.

- Introducing tree diagrams, we show that every contribution, $\mathfrak{b}$, to an effective potential—where $\mathfrak{b}$ stands for "branch-operator," a notion that is motived by the tree structure described below—of the re-expansion resulting from (3.34)–(3.37) is determined by a set, $\mathcal{R}_{\mathfrak{b}}$, of rectangles that are ordered and whose union is connected.
- We show that there is an injective map from $\{\mathcal{R}_{\mathfrak{b}}\}$ to a set, $\{\Gamma_{\mathfrak{b}}\}$, of paths of rectangles with certain properties.
- By assigning suitable weights to the paths $\Gamma_{\mathfrak{b}}$, we will be able to derive upper bounds on the norms of the contributions $\mathfrak{b}$. This will allow us to estimate the norm $\|V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})}\|$ by counting (weighted) paths belonging to the set $\{\Gamma_{\mathfrak{b}}\}$.

A. Tree expansion

In order to find an efficient description (see Definition 3.1) of the structure of contributions emerging from the re-expansion of $V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})}$, we study the type of terms we get after a few re-expansion steps. For example, if we assume that the relative positions of $J_{\mathbf{k},\mathbf{q}}$ and $J_{\mathbf{r},\mathbf{i}}$ are such that the first re-expansion step is of type (c), followed by a re-expansion step of type (a), then we get

$$V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}} = e^{S_{J_{\mathbf{k},\mathbf{q}}}} V^{(\mathbf{k},\mathbf{q})_{-2}}_{J_{\mathbf{r},\mathbf{i}}} e^{-S_{J_{\mathbf{k},\mathbf{q}}}} + \sum_{J_{\mathbf{k}',\mathbf{q}'} \in \mathcal{G}^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}} \mathcal{A}_{J_{\mathbf{k},\mathbf{q}}}(V^{(\mathbf{k},\mathbf{q})_{-2}}_{J_{\mathbf{k}',\mathbf{q}'}}) \tag{3.38}$$

$$= V^{(\mathbf{k},\mathbf{q})_{-2}}_{J_{\mathbf{r},\mathbf{i}}} + \mathcal{A}_{J_{\mathbf{k},\mathbf{q}}}(V^{(\mathbf{k},\mathbf{q})_{-2}}_{J_{\mathbf{r},\mathbf{i}}}) + \sum_{J_{\mathbf{k}',\mathbf{q}'} \in \mathcal{G}^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}} \mathcal{A}_{J_{\mathbf{k},\mathbf{q}}}(V^{(\mathbf{k},\mathbf{q})_{-2}}_{J_{\mathbf{k}',\mathbf{q}'}}). \tag{3.39}$$

Note that in (3.33) and, consequently, in (3.39), we interpret the sum over n as a single contribution. The re-expansion of every potential term alluded to above, iterated down either to the first level where case (b) of Definition 2.4 applies or, if this does not happen, to level $(\mathbf{0},\mathbf{N})$, can be described using an upside-down tree structure (see the first three levels in Fig. 2), following the list of prescriptions described in the next definition.

Definition 3.1.

1. The levels of a tree used to identify the contributions to the re-expansion of a potential $V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}$ are labeled by $(\mathbf{k}',\mathbf{q}')$ with $(\mathbf{k}',\mathbf{q}')$ such that $(\mathbf{k},\mathbf{q}) \geq (\mathbf{k}',\mathbf{q}') \geq (\mathbf{0},\mathbf{N})$. We say that such a tree is *rooted* at level $(\mathbf{k},\mathbf{q})$.
2. There is a single vertex at the top of a tree rooted at level $(\mathbf{k},\mathbf{q})$; it is labeled by the symbol $V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}$ of the potential.
3. The vertices at level $(\mathbf{k}',\mathbf{q}')_{-1}$ of a tree rooted at level $(\mathbf{k},\mathbf{q})$ are determined by the vertices of the tree at level $(\mathbf{k}',\mathbf{q}')$ in the following way: Each vertex $\mathfrak{v} \equiv \mathfrak{v}_{V^{(\mathbf{k}',\mathbf{q}')}_{J_{\mathbf{s},u}}}$ at level $(\mathbf{k}',\mathbf{q}')$, labeled by $V^{(\mathbf{k}',\mathbf{q}')}_{J_{\mathbf{s},u}}$, is linked to two sets of descendants (vertices) at level $(\mathbf{k}',\mathbf{q}')_{-1}$ with the following properties: The two sets of vertices are *empty* if $(\mathbf{s},\mathbf{u}) = (\mathbf{k}',\mathbf{q}')$; otherwise, we have the following:
   - the leftmost set of vertices actually consists of a single vertex, which is labeled by the potential $V^{(\mathbf{k}',\mathbf{q}')_{-1}}_{J_{\mathbf{s},\mathbf{u}}}$;
   - the rightmost set of vertices is empty if $J_{\mathbf{k}',\mathbf{q}'} \not\subset J_{\mathbf{s},\mathbf{u}}$; otherwise, it contains a vertex for each element $J_{\mathbf{s}',\mathbf{u}'}$ belonging to $\mathcal{G}^{(\mathbf{k}',\mathbf{q}')}_{J_{\mathbf{s},\mathbf{u}}} \cup \{J_{\mathbf{s},\mathbf{u}}\}$, and this vertex is labeled by $V^{(\mathbf{k}',\mathbf{q}')_{-1}}_{J_{\mathbf{s}',\mathbf{u}'}}$.
4. Each vertex $\mathfrak{v}$ at level $(\mathbf{k}',\mathbf{q}')$ is connected by an edge to its descendants at level $(\mathbf{k}',\mathbf{q}')_{-1}$. Edges are labeled by rectangles, or carry no label, in the following way:

(e-i) the edge connecting a vertex $\mathfrak{v}$ at level $(\mathbf{k}',\mathbf{q}')$ to its leftmost descendant at level $(\mathbf{k}',\mathbf{q}')_{-1}$ has no label. It stands for the map

$$V^{(\mathbf{k}',\mathbf{q}')}_{J_{\mathbf{s},\mathbf{u}}} \to V^{(\mathbf{k}',\mathbf{q}')_{-1}}_{J_{\mathbf{s},\mathbf{u}}},$$

where $V^{(\mathbf{k}',\mathbf{q}')}_{J_{\mathbf{s},\mathbf{u}}}$ is the potential labeling $\mathfrak{v}$ and $V^{(\mathbf{k}',\mathbf{q}')_{-1}}_{J_{\mathbf{s},\mathbf{u}}}$ labels its leftmost descendant at level $(\mathbf{k}',\mathbf{q}')_{-1}$;

(e-ii) each edge $\mathfrak{e}$ connecting the vertex $\mathfrak{v}$ at level $(\mathbf{k}',\mathbf{q}')$ to other descendants at level $(\mathbf{k}',\mathbf{q}')_{-1}$ is labeled by a rectangle $J_{\mathbf{k}',\mathbf{q}'}$. It stands for the map

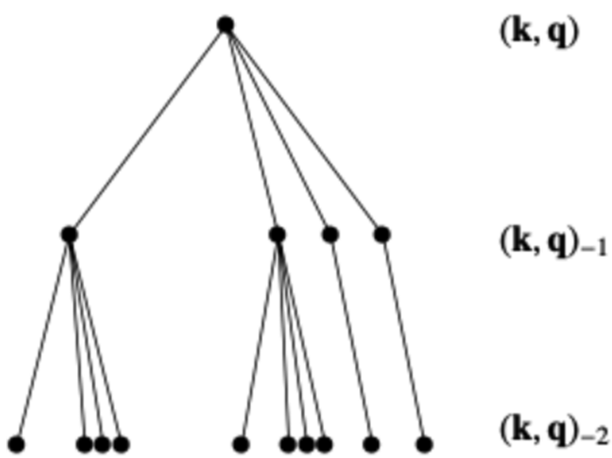

FIG. 2. Example of a tree associated with the first two steps of the re-expansion of $V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}$.

$$V_{J_{\mathbf{s},\mathbf{u}}}^{(\mathbf{k}',\mathbf{q}')} \to \mathcal{A}_{J_{\mathbf{k}',\mathbf{q}'}}(V_{J_{\mathbf{s}',\mathbf{u}'}}^{(\mathbf{k}',\mathbf{q}')_{-1}}),$$

where $V_{J_{\mathbf{s},\mathbf{u}}}^{(\mathbf{k}',\mathbf{q}')}$ labels the vertex $\mathfrak{v}$ and $V_{J_{\mathbf{s}',\mathbf{u}'}}^{(\mathbf{k}',\mathbf{q}')_{-1}}$ is the potential labeling the vertex connected to $\mathfrak{v}$ by the edge $\mathfrak{e}$.

5. A leaf of the tree is a vertex at some level $(\mathbf{k}',\mathbf{q}')$ that has no descendants, i.e., it is not connected to any vertex at level $(\mathbf{k}',\mathbf{q}')_{-1}$ by any edge. Note that a leaf of the tree is labeled by a potential of the type $V_{J_{\mathbf{k}'',\mathbf{q}''}}^{(\mathbf{k}'',\mathbf{q}'')}$ for some $(\mathbf{k}'',\mathbf{q}'') \geq (\mathbf{0},\mathbf{N})$.
6. A branch of a tree rooted at $(\mathbf{k},\mathbf{q})$ is an ordered connected set of edges with the following properties:
   - the first edge of a branch has the vertex at level $(\mathbf{k},\mathbf{q})$ as an endpoint,
   - the last edge of a branch has a leaf at some level $(\mathbf{k}'',\mathbf{q}'')$ as an endpoint (referred to as the leaf of the branch), and
   - there is a single edge connecting vertices at levels $(\mathbf{k}',\mathbf{q}')$ and $(\mathbf{k}',\mathbf{q}')_{-1}$ for every $(\mathbf{k}',\mathbf{q}')$ with $(\mathbf{k},\mathbf{q}) \geq (\mathbf{k}',\mathbf{q}') > (\mathbf{k}'',\mathbf{q}'')$.
7. With each branch $\mathfrak{b}$ of a tree, we associate a set, $\mathcal{R}_{\mathfrak{b}}$, of rectangles consisting of (i) those rectangles labeling the edges of $\mathfrak{b}$ and (ii) the rectangle $J_{\mathbf{k}'',\mathbf{q}''}$ indicating the support of the potential labeling the leaf of $\mathfrak{b}$.

   The set $\mathcal{R}_{\mathfrak{b}}$ inherits the ordering relation (2.14), and hence, its elements can be enumerated by a map

$$i \in \{1,\ldots,|\mathcal{R}_{\mathfrak{b}}|\} \to J_{\mathbf{k}^{(i)},\mathbf{q}^{(i)}} \in \mathcal{R}_{\mathfrak{b}}$$

   with $(\mathbf{k}^{(i)},\mathbf{q}^{(i)}) > (\mathbf{k}^{(i+1)},\mathbf{q}^{(i+1)})$, where $|\mathcal{R}_{\mathfrak{b}}|$ is the cardinality of the set $\mathcal{R}_{\mathfrak{b}}$. Note that $J_{\mathbf{k}^{(|\mathcal{R}_{\mathfrak{b}}|)},\mathbf{q}^{(|\mathcal{R}_{\mathfrak{b}}|)}}$ is the rectangle associated with the potential labeling the leaf of $\mathfrak{b}$.
8. To every branch $\mathfrak{b}$, we can associate the "*branch operator*," also denoted by $\mathfrak{b}$,

$$\mathfrak{b} := \mathcal{A}_{J_{\mathbf{k}^{(1)},\mathbf{q}^{(1)}}}(\mathcal{A}_{J_{\mathbf{k}^{(2)},\mathbf{q}^{(2)}}}(\cdots\mathcal{A}_{J_{\mathbf{k}^{(|\mathcal{R}_{\mathfrak{b}}|-1)},\mathbf{q}^{(|\mathcal{R}_{\mathfrak{b}}|-1)}}}(V_{\mathcal{L}_{\mathfrak{b}}})\cdots)), \tag{3.40}$$

   where $V_{\mathcal{L}_{\mathfrak{b}}}$ is the potential labeling the leaf of $\mathfrak{b}$; $V_{\mathcal{L}_{\mathfrak{b}}}$ can be either $V_{J_{\mathbf{k}^{(|\mathcal{R}_{\mathfrak{b}}|)},\mathbf{q}^{(|\mathcal{R}_{\mathfrak{b}}|)}}}^{(\mathbf{k}^{(|\mathcal{R}_{\mathfrak{b}}|)},\mathbf{q}^{(|\mathcal{R}_{\mathfrak{b}}|)})}$ or $V_{\mathcal{J}_{\mathbf{s},\mathbf{u}}}^{(\mathbf{0},\mathbf{N})}$.

The set of branches whose corresponding branch operators are non-zero is denoted by $\mathcal{B}_{V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})}}$.

Remark 3.2. We stress that the operators corresponding to most of the branches are actually zero, for example, when the corresponding leaf is an operator of the type $V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{0},\mathbf{N})}$ with $r > 1$, which is zero by definition.

1. Properties of the branches $\mathfrak{b} \in \mathcal{B}_{V_{J_{r,i}}^{(k,q)}}$

Definition 3.1 implies the following properties of the elements of the set $\mathcal{B}_{V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})}}$ defined above:

(P-i) For $\mathfrak{b} \in \mathcal{B}_{V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})}}$, the set

$$\bigcup_{i\in\{1,\ldots,|\mathcal{R}_{\mathfrak{b}}|\}} J_{\mathbf{k}^{(i)},\mathbf{q}^{(i)}}$$

is connected due to (3.40), although $J_{\mathbf{k}^{(i)},\mathbf{q}^{(i)}} \cap J_{\mathbf{k}^{(i+1)},\mathbf{q}^{(i+1)}}$ might be empty for some i. Likewise, for any fixed $n \in \{1\ldots|\mathcal{R}_{\mathfrak{b}}|\}$, the set $\bigcup_{n\leq i\leq|\mathcal{R}_{\mathfrak{b}}|} J_{\mathbf{k}^{(i)},\mathbf{q}^{(i)}}$ is connected. Indeed, for any operator $\mathcal{O}$ and for any m, $\mathcal{A}_{J_{\mathbf{k}^{(m)},\mathbf{q}^{(m)}}}(\mathcal{O}) = 0$ whenever the supports of $\mathcal{O}$ and $S_{J_{\mathbf{k}^{(m)},\mathbf{q}^{(m)}}}$ have empty intersection; see formula (3.33).

(P-ii) For $\mathfrak{b} \in \mathcal{B}_{V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})}}$, the cardinality, $|\mathcal{R}_{\mathfrak{b}}|$, of the set $\mathcal{R}_{\mathfrak{b}}$ of rectangles is such that $|\mathcal{R}_{\mathfrak{b}}| \geq \mathcal{O}(\frac{r}{k}) \geq \mathcal{O}(r^{\frac{3}{4}})$. This lower bound on $|\mathcal{R}_{\mathfrak{b}}|$ is a consequence of the restriction imposed on $k = |\mathbf{k}|$ and required in regime $\mathfrak{R}1$ [and it will turn out to be crucial to derive our estimate (5.105) and (5.106) in Theorem 5.1].

(P-iii) The set $J_{\mathbf{r},\mathbf{i}}$ is the minimal rectangle associated with $\bigcup_{i\in\{1,\ldots,|\mathcal{R}_{\mathfrak{b}}|\}} J_{\mathbf{k}^{(i)},\mathbf{q}^{(i)}}$ for any branch $\mathfrak{b} \in \mathcal{B}_{V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})}}$. Furthermore, if we amputate a branch at some vertex by keeping only the descendants of that vertex (i.e., the lower part only), then the same property holds for the rectangle associated with the potential labeling the (new) root vertex of the amputated branch that has been created.

(P-iv) Two different branches $\mathfrak{b},\mathfrak{b}' \in \mathcal{B}_{V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})}}$ are associated with two different (ordered) sets of rectangles $\mathcal{R}_{\mathfrak{b}}$ and $\mathcal{R}_{\mathfrak{b}'}$.

Sketch of proof:
(1) The two branches must cross at some vertex.
(2) Consider the first vertex (starting at the bottom of the tree) where they cross and the two (possibly) amputated branches corresponding to the two original branches that have this vertex as their root vertex.

(3) Now, note that there are two alternatives: (3-i) either the rectangles associated with the two edges linked to the root vertex (the vertex where they cross) are different in the sense that one edge is associated with a rectangle and the other to none (3-ii) or some of the remaining rectangles in the amputated branches must differ due to property (P-iii) since the potentials labeling the vertices at the level just below the common root vertex are different.

(P-v) Each term in the re-expansion is associated with a branch $\mathfrak{b}$ of the tree, and this correspondence is bijective by construction. Thus, by property (P-iv), two distinct non-zero terms in the re-expansion, corresponding to two different branches $\mathfrak{b}_1, \mathfrak{b}_2 \in \mathcal{B}_{V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}}$, are labeled by two different sets of rectangles, $\mathcal{R}_{\mathfrak{b}_1}$ and $\mathcal{R}_{\mathfrak{b}_2}$, respectively.

B. Summing over the norms of branch-operator: *Weights* and *paths*, $\Gamma_{\mathfrak{b}}$

Our task is to estimate the norms of the potentials $V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}$, which can be accomplished by taking the re-expansion of the potentials into account according to the prescriptions of Definition 3.1. More precisely, each potential $V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}$ can be expressed as the sum $\sum_{\mathfrak{b}\in\mathcal{B}_{V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}}}\mathfrak{b}$, where $\mathfrak{b}$ are the branch operators defined in point 8 of Definition 3.1. Therefore, we are led to estimating the sum over the norms of branch operators to wit

$$\sum_{\mathfrak{b}\in\mathcal{B}_{V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}}} \|\mathfrak{b}\|.$$

This can be done by assigning a "weight" to every set $\mathcal{R}_{\mathfrak{b}}$ of rectangles, the weight being proportional to the product of operator norms of the potentials associated [in step $(\mathbf{k},\mathbf{q})_{-1}$] with each rectangle $J_{\mathbf{k},\mathbf{q}}$ in the set $\mathcal{R}_{\mathfrak{b}}$, i.e.,

$$\sum_{\mathfrak{b}\in\mathcal{B}_{V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}}} (c\cdot t)^{|\mathcal{R}_{\mathfrak{b}}|-1}\,\|V_{\mathcal{L}_{\mathfrak{b}}}\| \prod_{i\in\{1,\ldots,|\mathcal{R}_{\mathfrak{b}}|-1\}} \|V^{(\mathbf{k}^{(i)},\mathbf{q}^{(i)})_{-1}}_{J_{\mathbf{k}^{(i)},\mathbf{q}^{(i)}}}\|, \tag{3.41}$$

where $V_{\mathcal{L}_{\mathfrak{b}}}$ is the potential labeling the leaf of $\mathfrak{b}$ since a factor $(c\cdot t)\cdot\|V^{(\mathbf{k}^{(i)},\mathbf{q}^{(i)})_{-1}}_{J_{\mathbf{k}^{(i)},\mathbf{q}^{(i)}}}\|$ is associated with the map $\mathcal{A}_{J_{\mathbf{k}^{(i)},\mathbf{q}^{(i)}}}$; here, $c>0$ is a universal constant.

In order to count the sets $\mathcal{R}_{\mathfrak{b}}$, we shall assign a path, $\Gamma_{\mathfrak{b}}$, to each $\mathfrak{b}$, where $\Gamma_{\mathfrak{b}}$ has the property to visit all the rectangles in the set $\mathcal{R}_{\mathfrak{b}}$. Since we must estimate the "weighted" number of sets $\mathcal{R}_{\mathfrak{b}}$, the paths must be weighted accordingly.

1. Paths of connected rectangles

The following definitions clarify what we mean by a path visiting rectangles.

Definition 3.3.

(i) A path Γ is a finite sequence of rectangles $\{J_{\mathbf{s}^{(i)},\mathbf{u}^{(i)}}\}_{i=1}^{n}$, for some $n\in\mathbb{N}$, with the property that $J_{\mathbf{s}^{(i)},\mathbf{u}^{(i)}}\neq J_{\mathbf{s}^{(i+1)},\mathbf{u}^{(i+1)}}$ and $J_{\mathbf{s}^{(i)},\mathbf{u}^{(i)}}\cap J_{\mathbf{s}^{(i+1)},\mathbf{u}^{(i+1)}}\neq\emptyset$ for every $i=1\ldots n-1$.
Warning: In contrast to item 7 in Definition 3.1, no relation is assumed here between the ordering labeled by the index i and the ordering $\prec$.

(ii) The set of ordered pairs,

$$\mathcal{S}_\Gamma := \left\{\left(J_{\mathbf{s}^{(i)},\mathbf{u}^{(i)}}, J_{\mathbf{s}^{(i+1)},\mathbf{u}^{(i+1)}}\right)\middle|\ i=1,\ldots,n-1\right\},$$

is called the set of steps of the path $\Gamma\equiv\left\{J_{\mathbf{s}^{(i)},\mathbf{u}^{(i)}}\right\}_{i=1}^{n}$.

(iii) The length, l_Γ, of the path $\Gamma\equiv\{J_{\mathbf{s}^{(i)},\mathbf{u}^{(i)}}\}_{i=1}^{n}$ is defined to be $l_\Gamma := n-1$.

(iv) The support, $\mathrm{supp}(\Gamma)$, of a path $\Gamma\equiv\{J_{\mathbf{s}^{(i)},\mathbf{u}^{(i)}}\}_{i=1}^{n}$ is defined to be

$$\mathrm{supp}(\Gamma) := \left\{J_{\mathbf{s}^{(i)},\mathbf{u}^{(i)}},\ i\in\{1\ \ldots n\}\right\}.$$

(v) A path $\Gamma\equiv\{J_{\mathbf{s}^{(i)},\mathbf{u}^{(i)}}\}_{i=1}^{n}$ is closed if $J_{\mathbf{s}^{(1)},\mathbf{u}^{(1)}} = J_{\mathbf{s}^{(n)},\mathbf{u}^{(n)}}$.

