TRUDINGER-MOSER TYPE INEQUALITY WITH LOGARITHMIC CONVOLUTION POTENTIALS

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Abstract. We establish Moser-Trudinger type inequalities in presence of a logarithmic convolution potential when the domain is a ball or the entire space $\mathbb{R}^2$. Moreover, we characterize critical nonlinear growth rates for these inequalities to hold and for the existence of corresponding extremal functions. In addition, we show that extremal functions satisfy corresponding Euler-Lagrange equations, and we derive general symmetry and uniqueness results for solutions of these equations.

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1. Introduction

The classical Trudinger-Moser inequality states that for any bounded domain $\Omega \subset \mathbb{R}^2$ we have

$$\sup_{u \in H_0^1(\Omega), |\nabla u|_2 \le 1} \int_\Omega e^{4\pi u^2} dx = c(\Omega) < +\infty \tag{1.1}$$

where, here and in the following, $|\cdot|_p$ denotes the usual L^p-norm for $1 \le p \le \infty$, so $u \mapsto |\nabla u|_2^2$ is the classical Dirichlet integral over Ω (see [22, 27]). Carleson and Chang [11] proved that the supremum in (1.1) is attained if $\Omega = B_1 := B_1(0)$ is the unit disc in $\mathbb{R}^2$, and successively Flucher [17] extended this result to arbitrary bounded domains in $\mathbb{R}^2$.

The inequality (1.1) does not extend to unbounded domains; in particular, it does not hold in the case $\Omega = \mathbb{R}^2$. Related inequalities for unbounded domains have been proposed first by Cao [10] and Adachi-Tanaka [1] under the assumption of subcritical growth, which corresponds to the case where 4π is replaced by $\alpha < 4\pi$ in (1.1). The case of critical nonlinear growth was considered by Ruf [23], who proved a Moser-Trudinger type inequality holds for unbounded domains under the assumption that the Dirichlet norm is replaced by the standard Sobolev norm and the nonlinearity $e^{4\pi u^2}$ is replaced by $e^{4\pi u^2} - 1$. More precisely, denoting by $\|u\| = \big(\int_\Omega (|\nabla u|^2 + u^2) dx \big)^{1/2}$ the usual Sobolev norm on $H_0^1(\Omega)$, Ruf proved that there exists a constant $d > 0$ such that

$$\sup_{u \in H_0^1(\Omega), \|u\| \le 1} \int_\Omega \big(e^{4\pi u^2} - 1 \big) dx \le d \qquad \text{for any domain } \Omega \subset \mathbb{R}^2. \tag{1.2}$$

Moreover, the supremum in (1.2) is attained if $\Omega = B_R(0)$ is any ball and $\Omega = \mathbb{R}^2$. Ruf also proved in [23] that the inequality (1.2) is sharp: for any growth $e^{\alpha u^2}$ with $\alpha > 4\pi$, the corresponding supremum equals $+\infty$. Further variants and generalizations of (1.2) are studied e.g. in [12].

In this paper, we wish to derive Moser-Trudinger type inequalities in the presence of a logarithmic convolution potential. More precisely, we aim to derive inequalities where the local terms $e^{4\pi u^2}$ and $e^{4\pi u^2} - 1$ in (1.1) and (1.2) are replaced with nonlocal ones, which move the Moser-Trudinger functional in (1.1) and (1.2) into a nonlocal interaction energy of the form

$$u \mapsto \int_\Omega \int_\Omega \ln \frac{1}{|x-y|} F(u(x)) F(u(y)) \, dx dy \tag{1.3}$$

with suitable nonlinearities $F : \mathbb{R} \to \mathbb{R}$. Here we restrict our attention to the cases $\Omega = B_1$ and $\Omega = \mathbb{R}^2$.

We recall that free energy functionals involving logarithmic kernel functions are considered in mathematical models for chemotaxis, see e.g. [15, 26]. There are many other two dimensional applications, such as the statistical mechanics of selfgravitating clouds [4, 28] and of point vortices in turbulent Euler flows [9]. In particular, when Ω is the entire space, the relevance of the logarithmic convolution kernel in (1.3) is due to the fact that it represents, up to a constant, the fundamental solution of $-\Delta$. Therefore, in the case where $\Omega = \mathbb{R}^2$, this kernel arises in a reduction of planar Schrödinger-Poisson systems to a single integro-differential equation, see e.g. [3, 7, 8, 13, 14, 16, 20, 21, 24, 25] and (1.16) below.

