Some Einstein nilmanifolds with skew torsion arising in CR geometry *

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Abstract

We describe some new examples of nilmanifolds admitting an Einstein with skew torsion invariant Riemannian metric. These are affine CR quadrics, whose CR structure is preserved by the characteristic connection.

Keywords: Riemannian CR manifold; characteristic connection; Einstein metric with torsion.

1 Introduction

In [7] Wolf proved that an Einstein Riemannian manifold admitting a nilpotent transitive Lie group of isometries must be flat. In the light of this, it is natural to ask for the existence of invariant metrics on nilmanifolds satisfying some different conditions of Einstein type. Recently, much attention has been devoted to the study of metric connections with totally skew-symmetric torsion (see e.g. [1]); in this context, in [2] Agricola and Ferreira carried a systematic investigation of Riemannian manifolds (M, g) admitting a characteristic connection whose Ricci tensor satisfies

$$S(Ric^\nabla) = \lambda g$$

for some function λ . In particular, several new examples have been discussed, for which ∇ also preserves a geometric structure compatible with the metric, like a complex, contact or G_2 structure. In this paper, we are concerned with characteristic connections on CR manifolds endowed with a compatible Riemannian metric. In [3], we provided necessary and sufficient conditions for a Riemannian (almost) CR manifold to admit a characteristic connection (Theorem 1); we also furnished an explicit description of the torsion of any characteristic connection. This result generalizes and unifies Theorems 10.1 and 8.2 in [4], concerning almost Hermitian and almost contact metric manifolds.

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In this paper, we apply the above result to provide new examples of nilpotent Lie groups endowed with a left invariant Einstein with skew torsion Riemannian metric. These groups are affine CR quadrics in complex Euclidean spaces and arise in a natural way in CR geometry, being flat models in Tanaka's theory of equivalence of regular CR manifolds (see for instance [6, 5]).

2 Characteristic connections on Riemannian CR manifolds

We start by recalling some basic information concerning metric connections with torsion. Let (M, g) be a Riemannian manifold; a metric connection ∇ with torsion T is said to have (totally) skew-symmetric torsion if the $(0, 3)$ -tensor field defined by $T(X, Y, Z) := g(T(X, Y), Z)$ is a 3-form. In this case the relation between ∇ and the Levi-Civita connection ∇^g is

$$\nabla_X Y = \nabla_X^g Y + \frac{1}{2}T(X, Y).$$

The manifold (M, g, ∇) is called *Einstein with skew torsion*, if the symmetric part $S(Ric^\nabla)$ of the Ricci tensor of ∇ is proportional to the metric (see [2]). The tensor $S(Ric^\nabla)$ is related to the Riemannian Ricci tensor Ric^g by

$$S(Ric^\nabla)(X, Y) = Ric^g(X, Y) - \frac{1}{4}S_T(X, Y). \quad (1)$$

Here S_T is the symmetric tensor field determined by T and g as follows. Given a Euclidean vector space (V, g) and a 3-form T on V , one can define the symmetric bilinear form

$$S_T : V \times V \rightarrow \mathbb{R} \quad S_T(X, Y) := \sum_{i=1}^n g(T(X, e_i), T(Y, e_i)),$$

where $\{e_i\}$ is an arbitrary orthonormal basis of V .

Next we recall that a CR structure of type (n, k) on a differentiable manifold M of dimension $2n + k$, $k \geq 0$, is a pair (HM, J) , where HM is a rank $2n$ vector subbundle of the tangent bundle TM , and $J : HM \rightarrow HM$ is a smooth fiber preserving bundle isomorphism, such that $J^2 = -Id$. It is required that J satisfies a formal integrability condition (see [6, 5] for more information).

Let (M, HM, J, g) be a Riemannian CR manifold, that is (HM, J) is a CR structure of type (n, k) and g is a compatible Riemannian metric, i.e. $g(JX, JY) = g(X, Y)$ for every $X, Y \in \Gamma HM$. A metric connection on M with totally skew-symmetric torsion which parallelizes the structure (HM, J) will be called *characteristic*.