Each rectangle $J_{\mathbf{k}^{(i)},\mathbf{q}^{(i)}}$ of the set $\mathcal{R}_{\mathfrak{b}}$ contributes to weight (3.41) of $\mathcal{R}_{\mathfrak{b}}$ through $c\cdot t\cdot\|V^{(\mathbf{k}^{(i)},\mathbf{q}^{(i)})_{-1}}_{J_{\mathbf{k}^{(i)},\mathbf{q}^{(i)}}}\|$ [except for $J_{\mathbf{k}^{(|\mathcal{R}_{\mathfrak{b}}|)},\mathbf{q}^{(|\mathcal{R}_{\mathfrak{b}}|)}}$ that contributes through $c\cdot\|V_{\mathcal{L}_{\mathfrak{b}}}\|$], which (as it will be shown) decreases with the size of the rectangle. Thus, we have to make sure that the path

$\Gamma_{\mathfrak{b}}$ does not visit small rectangles of $\mathcal{R}_{\mathfrak{b}}$, which have a "big" weight, repeatedly. This motivates the requirements imposed on the paths $\Gamma_{\mathfrak{b}}$ considered henceforth, in particular property (C) stated in Sec. III B 2.

2. Connected components, $\mathcal{Z}_\rho^{(j)}$, of rectangles and definition of $\Gamma_{\mathfrak{b}}$

Since the weight of a rectangle is a function of its size, it is convenient to write the connected set $\bigcup_{i\in\{1,\dots,|\mathcal{R}_{\mathfrak{b}}|\}} J_{\mathbf{k}^{(i)},\mathbf{q}^{(i)}}$ as the union

$$\bigcup_{\rho=k_0}^{k}\left(\bigcup_{j=1}^{j_\rho}\mathcal{Z}_\rho^{(j)}\right),$$

where $\{\mathcal{Z}_\rho^{(j)},\quad j=1,\dots,j_\rho\}$ are distinct connected components of (unions of) rectangles of a given size ρ, $k_0\le\rho\le k$, starting from the lowest one $k_0\ge 1$, with the following properties:

(1) $j_{k_0}=1$ (i.e., there is only one component for $\rho=k_0$);

(2) rectangles of the same size but belonging to different components do *not* overlap, i.e., for any ρ, $\mathcal{Z}_\rho^{(j)}\cap\mathcal{Z}_\rho^{(j')}=\emptyset$ for $j\ne j'$.

We call $\mathrm{supp}(\mathcal{Z}_\rho^{(j)})$, $\rho=k_0,\dots,k, j=1,\dots,j_\rho$, the set of rectangles of $\mathcal{Z}_\rho^{(j)}$, i.e.,

$$\mathrm{supp}(\mathcal{Z}_\rho^{(j)}):=\left\{J_{\mathbf{k}^{(i)},\mathbf{q}^{(i)}}:J_{\mathbf{k}^{(i)},\mathbf{q}^{(i)}}\subset\mathcal{Z}_\rho^{(j)}, i\in\{1,\dots,|\mathcal{R}_{\mathfrak{b}}|\}\right\}.$$

Starting from a branch $\mathfrak{b}\in\mathcal{B}_{V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})}}$, we shall inductively construct a path, $\Gamma_{\mathfrak{b}}$, of length $l_{\Gamma_{\mathfrak{b}}}$ bounded by

$$l_{\Gamma_{\mathfrak{b}}}\le 2\left(n_{k_0}+\sum_{j=1}^{j_2}n_{k_0+1}^{(j)}+\dots+\sum_{j=1}^{j_k}n_k^{(j)}\right)-2,$$

where $n_\rho^{(j)}:=|\mathrm{supp}(\mathcal{Z}_\rho^{(j)})|$, with the following properties:

(A) the support of $\Gamma_{\mathfrak{b}}$ is $\mathcal{R}_{\mathfrak{b}}$;

(B) for each component $\mathcal{Z}_\rho^{(j)}$ consisting of the union of $n_\rho^{(j)}$ rectangles, at most $2n_\rho^{(j)}-2$ steps are made [i.e., there are at most $2n_\rho^{(j)}-2$ steps $\sigma\in\mathcal{S}_{\Gamma_{\mathfrak{b}}}$ for which $\sigma\in\mathrm{supp}(\mathcal{Z}_\rho^{(j)})\times\mathrm{supp}(\mathcal{Z}_\rho^{(j)})$];

(C) there are at most two steps connecting rectangles in $\mathrm{supp}(\mathcal{Z}_\rho^{(j)})$ with rectangles of lower size: more precisely, for every connected component $\mathcal{Z}_\rho^{(j)}$, there is at most one $J_{\mathbf{s},\mathbf{u}}$ in $\mathrm{supp}(\mathcal{Z}_\rho^{(j)})$ such that $(J_{\mathbf{s}',\mathbf{u}'},J_{\mathbf{s},\mathbf{u}})\in\mathcal{S}_{\Gamma_{\mathfrak{b}}}$ with $s'<s$, $J_{\mathbf{s}',\mathbf{u}'}\in\mathcal{R}_{\mathfrak{b}}$ and at most one $J_{\mathbf{s},\mathbf{u}}\in\mathrm{supp}(\mathcal{Z}_\rho^{(j)})$ such that $(J_{\mathbf{s},\mathbf{u}},J_{\mathbf{s}',\mathbf{u}'})\in\mathcal{S}_{\Gamma_{\mathfrak{b}}}$ with $s>s'$, $J_{\mathbf{s}',\mathbf{u}'}\in\mathcal{R}_{\mathfrak{b}}$.

The precise construction is carried out by induction in k in Lemma A.5, combined with Lemma A.4, i.e., we assume that we have constructed a path $\Gamma_{\mathfrak{b}}^{(k'-1)}$, with $k_0+1\le k'\le k$, fulfilling (A)–(C) for the set $\cup_{\rho=k_0}^{k'-1}\cup_{j=1}^{j_\rho}\mathcal{Z}_\rho^{(j)}$, which is connected by property (P-i). Starting from this path, we construct a new one, denoted by $\Gamma_{\mathfrak{b}}^{(k')}$, with the desired properties.

3. Weighted sums of paths

The features specified by (A)–(C), above, are used to distribute the total weight available, as shown in (3.41), among the steps of the path $\Gamma_{\mathfrak{b}}$, in a way that is optimal to derive suitable bounds. In fact, we will associate a weight with the steps of the paths $\Gamma_{\mathfrak{b}}$ described in Sec. III B 2, so as to estimate (3.41) in terms of a weighted sum of paths. The mechanism, which we shall illustrate below, is essentially the one used in Theorem 5.1 to control regime $\mathfrak{R}1$, with some modifications that we omit here in order not to obscure the key ideas, which are related to the proof by induction of Theorem 5.1.

We observe that there are $n_\rho^{(j)}$ rectangles in the set $\mathrm{supp}(\mathcal{Z}_\rho^{(j)})$ and that, for the paths $\Gamma_{\mathfrak{b}}$, there are at most $2n_\rho^{(j)}-2$ steps between these rectangles; see property (B). In addition, there are at most two steps, from rectangles of lower size and back, to be taken into account; see property (C). Consequently, to each step $\sigma=(J_{\mathbf{s}^{(i)},\mathbf{u}^{(i)}},J_{\mathbf{s}^{(i+1)},\mathbf{u}^{(i+1)}})\in\mathcal{S}_{\Gamma_{\mathfrak{b}}}$, we can assign the weight

$$\mathfrak{W}_\sigma\equiv\mathfrak{W}_{s^{(i)}\to s^{(i+1)}}=((c+1)t)^{\frac12}\cdot\min\left\{\|V_{J_{\mathbf{s}^{(i)},\mathbf{u}^{(i)}}}^{(\mathbf{s}^{(i)},\mathbf{u}^{(i)})_{-1}}\|^{\frac12},\|V_{J_{\mathbf{s}^{(i+1)},\mathbf{u}^{(i+1)}}}^{(\mathbf{s}^{(i+1)},\mathbf{u}^{(i+1)})_{-1}}\|^{\frac12}\right\},$$

where t is sufficiently small such that $(c+1)t\cdot\|V_{J_{\mathbf{k}^{(i)},\mathbf{q}^{(i)}}}^{(\mathbf{k}^{(i)},\mathbf{q}^{(i)})_{-1}}\|<1$, and the following estimate holds:

$$(c\cdot t)^{|R_{\mathfrak{b}}|-1}\,\|V_{\mathcal{L}_{\mathfrak{b}}}\|\prod_{i=1}^{|R_{\mathfrak{b}}|-1}\|V_{J_{\mathbf{k}^{(i)},\mathbf{q}^{(i)}}}^{(\mathbf{k}^{(i)},\mathbf{q}^{(i)})_{-1}}\|\le\frac{1}{t}\prod_{\sigma\in\mathcal{S}_{\Gamma_{\mathfrak{b}}}}\mathfrak{w}_\sigma. \tag{3.42}$$

The previous inequality is true because, if we denote by $\mathcal{S}_{\mathcal{Z}_\rho^{(j)}}$ the set of at most $2n_\rho^{(j)}-2$ steps between rectangles of $\mathrm{supp}\mathcal{Z}_\rho^{(j)}$ and the additional at most two steps from rectangles of lower size and back, then we have

$$(c\cdot t)^{|\mathrm{supp}(\mathcal{Z}_\rho^{(j)})|}\prod_{J_{\mathbf{s},\mathbf{u}}\in\,\mathrm{supp}(\mathcal{Z}_\rho^{(j)})}\|V_{J_{\mathbf{s},\mathbf{u}}}^{(\mathbf{s},\mathbf{u})_{-1}}\|\le\prod_{\sigma\in\mathcal{S}_{\mathcal{Z}_\rho^{(j)}}}\mathfrak{w}_\sigma.$$

Finally, we use the estimate

$$\sum_{\mathfrak{b}\in\mathcal{B}_{V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})}}}c^{|\mathcal{R}_{\mathfrak{b}}|-1}t^{|R_{\mathfrak{b}}|}\,\|V_{\mathcal{L}_{\mathfrak{b}}}\|\prod_{i\in\{1,\ldots,|\mathcal{R}_{\mathfrak{b}}|-1\}}\|V_{J_{\mathbf{k}^{(i)},\mathbf{q}^{(i)}}}^{(\mathbf{k}^{(i)},\mathbf{q}^{(i)})_{-1}}\| \tag{3.43}$$

$$\le\sum_{\Gamma_{\mathfrak{b}},\,\mathfrak{b}\in\mathcal{B}_{V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})}}}\prod_{\sigma\in\mathcal{S}_{\Gamma_{\mathfrak{b}}}}\mathfrak{w}_\sigma\le C_d\cdot r^{2d-1}\cdot\sum_{j=\lfloor c_d\cdot\frac{r}{k}\rfloor}^{\infty}\left(\sum_{\rho,\rho'=1}^{k}\mathfrak{w}_{\rho\to\rho'}\,D_{\rho,\rho'}\right)^j, \tag{3.44}$$

where $\lfloor c_d\cdot\frac{r}{k}\rfloor$ is a lower bound for $|\mathcal{R}_{\mathfrak{b}}|$ and $C_d\cdot r^{2d-1}$ is an upper bound on the possible positions of the rectangle $J_{\mathbf{k}^{(|\mathcal{R}_{\mathfrak{b}}|)},\mathbf{q}^{(|\mathcal{R}_{\mathfrak{b}}|)}}$ of the path, where c_d, C_d are d-dependent constants; finally,

$$D_{s,s'}:=\mathfrak{C}_d\cdot s^d\cdot s'^{d-1}, \tag{3.45}$$

where $\mathfrak{C}_d$ is a d-dependent constant, is an upper bound on the number of possible directions of a path $\Gamma=\{J_{\mathbf{s}^{(i)},\mathbf{u}^{(i)}}\}_{i=1}^n$, extended by one more step as specified here: given the path $\Gamma=\{J_{\mathbf{s}^{(i)},\mathbf{u}^{(i)}}\}_{i=1}^n$, the number of paths $\Gamma^+=\{J_{\mathbf{s}'^{(i)},\mathbf{u}'^{(i)}}\}_{i=1}^{n+1}$ of length $l_{\Gamma^+}=n$, whose first n elements agree with Γ (i.e., $\{J_{\mathbf{s}^{(i)},\mathbf{u}^{(i)}}\}_{i=1}^n=\{J_{\mathbf{s}'^{(i)},\mathbf{u}'^{(i)}}\}_{i=1}^n$) and for which $s'^{(n+1)}:=s'$ and $s^{(n)}:=s$, is bounded from above by $D_{s,s'}$.

A minor modification of the inequality provided in (3.44) will enable us to prove the result of Theorem 5.1 concerning regime $\mathfrak{R}1$.

IV. THE UNITARY CONJUGATION $e^{S_{J_{\mathbf{k},\mathbf{q}}}}$ AND THE SPECTRAL GAP OF $G_{J_{\mathbf{k},\mathbf{q}}}$

The operator $e^{S_{J_{\mathbf{k},\mathbf{q}}}}$ is constructed so as to block-diagonalize the Hamiltonian $G_{J_{\mathbf{k},\mathbf{q}}}+tV_{J_{\mathbf{k},\mathbf{q}}}^{(\mathbf{k},\mathbf{q})_{-1}}$ with respect to the decomposition of the identity

$$\mathbb{1}=P_{J_{\mathbf{k},\mathbf{q}}}^{(+)}+P_{J_{\mathbf{k},\mathbf{q}}}^{(-)}. \tag{4.46}$$

The operator

$$G_{J_{\mathbf{k},\mathbf{q}}}:=\sum_{\mathbf{i}\subset J_{\mathbf{k},\mathbf{q}}}H_{\mathbf{i}}+t\sum_{J_{\mathbf{k}'_{(1)},\mathbf{q}'}\subset J_{\mathbf{k},\mathbf{q}}}V_{J_{\mathbf{k}'_{(1)},\mathbf{q}'}}^{(\mathbf{k},\mathbf{q})_{-1}}+\cdots+t\sum_{J_{\mathbf{k}'_{(|\mathbf{k}|-1)},\mathbf{q}'}\subset J_{\mathbf{k},\mathbf{q}}}V_{J_{\mathbf{k}'_{(|\mathbf{k}|-1)},\mathbf{q}'}}^{(\mathbf{k},\mathbf{q})_{-1}} \tag{4.47}$$

is already block-diagonal with respect to (4.46).

For this construction, we refer the reader to the notation and the results in Secs. 2 and 3 of Ref. 28. We add the definition of $E_{J_{\mathbf{k},\mathbf{q}}}$, which is, in fact, the ground-state energy of the operator $G_{J_{\mathbf{k},\mathbf{q}}}$,

$$E_{J_{\mathbf{k},\mathbf{q}}}:=\langle\bigotimes_{\mathbf{j}\in J_{\mathbf{k},\mathbf{q}}}\Omega_{\mathbf{j}},\,G_{J_{\mathbf{k},\mathbf{q}}}\bigotimes_{\mathbf{j}\in J_{\mathbf{k},\mathbf{q}}}\Omega_{\mathbf{j}}\rangle, \tag{4.48}$$

i.e.,

$$G_{J_{\mathbf{k},\mathbf{q}}}P_{J_{\mathbf{k},\mathbf{q}}}^{(-)}=E_{J_{\mathbf{k},\mathbf{q}}}P_{J_{\mathbf{k},\mathbf{q}}}^{(-)}.$$

We recall that

$$ad\,A\,(B):=[A,B], \tag{4.49}$$

where A and B are bounded operators, and for $n\ge 2$,

$$ad^n A\,(B) := [A,\, ad^{n-1} A\,(B)]. \tag{4.50}$$

To carry out the block-diagonalization step $(\mathbf{k},\mathbf{q})$, the operator $S_{J_{\mathbf{k},\mathbf{q}}}$ is defined by the series

$$S_{J_{\mathbf{k},\mathbf{q}}} := \sum_{j=1}^{\infty} t^j (S_{J_{\mathbf{k},\mathbf{q}}})_j \tag{4.51}$$

with

- $$(S_{J_{\mathbf{k},\mathbf{q}}})_j := ad^{-1}\, G_{J_{\mathbf{k},\mathbf{q}}}\left((V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}})^{od}_j\right) := \frac{1}{G_{J_{\mathbf{k},\mathbf{q}}} - E_{J_{\mathbf{k},\mathbf{q}}}} P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} (V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}})_j\, P^{(-)}_{J_{\mathbf{k},\mathbf{q}}} - h.c., \tag{4.52}$$
  where "*od*" means the *off-diagonal* with respect to the decomposition of identity (4.46);
- $(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}})_1 := V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}}$, and for $j \geq 2$,

$$\begin{aligned}(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}})_j := &\sum_{p\geq 2, r_1\geq 1\ldots,r_p\geq 1\,;\, r_1+\cdots+r_p=j} \frac{1}{p!} \mathrm{ad}\,(S_{J_{\mathbf{k},\mathbf{q}}})_{r_1}\Big(\mathrm{ad}\,(S_{J_{\mathbf{k},\mathbf{q}}})_{r_2} \cdots (\mathrm{ad}\,(S_{J_{\mathbf{k},\mathbf{q}}})_{r_p}(G_{J_{\mathbf{k},\mathbf{q}}}))\cdots\Big)\\ &+ \sum_{p\geq 1, r_1\geq 1\ldots,r_p\geq 1\,;\, r_1+\cdots+r_p=j-1} \frac{1}{p!} \mathrm{ad}\,(S_{J_{\mathbf{k},\mathbf{q}}})_{r_1}\Big(\mathrm{ad}\,(S_{J_{\mathbf{k},\mathbf{q}}})_{r_2} \cdots (\mathrm{ad}\,(S_{J_{\mathbf{k},\mathbf{q}}})_{r_p}(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}}))\cdots\Big).\end{aligned} \tag{4.53}$$

We recall that

$$K^{(\mathbf{k},\mathbf{q})}_{\Lambda^d_N} := e^{S_{J_{\mathbf{k},\mathbf{q}}}}\, K^{(\mathbf{k},\mathbf{q})_{-1}}_{\Lambda^d_N}\, e^{-S_{J_{\mathbf{k},\mathbf{q}}}}. \tag{4.54}$$

The algorithm described in Definition 2.4 can be motivated by inspecting the proof of the next theorem, which establishes the consistency property alluded to in Sec. II C before introducing Definition 2.4.

Theorem 4.1. *The Hamiltonian* $K^{(\mathbf{k},\mathbf{q})}_{\Lambda^d_N} := e^{S_{J_{\mathbf{k},\mathbf{q}}}}\, K^{(\mathbf{k},\mathbf{q})_{-1}}_{\Lambda^d_N}\, e^{-S_{J_{\mathbf{k},\mathbf{q}}}}$ *can be written in the form given in (2.19), where the terms* $\{V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{l},\mathbf{i}}}\}$ *are obtained from the terms* $\{V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{l},\mathbf{i}}}\}$ *according to the algorithm described in Definition 2.4.*

Proof. In the expression

$$\begin{aligned}e^{S_{J_{\mathbf{k},\mathbf{q}}}}\, K^{(\mathbf{k},\mathbf{q})_{-1}}_{\Lambda^d_N}\, e^{-S_{J_{\mathbf{k},\mathbf{q}}}} = e^{S_{J_{\mathbf{k},\mathbf{q}}}} \Bigg\{ &\sum_{\mathbf{i}\in\Lambda^{(d)}_N} H_{\mathbf{i}} + t \sum_{\mathbf{k}'_{(1)},\mathbf{q}'} V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{k}'_{(1)},\mathbf{q}'}} + t \sum_{\mathbf{k}'_{(2)},\mathbf{q}'} V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{k}'_{(2)},\mathbf{q}'}} + \cdots +\\ &+ t \sum_{\mathbf{k}'_{(|\mathbf{k}|)},\mathbf{q}'} V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{k}'_{(|\mathbf{k}|)},\mathbf{q}'}} + t \sum_{\mathbf{k}'_{(|\mathbf{k}|+1)},\mathbf{q}'} V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{k}'_{(|\mathbf{k}|+1)},\mathbf{q}'}} + \cdots + t V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{N},\mathbf{1}}} \Bigg\} e^{-S_{J_{\mathbf{k},\mathbf{q}}}},\end{aligned} \tag{4.55}$$

we observe that we have the following:

- For all rectangles $J_{\mathbf{l},\mathbf{i}}$ such that $J_{\mathbf{l},\mathbf{i}} \cap J_{\mathbf{k},\mathbf{q}} = \emptyset$, we have that
  $$e^{S_{J_{\mathbf{k},\mathbf{q}}}}\, V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{l},\mathbf{i}}}\, e^{-S_{J_{\mathbf{k},\mathbf{q}}}} = V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{l},\mathbf{i}}} = V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{l},\mathbf{i}}}, \tag{4.56}$$
  where the last identity is due to item (a) in Definition 2.4.
- Regarding the terms constituting $G_{J_{\mathbf{k},\mathbf{q}}}$ [see the definition in (2.26)], we note that if we add $V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}}$, we get

$$\begin{aligned}&e^{S_{J_{\mathbf{k},\mathbf{q}}}}\,(G_{J_{\mathbf{k},\mathbf{q}}} + tV^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}})\, e^{-S_{J_{\mathbf{k},\mathbf{q}}}}\\ &= \sum_{\mathbf{i}\subset J_{\mathbf{k},\mathbf{q}}} H_{\mathbf{i}} + t \sum_{J_{\mathbf{k}'_{(1)},\mathbf{q}'}\subset J_{\mathbf{k},\mathbf{q}}} V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'_{(1)},\mathbf{q}'}} + \cdots + t \sum_{J_{\mathbf{k}'_{(|\mathbf{k}|-1)},\mathbf{q}'}\subset J_{\mathbf{k},\mathbf{q}}} V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'_{(|\mathbf{k}|-1)},\mathbf{q}'}} + \sum_{j=1}^{\infty} t^j (V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}})^{diag}_j\\ &= \sum_{\mathbf{i}\subset J_{\mathbf{k},\mathbf{q}}} H_{\mathbf{i}} + t \sum_{J_{\mathbf{k}'_{(1)},\mathbf{q}'}\subset J_{\mathbf{k},\mathbf{q}}} V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'_{(1)},\mathbf{q}'}} + \cdots + t \sum_{J_{\mathbf{k}'_{(|\mathbf{k}|-1)},\mathbf{q}'}\subset J_{\mathbf{k},\mathbf{q}}} V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'_{(|\mathbf{k}|-1)},\mathbf{q}'}} + tV^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{k},\mathbf{q}}},\end{aligned} \tag{4.57}$$

where the first equation results from the Lie–Schwinger procedure and the second one follows from Definition 2.4, items (a) and (b).

