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A blow-up result for a semilinear wave equation with scale-invariant damping and mass and nonlinearity of derivative type

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Abstract In this note, we prove blow-up results for semilinear wave models with damping and mass in the scale-invariant case and with nonlinear terms of derivative type. We consider the single equation and the weakly coupled system. In the first case we get a blow-up result for exponents below a certain shift of the Glassey exponent. For the weakly coupled system we find as critical curve a shift of the corresponding curve for the weakly coupled system of semilinear wave equations with the same kind of nonlinearities. Our approach follows the one for the respective classical wave equation by Zhou. In particular, an explicit integral representation formula for a solution of the corresponding linear scale-invariant wave equation, which is derived by using Yagdjian's integral transform approach, is employed in the blow-up argument. While in the case of the single equation we may use a comparison argument, for the weakly coupled system an iteration argument is applied.

Keywords Blow-up · Glassey exponent · Nonlinearity of derivative type · Time-dependent and scale-invariant lower order terms · Integral representation formula · Upper bound estimates of the lifespan

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1 Introduction

In this work we prove a blow-up result for the semilinear wave equation with time-dependent damping and mass in the scale-invariant case and with nonlinearity of derivative type,

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namely,

$$\begin{cases} u_{tt} - \Delta u + \frac{\mu}{1+t}u_t + \frac{v^2}{(1+t)^2}u = |\partial_t u|^p, & x \in \mathbb{R}^n, t > 0, \\ u(0, x) = \varepsilon u_0(x), & x \in \mathbb{R}^n, \\ u_t(0, x) = \varepsilon u_1(x), & x \in \mathbb{R}^n, \end{cases} \quad (1)$$

where μ, v^2 are nonnegative constants, $p > 1$ and ε is a positive constant describing the smallness of Cauchy data. Let us introduce the quantity

$$\delta \doteq (\mu - 1)^2 - 4v^2. \quad (2)$$

Recently, semilinear wave equation with scale-invariant damping and mass terms and power nonlinearity $|u|^p$ has been studied in several papers. It turns out that if δ is “large”, that is, for $\delta \geq (n+1)^2$, the critical exponent for

$$\begin{cases} u_{tt} - \Delta u + \frac{\mu}{1+t}u_t + \frac{v^2}{(1+t)^2}u = |u|^p, & x \in \mathbb{R}^n, t > 0, \\ u(0, x) = \varepsilon u_0(x), & x \in \mathbb{R}^n, \\ u_t(0, x) = \varepsilon u_1(x), & x \in \mathbb{R}^n, \end{cases} \quad (3)$$

is given by the shift $p_{\text{Fuj}} \left(n + \frac{\mu-1}{2} - \frac{\sqrt{\delta}}{2} \right)$ of the *Fujita exponent* $p_{\text{Fuj}}(n) \doteq 1 + \frac{2}{n}$ (cf. [59, 4, 33, 39, 34]). This follows from the fact that the critical exponent of the semilinear classical damped wave equation with power nonlinearity is the Fujita exponent and for large δ (3) has similar properties to this model somehow. On the other hand, for “small” and nonnegative value of δ the critical exponent for (3) should reasonably be the shift $p_{\text{Str}}(n + \mu)$ of the *Strauss exponent* $p_{\text{Str}}(n)$ the critical exponent for the semilinear wave equations with power nonlinearity (named after the author of [50], where a conjecture for the critical exponent for the semilinear wave equation with $|u|^p$ as nonlinear term is done), which is the positive root of the quadratic equation $(n-1)p^2 - (n+1)p - 2 = 0$ (cf. [18, 21, 10, 49, 46, 25, 62, 63, 61, 65, 51, 66] for the necessity part and [18, 11, 26, 27, 9, 53, 17] for the sufficiency part or [22, 23] for the radial symmetric case). This conjecture for the scale-invariant model is still open for the sufficiency part (for the necessity part, that is the blow-up results, see [6, 56, 32, 14, 54, 40, 44, 20]), even though some partial results in the special case $\delta = 1$ have been proved for $n \geq 3$ in the radial symmetric case (see [5, 35] for the odd dimensional case and [36] for the even dimensional case, respectively). This peculiarity of a “parabolic-like” behavior for large values of δ and of “wave-like” behavior for small values of δ has been showed also for the corresponding weakly coupled system (cf. [3, 37]).

In the case of the Cauchy problem for the semilinear wave equation with nonlinearity of derivative type

$$\begin{cases} u_{tt} - \Delta u = |\partial_t u|^p, & x \in \mathbb{R}^n, t > 0, \\ u(0, x) = \varepsilon u_0(x), & x \in \mathbb{R}^n, \\ u_t(0, x) = \varepsilon u_1(x), & x \in \mathbb{R}^n, \end{cases} \quad (4)$$

the critical exponent is the so-called *Glassey exponent* $p_{\text{Gla}}(n) \doteq \frac{n+1}{n-1}$. We refer to the classical works [19, 48, 28, 47, 45, 1, 12, 55, 64, 13] for the proof of this conjecture, although up to the knowledge of the authors the global existence in the supercritical case for the not radial symmetric case in high dimensions is still open. Recently, in [30] a blow-up result for $1 < p \leq p_{\text{Gla}}(n)$ has been proved for a semilinear damped wave model in the scattering case,

that is, when the time-dependent coefficient of the damping term $b(t)u_t$ is nonnegative and summable.

Therefore, according to what happens for the semilinear Cauchy problem (3) it would be natural to find as critical exponent for (1) a suitable shift for the Glassey exponent. Purpose of this paper is to prove under certain sign assumptions for the Cauchy data a blow-up result for the Cauchy problem (1) provided that the exponent in the nonlinear term satisfies $1 < p \leq p_{\text{Gla}}(n + \sigma)$ with the shift σ defined by

$$\sigma \doteq \begin{cases} \mu + 1 - \sqrt{\delta} & \text{if } \delta \in [0, 1), \\ \mu & \text{if } \delta \geq 1. \end{cases} \quad (5)$$

As byproduct of the comparison argument that will be employed, we get an upper bound estimate for the lifespan in terms of ε as well.

Let us consider the weakly coupled system of semilinear wave equations with damping and mass in the scale-invariant case and nonlinearities of derivative type, that is,

$$\begin{cases} u_{tt} - \Delta u + \frac{\mu_1}{1+t} u_t + \frac{v_1^2}{(1+t)^2} u = |\partial_t v|^p, & x \in \mathbb{R}^n, t > 0, \\ v_{tt} - \Delta v + \frac{\mu_2}{1+t} v_t + \frac{v_2^2}{(1+t)^2} v = |\partial_t u|^q, & x \in \mathbb{R}^n, t > 0, \\ u(0, x) = \varepsilon u_0(x), \quad v(0, x) = \varepsilon v_0(x), & x \in \mathbb{R}^n, \\ u_t(0, x) = \varepsilon u_1(x), \quad v_t(0, x) = \varepsilon v_1(x), & x \in \mathbb{R}^n, \end{cases} \quad (6)$$

where $\mu_1, \mu_2, v_1^2, v_2^2$ are nonnegative constants, $p, q > 1$ and ε is a positive constant describing the smallness of Cauchy data. Similarly to the case of a single equation, we introduce the quantities $\delta_j \doteq (\mu_j - 1)^2 - 4v_j^2$ for $j = 1, 2$.

The machinery, that we are going to develop in the case of a single semilinear equation, works nicely also in the case of this weakly coupled system.

In order to understand our blow-up result for (6), we shall first recall some results which are known in the literature for the semilinear weakly coupled system of wave equation with nonlinear terms of derivative type, namely,

$$\begin{cases} u_{tt} - \Delta u = |\partial_t v|^p, & x \in \mathbb{R}^n, t > 0, \\ v_{tt} - \Delta v = |\partial_t u|^q, & x \in \mathbb{R}^n, t > 0, \\ u(0, x) = \varepsilon u_0(x), \quad v(0, x) = \varepsilon v_0(x), & x \in \mathbb{R}^n, \\ u_t(0, x) = \varepsilon u_1(x), \quad v_t(0, x) = \varepsilon v_1(x), & x \in \mathbb{R}^n. \end{cases} \quad (7)$$

The non-existence of global in time solutions to (7) (which corresponds to (6) in the case $\mu_1 = \mu_2 = 0$ and $v_1^2 = v_2^2 = 0$) has been studied in [8, 60], while the existence part has been proved in the three dimensional and radial case in [24]. Recently, in [15, Section 8] the upper bound for the lifespan has been derived. Summarizing the main results of these works, we can see that

$$\max\{\Lambda(n, p, q), \Lambda(n, q, p)\} = 0, \quad (8)$$

is the critical line in the (p, q) -plane for the semilinear weakly coupled system (7), where

$$\Lambda(n, p, q) \doteq \frac{p+1}{pq-1} - \frac{n-1}{2}. \quad (9)$$

Let us recall the meaning of critical curve for a weakly coupled system: if $p, q > 1$ satisfy $\max\{\Lambda(n, p, q), \Lambda(n, q, p)\} < 0$ (*supercritical case*), then, it is possible to prove a global

existence result for small data solutions; on the contrary, for $\max\{\Lambda(n, p, q), \Lambda(n, q, p)\} \geq 0$ it is possible to prove the nonexistence of global in time solutions regardless the smallness of the Cauchy data and under certain sign assumptions for them. Let us point out that, according to the results we quoted above, the conjecture that the critical line for (7) is given by (8) has been shown to be true only partially, as the global existence of small data solutions has been proved only in the 3-dimensional and radial symmetric case.

In the massless case ($v_1^2 = v_2^2 = 0$) and *scattering producing* case, that is, if we consider time-dependent, nonnegative and summable coefficients $b_1(t), b_2(t)$ instead of $\mu_1(1+t)^{-1}, \mu_2(1+t)^{-1}$, really recently in [42] a blow-up result has been proved in the same range for the exponents (p, q) as for the corresponding not damped case, namely, for (p, q) such that $\max\{\Lambda(n, p, q), \Lambda(n, q, p)\} \geq 0$ is satisfied.