Let $P : TM \rightarrow HM$ be the orthogonal projection. We introduce an operator $\Gamma : \Gamma HM \times \Gamma HM \rightarrow \Gamma HM$ by $\Gamma_X Y := P(\nabla_X^g Y)$; for every $X \in \Gamma HM$, we also denote by $\Gamma_X J : \Gamma HM \rightarrow \Gamma HM$ the $C^\infty(M)$ -linear operator such that

$$(\Gamma_X J)Y := \Gamma_X(JY) - J(\Gamma_X Y).$$

Theorem 1 [3] *Let (M, HM, J, g) be a Riemannian CR manifold. Then M admits a characteristic connection if and only if the following conditions are satisfied:*

- 1) $P[\xi, JX] - JP[\xi, X] = 0$,
- 2) $(\mathcal{L}_\xi g)(X, Y) = 0$,
- 3) $(\mathcal{L}_X g)(\xi, \xi') = 0$,

for every $X, Y \in \Gamma HM$, and $\xi, \xi' \in \Gamma HM^\perp$. Furthermore, the torsion of each characteristic connection satisfies:

$$T(X, Y, Z) = -\sigma_{XYZ}g((\Gamma_{JX}J)Y, Z), \quad (2)$$

$$T(X, Y, \xi) = -g([X, Y], \xi) =: -L_\xi(X, Y), \quad (3)$$

$$T(X, \xi, \xi') = -g([\xi, \xi'], X) =: -L'_X(\xi, \xi'). \quad (4)$$

Here $\mathcal{L}$ denotes the Lie derivative, and σ denotes a cyclic sum. The components of the torsion in (3) and (4) are determined by the Levi-Tanaka forms L and L' of the distributions HM and $HM^\perp$ (for more details see [3]). We point out that if $k < 3$ there exists a unique characteristic connection. For $k \geq 3$ the characteristic connections are in a one-to-one correspondence with the smooth sections of $\Lambda^3(HM^\perp)$. In fact, for each $A \in \Gamma \Lambda^3(HM^\perp)$, the corresponding characteristic connection is the one whose torsion satisfies $T(\xi, \xi', \xi'') = A(\xi, \xi', \xi'')$ for every $\xi, \xi', \xi'' \in \Gamma HM^\perp$.

3 Einstein metrics with skew torsion on CR quadrics

An affine CR quadric in $\mathbb{C}^{n+k}$ is a submanifold Q of type

$$Q = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^k \mid \operatorname{Im} w = F(z, z)\},$$

where $F : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}^k$ is a non vanishing vector valued Hermitian symmetric bilinear form. It is well known that Q is a CR manifold of type (n, k) [6]. Moreover, Q can be described as the simply connected nilpotent Lie group whose Lie algebra is the $\mathbb{Z}$ -graded real Lie algebra

$$\mathfrak{m} = \mathbb{C}^n \oplus \mathbb{R}^k = \mathfrak{m}_{-1} \oplus \mathfrak{m}_{-2}, \quad [X, Y] := \operatorname{Im} F(X, Y) \quad \text{for every } X, Y \in \mathbb{C}^n.$$

The CR structure (HQ, J) on Q is left invariant and determined by $H_e Q = \mathfrak{m}_{-1}$, and $J_e = J$, where J is the standard complex structure on $\mathbb{C}^n$. The Lie algebra $\mathfrak{m}$ is the so called Tanaka algebra of Q [5].

Fix an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{m}$ which is Hermitian on $\mathfrak{m}_{-1}$ with respect to J and such that $\mathfrak{m}_{-1}$ and $\mathfrak{m}_{-2}$ are orthogonal. Then taking the left invariant Riemannian metric g on Q determined by $\langle \cdot, \cdot \rangle$, one can readily verify, using left invariant vector fields, that Q satisfies conditions 1), 2), 3) in Theorem 1. In particular, Q admits a family of left invariant characteristic connections parametrized by the 3-forms on $\mathfrak{m}_{-2}$. Such a metric g will be called an *admissible* Riemannian metric on the quadric. In correspondence of g , we define a real quadratic form $q_F : \mathbb{C}^n \rightarrow \mathbb{R}$ by

$$q_F(X) := \sum_{i=1}^n \|F(X, e_i)\|_g^2.$$

Here $\{e_i, Je_i\}$ is an arbitrary g -orthonormal J -basis on $\mathbb{R}^{2n} \cong \mathbb{C}^n$. It is readily checked that the definition does not depend on the choice of the basis.

Theorem 2 *Let Q be an affine CR quadric of type (n, k) , endowed an admissible Riemannian metric. Let ∇ be a left invariant characteristic connection determined by a 3-form A on $\mathbb{R}^k$. The symmetric part of the Ricci tensor of ∇ is given by:*

$$S(Ric^\nabla)(X, X) = -q_F(X), \quad S(Ric^\nabla)(X, \xi) = 0, \quad S(Ric^\nabla)(\xi, \xi) = -\frac{1}{4}S_A(\xi, \xi)$$

for every $X \in \mathfrak{m}_{-1}$, $\xi \in \mathfrak{m}_{-2}$. Hence g is an Einstein metric with skew torsion with respect to ∇ if and only if there exists $\lambda > 0$ such that

$$q_F(X) = \lambda \langle X, X \rangle, \quad S_A(\xi, \xi) = \lambda \langle \xi, \xi \rangle. \quad (5)$$

