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On some hyperbolic equations of third order

Ferruccio Colombini[1] · Todor Gramchev[2] · Nicola Orrù[3] · Giovanni Taglialatela[4]

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Abstract
We give sufficient conditions for the well posedness in $\mathcal{C}^\infty$ of the Cauchy problem for third-order equations with time-dependent coefficients.

Keywords Hyperbolic equations · Logarithmic conditions · Levi conditions

Mathematics Subject Classification 35L30 (primary) · 35B45 (secondary)

1 Introduction

In this paper, we study the Cauchy Problem in $\mathcal{C}^\infty$ for some weakly hyperbolic equation of third order. We are interested in Levi conditions, that is in conditions on lower-order terms which ensure the well posedness of the Cauchy Problem.

In the case of strictly hyperbolic equations of order m (that is when the characteristic roots are real and distinct, m is a natural number), Petrowski [30] (see also [24]) proved well posedness of the Cauchy Problem in $\mathcal{C}^\infty$ for any lower-order term. Then, Oleinik [28] studied weakly hyperbolic equations of second order (that is the two characteristic roots are real but may coincide) with $\mathcal{C}^\infty$ coefficients and lower-order terms and gave some sufficient conditions for well posedness of the Cauchy Problem. Nishitani

Deceased: Todor Gramchev.

This paper was started before the demise of T. Gramchev and carried out in his memory.

✉ Giovanni Taglialatela
giovanni.taglialatela@uniba.it

Ferruccio Colombini
ferruccio.colombini@unipi.it

Nicola Orrù
orru.nicola@yahoo.it

1 Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy

2 Dipartimento di Matematica e Informatica, via Ospedale 72, 09124 Cagliari, Italy

3 Liceo Scientifico A. Pacinotti, Via Liguria 9, 09127 Cagliari, Italy

4 Dipartimento di Economia e Finanza, Università di Bari "Aldo Moro", Largo Abbazia S. Scolastica, 70124 Bari, Italy

[25] found necessary and sufficient conditions for second-order equations when there is only one space variable and the coefficients are analytic. In the papers [6] and [7], some second-order hyperbolic equations with coefficients depending only on t (in many space variables) were studied. They studied the Cauchy problem both in $\mathcal{C}^\infty$ and in Gevrey classes. They studied the case of $\mathcal{C}^\infty$ and of analytic coefficients and gave some sufficient conditions on the lower-order terms for the well posedness of the Cauchy problem (we call them *logarithmic conditions*). We generalize these conditions to equations of third order with time-dependent non-smooth coefficients.

Although there are many papers on higher-order equations, only few general result has been obtained (for a general framework see, e.g., [24]). We recall that a necessary and sufficient condition for the $\mathcal{C}^\infty$ well posedness has been obtained only for few classes of operators, in particular operators with constant coefficients [16, 17, 34] (or, more generally operators whose principal part has constant coefficients [14, 36]) and operators with characteristics of constant multiplicities [5, 13, 15].

Concerning third-order operators, which is the main topic of the present paper, many paper treat this subject (see, e.g., [1–4, 8, 10, 21–23, 26, 27, 37]). However, no completely satisfactory result has yet been obtained.

In particular, Wakabayashi [37] has studied this problem, obtaining results similar to ours. He considers operators with double characteristics and operators of third order. In this case, his sub-principal symbol is the same as ours, while the sub-sub-principal symbol (of order 1) is different. His conditions on lower-order terms are similar to ours in one space variable. In general, his conditions imply our logarithmic conditions. There are two important differences between our works: Wakabayashi supposes that the coefficients of the equations are analytic, while we suppose that they are $\mathcal{C}^2$ and his conditions are pointwise while our conditions are integral. We do not know if they are equivalent in many space variables, when the coefficients are analytic. Finally, our conditions can be expressed simply in terms of the coefficients of the operator (at least if the coefficients are analytic).

In [8], it is considered homogeneous higher-order operators with coefficients depending only on the time variable and finite degeneracy; and a necessary and sufficient condition for the $\mathcal{C}^\infty$ well posedness is stated. This result has been extended in [10], to non-homogeneous equations, and in [32] and [33] to equations with principal part depending *only* on one space variable.

Recently, Nishitani [26, Theorem 6.1] considered third-order equations with analytic coefficients in one space variable and generalizes the results of [8] and [32] to equations whose principal symbol depends on both t and x. We compare these results with our Theorem 2 in Example 3 of Sect. 7.

We have the following results:

Theorem 1 ([7, 11, 12]) *Let us consider a second-order equation*

$$\partial_t^2 u + \sum_{j=1}^d a_j(t)\partial_t\partial_{x_j} u + \sum_{j,h=1}^d b_{jh}(t)\partial_{x_j}\partial_{x_h} u + c_0(t)\partial_t u + \sum_{j=1}^d c_j(t)\partial_{x_j} u + d(t)u = f\,.$$

Suppose that the coefficients are real and $\mathcal{C}^\infty$ in t and do not depend on the space variables.

Suppose that the symbol of the principal part

$$L(t,\tau,\xi) = \tau^2 + a(t,\xi)\tau + b(t,\xi)\,,$$

where $a(t,\xi)=\sum_{j=1}^{d}a_j(t)\xi_j$, $b(t,\xi)=\sum_{j,h=1}^{d}b_{jh}(t)\xi_j\xi_h$ *has real zeros in* τ *for any* $\xi\in\mathbb{R}^d$, $t\in[0,T]$:

$$\Delta(t,\xi)=a^2(t,\xi)-4b(t,\xi)\geqslant 0$$

(*weak hyperbolicity*).

If

$$\int_0^T \frac{\left|\partial_t\Delta(t,\xi)\right|}{\Delta(t,\xi)+1}\,\mathrm{d}t\leqslant C\log|\xi| \tag{1.1}$$

for any $\xi\in\mathbb{R}^n$ *with* $|\xi|\geqslant C_1>1$ (*this condition is automatically satisfied if the coefficients are analytic*), *and if*

$$\int_0^T \frac{\left|-1/2c_0(t)\sum_{j=1}^{n}a_j(t)\xi_j+\sum_{j=1}^{n}c_j(t)\xi_j-1/2\sum_{j=1}^{n}\partial_t a_j(t)\xi_j\right|}{\sqrt{\Delta(t,\xi)+1}}\,\mathrm{d}t\leqslant C\log|\xi| \tag{1.2}$$

for any ξ *with* $|\xi|\geqslant C_1>1$ (*logarithmic condition*), *then the Cauchy problem is well posed in* $\mathcal{C}^\infty$.

Now, we consider a third-order equation

$$\partial_t^3 u+\sum_{j=0}^{2}\sum_{|\alpha|\leq 3-j}a_{j,\alpha}(t)\partial_t^j\partial_x^\alpha u=f\,, \tag{1.3}$$

with initial conditions

$$u(0,x)=u_0\,,\quad \partial_t u(0,x)=u_1\,,\quad \partial_t^2 u(0,x)=u_2\,. \tag{1.4}$$

Let

$$\begin{aligned}
L(t,\tau,\xi) &\overset{\text{def}}{=} \tau^3+\sum_{j+|\alpha|=3}a_{j,\alpha}(t)\tau^j\xi^\alpha,\\
M(t,\tau,\xi) &\overset{\text{def}}{=} \sum_{j+|\alpha|=2}a_{j,\alpha}(t)\tau^j\xi^\alpha,\\
N(t,\tau,\xi) &\overset{\text{def}}{=} \sum_{j+|\alpha|=1}a_{j,\alpha}(t)\tau^j\xi^\alpha,\\
p(t) &\overset{\text{def}}{=} a_{0,0}(t),
\end{aligned}$$

so that Eq. (1.3) can be rewritten as

$$L(t,\partial_t,\partial_x)u+M(t,\partial_t,\partial_x)u+N(t,\partial_t,\partial_x)u+p(t)u=f\,.$$

We assume that the coefficients of L belong to $\mathcal{C}^2\big([0,T]\big)$, those of M and N belong to $\mathcal{C}^1\big([0,T]\big)$, whereas $p(t)$ belongs to $L^\infty\big([0,T]\big)$.

The principal part $L(t,\tau,\xi)$, as a polynomial in τ, has only real roots:

$$\tau_1(t,\xi)\leqslant\tau_2(t,\xi)\leqslant\tau_3(t,\xi)$$

for any t,ξ, (weak hyperbolicity). This is equivalent to say that the *discriminant of* L is nonnegative:

$$\begin{aligned}\Delta_L(t,\xi) &\stackrel{\text{def}}{=} \big(\tau_1(t,\xi)-\tau_2(t,\xi)\big)^2\big(\tau_2(t,\xi)-\tau_3(t,\xi)\big)^2\big(\tau_3(t,\xi)-\tau_1(t,\xi)\big)^2\\ &= A_1^2(t,\xi)A_2^2(t,\xi) - 4A_2^3(t,\xi) - 4A_1^3(t,\xi)A_3(t,\xi)\\ &\quad + 18A_1(t,\xi)A_2(t,\xi)A_3(t,\xi) - 27A_3^2(t,\xi) \geqslant 0,\end{aligned}$$

where

$$A_j(t,\xi) \stackrel{\text{def}}{=} \sum_{|\alpha|=3-j} a_{j,\alpha}(t)\,\xi^\alpha\,. \tag{1.5}$$

We set also

$$\begin{aligned}\Delta_L^{(1)}(t,\xi) &\stackrel{\text{def}}{=} \big(\tau_1(t,\xi)-\tau_2(t,\xi)\big)^2 + \big(\tau_2(t,\xi)-\tau_3(t,\xi)\big)^2 + \big(\tau_3(t,\xi)-\tau_1(t,\xi)\big)^2\\ &= 2\,[A_1^2(t,\xi) - 3A_2(t,\xi)]\,.\end{aligned}$$

Note that, if $\Delta_L(\bar t,\bar\xi)=0$ and $\Delta_L^{(1)}(\bar t,\bar\xi)\neq 0$, then L has a double root (for example $\tau_1(\bar t,\bar\xi)=\tau_2(\bar t,\bar\xi)$ and $\tau_1(\bar t,\bar\xi)\neq\tau_3(\bar t,\bar\xi)$), whereas if $\Delta_L^{(1)}(\bar t,\bar\xi)=0$, then L has a triple root: $\tau_1(\bar t,\bar\xi)=\tau_2(\bar t,\bar\xi)=\tau_3(\bar t,\bar\xi)$.

Note that, (cf. Lemma A.1)

$$\Delta_L^{(1)}(t,\xi) = \frac{9}{2}\,\Delta_{\partial L}(t,\xi)\,,$$

where

$$\Delta_{\partial L}(t,\xi) = \big(\sigma_1(t,\xi)-\sigma_2(t,\xi)\big)^2$$

is the discriminant of the polynomial

$$\partial_\tau L(t,\tau,\xi) \stackrel{\text{def}}{=} 3\,\tau^2 + 2\,A_1(t,\xi)\,\tau + A_2(t,\xi) = 3\,\big(\tau-\sigma_1(t,\xi)\big)\,\big(\tau-\sigma_2(t,\xi)\big)\,.$$

Notations In the following, we note

$$\begin{aligned}\mathcal{S}_2 &= \Big\{\ (1,2)\,,\ (2,3)\,,\ (3,1)\ \Big\}\\ \mathcal{S}_3 &= \Big\{\ (1,2,3)\,,\ (2,3,1)\,,\ (3,1,2)\ \Big\}.\end{aligned}$$

Let $f(t,\xi)$ and $g(t,\xi)$ be positive functions, we will write $f\lesssim g$ (or, equivalently $g\gtrsim f$) to mean that there exists a positive constant C such that

$$f(t,\xi)\leqslant C\,g(t,\xi)\,,\quad \text{for any } (t,\xi)\in[0,T]\in\mathbb{R}^n\,.$$

Similarly, we will write $f\approx g$ to mean that $f\lesssim g$ and $g\lesssim f$.

These notations will make the formulas more readable and will allow us to focus only on the important terms of the estimates.

Consider the auxiliary polynomial

$$\mathcal{L}(t,\tau,\xi) \stackrel{\text{def}}{=} L(t,\tau,\xi) - \partial_t^2 L(t,\tau,\xi)\,, \tag{1.6}$$

we can prove (see Lemma 2.1 below) that its roots $\lambda_j(t,\xi)$ are real and distinct for $\xi \neq 0$, and, there exist positive constants C_1 and C_2 such that

$$\left|\lambda_j(t,\xi) - \tau_j(t,\xi)\right| \leqslant C_1,$$
$$\left|\lambda_j(t,\xi) - \lambda_k(t,\xi)\right| \geqslant C_2,$$

for all $(t,\xi) \in [0,T] \times \mathbb{R}^n \setminus \{0\}$ and $(j,k) \in \mathcal{S}_2$.

We denote by μ_1 and μ_2 the roots of $\partial_\tau \mathcal{L}(t,\tau,\xi)$.

Define the symbols

$$\check{M}(t,\tau,\xi) \overset{\text{def}}{=} M(t,\tau,\xi) - \frac{1}{2}\partial_t\partial_\tau L(t,\tau,\xi), \tag{1.7}$$

$$\check{N}(t,\tau,\xi) \overset{\text{def}}{=} N(t,\tau,\xi) - \frac{1}{2}\partial_t\partial_\tau M(t,\tau,\xi) + \frac{1}{12}\partial_t^2\partial_\tau^2 L(t,\tau,\xi)\,. \tag{1.8}$$

We can now state our main result.

Theorem 2 *Assume that*

$$\int_0^T \sum_{(j,k)\in\mathcal{S}_2} \frac{\left|\partial_t\lambda_j(t,\xi) - \partial_t\lambda_k(t,\xi)\right|}{\left|\lambda_j(t,\xi) - \lambda_k(t,\xi)\right|}\,\mathrm{d}t \lesssim \log\bigl(1+|\xi|\bigr)\,, \tag{1.9}$$

$$\int_0^T \sum_{(j,k)\in\mathcal{S}_2} \frac{\left|\partial_t^2\lambda_j(t,\xi) - \partial_t^2\lambda_k(t,\xi)\right|}{\left|\partial_t\lambda_j(t,\xi) - \partial_t\lambda_k(t,\xi)\right|+1}\,\mathrm{d}t \lesssim \log\bigl(1+|\xi|\bigr)\,, \tag{1.10}$$

$$\int_0^T \sum_{j=1}^{3} \frac{\left|\partial_t\check{M}\bigl(t,\lambda_j(t,\xi),\xi\bigr)\right|}{\left|\check{M}\bigl(t,\lambda_j(t,\xi),\xi\bigr)\right|+1}\,\mathrm{d}t \lesssim \log\bigl(1+|\xi|\bigr)\,, \tag{1.11}$$

$$\int_0^T \sum_{j=1}^{2} \frac{\left|\partial_t\check{N}\bigl(t,\mu_j(t,\xi),\xi\bigr)\right|}{\left|\check{N}\bigl(t,\mu_j(t,\xi),\xi\bigr)\right|+1}\,\mathrm{d}t \lesssim \log\bigl(1+|\xi|\bigr)\,, \tag{1.12}$$

$$\int_0^T \sum_{(j,k,l)\in\mathcal{S}_3} \frac{\left|\check{M}\bigl(t,\lambda_j(t,\xi),\xi\bigr)\right|}{\left|\lambda_j(t,\xi) - \lambda_k(t,\xi)\right|\cdot\left|\lambda_j(t,\xi) - \lambda_l(t,\xi)\right|}\,\mathrm{d}t \lesssim \log\bigl(1+|\xi|\bigr)\,, \tag{1.13}$$

$$\int_0^T \sum_{j=1}^{2} \sqrt{\frac{\left|\check{N}\bigl(t,\mu_j(t,\xi),\xi\bigr)\right|}{\left|\mu_2(t,\xi) - \mu_1(t,\xi)\right|}}\,\mathrm{d}t \lesssim \log\bigl(1+|\xi|\bigr)\,, \tag{1.14}$$

Then, the Cauchy problem (1.3, 1.4) *is well posed in* $\mathcal{C}^\infty$.

In the following, we will say that a function $f(t,\xi)$ verifies the *logarithmic condition* if

$$\int_0^T \big|f(t,\xi)\big|\,\mathrm{d}t \lesssim \log\big(1+|\xi|\big)\,.$$

Remark 1.1 Conditions (1.9)–(1.12) are hypothesis on the regularity of the coefficients. Indeed if the coefficients are analytic, then they are satisfied, see Sect. 4.

Conditions (1.13) and (1.14) are Levi conditions on the lower-order terms. They are necessary if the coefficients of the principal symbols are constant, see Sect. 6.

Remark 1.2 The hypothesis in Theorem 2 can be expressed in terms of the coefficients of the operator. This is possible either by expliciting the roots of $\mathcal{L}$ and $\partial_\tau\mathcal{L}$, or by transforming Hypothesis (1.9)–(1.14) into symmetric rational functions of the roots of $\mathcal{L}$. This will be developed in Sect. 3.

The plan of the paper is the following. In Sect. 2, we will prove Theorem 2. In Sect. 3, we give some different forms of the Levi conditions (1.13) and (1.14). In Sect. 4, we will show that if the coefficients are analytic, then (1.9)–(1.12) are satisfied. In Sect. 5, we give some sufficient pointwise conditions that are equivalent to ours in space dimension $n = 1$. We show also that the Levi conditions (1.13) and (1.14) are equivalent to the condition of *good decomposition* [13], which is necessary and sufficient for the well posedness for operators with characteristics of constant multiplicities [5, 13, 15]. In Sect. 6, we show that the Levi conditions (1.13) and (1.14) are necessary for the $\mathcal{C}^\infty$ well posedness if the coefficients of the principal symbols are constant. Finally, in Sect. 7, we give some examples.

In the proofs, for the sake of simplicity, we will omit the dependence on t and ξ in the notations.

2 Proof of theorem 2

Lemma 2.1 ([19]) *Consider the polynomial*

$$L_\varepsilon(t,\tau,\xi) = L(t,\tau,\xi) - \varepsilon^2|\xi|^2\partial_\tau^2 L(t,\tau,\xi)\,.$$

Its roots $\tau_{j,\varepsilon}(t,\xi)$ are real and distinct, moreover,

$$\big|\tau_{j,\varepsilon} - \tau_j\big| \lesssim \varepsilon|\xi|, \qquad j = 1,2,3,$$
$$\big|\tau_{j,\varepsilon} - \tau_{k,\varepsilon}\big| \gtrsim \varepsilon|\xi|, \qquad (j,k) \in \mathcal{S}_2\,.$$

Remark 2.2 By direct calculation (cf. [21, par. 1020]), the discriminant of L_ε is given by

$$\Delta_{L_\varepsilon} = \Delta_L + \frac{1}{2}\varepsilon^2|\xi|^2\,\Delta^2_{\partial_\tau L} + 36\,\varepsilon^4|\xi|^4\,\Delta_{\partial_\tau L} + 864\,\varepsilon^6|\xi|^6\,,$$

where $\Delta_{\partial_\tau L}$ is the discriminant of the polynomial $\partial_\tau L$.