Consequently, coming back to the weakly coupled system in the scale-invariant case (6), we may expect as critical curve in the (p, q) -plane a curve with branches that are shifts of the branches of the critical curve for (7). Indeed, due to the blow-up result for (6) which we are going to state in the next section, we may conjecture

$$\max\{\Lambda(n + \sigma_1, p, q), \Lambda(n + \sigma_2, q, p)\} = 0$$

as critical curve, where σ_1, σ_2 are defined analogously as in (5).

2 Main results

Theorem 1 *Let $n \geq 1$ and let μ, v^2 be nonnegative constants such that $\delta \geq 0$. We consider u_0, u_1 compactly supported in B_R such that u_0 and u_1 are nonnegative functions if $\delta \geq 1$; else, if $\delta \in [0, 1)$ we assume that $u_0 = 0$ and u_1 is a nonnegative function.*

Let $1 < p \leq p_{\text{Gla}}(n + \sigma)$ be the exponent of the nonlinearity of derive type, where σ is defined by (5). Then, there exists $\varepsilon_0 = \varepsilon_0(n, p, \mu, v^2, u_0, u_1, R) > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ if u is a local in time solution to (1), u blows up in finite time. Furthermore, the following upper bound estimate for the lifespan of the solution holds

$$T(\varepsilon) \leq \begin{cases} C\varepsilon^{-\left(\frac{1}{p-1} - \frac{n+\sigma-1}{2}\right)^{-1}} & \text{if } 1 < p < p_{\text{Gla}}(n + \sigma), \\ \exp(C\varepsilon^{-(p-1)}) & \text{if } p = p_{\text{Gla}}(n + \sigma), \end{cases} \quad (10)$$

where the positive constant C is independent of ε .

Remark 1 The sign assumptions on Cauchy data in the statement of Theorem 1 are done with the purpose to ensure a suitable control from below for a function which depends on the solution of the homogeneous linear problem related to (1).

Theorem 2 *Let $n \geq 1$ and let $\mu_1, \mu_2, v_1^2, v_2^2$ be nonnegative constants such that $\delta_1, \delta_2 \geq 0$. We consider data u_0, u_1, v_0, v_1 that are nonnegative and compactly supported in B_R functions. Moreover, we assume $u_0 = 0$ (respectively $v_0 = 0$) in the case $\delta_1 \in [0, 1)$ (respectively $\delta_2 \in [0, 1)$). Let us assume that the exponents of the nonlinearities of derive type $p, q > 1$ satisfy*

$$\max\left\{\frac{p+1}{pq-1} - \frac{n+\sigma_1-1}{2}, \frac{q+1}{pq-1} - \frac{n+\sigma_2-1}{2}\right\} \geq 0, \quad (11)$$

where

$$\sigma_k \doteq \begin{cases} \mu_k + 1 - \sqrt{\delta_k} & \text{if } \delta_k \in [0, 1), \\ \mu_k & \text{if } \delta_k \geq 1, \end{cases} \quad k = 1, 2. \quad (12)$$

Then, there exists $\varepsilon_0 = \varepsilon_0(n, p, q, \mu_1, \mu_2, v_1^2, v_2^2, u_0, u_1, v_0, v_1, R) > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ if (u, v) is a local in time solution to (6), (u, v) blows up in finite time. Furthermore, the following upper bound estimate for the lifespan of the solution holds

$$T(\varepsilon) \leq \begin{cases} C\varepsilon^{-(\Omega(n, \sigma_1, \sigma_2, p, q))^{-1}} & \text{if } \Omega(n, \sigma_1, \sigma_2, p, q) > 0, \\ \exp\left(C\varepsilon^{-(pq-1)}\right) & \text{if } \Omega(n, \sigma_1, \sigma_2, p, q) = 0, \\ \exp\left(C\varepsilon^{-\min\left\{\frac{pq-1}{p+1}, \frac{pq-1}{q+1}\right\}}\right) & \text{if } \Lambda(n + \sigma_1, p, q) = \Lambda(n + \sigma_2, q, p) = 0, \end{cases} \quad (13)$$

where the positive constant C is independent of ε , $\Lambda(n, p, q)$ is defined by (9) and

$$\Omega(n, \sigma_1, \sigma_2, p, q) \doteq \max\{\Lambda(n + \sigma_1, p, q), \Lambda(n + \sigma_2, q, p)\}.$$

Remark 2 In the cusp point of the critical line that we found in Theorem 2, that is, for (p, q) such that $\Lambda(n + \sigma_1, p, q) = \Lambda(n + \sigma_2, q, p) = 0$, we may specify more explicitly the condition on the lifespan. Indeed, if we denote

$$\eta \doteq \frac{n + \sigma_2 + 1}{2} - \frac{n + \sigma_1 - 1}{2}q, \quad \xi \doteq \frac{n + \sigma_1 + 1}{2} - \frac{n + \sigma_2 - 1}{2}p,$$

then, straightforward computations lead to

$$p\eta + \xi = (pq - 1)\Lambda(n + \sigma_1, p, q) \quad \text{and} \quad \eta + q\xi = (pq - 1)\Lambda(n + \sigma_2, q, p).$$

Therefore, on the cusp point of the critical line we get $\eta = \xi = 0$ due to the fact that $pq > 1$. From $\eta = 0$ and $\xi = 0$ we obtain the explicit expressions of (p, q) when $\Lambda(n + \sigma_1, p, q) = \Lambda(n + \sigma_2, q, p) = 0$, namely,

$$p = \tilde{p}(n, \sigma_1, \sigma_2) \doteq \frac{n + \sigma_1 + 1}{n + \sigma_2 - 1} \quad \text{and} \quad q = \tilde{q}(n, \sigma_1, \sigma_2) \doteq \frac{n + \sigma_2 + 1}{n + \sigma_1 - 1}.$$

Since $\tilde{p}(n, \sigma_1, \sigma_2) \geq \tilde{q}(n, \sigma_1, \sigma_2)$ if and only if $\sigma_1 \geq \sigma_2$, we can rewrite the last upper bound estimate for the lifespan in (13) as follows:

$$T(\varepsilon) \leq \begin{cases} \exp\left(C\varepsilon^{-\frac{pq-1}{p+1}}\right) & \text{if } \Lambda(n + \sigma_1, p, q) = \Lambda(n + \sigma_2, q, p) = 0 \text{ and } \sigma_1 \geq \sigma_2, \\ \exp\left(C\varepsilon^{-\frac{pq-1}{q+1}}\right) & \text{if } \Lambda(n + \sigma_1, p, q) = \Lambda(n + \sigma_2, q, p) = 0 \text{ and } \sigma_1 \leq \sigma_2. \end{cases}$$

Of course, when $\sigma_1 = \sigma_2$ these estimates coincide with the estimate for the critical case in (10) and $\tilde{p}(n, \sigma_1, \sigma_1) = \tilde{q}(n, \sigma_1, \sigma_1) = p_{\text{Gla}}(n + \sigma_1)$.

Let us illustrate our strategy in the proof of Theorems 1 and 2: our approach in the proof of the blow-up results is based on the work [64] for the classical wave equation with nonlinearity of derivative type; therefore, as main tool we need to employ an integral representation formula for the linear and one-dimensional problem associated to (1), which generalizes d'Alembert's formula in the case of the free wave equation. This formula has been proved really recently in [38]. Applying such formula, we end up with a nonlinear ordinary integral inequality (OII) for the single equation (1) and a system of OIIs for the weakly coupled system (6), respectively. Then, for (1) a simple comparison argument suffices to prove Theorem 1, while in order to prove Theorem 2 we shall employ an iteration argument. Furthermore, in the critical case we will combine it with the slicing method.

3 Proof of Theorem 1

This section is devoted to the proof of Theorem 1. Before introducing the suitable function that will allow us to prove the blow-dynamic in the case $1 < p \leq p_{\text{Gla}}(n + \sigma)$, we recall the previously mentioned generalization of D'Alembert's representation formula.

3.1 Integral representation formula for the 1-dimensional linear case

In this subsection, we recall a representation formula for the solution of the linear Cauchy problem for a scale-invariant wave equation, namely,

$$\begin{cases} u_{tt} - u_{xx} + \frac{\mu}{1+t}u_t + \frac{v^2}{(1+t)^2}u = f(t, x), & x \in \mathbb{R}, t > 0, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \\ u_t(0, x) = u_1(x), & x \in \mathbb{R}, \end{cases} \quad (14)$$

where μ, v^2 are nonnegative constants. For the proof of this formula one can see [38, Theorem 1.1].

Proposition 1 *Let $n = 1$ and let μ, v^2 be nonnegative constants. Let us assume $u_0 \in \mathcal{C}^2(\mathbb{R})$, $u_1 \in \mathcal{C}^1(\mathbb{R})$ and $f \in \mathcal{C}_{t,x}^{0,1}([0, \infty) \times \mathbb{R})$. Then, a representation formula for the solution of (14) is given by*

$$\begin{aligned} u(t, x) &= \frac{1}{2}(1+t)^{-\frac{\mu}{2}}(u_0(x+t) + u_0(x-t)) \\ &+ \frac{1}{2\sqrt{\delta}} \int_{x-t}^{x+t} \left[u_0(y)K_0(t, x; y; \mu, v^2) + (u_1(y) + \mu u_0(y))K_1(t, x; y; \mu, v^2) \right] dy \\ &+ \frac{1}{2\sqrt{\delta}} \int_0^t \int_{x-t+b}^{x+t-b} f(b, y)E(t, x; b, y; \mu, v^2) dy db, \end{aligned} \quad (15)$$

where the kernel functions are defined as follows

$$\begin{aligned} E(t, x; b, y; \mu, v^2) &\doteq (1+t)^{-\frac{\mu}{2}+\gamma}(1+b)^{\frac{\mu}{2}+\gamma}((t+b+2)^2 - (y-x)^2)^{-\gamma} \\ &\times F\left(\gamma, \gamma; 1; \frac{(t-b)^2 - (y-x)^2}{(t+b+2)^2 - (y-x)^2}\right), \end{aligned} \quad (16)$$

$$K_0(t, x; y; \mu, v^2) \doteq -\partial_b E(t, x; b, y; \mu, v^2) \Big|_{b=0}, \quad (17)$$

$$K_1(t, x; y; \mu, v^2) \doteq E(t, x; 0, y; \mu, v^2) \quad (18)$$

with parameter $\gamma \doteq \frac{1-\sqrt{\delta}}{2}$ and $F(a, b; c; z)$ Gauss hypergeometric function.