- For the terms $V^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{l},\mathbf{i}}}$ with $J_{\mathbf{l},\mathbf{i}} \cap J_{\mathbf{k},\mathbf{q}} \neq \emptyset$, but $J_{\mathbf{k},\mathbf{q}} \not\subseteq J_{\mathbf{l},\mathbf{i}}$ and $J_{\mathbf{l},\mathbf{i}} \not\subseteq J_{\mathbf{k},\mathbf{q}}$, we write

$$e^{S_{J_{\mathbf{k},\mathbf{q}}}}\, V^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{l},\mathbf{i}}}\, e^{-S_{J_{\mathbf{k},\mathbf{q}}}} = V^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{l},\mathbf{i}}} + \sum_{n=1}^{\infty} \frac{1}{n!}\, ad^n S_{J_{\mathbf{k},\mathbf{q}}}\big(V^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{l},\mathbf{i}}}\big), \tag{4.58}$$

where the first term on the right-hand side is $V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{l},\mathbf{i}}}$ by definition [see item (a) in Definition 2.4] and the second term contributes to the potential $V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{j}}}$, where $J_{\mathbf{r},\mathbf{j}} \equiv [J_{\mathbf{l},\mathbf{i}} \cup J_{\mathbf{k},\mathbf{q}}]$, along with analogous terms contained in the second sum on the right-hand side of formula (2.31) (where $\mathbf{i}$ is replaced by $\mathbf{j}$) and with

$$e^{S_{J_{\mathbf{k},\mathbf{q}}}}\, V^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{r},\mathbf{j}}}\, e^{-S_{J_{\mathbf{k},\mathbf{q}}}}. \tag{4.59}$$

Note that the term in (4.59) corresponds to the first term in (2.31) (where $\mathbf{i}$ is replaced by $\mathbf{j}$).

□

In the remainder of this section, we reproduce a key result, established in Ref. 6, which enables us to estimate the spectral gap above the ground-state energy of the Hamiltonian $G_{J_{\mathbf{k},\mathbf{q}}}$. The proof is included for the convenience of the reader, but the arguments are essentially identical to those used in Ref. 6. As for chains ($d = 1$; see Ref. 6), it is not difficult to prove that under the assumption that

$$\| V^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{l},\mathbf{i}}} \| \leq t^{\frac{l-1}{4}}, \quad l = |\mathbf{l}| := l_1 + l_2 + \cdots + l_d, \qquad \text{for } 0 \leq t < t_d, \tag{4.60}$$

the Hamiltonian $G_{J_{\mathbf{k},\mathbf{q}}}$ has a gap $\Delta_{J_{\mathbf{k},\mathbf{q}}} \geq \frac{1}{2}$ for all $t \in [0, t_d)$, where t_d depends on the lattice dimension but is independent of $(\mathbf{k},\mathbf{q})$ and N. The main ingredients for the proof can be found in Lemma A.1 and Corollary A.2, namely,

$$P^{(+)}_{J_{\mathbf{l},\mathbf{i}}} \leq \sum_{\mathbf{j}\in J_{\mathbf{l},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} \tag{4.61}$$

and

$$\sum_{\mathbf{i}\,:\, J_{\mathbf{l},\mathbf{i}} \subset J_{\mathbf{k},\mathbf{q}}} P^{(+)}_{J_{\mathbf{l},\mathbf{i}}} \leq \left\{ \prod_{j=1}^{d} (l_j + 1) \right\} \sum_{\mathbf{i}\in J_{\mathbf{k},\mathbf{q}}} P^{\perp}_{\Omega_{\mathbf{i}}} \tag{4.62}$$

$$\leq (l+1)^d \sum_{\mathbf{i}\in J_{\mathbf{k},\mathbf{q}}} P^{\perp}_{\Omega_{\mathbf{i}}}. \tag{4.63}$$

Remark 4.2. Observe that the number of shapes[29] of rectangles $J_{\mathbf{l},\mathbf{i}}$ at fixed $|\mathbf{l}| = l$ is bounded above by $(l+1)^{d-1} = O(l^{d-1})$. As a consequence, we have the following:

(a) the number of rectangles $J_{\mathbf{k},\mathbf{q}} \subset J_{\mathbf{r},\mathbf{i}}$ with fixed circumference k is bounded by $(r+1)^d (k+1)^{d-1} = \mathcal{O}(r^d k^{d-1})$,
(b) the number of rectangles $J_{\mathbf{k}',\mathbf{q}'} \subset J_{\mathbf{r},\mathbf{i}}$ is, then, bounded by $(r+1)^d \sum_{k=1}^{r} (k+1)^{d-1} = \mathcal{O}(r^{2d})$, and
(c) the number of rectangles in $\mathcal{G}^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}$ is bounded by $2d(r+1)^{d-1} \sum_{k=1}^{r} (k+1)^{d-1} = \mathcal{O}(r^{2d-1})$.[30]

Remark 4.3. Our block-diagonalization procedure relies on the following crucial property: If $V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{l},\mathbf{i}}}$ is block-diagonal with respect to the decomposition of the identity into

$$\mathbb{1} = P^{(+)}_{J_{\mathbf{l},\mathbf{i}}} + P^{(-)}_{J_{\mathbf{l},\mathbf{i}}},$$

i.e., if

$$V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{l},\mathbf{i}}} = P^{(+)}_{J_{\mathbf{l},\mathbf{i}}} V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{l},\mathbf{i}}} P^{(+)}_{J_{\mathbf{l},\mathbf{i}}} + P^{(-)}_{J_{\mathbf{l},\mathbf{i}}} V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{l},\mathbf{i}}} P^{(-)}_{J_{\mathbf{l},\mathbf{i}}},$$

then we have that

$$P^{(+)}_{J_{\mathbf{l}',\mathbf{i}'}} \Big[P^{(+)}_{J_{\mathbf{l},\mathbf{i}}} V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{l},\mathbf{i}}} P^{(+)}_{J_{\mathbf{l},\mathbf{i}}} + P^{(-)}_{J_{\mathbf{l},\mathbf{i}}} V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{l},\mathbf{i}}} P^{(-)}_{J_{\mathbf{l},\mathbf{i}}} \Big] P^{(-)}_{J_{\mathbf{l}',\mathbf{i}'}} = 0$$

for $J_{\mathbf{l}',\mathbf{i}'}$ with $J_{\mathbf{l},\mathbf{i}} \subset J_{\mathbf{l}',\mathbf{i}'}$. This is seen by using

$$P^{(+)}_{J_{\mathbf{l},\mathbf{i}}} P^{(-)}_{J_{\mathbf{l}',\mathbf{i}'}} = 0 \tag{4.64}$$

in the first term and

$$P^{(-)}_{J_{\mathbf{l},\mathbf{i}}} V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{l},\mathbf{i}}} P^{(-)}_{J_{\mathbf{l},\mathbf{i}}} P^{(-)}_{J_{\mathbf{l}',\mathbf{i}'}} = P^{(-)}_{J_{\mathbf{l}',\mathbf{i}'}} P^{(-)}_{J_{\mathbf{l},\mathbf{i}}} V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{l},\mathbf{i}}} P^{(-)}_{J_{\mathbf{l},\mathbf{i}}} P^{(-)}_{J_{\mathbf{l}',\mathbf{i}'}} \tag{4.65}$$

combined with

$$P^{(+)}_{J_{\mathbf{l}',\mathbf{i}'}} P^{(-)}_{J_{\mathbf{l}',\mathbf{i}'}} = 0 \tag{4.66}$$

in the second term.

Lemma 4.4. Assuming (4.60), the following bound on the operator $G_{J_{\mathbf{k},\mathbf{q}}}$ holds:

$$\begin{aligned} P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} G_{J_{\mathbf{k},\mathbf{q}}} P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} \geq \left(1 - 2\cdot C_d \cdot t \sum_{l=1}^{\infty} (l+1)^{2d} \cdot t^{\frac{l-1}{4}}\right) P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} \\ + P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} \left[t \sum_{J_{\mathbf{k}'_{(1)},\mathbf{q}'} \subset J_{\mathbf{k},\mathbf{q}}} \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'_{(1)},\mathbf{q}'}} \rangle + \cdots + t \sum_{J_{\mathbf{k}'_{(|\mathbf{k}|-1)},\mathbf{q}'} \subset J_{\mathbf{k},\mathbf{q}}} \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'_{(|\mathbf{k}|-1)},\mathbf{q}'}} \rangle \right] P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} \end{aligned} \tag{4.67}$$

for $t \in [0, t_d)$, with t_d independent of $(\mathbf{k}, \mathbf{q})$ and N, where C_d is the d-dependent constant implicit in the estimate of the number of shapes in Remark 4.2.

Proof. We observe that, due to Remark 4.3, for $1 \leq j \leq |\mathbf{k}| - 1$, we can write

$$P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} \sum_{J_{\mathbf{k}'_{(j)},\mathbf{q}'} \subset J_{\mathbf{k},\mathbf{q}}} V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'_{(j)},\mathbf{q}'}} P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} = P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} \sum_{J_{\mathbf{k}'_{(j)},\mathbf{q}'} \subset J_{\mathbf{k},\mathbf{q}}} P^{(+)}_{J_{\mathbf{k}'_{(j)},\mathbf{q}'}} V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'_{(j)},\mathbf{q}'}} P^{(+)}_{J_{\mathbf{k}'_{(j)},\mathbf{q}'}} P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} + P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} \sum_{J_{\mathbf{k}'_{(j)},\mathbf{q}'} \subset J_{\mathbf{k},\mathbf{q}}} \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'_{(j)},\mathbf{q}'}} \rangle P^{(-)}_{J_{\mathbf{k}'_{(j)},\mathbf{q}'}} P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} \tag{4.68}$$

with

$$\langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{l},\mathbf{i}}} \rangle := \left\langle \left(\bigotimes \prod_{\mathbf{j} \in J_{\mathbf{l},\mathbf{i}}} P_{\Omega_{\mathbf{j}}} \right), V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{l},\mathbf{i}}} \left(\bigotimes \prod_{\mathbf{j} \in J_{\mathbf{l},\mathbf{i}}} P_{\Omega_{\mathbf{j}}} \right) \right\rangle.$$

Furthermore, we can estimate

$$\pm P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} \sum_{J_{\mathbf{k}'_{(j)},\mathbf{q}'} \subset J_{\mathbf{k},\mathbf{q}}} P^{(+)}_{J_{\mathbf{k}'_{(j)},\mathbf{q}'}} V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'_{(j)},\mathbf{q}'}} P^{(+)}_{J_{\mathbf{k}'_{(j)},\mathbf{q}'}} P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} \tag{4.69}$$

$$\leq P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} \sum_{J_{\mathbf{k}'_{(j)},\mathbf{q}'} \subset J_{\mathbf{k},\mathbf{q}}} \| V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'_{(j)},\mathbf{q}'}} \| P^{(+)}_{J_{\mathbf{k}'_{(j)},\mathbf{q}'}} P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} \tag{4.70}$$

$$\leq P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} t^{\frac{j-1}{4}} \sum_{J_{\mathbf{k}'_{(j)},\mathbf{q}'} \subset J_{\mathbf{k},\mathbf{q}}} P^{(+)}_{J_{\mathbf{k}'_{(j)},\mathbf{q}'}} P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} \tag{4.71}$$

$$\leq P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} C_d \cdot (j+1)^{2d} \cdot t^{\frac{j-1}{4}} \sum_{\mathbf{i} \in J_{\mathbf{k},\mathbf{q}}} P^{\perp}_{\Omega_{\mathbf{i}}} P^{(+)}_{J_{\mathbf{k},\mathbf{q}}}, \tag{4.72}$$

where we have used the following:

1. the bound in (4.60) for the step from (4.70) to (4.71) and
2. the property in (4.62) combined with Remark 4.2 for the step from (4.71) to (4.72).

Hence, we can combine the inequality [due to (1.4)]

$$\sum_{\mathbf{i} \subset J_{\mathbf{k},\mathbf{q}}} H_{\mathbf{i}} \geq \sum_{\mathbf{i} \in J_{\mathbf{k},\mathbf{q}}} P^{\perp}_{\Omega_{\mathbf{i}}} \tag{4.73}$$

with (4.68)–(4.72), and we get

$$P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} G_{J_{\mathbf{k},\mathbf{q}}} P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} \tag{4.74}$$

$$\geq P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} \left[\left(1 - C_d \cdot t \sum_{l=1}^{\infty} (l+1)^{2d} \cdot t^{\frac{l-1}{4}} \right) \sum_{\mathbf{i} \in J_{\mathbf{k},\mathbf{q}}} P^{\perp}_{\Omega_{\mathbf{i}}} \right] P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} \tag{4.75}$$

$$+ P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} \left[t \sum_{J_{\mathbf{k}'_{(1)},\mathbf{q}'} \subset J_{\mathbf{k},\mathbf{q}}} \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'_{(1)},\mathbf{q}'}} \rangle P^{(-)}_{J_{\mathbf{k}'_{(1)},\mathbf{q}'}} + \cdots + t \sum_{J_{\mathbf{k}'_{(|\mathbf{k}|-1)},\mathbf{q}'} \subset J_{\mathbf{k},\mathbf{q}}} \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'_{(|\mathbf{k}|-1)}},\mathbf{q}'} \rangle P^{(-)}_{J_{\mathbf{k}'_{(|\mathbf{k}|-1)},\mathbf{q}'}} \right] P^{(+)}_{J_{\mathbf{k},\mathbf{q}}}. \tag{4.76}$$

Next, we use the identity

$$\mathbb{1} = P^{(-)}_{J_{\mathbf{k}'_{(j)},\mathbf{q}'}} + P^{(+)}_{J_{\mathbf{k}'_{(j)},\mathbf{q}'}}$$

in the right-hand side of (4.75), and we get

$$P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} G_{J_{\mathbf{k},\mathbf{q}}} P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} \tag{4.77}$$

$$\geq P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} \left[\left(1 - C_d \cdot t \sum_{l=1}^{\infty} (l+1)^{2d} \cdot t^{\frac{l-1}{4}} \right) \sum_{\mathbf{i} \in J_{\mathbf{k},\mathbf{q}}} P^{\perp}_{\Omega_{\mathbf{i}}} \right] P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} \tag{4.78}$$

$$- P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} \left[t \sum_{J_{\mathbf{k}'_{(1)},\mathbf{q}'} \subset J_{\mathbf{k},\mathbf{q}}} \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'_{(1)},\mathbf{q}'}} \rangle P^{(+)}_{J_{\mathbf{k}'_{(1)},\mathbf{q}'}} + \cdots + t \sum_{J_{\mathbf{k}'_{(|\mathbf{k}|-1)},\mathbf{q}'} \subset J_{\mathbf{k},\mathbf{q}}} \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'_{(|\mathbf{k}|-1)}},\mathbf{q}'} \rangle P^{(+)}_{J_{\mathbf{k}'_{(|\mathbf{k}|-1)},\mathbf{q}'}} \right] P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} \tag{4.79}$$

$$+ P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} \left[t \sum_{J_{\mathbf{k}'_{(1)},\mathbf{q}'} \subset J_{\mathbf{k},\mathbf{q}}} \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'_{(1)},\mathbf{q}'}} \rangle + \cdots + t \sum_{J_{\mathbf{k}'_{(|\mathbf{k}|-1)},\mathbf{q}'} \subset J_{\mathbf{k},\mathbf{q}}} \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'_{(|\mathbf{k}|-1)}},\mathbf{q}'} \rangle \right] P^{(+)}_{J_{\mathbf{k},\mathbf{q}}}. \tag{4.80}$$

By invoking the obvious bound

$$|\langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'_{(j)},\mathbf{q}'}} \rangle| \leq \| V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'_{(j)}},\mathbf{q}'} \|,$$

we finally get

$$P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} G_{J_{\mathbf{k},\mathbf{q}}} P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} \tag{4.81}$$

$$\geq P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} \left[\left(1 - 2 \cdot C_d \cdot t \sum_{l=1}^{\infty} (l+1)^{2d} \cdot t^{\frac{l-1}{4}} \right) \sum_{\mathbf{i} \in J_{\mathbf{k},\mathbf{q}}} P^{\perp}_{\Omega_{\mathbf{i}}} \right] P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} \tag{4.82}$$

$$+ P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} \left[t \sum_{J_{\mathbf{k}'_{(1)},\mathbf{q}'} \subset J_{\mathbf{k},\mathbf{q}}} \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'_{(1)},\mathbf{q}'}} \rangle + \cdots + t \sum_{J_{\mathbf{k}'_{(|\mathbf{k}|-1)},\mathbf{q}'} \subset J_{\mathbf{k},\mathbf{q}}} \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'_{(|\mathbf{k}|-1)}},\mathbf{q}'} \rangle \right] P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} \tag{4.83}$$

$$\geq \left(1 - 2 \cdot C_d \cdot t \sum_{l=1}^{\infty} (l+1)^{2d} \cdot t^{\frac{l-1}{4}} \right) P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} \tag{4.84}$$

$$+ P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} \left[t \sum_{J_{\mathbf{k}'_{(1)},\mathbf{q}'} \subset J_{\mathbf{k},\mathbf{q}}} \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'_{(1)},\mathbf{q}'}} \rangle + \cdots + t \sum_{J_{\mathbf{k}'_{(|\mathbf{k}|-1)},\mathbf{q}'} \subset J_{\mathbf{k},\mathbf{q}}} \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'_{(|\mathbf{k}|-1)}},\mathbf{q}'} \rangle \right] P^{(+)}_{J_{\mathbf{k},\mathbf{q}}}, \tag{4.85}$$

where Lemma A.1 is used for the last inequality and $t(>0)$ is assumed small enough such that

$$1 - 2 \cdot C_d \cdot t \sum_{l=1}^{\infty} (l+1)^{2d} \cdot t^{\frac{l-1}{4}} > 0. \tag{4.86}$$

□

Lemma 4.4 implies that under the assumption in (4.60), the Hamiltonian $G_{J_{\mathbf{k},\mathbf{q}}}$ has a gap that can be estimated from below by $\frac{1}{2}$ for $t > 0$ sufficiently small but independent of N and $(\mathbf{k},\mathbf{q})$, as stated in the corollary below.

Corollary 4.5. Assuming Lemma 4.4 for $t > 0$ sufficiently small, dependent on d but independent of N and $(\mathbf{k},\mathbf{q})$, the Hamiltonian $G_{J_{\mathbf{k},\mathbf{q}}}$ has a gap $\Delta_{J_{\mathbf{k},\mathbf{q}}} \geq \frac{1}{2}$ above the ground state energy

$$E_{J_{\mathbf{k},\mathbf{q}}} = t \sum_{J_{\mathbf{k}'_{(1)},\mathbf{q}'} \subset J_{\mathbf{k},\mathbf{q}}} \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'_{(1)},\mathbf{q}'}} \rangle + \cdots + t \sum_{J_{\mathbf{k}'_{(j_{\mathbf{k}}-1)},\mathbf{q}'} \subset J_{\mathbf{k},\mathbf{q}}} \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'_{(j_{\mathbf{k}}-1)},\mathbf{q}'}} \rangle$$

corresponding to the ground state vector $\otimes_{\mathbf{i}\in J_{\mathbf{k},\mathbf{q}}} \Omega_{\mathbf{i}}$ due to the identity

$$P^{(-)}_{J_{\mathbf{k},\mathbf{q}}} G_{J_{\mathbf{k},\mathbf{q}}} P^{(-)}_{J_{\mathbf{k},\mathbf{q}}} \tag{4.87}$$

$$= P^{(-)}_{J_{\mathbf{k},\mathbf{q}}} \left[t \sum_{J_{\mathbf{k}'_{(1)},\mathbf{q}'} \subset J_{\mathbf{k},\mathbf{q}}} \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'_{(1)},\mathbf{q}'}} \rangle P^{(-)}_{J_{\mathbf{k}'_{(1)},\mathbf{q}'}} + \cdots + t \sum_{J_{\mathbf{k}'_{(|\mathbf{k}|-1)},\mathbf{q}'} \subset J_{\mathbf{k},\mathbf{q}}} \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'_{(|\mathbf{k}|-1)},\mathbf{q}'}} \rangle P^{(-)}_{J_{\mathbf{k}'_{(|\mathbf{k}|-1)},\mathbf{q}'}} \right] P^{(-)}_{J_{\mathbf{k},\mathbf{q}}}$$

$$= P^{(-)}_{J_{\mathbf{k},\mathbf{q}}} \left[t \sum_{J_{\mathbf{k}'_{(1)},\mathbf{q}'} \subset J_{\mathbf{k},\mathbf{q}}} \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'_{(1)},\mathbf{q}'}} \rangle + \cdots + t \sum_{J_{\mathbf{k}'_{(|\mathbf{k}|-1)},\mathbf{q}'} \subset J_{\mathbf{k},\mathbf{q}}} \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'_{(|\mathbf{k}|-1)},\mathbf{q}'}} \rangle \right] P^{(-)}_{J_{\mathbf{k},\mathbf{q}}}.$$

Remark 4.6. Estimates (4.73) and (4.69)–(4.72) show that, after implementing our block-diagonalization procedure and subtracting the ground-state energy, the interaction terms of the *transformed* Hamiltonian are form-bounded by the unperturbed Hamiltonian *uniformly* in the size of the region $J_{\mathbf{k},\mathbf{q}}$, provided that the coupling constant $t > 0$ is small enough. Spectral calculus, then, implies that, for z outside the spectrum of the unperturbed Hamiltonian, $\sum_{\mathbf{i}\subset J_{\mathbf{k},\mathbf{q}}} H_{\mathbf{i}}$, i.e., for $\delta := \text{dist}(z\,;\,\sigma(\sum_{\mathbf{i}\subset J_{\mathbf{k},\mathbf{q}}} H_{\mathbf{i}})\,) > 0$, the resolvent

$$\frac{1}{P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} (G_{J_{\mathbf{k},\mathbf{q}}} - E_{J_{\mathbf{k},\mathbf{q}}} - z) P^{(+)}_{J_{\mathbf{k},\mathbf{q}}}} \tag{4.88}$$

is well defined on $P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} \mathcal{H}^{(N)}$, provided that $0 < t < t_{\delta,z}$, where the constant $t_{\delta,z}$ only depends on δ and z but is *independent* of $J_{\mathbf{k},\mathbf{q}}$.