Similarly,

$$\Delta_{\partial_\tau L_\varepsilon} = \Delta_{\partial_\tau L} + 72\,\varepsilon^2|\xi|^2\,. \tag{2.1}$$

We will take $\varepsilon = 1/|\xi|$.

We consider

$$
\begin{aligned}
L_{j,\varepsilon}(t,\tau,\xi) &= \tau - i\tau_{j,\varepsilon}(t,\xi), && j=1,2,3,\\
L_{jk,\varepsilon}(t,\tau,\xi) &= \big(\tau - i\tau_{j,\varepsilon}(t,\xi)\big)\big(\tau - i\tau_{k,\varepsilon}(t,\xi)\big)\,, && (j,k)\in\mathcal{S}_2\,,\\
L_{123,\varepsilon}(t,\tau,\xi) &= \big(\tau - i\tau_{1,\varepsilon}(t,\xi)\big)\big(\tau - i\tau_{2,\varepsilon}(t,\xi)\big)\big(\tau - i\tau_{3,\varepsilon}(t,\xi)\big)\\
&= L_\varepsilon(t,\tau,i\xi) = L(t,\tau,i\xi) + \varepsilon^2|\xi|^2\partial_\tau^2 L(t,\tau,i\xi)\,.
\end{aligned}
$$

We define also the operators

$$
\widetilde{L}_{jh,\varepsilon}(t,\partial_t,\xi) = \frac{1}{2}\left[L_{j,\varepsilon}(t,\partial_t,\xi)\circ L_{h,\varepsilon}(t,\partial_t,\xi) + L_{h,\varepsilon}(t,\partial_t,\xi)\circ L_{j,\varepsilon}(t,\partial_t,\xi)\right], \tag{2.2}
$$

for any $(j,h)\in\mathcal{S}_2$, and

$$
\widetilde{L}_{123,\varepsilon}(t,\partial_t,\xi) = \frac{1}{6}\sum_{\substack{j,h,l=1,2,3\\ j\neq h,\, j\neq l,\, h\neq l}} L_{j,\varepsilon}(t,\partial_t,\xi)\circ L_{h,\varepsilon}(t,\partial_t,\xi)\circ L_{l,\varepsilon}(t,\partial_t,\xi)\,. \tag{2.3}
$$

If $\varepsilon = 0$, we will write L_j, L_{jh}, $\widetilde{L}_{jh}$, ..., instead of $L_{j,0}$, $L_{jh,0}$, $\widetilde{L}_{jh,0}$, ...

Lemma 2.3 *For any $(j,h)\in\mathcal{S}_2$, we have*

$$
\widetilde{L}_{jh,\varepsilon} = L_{jh,\varepsilon} + \frac{1}{2}\,\partial_t\partial_\tau L_{jh,\varepsilon}\,, \tag{2.4}
$$

and

$$
L_{j,\varepsilon}\circ L_{h,\varepsilon} - \widetilde{L}_{jh,\varepsilon} = \frac{i}{2}\,(\tau'_{j,\varepsilon} - \tau'_{h,\varepsilon})\,. \tag{2.5}
$$

Proof As

$$
L_{j,\varepsilon}\circ L_{h,\varepsilon} = (\partial_t - i\tau_{j,\varepsilon})\circ(\partial_t - i\tau_{h,\varepsilon}) = L_{jh,\varepsilon} - i\tau'_{h,\varepsilon}\,, \tag{2.6}
$$

we have

$$
\begin{aligned}
&(\partial_t - i\tau_{j,\varepsilon})\circ(\partial_t - i\tau_{h,\varepsilon}) + (\partial_t - i\tau_{h,\varepsilon})\circ(\partial_t - i\tau_{j,\varepsilon})\\
&\quad = 2L_{jh,\varepsilon} - i(\tau'_{j,\varepsilon} + \tau'_{h,\varepsilon})\\
&\quad = 2L_{jh,\varepsilon} + (\partial_t\partial_\tau L_{jh,\varepsilon})\,,
\end{aligned} \tag{2.7}
$$

from which (2.4) follows.

Identity (2.5) follows from (2.6) and (2.7). □

Note that,

$$
L_{j,\varepsilon}v - L_{h,\varepsilon}v = -i(\tau_{j,\varepsilon} - \tau_{h,\varepsilon})v\,, \tag{2.8}
$$

whereas, from (2.7),

$$\begin{aligned}\widetilde{L}_{jh,\varepsilon} v - \widetilde{L}_{jl,\varepsilon} v &= L_{jh,\varepsilon} v - L_{jl,\varepsilon} v - \frac{i}{2}(\tau'_{j,\varepsilon} + \tau'_{h,\varepsilon}) v + \frac{i}{2}(\tau'_{j,\varepsilon} + \tau'_{l,\varepsilon}) v \\ &= -i(\tau_{h,\varepsilon} - \tau_{l,\varepsilon})\, L_{j,\varepsilon} v - \frac{i}{2}(\tau'_{h,\varepsilon} - \tau'_{l,\varepsilon}) v\,. \end{aligned} \tag{2.9}$$

Lemma 2.4 *We have*

$$\widetilde{L}_{123,\varepsilon} = L_{123,\varepsilon} + \frac{1}{2}\,\partial_t \partial_\tau L_{123,\varepsilon} + \frac{1}{6}\,\partial_t^2 \partial_\tau^2 L_{123,\varepsilon} \tag{2.10}$$

$$\begin{aligned} L_{1,\varepsilon} \circ L_{2,\varepsilon} \circ L_{3,\varepsilon} - \widetilde{L}_{123,\varepsilon} &= \frac{i}{2}(\tau'_1 - \tau'_2)\, L_{3,\varepsilon} + \frac{i}{2}(\tau'_2 - \tau'_3)\, L_{1,\varepsilon} - \frac{i}{2}(\tau'_3 - \tau'_1)\, L_{2,\varepsilon} \\ &\quad - \frac{1}{3}\, i\,(\tau''_3 - \tau''_1) - \frac{1}{3}\, i\,(\tau''_3 - \tau''_2) \end{aligned} \tag{2.11}$$

$$\widetilde{L}_{123,\varepsilon} - \widetilde{L}_{123,0} = 2\,\varepsilon^2 |\xi|^2 \sum_{j=1}^{3} L_{j,\varepsilon}\,. \tag{2.12}$$

Proof We have

$$\begin{aligned} L_{1,\varepsilon} \circ L_{2,\varepsilon} \circ L_{3,\varepsilon} &= (\partial_t - i\tau_{1,\varepsilon}) \circ (\partial_t - i\tau_{2,\varepsilon}) \circ (\partial_t - i\tau_{3,\varepsilon}) \\ &= L_{123,\varepsilon} + (-i\tau'_3) L_{1,\varepsilon} + \partial_t L_{23,\varepsilon} + (-i\tau''_3) \\ &= L_{123,\varepsilon} + (-i\tau'_3) L_{1,\varepsilon} + (-i\tau'_2) L_{3,\varepsilon} + (-i\tau'_3) L_{2,\varepsilon} + (-i\tau''_3)\,. \end{aligned} \tag{2.13}$$

Summing over all permutations, we get (2.10).

Identity (2.11) follows from (2.13).

As (2.10) with $\varepsilon = 0$ gives

$$\widetilde{L}_{123,0} = L + \frac{1}{2}\,\partial_t \partial_\tau L + \frac{1}{6}\,\partial_t^2 \partial_\tau^2 L\,, \tag{2.14}$$

we get

$$\begin{aligned} \widetilde{L}_{123,\varepsilon} - \widetilde{L}_{123,0} &= L_{123,\varepsilon} - L + \frac{1}{2}\,\partial_t \partial_\tau [L_{123,\varepsilon} - L] + \frac{1}{6}\,\partial_t^2 \partial_\tau^2 [L_{123,\varepsilon} - L] \\ &= \varepsilon^2 |\xi|^2 \partial_\tau^2 L = \varepsilon^2 |\xi|^2 \partial_\tau^2 L_{123,\varepsilon} = 2\,\varepsilon^2 |\xi|^2 \sum_{j=1}^{3} L_{j,\varepsilon}\,. \end{aligned}$$

□

We define an energy, after a Fourier transform with respect to the variables x ($v = \mathcal{F}_x u$):

$$E(t,\xi) \overset{\text{def}}{=} k(t,\xi) \left[\sum_{(j,h)\in \mathcal{S}_2} |\widetilde{L}_{jh,\varepsilon} v|^2 + \mathcal{H}^2(t,\xi) \left[\sum_{j=1}^{3} |L_{j,\varepsilon} v|^2 + |v|^2 \right] \right],$$

where the weight $k(t,\xi)$ is defined by

$$k(t,\xi) \stackrel{\text{def}}{=} \exp\left[-\eta \int_{-T}^{t} \mathcal{K}(s,\xi)\, ds\right],$$

with

$$\begin{aligned}
\mathcal{K}(t,\xi) \stackrel{\text{def}}{=} & \sum_{(j,h)\in\mathcal{S}_2} \frac{|\tau'_{j,\varepsilon}-\tau'_{h,\varepsilon}|}{|\tau_{j,\varepsilon}-\tau_{h,\varepsilon}|} + \sum_{(j,h)\in\mathcal{S}_2} \frac{|\tau''_{j,\varepsilon}-\tau''_{h,\varepsilon}|}{|\tau'_{j,\varepsilon}-\tau'_{h,\varepsilon}|+1} \\
& + \sum_{j=1}^{3} \frac{\left|\partial_t \check{M}(\tau_{j,\varepsilon})\right|}{\left|\check{M}(\tau_{j,\varepsilon})\right|+1} + \sum_{j=1}^{2} \frac{\left|\partial_t \check{N}(\sigma_{j,\varepsilon})\right|}{\left|\check{N}(\sigma_{j,\varepsilon})\right|+1} \\
& + \sum_{(j,h,l)\in\mathcal{S}_3} \frac{\left|\check{M}(\tau_{j,\varepsilon})\right|}{|\tau_{j,\varepsilon}-\tau_{h,\varepsilon}|\cdot|\tau_{j,\varepsilon}-\tau_{l,\varepsilon}|} + \sum_{j=1}^{2} \sqrt{\frac{\left|\check{N}(\sigma_{1,\varepsilon})\right|}{|\sigma_{2,\varepsilon}-\sigma_{1,\varepsilon}|}} + \log|\xi|, \\
\mathcal{H}(t,\xi) \stackrel{\text{def}}{=} & 1 + \sum_{(j,h)\in\mathcal{S}_2} \frac{|\tau'_{j,\varepsilon}-\tau'_{h,\varepsilon}|}{|\tau_{j,\varepsilon}-\tau_{h,\varepsilon}|} \\
& + \sum_{(j,h,l)\in\mathcal{S}_3} \frac{\left|\check{M}(\tau_{j,\varepsilon})\right|}{|\tau_{j,\varepsilon}-\tau_{h,\varepsilon}|\cdot|\tau_{j,\varepsilon}-\tau_{l,\varepsilon}|} + \sum_{j=1}^{2} \sqrt{\frac{\left|\check{N}(\sigma_{j,\varepsilon})\right|+1}{|\sigma_{2,\varepsilon}-\sigma_{1,\varepsilon}|}},
\end{aligned}$$

$\check{M}$ and $\check{N}$ are defined in (1.7) and (1.8).

Differentiating the energy with respect to time, we get

$$\begin{aligned}
E'(t,\xi) = & -\eta\, \mathcal{K}(t,\xi)\, E(t,\xi) \\
& + k(t,\xi)\Bigg[2 \sum_{(j,h)\in\mathcal{S}_2} \mathrm{Re}\left\langle \partial_t \widetilde{L}_{jh,\varepsilon} v\,,\, \widetilde{L}_{jh,\varepsilon} v \right\rangle \\
& + 2\,\mathcal{H}(t,\xi)\, \mathcal{H}'(t,\xi) \left[\sum_{j=1}^{3} |L_{j,\varepsilon} v|^2 + |v|^2 \right] \\
& + \mathcal{H}^2(t,\xi) \left[\sum_{j=1}^{3} 2\,\mathrm{Re}\left\langle \partial_t L_{j,\varepsilon} v\,,\, L_{j,\varepsilon} v \right\rangle + 2\,\mathrm{Re}\langle \partial_t v\,,\, v\rangle \right]\Bigg].
\end{aligned}$$

Now we show that the second, third and forth summand can be estimated by $C\,\mathcal{K}(t,\xi)\, E(t,\xi)$, for some suitable positive constant C.

2.1 Estimation of the terms 2 $\mathrm{Re}\left\langle \partial_t \widetilde{L}_{jh,\varepsilon} v\,,\, \widetilde{L}_{jh,\varepsilon} v \right\rangle$

As

$$\partial_t \widetilde{L}_{jh,\varepsilon} v = (L_l \circ \widetilde{L}_{jh,\varepsilon}) v + i\,\tau_{l,\varepsilon}\, \widetilde{L}_{jh,\varepsilon} v,$$

we have

$$\begin{aligned}
2\,\mathrm{Re}\left\langle \partial_t \widetilde{L}_{jh,\varepsilon} v\,,\, \widetilde{L}_{jh,\varepsilon} v \right\rangle & = 2\,\mathrm{Re}\left\langle (L_l \circ \widetilde{L}_{jh,\varepsilon}) v\,,\, \widetilde{L}_{jh,\varepsilon} v \right\rangle + 2\,\mathrm{Re}\left\langle i\,\tau_{l,\varepsilon}\, \widetilde{L}_{jh,\varepsilon} v\,,\, \widetilde{L}_{jh,\varepsilon} v \right\rangle \\
& = 2\,\mathrm{Re}\left\langle (L_l \circ \widetilde{L}_{jh,\varepsilon}) v\,,\, \widetilde{L}_{jh,\varepsilon} v \right\rangle.
\end{aligned}$$

Define

$$\widetilde{M} \overset{\text{def}}{=} \check{M} + \frac{1}{2}\partial_t\partial_\tau\check{M} \\ = M - \frac{1}{2}\partial_t\partial_\tau L + \frac{1}{2}\partial_t\partial_\tau M - \frac{1}{4}\partial_t^2\partial_\tau^2 L\,. \tag{2.15}$$

so that

$$\widetilde{L}_{123,0} + \widetilde{M} + \check{N} = L + M + N\,, \tag{2.16}$$

hence

$$\begin{aligned} 2\,\mathrm{Re}\Big\langle (L_l\circ\widetilde{L}_{jh,\varepsilon})v\,,\,\widetilde{L}_{jh,\varepsilon}v\Big\rangle &= 2\,\mathrm{Re}\Big\langle (L_l\circ\widetilde{L}_{jh,\varepsilon})v - \widetilde{L}_{123,\varepsilon}v\,,\,\widetilde{L}_{jh,\varepsilon}v\Big\rangle \\ &\quad + 2\,\mathrm{Re}\Big\langle \widetilde{L}_{123,\varepsilon}v - \widetilde{L}_{123,0}v\,,\,\widetilde{L}_{jh,\varepsilon}v\Big\rangle \\ &\quad + 2\,\mathrm{Re}\Big\langle Lv + Mv + Nv\,,\,\widetilde{L}_{jh,\varepsilon}v\Big\rangle \\ &\quad - 2\,\mathrm{Re}\Big\langle \widetilde{M}v\,,\,\widetilde{L}_{jh,\varepsilon}v\Big\rangle - 2\,\mathrm{Re}\Big\langle \check{N}v\,,\,\widetilde{L}_{jh,\varepsilon}v\Big\rangle \end{aligned}$$

2.1.1 Estimation of $2\,\mathrm{Re}\Big\langle (L_l\circ\widetilde{L}_{jh,\varepsilon})v - \widetilde{L}_{123,\varepsilon}v\,,\,\widetilde{L}_{jh,\varepsilon}v\Big\rangle$

First of all, we have

$$\Big|\mathrm{Re}\Big\langle (L_l\circ\widetilde{L}_{jh,\varepsilon})v - \widetilde{L}_{123,\varepsilon}v\,,\,\widetilde{L}_{jh,\varepsilon}v\Big\rangle\Big| \lesssim \big|(L_l\circ\widetilde{L}_{jh,\varepsilon})v - \widetilde{L}_{123,\varepsilon}v\big|\,|\widetilde{L}_{jh,\varepsilon}v|\,.$$

According to (2.11), $(L_l\circ\widetilde{L}_{jh,\varepsilon})v - \widetilde{L}_{123,\varepsilon}v$ is a linear combination, with constant coefficients, of terms like $(\tau'_{\alpha,\varepsilon} - \tau'_{\beta,\varepsilon})\,L_{\gamma,\varepsilon}v$, with $(\alpha,\beta,\gamma)\in\mathcal{S}_3$, and $(\tau''_{\alpha,\varepsilon} - \tau''_{\beta,\varepsilon})v$, with $(\alpha,\beta)\in\mathcal{S}_2$, hence:

$$\big|(L_l\circ\widetilde{L}_{jh,\varepsilon})v - \widetilde{L}_{123,\varepsilon}v\big| \lesssim \sum_{(\alpha,\beta,\gamma)\in\mathcal{S}_3} |\tau'_{\alpha,\varepsilon} - \tau'_{\beta,\varepsilon}|\,|L_{\gamma,\varepsilon}v| + \sum_{(\alpha,\beta)\in\mathcal{S}_2} |\tau''_{\alpha,\varepsilon} - \tau''_{\beta,\varepsilon}|\,|v|\,.$$