Remark 3 In the next sections, we will need to estimate from below the kernel function E . In particular, we use the lower bound estimate

$$F(a, a; c; z) \geq 1 \quad (19)$$

for any $z \in [0, 1)$ when $a \in \mathbb{R}$ and $c > 0$. This estimate follows trivially from the series expansion of $F(a, a; c; z)$.

In the next subsection, we will prove the blow-up result by using this representation formula for an auxiliary function related to a local solution to (1).

3.2 Comparison argument

Let us consider a local (in time) solution u of the Cauchy problem (1). Then, we introduce a new function which depends on the time variable and only on the first space variable, by integrating with respect to the remaining $(n-1)$ spatial variables. That is, if we denote $x = (z, w)$ with $z \in \mathbb{R}$ and $w \in \mathbb{R}^{n-1}$, then, we deal with the function

$$\mathcal{U}(t, z) \doteq \int_{\mathbb{R}^{n-1}} u(t, z, w) dw \quad \text{for any } t > 0, z \in \mathbb{R}.$$

Of course, in the one dimensional case we may work directly with u instead of \mathcal{U} . Hereafter, we will deal only with the case $n \geq 2$ for the sake of brevity, although one can proceed exactly in the same way for $n = 1$ by working with u in place of \mathcal{U} . Similarly, we introduce

$$\mathcal{U}_0(z) \doteq \int_{\mathbb{R}^{n-1}} u_0(z, w) dw, \quad \mathcal{U}_1(z) \doteq \int_{\mathbb{R}^{n-1}} u_1(z, w) dw \quad \text{for any } z \in \mathbb{R}.$$

Since we assume that u_0, u_1 are compactly supported with support contained in B_R , it follows that $\mathcal{U}_0, \mathcal{U}_1$ are compactly supported in $[-R, R]$. Analogously, as $\text{supp } u(t, \cdot) \subset B_{R+t}$ for any $t > 0$, due to the property of finite speed of propagation of perturbations, we have $\text{supp } \mathcal{U}(t, \cdot) \subset [-(R+t), R+t]$ for any $t > 0$.

Therefore, \mathcal{U} solves the following Cauchy problem

$$\begin{cases} \mathcal{U}_{tt} - \mathcal{U}_{zz} + \frac{\mu}{1+t} \mathcal{U}_t + \frac{v^2}{(1+t)^2} \mathcal{U} = \int_{\mathbb{R}^{n-1}} |\partial_t u(t, z, w)|^p dw, & z \in \mathbb{R}, t > 0, \\ \mathcal{U}(0, z) = \varepsilon \mathcal{U}_0(z), & z \in \mathbb{R}, \\ \mathcal{U}_t(0, z) = \varepsilon \mathcal{U}_1(z), & z \in \mathbb{R}. \end{cases}$$

By Proposition 1 we know an explicit representation for \mathcal{U} . Also,

$$\begin{aligned} \mathcal{U}(t, z) &= \frac{\varepsilon}{2} (1+t)^{-\frac{n}{2}} (\mathcal{U}_0(z+t) + \mathcal{U}_0(z-t)) \\ &\quad + \frac{\varepsilon}{2\sqrt{\delta}} \int_{z-t}^{z+t} [\mathcal{U}_0(y) K_0(t, z; y; \mu, v^2) + (\mathcal{U}_1(y) + \mu \mathcal{U}_0(y)) K_1(t, z; y; \mu, v^2)] dy \\ &\quad + \frac{1}{2\sqrt{\delta}} \int_0^t \int_{z-t+b}^{z+t-b} \int_{\mathbb{R}^{n-1}} |\partial_t u(b, y, w)|^p dw E(t, z; b, y; \mu, v^2) dy db, \end{aligned}$$

where the kernel functions E, K_0, K_1 are defined by (16), (17) and (18), respectively.

Due to the sign assumption for u_0 it follows that \mathcal{U}_0 is a nonnegative function. Consequently, from the last equality we get

$$\begin{aligned} \mathcal{U}(t, z) &\geq \frac{\varepsilon}{2\sqrt{\delta}} \int_{z-t}^{z+t} \mathcal{U}_0(y) [K_0(t, z; y; \mu, v^2) + \mu K_1(t, z; y; \mu, v^2)] dy \\ &\quad + \frac{\varepsilon}{2\sqrt{\delta}} \int_{z-t}^{z+t} \mathcal{U}_1(y) K_1(t, z; y; \mu, v^2) dy \\ &\quad + \frac{1}{2\sqrt{\delta}} \int_0^t \int_{z-t+b}^{z+t-b} \int_{\mathbb{R}^{n-1}} |\partial_t u(b, y, w)|^p dw E(t, z; b, y; \mu, v^2) dy db \doteq \varepsilon J + I. \end{aligned}$$

Let us estimate from below the two addends in the last inequality for $\mathcal{U}(t, z)$, which are denoted by J and I . According to Remark 3, for μ, v^2 such that $\delta \geq 0$ the hypergeometric function that appears in the kernel K_1 is estimated from below by a constant for $|z-y| \leq t$. Hence, using

$$4(t+1) \leq (t+2)^2 - (y-z)^2 \leq (t+2)^2$$

for $y \in [z-t, z+t]$, we obtain

$$K_1(t, z; y; \mu, \nu^2) \gtrsim (1+t)^{-\frac{\mu}{2}+\gamma} ((t+2)^2 - (y-z)^2)^{-\gamma} \gtrsim \begin{cases} (1+t)^{-\frac{\mu}{2}-\gamma} & \text{if } \delta \in [0, 1), \\ (1+t)^{-\frac{\mu}{2}} & \text{if } \delta \geq 1, \end{cases}$$

where in the second inequality we estimated the factor containing y with its minimum on $[z-t, z+t]$, that is,

$$K_1(t, z; y; \mu, \nu^2) \gtrsim (1+t)^{-\frac{\sigma}{2}} \quad (20)$$

with σ defined by (5). Elementary computations lead to

$$\begin{aligned} & \partial_b E(t, z; 0, y; \mu, \nu^2) \\ &= (1+t)^{-\frac{\mu}{2}+\gamma} ((t+2)^2 - (y-z)^2)^{-\gamma} \\ & \quad \times \left[4\gamma^2 (1+t) ((y-z)^2 - t(t+2)) ((t+2)^2 - (y-z)^2)^{-2} F(\gamma+1, \gamma+1; 2; \zeta) \right. \\ & \quad \left. + \left(\frac{\mu}{2} + \gamma\right) F(\gamma, \gamma; 1; \zeta) - 2\gamma ((t+2)^2 - (y-z)^2)^{-1} (t+2) F(\gamma, \gamma; 1; \zeta) \right], \end{aligned}$$

where $\zeta = \zeta(t, z; y) \doteq \frac{t^2 - (y-z)^2}{(t+2)^2 - (y-z)^2}$. Therefore, for $y \in [z-t, z+t]$ and $\delta \geq 1$, since $\gamma \leq 0$, we have

$$\begin{aligned} & K_0(t, z; y; \mu, \nu^2) + \mu K_1(t, z; y; \mu, \nu^2) \\ &= \mu E(t, z; 0, y; \mu, \nu^2) - \partial_b E(t, z; 0, y; \mu, \nu^2) \\ &\geq (1+t)^{-\frac{\mu}{2}+\gamma} ((t+2)^2 - (y-z)^2)^{-\gamma} \left[\left(\frac{\mu}{2} - \gamma\right) + \frac{2\gamma(t+2)}{(t+2)^2 - (y-z)^2} \right] F(\gamma, \gamma; 1; \zeta) \\ &\geq (1+t)^{-\frac{\mu}{2}+\gamma} ((t+2)^2 - (y-z)^2)^{-\gamma} \left[\left(\frac{\mu}{2} - \gamma\right) + \frac{\gamma(t+2)}{2(t+1)} \right] F(\gamma, \gamma; 1; \zeta), \end{aligned}$$

where in the last inequality we estimated the second addend in brackets by its minimum. Also, for $y \in [z-t, z+t]$ and $\delta \geq 1$ we get

$$\begin{aligned} & K_0(t, z; y; \mu, \nu^2) + \mu K_1(t, z; y; \mu, \nu^2) \\ &\geq (1+t)^{-\frac{\mu}{2}+\gamma} ((t+2)^2 - (y-z)^2)^{-\gamma} \left[\left(\frac{\mu}{2} - \gamma\right) + \frac{\gamma}{2} \left(2 - \frac{t}{t+1}\right) \right] F(\gamma, \gamma; 1; \zeta) \\ &\geq \frac{\mu}{2} (1+t)^{-\frac{\mu}{2}+\gamma} ((t+2)^2 - (y-z)^2)^{-\gamma} F(\gamma, \gamma; 1; \zeta) \\ &\gtrsim (1+t)^{-\frac{\mu}{2}} = (1+t)^{-\frac{\sigma}{2}}, \end{aligned} \quad (21)$$

where in the second step we used $\gamma \leq 0$ and in the last step we used the same estimate from below as in (20) and (19).

Let us remark that $[-R, R] \subset [z-t, z+t]$ if and only if $z \in [-t+R, t-R]$. Thus, we found for $z \in [-t+R, t-R]$ (in the case $\delta \geq 1$) the estimate

$$\begin{aligned} J &= \frac{1}{2\sqrt{\delta}} \int_{z-t}^{z+t} \left[\mathcal{W}_0(y) K_0(t, z; y; \mu, \nu^2) + (\mathcal{W}_1(y) + \mu \mathcal{W}_0(y)) K_1(t, z; y; \mu, \nu^2) \right] dy \\ &\gtrsim (1+t)^{-\frac{\sigma}{2}} \int_{z-t}^{z+t} (\mathcal{W}_1(y) + \mathcal{W}_0(y)) dy \\ &= (1+t)^{-\frac{\sigma}{2}} \int_{\mathbb{R}} (\mathcal{W}_1(y) + \mathcal{W}_0(y)) dy = \|u_0 + u_1\|_{L^1(\mathbb{R}^n)} (1+t)^{-\frac{\sigma}{2}}. \end{aligned} \quad (22)$$

In the case $\delta \in [0, 1)$, we may not prove the estimate in (21) for the term $K_0 + \mu K_1$ as in the previous case, due to the fact that γ is positive. Nonetheless, (20) still holds for K_1 . Hence, assuming $u_0 = 0$ in the latter case, we get once again the lower bound estimate for J in (22).