V. CONTROL OF $\|V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}\|$

The next theorem is the key result of this paper and is based on a lengthy analysis of the different regimes (outlined in Sec. II C) to control the potentials yielded, step by step, by the algorithm in Definition 2.4.

Theorem 5.1. *There exists $t_d > 0$ such that for $0 \leq t < t_d$, the Hamiltonians $G_{J_{\mathbf{k},\mathbf{q}}}$ and $K^{(\mathbf{k},\mathbf{q})}_{N^d}$ are well defined, and for any rectangle $J_{\mathbf{r},\mathbf{i}}$, with $r = |\mathbf{r}| \geq 1$, and for $x_d := 20d$, we have the following:*

(S1) *Let $(\mathbf{k},\mathbf{q})_* := (\mathbf{k}_*,\mathbf{q}_*)$ be defined for some $(\mathbf{k}_*,\mathbf{q}_*)$ such that $|\mathbf{k}_*| = \lfloor r^{\frac{1}{4}} \rfloor$, where $\lfloor \cdot \rfloor$ is the integer part. If $(\mathbf{k},\mathbf{q}) < (\mathbf{k},\mathbf{q})_*$, then*

$$\| V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}} \| \leq \frac{t^{\frac{r-1}{3}}}{r^{x_d+2d}}. \tag{5.89}$$

Let $(\mathbf{k},\mathbf{q})_{**} := (\mathbf{k}_{**},\mathbf{q}_{**})$ be defined for some $(\mathbf{k}_{**},\mathbf{q}_{**})$ such that $|\mathbf{k}_{**}| = r - \lfloor r^{\frac{1}{4}} \rfloor$. If $(\mathbf{k},\mathbf{q})_{**} > (\mathbf{k},\mathbf{q}) \geq (\mathbf{k},\mathbf{q})_*$, then

$$\| V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}} \| \leq 2 \cdot \frac{t^{\frac{r-1}{3}}}{r^{x_d+2d}}. \tag{5.90}$$

If $(\mathbf{r},\mathbf{i}) \succ (\mathbf{k},\mathbf{q}) \geq (\mathbf{k},\mathbf{q})_{**}$, then

$$\left\| \frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} P^{(\#)}_{J_{\mathbf{r},\mathbf{i}}} V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}} P^{(\hat{\#})}_{J_{\mathbf{r},\mathbf{i}}} \frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} \right\| \leq 3 \cdot \frac{t^{\frac{r-1}{3}}}{r^{x_d+2d}}, \quad \#, \hat{\#} = \pm \tag{5.91}$$

and

$$\| V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}} \| \leq 48 \cdot \frac{t^{\frac{r-1}{3}}}{r^{x_d}}. \tag{5.92}$$

If $(\mathbf{k},\mathbf{q}) \geq (\mathbf{r},\mathbf{i})$, *then*

$$\| V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}} \| \leq 96 \cdot \frac{t^{\frac{r-1}{3}}}{r^{x_d}}. \tag{5.93}$$

(S2) $G_{J_{(\mathbf{k},\mathbf{q})_{+1}}}$ *has spectral gap* $\Delta_{J_{(\mathbf{k},\mathbf{q})_{+1}}} \geq \frac{1}{2}$ *above its ground state energy, where* $G_{J_{\mathbf{k},\mathbf{q}}}$ *is defined in (2.26) for* $|\mathbf{k}| \geq 2$, *and*

$$G_{J_{(\mathbf{1}_j,\mathbf{q})_{+1}}} := H^{(0)}_{J_{(\mathbf{1}_j,\mathbf{q})_{+1}}} := \sum_{\mathbf{i} \in J_{(\mathbf{1}_j,\mathbf{q})_{+1}}} H_{\mathbf{i}},$$

provided that $(\mathbf{1}_j,\mathbf{q})_{+1}$ *is of the form* $(\mathbf{1}_{j'},\mathbf{q}')$ *for some* j' *and* $\mathbf{q}'$; $(\mathbf{1}_j,\mathbf{q})$ *is defined in (1.13).*

Proof. The proof is by induction in the diagonalization step $(\mathbf{k},\mathbf{q})$. Hence, for each $(\mathbf{r},\mathbf{i})$, we shall prove (S1) and (S2) from $(\mathbf{k},\mathbf{q}) = (\mathbf{0},\mathbf{N})$ up to $(\mathbf{k},\mathbf{q}) = (\mathbf{N}-\mathbf{1},\mathbf{1})$ (note that in step $(\mathbf{k},\mathbf{q})$, (S2) concerns the Hamiltonian $G_{J_{(\mathbf{k},\mathbf{q})_{+1}}}$, and it is not defined for $(\mathbf{k},\mathbf{q}) = (\mathbf{N}-\mathbf{1},\mathbf{1})$). That is, we assume that (S1) holds for all $V^{(\mathbf{k}',\mathbf{q}')}_{J_{\mathbf{r},\mathbf{i}}}$ with $(\mathbf{k}',\mathbf{q}') < (\mathbf{k},\mathbf{q})$ and (S2) for all $(\mathbf{k}',\mathbf{q}') < (\mathbf{k},\mathbf{q})$. Then, we show that they hold for all $V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}$ and for $G_{J_{(\mathbf{k},\mathbf{q})_{+1}}}$. By Lemma A.3, this implies that $S_{J_{\mathbf{k},\mathbf{q}}}$ and, consequently, that $K^{(\mathbf{k},\mathbf{q})}_{\Lambda^d_N}$ are well defined operators [see (4.54)].

For $(\mathbf{k},\mathbf{q}) = (\mathbf{0},\mathbf{N})$, (S1) can be verified by direct computation because

$$\| V^{(\mathbf{0},\mathbf{N})}_{J_{\mathbf{1}_j,\mathbf{q}}} \| = \| V_{J_{\mathbf{1}_j,\mathbf{q}}} \| \leq 1,$$

and $V^{(\mathbf{0},\mathbf{N})}_{J_{\mathbf{r},\mathbf{i}}} = 0$ otherwise; (S2) holds trivially since, by definition, $(\mathbf{0},\mathbf{N})_{+1} = (\mathbf{1}_1,\mathbf{1})$ and $G_{J_{\mathbf{1}_1,\mathbf{1}}} = H^{(0)}_{J_{\mathbf{1}_1,\mathbf{1}}}$ [recall that $\mathbf{1}_j$ is defined in (1.13)].

At each stage of our proof, we choose $t(\geq 0)$ in an interval such that the previous stages and Lemma A.3 are verified. Hence, by this procedure, we may progressively restrict such an interval until we determine $t_d > 0$ for which all the stages hold true for $0 \leq t < t_d$.

Warning: Throughout the proof, several positive constants are introduced. We shall use the symbols c, C for those that are universal and the symbols c_d, C_d for those that depend on the dimension d, and their value may change from line to line.

Induction step in the proof of (S1). Starting from Definition 2.4, we consider the following cases:

Case $r = 1$.

Let $k > 1(= r)$ or $k = 1(= r)$ but $J_{\mathbf{r},\mathbf{i}}$ such that $\mathbf{i} \neq \mathbf{q}$. Then, the possible cases are described in (a); see Definition 2.4, and we have that

$$\| V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}} \| = \| V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{r},\mathbf{i}}} \|. \tag{5.94}$$

Let $k = 1$, and assume that $J_{\mathbf{r},\mathbf{i}}$ is equal to $J_{\mathbf{k},\mathbf{q}}$. Then, we refer to case (b) and find that

$$\| V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{k},\mathbf{q}}} \| \leq 2 \| V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}} \| \leq 2, \tag{5.95}$$

where we have following:

(i) the inequality $\| V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{k},\mathbf{q}}} \| \leq 2 \| V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}} \|$ holds for $t(\geq 0)$ sufficiently small uniformly in $\mathbf{q}$ and N, thanks to Lemma A.3, which can be applied since we assume (S1) and (S2) in step $(\mathbf{k},\mathbf{q})_{-1}$;

(ii) we use $\| V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}} \| = \| V^{(\mathbf{k},\mathbf{q})_{-2}}_{J_{\mathbf{k},\mathbf{q}}} \| = \cdots = \| V^{(\mathbf{0},\mathbf{N})}_{J_{\mathbf{k},\mathbf{q}}} \| \leq 1$.

Inequality (5.91) follows trivially by using $\| \frac{1}{\sum_{\mathbf{j} \in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} \| \leq 1$ and $\| P^{(\#)}_{J_{\mathbf{r},\mathbf{i}}} V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}} P^{(\#)}_{J_{\mathbf{r},\mathbf{i}}} \| \leq \| V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}} \|$.

Case $r = 2$.

This case is not much different from the one corresponding to $r = 1$ with the exception that

$$\left\| V^{(\mathbf{k}',\mathbf{q}')}_{J_{\mathbf{r},\mathbf{i}}} \right\| \leq \left\| V^{(\mathbf{k}',\mathbf{q}')_{-1}}_{J_{\mathbf{r},\mathbf{i}}} \right\| + \left\| \sum_{J_{\mathbf{k}'',\mathbf{q}''} \in \mathcal{G}^{(\mathbf{k}',\mathbf{q}')}_{J_{\mathbf{r},\mathbf{i}}}} \sum_{n=1}^{\infty} \frac{1}{n!} ad^n S_{J_{\mathbf{k}',\mathbf{q}'}} (V^{(\mathbf{k}',\mathbf{q}')_{-1}}_{J_{\mathbf{k}'',\mathbf{q}''}}) \right\| \tag{5.96}$$

also must be used in the re-expansion, for some $(\mathbf{k}',\mathbf{q}')$ with $k'=1$, and then iterated for the first term of the right-hand side of (5.96) if the conditions of (c) in Definition 2.4 are fulfilled. The second term in (5.96) is a reminder that, however, is produced along the re-expansion only for a finite number of steps, and this number is bounded by a constant independent of $(\mathbf{k},\mathbf{q})$, $\mathbf{i}$, and N. Note also that, for $t>0$ sufficiently small, the norm of the last term in (5.96) can be bounded by a constant multiplied by a factor t using Lemma A.3 and inductive hypotheses (S1) and (S2) for $r=1$. For $t(\geq 0)$ sufficiently small, these observations suffice to state (S1) for rectangles with $r=2$, provided that (S1) and (S2) hold for $r=1$.

Case $r>2$.

As explained in Sec. II C, in order to control the norm $\|V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})}\|$, we distinguish three regimes depending on the relative magnitude between $k=|\mathbf{k}|$ and $r=|\mathbf{r}|$. They are associated with (5.89)–(5.93), respectively. For the convenience of the reader, we recall how the inductive hypotheses are used in the following analysis of the three regimes. By assuming that (5.89)–(5.93) are true for the potentials associated with any rectangle $J_{\mathbf{l}',\mathbf{i}'}$, with $(\mathbf{l}',\mathbf{i}')\leq(\mathbf{r},\mathbf{i})$, in steps $(\mathbf{k}',\mathbf{q}')\prec(\mathbf{k},\mathbf{q})$, we prove that, depending on the considered regime, (5.89)–(5.91) hold, respectively, in step $(\mathbf{k},\mathbf{q})$ for the potential associated with $J_{\mathbf{r},\mathbf{i}}$, but if (5.91) is verified, then, consequently, (5.92) and (5.93) also hold true [in step $(\mathbf{k},\mathbf{q})$].

A. *Regime* $(\mathfrak{R}1)$

Here, we apply the argument explained in Sec. III B in order to show that (S1) holds for $V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})}$ with $(\mathbf{k},\mathbf{q})$ belonging to the first regime, provided that (S1) and (S2) hold for all potentials in step $(\mathbf{k}',\mathbf{q}')\prec(\mathbf{k},\mathbf{q})$. Given the assumption, we can exploit (A7) in Lemma A.3 so as to conclude that for any (bounded) operator V,

$$\|\mathcal{A}_{J_{\mathbf{k}'',\mathbf{q}''}}(V)\|\leq c\cdot t\cdot\|V_{J_{\mathbf{k}'',\mathbf{q}''}}^{(\mathbf{k}'',\mathbf{q}'')_{-1}}\|\cdot\|V\| \tag{5.97}$$

if $(\mathbf{k}'',\mathbf{q}'')\leq(\mathbf{k},\mathbf{q})$, where c is a universal constant. We recall that, as explained in Sec. III, the strategy is to re-expand the potential $V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})}$ according to the prescriptions of Definition 3.1. Consequently, the potential can be expressed as the sum $\sum_{\mathfrak{b}\in\mathcal{B}_{V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})}}}\mathfrak{b}$, where $\mathfrak{b}$ are the branch operators defined in point 8 of Definition 3.1. Due to property (P-v) in Sec. III A 1, the number of summands coincides with the number of sets $\mathcal{R}_{\mathfrak{b}}$ that are associated with $V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})}$. Furthermore, in order to estimate the norm of the sum of the operators resulting from the re-expansion, it is enough to use (5.97) repeatedly, i.e.,

$$\|\mathcal{A}_{J_{\mathbf{k}^{(1)},\mathbf{q}^{(1)}}}(\mathcal{A}_{J_{\mathbf{k}^{(2)},\mathbf{q}^{(2)}}}(\cdots\mathcal{A}_{J_{\mathbf{k}^{(|\mathcal{R}_{\mathfrak{b}}|-1)},\mathbf{q}^{(|\mathcal{R}_{\mathfrak{b}}|-1)}}}(V_{\mathcal{L}_{\mathfrak{b}}})\cdots))\|\leq(c\cdot t)^{|\mathcal{R}_{\mathfrak{b}}|-1}\|V_{\mathcal{L}_{\mathfrak{b}}}\|\prod_{i=1}^{|\mathcal{R}_{\mathfrak{b}}|-1}\|V_{J_{\mathbf{k}^{(i)},\mathbf{q}^{(i)}}}^{(\mathbf{k}^{(i)},\mathbf{q}^{(i)})_{-1}}\|, \tag{5.98}$$

where $V_{\mathcal{L}_{\mathfrak{b}}}$ is the potential labeling the leaf of $\mathfrak{b}$, and compute the "weighted" number of sets $\{J_{\mathbf{k}^{(i)},\mathbf{q}^{(i)}},\ i\in\{1,\ldots,|\mathcal{R}_{\mathfrak{b}}|\}\}$, weighted in the sense that each rectangle $J_{\mathbf{k}^{(i)},\mathbf{q}^{(i)}}$ is given the weight $c\cdot t\cdot\|V_{J_{\mathbf{k}^{(i)},\mathbf{q}^{(i)}}}^{(\mathbf{k}^{(i)},\mathbf{q}^{(i)})_{-1}}\|$ except for the one associated with the leaf of the branch, that is, given the weight $\|V_{\mathcal{L}_{\mathfrak{b}}}\|$.

Following the scheme described in Sec. III B, we estimate the weighted sum of sets $\mathcal{R}_{\mathfrak{b}}$ in terms of a weighted sum of paths $\Gamma_{\mathfrak{b}}$. Different from Sec. III B 3, here, we assign the weight to each step after extracting from (5.98) what is needed to provide the bound in (5.89). The overall control will be ensured by the pre-factor $(c\cdot t)^{|\mathcal{R}_{\mathfrak{b}}|-1}$ that is small enough due to the upper bound on k, $k\leq\lfloor r^{\frac{1}{4}}\rfloor$, in regime $\mathfrak{R}1$. Indeed, the latter implies the lower bound $|\mathcal{R}_{\mathfrak{b}}|\geq\lfloor c_d\cdot r/k\rfloor$.

In detail, concerning the powers of t, note that from the product

$$(c\cdot t)^{|\mathcal{R}_{\mathfrak{b}}|-1}\|V_{\mathcal{L}_{\mathfrak{b}}}\|\prod_{i=1}^{|\mathcal{R}_{\mathfrak{b}}|-1}\|V_{J_{\mathbf{k}^{(i)},\mathbf{q}^{(i)}}}^{(\mathbf{k}^{(i)},\mathbf{q}^{(i)})_{-1}}\|, \tag{5.99}$$

we get at least $t^{\frac{r-1}{3}}$ due to (1) the requirement that $J_{\mathbf{r},\mathbf{i}}$ is the minimal rectangle associated with $\cup_{i\in\{1,\ldots,|\mathcal{R}_{\mathfrak{b}}|\}}J_{\mathbf{k}^{(i)},\mathbf{q}^{(i)}}$ and (2) borrowing a power $t^{\frac{2}{3}}$ from each factor t in $(c\cdot t)^{|\mathcal{R}_{\mathfrak{b}}|-1}$. Hence, in product (5.99), we can factor out $t^{\frac{r-1}{3}}$ and keep a power $t^{\frac{1}{3}}$ for each rectangle of $\mathcal{R}_{\mathfrak{b}}$ except the one associated with the leaf of the branch. This also means that we can assign at least a factor

$$(c+1)\frac{t^{1/6}}{\rho^{x_d}} \tag{5.100}$$

to each rectangle of size ρ in $\mathcal{R}_{\mathfrak{b}}$.

Consider the rectangles of the set $\mathrm{supp}(\mathcal{Z}_{\rho}^{(j)})$ (see Sec. III B 2): there are $n_{\rho}^{(j)}$ such rectangles, and for the constructed paths $\Gamma_{\mathfrak{b}}$, there are at most $2n_{\rho}^{(j)}-2$ steps between them. In addition, there are at most two steps, from rectangles of lower size and back, to be taken into account. To each step $\mathcal{S}_{\Gamma_{\mathfrak{b}}} \ni \sigma = (J_{\mathbf{s}^{(i)},\mathbf{u}^{(i)}}, J_{\mathbf{s}^{(i+1)},\mathbf{u}^{(i+1)}})$ we assign the weight

$$w_\sigma := \left((c+1)\frac{t^{1/6}}{s_\sigma^{x_d}}\right)^{1/2}$$

with $s_\sigma := \max\{s^{(i)}, s^{(i+1)}\}$, with $w_\sigma < 1$ for $t > 0$ sufficiently small.

From the considerations regarding (5.99) and (5.100), we get the first inequality in the following formula:

$$(c\cdot t)^{|\mathcal{R}_{\mathfrak{b}}|-1}\|V_{\mathcal{L}_{\mathfrak{b}}}\|\prod_{i=1}^{|\mathcal{R}_{\mathfrak{b}}|-1}\|V_{J_{\mathbf{k}^{(i)},\mathbf{q}^{(i)}}}^{(\mathbf{k}^{(i)},\mathbf{q}^{(i)})_{-1}}\| \le t^{\frac{r-1}{3}}\prod_{\rho=1\,;j_\rho\neq 0}^{k}\prod_{j=1}^{j_\rho}\left((c+1)\frac{t^{1/6}}{\rho^{x_d}}\right)^{n_\rho^{(j)}} \le t^{\frac{r-1}{3}}\cdot\prod_{\sigma\in\mathcal{S}_{\Gamma_{\mathfrak{b}}}} w_\sigma, \tag{5.101}$$

whereas for the second inequality, we use the following observation: if we denote by $\mathcal{S}_{\mathcal{Z}_\rho^{(j)}}$ the set consisting of at most $2n_\rho^{(j)}-2$ steps between rectangles of $\mathrm{supp}\mathcal{Z}_\rho^{(j)}$ and the additional at most two steps from rectangles of lower size and back, then we have

$$\left((c+1)\frac{t^{\frac{1}{6}}}{\rho^{x_d}}\right)^{n_\rho^{(j)}} \le \prod_{\sigma\in\mathcal{S}_{\mathcal{Z}_\rho^{(j)}}} w_\sigma$$

since w_σ, $\sigma\in\mathcal{S}_{\mathcal{Z}_\rho^{(j)}}$, coincides with $\left((c+1)\frac{t^{\frac{1}{6}}}{\rho^{x_d}}\right)^{\frac{1}{2}} < 1$ and $|\mathcal{S}_{\mathcal{Z}_\rho^{(j)}}| \le 2n_\rho^{(j)}$, by construction.

Hence, the total weighted number of rectangles

$$\sum_{\mathfrak{b}\in\mathcal{B}_{V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})}}}(c\cdot t)^{|\mathcal{R}_{\mathfrak{b}}|-1}\|V_{\mathcal{L}_{\mathfrak{b}}}\|\prod_{i=1}^{|\mathcal{R}_{\mathfrak{b}}|-1}\|V_{J_{\mathbf{k}^{(i)},\mathbf{q}^{(i)}}}^{(\mathbf{k}^{(i)},\mathbf{q}^{(i)})_{-1}}\| \le \sum_{\Gamma_{\mathfrak{b}},\,\mathfrak{b}\in\mathcal{B}_{V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})}}} t^{\frac{r-1}{3}}\cdot\prod_{\sigma\in\mathcal{S}_{\Gamma_{\mathfrak{b}}}} w_\sigma \tag{5.102}$$

can be bounded from above by estimating the number of weighted paths $\Gamma_{\mathfrak{b}}$ as follows:

$$\sum_{\Gamma_{\mathfrak{b}},\,\mathfrak{b}\in\mathcal{B}_{V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})}}} t^{\frac{r-1}{3}}\cdot\prod_{\sigma\in\mathcal{S}_{\Gamma_{\mathfrak{b}}}} w_\sigma \tag{5.103}$$

$$\le C_d\cdot r^{2d-1}\cdot t^{\frac{r-1}{3}}\cdot\sum_{j=\lfloor c_d\cdot r/k\rfloor}^{\infty}\left(\sum_{\rho,\rho'=1}^{k}\left((c+1)\frac{t^{1/6}}{(\max\{\rho,\rho'\})^{x_d}}\right)^{1/2} D_{\rho,\rho'}\right)^j, \tag{5.104}$$

where we have the following:

- $$\sum_{\rho'=1}^{k}\left((c+1)\frac{t^{1/6}}{(\max\{\rho,\rho'\})^{x_d}}\right)^{1/2} D_{\rho,\rho'}$$
  accounts for all the weighted directions for a step from a rectangle of size ρ, where $D_{\rho,\rho'}$ is defined in (3.45); note that the weight for the number of directions is due to the restriction of the class of paths used in the argument that culminates in (5.101);
- the term $C_d\cdot r^{2d-1}$ is a bound[31] on the number of possible initial rectangles of a fixed path $\Gamma_{\mathfrak{b}}$;
- the sum over j is the sum over the number of steps of $\Gamma_{\mathfrak{b}}$, which by construction is bounded from below by $\lfloor c_d\cdot r/k\rfloor$.