Concerning the first sum, from (2.9) and (2.8), we have

$$\begin{aligned} L_{\gamma,\varepsilon}v &= i\,\frac{\widetilde{L}_{\gamma\alpha,\varepsilon}v - \widetilde{L}_{\gamma\beta,\varepsilon}v}{\tau_{\alpha,\varepsilon} - \tau_{\beta,\varepsilon}} - \frac{1}{2}\,\frac{\tau'_{\alpha,\varepsilon} - \tau'_{\beta,\varepsilon}}{\tau_{\alpha,\varepsilon} - \tau_{\beta,\varepsilon}}\,v \\ &= i\,\frac{\widetilde{L}_{\gamma\alpha,\varepsilon}v - \widetilde{L}_{\gamma\beta,\varepsilon}v}{\tau_{\alpha,\varepsilon} - \tau_{\beta,\varepsilon}} - \frac{i}{2}\,\frac{\tau'_{\alpha,\varepsilon} - \tau'_{\beta,\varepsilon}}{\tau_{\alpha,\varepsilon} - \tau_{\beta,\varepsilon}}\,\frac{L_{\alpha,\varepsilon}v - L_{\beta,\varepsilon}v}{\tau_{\alpha,\varepsilon} - \tau_{\beta,\varepsilon}}\,, \end{aligned}$$

hence

$$
\begin{aligned}
|\tau'_{\alpha,\varepsilon} - \tau'_{\beta,\varepsilon}|\,|L_{\gamma,\varepsilon} v| &\lesssim \left|\frac{\tau'_{\alpha,\varepsilon} - \tau'_{\beta,\varepsilon}}{\tau_{\alpha,\varepsilon} - \tau_{\beta,\varepsilon}}\right| \big[|\widetilde{L}_{\gamma\alpha,\varepsilon} v| + |\widetilde{L}_{\gamma\beta,\varepsilon} v|\big] \\
&\quad + \left|\frac{\tau'_{\alpha,\varepsilon} - \tau'_{\beta,\varepsilon}}{\tau_{\alpha,\varepsilon} - \tau_{\beta,\varepsilon}}\right|^2 \big[|L_{\alpha,\varepsilon} v| + |L_{\beta,\varepsilon} v|\big] \\
&\lesssim \mathcal{K}(t,\xi) \left[\sum_{(j,h)\in\mathcal{S}_2} |\widetilde{L}_{jh,\varepsilon} v| + \mathcal{H}(t,\xi) \sum_{j=1}^{3} |L_{j,\varepsilon} v|\right].
\end{aligned}
$$

Concerning the second sum, from (2.8), we have

$$
\begin{aligned}
\big|(\tau''_{\alpha,\varepsilon} - \tau''_{\beta,\varepsilon})v\big| &= \frac{|\tau''_{\alpha,\varepsilon} - \tau''_{\beta,\varepsilon}|}{|\tau_{\alpha,\varepsilon} - \tau_{\beta,\varepsilon}|} \, |L_{\beta,\varepsilon} v - L_{\alpha,\varepsilon} v| \\
&= \frac{|\tau''_{\alpha,\varepsilon} - \tau''_{\beta,\varepsilon}|}{|\tau'_{\alpha,\varepsilon} - \tau'_{\beta,\varepsilon}| + 1} \, \frac{|\tau'_{\alpha,\varepsilon} - \tau'_{\beta,\varepsilon}| + 1}{|\tau_{\alpha,\varepsilon} - \tau_{\beta,\varepsilon}|} \, |L_{\beta,\varepsilon} v - L_{\alpha,\varepsilon} v| \\
&\leqslant \mathcal{K}(t,\xi)\, \mathcal{H}(t,\xi) \sum_{j=1}^{3} |L_{j,\varepsilon} v|\,.
\end{aligned}
$$

Combining the above estimates, we get

$$
\Big|\mathrm{Re}\Big\langle (L_l \circ \widetilde{L}_{jh,\varepsilon}) v - \widetilde{L}_{123,\varepsilon} v \,,\, \widetilde{L}_{jh,\varepsilon} v \Big\rangle\Big| \lesssim \mathcal{K}(t,\xi) \left[\sum_{(j,h)\in\mathcal{S}_2} |\widetilde{L}_{jh,\varepsilon} v|^2 + \mathcal{H}^2(t,\xi) \sum_{j=1}^{3} |L_{j,\varepsilon} v|^2\right].
$$

2.1.2 Estimation of $2\,\mathrm{Re}\Big\langle \widetilde{L}_{123,\varepsilon} v - \widetilde{L}_{123,0} v \,,\, \widetilde{L}_{jh,\varepsilon} v \Big\rangle$

Using (2.12), we have

$$
2\,\mathrm{Re}\Big\langle \widetilde{L}_{123,\varepsilon} v - \widetilde{L}_{123,0} v \,,\, \widetilde{L}_{jh,\varepsilon} v \Big\rangle \lesssim \sum_{j=1}^{3} |L_{j,\varepsilon} v|^2 + |\widetilde{L}_{jh,\varepsilon} v|^2 \,.
$$

2.1.3 Estimation of $2\,\mathrm{Re}\Big\langle Lv + Mv + Nv \,,\, \widetilde{L}_{jh,\varepsilon} v \Big\rangle$

As

$$
Lv + Mv + Nv = -pv
$$

$$
2\,\mathrm{Re}\Big\langle Lv + Mv + Nv \,,\, \widetilde{L}_{jh,\varepsilon} v \Big\rangle = -2\,\mathrm{Re}\Big\langle pv \,,\, \widetilde{L}_{jh,\varepsilon} v \Big\rangle \lesssim |v|^2 + |\widetilde{L}_{jh,\varepsilon} v|^2 \,.
$$

2.1.4 Estimation of $2\,\mathrm{Re}\Big\langle \widetilde{M} v \,,\, \widetilde{L}_{jh,\varepsilon} v \Big\rangle$

To estimate $\check{M} = M - \frac{1}{2}\partial_t \partial_\tau L$, we can use the Lagrange interpolation formula

$$\check{M} = \sum_{(j,h,l)\in\mathcal{S}_3} \ell_j(t,\xi)\, L_{hl,\varepsilon}\,, \quad \text{where} \quad \ell_j(t,\xi) \overset{\text{def}}{=} \frac{\check{M}(\tau_{j,\varepsilon})}{(\tau_{j,\varepsilon}-\tau_{h,\varepsilon})(\tau_{j,\varepsilon}-\tau_{l,\varepsilon})}\,. \tag{2.17}$$

If $\widetilde{M}$ was a linear combination of the $\widetilde{L}_{hl,\varepsilon}$, then we could estimate it.

Now we observe that from $\check{M} = \sum \ell_j L_{hl,\varepsilon}$, it does not follow $\widetilde{M} = \sum \ell_j \widetilde{L}_{hl,\varepsilon}$, but it follows

$$\begin{aligned}\widetilde{M} &= \check{M} + \frac{1}{2}\partial_t\partial_\tau \check{M}\\ &= \sum_{(j,h,l)\in\mathcal{S}_3} \ell_j L_{hl,\varepsilon} + \frac{1}{2}\sum_{(j,h,l)\in\mathcal{S}_3} \ell_j \partial_t\partial_\tau L_{hl,\varepsilon} + \frac{1}{2}\sum_{(j,h,l)\in\mathcal{S}_3} \partial_t\ell_j\partial_\tau L_{hl,\varepsilon}\\ &= \sum_{(j,h,l)\in\mathcal{S}_3} \ell_j \widetilde{L}_{hl,\varepsilon} + \frac{1}{2}\sum_{(j,h,l)\in\mathcal{S}_3} \partial_t\ell_j [L_{h,\varepsilon}+L_{l,\varepsilon}]\,.\end{aligned}$$

We have

$$\begin{aligned}\partial_t\ell_j &= \partial_t \frac{\check{M}(\tau_{j,\varepsilon})}{(\tau_{j,\varepsilon}-\tau_{h,\varepsilon})\,(\tau_{j,\varepsilon}-\tau_{l,\varepsilon})}\\ &= \frac{\partial_t\check{M}(\tau_{j,\varepsilon})}{(\tau_{j,\varepsilon}-\tau_{h,\varepsilon})\,(\tau_{j,\varepsilon}-\tau_{l,\varepsilon})} - \frac{\check{M}(\tau_{j,\varepsilon})}{(\tau_{j,\varepsilon}-\tau_{h,\varepsilon})\,(\tau_{j,\varepsilon}-\tau_{l,\varepsilon})}\cdot\left(\frac{\tau'_{j,\varepsilon}-\tau'_{h,\varepsilon}}{\tau_{j,\varepsilon}-\tau_{h,\varepsilon}} + \frac{\tau'_{j,\varepsilon}-\tau'_{l,\varepsilon}}{\tau_{j,\varepsilon}-\tau_{l,\varepsilon}}\right),\end{aligned}$$

and

$$\begin{aligned}\left|\frac{\partial_t\check{M}(\tau_{j,\varepsilon})}{(\tau_{j,\varepsilon}-\tau_{h,\varepsilon})\,(\tau_{j,\varepsilon}-\tau_{l,\varepsilon})}\right| &= \frac{\left|\partial_t\check{M}(\tau_{j,\varepsilon})\right|}{\left|\check{M}(\tau_{j,\varepsilon})\right|+1}\cdot\frac{\left|\check{M}(\tau_{j,\varepsilon})\right|+1}{|\tau_{j,\varepsilon}-\tau_{h,\varepsilon}|\,|\tau_{j,\varepsilon}-\tau_{l,\varepsilon}|}\\ &\lesssim \frac{\left|\partial_t\check{M}(\tau_{j,\varepsilon})\right|}{\left|\check{M}(\tau_{j,\varepsilon})\right|+1}\left(\frac{\left|\check{M}(\tau_{j,\varepsilon})\right|}{|\tau_{j,\varepsilon}-\tau_{h,\varepsilon}|\,|\tau_{j,\varepsilon}-\tau_{l,\varepsilon}|}+1\right)\end{aligned}$$

since $|\tau_{j,\varepsilon}-\tau_{h,\varepsilon}| \geqslant C > 0$. Hence,

$$\left|\partial_t \frac{\check{M}(\tau_{j,\varepsilon})}{(\tau_{j,\varepsilon}-\tau_{h,\varepsilon})\,(\tau_{j,\varepsilon}-\tau_{l,\varepsilon})}\right| \leqslant \mathcal{K}(t,\xi)\,\mathcal{H}(t,\xi)\,. \tag{2.18}$$

Thus, we get

$$|\widetilde{M}v| \lesssim \mathcal{K}(t,\xi)\left[\sum_{(j,h)\in\mathcal{S}_2} |\widetilde{L}_{jh,\varepsilon}v| + \mathcal{H}(t,\xi)\sum_{j=1}^{3}|L_{j,\varepsilon}v|\right],$$

hence

$$\left|2\,\mathrm{Re}\left\langle \widetilde{M}v\,,\,\widetilde{L}_{jh,\varepsilon}v\right\rangle\right| \lesssim \mathcal{K}(t,\xi)\left[\sum_{(j,h)\in\mathcal{S}_2} |\widetilde{L}_{jh,\varepsilon}v|^2 + \mathcal{H}^2(t,\xi)\sum_{j=1}^{3}|L_{j,\varepsilon}v|^2\right].$$

2.1.5 Estimation of $2\,\mathrm{Re}\left\langle \check{N}v\,,\,\widetilde{L}_{jh,\varepsilon}v\right\rangle$

Using Lagrange's interpolation formula:

$$\check{N}(\tau) = \frac{\check{N}(\sigma_{1,\varepsilon})}{\sigma_{1,\varepsilon} - \sigma_{2,\varepsilon}} (\tau - \sigma_{2,\varepsilon}) + \frac{\check{N}(\sigma_{2,\varepsilon})}{\sigma_{2,\varepsilon} - \sigma_{1,\varepsilon}} (\tau - \sigma_{1,\varepsilon}) .$$

Now, since $\tau_{1,\varepsilon} \leqslant \sigma_{1,\varepsilon} \leqslant \tau_{2,\varepsilon} \leqslant \sigma_{2,\varepsilon} \leqslant \tau_{3,\varepsilon}$, we can find $\theta_1, \theta_2 \in [0, 1]$ such that

$$\sigma_{j,\varepsilon} = \theta_j \, \tau_{1,\varepsilon} + (1 - \theta_j) \, \tau_{3,\varepsilon} ,$$

hence

$$(\tau - \sigma_{j,\varepsilon}) = \theta_j \, (\tau - \tau_{1,\varepsilon}) + (1 - \theta_j) \, (\tau - \tau_{3,\varepsilon}) ,$$

thus $\check{N}(\tau)$ is a linear combination, with bounded coefficients, of terms of the form

$$\frac{\check{N}(\sigma_{j,\varepsilon})}{\sigma_{1,\varepsilon} - \sigma_{2,\varepsilon}} (\tau - \tau_{h,\varepsilon}) ,$$

$j = 1, 2$, $h = 1, 3$.

We get

$$\begin{aligned} 2 \operatorname{Re} \left\langle \check{N} v , \widetilde{L}_{jh,\varepsilon} v \right\rangle &\lesssim \frac{\left|\check{N}(\sigma_{1,\varepsilon})\right| + \left|\check{N}(\sigma_{2,\varepsilon})\right|}{|\sigma_{2,\varepsilon} - \sigma_{1,\varepsilon}|} \Big[|L_{1,\varepsilon} v| + |L_{3,\varepsilon} v| \Big] \, |\widetilde{L}_{jh,\varepsilon} v| \\ &\lesssim \sqrt{\frac{\left|\check{N}(\sigma_{1,\varepsilon})\right| + \left|\check{N}(\sigma_{2,\varepsilon})\right|}{|\sigma_{2,\varepsilon} - \sigma_{1,\varepsilon}|}} \Bigg[|\widetilde{L}_{jh,\varepsilon} v|^2 \\ &\quad + \frac{\left|\check{N}(\sigma_{1,\varepsilon})\right| + \left|\check{N}(\sigma_{2,\varepsilon})\right|}{|\sigma_{2,\varepsilon} - \sigma_{1,\varepsilon}|} \Big[|L_{1,\varepsilon} v|^2 + |L_{3,\varepsilon} v|^2 \Big] \Bigg] \\ &\lesssim \mathcal{K}(t, \xi) \Bigg[\sum_{(j,h) \in \mathcal{S}_2} |\widetilde{L}_{jh,\varepsilon} v|^2 + \mathcal{H}^2(t, \xi) \sum_{j=1}^{3} |L_{j,\varepsilon} v|^2 \Bigg] . \end{aligned}$$

2.2 Estimation of the term $2\,\mathcal{H}(t, \xi)\,\mathcal{H}'(t, \xi)$

Lemma 2.5 *There exists $C_H > 0$ such that*

$$\mathcal{H}'(t, \xi) \leqslant C_H \, \mathcal{K}(t, \xi) \, \mathcal{H}(t, \xi) .$$

Proof For any $(j, h) \in \mathcal{S}_2$, we have

$$\partial_t \frac{\tau'_{j,\varepsilon} - \tau'_{h,\varepsilon}}{\tau_{j,\varepsilon} - \tau_{h,\varepsilon}} = \frac{\tau''_{j,\varepsilon} - \tau''_{h,\varepsilon}}{\tau_{j,\varepsilon} - \tau_{h,\varepsilon}} - \frac{(\tau'_{j,\varepsilon} - \tau'_{h,\varepsilon})^2}{(\tau_{j,\varepsilon} - \tau_{h,\varepsilon})^2} ,$$

hence

$$\begin{aligned} \partial_t \frac{|\tau'_{j,\varepsilon} - \tau'_{h,\varepsilon}|}{|\tau_{j,\varepsilon} - \tau_{h,\varepsilon}|} &\leqslant \frac{|\tau''_{j,\varepsilon} - \tau''_{h,\varepsilon}|}{|\tau'_{j,\varepsilon} - \tau'_{h,\varepsilon}| + 1} \cdot \frac{|\tau'_{j,\varepsilon} - \tau'_{h,\varepsilon}| + 1}{|\tau_{j,\varepsilon} - \tau_{h,\varepsilon}|} + \frac{|\tau'_{j,\varepsilon} - \tau'_{h,\varepsilon}|^2}{|\tau_{j,\varepsilon} - \tau_{h,\varepsilon}|^2} \\ &\lesssim \mathcal{K}(t, \xi) \, \mathcal{H}(t, \xi) . \end{aligned}$$

The terms $\partial_t \frac{\check{M}(\tau_{j,\varepsilon})}{(\tau_{j,\varepsilon}-\tau_{h,\varepsilon})(\tau_{j,\varepsilon}-\tau_{l,\varepsilon})}$ are estimated as in (2.18).

For any $(j,h) \in \mathcal{S}_2$, we have

$$\partial_t \frac{|\check{N}(\sigma_{j,\varepsilon})|+1}{|\sigma_{2,\varepsilon}-\sigma_{1,\varepsilon}|} = \frac{1}{|\sigma_{2,\varepsilon}-\sigma_{1,\varepsilon}|} \cdot \frac{\check{N}(\sigma_{j,\varepsilon})}{|\check{N}(\sigma_{j,\varepsilon})|} \cdot \partial_t \check{N}(\sigma_{j,\varepsilon}) - \frac{|\check{N}(\sigma_{j,\varepsilon})|+1}{|\sigma_{2,\varepsilon}-\sigma_{1,\varepsilon}|} \cdot \frac{\sigma'_{2,\varepsilon}-\sigma'_{1,\varepsilon}}{\sigma_{2,\varepsilon}-\sigma_{1,\varepsilon}}$$
$$= \frac{|\check{N}(\sigma_{j,\varepsilon})|+1}{|\sigma_{2,\varepsilon}-\sigma_{1,\varepsilon}|}\left[\frac{\check{N}(\sigma_{j,\varepsilon})}{|\check{N}(\sigma_{j,\varepsilon})|} \cdot \frac{\partial_t \check{N}(\sigma_{j,\varepsilon})}{|\check{N}(\sigma_{j,\varepsilon})|+1} - \frac{\sigma'_{2,\varepsilon}-\sigma'_{1,\varepsilon}}{\sigma_{2,\varepsilon}-\sigma_{1,\varepsilon}}\right],$$

hence

$$\partial_t \sqrt{\frac{|\check{N}(\sigma_{j,\varepsilon})|+1}{|\sigma_{2,\varepsilon}-\sigma_{1,\varepsilon}|}} = \frac{1}{2\sqrt{\frac{|\check{N}(\sigma_{j,\varepsilon})|+1}{|\sigma_{2,\varepsilon}-\sigma_{1,\varepsilon}|}}}\, \partial_t \frac{|\check{N}(\sigma_{j,\varepsilon})|+1}{|\sigma_{2,\varepsilon}-\sigma_{1,\varepsilon}|}$$
$$= \frac{1}{2}\sqrt{\frac{|\check{N}(\sigma_{j,\varepsilon})|+1}{|\sigma_{2,\varepsilon}-\sigma_{1,\varepsilon}|}}\left[\frac{\check{N}(\sigma_{j,\varepsilon})}{|\check{N}(\sigma_{j,\varepsilon})|} \cdot \frac{\partial_t \check{N}(\sigma_{j,\varepsilon})}{|\check{N}(\sigma_{j,\varepsilon})|+1} - \frac{\sigma'_{2,\varepsilon}-\sigma'_{1,\varepsilon}}{\sigma_{2,\varepsilon}-\sigma_{1,\varepsilon}}\right].$$

All the terms but the last can be estimated by $\mathcal{K}(t,\xi)$ or $\mathcal{H}(t,\xi)$. For the last, we remark that (cf. Lemma A.1)

$$(\tau_{1,\varepsilon}-\tau_{2,\varepsilon})^2 + (\tau_{2,\varepsilon}-\tau_{3,\varepsilon})^2 + (\tau_{3,\varepsilon}-\tau_{1,\varepsilon})^2 = \frac{9}{2}\,(\sigma_{2,\varepsilon}-\sigma_{1,\varepsilon})^2,$$

thus

$$\frac{\sigma'_{2,\varepsilon}-\sigma'_{1,\varepsilon}}{\sigma_{2,\varepsilon}-\sigma_{1,\varepsilon}} = \frac{1}{2}\,\frac{[(\sigma_{2,\varepsilon}-\sigma_{1,\varepsilon})^2]'}{(\sigma_{2,\varepsilon}-\sigma_{1,\varepsilon})^2} = \frac{1}{2}\,\frac{\left[\sum_{j,h\in\mathcal{S}_2}(\tau_{j,\varepsilon}-\tau_{h,\varepsilon})^2\right]'}{\sum_{j,h\in\mathcal{S}_2}(\tau_{j,\varepsilon}-\tau_{h,\varepsilon})^2}$$
$$= \sum_{j,h\in\mathcal{S}_2} \frac{(\tau_{j,\varepsilon}-\tau_{h,\varepsilon})(\tau'_{j,\varepsilon}-\tau'_{h,\varepsilon})}{\sum_{j,h\in\mathcal{S}_2}(\tau_{j,\varepsilon}-\tau_{h,\varepsilon})^2},$$

which gives

$$\left|\frac{\sigma'_{2,\varepsilon}-\sigma'_{1,\varepsilon}}{\sigma_{2,\varepsilon}-\sigma_{1,\varepsilon}}\right| \leqslant \sum_{j,h\in\mathcal{S}_2} \frac{|\tau'_{j,\varepsilon}-\tau'_{h,\varepsilon}|}{|\tau_{j,\varepsilon}-\tau_{h,\varepsilon}|},$$

which can be estimated by $\mathcal{K}(t,\xi)$ or $\mathcal{H}(t,\xi)$.