Next we estimate the term I . Since $\text{supp } u(t, \cdot) \subset B_{R+t}$ implies

$$\text{supp } \partial_t u(t, z, \cdot) \subset \{w \in \mathbb{R}^{n-1} : |w| \leq ((R+t)^2 - z^2)^{1/2}\} \quad \text{for any } t > 0, z \in \mathbb{R}$$

by Hölder's inequality we get

$$\begin{aligned} |\mathcal{Z}_t(b, y)| &= \left| \int_{\mathbb{R}^{n-1}} \partial_t u(b, y, w) \, dw \right| \\ &\leq \left(\int_{\mathbb{R}^{n-1}} |\partial_t u(b, y, w)|^p \, dw \right)^{\frac{1}{p}} (\text{meas}(\text{supp } \partial_t u(b, y, \cdot)))^{1-\frac{1}{p}} \\ &\lesssim ((R+b)^2 - y^2)^{\frac{n-1}{2}(1-\frac{1}{p})} \left(\int_{\mathbb{R}^{n-1}} |\partial_t u(b, y, w)|^p \, dw \right)^{\frac{1}{p}}. \end{aligned}$$

Hence,

$$\int_{\mathbb{R}^{n-1}} |\partial_t u(b, y, w)|^p \, dw \gtrsim ((R+b)^2 - y^2)^{-\frac{n-1}{2}(p-1)} |\mathcal{Z}_t(b, y)|^p,$$

which implies in turn

$$\begin{aligned} I &\gtrsim \int_0^t \int_{z-t+b}^{z+t-b} ((R+b)^2 - y^2)^{-\frac{n-1}{2}(p-1)} |\mathcal{Z}_t(b, y)|^p E(t, z; b, y; \mu, \nu^2) \, dy \, db \\ &= \int_{z-t}^{z+t} \int_0^{t-|y-z|} ((R+b)^2 - y^2)^{-\frac{n-1}{2}(p-1)} |\mathcal{Z}_t(b, y)|^p E(t, z; b, y; \mu, \nu^2) \, db \, dy, \end{aligned}$$

where we used Fubini's theorem in the last equality. We work now on the characteristic $t - z = R$. Then, shrinking the domain of integration, we have

$$\begin{aligned} I &\gtrsim \int_R^z \int_{y-R}^{y+R} ((R+b)^2 - y^2)^{-\frac{n-1}{2}(p-1)} |\mathcal{Z}_t(b, y)|^p E(t, z; b, y; \mu, \nu^2) \, db \, dy \\ &\gtrsim \int_R^z (R+y)^{-\frac{n-1}{2}(p-1)} \int_{y-R}^{y+R} |\mathcal{Z}_t(b, y)|^p E(t, z; b, y; \mu, \nu^2) \, db \, dy. \end{aligned}$$

Note that the unexpressed multiplicative constant in the previous chain of inequalities depends on R . Now we estimate from below the kernel function E . Consequently, using again (19), for $b \in [0, t], y \in [z - (t - b), z + t - b]$ we may estimate

$$\begin{aligned} E(t, z; b, y; \mu, \nu^2) &\gtrsim (1+t)^{-\frac{n}{2}+\gamma} (1+b)^{\frac{n}{2}+\gamma} ((t+b+2)^2 - (y-z)^2)^{-\gamma} \\ &\gtrsim (1+t)^{-\frac{\sigma}{2}} (1+b)^{\frac{\sigma}{2}}. \end{aligned}$$

Also, on the characteristic $t - z = R$ we get

$$\begin{aligned} I &\gtrsim (1+t)^{-\frac{\sigma}{2}} \int_R^z (R+y)^{-\frac{n-1}{2}(p-1)} \int_{y-R}^{y+R} |\mathcal{Z}_t(b, y)|^p (1+b)^{\frac{\sigma}{2}} \, db \, dy \\ &\gtrsim (1+t)^{-\frac{\sigma}{2}} \int_R^z (R+y)^{-\frac{n-1}{2}(p-1)+\frac{\sigma}{2}} \int_{y-R}^{y+R} |\mathcal{Z}_t(b, y)|^p \left(\frac{1+b}{R+y} \right)^{\frac{\sigma}{2}} \, db \, dy. \end{aligned}$$

We notice that the quotient in the b -integral in the last line is bounded from below on the domain of integration by a positive constant, that depends on R . Clearly, we can assume without loss of generality $R > 1$. We take $y \in [R, z]$ and $b \in [y - R, y + R]$. Then,

$$\left(\frac{1+b}{R+y}\right)^{\frac{\sigma}{2}} \geq \left(\frac{1+y-R}{R+y}\right)^{\frac{\sigma}{2}} \geq C_R.$$

The last inequality can be proved by splitting the cases $y \in [R, 3R - 2]$ and $y \geq 3R - 2$, as follows:

$$\begin{aligned} \left(\frac{1+y-R}{R+y}\right)^{\frac{\sigma}{2}} &\geq (R+y)^{-\frac{\sigma}{2}} \geq (2(2R-1))^{-\frac{\sigma}{2}} && \text{for } y \in [R, 3R-2], \\ \left(\frac{1+y-R}{R+y}\right)^{\frac{\sigma}{2}} &\geq 2^{-\frac{\sigma}{2}} && \text{for } y \geq 3R-2. \end{aligned}$$

Therefore, by using Jensen's inequality and the fundamental theorem of calculus we arrive at

$$\begin{aligned} I &\gtrsim (1+t)^{-\frac{\sigma}{2}} \int_R^z (R+y)^{-\frac{n-1}{2}(p-1)+\frac{\sigma}{2}} \int_{y-R}^{y+R} |\mathcal{U}_t(b, y)|^p db dy \\ &\gtrsim (1+t)^{-\frac{\sigma}{2}} \int_R^z (R+y)^{-\frac{n-1}{2}(p-1)+\frac{\sigma}{2}} \left| \int_{y-R}^{y+R} \mathcal{U}_t(b, y) db \right|^p dy \\ &= (1+t)^{-\frac{\sigma}{2}} \int_R^z (R+y)^{-\frac{n-1}{2}(p-1)+\frac{\sigma}{2}} |\mathcal{U}(y+R, y)|^p dy \\ &= (1+t)^{-\frac{\sigma}{2}} \int_R^z (R+y)^{-\frac{n+\sigma-1}{2}(p-1)} \left| (R+y)^{\frac{\sigma}{2}} \mathcal{U}(y+R, y) \right|^p dy, \end{aligned}$$

where in the third step we used $\mathcal{U}(y-R, y) = 0$. Combining the lower bound estimates for J and I , on the characteristic $t - z = R$ and for $t \geq 2R$ we found

$$(R+z)^{\frac{\sigma}{2}} \mathcal{U}(z+R, z) \gtrsim \varepsilon \|u_0 + u_1\|_{L^1(\mathbb{R}^n)} + \int_R^z (R+y)^{-\frac{n+\sigma-1}{2}(p-1)} \left| (R+y)^{\frac{\sigma}{2}} \mathcal{U}(y+R, y) \right|^p dy.$$

If we introduce the function $U(z) \doteq (R+z)^{\frac{\sigma}{2}} \mathcal{U}(z+R, z)$ and we denote by C the unexpressed multiplicative constant in the last inequality, we may rewrite

$$U(z) \geq M\varepsilon + C \int_R^z (R+y)^{-\frac{n+\sigma-1}{2}(p-1)} |U(y)|^p dy \quad \text{for any } z \geq R, \quad (23)$$

where $M \doteq C \|u_0 + u_1\|_{L^1(\mathbb{R}^n)}$. Let us introduce the function

$$G(z) \doteq M\varepsilon + C \int_R^z (R+y)^{-\frac{n+\sigma-1}{2}(p-1)} |U(y)|^p dy \quad \text{for any } z \geq R.$$

Clearly, by (23) we obtain $U \geq G$. Moreover, G solves the differential inequality

$$\begin{aligned} G'(z) &= C(R+z)^{-\frac{n+\sigma-1}{2}(p-1)} |U(z)|^p \\ &\geq C(R+z)^{-\frac{n+\sigma-1}{2}(p-1)} (G(z))^p. \end{aligned}$$

As G is a positive function, then, separation of variables leads to

$$\frac{(M\varepsilon)^{1-p} - G(z)^{1-p}}{p-1} \geq \frac{C}{1 - \frac{n+\sigma-1}{2}(p-1)} \left((R+z)^{-\frac{n+\sigma-1}{2}(p-1)+1} - (2R)^{-\frac{n+\sigma-1}{2}(p-1)+1} \right)$$

in the subcritical case $p \in (1, p_{\text{Gla}}(n + \sigma))$ and

$$\frac{(M\varepsilon)^{1-p} - G(z)^{1-p}}{p-1} \geq C \log \left(\frac{R+z}{2R} \right)$$

if $p = p_{\text{Gla}}(n + \sigma)$. In the subcritical case $p \in (1, p_{\text{Gla}}(n + \sigma))$, choosing $\varepsilon \in (0, \varepsilon_0]$ sufficiently small with $\varepsilon_0 = \varepsilon_0(n, p, \mu, v^2, u_0, u_1, R)$, we get

$$\begin{aligned} G(z) &\geq \left[(M\varepsilon)^{1-p} + \frac{C}{\frac{1}{p-1} - \frac{n+\sigma-1}{2}} \left((2R)^{-\frac{n+\sigma-1}{2}(p-1)+1} - (R+z)^{-\frac{n+\sigma-1}{2}(p-1)+1} \right) \right]^{-\frac{1}{p-1}} \\ &\geq \left[2(M\varepsilon)^{1-p} - \frac{C}{\frac{1}{p-1} - \frac{n+\sigma-1}{2}} (R+z)^{-\frac{n+\sigma-1}{2}(p-1)+1} \right]^{-\frac{1}{p-1}}. \end{aligned}$$

From this last estimate we see that for $t = R + z \simeq \varepsilon^{-\left(\frac{1}{p-1} - \frac{n+\sigma-1}{2}\right)^{-1}}$ the lower bound for G blows up. Then, G (and U in turn) blows up in finite time and the upper bound for the lifespan

$$T(\varepsilon) \lesssim \varepsilon^{-\left(\frac{1}{p-1} - \frac{n+\sigma-1}{2}\right)^{-1}}$$

is fulfilled in the subcritical case. Analogously, in the critical case $p = p_{\text{Gla}}(n + \sigma)$ we have that

$$G(z) \geq \left[(M\varepsilon)^{1-p} - C(p-1) \log \left(\frac{R+z}{2R} \right) \right]^{-\frac{1}{p-1}}$$

implies the blow-up in finite time of $U(z)$ and the lifespan estimate

$$T(\varepsilon) \lesssim \exp \left(\tilde{C} \varepsilon^{-(p-1)} \right).$$

So, the proof of Theorem 1 is complete.