When Ω is a bounded domain, the logarithmic Newtonian kernel is replaced by the Green function of $-\Delta$ with Dirichlet/Neumann boundary conditions.

To proceed, we make the following general assumption on the function $F : \mathbb{R} \to \mathbb{R}$ considered in (1.3).

(A) $F : \mathbb{R} \to [0, \infty)$ is even, continuous, and strictly increasing on $[0, \infty)$. Moreover, there exist constants $\alpha, c > 0$ with

$$F(t) \le c e^{\alpha t^2} \qquad \text{for } t \in \mathbb{R}. \tag{1.4}$$

In the case where $\Omega = B_1$ is the unit ball, we then wish to analyze the problem of maximizing the quantity (1.3) among functions in the set

$$\mathcal{B}_1 := \{u \in H_0^1(B_1) \;:\; |\nabla u|_2 \le 1\}.$$

However, since the logarithmic kernel function in (1.3) changes sign, it is not a priori clear that the double integral in (1.3) has a well defined value. To clarify this point, we split the kernel $\ln \frac{1}{|\cdot|}$ into its positive and negative part and define functionals $\Phi_\pm : \mathcal{M}(B_1) \to [0, \infty]$ by

$$\Phi_+(u) = \int_{B_1} \int_{B_1} \ln^+ \frac{1}{|x-y|} F(u(x))F(u(y))\, dxdy; \tag{1.5}$$

$$\Phi_-(u) = \int_{B_1} \int_{B_1} \ln^+ |x-y|\, F(u(x))F(u(y))\, dxdy, \tag{1.6}$$

where $\ln^+ = \max\{\ln, 0\}$. Here, for a measurable subset $\Omega \subset \mathbb{R}^2$, we let $\mathcal{M}(\Omega)$ denotes the space of (Lebesgue-)measurable functions $\Omega \to \mathbb{R}$. As we shall see in Section 2 below, it follows from (1.4) that $\Phi_\pm(u) < \infty$ for every $u \in H_0^1(B_1)$, and therefore the quantity in (1.3) has a well-defined value

$$\Phi(u) := \int_{B_1} \int_{B_1} \ln \frac{1}{|x-y|} F(u(x))F(u(y))\, dxdy = \Phi_+(u) - \Phi_-(u) \quad \in (-\infty, \infty) \tag{1.7}$$

for every $u \in H_0^1(B_1)$. In our first main result, we provide a sharp borderline condition for the maximization problem related to maximizing Φ within the set $\mathcal{B}_1$. We need to distinguish different forms of asymptotic growth of the nonlinearity F. As in (1.1) and (1.2), the value 4π will play a key role.

Definition 1.1. *Let $\beta \in \mathbb{R}$, and let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary function. We say that f has*

(i) *at most β-critical growth if $|f(s)| \le c\, e^{4\pi s^2}(1 + |s|)^\beta$ for $s \in \mathbb{R}$ with some constant $c > 0$.*

(ii) *at least β-critical growth if there exist $s_0, c > 0$ with the property that*

$$|f(s)| \ge c\, e^{4\pi s^2} |s|^\beta \qquad \textit{for } |s| \ge s_0.$$

The following is our first main result of the paper.

Theorem 1.2. *Suppose that F satisfies (A).*

(i) If F has at most β-critical growth for some $\beta \le -1$, then

$$m_1(F) := \sup_{u\in\mathcal{B}_1} \Phi(u) < \infty.$$

(ii) If F has at most β-critical growth for some $\beta < -1$, then $m_1(F)$ is attained, and every maximizer for Φ in $\mathcal{B}_1$ is, up to sign, a radial and radially decreasing function in $\mathcal{B}_1$.

(iii) If F has at least β critical growth for some $\beta > -1$, then $m_1(F) = \infty$.

Remark 1.3. (i) In the case where F satisfies (A) and

$$F(s) \le c_1 e^{\alpha s^2} \qquad \text{for } s \in \mathbb{R} \text{ with constants } \alpha < 4\pi,\ c_1 > 0, \tag{1.8}$$

then F obviously has at most β-critical growth for any $\beta \in \mathbb{R}$ and therefore the assumptions of Theorem 1.2(i) are satisfied. In this subcritical case, an easy proof of the claim $m_1(F) < \infty$ can be given by combining (1.1) with the logarithmic Hardy-Littlewood-Sobolev inequality in [5]. Indeed, let $u \in \mathcal{B}_1 \cap L^\infty(B_1)$, and let $v := F(u)$. Then $v \le c_1 e^{\alpha u^2}$ and

$$v \ln v \le c_1 e^{\alpha u^2}\Big(\alpha u^2 + \ln c_1\Big) \le c_2 e^{4\pi u^2}$$

with a constant $c_2 > 0$. By the Trudinger-Moser inequality (1.1), it thus follows that

$$|v|_1 \le c_1 \int_{B_1} e^{4\pi u^2} dx \le c_1 c(B_1) =: c_2 \qquad \text{and} \qquad \int_{\mathbb{R}^2} v \ln v \, dx \le c_2 c(B_1) =: c_3. \tag{1.9}$$