Finally, we get

$$\partial_t \sqrt{\frac{|\check{N}(\sigma_{j,\varepsilon})|+1}{|\sigma_{2,\varepsilon}-\sigma_{1,\varepsilon}|}} \lesssim \mathcal{K}(t,\xi)\,\mathcal{H}(t,\xi).$$

□

2.3 Estimation of the terms $2\,\mathcal{H}^2\mathrm{Re}\big\langle \partial_t L_{j,\varepsilon} v\,,\, L_{j,\varepsilon} v\big\rangle$

As

$$\begin{aligned}\partial_t(L_{j,\varepsilon} v) &= [L_{h,\varepsilon} + i\tau_{h,\varepsilon}] L_{j,\varepsilon} v \\ &= (L_{h,\varepsilon} \circ L_{j,\varepsilon}) v + i\tau_{h,\varepsilon} L_{j,\varepsilon} v\,,\end{aligned}$$

we have

$$\begin{aligned}2\mathrm{Re}\big\langle \partial_t L_{j,\varepsilon} v\,,\, L_{j,\varepsilon} v\big\rangle &= 2\mathrm{Re}\big\langle (L_{h,\varepsilon} \circ L_{j,\varepsilon}) v\,,\, L_{j,\varepsilon} v\big\rangle + 2\mathrm{Re}\big\langle i\tau_{h,\varepsilon} L_{j,\varepsilon}\,,\, L_{j,\varepsilon} v\big\rangle \\ &= 2\mathrm{Re}\big\langle (L_{h,\varepsilon} \circ L_{j,\varepsilon}) v\,,\, L_{j,\varepsilon} v\big\rangle\,,\end{aligned}$$

as $\tau_{h,\varepsilon}$ is a real function, hence

$$2\,\mathcal{H}^2\mathrm{Re}\big\langle \partial_t L_{j,\varepsilon} v\,,\, L_{j,\varepsilon} v\big\rangle \leqslant \mathcal{K}\left[\big|(L_{h,\varepsilon} \circ L_{j,\varepsilon}) v\big|^2 + \mathcal{H}^2\,|L_{j,\varepsilon}|^2\right].$$

Now, using (2.5), we have

$$\begin{aligned}L_{j,\varepsilon} \circ L_{h,\varepsilon} - \widetilde{L}_{jh,\varepsilon} &= \frac{i}{2}\,(\tau'_{j,\varepsilon} - \tau'_{h,\varepsilon}) \\ &= -\frac{i}{2}\,\frac{\tau'_{j,\varepsilon} - \tau'_{h,\varepsilon}}{\tau_{j,\varepsilon} - \tau_{h,\varepsilon}}(L_{j,\varepsilon} v - L_{h,\varepsilon} v)\,,\end{aligned}$$

hence

$$\begin{aligned}\big|(L_{h,\varepsilon} \circ L_{j,\varepsilon}) v\big|^2 &\leqslant 2\big|\widetilde{L}_{jh,\varepsilon} v\big|^2 + 2\big|(L_{j,\varepsilon} \circ L_{h,\varepsilon} - \widetilde{L}_{jh,\varepsilon}) v\big|^2 \\ &\leqslant 2\big|\widetilde{L}_{jh,\varepsilon} v\big|^2 + \left|\frac{\tau'_{j,\varepsilon} - \tau'_{h,\varepsilon}}{\tau_{j,\varepsilon} - \tau_{h,\varepsilon}}\right|^2 \Big(|L_{j,\varepsilon} v|^2 + |L_{h,\varepsilon} v|^2\Big)\,.\end{aligned}$$

2.4 Estimation of the terms $2\mathrm{Re}\langle \partial_t v\,,\, v\rangle$

As

$$\partial_t v = L_{1,\varepsilon} v + i\tau_{1,\varepsilon} v\,,$$

we have

$$\begin{aligned}2\,\mathrm{Re}\langle \partial_t v\,,\, v\rangle &= 2\,\mathrm{Re}\big\langle L_{1,\varepsilon} v\,,\, v\big\rangle + 2\mathrm{Re}\big\langle i\tau_{1,\varepsilon} v\,,\, v\big\rangle \\ &\leqslant |L_{1,\varepsilon} v|^2 + |v|^2\,.\end{aligned}$$

Taking η large, we arrive to the estimates

$$E'(t) \leqslant C\,E(t)\,,$$

$$E(t) \leqslant \exp\big(C(t - t_0)\big)\,E(t_0)\,,$$

from this taking into account the inequality

$$\mathcal{K}(t,\xi) \geqslant \big(2+|\xi|\big)^{-c_0}$$

it follows that the Cauchy problem for the given equation is well posed.

This concludes the proof of Theorem 2.

3 Equivalent forms of the Levi conditions

In this paragraph, we can give some alternative forms of the Levi conditions. In particular, we express these conditions in terms of the roots of L.

We recall that

$$\big|\tau_{j,\varepsilon}-\tau_{h,\varepsilon}\big| \approx \big|\tau_j-\tau_h\big|+1, \quad \text{for any } (j,h)\in\mathcal{S}_2\,, \tag{3.1}$$

and

$$\big|\tau_j-\tau_{j,\varepsilon}\big| \lesssim 1\,, \quad \text{for any } j=1,2,3, \tag{3.2}$$

Proposition 3.1 *Hypothesis* (1.13) *is equivalent to the conditions*

$$\int_0^T \sum_{(j,h,l)\in\mathcal{S}_3} \frac{\big|\check{M}\big(t,\tau_j(t,\xi),\xi\big)\big|}{\Big(\big|\tau_j(t,\xi)-\tau_h(t,\xi)\big|+1\Big)\Big(\big|\tau_j(t,\xi)-\tau_l(t,\xi)\big|+1\Big)}\,\mathrm{d}t \lesssim \log\big(1+|\xi|\big)\,, \tag{3.3a}$$

$$\int_0^T \frac{\big|\partial_\tau\check{M}\big(t,\tau_1(t,\xi),\xi\big)\big|+\big|\partial_\tau\check{M}\big(t,\tau_3(t,\xi),\xi\big)\big|}{\big|\tau_1(t,\xi)-\tau_3(t,\xi)\big|+1}\,\mathrm{d}t \lesssim \log\big(1+|\xi|\big)\,. \tag{3.3b}$$

Proof For the sake of simplicity, we omit the t and ξ variables.

We start by proving that (1.13) implies (3.3b).

By the Lagrange interpolation formula, we have

$$\check{M}(\tau)=\sum_{(j,h,l)\in\mathcal{S}_3}\ell_{j,\varepsilon}\,L_{hl,\varepsilon}(\tau)\,, \quad \text{where} \quad \ell_{j,\varepsilon}\overset{\text{def}}{=}\frac{\check{M}(\tau_{j,\varepsilon})}{(\tau_{j,\varepsilon}-\tau_{h,\varepsilon})(\tau_{j,\varepsilon}-\tau_{l,\varepsilon})}\,.$$

Differentiating with respect to τ:

$$\begin{aligned}\partial_\tau\check{M}(\tau)&=\sum_{(j,h,l)\in\mathcal{S}_3}\ell_{j,\varepsilon}\,\big[L_{k,\varepsilon}(\tau)+L_{l,\varepsilon}(\tau)\big]\\&=\sum_{(j,h,l)\in\mathcal{S}_3}\big[\ell_{j,\varepsilon}+\ell_{k,\varepsilon}\big]\,L_{l,\varepsilon}(\tau)\,,\end{aligned}$$

hence $\partial_\tau\check{M}(\tau)$ is a linear combination of $L_{1,\varepsilon}(\tau)$, $L_{2,\varepsilon}(\tau)$ and $L_{3,\varepsilon}(\tau)$ with coefficients verifying the logarithmic condition.

To prove that $\partial_\tau\check{M}(\tau)$ is a linear combination of only $L_{1,\varepsilon}$ and $L_{3,\varepsilon}$ we note that, since $\tau_{1,\varepsilon}\leqslant\tau_{2,\varepsilon}\leqslant\tau_{3,\varepsilon}$, we can find $\theta\in[0,1]$ such that

$$\tau_{2,\varepsilon}=\theta\,\tau_{1,\varepsilon}+(1-\theta)\,\tau_{3,\varepsilon}\,,$$

hence

$$L_{2,\varepsilon} = \theta\, L_{1,\varepsilon} + (1-\theta)\, L_{3,\varepsilon}\,,$$

, and then,

$$\partial_\tau \check{M}(\tau) = b_{1,\varepsilon}\, L_{1,\varepsilon}(\tau) + b_{3,\varepsilon}\, L_{3,\varepsilon}(\tau)\,, \tag{3.4}$$

where $b_{1,\varepsilon}$ and $b_{3,\varepsilon}$ are some linear combination of the $\ell_{1,\varepsilon}$, $\ell_{2,\varepsilon}$ and $\ell_{3,\varepsilon}$, hence they verify the logarithmic condition.

Substituting τ with $\tau_{1,\varepsilon}$ and $\tau_{3,\varepsilon}$ in (3.4), we get

$$\int_0^T \frac{\left|\partial_\tau \check{M}(\tau_{1,\varepsilon})\right| + \left|\partial_\tau \check{M}(\tau_{3,\varepsilon})\right|}{\left|\tau_{1,\varepsilon} - \tau_{3,\varepsilon}\right|}\, \mathrm{d}t \lesssim \log\big(1+|\xi|\big)\,. \tag{3.5}$$

Since

$$\partial_\tau \check{M}(\tau_1) = \partial_\tau \check{M}(\tau_{1,\varepsilon}) + \partial_\tau^2 \check{M}(\tau_{1,\varepsilon})\,(\tau_1 - \tau_{1,\varepsilon})$$

we get

$$\left|\partial_\tau \check{M}(\tau_1)\right| \lesssim \left|\partial_\tau \check{M}(\tau_{1,\varepsilon})\right| + 1\,,$$

hence

$$\frac{\left|\partial_\tau \check{M}(\tau_1)\right|}{|\tau_3 - \tau_1| + 1} \lesssim \frac{\left|\partial_\tau \check{M}(\tau_{1,\varepsilon})\right|}{|\tau_{3,\varepsilon} - \tau_{1,\varepsilon}|} + 1\,.$$

An analogous estimate holds true with τ_1 and $\tau_{1,\varepsilon}$ replaced by τ_3 and $\tau_{3,\varepsilon}$. Combining such inequalities with (3.5), we get (3.3b).

To prove that (1.13) implies (3.3a), we remark that, since $\check{M}$ is a polynomial of degree 2, it coincides with its Taylor's expansion of order 2, hence, we have

$$\check{M}(\tau_1) = \check{M}(\tau_{1,\varepsilon}) + \partial_\tau \check{M}(\tau_{1,\varepsilon})\,(\tau_1 - \tau_{1,\varepsilon}) + \frac{1}{2}\,\partial_\tau^2 \check{M}(\tau_{1,\varepsilon})\,(\tau_1 - \tau_{1,\varepsilon})^2\,.$$

Taking into account (3.1) and (3.2), we have

$$\frac{\left|\check{M}(\tau_1)\right|}{\big(|\tau_2 - \tau_1| + 1\big)\,\big(|\tau_3 - \tau_1| + 1\big)} \lesssim \frac{\left|\check{M}(\tau_{1,\varepsilon})\right|}{|\tau_{2,\varepsilon} - \tau_{1,\varepsilon}|\,|\tau_{3,\varepsilon} - \tau_{1,\varepsilon}|} + \frac{\left|\partial_\tau \check{M}(\tau_{1,\varepsilon})\right|}{|\tau_{3,\varepsilon} - \tau_{1,\varepsilon}|} + 1\,.$$

An analogous estimate holds true for $\check{M}(\tau_3)$.

To prove the estimate for $\check{M}(\tau_2)$, we split the phase space $[0,T]\times\mathbb{R}^n$ in two sub-zones:

$$Z_1 = \Big\{\, (t,\xi) \in [0,T]\times\mathbb{R}^n \;\Big|\; |\tau_2 - \tau_1| \leqslant |\tau_3 - \tau_2| \,\Big\}$$
$$Z_2 = \Big\{\, (t,\xi) \in [0,T]\times\mathbb{R}^n \;\Big|\; |\tau_3 - \tau_2| \leqslant |\tau_2 - \tau_1| \,\Big\}\,.$$

For $(t,\xi) \in Z_1$, we write

$$\check{M}(\tau_2) = \check{M}(\tau_{1,\varepsilon}) + \partial_\tau \check{M}(\tau_{1,\varepsilon})\,(\tau_2 - \tau_{1,\varepsilon}) + \frac{1}{2}\,\partial_\tau^2 \check{M}(\tau_{1,\varepsilon})\,(\tau_2 - \tau_{1,\varepsilon})^2\,,$$

and, since

$$|\tau_2 - \tau_{1,\varepsilon}| \lesssim |\tau_2 - \tau_1| + 1 \leqslant |\tau_3 - \tau_2| + 1\,,$$

we get

$$\begin{aligned}
&\frac{|\check{M}(\tau_2)|}{\big(|\tau_2-\tau_1|+1\big)\big(|\tau_3-\tau_2|+1\big)} \\
&\quad\lesssim \frac{|\check{M}(\tau_{1,\varepsilon})|}{\big(|\tau_2-\tau_1|+1\big)\big(|\tau_3-\tau_2|+1\big)} + \frac{|\partial_\tau \check{M}(\tau_{1,\varepsilon})|\,|\tau_2-\tau_{1,\varepsilon}|}{\big(|\tau_2-\tau_1|+1\big)\big(|\tau_3-\tau_2|+1\big)} \\
&\qquad + \frac{1}{2}\,|\partial_\tau^2 \check{M}(\tau_{1,\varepsilon})|\,\frac{|\tau_2-\tau_{1,\varepsilon}|^2}{\big(|\tau_2-\tau_1|+1\big)\big(|\tau_3-\tau_2|+1\big)} \\
&\quad\lesssim \frac{|\check{M}(\tau_{1,\varepsilon})|}{\big(|\tau_2-\tau_1|+1\big)\big(|\tau_3-\tau_2|+1\big)} + \frac{|\partial_\tau \check{M}(\tau_{1,\varepsilon})|}{|\tau_3-\tau_2|+1} + 1\,.
\end{aligned}$$

Now, using the fact that $(t,\xi) \in Z_1$ if, and only if $|\tau_3 - \tau_1| \leqslant 2|\tau_3 - \tau_2|$, we can estimate the second term by $\frac{|\partial_\tau \check{M}(\tau_{1,\varepsilon})|}{|\tau_3-\tau_1|+1}$. Finally, by (3.1), we get

$$\frac{|\check{M}(\tau_2)|}{\big(|\tau_2-\tau_1|+1\big)\big(|\tau_3-\tau_2|+1\big)} \lesssim \frac{|\check{M}(\tau_{1,\varepsilon})|}{|\tau_{2,\varepsilon}-\tau_{1,\varepsilon}|\,|\tau_{3,\varepsilon}-\tau_{1,\varepsilon}|} + \frac{|\partial_\tau \check{M}(\tau_{1,\varepsilon})|}{|\tau_{3,\varepsilon}-\tau_{1,\varepsilon}|} + 1\,. \tag{3.6}$$

For $(t,\xi) \in Z_2$, we repeat the above calculation, with $\tau_{1,\varepsilon}$ and $\tau_{3,\varepsilon}$ exchanged, and we get

$$\begin{aligned}
&\frac{|\check{M}(\tau_2)|}{\big(|\tau_2-\tau_1|+1\big)\big(|\tau_3-\tau_2|+1\big)} \\
&\quad\lesssim \frac{|\check{M}(\tau_{3,\varepsilon})|}{|\tau_{3,\varepsilon}-\tau_{2,\varepsilon}|\,|\tau_{3,\varepsilon}-\tau_{1,\varepsilon}|} + \frac{|\partial_\tau \check{M}(\tau_{3,\varepsilon})|}{|\tau_{3,\varepsilon}-\tau_{1,\varepsilon}|} + 1\,.
\end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7), we get

$$\begin{aligned}
&\frac{|\check{M}(\tau_2)|}{\big(|\tau_2-\tau_1|+1\big)\big(|\tau_3-\tau_2|+1\big)} \\
&\quad\lesssim \frac{|\check{M}(\tau_{3,\varepsilon})|}{|\tau_{3,\varepsilon}-\tau_{2,\varepsilon}|\,|\tau_{3,\varepsilon}-\tau_{1,\varepsilon}|} + \frac{|\check{M}(\tau_{1,\varepsilon})|}{|\tau_{2,\varepsilon}-\tau_{1,\varepsilon}|\,|\tau_{3,\varepsilon}-\tau_{1,\varepsilon}|} + \frac{|\partial_\tau \check{M}(\tau_{1,\varepsilon})| + |\partial_\tau \check{M}(\tau_{3,\varepsilon})|}{|\tau_{3,\varepsilon}-\tau_{1,\varepsilon}|} + 1\,.
\end{aligned}$$

Thanks to (3.5) we see that (1.13) implies (3.3a).