Next, we bound

$$(5.104) \le C_d\cdot r^{2d-1}\cdot t^{\frac{r-1}{3}}\cdot\sum_{j=\lfloor c_d\cdot r/k\rfloor}^{\infty}\left((c+1)^{1/2}\cdot t^{\frac{1}{12}}\cdot 2\sum_{\rho=1}^{k}\frac{\rho\cdot D_{\rho,\rho}}{\rho^{x_d/2}}\right)^j \tag{5.105}$$

$$\le C_d\cdot r^{2d-1}\cdot t^{\frac{r-1}{3}}\cdot t^{\frac{1}{24}\cdot\lfloor c_d\cdot r/k\rfloor}\cdot\sum_{j=\lfloor c_d\cdot r/k\rfloor}^{\infty}\left((c+1)\cdot t^{\frac{1}{24}}\cdot 2\sum_{\rho=1}^{k}\frac{\rho\cdot D_{\rho,\rho}}{\rho^{x_d/2}}\right)^j$$

$$\le \frac{t^{\frac{r-1}{3}}}{r^{x_d+2d}}, \tag{5.106}$$

where $t \geq 0$ has been chosen small enough such that (recall $k \leq \lfloor r^{\frac{1}{4}} \rfloor$)

$$C_d \cdot r^{4d-1+x_d} \cdot t^{\frac{1}{24} \cdot \lfloor c_d \cdot r/k \rfloor} \cdot \sum_{j=\lfloor c_d \cdot r/k \rfloor}^{\infty} \left((c+1)^{1/2} \cdot t^{\frac{1}{24}} \cdot 2 \sum_{\rho=1}^{k} \frac{\rho \cdot D_{\rho,\rho}}{\rho^{x_d/2}} \right)^j < 1. \tag{5.107}$$

B. *Regime* ($\mathfrak{R}2$)

For $(\mathbf{k}, \mathbf{q})$ in this regime, starting from the inequality

$$\| V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})} \| \leq \| V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})_{-1}} \| + \| \sum_{J_{\mathbf{k}',\mathbf{q}'} \in \mathcal{G}_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})}} \sum_{n=1}^{\infty} \frac{1}{n!} ad^n S_{J_{\mathbf{k},\mathbf{q}}} (V_{J_{\mathbf{k}',\mathbf{q}'}}^{(\mathbf{k},\mathbf{q})_{-1}}) \|, \tag{5.108}$$

we only keep expanding the first potential on the right-hand side. Then, using inductive hypotheses (5.89), (5.90), (5.92), and (5.93), for $t \geq 0$ sufficiently small, we can estimate

$$\| V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})} \| \leq \| V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k}_*,\mathbf{q}_*)} \| \tag{5.109}$$

$$+ \sum_{s=\lfloor r^{\frac{1}{4}} \rfloor}^{r-\lfloor r^{\frac{1}{4}} \rfloor} \sum_{s_1=0}^{s} \sum_{s_2=0}^{s-s_1} \cdots \sum_{s_d=0}^{s-s_1-\cdots-s_{d-1}} \delta_{s_1+s_2+\cdots+s_d-s} \cdot c_d \cdot r^{2d} \cdot t \cdot \frac{t^{\frac{s-1}{3}}}{s^{x_d}} \cdot \frac{t^{\frac{r-s-1}{3}}}{(r-s)^{x_d}}, \tag{5.110}$$

where we have the following:

- $(\mathbf{k}_*, \mathbf{q}_*)$ is the greatest rectangle of regime $\mathfrak{R}1$ with respect to the ordering $\succ$, and by construction, $k_* = \lfloor r^{\frac{1}{4}} \rfloor$;
- the factor

$$c_d \cdot r^{2d-1} \cdot t \cdot \frac{t^{\frac{s-1}{3}}}{s^{x_d}} \cdot \frac{t^{\frac{r-s-1}{3}}}{(r-s)^{x_d}}$$

  is an upper bound to the sum of the products of the type $\| \mathcal{A}_{J_{(\mathbf{k},\mathbf{q})_{-j}}} (V_{J_{\mathbf{k}',\mathbf{q}'}}^{(\mathbf{k},\mathbf{q})_{-j-1}}) \|$ for some j, where the size of the rectangle associated with $(\mathbf{k}, \mathbf{q})_{-j}$ is equal to s;
- the multiplicative factor $\mathcal{O}(r^{2d-1})$ is an upper bound estimate (see Remark 4.2) to the number of rectangles $J_{\mathbf{k}',\mathbf{q}'} \subset J_{\mathbf{r},\mathbf{i}}$ such that $[J_{\mathbf{k}',\mathbf{q}'} \cup J_{\mathbf{k},\mathbf{q}}] = J_{\mathbf{r},\mathbf{i}}$.

Now, for any s with $\lfloor r^{\frac{1}{4}} \rfloor \leq s \leq r - \lfloor r^{\frac{1}{4}} \rfloor$, we have

$$\sum_{s_1=0}^{s} \sum_{s_2=0}^{s-s_1} \cdots \sum_{s_d=0}^{s-s_1-\cdots-s_{d-1}} \delta_{s_1+s_2+\cdots+s_d-s} \cdot c_d \cdot r^{2d} \cdot t \cdot \frac{t^{\frac{s-1}{3}} \cdot}{s^{x_d}} \frac{t^{\frac{r-s-1}{3}}}{(r-s)^{x_d}} \tag{5.111}$$

$$\leq s^d \cdot c_d \cdot r^{2d-1} \cdot t^{\frac{2}{3}} \cdot \frac{t^{\frac{r-1}{3}}}{s^{x_d} \cdot (r-s)^{x_d}} \tag{5.112}$$

$$\leq r^d \cdot c_d \cdot r^{2d-1} \cdot t^{\frac{2}{3}} \cdot \frac{t^{\frac{r-1}{3}}}{s^{x_d} \cdot (r-s)^{x_d}} \tag{5.113}$$

$$\leq 2^{x_d} \cdot c_d \cdot r^{2d-1} \cdot t^{\frac{2}{3}} \cdot \frac{t^{\frac{r-1}{3}}}{r^{x_d} \cdot r^{x_d/4}} \tag{5.114}$$

as

$$\max_{\lfloor r^{\frac{1}{4}} \rfloor \leq s \leq r - \lfloor r^{\frac{1}{4}} \rfloor} \frac{1}{s^{x_d} \cdot (r-s)^{x_d}} \leq \frac{1}{r^{x_d/4} \cdot \left(r - r^{\frac{1}{4}} \right)^{x_d}} \leq \frac{2^{x_d}}{r^{x_d} \cdot r^{x_d/4}}$$

since $r - \lfloor r^{\frac{1}{4}} \rfloor \geq \frac{r}{2}$. However, then, using the inductive hypothesis for $\| V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})_*} \|$,

$$\| V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})} \| \leq \| V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})_*} \| + \sum_{s=\lfloor r^{\frac{1}{4}} \rfloor}^{r-\lfloor r^{\frac{1}{4}} \rfloor} r^d \cdot 2^{x_d} \cdot c_d \cdot r^{2d-1} \cdot t^{\frac{2}{3}} \cdot \frac{t^{\frac{r-1}{3}}}{r^{x_d} \cdot r^{x_d/4}} \tag{5.115}$$

$$\leq \frac{t^{\frac{r-1}{3}}}{r^{x_d+2d}} + 2^{x_d} \cdot c_d \cdot t^{\frac{2}{3}} \cdot \frac{t^{\frac{r-1}{3}}}{r^{\frac{5x_d}{4}-3d}} \tag{5.116}$$

$$\leq 2 \cdot \frac{t^{\frac{r-1}{3}}}{r^{x_d+2d}} \tag{5.117}$$

since $x_d = 20d$ and $t \geq 0$ is small enough.

C. *Regime* ($\mathfrak{R}3$)

Proof of (5.91). For $(\mathbf{k},\mathbf{q})_{**} \prec (\mathbf{k},\mathbf{q}) \prec (\mathbf{r},\mathbf{i})$, we first consider

$$\frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} P^{(+)}_{J_{\mathbf{r},\mathbf{i}}} V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}} P^{(-)}_{J_{\mathbf{r},\mathbf{i}}} \frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} \tag{5.118}$$

$$= \frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} P^{(+)}_{J_{\mathbf{r},\mathbf{i}}} V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}} P^{(-)}_{J_{\mathbf{r},\mathbf{i}}}. \tag{5.119}$$

We recall that for $(\mathbf{k},\mathbf{q}) \prec (\mathbf{r},\mathbf{i})$, the two types of re-expansion that have to be considered correspond to (a) and (c) in Definition 2.4. Note that the re-expansion of type (a) is trivial since it does not change the potential. Using the re-expansion of type (c), which is associated with formulas (3.34)–(3.36), we get

$$\frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} P^{(+)}_{J_{\mathbf{r},\mathbf{i}}} V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}} P^{(-)}_{J_{\mathbf{r},\mathbf{i}}} \tag{5.120}$$

$$= \frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} P^{(+)}_{J_{\mathbf{r},\mathbf{i}}} V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{r},\mathbf{i}}} P^{(-)}_{J_{\mathbf{r},\mathbf{i}}} \tag{5.121}$$

$$+ \frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} P^{(+)}_{J_{\mathbf{r},\mathbf{i}}} \left\{ \sum_{n=1}^{\infty} \frac{1}{n!} ad^n S_{J_{\mathbf{k},\mathbf{q}}} (V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{r},\mathbf{i}}}) \right\} P^{(-)}_{J_{\mathbf{r},\mathbf{i}}} \tag{5.122}$$

$$+ \frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} P^{(+)}_{J_{\mathbf{r},\mathbf{i}}} \left\{ \sum_{J_{\mathbf{k}',\mathbf{q}'} \in \mathcal{G}^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}} \sum_{n=1}^{\infty} \frac{1}{n!} ad^n S_{J_{\mathbf{k},\mathbf{q}}} (V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}}) \right\} P^{(-)}_{J_{\mathbf{r},\mathbf{i}}}. \tag{5.123}$$

We shall keep re-expanding the terms analogous to $V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{r},\mathbf{i}}}$ in (5.121) from $(\mathbf{k},\mathbf{q})_{-1}$ down to $(\mathbf{k}_{**},\mathbf{q}_{**})$. The pair $(\mathbf{k}_{**},\mathbf{q}_{**})$ represents the greatest rectangle with respect to the ordering $\succ$ in regime $\mathfrak{R}2$ and by construction has $k_{**} = r - \lfloor r^{\frac{1}{4}} \rfloor$.

On the contrary, at each step, we estimate the terms of types (5.122) and (5.123) that are produced by the iteration, without further expanding the potentials analogous to $V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{r},\mathbf{i}}}$ and $V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}}$ that are contained in them.

Estimate of (5.122). Concerning (5.122), we observe that using inductive hypotheses (5.90)–(5.93) along with Lemma (A.3), we can bound

$$\|(5.122)\| \leq c \cdot t \cdot \frac{t^{\frac{k-1}{3}}}{k^{x_d}} \cdot \frac{t^{\frac{r-1}{3}}}{r^{x_d}}. \tag{5.124}$$

At fixed k, the number of contributions of type (5.122) can be estimated from above by $\mathcal{O}(r^d \cdot k^{d-1})$; see Remark 4.2. Being $k \geq r - \lfloor r^{\frac{1}{4}} \rfloor$ in regime $\mathfrak{R}3$, the power $t^{\frac{k-1}{3}}$ will be used to control the number of these types of contributions produced along the way down to $(\mathbf{k}_{**},\mathbf{q}_{**})$.

Estimate of (5.123). It is convenient to split the corresponding term, (5.123), into

$$(5.123) \tag{5.125}$$

$$= \frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} \left\{ P^{+}_{\mathcal{J}_{\mathbf{r},\mathbf{i}}} \sum_{J_{\mathbf{k}',\mathbf{q}'} \in \mathcal{G}^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}} ad\, S_{J_{\mathbf{k},\mathbf{q}}} (V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}}) \right\} P^{(-)}_{J_{\mathbf{r},\mathbf{i}}} \tag{5.126}$$

$$+ \frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} \left\{ P^{+}_{\mathcal{J}_{\mathbf{r},\mathbf{i}}} \sum_{J_{\mathbf{k}',\mathbf{q}'} \in \mathcal{G}^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}} \sum_{n=2}^{\infty} \frac{1}{n!} ad^n S_{J_{\mathbf{k},\mathbf{q}}} (V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}}) \right\} P^{(-)}_{J_{\mathbf{r},\mathbf{i}}}. \tag{5.127}$$

In (5.126), we distinguish $J_{\mathbf{k}',\mathbf{q}'}$ *small* and *large* depending on whether $(\mathbf{k}',\mathbf{q}') \prec (\mathbf{k},\mathbf{q})$ or $(\mathbf{k},\mathbf{q}) \preceq (\mathbf{k}',\mathbf{q}')$, respectively, and denote by $(\mathcal{G}^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}})_{small}$ the subset formed by the *small* $J_{\mathbf{k}',\mathbf{q}'}$ belonging to the set $\mathcal{G}^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}$. We call

$$(5.126)_{small} \quad \text{and} \quad (5.126)_{large},$$

respectively, the corresponding contributions to (5.126). Next, we study some commutators that enter the expression $(5.126)_{small}$ estimated below. We observe that

$$[S_{J_{\mathbf{k},\mathbf{q}}}, V^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{k}',\mathbf{q}'}}] \tag{5.128}$$

$$= [S_{J_{\mathbf{k},\mathbf{q}}}, P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} V^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{k}',\mathbf{q}'}} P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} + P^{(-)}_{J_{\mathbf{k}',\mathbf{q}'}} V^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{k}',\mathbf{q}'}} P^{(-)}_{J_{\mathbf{k}',\mathbf{q}'}}] \tag{5.129}$$

$$= [S_{J_{\mathbf{k},\mathbf{q}}}, P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} V^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{k}',\mathbf{q}'}} P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} + \langle V^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{k}',\mathbf{q}'}} \rangle P^{(-)}_{J_{\mathbf{k}',\mathbf{q}'}}] \tag{5.130}$$

$$= [S_{J_{\mathbf{k},\mathbf{q}}}, P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} V^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{k}',\mathbf{q}'}} P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}}] - [S_{J_{\mathbf{k},\mathbf{q}}}, < V^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{k}',\mathbf{q}'}} > P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}}], \tag{5.131}$$

where we have exploited that $V^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{k}',\mathbf{q}'}}$ is block-diagonalized since, by definition, *small* means $(\mathbf{k}',\mathbf{q}') \prec (\mathbf{k},\mathbf{q})$. We also observe that $P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} P^{(-)}_{J_{\mathbf{r},\mathbf{i}}} = 0$ since $J_{\mathbf{k}',\mathbf{q}'} \subset J_{\mathbf{r},\mathbf{i}}$ by construction; hence,

$$P^{(+)}_{J_{\mathbf{r},\mathbf{i}}} [S_{J_{\mathbf{k},\mathbf{q}}}, P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} V^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{k}',\mathbf{q}'}} P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}}] P^{(-)}_{J_{\mathbf{r},\mathbf{i}}} \tag{5.132}$$

$$-P^{(+)}_{J_{\mathbf{r},\mathbf{i}}} [S_{J_{\mathbf{k},\mathbf{q}}}, < V^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{k}',\mathbf{q}'}} > P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}}] P^{(-)}_{J_{\mathbf{r},\mathbf{i}}} \tag{5.133}$$

$$= -P^{(+)}_{J_{\mathbf{r},\mathbf{i}}} P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} V^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{k}',\mathbf{q}'}} P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} S_{J_{\mathbf{k},\mathbf{q}}} P^{(-)}_{J_{\mathbf{r},\mathbf{i}}} \tag{5.134}$$

$$+P^{(+)}_{J_{\mathbf{r},\mathbf{i}}} < V^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{k}',\mathbf{q}'}} > P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} S_{J_{\mathbf{k},\mathbf{q}}} P^{(-)}_{J_{\mathbf{r},\mathbf{i}}}. \tag{5.135}$$

We recall that for $j \geq 1$,

$$(S_{J_{\mathbf{k},\mathbf{q}}})_j := \frac{1}{G_{J_{\mathbf{k},\mathbf{q}}} - E_{J_{\mathbf{k},\mathbf{q}}}} P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} (V^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{k},\mathbf{q}}})_j P^{(-)}_{J_{\mathbf{k},\mathbf{q}}} - h.c., \tag{5.136}$$

and, from Lemma A.3, we get

$$\|\sum_{j=2}^{\infty} t^j (S_{J_{\mathbf{k},\mathbf{q}}})_j\| \leq C \cdot t^2 \cdot \|(V^{(k,q-1)}_{J_{\mathbf{k},\mathbf{q}}})_1\|^2 \tag{5.137}$$

for $t \geq 0$ sufficiently small. Hence, we split $(5.126)_{small}$ into two contributions:

(1) the leading order term

$$-\frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} P^{(+)}_{J_{\mathbf{r},\mathbf{i}}} \times \left\{ \sum_{J_{\mathbf{k}',\mathbf{q}'} \in (\mathcal{G}^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}})_{small}} P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} \left(V^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{k}',\mathbf{q}'}} - \langle V^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{k}',\mathbf{q}'}} \rangle \right) P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} \left(\frac{t}{G_{J_{\mathbf{k},\mathbf{q}}} - E_{J_{\mathbf{k},\mathbf{q}}}} P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} V^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{k},\mathbf{q}}} P^{(-)}_{J_{\mathbf{k},\mathbf{q}}} - h.c. \right) \right\} P^{(-)}_{J_{\mathbf{r},\mathbf{i}}} \tag{5.138}$$

$$= -\frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} P^{(+)}_{J_{\mathbf{r},\mathbf{i}}} \times \left\{ \sum_{J_{\mathbf{k}',\mathbf{q}'} \in (\mathcal{G}^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}})_{small}} P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} \left(V^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{k}',\mathbf{q}'}} - \langle V^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{k}',\mathbf{q}'}} \rangle \right) P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} \frac{t}{G_{J_{\mathbf{k},\mathbf{q}}} - E_{J_{\mathbf{k},\mathbf{q}}}} P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} V^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{k},\mathbf{q}}} P^{(-)}_{J_{\mathbf{k},\mathbf{q}}} \right\} P^{(-)}_{J_{\mathbf{r},\mathbf{i}}}, \tag{5.139}$$

where we have used $P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} P^{(-)}_{J_{\mathbf{r},\mathbf{i}}} = 0$;

(2) the remainder term

$$-\frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} P^{(+)}_{J_{\mathbf{r},\mathbf{i}}} \left\{ \sum_{J_{\mathbf{k}',\mathbf{q}'} \in (\mathcal{G}^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}})_{small}} P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} \left(V^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{k}',\mathbf{q}'}} - \langle V^{(\mathbf{k},\mathbf{q})-1}_{J_{\mathbf{k}',\mathbf{q}'}} \rangle \right) P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} \sum_{j=2}^{\infty} t^j (S_{J_{\mathbf{k},\mathbf{q}}})_j \right\} P^{(-)}_{J_{\mathbf{r},\mathbf{i}}}. \tag{5.140}$$

In order to estimate the leading order term (5.138), we make use of the inequality

$$\left\| \frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} \left\{ \sum_{J_{\mathbf{k}',\mathbf{q}'} \in (\mathcal{G}^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r};\mathbf{i}}})_{small}} P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} \left(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}} - \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}} \rangle \right) P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} \frac{t}{G_{J_{\mathbf{k},\mathbf{q}}} - E_{J_{\mathbf{k},\mathbf{q}}}} P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}} P^{(-)}_{J_{\mathbf{k},\mathbf{q}}} \right\} P^{(-)}_{J_{\mathbf{r},\mathbf{i}}} \right\|$$

$$\leq \left\| \sum_{J_{\mathbf{k}',\mathbf{q}'} \in (\mathcal{G}^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r};\mathbf{i}}})_{small}} \frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} P^{(+)}_{J_{\mathbf{r},\mathbf{i}}} P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} \left(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}} - \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}} \rangle \right) P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} \right\| \tag{5.141}$$

$$\times t \cdot \left\| \frac{1}{G_{J_{\mathbf{k},\mathbf{q}}} - E_{J_{\mathbf{k},\mathbf{q}}}} P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} \left(\sum_{\mathbf{j}\in J_{\mathbf{k},\mathbf{q}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1 \right) \right\| \cdot \left\| \frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{k},\mathbf{q}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}} P^{(-)}_{J_{\mathbf{k},\mathbf{q}}} \right\|. \tag{5.142}$$

Now, we introduce the notation

$$\overline{\sum}_{J_{\mathbf{k}',\mathbf{q}'},\, J_{\mathbf{k}'',\mathbf{q}''}} := \sum_{J_{\mathbf{k}',\mathbf{q}'},\, J_{\mathbf{k}'',\mathbf{q}''} \in (\mathcal{G}^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r};\mathbf{i}}})_{small}\, ;\, J_{\mathbf{k}',\mathbf{q}'} \cap J_{\mathbf{k}'',\mathbf{q}''} = \emptyset} \tag{5.143}$$

and

$$\sideset{}{'}\sum_{J_{\mathbf{k}',\mathbf{q}'},\, J_{\mathbf{k}'',\mathbf{q}''}} := \sum_{J_{\mathbf{k}',\mathbf{q}'},\, J_{\mathbf{k}'',\mathbf{q}''} \in (\mathcal{G}^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r};\mathbf{i}}})_{small}\, ;\, J_{\mathbf{k}',\mathbf{q}'} \cap J_{\mathbf{k}'',\mathbf{q}''} \neq \emptyset}. \tag{5.144}$$