Combining (1.9) and the logarithmic Hardy-Littlewood-Sobolev inequality [5, Theorem 2], we infer that

$$\begin{aligned}\Phi(u) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\frac{1}{|x-y|} v(x)v(y)\,dxdy &\le \frac{|v|_1}{2}\Big(|v|_1\big(c_4 + \big|\ln|v|_1\big|\big) + \int_{\mathbb{R}^2} v\ln v\,dx\Big)\\ &\le \frac{c_2}{2}\big(c_2\big(c_4 + |\ln c_2|\big) + c_3\big) < \infty,\end{aligned}$$

where $c_4 > 0$ is an explicit constant given in [5, Theorem 2]. For general $u \in \mathcal{B}_1$, the same inequality follows by approximation, and hence we have $m_1(F) < \infty$. We emphasize that this argument does not apply in the case where F has critical growth, so it does not give the sharp bound stated in Theorem 1.2.
(ii) It remains an open problem whether the functional Φ attains a maximum in $\mathcal{B}_1$ if F has at most β-critical growth for $\beta = -1$.

Next we wish to state a corresponding logarithmic Trudinger-Moser type inequality in the entire space $\mathbb{R}^2$. In view of (1.2), it is natural to make an additional assumption on F. We shall assume the following.

(A_1) $F : \mathbb{R} \to [0,\infty)$ satisfies (A), and $F(t) = O(|t|)$ as $t \to 0$.

Moreover, we define functionals $\Psi_\pm : \mathcal{M}(\mathbb{R}^2) \to [0,\infty]$ by

$$\Psi_+(u) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln^+\frac{1}{|x-y|}\, F(u(x))F(u(y))\,dxdy; \tag{1.10}$$

$$\Psi_-(u) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln^+|x-y|\, F(u(x))F(u(y))\,dxdy,. \tag{1.11}$$

We shall see in Section 2 below that, as a consequence of assumption (A_1), we have $\Psi_+(u) < \infty$ for every $u \in H^1(\mathbb{R}^2)$. Therefore

$$\Psi(u) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\frac{1}{|x-y|}\, F(u(x))F(u(y))\,dxdy = \Psi_+(u) - \Psi_-(u) \quad \in [-\infty,\infty) \tag{1.12}$$

is well-defined for every $u \in H^1(\mathbb{R}^2)$. We let $\|u\| = \big(\int_\Omega (|\nabla u|^2 + u^2)dx\big)^{1/2}$ denote the usual H^1-norm of $u \in H^1(\mathbb{R}^2)$, and we define the set

$$\mathcal{B}_\infty := \{u \in H^1(\mathbb{R}^2) \,:\, \|u\| \le 1\}.$$

Theorem 1.4. *Suppose that F satisfies (A_1).*

(i) If F has at most β-critical growth for some $\beta \le -1$, then

$$m_\infty(F) := \sup_{u \in \mathcal{B}_\infty} \Psi < \infty.$$

(ii) If F has at most β-critical growth for some $\beta < -1$, then the value $m_\infty(F)$ is attained, and every maximizer for Ψ in $\mathcal{B}_\infty$ is, up to sign and translation, a radial and radially decreasing function in $\mathcal{B}_\infty$.

(iii) If F has at least β critical growth for some $\beta > -1$, then

$$m_\infty(F) = \infty$$

in the sense that there exists a sequence $(u_n)_n$ in $\mathcal{B}_\infty$ with $\Psi_\pm(u_n) < \infty$ for every $n \in \mathbb{N}$ and $\Psi(u_n) \to \infty$ as $n \to \infty$.

As Theorem 1.2(i) and Theorem 1.4(i) yield the existence of maximizers, the question arises whether these maximizers satisfy a corresponding Euler-Lagrange equation. Due to the weak growth conditions imposed on the nonlinearity F, this is not immediate. We can give an answer to this problem under natural additional assumptions.