Finally, we prove that (3.3a) and (3.3b) imply (1.13). Using Taylor expansion as before, we have

$$\check{M}(\tau_{j,\varepsilon}) = \check{M}(\tau_j) + \partial_\tau \check{M}(\tau_j)\,(\tau_{j,\varepsilon}-\tau_j) + \frac{1}{2}\,\partial_\tau^2 \check{M}(\tau_j)\,(\tau_{j,\varepsilon}-\tau_j)^2\,, \quad \text{for } j = 1,2,3\,. \tag{3.8}$$

Using (3.8) with $j = 1$, we have

$$\frac{|\check{M}(\tau_{1,\varepsilon})|}{|\tau_{2,\varepsilon}-\tau_{1,\varepsilon}|\,|\tau_{3,\varepsilon}-\tau_{1,\varepsilon}|} \lesssim \frac{|\check{M}(\tau_1)|}{|\tau_{2,\varepsilon}-\tau_{1,\varepsilon}|\,|\tau_{3,\varepsilon}-\tau_{1,\varepsilon}|} + \frac{|\partial_\tau\check{M}(\tau_1)|\,|\tau_{1,\varepsilon}-\tau_1|}{|\tau_{2,\varepsilon}-\tau_{1,\varepsilon}|\,|\tau_{3,\varepsilon}-\tau_{1,\varepsilon}|} + \frac{|\tau_{1,\varepsilon}-\tau_1|^2}{|\tau_{2,\varepsilon}-\tau_{1,\varepsilon}|\,|\tau_{3,\varepsilon}-\tau_{1,\varepsilon}|},$$

and, using (3.1) and (3.2), we get

$$\frac{|\check{M}(\tau_{1,\varepsilon})|}{|\tau_{2,\varepsilon}-\tau_{1,\varepsilon}|\,|\tau_{3,\varepsilon}-\tau_{1,\varepsilon}|} \lesssim \frac{|\check{M}(\tau_1)|}{\left(|\tau_2-\tau_1|+1\right)\left(|\tau_3-\tau_1|+1\right)} + \frac{|\partial_\tau\check{M}(\tau_1)|}{|\tau_3-\tau_1|+1} + 1\,.$$

Analogously, exchanging τ_1 with τ_3 and $\tau_{1,\varepsilon}$ with $\tau_{3,\varepsilon}$:

$$\frac{|\check{M}(\tau_{3,\varepsilon})|}{|\tau_{2,\varepsilon}-\tau_{3,\varepsilon}|\,|\tau_{1,\varepsilon}-\tau_{3,\varepsilon}|} \lesssim \frac{|\check{M}(\tau_3)|}{\left(|\tau_3-\tau_2|+1\right)\left(|\tau_3-\tau_1|+1\right)} + \frac{|\partial_\tau\check{M}(\tau_3)|}{|\tau_3-\tau_1|+1} + 1\,.$$

Using (3.8) with $j=2$, (3.1) and (3.2), we have

$$\begin{aligned}\frac{|\check{M}(\tau_{2,\varepsilon})|}{|\tau_{2,\varepsilon}-\tau_{1,\varepsilon}|\,|\tau_{3,\varepsilon}-\tau_{2,\varepsilon}|} &\lesssim \frac{|\check{M}(\tau_2)|}{|\tau_{2,\varepsilon}-\tau_{1,\varepsilon}|\,|\tau_{3,\varepsilon}-\tau_{2,\varepsilon}|} + \frac{|\partial_\tau\check{M}(\tau_2)|\,|\tau_{2,\varepsilon}-\tau_2|}{|\tau_{2,\varepsilon}-\tau_{1,\varepsilon}|\,|\tau_{3,\varepsilon}-\tau_{2,\varepsilon}|} + \frac{|\tau_{2,\varepsilon}-\tau_2|^2}{|\tau_{2,\varepsilon}-\tau_{1,\varepsilon}|\,|\tau_{3,\varepsilon}-\tau_{2,\varepsilon}|}\\ &\lesssim \frac{|\check{M}(\tau_2)|}{\left(|\tau_3-\tau_2|+1\right)\left(|\tau_2-\tau_1|+1\right)} + \frac{|\partial_\tau\check{M}(\tau_2)|}{\left(|\tau_3-\tau_2|+1\right)\left(|\tau_2-\tau_1|+1\right)} + 1\,.\end{aligned}$$

To estimate the second term, we note that, since $\tau_1 \leqslant \tau_2 \leqslant \tau_3$, we can find $\theta \in [0,1]$ such that

$$\tau_2 = \theta\,\tau_1 + (1-\theta)\,\tau_3\,,$$

hence

$$\partial_\tau\check{M}(\tau_2) = \theta\,\partial_\tau\check{M}(\tau_1) + (1-\theta)\,\partial_\tau\check{M}(\tau_3)\,,$$

, and then,

$$|\partial_\tau\check{M}(\tau_2)| \leqslant |\partial_\tau\check{M}(\tau_1)| + |\partial_\tau\check{M}(\tau_3)|\,.$$

We note also that if $(t,\xi)\in Z_1$, then $|\tau_3-\tau_1| \leqslant 2|\tau_3-\tau_2|$, whereas if $(t,\xi)\in Z_2$, then $|\tau_3-\tau_1| \leqslant 2|\tau_2-\tau_1|$, thus

$$\left(|\tau_3-\tau_2|+1\right)\left(|\tau_2-\tau_1|+1\right) \gtrsim |\tau_3-\tau_1|+1\,.$$

Combining the above estimates, we get

$$\frac{|\partial_\tau\check{M}(\tau_2)|}{\left(|\tau_3-\tau_2|+1\right)\left(|\tau_2-\tau_1|+1\right)} \leqslant \frac{|\partial_\tau\check{M}(\tau_1)| + |\partial_\tau\check{M}(\tau_3)|}{|\tau_3-\tau_1|+1}\,.$$

□

Proposition 3.2 *Hypothesis* (1.14) *is equivalent to the condition*

$$\int_0^T \sum_{j=1}^{2} \sqrt{\frac{\left|\check{N}\big(t,\sigma_j(t,\xi),\xi\big)\right|}{\left|\sigma_2(t,\xi)-\sigma_1(t,\xi)\right|+1}}\,\mathrm{d}t \lesssim \log\big(1+|\xi|\big)\,, \tag{3.9}$$

where $\sigma_1(t,\xi)$ and $\sigma_2(t,\xi)$ are the roots in τ of $\partial_\tau L(t,\tau,\xi)$.

Proof First of all, we remark that, since

$$\left|\left|\check{N}(\mu_1)\right|-\left|\check{N}(\mu_2)\right|\right| \lesssim |\mu_2-\mu_1|\,,$$

condition (1.14) is equivalent to

$$\int_0^T \sqrt{\frac{\left|\check{N}\big(t,\mu_1(t,\xi),\xi\big)\right|}{\left|\mu_2(t,\xi)-\mu_1(t,\xi)\right|}}\,\mathrm{d}t \lesssim \log\big(1+|\xi|\big)\,. \tag{3.10}$$

Analogously, since

$$\left|\left|\check{N}(\sigma_1)\right|-\left|\check{N}(\sigma_2)\right|\right| \lesssim |\sigma_2-\sigma_1|\,,$$

condition (3.9) is equivalent to

$$\int_0^T \sqrt{\frac{\left|\check{N}\big(t,\sigma_1(t,\xi),\xi\big)\right|}{\left|\sigma_2(t,\xi)-\sigma_1(t,\xi)\right|+1}}\,\mathrm{d}t \lesssim \log\big(1+|\xi|\big)\,. \tag{3.11}$$

Now, by (2.1), we see that

$$|\mu_2-\mu_1| \approx |\sigma_2-\sigma_1|+1\,,$$

whereas, by direct computation, we see that

$$|\mu_1-\sigma_1| \lesssim 1\,,$$

from which we deduce the equivalence of (3.10) and (3.11). □

Remark 3.3 More generally, Hypothesis (1.14) is equivalent to each of the conditions

$$\int_0^T \sqrt{\frac{\mathscr{M}(t,\xi)}{\mathscr{D}(t,\xi)}}\,\mathrm{d}t \lesssim \log\big(1+|\xi|\big)\,, \tag{3.12}$$

where $\mathscr{N}$ can be any of the symbol

$$\left|\check{N}\big(t,\sigma_j(t,\xi),\xi\big)\right|,\quad \text{for } j=1,2\,,\qquad \left|\check{N}\big(t,\mu_j(t,\xi),\xi\big)\right|,\quad \text{for } j=1,2\,,$$
$$\left|\check{N}\big(t,\tau_j(t,\xi),\xi\big)\right|,\quad \text{for } j=1,2,3\,,\quad \left|\check{N}\big(t,\lambda_j(t,\xi),\xi\big)\right|,\quad \text{for } j=1,2,3\,,$$

and $\mathscr{D}$ can be any of the symbol

$$
\begin{array}{ll}
\left|\sigma_2(t,\xi)-\sigma_1(t,\xi)\right|+1 & \left|\mu_2(t,\xi)-\mu_1(t,\xi)\right| \\
\sqrt{\Delta_L^{(1)}(t,\xi)}+1 & \sqrt{\Delta_{\mathcal{L}}^{(1)}(t,\xi)} \\
\left|\tau_3(t,\xi)-\tau_1(t,\xi)\right|+1 & \left|\lambda_3(t,\xi)-\lambda_1(t,\xi)\right|.
\end{array}
$$

Indeed, from Lemma A.1, we have

$$
(\tau_1-\tau_2)^2+(\tau_2-\tau_3)^2+(\tau_3-\tau_1)^2=\frac{9}{2}(\sigma_2-\sigma_1)^2\,,
$$

hence

$$
|\sigma_2-\sigma_1|+1\approx\sqrt{\Delta_L^{(1)}}+1\,.
$$

Next, from the elementary inequality

$$
\max(\alpha,\beta,\gamma)\leqslant\sqrt{\alpha^2+\beta^2+\gamma^2}\leqslant\sqrt{3}\,\max(\alpha,\beta,\gamma)\,,\qquad\text{for any }\alpha,\beta,\gamma\geqslant 0\,,
$$

we see that

$$
|\tau_3-\tau_1|\leqslant\sqrt{(\tau_3-\tau_1)^2+(\tau_3-\tau_2)^2+(\tau_2-\tau_1)^2}\leqslant\sqrt{3}\,|\tau_3-\tau_1|, \tag{3.13}
$$

and, consequently,

$$
\sqrt{\Delta_L^{(1)}}+1\approx|\tau_3-\tau_1|+1\,.
$$

We prove that

$$
|\mu_2-\mu_1|\approx\sqrt{\Delta_{\mathcal{L}}^{(1)}}\approx|\lambda_3-\lambda_1|
$$

in a similar way.

Since

$$
\left|\left|\check{N}(\sigma_1)\right|-\left|\check{N}(\tau_1)\right|\right|\lesssim|\sigma_1-\tau_1|\,,
$$

and

$$
|\sigma_1-\tau_1|\leqslant|\tau_3-\tau_1|\,,
$$

we see that

$$
\frac{\left|\check{N}(\sigma_1)\right|}{|\tau_3-\tau_1|+1}\approx\frac{\left|\check{N}(\tau_1)\right|}{|\tau_3-\tau_1|+1}+1\,.
$$

We can prove the other equivalences in a similar way.

Remark 3.4 By similar arguments, we can prove that (3.3b) is equivalent to

$$
\int_0^T\frac{\left|\partial_\tau\check{M}\big(t,\sigma_1(t,\xi),\xi\big)\right|+\left|\partial_\tau\check{M}\big(t,\sigma_2(t,\xi),\xi\big)\right|}{\left|\sigma_1(t,\xi)-\sigma_2(t,\xi)\right|+1}\,\mathrm{d}t\lesssim\log\big(1+|\xi|\big)\,, \tag{3.14}
$$

where σ_1 and σ_2 are the roots of $\partial_\tau L(t,\tau,\xi)$.

More generally, Hypothesis (3.3b) is equivalent to each of the conditions

$$\int_0^T \sqrt{\frac{\mathscr{M}(t,\xi)}{\mathscr{D}(t,\xi)}}\, \mathrm{d}t \lesssim \log\big(1+|\xi|\big)\,,$$

where $\mathscr{M}$ can be any of the symbol

$$\begin{aligned} &\big|\partial_\tau \check{M}\big(t,\sigma_j(t,\xi),\xi\big)\big|\,, \quad \text{for } j=1,2\,, &&\big|\partial_\tau \check{M}\big(t,\mu_j(t,\xi),\xi\big)\big|\,, \quad \text{for } j=1,2\,,\\ &\big|\partial_\tau \check{M}\big(t,\tau_j(t,\xi),\xi\big)\big|\,, \quad \text{for } j=1,2,3\,, &&\big|\partial_\tau \check{M}\big(t,\lambda_j(t,\xi),\xi\big)\big|\,, \quad \text{for } j=1,2,3\,,\end{aligned}$$

and $\mathscr{D}$ is as in the previous Remark.

4 The case of analytic coefficients

In this section, we show that if the coefficients are analytic, then hypothesis (1.9)–(1.12) are satisfied.

Lemma 4.1 ([7, 21, 29]) *Let $f_1,\dots,f_d$ be analytic functions on an open set $\mathscr{O}\subset\mathbb{C}$, $f_j\not\equiv 0$ for $j=1,\dots,d$. For any $\alpha=(\alpha_1,\dots,\alpha_d)\in\mathbb{R}^d$ set*

$$\varphi_\alpha(x) := \sum_{j=1}^d \alpha_j f_j(x)\,.$$

Then, for any compact set $\mathscr{K}\subset\mathscr{O}$, there exists $\nu\in\mathbb{N}$ such that either $\varphi_\alpha(x)\equiv 0$ or $\varphi_\alpha(x)$ has at most ν zeros in $\mathscr{K}$, if counted with their multiplicity.

Proof With no loss of generality, we can assume that $f_1,\dots,f_d$ are linearly independent.

If, for any $k\in\mathbb{N}$, there exists $\alpha^{(k)}$ with $\|\alpha^{(k)}\|=1$ such that $\varphi_\alpha(x)$ has at least k zeros in $\mathscr{K}$, by passing to a suitable subsequence, we can assume that $\alpha^{(k)}$ converges to some α^*. Hence, φ_{α^*} must have an infinite number of zeros, and hence, it is identically zero. This contradicts the fact that the $f_1,\dots,f_d$ are linearly independent. □

As any polynomial $\Phi(t,\xi)$ in ξ with analytic coefficients can be regarded as a linear combination of its coefficients, we deduce from Lemma 4.1.

Corollary 4.2 *Let $\Phi(t,\xi)$ be a polynomial in ξ with analytic coefficients on an open set $\mathscr{O}\subset\mathbb{C}$.*

Then, for any compact set $\mathscr{K}\subset\mathscr{O}$, there exists $\nu\in\mathbb{N}$ such that either $\Phi(t,\xi)\equiv 0$ or $\Phi(\cdot,\xi)$ has at most ν zeros in $\mathscr{K}$, if counted with their multiplicity.

Proposition 4.3 *Let $P(t,\tau,\xi)$ be a third-order monic hyperbolic polynomial in τ, whose coefficients are polynomial in ξ and analytic in $t\in\mathscr{O}$, $\mathscr{O}$ open set in $\mathbb{C}$. Assume that the roots in τ of $P(t,\tau,\xi)$ are distinct for ξ large:*

$$\tau_1(t,\xi) < \tau_2(t,\xi) < \tau_3(t,\xi)\,, \qquad \text{if } |\xi| \geqslant R\,.$$

Let $Q(t,\tau,\xi)$ be another polynomial in τ whose coefficients are polynomial in ξ and analytic in $t \in \mathscr{O}$.

Then, for any compact set $\mathscr{K} \subset \mathscr{O}$, there exists $\nu \in \mathbb{N}$ such that for any $\xi \in \mathbb{R}^n$, with $|\xi| \geqslant R$, the functions

$$t \mapsto Q\big(t, \tau_j(t,\xi), \xi\big)\,, \qquad j = 1,2,3\,,$$

are either identically zero or have at most ν zeros in $\mathscr{K}$.

Remark 4.4 If a root $\tau_j(t,\xi)$ was a linear function of ξ, then $Q\big(t,\tau_j(t,\xi),\xi\big)$ would be a polynomial in ξ, and, by Corollary 4.2, we can get easily the result.

Unfortunately, in general, the roots $\tau_j(t,\xi)$ are not linear function of ξ, hence, we cannot apply Corollary 4.2 directly.

Proof First of all, we remark that, by the implicit function theorem, the roots $\tau_j = \tau_j(t,\xi)$ are analytic functions for ξ large, thus so are the functions $Q\big(t,\tau_j(t,\xi),\xi\big)$, $j = 1,2,3$.

Let

$$\begin{aligned}
\Phi_1(t,\xi) &= Q\big(t,\tau_1(t,\xi),\xi\big) + Q\big(t,\tau_2(t,\xi),\xi\big) + Q\big(t,\tau_3(t,\xi),\xi\big)\\
\Phi_2(t,\xi) &= Q\big(t,\tau_1(t,\xi),\xi\big)\, Q\big(t,\tau_2(t,\xi),\xi\big)\\
&\qquad + Q\big(t,\tau_2(t,\xi),\xi\big)\, Q\big(t,\tau_3(t,\xi),\xi\big) + Q\big(t,\tau_3(t,\xi),\xi\big)\, Q\big(t,\tau_1(t,\xi),\xi\big)\\
\Phi_3(t,\xi) &= Q\big(t,\tau_1(t,\xi),\xi\big)\, Q\big(t,\tau_2(t,\xi),\xi\big)\, Q\big(t,\tau_3(t,\xi),\xi\big)\,.
\end{aligned}$$

These functions are symmetric in the τ_j's and so they are polynomials in ξ, with analytic coefficients in t.

For a given compact $\mathscr{K} \subset \mathscr{O}$, consider another compact $\mathscr{L} \subset \mathscr{O}$ such that $\mathscr{K} \subset \mathring{\mathscr{L}}$. By Corollary 4.2, there exists $\nu \in \mathbb{N}$ such that, for any $j = 1,2,3$ and $\xi \in \mathbb{R}^n$ either $\Phi_j(\cdot,\xi)$ is identically zero or $\Phi_j(\cdot,\xi)$ has at most ν zeros in $\mathscr{L}$.