Assume $k = 3, 6, 7$. Then g is an Einstein metric with skew torsion with respect to a suitable left-invariant characteristic connection if and only if

$$q_F(X) = \lambda \langle X, X \rangle. \quad (6)$$

PROOF: Using the Koszul formula, we see that the Levi-Civita connection of g is determined by

$$g(\nabla_X^g Y, \xi) = \frac{1}{2}L_\xi(X, Y), \quad g(\nabla_X^g Y, Z) = 0, \quad g(\nabla_X^g \xi, \xi') = 0, \quad g(\nabla_\xi^g \xi', \xi'') = 0$$

for every $X, Y, Z \in \mathfrak{m}_{-1}$, $\xi, \xi', \xi'' \in \mathfrak{m}_{-2}$. Then, fix a g -orthonormal J -basis $\{e_i, Je_i\}$ on $\mathfrak{m}_{-1}$ and a g -orthonormal basis $\{\xi_r\}$ on $\mathfrak{m}_{-2}$. Setting $E_i := e_i$, $E_{n+i} := Je_i$, $i = 1, \dots, n$, we get

$$\nabla_X^g Y = \frac{1}{2} \sum_{r=1}^k L_{\xi_r}(X, Y) \xi_r, \quad \nabla_X^g \xi = \nabla_\xi^g X = -\frac{1}{2} \sum_{i=1}^{2n} L_\xi(X, E_i) E_i, \quad \nabla_\xi^g \xi' = 0. \quad (7)$$

A standard computation using (7) implies that the Riemannian Ricci tensor satisfies $Ric^g(X, \xi) = 0$ and

$$Ric^g(X, X) = -\frac{1}{2} \sum_{i=1}^{2n} \sum_{r=1}^k L_{\xi_r}(X, E_i)^2, \quad Ric^g(\xi, \xi) = \frac{1}{4} \sum_{i,j=1}^{2n} L_\xi(E_i, E_j)^2.$$

Now, the torsion T of ∇ satisfies

$$T(X, Y, Z) = 0, \quad T(X, Y, \xi) = -L_\xi(X, Y), \quad T(X, \xi, \xi') = 0.$$

Indeed, the first equality holds according to (2), since by (7) we have $\Gamma_X Y = 0$ for every $X, Y \in \mathfrak{m}_{-1}$. The last two equalities are a direct application of (3) and (4). A straightforward computation for the tensor S_T yields $S_T(X, \xi) = 0$ and

$$S_T(X, X) = 2 \sum_{i=1}^{2n} \sum_{r=1}^k L_{\xi_r}(X, E_i)^2, \quad S_T(\xi, \xi) = \sum_{i,j=1}^{2n} L_\xi(E_i, E_j)^2 + \sum_{r=1}^k \|T(\xi_r, \xi)\|^2.$$

Therefore, applying (1) we obtain the required expressions for $S(Ric^\nabla)$. Concerning the last statement, when $k = 3, 6, 7$ it is known that the Euclidean space $\mathbb{R}^k$ always admits a 3-form A of Einstein type, i.e. S_A is proportional to the scalar product ([2]). Hence, if (6) holds, we can choose a suitable A so that (5) holds. $\square$

Finally, we provide some examples of quadrics admitting an Einstein with skew torsion admissible metric. In all that follows we choose as $\langle, \rangle$ the standard inner product on $\mathbb{C}^n \oplus \mathbb{R}^k$.

Example 1 A real k -dimensional subspace W of the space $\mathcal{H}_n$ of $n \times n$ Hermitian symmetric matrices determines an affine CR quadric Q of type (n, k) : one defines the corresponding F by

$$F(z, w) = ({}^t\bar{z} a_1 w, \dots, {}^t\bar{z} a_k w),$$

where $\{a_1, \dots, a_k\}$ is a basis of W . Then one gets $q_F(z) = \sum_{i=1}^k \|a_i z\|^2$ for every $z \in \mathbb{C}^n$. Hence it is not hard to see that for every $n \geq 2$ and $k \in \{3, 6, 7\}$, $k \leq n^2$, there exist quadrics of type (n, k) , which satisfy (6). For instance, examples of type $(2, 3)$ can be obtained choosing as W one of the following

$$\{a \in \mathcal{H}_2 \mid a = \bar{a}\}, \{a \in \mathcal{H}_2 \mid \text{Tr}(a) = 0\}, \left\{ \begin{pmatrix} \alpha & z \\ -\bar{z} & \beta \end{pmatrix} \mid z + \bar{z} = 0 \right\}.$$

Choosing $W = \{a \in \mathcal{H}_3 \mid a = \bar{a}\}$ or $W = \{a \in \mathcal{H}_3 \mid a_{11} = a_{22} = a_{33}\}$ we obtain an example of type $(3, 6)$ and of type $(3, 7)$, respectively.

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