We can write

$$\left\| \sum_{J_{\mathbf{k}',\mathbf{q}'} \in (\mathcal{G}^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r};\mathbf{i}}})_{small}} \frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} P^{(+)}_{J_{\mathbf{r},\mathbf{i}}} P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} \left(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}} - \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}} \rangle \right) P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} \right\|^2 \tag{5.145}$$

$$\leq \sup_{\|\psi\|=1} \overline{\sum}_{J_{\mathbf{k}',\mathbf{q}'},\, J_{\mathbf{k}'',\mathbf{q}''}} \Bigg\langle \frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} \psi, \tag{5.146}$$

$$P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} \left(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}} - \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}} \rangle \right) P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} P^{(+)}_{J_{\mathbf{k}'',\mathbf{q}''}} \left(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'',\mathbf{q}''}} - \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'',\mathbf{q}''}} \rangle \right) P^{(+)}_{J_{\mathbf{k}'',\mathbf{q}''}} \frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} \psi \Bigg\rangle$$

$$+ \sup_{\|\psi\|=1} \sideset{}{'}\sum_{J_{\mathbf{k}',\mathbf{q}'},\, J_{\mathbf{k}'',\mathbf{q}''}} \Bigg\langle \frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} \psi, \tag{5.147}$$

$$P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} \left(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}} - \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}} \rangle \right) P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} P^{(+)}_{J_{\mathbf{k}'',\mathbf{q}''}} \left(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'',\mathbf{q}''}} - \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'',\mathbf{q}''}} \rangle \right) P^{(+)}_{J_{\mathbf{k}'',\mathbf{q}''}} \frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} \psi \Bigg\rangle.$$

Leading terms in (5.123): Contribution proportional to (5.146)
We observe that for $J_{\mathbf{k}',\mathbf{q}'} \cap J_{\mathbf{k}'',\mathbf{q}''} = \emptyset$, we have

$$P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} \left(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}} - \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}} \rangle \right) P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} P^{(+)}_{J_{\mathbf{k}'',\mathbf{q}''}} \left(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'',\mathbf{q}''}} - \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'',\mathbf{q}''}} \rangle \right) P^{(+)}_{J_{\mathbf{k}'',\mathbf{q}''}} \tag{5.148}$$

$$= P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} P^{(+)}_{J_{\mathbf{k}'',\mathbf{q}''}} \tag{5.149}$$

$$\times \left(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}} - \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}} \rangle \right) P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} P^{(+)}_{J_{\mathbf{k}'',\mathbf{q}''}} \left(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'',\mathbf{q}''}} - \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'',\mathbf{q}''}} \rangle \right) \tag{5.150}$$

$$\times P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} P^{(+)}_{J_{\mathbf{k}'',\mathbf{q}''}} \tag{5.151}$$

since

$$[P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}},\, V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'',\mathbf{q}''}} - \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'',\mathbf{q}''}} \rangle] = [V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}} - \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}} \rangle,\, P^{(+)}_{J_{\mathbf{k}'',\mathbf{q}''}}] = 0.$$

On the contrary, we note that

$$[P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}},\, P^{(+)}_{J_{\mathbf{k}'',\mathbf{q}''}}] = [P^{(-)}_{J_{\mathbf{k}',\mathbf{q}'}},\, P^{(-)}_{J_{\mathbf{k}'',\mathbf{q}''}}] = 0 \tag{5.152}$$

even if $J_{\mathbf{k}',\mathbf{q}'} \cap J_{\mathbf{k}'',\mathbf{q}''} \neq \emptyset$. Indeed,

$$P^{(-)}_{J_{\mathbf{k}',\mathbf{q}'}} = P^{(-)}_{J_{\mathbf{k}',\mathbf{q}'}\setminus J_{\mathbf{k}'',\mathbf{q}''}} \bigotimes P^{(-)}_{J_{\mathbf{k}',\mathbf{q}'}\cap J_{\mathbf{k}'',\mathbf{q}''}}, \quad P^{(-)}_{J_{\mathbf{k}'',\mathbf{q}''}} = P^{(-)}_{J_{\mathbf{k}'',\mathbf{q}''}\setminus J_{\mathbf{k}',\mathbf{q}'}} \bigotimes P^{(-)}_{J_{\mathbf{k}',\mathbf{q}'}\cap J_{\mathbf{k}'',\mathbf{q}''}}.$$

Hence,

$$P^{(-)}_{J_{\mathbf{k}',\mathbf{q}'}} P^{(-)}_{J_{\mathbf{k}'',\mathbf{q}''}} = (P^{(-)}_{J_{\mathbf{k}',\mathbf{q}'}\setminus J_{\mathbf{k}'',\mathbf{q}''}} \bigotimes P^{(-)}_{J_{\mathbf{k}',\mathbf{q}'}\cap J_{\mathbf{k}'',\mathbf{q}''}})(P^{(-)}_{J_{\mathbf{k}'',\mathbf{q}''}\setminus J_{\mathbf{k}',\mathbf{q}'}} \bigotimes P^{(-)}_{J_{\mathbf{k}',\mathbf{q}'}\cap J_{\mathbf{k}'',\mathbf{q}''}}) \tag{5.153}$$

$$= P^{(-)}_{J_{\mathbf{k}',\mathbf{q}'}\setminus J_{\mathbf{k}'',\mathbf{q}''}} \bigotimes P^{(-)}_{J_{\mathbf{k}',\mathbf{q}'}\cap J_{\mathbf{k}'',\mathbf{q}''}} \bigotimes P^{(-)}_{J_{\mathbf{k}'',\mathbf{q}''}\setminus J_{\mathbf{k}',\mathbf{q}'}} \tag{5.154}$$

$$= (P^{(-)}_{J_{\mathbf{k}'',\mathbf{q}''}\setminus J_{\mathbf{k}',\mathbf{q}'}} \bigotimes P^{(-)}_{J_{\mathbf{k}',\mathbf{q}'}\cap J_{\mathbf{k}'',\mathbf{q}''}})(P^{(-)}_{J_{\mathbf{k}',\mathbf{q}'}\setminus J_{\mathbf{k}'',\mathbf{q}''}} \bigotimes P^{(-)}_{J_{\mathbf{k}',\mathbf{q}'}\cap J_{\mathbf{k}'',\mathbf{q}''}}) \tag{5.155}$$

$$= P^{(-)}_{J_{\mathbf{k}'',\mathbf{q}''}} P^{(-)}_{J_{\mathbf{k}',\mathbf{q}'}}. \tag{5.156}$$

Hence, we can estimate

$$\sup_{\|\psi\|=1} \overline{\sum}_{J_{\mathbf{k}',\mathbf{q}'},\, J_{\mathbf{k}'',\mathbf{q}''}} \Bigg| \langle \frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} \psi, \tag{5.157}$$

$$P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} \Big(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}} - \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}} \rangle \Big) P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} P^{(+)}_{J_{\mathbf{k}'',\mathbf{q}''}} \Big(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'',\mathbf{q}''}} - \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'',\mathbf{q}''}} \rangle \Big) P^{(+)}_{J_{\mathbf{k}'',\mathbf{q}''}} \frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} \psi \rangle \Bigg|$$

$$\leq \sup_{\|\psi\|=1} \overline{\sum}_{J_{\mathbf{k}',\mathbf{q}'},\, J_{\mathbf{k}'',\mathbf{q}''}} \Bigg\| \frac{P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} P^{(+)}_{J_{\mathbf{k}'',\mathbf{q}''}}}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} \psi \Bigg\|^2 \cdot \| V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}} - \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}} \rangle \| \cdot \| V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'',\mathbf{q}''}} - \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'',\mathbf{q}''}} \rangle \|$$

$$\leq \sup_{\|\psi\|=1} \overline{\sum}_{J_{\mathbf{k}',\mathbf{q}'},\, J_{\mathbf{k}'',\mathbf{q}''}} 4 \| V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}} \| \cdot \| V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'',\mathbf{q}''}} \| \cdot \Bigg\| \frac{P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} P^{(+)}_{J_{\mathbf{k}'',\mathbf{q}''}}}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} \psi \Bigg\|^2 \tag{5.158}$$

$$\leq \sup_{\|\psi\|=1} \sum_{J_{\mathbf{k}',\mathbf{q}'},\, J_{\mathbf{k}'',\mathbf{q}''} \in \mathcal{G}^{(\mathbf{k},\mathbf{q})}_{\mathbf{r},\mathbf{i}}} 4 \| V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}} \| \cdot \| V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'',\mathbf{q}''}} \| \cdot \Bigg\| \frac{P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} P^{(+)}_{J_{\mathbf{k}'',\mathbf{q}''}}}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} \psi \Bigg\|^2 \tag{5.159}$$

$$= \sup_{\|\psi\|=1} \Bigg\langle \sum_{J_{\mathbf{k}',\mathbf{q}'} \in \mathcal{G}^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}} 2 \| V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}} \| \frac{P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}}}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} \psi, \sum_{J_{\mathbf{k}'',\mathbf{q}''} \in \mathcal{G}^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}} 2 \| V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'',\mathbf{q}''}} \| \frac{P^{(+)}_{J_{\mathbf{k}'',\mathbf{q}''}}}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} \psi \Bigg\rangle \tag{5.160}$$

$$= \Bigg\| \sum_{J_{\mathbf{k}',\mathbf{q}'} \in \mathcal{G}^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}} 2 \| V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}} \| \frac{P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}}}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} \Bigg\|^2, \tag{5.161}$$

where in the step from (5.159) and (5.160) we have used $[P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}}, P^{(+)}_{J_{\mathbf{k}'',\mathbf{q}''}}] = 0$.

Now, suppose that there are $1 \leq l \leq d$ components of $\mathbf{k}$ different from the corresponding ones in $\mathbf{r}$. Without loss of generality, we can assume that they are the first l components; for $l \leq d-1$, we get

$$\Bigg\| \sum_{J_{\mathbf{k}',\mathbf{q}'} \in \mathcal{G}^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}} 2 \| V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}} \| \frac{P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}}}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} \Bigg\| \tag{5.162}$$

$$= \Bigg\| \sum_{\mathbf{s}:\exists \mathbf{u} \text{ with } J_{\mathbf{s},\mathbf{u}} \in \mathcal{G}^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}} \ \sum_{\mathbf{u}:\ J_{\mathbf{s},\mathbf{u}} \in \mathcal{G}^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}} 2 \| V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{s},\mathbf{u}}} \| \frac{P^{(+)}_{J_{\mathbf{s},\mathbf{u}}}}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} \Bigg\| \tag{5.163}$$

$$\leq C \cdot \Bigg\{ \sum_{s_1 = r_1 - k_1}^{r} \cdots \sum_{s_l = r_l - k_l}^{r} \sum_{s_{l+1}=0}^{r} \cdots \sum_{s_d = 0}^{r} \frac{t^{\left(\sum_{j=1}^{d} \frac{s_j}{3}\right) - \frac{1}{3}}}{(s_1 + \cdots + s_d)^{x_d}} \cdot \Bigg[\prod_{j=l+1}^{d} (s_j + 1) \Bigg] \Bigg\} \tag{5.164}$$

(we call $s_1, \ldots, s_d$ the components of $\mathbf{k}'$) where in the step from (5.163) and (5.164) we use the following:

- an upper bound for $\| V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{s},\mathbf{u}}} \|$ that is independent of $\mathbf{u}$ by means of inductive hypothesis (5.93),

$$\| V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{s},\mathbf{u}}} \| \leq 96 \cdot \frac{t^{\frac{s-1}{3}}}{s^{x_d}}; \tag{5.165}$$

- the fact that, for fixed $\mathbf{k}'$, if $k_j \neq r_j$ for $j = 1, \ldots, l$, then $q_1', \ldots, q_l'$ are uniquely[32] determined by the condition $[J_{\mathbf{k}',\mathbf{q}'} \cup J_{\mathbf{k},\mathbf{q}}] = J_{\mathbf{r},\mathbf{i}}$;
- the estimate

$$\sum_{\mathbf{u}\,:\, u_1,\ldots,u_l=\text{fixed},\, J_{\mathbf{s},\mathbf{u}} \in \mathcal{G}^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}} P^{(+)}_{J_{\mathbf{s},\mathbf{u}}} \leq \left\{ \prod_{j=l+1}^{d} (s_j+1) \right\} \sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}}$$

  that can be proved following the same reasoning of Corollary A.2.

When $l = d$, the estimate of (5.162) written above holds with the product $\prod_{j=l+1}^{d}(s_j+1)$ replaced by 1.

Next, for $j = 1, \ldots, l$, we set

$$\rho_j := s_j - (r_j - k_j) \quad \Rightarrow \quad s_j = \rho_j + (r_j - k_j), \tag{5.166}$$

and we observe that since $s_j \geq r_j - k_j$ for $j = 1, \ldots, l$ and $s_j \geq 0$ for $j = l+1, \ldots, d$, we have

$$(s_1 + \cdots + s_d)^{x_d} \geq (r_1 - k_1 + \cdots + r_l - k_l)^{x_d}. \tag{5.167}$$

Hence, we can estimate

$$\begin{aligned}
&(5.164)\\
&\leq C \cdot t^{-1/3} \cdot t^{\sum_{j=1}^{l} \frac{(r_j-k_j)}{3}} \cdot \left\{ \sum_{s_{l+1}=0}^{r} \cdots \sum_{s_d=0}^{r} \frac{1}{(r_1-k_1+\cdots+r_l-k_l)^{x_d}} \cdot \left[t^{\sum_{j=l+1}^{d} \frac{s_j}{3}} \cdot \prod_{j=l+1}^{d} (s_j+1) \right] \sum_{\rho_1=0}^{\infty} \cdots \sum_{\rho_l=0}^{\infty} t^{\sum_{j=1}^{l} \frac{\rho_j}{3}} \right\}\\
&= C \cdot t^{-1/3} \cdot t^{\sum_{j=1}^{l} \frac{(r_j-k_j)}{3}} \cdot \frac{1}{(r_1-k_1+\cdots+r_l-k_l)^{x_d}} \qquad (5.168)\\
&\quad \times \left\{ \sum_{s_{l+1}=0}^{r} \cdots \sum_{s_d=0}^{r} \left[t^{\sum_{j=l+1}^{d} \frac{s_j}{3}} \cdot \prod_{j=l+1}^{d} (s_j+1) \right] \sum_{\rho_1=0}^{\infty} \cdots \sum_{\rho_l=0}^{\infty} t^{\sum_{j=1}^{l} \frac{\rho_j}{3}} \right\}\\
&\leq C_d \cdot t^{-1/3} \left(\frac{t^{\frac{r-1}{3}}}{t^{\frac{k-1}{3}}} \right) \cdot \frac{1}{(r_1-k_1+\cdots+r_l-k_l)^{x_d}}, \qquad (5.169)
\end{aligned}$$

where we have exploited the following:

- the quantity

$$\sum_{s_{l+1}=0}^{r} \cdots \sum_{s_d=0}^{r} \left[t^{\sum_{j=l+1}^{d} \frac{s_j}{3}} \cdot \prod_{j=l+1}^{d} (s_j+1) \right] \sum_{\rho_1=0}^{\infty} \cdots \sum_{\rho_l=0}^{\infty} t^{\sum_{j=1}^{l} \frac{\rho_j}{3}} \tag{5.170}$$

  is bounded from above by a d-dependent constant;
- for the considered $\mathbf{k}$,

$$t^{\sum_{j=1}^{l} \frac{(r_j-k_j)}{3}} = t^{\frac{r-k}{3}} \tag{5.171}$$

  since $k_j = r_j$ for $j = l+1, \ldots, d$ by assumption.

Leading terms in (5.123): Contribution proportional to (5.147)

By the Schwarz inequality and the trivial bound $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$, we estimate [recall the notation $\sum'$ in (5.144)]

$$\begin{aligned}
&\sup_{\|\psi\|=1} \sum_{J_{\mathbf{k}',\mathbf{q}'},\, J_{\mathbf{k}'',\mathbf{q}''}}{}' \Bigg| \Bigg\langle \frac{1}{\sum_{\mathbf{j}\in\mathcal{J}_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} \psi, \qquad (5.172)\\
&\qquad P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} \left(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}} - \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}} \rangle \right) P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}} P^{(+)}_{J_{\mathbf{k}'',\mathbf{q}''}} \left(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'',\mathbf{q}''}} - \langle V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'',\mathbf{q}''}} \rangle \right) P^{(+)}_{J_{\mathbf{k}'',\mathbf{q}''}} \frac{1}{\sum_{\mathbf{j}\in\mathcal{J}_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} \psi \Bigg\rangle \Bigg|\\
&\leq \sup_{\|\psi\|=1} \sum_{J_{\mathbf{k}',\mathbf{q}'},\, J_{\mathbf{k}'',\mathbf{q}''}}{}' 4 \| V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}} \| \cdot \| V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'',\mathbf{q}''}} \| \cdot \left\{ \frac{1}{2} \left\| \frac{P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}}}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} \psi \right\|^2 + \frac{1}{2} \left\| \frac{P^{(+)}_{J_{\mathbf{k}'',\mathbf{q}''}}}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} \psi \right\|^2 \right\}. \qquad (5.173)
\end{aligned}$$

Since the expression in (5.173) is symmetric under the permutation of $J_{\mathbf{k}',\mathbf{q}'}$ with $J_{\mathbf{k}'',\mathbf{q}''}$, we can write

$$\begin{aligned}
&(5.172) \qquad\qquad (5.174)\\
&\le \sup_{\|\psi\|=1} \sum_{J_{\mathbf{k}',\mathbf{q}'},J_{\mathbf{k}'',\mathbf{q}''}}{}' 4\|V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}}\|\cdot\|V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'',\mathbf{q}''}}\|\cdot\left\{\left\|\frac{P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}}}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}}P^{\perp}_{\Omega_{\mathbf{j}}}+1}\psi\right\|^2\right\}\\
&= \sup_{\|\psi\|=1} \sum_{J_{\mathbf{k}',\mathbf{q}'},J_{\mathbf{k}'',\mathbf{q}''}}{}' 4\|V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}}\|\cdot\|V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'',\mathbf{q}''}}\|\cdot\left\{\left\langle\frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}}P^{\perp}_{\Omega_{\mathbf{j}}}+1}\psi,\frac{P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}}}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}}P^{\perp}_{\Omega_{\mathbf{j}}}+1}\psi\right\rangle\right\}\\
&= \sup_{\|\psi\|=1}\left\{\left\langle\frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}}P^{\perp}_{\Omega_{\mathbf{j}}}+1}\psi,\sum_{J_{\mathbf{k}',\mathbf{q}'},J_{\mathbf{k}'',\mathbf{q}''}}{}' 4\|V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}}\|\cdot\|V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'',\mathbf{q}''}}\|\frac{P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}}}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}}P^{\perp}_{\Omega_{\mathbf{j}}}+1}\psi\right\rangle\right\}\\
&\le \left\|\sum_{J_{\mathbf{k}',\mathbf{q}'},J_{\mathbf{k}'',\mathbf{q}''}}{}' 4\|V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}}\|\cdot\|V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'',\mathbf{q}''}}\|\frac{P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}}}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}}P^{\perp}_{\Omega_{\mathbf{j}}}+1}\right\|. \qquad (5.175)
\end{aligned}$$

With steps similar to (5.163)–(5.169), assume that there are $1 \le l \le d$ components of $\mathbf{k}$ different from the corresponding ones in $\mathbf{r}$ (without loss of generality, we identify them with the first l components). Then, we can bound (5.175) as [warning: for $l = d$, $\prod_{j=l+1}^{d}(s_j+1)$, $\prod_{j=l+1}^{d}s_j$ must be replaced by 1 in (5.176) and related formulas]

$$\begin{aligned}
(5.175) &\le \left\|\sum_{J_{\mathbf{k}',\mathbf{q}'}\in\mathcal{G}^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}} 4\|V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}',\mathbf{q}'}}\|\frac{P^{(+)}_{J_{\mathbf{k}',\mathbf{q}'}}}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}}P^{\perp}_{\Omega_{\mathbf{j}}}+1}\sum_{J_{\mathbf{k}'',\mathbf{q}''}\in\mathcal{G}^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}\,:\,J_{\mathbf{k}',\mathbf{q}'}\cap J_{\mathbf{k}'',\mathbf{q}''}\neq\emptyset}\|V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'',\mathbf{q}''}}\|\right\|\\
&\le C\cdot\left\{\sum_{s_1=r_1-k_1}^{r}\cdots\sum_{s_l=r_l-k_l}^{r}\sum_{s_{l+1}=0}^{r}\cdots\sum_{s_d=0}^{r}\frac{t^{\left(\sum_{j=1}^{d}\frac{s_j}{3}\right)-\frac{1}{3}}}{(s_1+\cdots+s_d)^{x_d}}\cdot\left[\prod_{j=l+1}^{d}(s_j+1)\right]\right\} \qquad (5.176)\\
&\quad\times\left(\sum_{w=r-k}^{r}\frac{t^{\frac{w-1}{3}}}{w^{x_d}}\cdot\left(\prod_{j=1}^{d}s_j\right)\cdot w^{d-1}\right)
\end{aligned}$$

due to the estimate

$$\sum_{J_{\mathbf{k}'',\mathbf{q}''}\in\mathcal{G}^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}\,:\,J_{\mathbf{s},\mathbf{q}'}\cap J_{\mathbf{k}'',\mathbf{q}''}\neq\emptyset}\|V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k}'',\mathbf{q}''}}\|\le\mathcal{O}\left(\sum_{w=r-k}^{r}\frac{t^{\frac{w-1}{3}}}{w^{x_d}}\cdot\left(\prod_{j=1}^{d}s_j\right)\cdot w^{d-1}\right),$$

where we have the following:

(i) $\mathcal{O}\left(\left(\prod_{j=1}^{d}s_j\right)\cdot w^{d-1}\right)$ bounds from above the number of rectangles $J_{\mathbf{w},\mathbf{q}''}$ overlapping with the rectangle $J_{\mathbf{s},\mathbf{q}'}$;

(ii) $\mathcal{O}\left(\frac{t^{\frac{w-1}{3}}}{w^{x_d}}\right)$ is the bound to $\|V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{w},\mathbf{q}''}}\|$, provided by the inductive hypotheses.