4 Proof of Theorem 2

In this section we prove the blow-up result for the weakly coupled system (6). The section is organized as follows: in Subsection 4.1 we introduce two suitable functions which are related to the components of a local in time solution of (6) and we derive the corresponding iteration frame by using the same ideas from Subsection 3.2; then, in order to prove Theorem 2 we apply an iteration argument both in the subcritical case (Subsection 4.2) and in the critical case (Subsection 4.3). In particular, in the critical case we employ the so-called *slicing method* in order to deal with logarithmic factors. For further details on the slicing method see [2], where this method was introduced for the first time or [51, 52, 57, 41, 42, 43] where the slicing method is used in critical cases in order to manage factors of logarithmic type.

4.1 Iteration frame

Let (u, v) be a local in time solution to (6). If we denote $x = (z, w)$ with $z \in \mathbb{R}$ and $w \in \mathbb{R}^{n-1}$ as in Subsection 3.2, then, we may introduce the functions

$$\mathcal{U}(t, z) \doteq \int_{\mathbb{R}^{n-1}} u(t, z, w) \, dw, \quad \mathcal{V}(t, z) \doteq \int_{\mathbb{R}^{n-1}} v(t, z, w) \, dw$$

for any $t > 0, z \in \mathbb{R}$ in the case $n \geq 2$. Clearly, also in this case we can simply work with u, v instead of \mathcal{U}, \mathcal{V} for $n = 1$. Repeating the same steps as in the case of the single semilinear equation, we end up with the estimates

$$\begin{aligned} (R+z)^{\frac{\sigma_1}{2}} \mathcal{U}(z+R, z) &\gtrsim \varepsilon \|u_0 + u_1\|_{L^1(\mathbb{R}^n)} \\ &\quad + \int_R^z (R+y)^{-\frac{n-1}{2}(p-1) + \frac{\sigma_1}{2} - \frac{\sigma_2}{2} p} \left| (R+y)^{\frac{\sigma_2}{2}} \mathcal{V}(y+R, y) \right|^p dy, \\ (R+z)^{\frac{\sigma_2}{2}} \mathcal{V}(z+R, z) &\gtrsim \varepsilon \|v_0 + v_1\|_{L^1(\mathbb{R}^n)} \\ &\quad + \int_R^z (R+y)^{-\frac{n-1}{2}(q-1) + \frac{\sigma_2}{2} - \frac{\sigma_1}{2} q} \left| (R+y)^{\frac{\sigma_1}{2}} \mathcal{U}(y+R, y) \right|^q dy \end{aligned} \quad (24)$$

on the characteristic $t = z + R$ for $z \geq R$. Let us point out that the assumptions on the Cauchy data in the statement of Theorem 2 allow us to proceed exactly as the proof of Theorem 1 when we estimate from below the terms which are related to the solution of the corresponding linear homogeneous problem. We define the functions $U(z) \doteq (R+z)^{\frac{\sigma_1}{2}} \mathcal{U}(z+R, z)$ and $V(z) \doteq (R+z)^{\frac{\sigma_2}{2}} \mathcal{V}(z+R, z)$. Hence, denoting by C and K the unexpressed multiplicative constants in (24), we obtain the iteration frame

$$U(z) \geq M\varepsilon + C \int_R^z (R+y)^{-\frac{n-1}{2}(p-1) + \frac{\sigma_1}{2} - \frac{\sigma_2}{2} p} |V(y)|^p dy, \quad (25)$$

$$V(z) \geq N\varepsilon + K \int_R^z (R+y)^{-\frac{n-1}{2}(q-1) + \frac{\sigma_2}{2} - \frac{\sigma_1}{2} q} |U(y)|^q dy \quad (26)$$

for any $z \geq R$, where $M \doteq C \|u_0 + u_1\|_{L^1(\mathbb{R}^n)}$ and $N \doteq K \|v_0 + v_1\|_{L^1(\mathbb{R}^n)}$. Note that (25) and (26) provide not only the iteration frame for the pair (U, V) , but also the base step of the inductive argument. Indeed, in the base case we will simply estimate U, V from below by the two quantities $M\varepsilon, N\varepsilon$, respectively.

4.2 Iteration argument: subcritical case

In this section we prove that a local in time solution (u, v) to (6) blows up in finite time in the subcritical case

$$\begin{aligned} \Omega(n, \sigma_1, \sigma_2, p, q) &= \max \{ \Lambda(n + \sigma_1, p, q), \Lambda(n + \sigma_2, q, p) \} \\ &= \max \left\{ \frac{p+1}{pq-1} - \frac{n + \sigma_1 - 1}{2}, \frac{q+1}{pq-1} - \frac{n + \sigma_2 - 1}{2} \right\} > 0, \end{aligned}$$

of course, provided that u_0, u_1, v_0, v_1 satisfy the assumptions of Theorem 2.

Let us assume that $\Omega(n, \sigma_1, \sigma_2, p, q) = \Lambda(n + \sigma_1, p, q)$. First we prove the sequence of lower bound estimates for U

$$U(z) \geq C_j(R+z)^{-\alpha_j}(z-R)^{\beta_j} \quad \text{for any } z \geq R, \quad (27)$$

where $\{\alpha_j\}_{j \in \mathbb{N}}$, $\{\beta_j\}_{j \in \mathbb{N}}$ and $\{C_j\}_{j \in \mathbb{N}}$ are sequences of nonnegative real numbers that we will determine afterwards via an inductive procedure. Clearly, from (25) we see that (27) is true for $j = 0$, provided that $\alpha_0 \doteq 0$, $\beta_0 \doteq 0$ and $C_0 \doteq M\varepsilon$. We prove now the inductive step. We assume that (27) is satisfied for $j \geq 0$. Plugging (27) in (26), we get

$$\begin{aligned} V(z) &\geq K \int_R^z (R+y)^{-\frac{n-1}{2}(q-1) + \frac{\sigma_2}{2} - \frac{\sigma_1}{2}q} |U(y)|^q dy \\ &\geq KC_j^q \int_R^z (R+y)^{-\frac{n-1}{2}(q-1) + \frac{\sigma_2}{2} - \frac{\sigma_1}{2}q - q\alpha_j} (y-R)^{q\beta_j} dy \\ &\geq KC_j^q (R+z)^{-\frac{n-1}{2}(q-1) - \frac{\sigma_1}{2}q - q\alpha_j} \int_R^z (y-R)^{q\beta_j + \frac{\sigma_2}{2}} dy \\ &= KC_j^q (q\beta_j + \frac{\sigma_2}{2} + 1)^{-1} (R+z)^{-\frac{n-1}{2}(q-1) - \frac{\sigma_1}{2}q - q\alpha_j} (z-R)^{q\beta_j + \frac{\sigma_2}{2} + 1} \end{aligned}$$

for $z \geq R$. Combining the above lower bound for V and (25), we arrive at

$$\begin{aligned} U(z) &\geq C \int_R^z (R+y)^{-\frac{n-1}{2}(p-1) + \frac{\sigma_1}{2} - \frac{\sigma_2}{2}p} |V(y)|^p dy \\ &\geq \frac{CK^p C_j^{pq}}{(q\beta_j + \frac{\sigma_2}{2} + 1)^p} \int_R^z (R+y)^{-\frac{n-1}{2}(pq-1) - \frac{\sigma_1}{2}pq - \frac{\sigma_2}{2}p - pq\alpha_j} (y-R)^{pq\beta_j + \frac{\sigma_2}{2}p + p + \frac{\sigma_1}{2}} dy \\ &\geq \frac{CK^p C_j^{pq}}{(q\beta_j + \frac{\sigma_2}{2} + 1)^p} (R+z)^{-\frac{n-1}{2}(pq-1) - \frac{\sigma_1}{2}pq - \frac{\sigma_2}{2}p - pq\alpha_j} \int_R^z (y-R)^{pq\beta_j + \frac{\sigma_2}{2}p + \frac{\sigma_1}{2} + p} dy \\ &= \frac{CK^p C_j^{pq}}{(q\beta_j + \frac{\sigma_2}{2} + 1)^p (pq\beta_j + \frac{\sigma_2}{2}p + \frac{\sigma_1}{2} + p + 1)} (R+z)^{-\frac{n-1}{2}(pq-1) - \frac{\sigma_1}{2}pq - \frac{\sigma_2}{2}p - pq\alpha_j} \\ &\quad \times (z-R)^{pq\beta_j + \frac{\sigma_2}{2}p + \frac{\sigma_1}{2} + p + 1} \end{aligned}$$

for $z \geq R$. So, we proved (27) for $j+1$, provided that

$$\alpha_{j+1} \doteq \frac{n+\sigma_1-1}{2}(pq-1) + \frac{\sigma_2}{2}p + \frac{\sigma_1}{2} + pq\alpha_j, \quad (28)$$

$$\beta_{j+1} \doteq \frac{\sigma_2}{2}p + \frac{\sigma_1}{2} + p + 1 + pq\beta_j, \quad (29)$$

$$C_{j+1} \doteq CK^p \left(\frac{\sigma_2}{2} + 1 + q\beta_j\right)^{-p} \left(\frac{\sigma_2}{2}p + \frac{\sigma_1}{2} + p + 1 + pq\beta_j\right)^{-1} C_j^{pq}. \quad (30)$$