Assume at first that $\Phi_3(\cdot,\xi) \not\equiv 0$.

Let $\xi_0 \in \mathbb{R}^n \setminus \{0\}$ be fixed, if $\Phi_3(\cdot,\xi_0) \not\equiv 0$, then $\Phi_3(\cdot,\xi_0)$ has at most ν zeros in $\mathscr{L}$, hence, also the $Q\big(t,\tau_j(t,\xi_0),\xi_0\big)$s have at most ν zeros in $\mathscr{L}$. If $\Phi_3(\cdot,\xi_0) \equiv 0$ but one of the $Q\big(t,\tau_j(t,\xi_0),\xi_0\big)$ is not identically zero, then it has η zeros in $\mathscr{L}$. Since $\Phi_3(\cdot,\xi) \not\equiv 0$, we can find a sequence $\{\xi^{(k)}\}_{k\in\mathbb{N}}$ convergent to ξ_0 and such that $\Phi_3(\cdot,\xi^{(k)}) \not\equiv 0$. If k is large enough, by Rouché Theorem, $Q\big(t,\tau_j(t,\xi^{(k)}),\xi^{(k)}\big)$ has at most ν zeros in $\mathscr{L}$, then $Q\big(t,\tau_j(t,\xi_0),\xi_0\big)$ has $\eta \le \nu$ zeros in $\mathscr{K}$.

If $\Phi_3(t,\xi) \equiv 0$, then at least one of the $Q\big(t,\tau_j(t,\xi),\xi\big)$ (say $Q\big(t,\tau_3(t,\xi),\xi\big)$) is identically zero, thus $\Phi_2(\cdot,\xi)$ reduces to $Q\big(t,\tau_1(t,\xi),\xi\big)\, Q\big(t,\tau_2(t,\xi),\xi\big)$.

We can repeat the same argument as before: If $\Phi_2(\cdot,\xi)$ is not identical zero, then $Q\big(t,\tau_1(t,\xi),\xi\big)$ and $Q\big(t,\tau_2(t,\xi),\xi\big)$ have at most ν zeros in $\mathscr{K}$, whereas if $\Phi_2(\cdot,\xi)$ is identical zero, at least one between $Q\big(t,\tau_1(t,\xi),\xi\big)$ and $Q\big(t,\tau_2(t,\xi),\xi\big)$ vanishes identically. If $Q\big(t,\tau_2(t,\xi),\xi\big)$ is identically zero, then $\Phi_1(t,\xi) = Q\big(t,\tau_1(t,\xi),\xi\big)$, and we get immediately that $Q\big(t,\tau_1(t,\xi),\xi\big)$ is either identically zero or it has at most ν zeros in $\mathscr{K}$. □

Proposition 4.5 *Let $P(t,\tau,\xi)$ be as in Proposition 4.3 and let $Q(t,\tau,\sigma,\xi)$ be a symmetric polynomial in (τ,σ) whose coefficients are polynomial in ξ and analytic in $t \in \mathscr{O}$.*

Then, for any compact set $\mathscr{K} \subset \mathscr{O}$, *there exists* $\nu \in \mathbb{N}$ *such that for any* ξ, *the functions*

$$t \mapsto Q\big(t, \tau_j(t,\xi), \tau_k(t,\xi), \xi\big)\,, \qquad (j,k) \in \mathcal{S}_2\,,$$

are either identically zero or have at most ν *zeros in* $\mathscr{K}$.

The proof is similar to that of Proposition 4.3 and is obtained by considering the functions

$$\begin{aligned}
\Phi_1(t,\xi) &= Q_{1,2}(t,\xi) + Q_{2,3}(t,\xi) + Q_{3,1}(t,\xi)\\
\Phi_2(t,\xi) &= Q_{1,2}(t,\xi)\,Q_{2,3}(t,\xi) + Q_{2,3}(t,\xi)\,Q_{3,1}(t,\xi) + Q_{3,1}(t,\xi)\,Q_{1,2}(t,\xi)\\
\Phi_3(t,\xi) &= Q_{1,2}(t,\xi)\,Q_{2,3}(t,\xi)\,Q_{3,1}(t,\xi)\,,
\end{aligned}$$

where for the sake of brevity, we have set

$$Q_{j,k}(t,\xi) = Q\big(t, \tau_j(t,\xi), \tau_k(t,\xi), \xi\big)\,.$$

Proposition 4.6 *Let*

$$\Xi = \Big\{\ \xi \in \mathbb{R}^n\ \Big|\ |\xi| \geqslant R\ \Big\}\,,$$

for some $R > 0$ *and let* $f\,:\,[0,T] \times \Xi \to \mathbb{R}$ *be such that*

(1) *f is Lipschitz continuous in t, uniformly with respect to* ξ, *that is there exists* $C_0 > 0$ *such that*

$$\big|f(t_1,\xi) - f(t_2,\xi)\big| \leqslant C_0 |t_1 - t_2|\,, \qquad \text{for any } t_1, t_2 \in [0,T] \text{ and } \xi \in \Xi\,;$$

(2) *there exist positive constants* A, C_1, C_2 *such that*

$$C_1\,\big(1 + |\xi|\big)^{-A} \leqslant f(t,\xi) \leqslant C_2\,\big(1 + |\xi|\big)^{A}$$

for any $t \in [0,T]$ and $\xi \in \Xi$;

(3) *there exists* $\nu \in \mathbb{N}$ *such that for any* $\xi \in \Xi$, *there exists a partition* $0 = t_1 < t_2 < \cdots < t_{\mu-1} < t_\mu = T$ *of* $[0,T]$, *with* $\mu \leq \nu$ *such that*

- $f(\cdot,\xi) \in \mathcal{C}^1\big(\big]t_j, t_{j+1}\big[\big)$, for $j = 1, 2, \ldots, \mu - 1$;
- $\partial_t f(t,\xi) \neq 0$ for any $t \in \big]t_j, t_{j+1}\big[$, for $j = 1, 2, \ldots, \mu - 1$.

Then, f satisfies the logarithmic condition

$$\int_0^T \frac{\big|\partial_t f(t,\xi)\big|}{f(t,\xi)}\,\mathrm{d}t \lesssim \log\big(1 + |\xi|\big)\,, \quad \text{for any } \xi \in \Xi\,.$$

Proof Let us fix ξ. As $\partial_t f(t,\xi)$ does not change sign in $\big]t_j, t_{j+1}\big[$, we have

$$\int_{t_j}^{t_{j+1}} \frac{|\partial_t f(t,\xi)|}{f(t,\xi)}\,\mathrm{d}t = \left|\int_{t_j}^{t_{j+1}} \frac{\partial_t f(t,\xi)}{f(t,\xi)}\,\mathrm{d}t\right| = \left|\log f(t_{j+1},\xi) - \log f(t_j,\xi)\right| \leqslant C^*\ \log\big(1+|\xi|\big)\,,$$

where C^* depends on A, C_1, C_2. Hence,

$$\int_0^T \frac{|\partial_t f(t,\xi)|}{f(t,\xi)}\,\mathrm{d}t \leqslant \nu\, C^*\ \log\big(1+|\xi|\big)\,,$$

where ν and C^* are independent of ξ. □

4.1 Study of the condition (1.9)

Let

$$f(t,\xi) = \big|\lambda_j(t,\xi) - \lambda_h(t,\xi)\big|\,.$$

The function $f(t,\xi)$ never vanishes and its critical points verify

$$\partial_t\big(\lambda_j(t,\xi) - \lambda_h(t,\xi)\big) = 0\,.$$

Now, by the implicit function Theorem, we have

$$\partial_t\lambda_j(t,\xi) = -\frac{(\partial_t L_\varepsilon)(\lambda_j)}{(\partial_\tau L_\varepsilon)(\lambda_j)}\,, \tag{4.1}$$

where for the sake of brevity, we write

$$(\partial_t L_\varepsilon)(\lambda_j) = \partial_t L_\varepsilon(t,\tau,\xi)\ \big|_{\tau=\lambda_j(t,\xi)}\,,$$
$$(\partial_\tau L_\varepsilon)(\lambda_j) = \partial_\tau L_\varepsilon(t,\tau,\xi)\ \big|_{\tau=\lambda_j(t,\xi)}\ .$$

From (4.1), we get

$$\Big[\partial_t\big(\lambda_j(t_1,\xi) - \lambda_h(t_1,\xi)\big)\Big]^2 = \frac{Q(x,\lambda_j,\lambda_h,\xi)}{\big[(\partial_\tau L_\varepsilon)(\lambda_j)\big]^2\ \big[(\partial_\tau L_\varepsilon)(\lambda_h)\big]^2}\,,$$

where

$$Q(x,\tau,\sigma,\xi) = \Big[(\partial_t L_\varepsilon)(\tau)\,(\partial_\tau L_\varepsilon)(\sigma) - (\partial_t L_\varepsilon)(\sigma)\,(\partial_\tau L_\varepsilon)(\tau)\Big]^2\,.$$

The polynomial Q verifies the hypothesis of Proposition 4.5, hence the number of zeros of the function $t \mapsto \partial_t f(t,\xi)$ is bounded w.r.t. $\xi \in \Xi$, and, applying Proposition 4.6 to f, we see that (1.9) holds true.

4.2 Study of the condition (1.10)

Let

$$f(t,\xi)=\left|\partial_t\lambda_j(t_1,\xi)-\partial_t\lambda_h(t_1,\xi)\right|+1\,.$$

If $\partial_t f(t,\xi)$ changes sign at t_1, then either $\partial_t\lambda_j(t_1,\xi)-\partial_t\lambda_h(t_1,\xi)=0$ or $\partial_t^2\lambda_j(t_1,\xi)-\partial_t^2\lambda_h(t_1,\xi)=0$.

The first case can be treated as before. For the second, from (4.1), we get

$$\partial_{tt}^2\lambda_j(t,\xi)=\frac{\psi(t,\lambda_j,\xi)}{\left[(\partial_\tau L_\varepsilon)(\lambda_j)\right]^3}\,,$$

where

$$\begin{aligned}\psi(t,\tau,\xi)=\;&2\partial_{t\tau}^2L_\varepsilon(\tau)\,\partial_tL_\varepsilon(\tau)\,\partial_\tau L_\varepsilon(\tau)\\&-\partial_{\tau\tau}^2L_\varepsilon(\tau)\left[\partial_tL_\varepsilon(\tau)\right]^2-\partial_{tt}^2L_\varepsilon(\tau)\left[\partial_\tau L_\varepsilon(\tau)\right]^2,\end{aligned}$$

hence

$$\left[\partial_{tt}^2\big(\lambda_j(t_1,\xi)-\lambda_h(t_1,\xi)\big)\right]^2=\frac{Q(x,\lambda_j,\lambda_h,\xi)}{\left[(\partial_\tau L_\varepsilon)(\lambda_j)\right]^6\left[(\partial_\tau L_\varepsilon)(\lambda_h)\right]^6}\,,$$

where

$$Q(x,\tau,\sigma,\xi)=\left[\psi(t,\tau,\xi)\left[\partial_\tau L_\varepsilon(\sigma)\right]^3-\psi(t,\sigma,\xi)\left[\partial_\tau L_\varepsilon(\tau)\right]^3\right]^2.$$

The polynomial Q verifies the hypothesis of Proposition 4.5, hence the number of zeros of the function $t\mapsto\partial_t f(t,\xi)$ is bounded w.r.t. $\xi\in\Xi$, and, applying Proposition 4.6 to f, we see that (1.10) holds true.

4.3 Study of the condition (1.11)

If M has real coefficients, we consider

$$f(t,\xi)=\left|\check{M}\big(t,\lambda_j(t,\xi),\xi\big)\right|+1\,.$$

For fixed ξ, the oscillations of $f(t,\xi)$ are zeros in t of either $\check{M}\big(t,\lambda_j(t,\xi),\xi\big)$ or $\partial_t\check{M}\big(t,\lambda_j(t,\xi),\xi\big)$.

Now

$$\partial_t\check{M}\big(t,\lambda_j(t,\xi),\xi\big)=\partial_t\check{M}(t,\tau,\xi)\,\big|_{\tau=\lambda_j(t,\xi)}+\partial_\tau\check{M}(t,\tau,\xi)\,\big|_{\tau=\lambda_j(t,\xi)}\,\partial_t\lambda_j(t,\xi)\,,$$

and, by (4.1), we see that $\partial_t\check{M}\big(t,\lambda_j(t,\xi),\xi\big)=0$ if, and only if, $Q\big(t,\lambda_j(t,\xi),\xi\big)=0$, where

$$Q(t,\tau,\xi)=\partial_t\check{M}(t,\tau,\xi)\partial_\tau L_\varepsilon(t,\tau,\xi)-\partial_\tau\check{M}(t,\tau,\xi)\partial_t L_\varepsilon(t,\tau,\xi)\,.$$

The polynomial Q verifies the hypothesis of Proposition 4.5, hence the number of zeros of the function $t\mapsto\partial_t f(t,\xi)$ is bounded w.r.t. $\xi\in\Xi$, and, applying Proposition 4.6 to f, we see that (1.11) holds true.

If the coefficients of $\check{M}$ are complex, we consider the zeros of

$$\partial_t|\check{M}|^2 = 2\mathrm{Re}\big(\partial_t\check{M}\,\overline{\check{M}}\big) = 2\partial_t\mathrm{Re}(\check{M})\,\mathrm{Re}(\check{M}) + 2\partial_t\Im(\check{M})\,\Im(\check{M})$$

and by the same argument, we get (1.11) again.

The proof that condition (1.12) is satisfied is similar to that of condition (1.11), so we omit it.

5 Pointwise Levi conditions

Throughout this section, we assume that the coefficients of the operator are analytic, and we express the Levi conditions (1.13) and (1.14) as pointwise conditions.

We have to distinguish three cases:

Case I $\Delta_L \not\equiv 0$;
Case II $\Delta_L \equiv 0$ and $\Delta_L^{(1)} \not\equiv 0$;
Case III $\Delta_L \equiv \Delta_L^{(1)} \equiv 0$.

Case I: $\Delta_L \not\equiv 0$
We consider at first the terms of order 2.

Proposition 5.1 *Assume that*

$$\left|\tau_k(t,\xi) - \tau_l(t,\xi)\right| \left|\check{M}\big(t,\tau_j(t,\xi),\xi\big)\right| \lesssim \sqrt{\Delta_L(t,\xi)} + \left|\partial_t\sqrt{\Delta_L(t,\xi)}\right|, \tag{5.1}$$

for all $(j,k,l) \in \mathcal{S}_3$, *then condition* (1.13) *is verified.*

For the proof, we need the following Lemma [20, Proposition 4.1].

Lemma 5.2 *Let* $\Delta(t,\xi)$ *be an homogeneous polynomial in* ξ *with coefficient analytic in* t *and assume that* $\Delta(t,\xi) \not\equiv 0$. *Then*:

1. *there exists* $X \subset \mathbb{S}^n := \left\{\, \xi \in \mathbb{R}^n \;\middle|\; |\xi| = 1 \,\right\}$ *such that* $\Delta(t,\xi) \not\equiv 0$ *in* $]-\delta, T+\delta[$ *for any* $\xi \in X$, *and the set* $\mathbb{S}^n \setminus X$ *is negligible with respect to the Hausdorff* $(n-1)$*–measure*;
2. *for any* $[a,b] \subset\,]-\delta, T+\delta[$, *we can find constants* $c_1, c_2 > 0$ *and* $p, q \in \mathbb{N}$ *such that for any* $\xi \in X$ *and any* $\varepsilon \in (0, 1/e]$ *there exists* $A_{\xi,\varepsilon} \subset [a,b]$ *such that*

1. $A_{\xi,\varepsilon}$ *is a union of at most* p *disjoint intervals*,
2. $m(A_{\xi,\varepsilon}) \leqslant \varepsilon$,
3. $\min_{t\in[a,b]\setminus A_{\xi,\varepsilon}} \Delta(t,\xi) \geqslant c_1\,\varepsilon^{2q}\left\|\Delta(\cdot,\xi)\right\|_{L^\infty([a,b])}$
4. $\displaystyle\int_{[a,b]\setminus A_{\xi,\varepsilon}} \frac{|\Delta'(t,\xi)|}{\Delta(t,\xi)}\,\mathrm{d}t \leqslant c_2\,\log\frac{1}{\varepsilon}$.

Proof (Proof of Proposition 5.1) We will prove that (5.1) implies (3.3a) and (3.3b) hence, thanks to Proposition 3.1, we get (1.13).

Let $\varepsilon = |\xi|^{-2}$, with $|\xi|$ large enough, and let $A_{\xi,\varepsilon}$ be the set given by Lemma 5.2 with $\Delta(t,\xi) = \Delta_L(t,\xi)$; we have

$$\int_{A_{\xi,\varepsilon}} \frac{\big|\check{M}(\tau_j)\big|}{\big(|\tau_j - \tau_k| + 1\big)\big(|\tau_j - \tau_l| + 1\big)}\,\mathrm{d}t \lesssim \int_{A_{\xi,\varepsilon}} |\xi|^2\,\mathrm{d}t \lesssim \varepsilon\,|\xi|^2 = 1\,. \tag{5.2}$$

On the other side, as

$$\frac{\big|\check{M}(\tau_j)\big|}{\big(|\tau_j - \tau_k| + 1\big)\big(|\tau_j - \tau_l| + 1\big)} \leqslant \frac{|\tau_k - \tau_l|\,\big|\check{M}(\tau_j)\big|}{\sqrt{\Delta_L(t,\xi)}}\,,$$

assuming (5.1), thanks to Lemma 5.2, we have

$$\begin{aligned}\int_{]-\delta,T+\delta[\setminus A_{\xi,\varepsilon}} &\frac{\big|\check{M}(\tau_j)\big|}{\big(|\tau_j - \tau_k| + 1\big)\big(|\tau_j - \tau_l| + 1\big)}\,\mathrm{d}t\\ &\lesssim \int_{]-\delta,T+\delta[\setminus A_{\xi,\varepsilon}} 1 + \frac{\big|\partial_t \Delta_L(t,\xi)\big|}{\Delta_L(t,\xi)}\,\mathrm{d}t \lesssim \log\big(1 + |\xi|\big)\,.\end{aligned} \tag{5.3}$$

Combining (5.2) and (5.3), we get (3.3a).