Next, using the definition in (5.166) and arguments as in (5.168) and (5.169), we write

$$\begin{aligned}
&(5.176)\\
&\le C\cdot\left(t^{-1/3}\frac{t^{\frac{r-1}{3}}}{t^{\frac{k-1}{3}}}\right)^2\cdot\left\{\frac{1}{(r_1-k_1+\cdots+r_l-k_l)^{x_d}}\cdot\sum_{s_{l+1}=0}^{r}\cdots\sum_{s_d=0}^{r}\sum_{w=r-k}^{r}\left[w^{d-1}\frac{1}{w^{x_d}}\right]\right. \qquad (5.177)\\
&\quad\left.\times\left[\left(\prod_{j=l+1}^{d}s_j\right)\cdot t^{\sum_{j=l+1}^{d}s_j/3}\cdot\prod_{j=l+1}^{d}(s_j+1)\right]\sum_{\rho_1=0}^{\infty}\cdots\sum_{\rho_l=0}^{\infty}t^{\sum_{j=1}^{l}\frac{\rho_j}{3}}\prod_{j=1}^{l}(\rho_j+r_j-k_j)\right\}.
\end{aligned}$$

Now, we multiply the right-hand side of (5.177) by

$$\frac{(r_1-k_1+\cdots+r_l-k_l)^d}{(r_1-k_1+\cdots+r_l-k_l)^l}\ge 1,$$

and we get

$$(5.176) \tag{5.178}$$

$$\leq C\cdot\left(t^{-1/3}\frac{t^{\frac{r-1}{3}}}{t^{\frac{k-1}{3}}}\right)^2\cdot\left\{\frac{(r_1-k_1+\cdots+r_l-k_l)^d}{(r_1-k_1+\cdots+r_l-k_l)^{x_d}}\cdot\sum_{s_{l+1}=0}^{r}\cdots\sum_{s_d=0}^{r}\sum_{w=r-k}^{r}\left[w^{d-1}\frac{1}{w^{x_d}}\right]\right. \tag{5.179}$$

$$\left.\times\left[\left(\prod_{j=l+1}^{d}s_j\right)\cdot t^{\sum_{j=l+1}^{d}s_j/3}\cdot\prod_{j=l+1}^{d}(s_j+1)\right]\sum_{\rho_1=0}^{\infty}\cdots\sum_{\rho_l=0}^{\infty}t^{\sum_{j=1}^{l}\frac{\rho_j}{3}}\prod_{j=1}^{l}\left(\frac{\rho_j+r_j-k_j}{(r_1-k_1+\cdots+r_l-k_l)}\right)\right\}$$

$$\leq C_d\cdot\left(t^{-1/3}\frac{t^{\frac{r-1}{3}}}{t^{\frac{k-1}{3}}}\right)^2\cdot\left(\frac{1}{(r_1-k_1+\cdots+r_l-k_l)^{x_d-d}}\right)^2, \tag{5.180}$$

where, in the step from (5.179) and (5.180), we have used that $x_d \geq d+1$ and that all the following quantities are bounded from above by a d-dependent constant:

- $$(r_1-k_1+\cdots+r_l-k_l)^{x_d-d}\cdot\sum_{w=r-k}^{r}w^{d-1}\frac{1}{w^{x_d}}$$

- $$\sum_{s_{l+1}=0}^{r}\cdots\sum_{s_d=0}^{r}\left(\prod_{j=l+1}^{d}s_j\right)\cdot t^{\sum_{j=l+1}^{d}s_j/3}\cdot\prod_{j=l+1}^{d}(s_j+1)$$

- $$\sum_{\rho_1=0}^{\infty}\cdots\sum_{\rho_l=0}^{\infty}t^{\sum_{j=1}^{l}\frac{\rho_j}{3}}\prod_{j=1}^{l}\left(\frac{\rho_j+r_j-k_j}{(r_1-k_1+\cdots+r_l-k_l)}\right).$$

Leading terms in (5.123): Contribution proportional to (5.142)
Finally, we estimate (5.142) by exploiting the inequality

$$\left\|\frac{1}{G_{J_{\mathbf{k},\mathbf{q}}}-E_{J_{\mathbf{k},\mathbf{q}}}}\left(\sum_{\mathbf{j}\in J_{\mathbf{k},\mathbf{q}}}P^{\perp}_{\Omega_{\mathbf{j}}}+1\right)\frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{k},\mathbf{q}}}P^{\perp}_{\Omega_{\mathbf{j}}}+1}P^{(+)}_{J_{\mathbf{k},\mathbf{q}}}V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}}P^{(-)}_{J_{\mathbf{k},\mathbf{q}}}\right\| \tag{5.181}$$

$$\leq\left\|\frac{1}{G_{J_{\mathbf{k},\mathbf{q}}}-E_{J_{\mathbf{k},\mathbf{q}}}}\left(\sum_{\mathbf{j}\in J_{\mathbf{k},\mathbf{q}}}P^{\perp}_{\Omega_{\mathbf{j}}}+1\right)\right\|\left\|\frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{k},\mathbf{q}}}P^{\perp}_{\Omega_{\mathbf{j}}}+1}P^{(+)}_{J_{\mathbf{k},\mathbf{q}}}V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}}P^{(-)}_{J_{\mathbf{k},\mathbf{q}}}\right\| \tag{5.182}$$

$$\leq 3\cdot\frac{t^{\frac{k-1}{3}}}{k^{x_d+2d}}, \tag{5.183}$$

where the first factor can be estimated to be less than 3, provided that t_d is small enough, by using the bound in (4.67) (see Lemma 4.4) that holds due to (S2) in the previous step; for the second factor, we invoke the inductive hypothesis in (5.91).

Hence, we conclude that at fixed $\mathbf{k}$ and with l components different from the corresponding components of $\mathbf{r}$,

$$\|(5.138)\|\leq C_d\cdot t^{2/3}\cdot\frac{t^{\frac{r-1}{3}}}{(r_1-k_1+\cdots+r_l-k_l)^{x_d-d}\cdot k^{x_d+2d}}. \tag{5.184}$$

Higher order terms in (5.123)
In order to show the bound in (5.91), with regard to (5.123), we have still to estimate the following:

- remainder (5.140) [coming from the study of $(5.126)_{small}$] and those corresponding to $(5.126)_{large}$, i.e., proportional to terms with $J_{\mathbf{k}',\mathbf{q}'}$ such that $(\mathbf{k}',\mathbf{q}') \succ \mathbf{k},\mathbf{q}$);
- the contribution due to (5.127).

We observe that we have the following:

(i) in all these terms, there are either two factors $S_{\mathbf{k},\mathbf{q}}$ or two factors $\|(V_{J_{\mathbf{k},\mathbf{q}}}^{(\mathbf{k},\mathbf{q})_{-1}})_1\|$ [see (5.137)] or $J_{\mathbf{k}',\mathbf{q}'}$ is large such that $(\mathbf{k}',\mathbf{q}')>(\mathbf{k},\mathbf{q})$; thus, we get at least an extra factor $\mathcal{O}(t^{\frac{r-2r^{1/4}-1}{3}})$;

(ii) the bound from above, $\mathcal{O}(r^{2d-1})$, of the number of the elements of $\mathcal{G}_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})}$ (see Remark 4.2). Hence, just using inductive hypotheses (5.92) and (5.93), we can estimate

$$\|(5.140)\|+\|(5.126)_{large}\|+\|(5.127)\| \tag{5.185}$$

$$\leq C_d\cdot t\cdot r^{4d-1}\cdot t^{\frac{r-2\cdot r^{1/4}-1}{3}}\cdot\frac{t^{\frac{r-1}{3}}}{(r-k)^{x_d}\cdot k^{x_d}}. \tag{5.186}$$

At fixed k, there are at most $\mathcal{O}(r^d\cdot k^{d-1})$ contributions of type (5.185).

Complete estimate of (5.91)

Finally, by the re-expansion outlined above and due to the estimates of (5.122), (5.138), (5.140), $(5.126)_{large}$, and (5.127) that have been derived [see (5.124), (5.184), and (5.185)], we can conclude that

$$\Big\|\frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}}P^{\perp}_{\Omega_{\mathbf{j}}}+1}P^{(+)}_{J_{\mathbf{r},\mathbf{i}}}\,V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}\,P^{(-)}_{J_{\mathbf{r},\mathbf{i}}}\Big\| \tag{5.187}$$

$$\leq\Big\|\frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}}P^{\perp}_{\Omega_{\mathbf{j}}}+1}P^{(+)}_{J_{\mathbf{r},\mathbf{i}}}\,V^{(\mathbf{k},\mathbf{q})_{**}}_{J_{\mathbf{r},\mathbf{i}}}\,P^{(-)}_{J_{\mathbf{r},\mathbf{i}}}\Big\| \tag{5.188}$$

$$+C_d\cdot t\cdot\sum_{l=1}^{d}\binom{d}{l}\cdot\sum_{k_1=0}^{r_1-1}\cdots\sum_{k_l=0}^{r_l-1}\Theta\Big(k-r+\lfloor r^{\frac14}\rfloor\Big) \tag{5.189}$$

$$\times\frac{t^{\frac{r-1}{3}}}{(r_1-k_1+\cdots+r_l-k_l)^{x_d-d}\cdot k^{x_d+2d}} \tag{5.190}$$

$$+\sum_{k=r-\lfloor r^{\frac14}\rfloor}^{k=r-1}\Bigg\{c_d\cdot r^d\cdot k^{d-1}\cdot t\cdot\frac{t^{\frac{k-1}{3}}\cdot t^{\frac{r-1}{3}}}{k^{x_d}\cdot r^{x_d}} \tag{5.191}$$

$$+C_d\cdot t\cdot r^{5d-1}\cdot k^{d-1}\cdot t^{\frac{r-2r^{\frac14}-1}{3}}\cdot\frac{t^{\frac{r-1}{3}}}{(r-k)^{x_d}\cdot k^{x_d}}\Bigg\}, \tag{5.192}$$

where Θ is the characteristic function of $\mathbb{R}^+$; indeed, $k\geq r-\lfloor r^{\frac14}\rfloor$ in regime $\mathfrak{R}3$. In addition to summand (5.188) that is smaller than $2\cdot\frac{t^{\frac{r-1}{3}}}{r^{x_d+2d}}$ by the inductive hypothesis (5.90), on the right-hand side of the estimate above, we have three summands that we shall discuss in detail. Prior to this discussion, we explain why the final estimate in (5.205) works.

Remark 5.2. We point out that we have the following:

(i) regarding the expression in (5.189) and (5.190), the factor $\frac{1}{k^{x_d+2d}}$ [coming from the inductive hypothesis used to estimate (5.142)] provides the expected behavior since $k\geq r-\lfloor r^{\frac14}\rfloor$ in regime $\mathfrak{R}3$, and the rest can be made less than $\frac{t^{\frac{r-1}{3}}}{3}$ due to the definition of x_d, as we explain below;

(ii) regarding the expressions in (5.191) and (5.192), we exploit the extra powers $t^{\frac{k-1}{3}}$ and $t^{\frac{r-2r^{\frac14}-1}{3}}$, respectively, in order to control the sum over k and provide the desired behavior.

As for (5.189) and (5.190), we first observe that we have

$$\sum_{k_1=0}^{r_1-1}\cdots\sum_{k_l=0}^{r_l-1}\Theta\Big(k-r+r^{\frac14}\Big)\times\frac{t^{\frac{r-1}{3}}}{(r_1-k_1+\cdots+r_l-k_l)^{x_d-d}\cdot k^{x_d+2d}} \tag{5.193}$$

$$\leq C_d\cdot\sum_{s_1=1}^{r_1}\cdots\sum_{s_l=1}^{r_l}\frac{t^{\frac{r-1}{3}}}{(s_1+s_2+\cdots+s_l)^{x_d-d}\cdot r^{x_d+2d}} \tag{5.194}$$

$$\leq C_d\cdot\frac{t^{\frac{r-1}{3}}}{r^{x_d+2d}}\sum_{s_1=1}^{\infty}\cdots\sum_{s_l=1}^{\infty}\frac{1}{(s_1+s_2+\cdots+s_l)^{x_d-d}} \tag{5.195}$$

$$\leq C_d\cdot\frac{t^{\frac{r-1}{3}}}{r^{x_d+2d}}, \tag{5.196}$$

where, since $x_d - d > l$, $\sum_{s_1=1}^{\infty}\cdots\sum_{s_l=1}^{\infty}\frac{1}{(s_1+s_2+\cdots+s_l)^{x_d-d}}$ is bounded by a d-dependent constant. Therefore, since

$$x_d > d + d - 1 = 2d - 1,$$

we see that the overall quantity can be made less than $\frac{1}{3}\cdot\frac{t^{\frac{r-1}{3}}}{r^{x_d+2d}}$, provided that $t \geq 0$ is small enough.

As for (5.191), we have

$$\sum_{k=r-\lfloor r^{\frac{1}{4}}\rfloor}^{k=r-1} c_d\cdot r^d\cdot k^{d-1}\cdot t\cdot\frac{t^{\frac{k-1}{3}}\cdot t^{\frac{r-1}{3}}}{k^{x_d}\cdot r^{x_d}} \tag{5.197}$$

$$\leq r^{\frac{1}{4}}\cdot c_d\cdot r^d\cdot r^{d-1}\cdot t^{\frac{r-r^{\frac{1}{4}}-1}{3}}\cdot t^{\frac{r-1}{3}}\cdot\frac{1}{\left(r-r^{\frac{1}{4}}\right)^{x_d}\cdot r^{x_d}} \tag{5.198}$$

$$\leq 2^{x_d}\cdot c_d\cdot t^{\frac{r-r^{\frac{1}{4}}-1}{3}}\cdot\frac{t^{\frac{r-1}{3}}}{r^{2x_d-2d+\frac{3}{4}}} \tag{5.199}$$

$$\leq \frac{1}{3}\cdot\frac{t^{\frac{r-1}{3}}}{r^{x_d+2d}} \tag{5.200}$$

since $x_d \geq 4d - \frac{3}{4}$ and $t \geq 0$ is small enough.

As for (5.192), this quantity can be estimated in the following way:

$$\sum_{k=r-\lfloor r^{\frac{1}{4}}\rfloor}^{r-1} C_d\cdot t\cdot r^{5d-1}\cdot k^{d-1}\cdot t^{\frac{r-2r^{\frac{1}{4}}-1}{3}}\cdot\frac{t^{\frac{r-1}{3}}}{(r-k)^{x_d}\cdot k^{x_d}} \tag{5.201}$$

$$\leq r^{\frac{1}{4}}\cdot 2^{x_d}\cdot C_d\cdot t\cdot t^{\frac{r-2r^{\frac{1}{4}}-1}{3}}\frac{t^{\frac{r-1}{3}}}{r^{x_d-6d+2}} \tag{5.202}$$

$$= 2^{x_d}\cdot C_d\cdot t\cdot t^{\frac{r-2r^{\frac{1}{4}}-1}{3}}\cdot\frac{t^{\frac{r-1}{3}}}{r^{x_d-6d+\frac{7}{4}}} \tag{5.203}$$

$$\leq \frac{1}{3}\cdot\frac{t^{\frac{r-1}{3}}}{r^{x_d+2d}}, \tag{5.204}$$

where the last inequality holds, provided that $t \geq 0$ is so small as to fulfill the inequality

$$2^{x_d}\cdot C_d\cdot t\cdot t^{\frac{r-2r^{\frac{1}{4}}-1}{3}} \leq \frac{1}{3\cdot r^{8d-\frac{7}{4}}}$$

uniformly in r.

Finally, for $t \geq 0$ small enough, we find

$$\Big\|\frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}}P^{\perp}_{\Omega_{\mathbf{j}}}+1}P^{(+)}_{J_{\mathbf{r},\mathbf{i}}}\,V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}\,P^{(-)}_{J_{\mathbf{r},\mathbf{i}}}\Big\| \tag{5.205}$$

$$\leq 2\cdot\frac{t^{\frac{r-1}{3}}}{r^{x_d+2d}}+3\cdot\frac{1}{3}\cdot\frac{t^{\frac{r-1}{3}}}{r^{x_d+2d}} \tag{5.206}$$

$$= 3\cdot\frac{t^{\frac{r-1}{3}}}{r^{x_d+2d}} \tag{5.207}$$

as claimed.

The control of

$$\frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}}P^{\perp}_{\Omega_{\mathbf{j}}}+1}\,P^{(+)}_{J_{\mathbf{r},\mathbf{i}}}\,V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}}\,P^{(+)}_{J_{\mathbf{r},\mathbf{i}}}\,\frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}}P^{\perp}_{\Omega_{\mathbf{j}}}+1} \tag{5.208}$$

is analogous. The study of

$$\frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} P^{(-)}_{J_{\mathbf{r},\mathbf{i}}} V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}} P^{(-)}_{J_{\mathbf{r},\mathbf{i}}} \frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} \tag{5.209}$$

is actually simpler since in the analogous re-expansion, the terms proportional to *small* $J_{\mathbf{k}',\mathbf{q}'}$ are identically zero.

Proof of (5.92) and (5.93). Concerning (5.92), we observe that

$$\| P^{(\#)}_{J_{\mathbf{r},\mathbf{i}}} V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}} P^{(\hat{\#})}_{J_{\mathbf{r},\mathbf{i}}} \| \tag{5.210}$$

$$\le \| \sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1 \|^2 \cdot \| \frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} P^{(\#)}_{J_{\mathbf{r},\mathbf{i}}} V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{r},\mathbf{i}}} P^{(\hat{\#})}_{J_{\mathbf{r},\mathbf{i}}} \frac{1}{\sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1} \|, \tag{5.211}$$

where $\#, \hat{\#} = \pm$ and $\| \sum_{\mathbf{j}\in J_{\mathbf{r},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + 1 \|^2 \le (r^d + 1)^2 \le 4 \cdot r^{2d}$. Then, we use the estimates in (5.91) proven above.

In order to prove (5.93), it is enough to exploit inequality (A6) and (5.92).

Inductive step to prove (S2). Since we have already proven (S1), the bound in (4.60) is fulfilled for $t \ge 0$ sufficiently small, and we can use Lemma (4.4) and Corollary (4.5). Hence, (S2) holds for $t \ge 0$ sufficiently small but independent of N, $\mathbf{k}$, and $\mathbf{q}$. □

Analogously to the treatment of the one-dimensional systems in Ref. 6, we can now derive the main result of this paper.

Theorem 5.3. *Under the assumption that* (1.4) and (1.7) *hold, the Hamiltonian K_N defined in (1.5) has the following properties: There exists some $t_d > 0$ such that for any $t \in \mathbb{R}$ with $|t| < t_d$ and for all $N < \infty$, we have the following:*

- (i) *$K_N \equiv K_N(t)$ has a unique ground-state, and*
- (ii) *the energy spectrum of K_N has a strictly positive gap, $\Delta_N(t) \ge \frac{1}{2}$, above the ground-state energy.*

Proof. The final transformed Hamiltonian is $K^{(\mathbf{N-1,1})}_{\Lambda^d_N} \equiv G_{J_{\mathbf{N-1,1}}} + tV^{(\mathbf{N-1,1})}_{J_{\mathbf{N-1,1}}}$. Hence, the composition of the unitary conjugations associated with each block-diagonalization step yields the unitary operator $\exp(S_N(t))$ [see (1.9)] such that the operator

$$e^{S_N(t)} K_N(t) e^{-S_N(t)} = G_{J_{\mathbf{N-1,1}}} + tV^{(\mathbf{N-1,1})}_{J_{\mathbf{N-1,1}}} =: \widetilde{K}_N(t)$$

enjoys the properties in (1.10) and (1.11), which follow from Theorem 5.1 and from (4.81)–(4.85) for $(\mathbf{k},\mathbf{q}) = (\mathbf{N-1},\mathbf{1})$, where we also include the block-diagonalized potential $V^{(\mathbf{N-1,1})}_{J_{\mathbf{N-1,1}}}$. □

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Simone Del Vecchio: Conceptualization (equal); Formal analysis (equal); Writing – original draft (equal). **Jürg Fröhlich:** Conceptualization (supporting); Formal analysis (supporting); Writing – original draft (supporting). **Alessandro Pizzo:** Conceptualization (equal); Formal analysis (equal); Writing – original draft (equal); Writing – review & editing (equal). **Stefano Rossi:** Conceptualization (equal); Formal analysis (equal); Writing – original draft (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

APPENDIX: AUXILIARY RESULTS

Lemma A.1. For any $J_{\mathbf{l},\mathbf{i}}$, recalling that

$$P^{(+)}_{J_{\mathbf{l},\mathbf{i}}} := \left(\bigotimes_{\mathbf{j}\in J_{\mathbf{l},\mathbf{i}}} P_{\Omega_{\mathbf{j}}}\right)^{\perp}, \tag{A1}$$

we have that the inequality

$$\sum_{\mathbf{j}\in J_{\mathbf{l},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} \geq P^{(+)}_{J_{\mathbf{l},\mathbf{i}}} \tag{A2}$$

holds true where $P^{\perp}_{\Omega_{\mathbf{j}}} := \mathbb{1}_{\mathbf{j}} - P_{\Omega_{\mathbf{j}}}$.

Proof. Let us define the self-adjoint operator A acting on $\mathcal{H}^{(N)}$ as

$$A := \sum_{\mathbf{j}\in J_{\mathbf{l},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + \left(\bigotimes_{\mathbf{j}\in J_{\mathbf{l},\mathbf{i}}} P_{\Omega_{\mathbf{j}}}\right).$$

Since the A is the sum of $(l_1+1)\ (l_2+1)\dots(l_d+1)+1$ orthogonal projections that commute with one another, its spectrum must be contained in the set

$$\{0,1,2,\dots,(l_1+1)(l_2+1)\cdots(l_d+1)+1\}.$$

Next, we intend to prove that A is invertible, so the inequality $A \geq \mathbb{1}_{\mathcal{H}^{(N^d)}}$ will follow, which is exactly the sought inequality since by definition,

$$\mathbb{1}_{\mathcal{H}^{(N^d)}} - \bigotimes_{\mathbf{j}\in J_{\mathbf{l},\mathbf{i}}} P_{\Omega_{\mathbf{j}}} = P^{(+)}_{J_{\mathbf{l},\mathbf{i}}}.$$

Prove that the invertibility of A is equivalent to showing its injectivity. Decomposing the Hilbert space $\mathcal{H}^{(N^d)}$ as $\mathcal{H}_1\otimes\mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are given by $\otimes_{\mathbf{j}\in\Lambda_N^d\setminus J_{\mathbf{l},\mathbf{i}}}\mathcal{H}_{\mathbf{j}}$ and $\otimes_{\mathbf{j}\in J_{\mathbf{l},\mathbf{i}}}\mathcal{H}_{\mathbf{j}}$, respectively, yields a factorization of A as $A_1\otimes A_2$, with A_1 and A_2 acting on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Therefore, the injectivity of A is equivalent to the injectivity of both A_1 and A_2. However, only A_2 needs to be dealt with for A_1 is just a multiple of the identity. Thanks to the definition of A, A_2 is seen to coincide with

$$\sum_{\mathbf{j}\in J_{\mathbf{l},\mathbf{i}}} P^{\perp}_{\Omega_{\mathbf{j}}} + \prod_{\mathbf{j}\in J_{\mathbf{l},\mathbf{i}}} P_{\Omega_{\mathbf{j}}}.$$

Thus, A_2 is also the sum of projections that commute with one another.