Next we derive the explicit expressions for α_j and β_j . Applying (28) iteratively, we get

$$\alpha_j = A + pq\alpha_{j-1} = A(1+pq) + (pq)^2\alpha_{j-2} = A \sum_{k=0}^{j-1} (pq)^k + (pq)^j\alpha_0 = A \frac{(pq)^j - 1}{pq-1}, \quad (31)$$

where $A \doteq \frac{n+\sigma_1-1}{2}(pq-1) + \frac{\sigma_2}{2}p + \frac{\sigma_1}{2}$ and we used $\alpha_0 = 0$. Similarly, from (29) we find

$$\beta_j = B \frac{(pq)^j - 1}{pq-1}, \quad (32)$$

where $B \doteq \frac{\sigma_2}{2}p + \frac{\sigma_1}{2} + p + 1$. Now we use the explicit expression of β_j to get a lower bound estimate for C_j . Since $\beta_j \leq \frac{B}{pq-1}(pq)^j$, by (29) and (30) we get

$$\begin{aligned} C_j &= CK^p \left(\frac{\sigma_2}{2} + 1 + q\beta_{j-1} \right)^{-p} \left(\frac{\sigma_2}{2}p + \frac{\sigma_1}{2} + p + 1 + pq\beta_{j-1} \right)^{-1} C_{j-1}^{pq} \\ &\geq CK^p \left(\frac{\sigma_2}{2}p + \frac{\sigma_1}{2} + p + 1 + pq\beta_{j-1} \right)^{-(p+1)} C_{j-1}^{pq} \\ &= CK^p \beta_j^{-(p+1)} C_{j-1}^{pq} \geq \underbrace{CK^p \left(\frac{B}{pq-1} \right)^{-(p+1)}}_{\doteq \tilde{C}} (pq)^{-(p+1)j} C_{j-1}^{pq} = \tilde{C} (pq)^{-(p+1)j} C_{j-1}^{pq}. \end{aligned}$$

Applying the logarithmic function to both sides of the last inequality and using iteratively the resulting inequality, we get

$$\begin{aligned} \log C_j &\geq pq \log C_{j-1} - j \log(pq)^{p+1} + \log \tilde{C} \\ &\geq (pq)^2 \log C_{j-2} - (j + (j-1)(pq)) \log(pq)^{p+1} + (1 + pq) \log \tilde{C} \\ &\geq \dots \geq (pq)^j \log C_0 - \left(\sum_{k=0}^{j-1} (j-k)(pq)^k \right) \log(pq)^{p+1} + \left(\sum_{k=0}^{j-1} (pq)^k \right) \log \tilde{C}. \end{aligned}$$

Using the formula

$$\sum_{k=0}^{j-1} (j-k)(pq)^k = \frac{1}{pq-1} \left(\frac{(pq)^{j+1} - 1}{pq-1} - (j+1) \right),$$

which can be proved with an inductive argument, we have

$$\begin{aligned} \log C_j &\geq (pq)^j \log C_0 - \frac{1}{pq-1} \left(\frac{(pq)^{j+1} - 1}{pq-1} - (j+1) \right) \log(pq)^{p+1} + \frac{(pq)^j - 1}{pq-1} \log \tilde{C} \\ &= (pq)^j \left(\log C_0 - \frac{pq}{(pq-1)^2} \log(pq)^{p+1} + \frac{\log \tilde{C}}{pq-1} \right) + (j+1) \frac{\log(pq)^{p+1}}{pq-1} \\ &\quad + \frac{\log(pq)^{p+1}}{(pq-1)^2} - \frac{\log \tilde{C}}{pq-1}. \end{aligned}$$

For $j \geq j_0 \doteq \max \left\{ 0, \frac{\log \tilde{C}}{\log(pq)^{p+1}} - \frac{pq}{pq-1} \right\}$ from the last inequality we get

$$\log C_j \geq (pq)^j \log(\widehat{C}\varepsilon), \quad (33)$$

where $\widehat{C} \doteq M(pq)^{-\frac{(pq)(p+1)}{(pq-1)^2}} \tilde{C}^{\frac{1}{pq-1}}$. Finally, we combine (27), (31), (32) and (33) and it results

$$\begin{aligned} U(z) &\geq C_j (R+z)^{-A \frac{(pq)^j - 1}{pq-1}} (z-R)^{B \frac{(pq)^j - 1}{pq-1}} \\ &\geq \exp \left((pq)^j \log(\widehat{C}\varepsilon) \right) (R+z)^{-A \frac{(pq)^j - 1}{pq-1}} (z-R)^{B \frac{(pq)^j - 1}{pq-1}} \\ &= \exp \left((pq)^j \left(\log(\widehat{C}\varepsilon) - \frac{A}{pq-1} \log(R+z) + \frac{B}{pq-1} \log(z-R) \right) \right) (R+z)^{\frac{A}{pq-1}} (z-R)^{-\frac{B}{pq-1}} \end{aligned} \quad (34)$$

for $z \geq R$ and $j \geq j_0$. If we require $z \geq 3R$, then, it holds $2(z-R) \geq R+z$. So, from (34) we get

$$\begin{aligned} U(z) &\geq \exp\left((pq)^j \log\left(2^{-\frac{B}{pq-1}} \widehat{C} \varepsilon (R+z)^{\frac{B-A}{pq-1}}\right)\right) (R+z)^{\frac{A}{pq-1}} (z-R)^{-\frac{B}{pq-1}} \\ &= \exp\left((pq)^j \log\left(\bar{C} \varepsilon (R+z)^{\frac{p+1}{pq-1} - \frac{n+\sigma_1-1}{2}}\right)\right) (R+z)^{\frac{A}{pq-1}} (z-R)^{-\frac{B}{pq-1}} \end{aligned}$$

for $z \geq 3R$ and $j \geq j_0$, where $\bar{C} \doteq 2^{-\frac{B}{pq-1}} \widehat{C}$. We recall that we are working for (t, z) on the characteristic $t = z + R$, so we may rewrite the last inequality as

$$U(z) \geq \exp\left((pq)^j \log\left(\bar{C} \varepsilon t^{\Lambda(n+\sigma_1, p, q)}\right)\right) (R+z)^{\frac{A}{pq-1}} (z-R)^{-\frac{B}{pq-1}} \quad (35)$$

for $t \geq 4R$ and $j \geq j_0$. We choose $\varepsilon_0 = \varepsilon_0(n, \mu_1, \mu_2, v_1^2, v_2^2, u_0, u_1, v_0, v_1, R) > 0$ such that

$$(\bar{C} \varepsilon_0)^{-(\Lambda(n+\sigma_1, p, q))^{-1}} \geq 4R.$$

Then, for any $\varepsilon \in (0, \varepsilon_0]$ and $t > (\bar{C} \varepsilon)^{-(\Lambda(n+\sigma_1, p, q))^{-1}}$ we obtain

$$t \geq 4R \text{ and } \bar{C} \varepsilon t^{\Lambda(n+\sigma_1, p, q)} > 1$$

and, hence, letting $j \rightarrow \infty$ in (35) the lower bound for $U(z)$ blows up. Therefore, in order to get a finite value of $U(z)$, it must hold the converse inequality for t . So, we have showed the upper bound for the lifespan

$$T(\varepsilon) \lesssim \varepsilon^{-(\Lambda(n+\sigma_1, p, q))^{-1}}.$$

In the case $\Omega(n, \sigma_1, \sigma_2, p, q) = \Lambda(n + \sigma_2, q, p)$ it suffices to switch the role of U and V in order to show the estimate $T(\varepsilon) \lesssim \varepsilon^{-(\Lambda(n+\sigma_2, q, p))^{-1}}$ in an analogous way. Also, we completed the proof of Theorem 2 in the subcritical case. In the critical case $\Omega(n, \sigma_1, \sigma_2, p, q) = 0$ we need to modify our approach. As we have already announced, we will employ the slicing method in order to deal with logarithmic factors in the sequence of lower bounds for U .

4.3 Iteration argument: critical case

In this subsection we prove Theorem 2 in the critical case

$$\Omega(n, \sigma_1, \sigma_2, p, q) = \max\{\Lambda(n + \sigma_1, p, q), \Lambda(n + \sigma_2, q, p)\} = 0.$$

We begin with the case $\Lambda(n + \sigma_1, p, q) = 0 > \Lambda(n + \sigma_2, q, p)$.

Let us introduce the sequence $\{\ell_j\}_{j \in \mathbb{N}}$, where $\ell_j \doteq 2 - 2^{-(j+1)}$. Our goal is to prove the sequence of lower bound estimate for U

$$U(z) \geq D_j \left(\log \left(\frac{z}{\ell_j R} \right) \right)^{\theta_j} \quad \text{for } z \geq \ell_j R, \quad (36)$$

where $\{D_j\}_{j \in \mathbb{N}}$, $\{\theta_j\}_{j \in \mathbb{N}}$ are suitable sequences of nonnegative real numbers that we shall determine throughout the iteration procedure. Obviously, (36) is true for $j = 0$ provided that $D_0 \doteq M\varepsilon$ and $\theta_0 \doteq 0$. Also, we proved the base case. It remains to prove the inductive step. Before starting, we remark that $\{\ell_j\}_{j \in \mathbb{N}}$ is an increasing and bounded sequence. In particular, $\ell_j \geq \ell_0 = \frac{3}{2}$. Consequently, for any $j \in \mathbb{N}$ and any $z \geq \ell_j R$ we may use the inequality