Now we prove that (3.3b) holds true. As the roots of L are distinct for a.e. (t,ξ), by the Lagrange interpolation formula, we have

$$\check{M}(t,\tau,\xi) = \ell_1(t,\xi)\,L_{23}(t,\tau,\xi) + \ell_2(t,\xi)\,L_{13}(t,\tau,\xi) + \ell_3(t,\xi)\,L_{12}(t,\tau,\xi)\,, \tag{5.4}$$

where the operators L_{jk} are the operators $L_{jk,\varepsilon}$ defined in Sect. 2 with $\varepsilon = 0$, the ℓ_j are given by

$$\ell_j(t,\xi) := \frac{\check{M}\big(t,\tau_j(t,\xi),\xi\big)}{\big(\tau_j(t,\xi) - \tau_h(t,\xi)\big)\big(\tau_j(t,\xi) - \tau_l(t,\xi)\big)} \tag{5.5}$$

hence, by Hypothesis (5.1), they verify

$$\big|\ell_j(t,\xi)\big| \lesssim 1 + \frac{\big|\partial_t \Delta_L(t,\xi)\big|}{\Delta_L(t,\xi)}\,. \tag{5.6}$$

On the other side, differentiating (5.4) w.r.t τ, we get

$$\begin{aligned}\partial_\tau \check{M}(\tau) &= \ell_1\,(L_2 + L_3) + \ell_2\,(L_3 + L_1) + \ell_3\,(L_1 + L_2)\\ &= (\ell_2 + \ell_3)\,L_1 + (\ell_3 + \ell_1)\,L_2 + (\ell_1 + \ell_2)\,L_3\,,\end{aligned}$$

where the operators L_j are the operators $L_{j,\varepsilon}$ defined in Sect. 2 with $\varepsilon = 0$,

Now, since $\tau_1 \leqslant \tau_2 \leqslant \tau_3$, we can find $\theta \in [0,1]$ such that

$$\tau_2 = \theta\,\tau_1 + (1-\theta)\,\tau_3\,,$$

hence

$$L_2 = \theta\,L_1 + (1-\theta)\,L_3\,,$$

consequently, we can write

$$\partial_\tau \check{M}(\tau) = \widetilde{\ell}_3\, L_1(\tau) + \widetilde{\ell}_1\, L_3(\tau)\,, \tag{5.7}$$

where $\widetilde{\ell}_1$ and $\widetilde{\ell}_3$ are some linear combination with bounded coefficients of ℓ_1, ℓ_2 and ℓ_3, hence the $\widetilde{\ell}_j$s verify (5.6) too.

Let $\varepsilon = |\xi|^{-1}$, with $|\xi|$ large enough, and let $A_{\xi,\varepsilon}$ be as above, we have

$$\int_{A_{\xi,\varepsilon}} \frac{\left|\partial_\tau \check{M}(\tau_1)\right| + \left|\partial_\tau \check{M}(\tau_3)\right|}{|\tau_1 - \tau_3| + 1}\, \mathrm{d}t \lesssim \int_{A_{\xi,\varepsilon}} |\xi|\, \mathrm{d}t \lesssim \varepsilon\, |\xi| = 1\,,$$

whereas, thanks to (5.7):

$$\begin{aligned}
\int_{]-\delta,T+\delta[\setminus A_{\xi,\varepsilon}} \frac{\left|\partial_\tau \check{M}(\tau_1)\right| + \left|\partial_\tau \check{M}(\tau_3)\right|}{|\tau_1 - \tau_3| + 1}\, \mathrm{d}t &\lesssim \int_{]-\delta,T+\delta[\setminus A_{\xi,\varepsilon}} \left|\widetilde{\ell}_1(t,\xi)\right| + \left|\widetilde{\ell}_3(t,\xi)\right| \mathrm{d}t \\
&\lesssim \int_{]-\delta,T+\delta[\setminus A_{\xi,\varepsilon}} 1 + \frac{\left|\partial_t \Delta_L(t,\xi)\right|}{\Delta_L(t,\xi)}\, \mathrm{d}t \\
&\lesssim \log\bigl(1 + |\xi|\bigr)\,.
\end{aligned}$$

Combining the above estimates, we get (3.3b). □

Condition (5.1) means that if, for fixed ξ, the function

$$t \mapsto \bigl(\tau_j(t,\xi) - \tau_k(t,\xi)\bigr)\,\bigl(\tau_j(t,\xi) - \tau_l(t,\xi)\bigr)$$

vanishes of order ν at $t = t_0$, then the function

$$t \mapsto \check{M}\bigl(t, \tau_j(t,\xi), \xi\bigr)$$

must vanish (at least) at order $\nu - 1$ at $t = t_0$. Thus, in space dimension $n = 1$, Proposition 5.1 can be precised.

Proposition 5.3 *In space dimension $n = 1$, (1.13) is equivalent to the following condition:*

there exist $t_1, \dots, t_\nu \in\,]-\delta, T + \delta[$ such that

$$\prod_{h=1}^{\nu} |t - t_h|\, \left|\check{M}\bigl(t, \tau_j(t,\xi), \xi\bigr)\right| \lesssim \left|\tau_j(t,\xi) - \tau_k(t,\xi)\right| \left|\tau_j(t,\xi) - \tau_l(t,\xi)\right|, \tag{5.8}$$

with j, k, l such that $(j,k,l) \in \mathcal{S}_3$.

Remark 5.4 In the special case $\nu = 1$ and $t_1 = 0$, condition (5.8) reduces to

$$|t|\, \left|\check{M}\bigl(t, \tau_j(t,\xi), \xi\bigr)\right| \lesssim \left|\tau_j(t,\xi) - \tau_k(t,\xi)\right| \left|\tau_j(t,\xi) - \tau_l(t,\xi)\right|, \tag{5.9}$$

for any $(j,k,l) \in \mathcal{S}_3$.

Proof By the previous remark, we see that (5.8) is equivalent to (5.1) and implies (3.3a) and (3.3b).

Now we prove that (3.3a) implies (5.8) by contradiction. First of all, as the zeros of Δ_L are isolated, we can decompose $]-\delta, T+\delta[$ into a finite number of contiguous subintervals each containing a zero of Δ_L. Thus, with no loss of generality, we can restrict to the case in which $\Delta_L(t)$ vanishes only at $t=0$; in this case, condition (5.8) reduces to (5.9).

Suppose that (5.9) is violated, hence, with no loss of generality, we have

$$\frac{\left|\check{M}\big(t, \tau_3(t)\,\xi, \xi\big)\right|}{\left|\tau_3(t)-\tau_1(t)\right|\left|\tau_3(t)-\tau_2(t)\right|\,|\xi|^2} \gtrsim \frac{1}{t^m}$$

for some $m \geqslant 2$.

As $\Delta_L(0)=0$ and $\Delta_L(t) \neq 0$ for $t \neq 0$, there exist r_1, r_2 such that

$$\left|\tau_3(t)-\tau_1(t)\right| \gtrsim t^{r_1} \qquad \left|\tau_3(t)-\tau_2(t)\right| \gtrsim t^{r_2}$$

and $r \overset{\text{def}}{=} \max(r_1, r_2) \geqslant 1$. Note that, $\min(r_1, r_2) > 0$ if and only if $t=0$ is a triple point.

For $t \geqslant \varepsilon^{1/r}$, $\varepsilon = \frac{1}{|\xi|}$ and $|\xi| \geqslant 1$, we have

$$\left|\tau_3(t)-\tau_1(t)\right|\,|\xi| \gtrsim 1 \qquad \left|\tau_3(t)-\tau_2(t)\right|\,|\xi| \gtrsim 1\,,$$

hence

$$\begin{aligned}
&\int_{\varepsilon^{1/r}}^{T} \frac{\left|\check{M}\big(t, \tau_3(t)\,\xi, \xi\big)\right|}{\Big(\left|\tau_3(t)-\tau_1(t)\right|\,|\xi|+1\Big)\Big(\left|\tau_3(t)-\tau_2(t)\right|\,|\xi|+1\Big)}\,\mathrm{d}t \\
&\quad \gtrsim \int_{\varepsilon^{1/r}}^{T} \frac{1}{t^m}\,\frac{\left|\tau_3(t)-\tau_1(t)\right|\,|\xi|}{\left|\tau_3(t)-\tau_1(t)\right|\,|\xi|+1}\,\frac{\left|\tau_3(t)-\tau_2(t)\right|\,|\xi|}{\left|\tau_3(t)-\tau_2(t)\right|\,|\xi|+1}\,\mathrm{d}t \gtrsim \int_{\varepsilon^{1/r}}^{T} \frac{1}{t^m}\,\mathrm{d}t \approx |\xi|^{\frac{m-1}{r}}\,.
\end{aligned}$$

This shows that (3.3a) cannot hold true if (5.9) is violated. □

Now we consider the terms of order 1.

Proposition 5.5 *Assume that*

$$\left|\check{N}\big(t, \sigma_j(t,\xi), \xi\big)\right| \lesssim \sqrt{\Delta_{\partial_t L}(t,\xi)} + \frac{\big(\partial_t \Delta_{\partial_t L}(t,\xi)\big)^2}{\big[\Delta_{\partial_t L}(t,\xi)\big]^{3/2}}\,, \qquad j=1,2\,, \tag{5.10}$$

then condition (1.14) *is verified.*

Proof First of all, we recall that, by Proposition 3.2, (1.14) is equivalent to (3.9), hence we will prove that (5.10) implies (3.9).

Let $\varepsilon = |\xi|^{-1/2}$, with $|\xi|$ large enough, and let $A_{\xi,\varepsilon}$ be the set given by Lemma 5.2 with $\Delta(t,\xi) = \Delta_{\partial_t L}(t,\xi)$; we have

$$\int_{A_{\xi,\varepsilon}} \sqrt{\frac{\left|\check{N}\big(t,\sigma_j(t,\xi),\xi\big)\right|}{\left|\sigma_2(t,\xi)-\sigma_1(t,\xi)\right|+1}}\,\mathrm{d}t \lesssim \int_{A_{\xi,\varepsilon}} |\xi|^{1/2}\,\mathrm{d}t \lesssim \varepsilon\,|\xi|^{1/2} = 1\,.$$

On the other side, assuming (5.10) and thanks to Lemma 5.2, we have

$$\begin{aligned}\int_{]-\delta,T+\delta[\setminus A_{\xi,\varepsilon}} &\sqrt{\frac{\left|\check{N}\big(t,\sigma_j(t,\xi),\xi\big)\right|}{\left|\sigma_2(t,\xi)-\sigma_1(t,\xi)\right|+1}}\,\mathrm{d}t \\ &\lesssim \int_{]-\delta,T+\delta[\setminus A_{\xi,\varepsilon}} 1+\frac{\left|\partial_t\Delta_{\partial_t L}(t,\xi)\right|}{\Delta_{\partial_t L}(t,\xi)}\,\mathrm{d}t \lesssim \log\big(1+|\xi|\big)\,.\end{aligned}$$

Combining the above estimates, we get (3.9). □

Condition (5.10) means that if, for fixed ξ, the function $t\mapsto\Delta_{\partial_t L}(t,\xi)$ vanishes of order 2ν, with $\nu>2$, at $t=t_0$, then the function $t\mapsto\check{N}\big(t,\tau_j(t,\xi),\xi\big)$ must vanish (at least) at order $\nu-2$ at $t=t_0$. Thus, in space dimension $n=1$, Proposition 5.5 can be precised.

Proposition 5.6 *In space dimension $n=1$, (5.10) is equivalent to (1.14) and can be written in the following form: there exist $t_1,\dots,t_\nu$ such that*

$$\prod_{j=1}^{\nu}(t-t_j)^2\left|\check{N}\big(t,\sigma_1(t,\xi),\xi\big)\right| \lesssim \left|\sigma_2(t,\xi)-\sigma_1(t,\xi)\right|. \tag{5.11}$$

Proof We prove that if $n=1$, (3.9) implies (5.11). As before, with no loss of generality, we can assume that $\Delta_{\partial_t L}$ vanishes only in 0, and there exists $r\geqslant 1$ such that

$$\left|\sigma_2(t)-\sigma_1(t)\right|\gtrsim t^r\,.$$

Hence, for $t\geqslant\varepsilon^{1/r}$, $\varepsilon=\frac{1}{|\xi|}$ and $|\xi|\geqslant 1$, we have

$$\left|\sigma_2(t)-\sigma_1(t)\right|\,|\xi|\gtrsim 1\,,$$

and, consequently,

$$\frac{\left|\sigma_2(t)-\sigma_1(t)\right|\,|\xi|}{\left|\sigma_2(t)-\sigma_1(t)\right|\,|\xi|+1}\gtrsim 1\,.$$

If (5.11) fails to hold, then there exists $m\geqslant 3$ such that

$$\frac{\check{N}\big(t,\sigma_j(t,\xi),\xi\big)}{\sigma_2(t,\xi)-\sigma_1(t,\xi)}\approx\frac{1}{t^m}\quad\text{for } j=1 \text{ or } j=2\,,$$

and we have

$$\int_{\varepsilon^{1/r}}^{T} \sqrt{\frac{\left|\check{N}\big(t,\sigma_j(t,\xi),\xi\big)\right|}{\left|\sigma_3(t,\xi)-\sigma_1(t,\xi)\right|+1}}\,\mathrm{d}t \gtrsim \int_{\varepsilon^{1/r}}^{T}\frac{1}{t^{m/2}}\,\mathrm{d}t\approx|\xi|^{\frac{m/2-1}{r}}\,,$$

thus (3.9) cannot be satisfied. □

Case II: $\Delta_L \equiv 0$ *and* $\Delta_L^{(1)} \not\equiv 0$
With no loss of generality, we can assume that $\tau_1 \equiv \tau_2$ and $\tau_3 \not\equiv \tau_1$.

Proposition 5.7 *Assume that* $\tau_1 \equiv \tau_2$ *and* $\tau_3 \not\equiv \tau_1$. *If*

$$\check{M}\big(t, \tau_1(t,\xi), \xi\big) \equiv 0 \tag{5.12a}$$

and

$$\Big|Q\big(t, \tau_j(t,\xi), \xi\big)\Big| \lesssim \sqrt{\Delta_L^{(1)}(t,\xi)} + \Big|\partial_t \sqrt{\Delta_L^{(1)}(t,\xi)}\Big|, \tag{5.12b}$$

for $j = 1, 3$, *where* $Q(t,\tau,\xi) = \frac{\check{M}(t,\tau,\xi)}{\tau - \tau_1(t,\xi)}$, *then condition* (1.13) *is verified.*

Proof The proof is similar to that of Proposition 5.1: We prove that (5.12a) and (5.12b) imply (3.3a) and (3.3b), hence, by Proposition 3.1, we get (1.13).

Note that, by (5.12a), (3.3a) with $j = 1$ or $j = 2$ is trivially satisfied, thus we need only to prove (3.3a) with $j = 3$.

Let $\varepsilon = |\xi|^{-2}$ with $|\xi|$ large enough, and let $A_{\xi,\varepsilon}$ be the set given by Lemma 5.2 with $\Delta(t,\xi) = \Delta_L^{(1)}(t,\xi)$; we have

$$\int_{A_{\xi,\varepsilon}} \frac{\big|\check{M}(\tau_3)\big|}{\big(|\tau_3 - \tau_1| + 1\big)^2}\, \mathrm{d}t \lesssim \int_{A_{\xi,\varepsilon}} |\xi|^2\, \mathrm{d}t \lesssim \varepsilon\, |\xi|^2 = 1\,.$$

On the other side, as

$$\frac{\big|\check{M}(\tau_3)\big|}{\big(|\tau_3 - \tau_1| + 1\big)^2} \le \frac{\big|Q(\tau_3)\big|}{\sqrt{\Delta_L^{(1)}}},$$

assuming (5.12b), thanks to Lemma 5.2, we have

$$\int_{]-\delta, T+\delta[\setminus A_{\xi,\varepsilon}} \frac{\big|\check{M}(\tau_3)\big|}{\big(|\tau_3 - \tau_1| + 1\big)^2}\, \mathrm{d}t \lesssim \int_{]-\delta, T+\delta[\setminus A_{\xi,\varepsilon}} 1 + \frac{\big|\partial_t \Delta_L^{(1)}(t,\xi)\big|}{\Delta_L^{(1)}(t,\xi)}\, \mathrm{d}t \lesssim \log\big(1 + |\xi|\big)\,.$$

Now we prove that (3.3b) holds true.

As $\tau_3 \not\equiv \tau_1$, by Lagrange interpolation formula, we get

$$Q(\tau) = \frac{Q(\tau_3)}{\tau_3 - \tau_1}\,(\tau - \tau_1) + \frac{Q(\tau_1)}{\tau_1 - \tau_3}\,(\tau - \tau_3)$$

for a.e. (t,ξ), hence

$$\check{M}(\tau) = \frac{Q(\tau_3)}{\tau_3 - \tau_1}\, L_{12} + \frac{Q(\tau_1)}{\tau_1 - \tau_3}\, L_{13}\,, \tag{5.13}$$

and, consequently,

$$\partial_\tau \check{M}(\tau) = \frac{Q(\tau_3)}{\tau_3 - \tau_1}(L_1 + L_2) + \frac{Q(\tau_1)}{\tau_1 - \tau_3}(L_1 + L_3)\,. \tag{5.14}$$

Let $\varepsilon = |\xi|^{-1}$, with $|\xi|$ large enough, and let $A_{\xi,\varepsilon}$ be as above, we have

$$\int_{A_{\xi,\varepsilon}} \frac{\left|\partial_\tau \check{M}(\tau_1)\right| + \left|\partial_\tau \check{M}(\tau_3)\right|}{|\tau_1 - \tau_3| + 1}\,\mathrm{d}t \lesssim \int_{A_{\xi,\varepsilon}} |\xi|\,\mathrm{d}t \lesssim \varepsilon\,|\xi| = 1\,,$$

whereas, thanks to (5.14) and Lemma 5.2 we have

$$\int_{]-\delta,T+\delta[\setminus A_{\xi,\varepsilon}} \frac{\left|\partial_\tau \check{M}(\tau_1)\right| + \left|\partial_\tau \check{M}(\tau_3)\right|}{|\tau_1 - \tau_3| + 1}\,\mathrm{d}t \lesssim \int_{]-\delta,T+\delta[\setminus A_{\xi,\varepsilon}} \frac{\left|Q(\tau_1)\right| + \left|Q(\tau_3)\right|}{|\tau_1 - \tau_3| + 1}\,\mathrm{d}t$$

hence, by (5.12b),

$$\begin{aligned}\int_{]-\delta,T+\delta[\setminus A_{\xi,\varepsilon}} \frac{\left|\partial_\tau \check{M}(\tau_1)\right| + \left|\partial_\tau \check{M}(\tau_3)\right|}{|\tau_1 - \tau_3| + 1}\,\mathrm{d}t &\lesssim \int_{]-\delta,T+\delta[\setminus A_{\xi,\varepsilon}} 1 + \frac{\left|\partial_\tau \Delta_L^{(1)}(t,\xi)\right|}{\Delta_L^{(1)}(t,\xi)}\,\mathrm{d}t \\ &\lesssim \log\big(1 + |\xi|\big)\,.\end{aligned}$$

□

Proposition 5.8 *In space dimension $n = 1$, (1.13) is equivalent to (5.12a) and (5.12b).*

Moreover, (5.12b) can be written in the following form: There exist $t_1, \dots, t_\nu$ such that

$$\prod_{j=1}^{\nu} |t - t_j| \left| Q\big(t, \tau_k(t,\xi), \xi\big)\right| \lesssim \left|\tau_3(t,\xi) - \tau_1(t,\xi)\right|, \qquad k = 1, 3\,. \tag{5.15}$$

Remark 5.9 In the special case $\nu = 1$ and $t_1 = 0$, condition (5.15) reduces to

$$|t| \left| Q\big(t, \tau_k(t,\xi), \xi\big)\right| \lesssim \left|\tau_3(t,\xi) - \tau_1(t,\xi)\right|, \qquad k = 1, 3\,. \tag{5.16}$$

Proof We prove that if $\tau_1 \equiv \tau_2$, then (5.12a) is necessary in order to have (1.13).