Let Ψ be a vector in $\mathcal{H}_2$ such that $A_2\Psi = 0$. From the equality $(\Psi, A_2\Psi) = 0$, we see that

$$(\Psi, P^{\perp}_{\Omega_{\mathbf{j}}}\Psi) = 0 \quad \forall \mathbf{j}\in J_{\mathbf{l},\mathbf{i}} \quad \text{and} \quad \prod_{\mathbf{j}\in J_{\mathbf{l},\mathbf{i}}} P_{\Omega_{\mathbf{j}}}\Psi = 0. \tag{A3}$$

The first equalities in (A3) imply $\Psi = P_{\Omega_{\mathbf{j}}}\Psi$ for every $\mathbf{j}\in J_{\mathbf{l},\mathbf{i}}$. However, then, the second equality reads $\Psi = \prod_{\mathbf{j}\in J_{\mathbf{l},\mathbf{i}}} P_{\Omega_{\mathbf{j}}}\Psi = 0$, which is what we wanted to prove. □

From Lemma A.1, we derive the following:

Corollary A.2. For any $J_{\mathbf{k},\mathbf{q}}$, the following inequality holds:

$$\sum_{\mathbf{i}\,:\,J_{\mathbf{l},\mathbf{i}}\subset J_{\mathbf{k},\mathbf{q}}} P^{(+)}_{J_{\mathbf{l},\mathbf{i}}} \leq (l+1)^d \sum_{\mathbf{j}\in J_{\mathbf{k},\mathbf{q}}} P^{\perp}_{\Omega_{\mathbf{j}}} \tag{A4}$$

with $l = |\mathbf{l}|$.

Proof. For fixed $\mathbf{l}$, we sum the left-hand side of the inequality (see Lemma A.1),

$$P^{(+)}_{J_{\mathbf{l},\mathbf{i}}} \leq \sum_{\mathbf{j}\in J_{\mathbf{k},\mathbf{q}}} P^{\perp}_{\Omega_{\mathbf{j}}}, \tag{A5}$$

over all $J_{\mathbf{l},\mathbf{i}}$ contained in $J_{\mathbf{k},\mathbf{q}}$. Then, for each site $\mathbf{j}\in J_{\mathbf{k},\mathbf{q}}$, we get at most

$$(l_1+1)(l_2+1)\cdots(l_d+1)$$

terms of the type $P^{\perp}_{\Omega_j}$. Thus, the inequality in (A4) is proven. □

Lemma A.3. Assume $t>0$ sufficiently small, $\|V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}}\| \le 48\cdot \frac{t^{\frac{r-1}{3}}}{r^{x_d}}$ with $x_d = 20d$, and $\Delta_{J_{k,q}} \ge \frac{1}{2}$. Then, for any N and $(\mathbf{k},\mathbf{q})$, the inequalities

$$\|V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{k},\mathbf{q}}}\| \le 2\|V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}}\|, \tag{A6}$$

$$\|S_{J_{\mathbf{k},\mathbf{q}}}\| \le C\cdot t\cdot \|V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}}\|, \tag{A7}$$

and

$$\|\sum_{j=2}^{\infty} t^j (S_{J_{\mathbf{k},\mathbf{q}}})_j\| \le C\cdot t^2\cdot \|V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}}\|^2 \tag{A8}$$

hold true for a universal constant C.

Proof. We recall that

$$V^{(\mathbf{k},\mathbf{q})}_{J_{\mathbf{k},\mathbf{q}}} := \sum_{j=1}^{\infty} t^{j-1} (V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}})^{diag}_j \tag{A9}$$

and

$$S_{J_{\mathbf{k},\mathbf{q}}} := \sum_{j=1}^{\infty} t^j (S_{J_{\mathbf{k},\mathbf{q}}})_j \tag{A10}$$

with

$$(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}})_j \tag{A11}$$

$$:= \sum_{p\ge 2, v_1\ge 1\ldots, v_p\ge 1\,;\, v_1+\cdots+v_p=j} \frac{1}{p!}\mathrm{ad}\,(S_{J_{\mathbf{k},\mathbf{q}}})_{v_1}\Big(\mathrm{ad}\,(S_{J_{\mathbf{k},\mathbf{q}}})_{v_2}\cdots(\mathrm{ad}\,(S_{J_{\mathbf{k},\mathbf{q}}})_{v_p}(G_{J_{\mathbf{k},\mathbf{q}}}))\cdots\Big) \tag{A12}$$

$$+ \sum_{p\ge 1, v_1\ge 1\ldots, v_p\ge 1\,;\, v_1+\cdots+v_p=j-1} \frac{1}{p!}\mathrm{ad}\,(S_{J_{\mathbf{k},\mathbf{q}}})_{v_1}\Big(\mathrm{ad}\,(S_{J_{\mathbf{k},\mathbf{q}}})_{v_2}\cdots(\mathrm{ad}\,(S_{J_{\mathbf{k},\mathbf{q}}})_{v_p}(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}}))\cdots\Big) \tag{A13}$$

and

$$(S_{J_{\mathbf{k},\mathbf{q}}})_j := ad^{-1}\, G_{J_{\mathbf{k},\mathbf{q}}}\,((V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}})^{od}_j) := \frac{1}{G_{J_{\mathbf{k},\mathbf{q}}} - E_{J_{\mathbf{k},\mathbf{q}}}} P^{(+)}_{J_{\mathbf{k},\mathbf{q}}}\,(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}})_j\,P^{(-)}_{J_{\mathbf{k},\mathbf{q}}} - h.c. \tag{A14}$$

From (A14), we get

$$\mathrm{ad}\,(S_{J_{\mathbf{k},\mathbf{q}}})_{r_p}(G_{J_{\mathbf{k},\mathbf{q}}}) \tag{A15}$$

$$= \mathrm{ad}\,(S_{J_{\mathbf{k},\mathbf{q}}})_{r_p}(G_{J_{\mathbf{k},\mathbf{q}}} - E_{J_{\mathbf{k},\mathbf{q}}}) \tag{A16}$$

$$= \left[\frac{1}{G_{J_{\mathbf{k},\mathbf{q}}} - E_{J_{\mathbf{k},\mathbf{q}}}} P^{(+)}_{J_{\mathbf{k},\mathbf{q}}}\,(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}})_{r_p}\,P^{(-)}_{J_{\mathbf{k},\mathbf{q}}},\, G_{J_{\mathbf{k},\mathbf{q}}} - E_{J_{\mathbf{k},\mathbf{q}}}\right] + h.c. \tag{A17}$$

$$= -P^{(+)}_{J_{\mathbf{k},\mathbf{q}}}\,(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}})_{r_p}\,P^{(-)}_{J_{\mathbf{k},\mathbf{q}}} - P^{(-)}_{J_{\mathbf{k},\mathbf{q}}}\,(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}})_{r_p}\,P^{(+)}_{J_{\mathbf{k},\mathbf{q}}} \tag{A18}$$

and

$$\|(S_{J_{\mathbf{k},\mathbf{q}}})_j\| \le 2\frac{\|(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}})_j\|}{\Delta_{J_{\mathbf{k},\mathbf{q}}}} \le 4\|(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}})_j\| \tag{A19}$$

since we have assumed $\Delta_{J_{\mathbf{k},\mathbf{q}}} \ge \frac{1}{2}$. Next, using the definition in (A11), we can estimate

$$\|(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}})_j\| \tag{A20}$$

$$\leq \sum_{p=2}^{j}\frac{8^p}{p!}\sum_{\nu_1\geq 1\dots,\nu_p\geq 1\,;\,\nu_1+\dots+\nu_p=j}\|(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}})_{\nu_1}\|\|(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}})_{\nu_2}\|\cdots\|(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}})_{\nu_p}\| \tag{A21}$$

$$+2\|V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}}\|\sum_{p=2}^{j-1}\frac{8^p}{p!}\sum_{p\geq 1,\nu_1\geq 1\dots,\nu_p\geq 1\,;\,\nu_1+\dots+\nu_p=j-1}\|(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}})_{\nu_1}\|\|(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}})_{\nu_2}\|\cdots\|(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}})_{\nu_p}\|. \tag{A22}$$

In order to estimate (A20), we refer to Theorem 3.2 in Ref. 28; hence, we consider the numbers B_j, $j\geq 1$, recursively defined by

$$B_1 := \|V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}}\|, \tag{A23}$$

$$B_j := \frac{1}{a}\sum_{l=1}^{j-1}B_{j-l}B_l, \quad j\geq 2, \tag{A24}$$

with a such that

$$\frac{e^{8a}-8a-1}{a}+e^{8a}-1=1. \tag{A25}$$

Following Ref. 28, by induction, we get

$$\|(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}})_j\|\leq B_j\left(\frac{e^{8a}-8a-1}{a}\right)+2\|V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}}\|B_{j-1}\left(\frac{e^{8a}-1}{a}\right) \tag{A26}$$

and

$$B_j\geq\frac{2B_{j-1}\|V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}}\|}{a} \tag{A27}$$

that combined with (A25) yields

$$B_j\geq\|(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}})_j\|. \tag{A28}$$

The numbers B_j are seen to be Taylor's coefficients of

$$f(u) := \frac{a}{2}\cdot\left(1-\sqrt{1-\frac{4}{a}\cdot\|V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}}\|u}\right). \tag{A29}$$

See Ref. 28. Therefore, if we consider the norms $\|(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}})^{diag}_j\|$ as u-independent, the radius of analyticity, t_0, of

$$\sum_{j=1}^{\infty}u^{j-1}\|(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}})^{diag}_j\| = \frac{1}{u}\left(\sum_{j=1}^{\infty}u^j\|(V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}})^{diag}_j\|\right) \tag{A30}$$

is bounded below by that of $\sum_{j=1}u^jB_j$; hence,

$$t_0\geq\frac{a}{4\|V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}}\|}\geq\frac{a}{192}, \tag{A31}$$

where, in the last inequality, we use the assumption on $\|V^{(\mathbf{k},\mathbf{q})_{-1}}_{J_{\mathbf{k},\mathbf{q}}}\|$. The same bound holds for the radius of convergence of the series $S_{J_{\mathbf{k},\mathbf{q}}} := \sum_{j=1}^{\infty}t^j(S_{J_{\mathbf{k},\mathbf{q}}})_j$ as a consequence of inequality in (A19).

For $0<t<1$ and in the interval $(0,\frac{1}{2}\cdot\frac{a}{192})$ due to (A28) and (A29),

$$\sum_{j=1}^{\infty} t^{j-1} \| (V_{I_{k,q}}^{(\mathbf{k},\mathbf{q})_{-1}})_j^{diag} \| \le \frac{1}{t}\sum_{j=1}^{\infty} t^j B_j \tag{A32}$$

$$= \frac{1}{t}\cdot\frac{a}{2}\cdot\left(1-\sqrt{1-\left(\frac{4}{a}\cdot\|V_{I_{k,q}}^{(\mathbf{k},\mathbf{q})_{-1}}\|\right)t}\right) \tag{A33}$$

$$\le (1+C_a\cdot t)\,\|V_{I_{k,q}}^{(\mathbf{k},\mathbf{q})_{-1}}\| \tag{A34}$$

holds true for some a-dependent constant $C_a > 0$. This implies the inequality in (A6) by assuming $t > 0$ sufficiently small but independent of N, k, and q. Likewise, we derive (A7).

As for (A8), we start from

$$\|\sum_{j=2}^{\infty} t^j (S_{J_{\mathbf{k},\mathbf{q}}})_j\| \le \sum_{j=2}^{\infty} t^j \|(S_{J_{\mathbf{k},\mathbf{q}}})_j\| \le 4\sum_{j=2}^{\infty} t^j \|(V_{J_{\mathbf{k},\mathbf{q}}}^{(\mathbf{k},\mathbf{q})_{-1}})_j\| \le 4\sum_{j=2}^{\infty} t^j B_j. \tag{A35}$$

Then, using $B_1 \equiv \|V_{J_{\mathbf{k},\mathbf{q}}}^{(\mathbf{k},\mathbf{q})_{-1}}\|$ and a Taylor expansion, for t in the interval where (A34) holds, we estimate

$$\sum_{j=2}^{\infty} t^j B_j = \frac{a}{2}\cdot\left(1-\sqrt{1-\frac{4}{a}\cdot\|V_{J_{\mathbf{k},\mathbf{q}}}^{(\mathbf{k},\mathbf{q})_{-1}}\|t}\right) - t\cdot\|V_{J_{\mathbf{k},\mathbf{q}}}^{(\mathbf{k},\mathbf{q})_{-1}}\| \tag{A36}$$

$$\le D_a\cdot t^2\cdot\|V_{J_{\mathbf{k},\mathbf{q}}}^{(\mathbf{k},\mathbf{q})_{-1}}\|^2, \tag{A37}$$

where D_a only depends on a. □

Lemma A.4. Let $\{J_{\mathbf{s}^{(i)},\mathbf{u}^{(i)}}\,;\,|\mathbf{s}^{(i)}| = k, i\in\{1\,\ldots n\}\}$ *for some* $k\in\mathbb{N}$ *be such that* $\cup_i J_{\mathbf{s}^{(i)},\mathbf{u}^{(i)}}$ *is connected. Then, there is a closed path (see Definition* 3.3) γ_k *with* $supp(\gamma_k) = \{J_{\mathbf{s}^{(i)},\mathbf{u}^{(i)}}\,;\,|\mathbf{s}^{(i)}| = k, i\in\{1\,\ldots n\}\}$ *and length* $l_{\gamma_k} = 2n-2$.

Proof. We proceed by induction on the number, n, of elements of a collection of rectangles as in the statement. The statement is clearly true for $n = 2$ rectangles. We assume that it holds for collections of $n \ge 2$ rectangles and prove it continues to hold for collections of $n + 1$ rectangles as well. Given any such collection, without loss of generality, we can suppose that the union of the first n rectangles is still a connected set of $\mathbb{R}^d$. Clearly, the $n + 1$th rectangle must intersect at least one of the previous n rectangles. We can, then, pick one of those rectangles, say, $J_{\mathbf{k}^{(i_*)},\mathbf{q}^{(i_*)}}$, from the set and consider the step

$$(J_{\mathbf{k}^{(i_*)},\mathbf{q}^{(i_*)}}, J_{\mathbf{k}^{(n+1)},\mathbf{q}^{(n+1)}})$$

and the step back $(J_{\mathbf{k}^{(n+1)},\mathbf{q}^{(n+1)}}, J_{\mathbf{k}^{(i_*)},\mathbf{q}^{(i_*)}})$. Hence, we get a path with two more steps and enjoying the required properties. □

Lemma A.5. For $\mathfrak{b}\in\mathcal{B}_{V_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})}}$, *assume that*

$$\cup_{i\in\{1,\ldots,|\mathcal{R}_\mathfrak{b}|\}} J_{\mathbf{k}^{(i)},\mathbf{q}^{(i)}} = \cup_{\rho=k_0}^{k}\cup_{j=1}^{j_\rho}\mathcal{Z}_\rho^{(j)},$$

where $\{\mathcal{Z}_\rho^{(j)},\quad j = 1,\ldots,j_\rho\}$ *are distinct connected components of (unions of) rectangles of same size* ρ. *Then, there is a path,* $\Gamma_\mathfrak{b}$, *of length* $l_{\Gamma_\mathfrak{b}}$ *such that*

$$l_{\Gamma_\mathfrak{b}} \le 2\left(n_{k_0} + \sum_{j=1}^{j_2} n_{k_0+1}^{(j)} + \cdots + \sum_{j=1}^{j_k} n_k^{(j)}\right) - 2,$$

where $n_\rho^{(j)} := |\mathrm{supp}(\mathcal{Z}_\rho^{(j)})|$, *with the following properties:*

- (A) *the support of* $\Gamma_\mathfrak{b}$ *is* $\mathcal{R}_\mathfrak{b}$;
- (B) *for each component* $\mathcal{Z}_\rho^{(j)}$ *consisting of the union of* $n_\rho^{(j)}$ *rectangles, at most* $2n_\rho^{(j)} - 2$ *steps are implemented [i.e., there are at most* $2n_\rho^{(j)} - 2$ *steps* $\sigma\in\mathcal{S}_{\Gamma_\mathfrak{b}}$ *for which* $\sigma\in\mathrm{supp}(\mathcal{Z}_\rho^{(j)})\times\mathrm{supp}(\mathcal{Z}_\rho^{(j)})$*];*
- (C) *there are at most two steps connecting rectangles in* $\mathrm{supp}(\mathcal{Z}_\rho^{(j)})$ *with rectangles of lower size: more precisely, for every connected component* $\mathcal{Z}_\rho^{(j)}$, *there is at most one* $J_{\mathbf{s},\mathbf{u}}$ *in* $\mathrm{supp}(\mathcal{Z}_\rho^{(j)})$ *such that* $(J_{\mathbf{s}',\mathbf{u}'}, J_{\mathbf{s},\mathbf{u}})\in\mathcal{S}_{\Gamma_\mathfrak{b}}$ *with* $s' < s$ *and one* $J_{\mathbf{s},\mathbf{u}}$ *such that* $(J_{\mathbf{s},\mathbf{u}}, J_{\mathbf{s}',\mathbf{u}'})\in\mathcal{S}_{\Gamma_\mathfrak{b}}$ *with* $s < s'$.

Proof. The construction is by induction in the size, k, of the rectangles. We call $\gamma_\rho^{(j)}$ the closed path that visits the rectangles of the component $\mathcal{Z}_\rho^{(j)}$ and constructed according to Lemma A.4. Hence, for $k = k_0$, we just refer to Lemma A.4. Note that property (c) does not apply for $k = k_0$.

Next, we assume that we have constructed the path, say, $\Gamma_b^{(k'-1)}$, with $k_0 + 1 \leq k' \leq k$, fulfilling (A)–(C) for the set $\cup_{\rho=k_0}^{k'-1} \cup_{j=1}^{j_\rho} \mathcal{Z}_\rho^{(j)}$, which is connected by property (P-i). Then, from this path, we derive a new one, which we call $\Gamma_b^{(k')}$, with the desired properties for the set $\cup_{\rho=k_0}^{k'} \cup_{j=1}^{j_\rho} \mathcal{Z}_\rho^{(j)}$. The path is constructed using the following prescriptions:

- we follow $\Gamma_{\mathfrak{b}}^{(k'-1)}$ until it reaches a rectangle that has an overlap with a rectangle of one of the components $\mathcal{Z}_{k'}^{(j)}$; then, we implement a "turning step" that means we stop proceeding along $\Gamma_{\mathfrak{b}}^{(k'-1)}$ and start to follow the closed path $\gamma_{k'}^{(j)}$ along the component $\mathcal{Z}_{k'}^{(j)}$ by starting and ending at the rectangle of the turning step;
- we proceed in the same way along the remaining part of the path $\Gamma_{\mathfrak{b}}^{(k'-1)}$, which means we implement a turning step as soon as the rectangle that has been reached has an overlap with another component, say, $\mathcal{Z}_{k'}^{(j')}$, not visited yet;
- we iterate this procedure until all the components $\mathcal{Z}_{k'}^{(j)}$ have been visited, and the path $\Gamma_{\mathfrak{b}}^{(k'-1)}$ has been completed to the initial rectangle. □

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[27] For example, in dimension $d = 2$, in order to determine the successor of $(\mathbf{k}, \mathbf{q}) = (k_1, k_2;\ q_1, q_2)$ we observe that we have the following: (a) The elements $(k_1, k_2;\ q_1 + 1, q_2)$ and $(k_1, k_2;\ q_1, q_2 + 1)$ are both successors of $(k_1, k_2;\ q_1, q_2)$ but $(k_1, k_2;\ q_1, q_2 + 1) \succ (k_1, k_2;\ q_1 + 1, q_2)$. (b) For the elements $(k_1', k_2'; q_1', q_2')$, $(k_1'', k_2''; q_1'', q_2'')$ such that $k_1' + k_2' = k_1'' + k_2'' = k_1 + k_2$, if $k_1' > k_1''$, then $(k_1'', k_2''; q_1'', q_2'') \succ (k_1', k_2'; q_1', q_2')$.

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[29] By shape, we mean an equivalence class of rectangles that can be obtained from one another by translation on the lattice.

[30] The factor $2d(r+1)^{d-1}$ is an upper bound to the number of sites that sit on one of the faces of the rectangle $J_{\mathbf{r},\mathbf{i}}$. By definition, the rectangles in $\mathcal{G}_{J_{\mathbf{r},\mathbf{i}}}^{(\mathbf{k},\mathbf{q})}$ have non-empty intersection with at least one of the faces of $J_{\mathbf{r},\mathbf{i}}$.

[31] It is enough to consider the volume of the rectangle $J_{\mathbf{r},\mathbf{i}}$ and Remark 4.2.

[32] Note that, for $1 \leq j \leq l$, if $q_j \neq i_j$ and $q_j \neq i_j + k_j$, then k_j' must coincide with r_j; thus, q_j' is fixed. Otherwise, if $q_j = i_j$ or $q_j = i_j + k_j$, then, for fixed k_j', $q_j' = i_j + r_j - k_j'$ or $q_j' = i_j$, respectively.