$z \geq \frac{3}{5}(R+z)$. Let us assume that (36) holds, we shall prove that (36) is satisfied also for $j+1$. Combining (26) and (36), we get

$$\begin{aligned}
V(z) &\geq K \int_{\ell_{jR}}^z (R+y)^{-\frac{n-1}{2}(q-1) + \frac{\sigma_2}{2} - \frac{\sigma_1}{2}q} |U(y)|^q dy \\
&\geq KD_j^q \int_{\ell_{jR}}^z (R+y)^{-\frac{n-1}{2}(q-1) + \frac{\sigma_2}{2} - \frac{\sigma_1}{2}q} \left(\log \left(\frac{y}{\ell_{jR}} \right) \right)^{q\theta_j} dy \\
&\geq KD_j^q (R+z)^{-\frac{n-1}{2}(q-1) - \frac{\sigma_1}{2}q} \int_{\ell_{jR}}^z (R+y)^{\frac{\sigma_2}{2}} \left(\log \left(\frac{y}{\ell_{jR}} \right) \right)^{q\theta_j} dy \\
&\geq KD_j^q (R+z)^{-\frac{n-1}{2}(q-1) - \frac{\sigma_1}{2}q} \int_{\frac{\ell_j z}{\ell_{j+1}}}^z (R+y)^{\frac{\sigma_2}{2}} \left(\log \left(\frac{y}{\ell_{jR}} \right) \right)^{q\theta_j} dy \\
&\geq KD_j^q (R+z)^{-\frac{n-1}{2}(q-1) - \frac{\sigma_1}{2}q} \left(R + \frac{\ell_j z}{\ell_{j+1}} \right)^{\frac{\sigma_2}{2}} \left(\log \left(\frac{z}{\ell_{j+1}R} \right) \right)^{q\theta_j} \left(1 - \frac{\ell_j}{\ell_{j+1}} \right) z \\
&\geq \frac{3}{5} KD_j^q \left(\frac{\ell_j}{\ell_{j+1}} \right)^{\frac{\sigma_2}{2}} \left(1 - \frac{\ell_j}{\ell_{j+1}} \right) (R+z)^{-\frac{n-1}{2}(q-1) - \frac{\sigma_1}{2}q + \frac{\sigma_2}{2} + 1} \left(\log \left(\frac{z}{\ell_{j+1}R} \right) \right)^{q\theta_j}
\end{aligned}$$

for any $z \geq \ell_{j+1}R$. If we plug this lower bound for V in (25) and we use the critical condition $\Lambda(n + \sigma_1, p, q) = 0$, it results

$$\begin{aligned}
U(z) &\geq C \int_{\ell_{j+1}R}^z (R+y)^{-\frac{n-1}{2}(p-1) + \frac{\sigma_1}{2} - \frac{\sigma_2}{2}p} |V(y)|^p dy \\
&\geq \left(\frac{3}{5} \right)^p CK^p D_j^{pq} \left(\frac{\ell_j}{\ell_{j+1}} \right)^{\frac{\sigma_2 p}{2}} \left(1 - \frac{\ell_j}{\ell_{j+1}} \right)^p \int_{\ell_{j+1}R}^z (R+y)^{-\frac{n+\sigma_1-1}{2}(pq-1)+p} \left(\log \left(\frac{y}{\ell_{j+1}R} \right) \right)^{pq\theta_j} dy \\
&\geq \left(\frac{3}{5} \right)^{p+1} CK^p D_j^{pq} \left(\frac{\ell_j}{\ell_{j+1}} \right)^{\frac{\sigma_2 p}{2}} \left(1 - \frac{\ell_j}{\ell_{j+1}} \right)^p \int_{\ell_{j+1}R}^z \frac{1}{y} \left(\log \left(\frac{y}{\ell_{j+1}R} \right) \right)^{pq\theta_j} dy \\
&= \left(\frac{3}{5} \right)^{p+1} CK^p D_j^{pq} \left(\frac{\ell_j}{\ell_{j+1}} \right)^{\frac{\sigma_2 p}{2}} \left(1 - \frac{\ell_j}{\ell_{j+1}} \right)^p (pq\theta_j + 1)^{-1} \left(\log \left(\frac{z}{\ell_{j+1}R} \right) \right)^{pq\theta_j + 1}
\end{aligned}$$

for any $z \geq \ell_{j+1}R$. Since $1 - \frac{\ell_j}{\ell_{j+1}} \geq 2^{-(j+3)}$ and $2\ell_j > \ell_{j+1}$ for any $j \in \mathbb{N}$, we find

$$U(z) \geq CK^p 2^{-jp - ((4 + \frac{\sigma_2}{2})p+1)} D_j^{pq} (pq\theta_j + 1)^{-1} \left(\log \left(\frac{z}{\ell_{j+1}R} \right) \right)^{pq\theta_j + 1}$$

for any $z \geq \ell_{j+1}R$. Also, we proved (36) for $j+1$, provided that

$$\theta_{j+1} \doteq 1 + pq\theta_j, \quad (37)$$

$$D_{j+1} \doteq CK^p 2^{-jp - ((4 + \frac{\sigma_2}{2})p+1)} (pq\theta_j + 1)^{-1} D_j^{pq}. \quad (38)$$

Applying recursively (37), we obtain

$$\theta_j = 1 + pq\theta_{j-1} = 1 + pq + (pq)^2\theta_{j-2} = \dots = \sum_{k=0}^{j-1} (pq)^k + \theta_0 (pq)^j = \frac{(pq)^j - 1}{pq - 1}. \quad (39)$$

Therefore, in (38) we estimate D_j as follows:

$$\begin{aligned}
D_j &= CK^p 2^{-jp - ((4 + \frac{\sigma_2}{2})p+1)} \theta_j^{-1} D_{j-1}^{pq} \geq CK^p 2^{-jp - ((4 + \frac{\sigma_2}{2})p+1)} (pq-1)(pq)^{-j} D_{j-1}^{pq} \\
&= \tilde{D} (2^p pq)^{-j} D_{j-1}^{pq},
\end{aligned}$$

where $\tilde{D} \doteq 2^{-(4+\frac{\sigma_2}{2})p+1} CK^p(pq-1)$. Repeating the same argument as in Subsection 4.2 (application of the logarithmic function and iterative use of the resulting inequality), we find that

$$\log D_j \geq (pq)^j \log(\widehat{D}\varepsilon), \quad (40)$$

for $j \geq j_1 \doteq \max\left\{0, \frac{\log \tilde{D}}{\log(2^p pq)} - \frac{pq}{pq-1}\right\}$, where $\widehat{D} \doteq M(2^p pq)^{-\frac{(pq)}{(pq-1)^2}} \tilde{D}^{\frac{1}{pq-1}}$. Combining (36), (39) and (40) we get

$$U(z) \geq \exp\left((pq)^j \log(\widehat{D}\varepsilon)\right) \left(\log\left(\frac{z}{\ell_j R}\right)\right)^{\frac{(pq)^j - 1}{pq-1}}$$

for any $j \geq j_1$ and any $z \geq \ell_j R$. Since $\ell_j < 2$ for any $j \in \mathbb{N}$, then, from the previous inequality we get

$$\begin{aligned} U(z) &\geq \exp\left((pq)^j \log(\widehat{D}\varepsilon)\right) \left(\log\left(\frac{z}{2R}\right)\right)^{\frac{(pq)^j - 1}{pq-1}} \\ &= \exp\left((pq)^j \left(\log(\widehat{D}\varepsilon) + \frac{1}{pq-1} \log\left(\log\left(\frac{z}{2R}\right)\right)\right)\right) \left(\log\left(\frac{z}{2R}\right)\right)^{-\frac{1}{pq-1}} \\ &= \exp\left((pq)^j \log\left(\widehat{D}\varepsilon \left(\log\left(\frac{z}{2R}\right)\right)^{\frac{1}{pq-1}}\right)\right) \left(\log\left(\frac{z}{2R}\right)\right)^{-\frac{1}{pq-1}} \end{aligned} \quad (41)$$

for any $z \geq 2R$ and any $j \geq j_1$.

We can choose $\varepsilon_0 = \varepsilon_0(n, \mu_1, \mu_2, v_1^2, v_2^2, u_0, u_1, v_0, v_1, R) > 0$ sufficiently small such that

$$\exp\left((\widehat{D}\varepsilon_0)^{-(pq-1)}\right) \geq 1.$$

Consequently, for any $\varepsilon \in (0, \varepsilon_0]$ and $z > 2R \exp\left((\widehat{D}\varepsilon)^{-(pq-1)}\right)$ we obtain

$$z \geq 2R \quad \text{and} \quad \widehat{D}\varepsilon \left(\log\left(\frac{z}{2R}\right)\right)^{\frac{1}{pq-1}} > 1$$

and, hence, as $j \rightarrow \infty$ in (41) we see that the lower bound for $U(z)$ is not finite. Therefore, in order to guarantee the existence of $U(z)$, it must hold the converse inequality for z . Finally, since we are on the characteristic $t = z + R \lesssim z$, we derive the upper bound for the lifespan

$$T(\varepsilon) \leq \exp\left(\widehat{D}\varepsilon^{-(pq-1)}\right),$$

for a suitable positive constant \widehat{D} .

Finally, in the case $\Lambda(n + \sigma_2, q, p) = 0 > \Lambda(n + \sigma_1, p, q)$, by switching the role of U and V we get the same kind of upper bound estimate for the lifespan.

Case $\Lambda(n + \sigma_1, p, q) = \Lambda(n + \sigma_2, q, p) = 0$

In the cusp point of the critical curve we can improve the upper bound of the lifespan further. According to Remark 2 in this case $p = \tilde{p}(n, \sigma_1, \sigma_2)$ and $q = \tilde{q}(n, \sigma_1, \sigma_2)$. Therefore,

$$\begin{aligned} -\frac{n-1}{2}(p-1) + \frac{\sigma_1}{2} - \frac{\sigma_2}{2} p &= -\frac{n+\sigma_2-1}{2}(p-1) + \frac{\sigma_1}{2} - \frac{\sigma_2}{2} \\ &= -\frac{n+\sigma_2-1}{2} \left(\frac{n+\sigma_1+1}{n+\sigma_2-1} - 1\right) + \frac{\sigma_1}{2} - \frac{\sigma_2}{2} = -1 \end{aligned}$$

and, similarly, $-\frac{n-1}{2}(q-1) + \frac{\sigma_2}{2} - \frac{\sigma_1}{2}q = -1$.

Therefore, in the case $\Lambda(n + \sigma_1, p, q) = \Lambda(n + \sigma_2, q, p) = 0$ the iteration frame is simply

$$U(z) \geq M\varepsilon + C \int_R^z (R+y)^{-1} |V(y)|^p dy, \quad (42)$$

$$V(z) \geq N\varepsilon + K \int_R^z (R+y)^{-1} |U(y)|^q dy \quad (43)$$

for any $z \geq R$. Due to the special structure of (42) and (43), in this case is not necessary to applying the slicing procedure in order to restrict step by step the domain of integration.