Indeed if (5.12a) fails to hold then $\check{M}\big(t, \tau_1(t,\xi), \xi\big) \approx |\xi|^2$ and we have

$$\frac{\check{M}(\tau_1)}{\big(|\tau_1 - \tau_2| + 1\big)\big(|\tau_1 - \tau_3| + 1\big)} = \frac{\check{M}(\tau_1)}{|\tau_1 - \tau_3| + 1} \approx \frac{|\xi|^2}{|\xi| + 1}\,,$$

hence (3.3a) cannot be verified.

As before, we can assume that $\Delta_L^{(1)}$ vanishes only in 0, so that (5.15) reduces to (5.16). Moreover, we can assume that there exists $r \geqslant 1$ such that for $t \geqslant \varepsilon^{1/r}$, $\varepsilon = \frac{1}{|\xi|}$ and $|\xi| \geqslant 1$ we have

$$\left|\tau_3(t) - \tau_1(t)\right| |\xi| \gtrsim 1\,.$$

If (5.16) fails to hold, then there exists $m \geqslant 2$ such that

$$\frac{\check{Q}\big(t,\tau_3(t,\xi),\xi\big)}{\tau_3(t,\xi)-\tau_1(t,\xi)} \approx \frac{1}{t^m}\,,$$

and, as

$$\check{M}(t,\tau,\xi) = Q(t,\tau,\xi)\left(\tau-\tau_1(t,\xi)\right)$$

we have

$$\partial_\tau \check{M}(t,\tau_1,\xi) = Q(t,\tau_1,\xi)\,,$$

hence

$$\int_{\varepsilon^{1/r}}^{T} \frac{\left|\partial_\tau \check{M}(\tau_1)\right|}{|\tau_3-\tau_1|+1}\,\mathrm{d}t = \int_{\varepsilon^{1/r}}^{T} \frac{\left|Q(\tau_1)\right|}{|\tau_3-\tau_1|+1}\,\mathrm{d}t \gtrsim \int_{\varepsilon^{1/r}}^{T} \frac{1}{t^m}\,\mathrm{d}t \approx |\xi|^{\frac{m-1}{r}}\,.$$

Thus, (3.3b) cannot be satisfied. □

Note that, we need not to assume $n=1$ to prove the necessity of (5.12a).

The term of order 1 is treated as in the previous case.

Case III: $\Delta_L \equiv \Delta_L^{(1)} \equiv 0$ (operators with triple characteristics of constant multiplicity)

Proposition 5.10 *If $\Delta_L \equiv \Delta_L^{(1)} \equiv 0$, then L has a unique triple root*:

$$L(t,\tau,\xi) = \left(\tau-\tau_1(t,\xi)\right)^3\,, \tag{5.17}$$

and Hypothesis (1.9) *and* (1.10) *are satisfied. Moreover, Hypothesis* (1.11), (1.12), (1.13) *and* (1.14) *are satisfied if, and only if,*

$$\check{M}(\tau_1) \equiv 0\,, \quad \partial_\tau \check{M}(\tau_1) \equiv 0\,, \quad \check{N}(\tau_1) \equiv 0. \tag{5.18}$$

Proof It's clear that if $\Delta_L \equiv \Delta_L^{(1)} \equiv 0$, then L reduces to (5.17).

If L is as in (5.17), we have

$$\mathcal{L}(t,\tau,\xi) = \left(\tau-\tau_1(t,\xi)\right)^3 - 6\left(\tau-\tau_1(t,\xi)\right)$$

and

$$\lambda_1(t,\xi) = \tau_1(t,\xi)\,, \qquad \lambda_2(t,\xi) = \tau_1(t,\xi)+\sqrt{6}\,, \qquad \lambda_3(t,\xi) = \tau_1(t,\xi)-\sqrt{6}\,,$$

and it is clear that (1.9) and (1.10) are satisfied.

It is also clear that (5.18) holds true if, and only if,

$$\check{M}(t,\tau,\xi) = m_0(t)\left(\tau-\tau_1(t,\xi)\right)^2 \quad \text{and} \quad \check{N}(t,\tau,\xi) = n_0(t)\left(\tau-\tau_1(t,\xi)\right).$$

Thus, (1.11), (1.12), (1.13) and (1.14) hold true.

On the converse, if $\check{M}(\tau_1) \not\equiv 0$, then (3.3a) is not satisfied since

$$\int_0^T \left|\check{M}\big(t,\tau_1(t,\xi),\xi\big)\right|\mathrm{d}t \approx |\xi|^2\,.$$

Analogously, if $\partial_\tau \check{M}(\tau_1) \not\equiv 0$, then (3.3b) is not satisfied since

$$\int_0^T \left|\partial_\tau \check{M}\big(t, \tau_1(t,\xi), \xi\big)\right| \mathrm{d}t \approx |\xi| \,.$$

Finally, if $\check{N}(\mu_1) \not\equiv 0$, then (3.9) is not satisfied since

$$\int_0^T \sqrt{\left|\check{N}\big(t, \tau_1(t,\xi), \xi\big)\right|}\, \mathrm{d}t \approx |\xi|^{1/2} \,.$$

□

Conditions (5.18) correspond to the condition of *good decomposition* [13]: The operator P can be written as

$$P = L_1^3 + m_0(t)\, L_1^2 + n_0(t)\, L_1 + p_0 \,, \tag{5.19}$$

where $L_1^3 = L_1 \circ L_1 \circ L_1$ and $L_1^2 = L_1 \circ L_1$.

To see this, following [35], we have to check two conditions. The first is

$$\text{the principal symbol of } P - L_1^3 \text{ is divisible by } m_0(\tau - \tau_1)^2 \,, \tag{5.20}$$

and the second condition is

$$\text{the principal symbol of } \; P - (L_1^3 + m_0(t)\, L_1^2) \text{ is divisible by } (\tau - \tau_1) \,. \tag{5.21}$$

As $\tau_1 \equiv \tau_2 \equiv \tau_3$, the operator $\widetilde{L}_{123,\varepsilon}$ in (2.3) reduces to L_1^3 if $\varepsilon = 0$. According to (2.16), we have

$$P = L_1^3 + \check{M} + \frac{1}{2}\partial_t \partial_\tau \check{M} + \check{N} + p \,,$$

thus (5.20) holds true if, and only if, the first two conditions in (5.18) hold true. In this case,

$$\check{M}(t,\tau,\xi) = m_0(t)\, L_{1,1}(t,\tau,\xi) = m_0(t)\, \big(\tau - \tau_1(t,\xi)\big)^2 \,.$$

As $\tau_1 \equiv \tau_2$, the operator $\widetilde{L}_{12,\varepsilon}$ in (2.2) reduces to L_1^2 if $\varepsilon = 0$. By (2.4) with $\varepsilon = 0$, we have

$$L_1^2 = L_{11} + \frac{1}{2}\, \partial_t \partial_\tau L_{11} \,,$$

and, multiplying by m_0, we get

$$m_0\, L_1^2 = \check{M} + \frac{1}{2}\, \partial_t \partial_\tau \check{M} - m_0'(t)\, L_1 \,,$$

hence

$$P = L_1^3 + m_0\, L_1^2 + m_0'(t)\, L_1 + \check{N} + p \,,$$

thus (5.21) holds true if, and only if, the third condition in (5.18) holds true.

6 Operators with constant coefficients principal part

In this section, we prove the following Proposition

Proposition 6.1 *Assume that the coefficients are analytic and those of the principal symbol are constant.*

Then, Hypothesis (1.9), (1.10), (1.11) *and* (1.12) *are satisfied, whereas* (1.13) *and* (1.14) *are necessary and sufficient for the* $\mathcal{C}^\infty$ *well posedness.*

Proof If the coefficients are analytic, then Hypothesis (1.9), (1.10), (1.11) and (1.12) are satisfied (cf. Sect. 4).

Now, we recall that if the coefficients are constant, the necessary and sufficient conditions for the $\mathscr{C}^\infty$ well posedness is well known, see [16, 17, 34]:

$$\text{there exists } C > 0 \text{ such that } \tau^3 + \sum_{j+|\alpha|\le 3} a_{j,\alpha}(t)\tau^j\xi^\alpha \neq 0 \; \textit{if} \; \xi \in \mathbb{R} \text{ and } |\Im\tau| > C\,. \tag{6.1}$$

Various equivalent conditions have been given. According to [34], (6.1) is equivalent to the following conditions (cf. [31]): there exist bounded functions $\mathsf{m}_1, \mathsf{m}_2, \mathsf{m}_3, \mathsf{n}_1, \mathsf{n}_2$ such that

$$\begin{aligned} M(t,\tau,\xi) = {} & \mathsf{m}_1(t,\xi)\left(\tau - \tau_2(t,\xi)\right)\left(\tau - \tau_3(t,\xi)\right) + \mathsf{m}_2(t,\xi)\left(\tau - \tau_3(t,\xi)\right)\left(\tau - \tau_1(t,\xi)\right) \\ & + \mathsf{m}_3(t,\xi)\left(\tau - \tau_1(t,\xi)\right)\left(\tau - \tau_2(t,\xi)\right) \end{aligned} \tag{6.2}$$

$$N(t,\tau,\xi) = \mathsf{n}_1(t,\xi)\left(\tau - \sigma_2(t,\xi)\right) + \mathsf{n}_2(t,\xi)\left(\tau - \sigma_1(t,\xi)\right) \tag{6.3}$$

for a.e. (t,ξ).

We recall that the above conditions are also necessary and sufficient for the $\mathcal{C}^\infty$ well posedness if the coefficients of the lower-order terms can vary [14, 36] (also if the coefficients of the lower-order terms are only $\mathcal{C}^\infty$).

As we have proved in Sect. 2 that our conditions are sufficient for the well posedness, whereas conditions (6.2) and (6.3) are necessary, it remains to prove that if (6.2) and (6.3) hold true, then (1.13) and (1.14) are satisfied.

We prove at first that (6.2) implies (1.13). We have to distinguish three cases as in the previous section.

If $\Delta_L \not\equiv 0$, then (6.2) is (5.4) and the m_j is given by (5.5). The boundness of the m_j is equivalent to (5.1), hence, by Proposition 5.1, (1.13) holds true.

If $\Delta_L \equiv 0$ and $\Delta_{\partial_\tau} L \not\equiv 0$, then with no loss of generality, we can assume that $\tau_1 \equiv \tau_2$ and $\tau_3 \not\equiv \tau_1$. In this case, the right hand side of (6.2) is divisible by $\tau - \tau_1$, hence M must verify (5.12a). Moreover, (6.2) reduces to (5.13), thus the boundness of the m_j is equivalent to (5.12b). By Proposition 5.7, we get (1.13).

If $\Delta_{\partial_\tau} L \equiv 0$, then $\tau_1 \equiv \tau_2 \equiv \tau_3$ and the right hand side of (6.2) is divisible by $(\tau - \tau_1)^2$, hence M must verify the first two conditions in (5.18), and, by Proposition 5.10, we get (1.13).

Now we prove that (6.3) implies (1.14).

From (6.3), we have

$$N(\sigma_1) = \mathsf{n}_2\left(\sigma_1 - \sigma_2\right), \qquad N(\sigma_2) = \mathsf{n}_1\left(\sigma_2 - \sigma_1\right),$$

from which we deduce that if $\Delta_{\partial_\tau L} \not\equiv 0$, then (5.10) is satisfied; by Proposition 5.5, (1.14) holds true. On the other side, if $\Delta_{\partial_\tau L} \equiv 0$, then the third condition in (5.18) holds true, hence, by Proposition 5.10, we get (1.13). □

7 Some examples

In this section, we discuss some examples. In particular, we compare our Theorem 2 with Theorem 6.1 in [26] (see Example 3 below).

First of all, we remark that any positive or negative result for second-order equations can give a positive or negative result for third-order equations, just replacing the unknown function u by its time derivative $\partial_t u$.

Example 1 In [9], it is constructed a function $a(t) \in \mathcal{C}^\infty$ verifying $a(0) = 0$, $a(t) > 0$ for $t > 0$ having an infinite number of oscillations for $t \mapsto 0^+$, so that condition (1.1) is not verified and the Cauchy problem for

$$\partial_t^2 u - a(t)\, \partial_x^2 u = 0$$

is not well posed in $\mathcal{C}^\infty$. The same function can be used to show that the Cauchy problem for the equation

$$\partial_t^3 u - a(t)\, \partial_t \partial_x^2 u = 0$$

may be ill-posed. This shows that some control on the oscillations of the coefficients of the principal symbol is needed, as required by Hypothesis (1.9) and (1.10).

Example 2 It is well known that the Cauchy problem for the equation

$$\partial_t^2 u - \partial_x u = 0$$

is not well posed in $\mathcal{C}^\infty$ (see, e.g., [18]). Thus, the Cauchy problem for the equation

$$\partial_t^3 u - \partial_t \partial_x u = 0$$

is not well posed in $\mathcal{C}^\infty$ (see, e.g., [18]). This show that some Levi conditions on the lower-order terms are needed.

Example 3 Consider the homogeneous equation

$$\partial_t^3 u + a_1(t,x)\partial_t^2 \partial_x u + a_2(t,x)\partial_t \partial_x^2 u + a_3(t,x)\, \partial_x^3 u = 0\,.$$

Assume, for simplicity $a_1(t,x) \equiv 0$ and a_2 and a_3 analytic. The hyperbolicity assumption is then

$$\Delta_L(t,x) = -4\, a_2^3(t,x) - 27\, a_3^2(t,x) \geqslant 0\,.$$

Assume also that a_2 and a_3 vanish at the origin, so that we have a triple characteristic root, whereas $\Delta_L(t,x) > 0$ for $(t,\xi) \neq (0,0)$, so that the characteristic roots are simple for $(t,\xi) \neq (0,0)$.

According to [8], if $a_2(t,x) = a_2(t)$ and $a_3(t,x) = a_3(t)$ depend only on the time variable t, a necessary condition for the $\mathcal{C}^\infty$ well posedness is the following

$$-a_2^3(t) \lesssim \Delta(t)\,. \tag{7.1}$$

Condition (7.1) is sufficient also if $a_2(t,x) = a_2(x)$ and $a_3(t,x) = a_3(x)$ depend only on the space variable x [32], whereas if a_2 and a_3 depend on both variables the following condition should be also considered [26, Theorem 6.1]:

$$\left|\partial_t a_3\right| \lesssim \sqrt{a_2}\,\left|\partial_t a_2\right|\,.$$

In order to compare the above results with Theorem 2, we have seen that (see Sect. 4 for details), if the coefficients are analytic, then conditions (1.9)–(1.12) are satisfied.

Next, as the equation is homogeneous and $a_1 \equiv 0$, then

$$\check{M}(t,\tau,\xi) = -\frac{1}{2}\,a_2'(t)\,\xi^2$$

and $\check{N}(t,\tau,\xi) \equiv 0$ so that (1.14) is trivially satisfied.

To prove the equivalence between (7.1) and (1.14), we recall (cf. [8]) that if $a_1 \equiv 0$ then condition (7.1) is equivalent to the following condition

$$\tau_j^2(t) + \tau_k^2(t) \lesssim \left(\tau_j(t) - \tau_k(t)\right)^2, \qquad \text{for } j \neq k\,, \tag{7.2}$$

or, if the coefficients are analytic and the discriminant $\Delta(t)$ vanishes only at $t = 0$, to the following:

$$|t|^2\left[\left(\tau_j'(t)\right)^2 + \left(\tau_k'(t)\right)^2\right] \lesssim \left(\tau_j(t) - \tau_k(t)\right)^2, \qquad \text{for } j \neq k\,. \tag{7.3}$$

On the other side, thanks to Viete's formulas, as $a_1 \equiv 0$, we have

$$a_2(t) = -\frac{1}{2}\left[\tau_1^2(t) + \tau_2^2(t) + \tau_3^2(t)\right].$$

Combining (7.2) and (7.3), we see that (7.1) is equivalent to (5.9), which is equivalent to condition (1.13).

Appendix A

Lemma A.1 *Let*

$$p(\tau) = \tau^3 + A_1\tau^2 + A_2\tau + A_3 = (\tau - \tau_1)(\tau - \tau_2)(\tau - \tau_3),$$

and

$$p'(\tau) = 3\,\tau^2 + 2A_1\tau + A_2 = 3\,(\tau - \sigma_1)(\tau - \sigma_2),$$

then

$$(\tau_1 - \tau_2)^2 + (\tau_2 - \tau_3)^2 + (\tau_3 - \tau_1)^2 = \frac{9}{2}\,(\sigma_2 - \sigma_1)^2\,. \tag{A.1}$$

Proof Using Vieta's formulas

$$\begin{aligned}(\tau_1 - \tau_2)^2 + &(\tau_2 - \tau_3)^2 + (\tau_3 - \tau_1)^2 \\ &= 2\left[\tau_1^2 + \tau_2^2 + \tau_3^2 - \tau_1\tau_2 - \tau_2\tau_3 - \tau_3\tau_1\right] \\ &= 2\left[(\tau_1 + \tau_2 + \tau_3)^2 - 3(\tau_1\tau_2 + \tau_2\tau_3 + \tau_3\tau_1)\right] \\ &= 2\,[A_1^2 - 3A_2]\,,\end{aligned}$$

whereas

$$(\sigma_2 - \sigma_1)^2 = \frac{4}{9}A_1^2 - \frac{4}{3}A_2\,,$$

from which we get the result. □

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