Thus, the first step will be to prove

$$U(z) \geq E_j \left(\log \left(\frac{z}{R} \right) \right)^{\rho_j} \quad \text{for } z \geq R, \quad (44)$$

where as usual $\{E_j\}_{j \in \mathbb{N}}$, $\{\rho_j\}_{j \in \mathbb{N}}$ are suitable sequences of nonnegative real numbers. For $j = 0$ we get that (44) is fulfilled provided that $E_0 \doteq M\varepsilon$ and $\rho_0 \doteq 0$. We prove now the inductive step. Noticing that $R+y \leq 2y$ for $y \geq R$, if we plug (44) in (43), then, it follows

$$\begin{aligned} V(z) &\geq K \int_R^z (R+y)^{-1} |U(y)|^q dy \geq KE_j^q \int_R^z (R+y)^{-1} \left(\log \left(\frac{y}{R} \right) \right)^{q\rho_j} dy \\ &\geq 2^{-1} KE_j^q \int_R^z \frac{1}{y} \left(\log \left(\frac{y}{R} \right) \right)^{q\rho_j} dy = 2^{-1} KE_j^q (q\rho_j + 1)^{-1} \left(\log \left(\frac{z}{R} \right) \right)^{q\rho_j + 1} \end{aligned}$$

for any $z \geq R$. Next we use this lower bound for V in (42), obtaining

$$\begin{aligned} U(z) &\geq C \int_R^z (R+y)^{-1} |V(y)|^p dy \\ &\geq 2^{-p} CK^p (q\rho_j + 1)^{-p} E_j^{pq} \int_R^z (R+y)^{-1} \left(\log \left(\frac{y}{R} \right) \right)^{pq\rho_j + p} dy \\ &\geq 2^{-(p+1)} CK^p (q\rho_j + 1)^{-p} E_j^{pq} \int_R^z \frac{1}{y} \left(\log \left(\frac{y}{R} \right) \right)^{pq\rho_j + p} dy \\ &= 2^{-(p+1)} CK^p (q\rho_j + 1)^{-p} (pq\rho_j + p + 1)^{-1} E_j^{pq} \left(\log \left(\frac{z}{R} \right) \right)^{pq\rho_j + p + 1} \end{aligned}$$

for any $z \geq R$. Hence, we proved (44) for $j+1$ provided that

$$\rho_{j+1} \doteq p + 1 + pq\rho_j, \quad (45)$$

$$E_{j+1} \doteq 2^{-(p+1)} CK^p (q\rho_j + 1)^{-p} (pq\rho_j + p + 1)^{-1} E_j^{pq}. \quad (46)$$

We determine the value of ρ_j by using (45) iteratively

$$\begin{aligned} \rho_j &= p + 1 + pq\rho_{j-1} = (p+1)(1 + pq) + (pq)\rho_{j-2} = \dots = (p+1) \sum_{k=0}^{j-1} (pq)^k + (pq)^j \rho_0 \\ &= \frac{p+1}{pq-1} ((pq)^j - 1). \end{aligned} \quad (47)$$

Using (47), from (46) we find

$$\begin{aligned} E_j &= 2^{-(p+1)} CK^p (q\rho_{j-1} + 1)^{-p} (pq\rho_{j-1} + p + 1)^{-1} E_{j-1}^{pq} \\ &\geq 2^{-(p+1)} CK^p (pq\rho_{j-1} + p + 1)^{-(p+1)} E_{j-1}^{pq} \\ &= 2^{-(p+1)} CK^p \rho_j^{-(p+1)} E_{j-1}^{pq} \geq 2^{-(p+1)} \underbrace{CK^p \left(\frac{p+1}{pq-1} \right)^{-(p+1)}}_{\doteq \tilde{E}} (pq)^{-(p+1)j} E_{j-1}^{pq}. \end{aligned}$$

As in the previous cases, from the inequality $E_j \geq \widetilde{E}(pq)^{-(p+1)j} E_{j-1}^{pq}$ it follows the estimate

$$E_j \geq \exp\left((pq)^j \log(\widehat{E}\varepsilon)\right) \quad (48)$$

for any $j \geq j_2 \doteq \max\{0, \frac{\log \widehat{E}}{\log(pq)^{p+1}} - \frac{pq}{pq-1}\}$, where $\widehat{E} \doteq M(pq)^{-\frac{(pq)(p+1)}{(pq-1)^2}} \widetilde{E}^{\frac{1}{pq-1}}$. Therefore, combining (44), (47) and (48), we have

$$\begin{aligned} U(z) &\geq E_j \left(\log\left(\frac{z}{R}\right)\right)^{\frac{p+1}{pq-1}((pq)^j-1)} \geq \exp\left((pq)^j \log(\widehat{E}\varepsilon)\right) \left(\log\left(\frac{z}{R}\right)\right)^{\frac{p+1}{pq-1}((pq)^j-1)} \\ &\geq \exp\left((pq)^j \log\left(\widehat{E}\varepsilon \left(\log\left(\frac{z}{R}\right)\right)^{\frac{p+1}{pq-1}}\right)\right) \left(\log\left(\frac{z}{R}\right)\right)^{-\frac{p+1}{pq-1}} \end{aligned} \quad (49)$$

for $z \geq R$ and $j \geq j_2$. We pick $\varepsilon_0 = \varepsilon_0(n, \mu_1, \mu_2, v_1^2, v_2^2, u_0, u_1, v_0, v_1, R) > 0$ enough small such that

$$\exp\left(\left(\widehat{E}\varepsilon_0\right)^{-\frac{pq-1}{p+1}}\right) \geq 1.$$

Also, for any $\varepsilon \in (0, \varepsilon_0]$ and $z > R \exp\left(\left(\widehat{E}\varepsilon_0\right)^{-\frac{pq-1}{p+1}}\right)$ we find

$$z \geq R \text{ and } \widehat{E}\varepsilon \left(\log\left(\frac{z}{R}\right)\right)^{\frac{p+1}{pq-1}} > 1$$

and, hence, as $j \rightarrow \infty$ in (49) the lower bound for $U(z)$ blows up. Therefore, in order to guarantee the finiteness of $U(z)$, we have to require the opposite inequality for z . Moreover, we are on the characteristic $t = z + R \lesssim z$, so from the inequality $z \leq R \exp\left(\left(\widehat{E}\varepsilon_0\right)^{-\frac{pq-1}{p+1}}\right)$ we deduce for a suitable constant $\bar{E} > 0$ the upper bound for the lifespan

$$T(\varepsilon) \leq \exp\left(\bar{E}\varepsilon^{-\frac{pq-1}{p+1}}\right).$$

Finally, switching the role between U and V (that is, working with lower bound estimates for V and applying the iteration frame (25)-(26) in the reverse order) we end up with the estimate $T(\varepsilon) \leq \exp\left(E\varepsilon^{-\frac{pq-1}{q+1}}\right)$, for some constant $E > 0$. Summarizing, we proved the last estimate in (13) too.

5 Final remarks and open problems

Let us compare our results with other results known in the literature. In [30] a blow-up result is proved for (1) in the massless case by using the unbounded multiplier $m_1(t) \doteq (1+t)^\mu$ when $1 < p \leq p_{\text{Gla}}(n+2\mu)$. For the massless case (i.e. for $v^2 = 0$) in Theorem 1 we proved a blow-up result in the range $1 < p \leq p_{\text{Gla}}(n+\sigma)$, where

$$\sigma = \begin{cases} 2\mu & \text{for } \mu \in [0, 1), \\ 2 & \text{for } \mu \in [1, 2), \\ \mu & \text{for } \mu \in [2, \infty). \end{cases}$$

Since the Glassey exponent is decreasing with respect its argument, then, we improved the above cite result from [30] in the case $\mu > 1$ by enlarging the range of p for which the

nonexistence of global in time solutions can be proved under suitable assumptions for the data. Notice that for $\mu \in [0, 1)$ we found exactly the same result as in [30].

Up to the knowledge of the author the weakly coupled system (6) has not been studied so far in the literature. Nonetheless, even in this case we obtained as possible candidate for the critical curve

$$\max \left\{ \frac{p+1}{pq-1} - \frac{n+\sigma_1-1}{2}, \frac{q+1}{pq-1} - \frac{n+\sigma_2-1}{2} \right\} = 0,$$

which is a curve that presents a typical peculiarity of scale-invariant models: its branches are shifts of the branches of the critical curve for (7). The presence of these shifts is due to the fact that we deal exactly with scale-invariant lower order terms. In the massless case, if we take into account a weaker kind of damping term, namely, scattering producing damping terms (which means that the time-dependent coefficients for the damping terms are nonnegative and summable functions), then, the critical condition for the powers in the nonlinear terms is exactly the same as in the corresponding classical not-damped case (cf. [29, 30, 31, 58, 41, 42, 43]). Really recently, in [16] the blow-up dynamic for the semilinear wave equation with time-dependent and scattering producing damping and mass terms has been studied both in the subcritical and critical case.

Let us point out that we assumed throughout the paper that the parameters μ, v^2 satisfy $\delta \geq 0$ for the single semilinear equation (respectively, $\mu_1, \mu_2, v_1^2, v_2^2$ satisfy $\delta_1, \delta_2 \geq 0$ for the weakly coupled system). This assumptions are made in order to guarantee that the kernel functions defined by (16), (17) and (18) are nonnegative functions. Moreover, in the blow-up argument we estimate from below the hypergeometric functions by positive constants. This choice is also sharp in the case $\delta > 0$ when we consider an estimate from above. In the limit case $\delta = 0$ though we have to include a logarithmic factor in the upper bound for the hypergeometric function in (16). In any case, this is not an issue as in the global existence part one would deal in this specific case with a strict lower bound for the exponents. A similar situation is present in [7] in the case $\delta = 0$ but for study of the semilinear Cauchy problem with power nonlinearity (3).

Clearly, in order to prove the fact that the conditions for the exponents that we get in this paper are actually sharp, the sufficient part has to be studied.

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