Theorem 1.5. *Suppose that $F \in C^1(\mathbb{R})$, and suppose that $f := F'$ satisfies*

$$f(t) \le c e^{\alpha t^2} \qquad \text{for } t \in \mathbb{R} \text{ with some constants } \alpha, c > 0. \tag{1.13}$$

(i) If F satisfies (A) and u is a maximizer of Φ on $\mathcal{B}_1$, then there exists $\theta \in \mathbb{R}$ such that u satisfies the associated Euler-Lagrange equation in weak sense, i.e.,

$$\int_{B_1} \nabla u \nabla \varphi \, dx = \theta \int_{B_1} \ln \frac{1}{|\cdot|} * \big(1_{B_1} F(u)\big) f(u) \varphi \, dx \qquad \text{for all } \varphi \in H^1_0(B_1). \tag{1.14}$$

Moreover, $u \in W^{2,p}(B_1) \cap C^{1,\sigma}(\overline{B_1})$ for all $p < \infty$, $\sigma \in (0,1)$.

(ii) If F satisfies (A_1) and u is a maximizer of Ψ on $\mathcal{B}_\infty$, then there exists $\theta \in \mathbb{R}$ such that u satisfies the associated Euler-Lagrange equation weak sense, i.e.,

$$\int_{\mathbb{R}^2} \big(\nabla u \nabla \varphi + u\varphi\big) \, dx = \theta \int_{\mathbb{R}^2} (\ln \frac{1}{|\cdot|} * F(u)) f(u) \varphi \, dx \tag{1.15}$$

for all $\varphi \in H^1_0(\mathbb{R}^2)$ with bounded support. Moreover, $u \in W^{2,p}_{loc}(\mathbb{R}^2) \cap C^{1,\sigma}_{loc}(\mathbb{R}^2)$ for all $p < \infty$, $\sigma \in (0,1)$.

We note that related integro-differential equations with exponentially growing nonlinearities in $\mathbb{R}^2$ have been studied in [2] with Riesz convolution kernels but not with the logarithmic kernel. Given the fact that the function $\ln \frac{1}{|\cdot|}$ in (1.15) is, up to a constant factor 2π, the fundamental solution of the operator $-\Delta$, it follows that sufficiently regular solutions of (1.15) correspond to classical solutions (u, w) of the nonlinear system

$$\begin{cases} -\Delta u + u = \theta\, w f(u) & \text{in } \mathbb{R}^2, \\ \quad -\Delta w = 2\pi F(u) & \text{in } \mathbb{R}^2, \end{cases} \tag{1.16}$$

subject to the conditions

$$u \in L^\infty(\mathbb{R}^2) \qquad \text{and} \qquad w(x) \to -\infty \quad \text{as } |x| \to \infty \tag{1.17}$$

We have stated in Theorem 1.4 that, if F satisfies (A_1) and has at most β-critical growth for some $\beta < -1$, then $\Psi\big|_{\mathcal{B}_\infty}$ admits maximizers, and every maximizer is, up to sign and translation, radial and radially decreasing. In the following Theorem, we show that, under additional assumptions, the latter property is shared by any positive solution of (1.16), (1.17).

Theorem 1.6. *Suppose that $F \in C^1(\mathbb{R})$ with $f = F'$, suppose that $F(0) = f(0) = 0$, and suppose that f is a nondecreasing Lipschitz function on $[0, \infty)$. Then every classical solution (u, w) of (1.16), (1.17) with $u > 0$ in $\mathbb{R}^2$ is radially symmetric up to translation and strictly decreasing in the distance from the symmetry center.*

We remark that the uniqueness of positive solutions, up to translation, of (1.16), (1.17) remains an open problem. The same is true regarding the uniqueness, up to sign and translation, of maximizers of $\Psi|_{\mathcal{B}_\infty}$.

The paper is organized as follows. In Section 2, we establish some preliminaries related to some nonlocal interaction energies. In Section 3 we study Moser-Trudinger inequalities with logarithmic convolution potentials when the domain is a ball and we establish Theorem 1.2. Section 4 is concerned with the maximization problem when the domain is the entire space $\mathbb{R}^2$ and contains the proof of Theorem 1.4. In Section 5 we show that such extremal functions satisfy Euler Lagrange equations in a weak sense, and we prove Theorem 1.5. Finally Section 6 deals with the radial symmetry of the positive solutions corresponding to the maximization problem in $\mathbb{R}^2$ and contains the proof of Theorem 1.6.

Notation. Throughout this paper, if $v : \mathbb{R}^N \to \mathbb{R}$ is a radially symmetric function, we let v also denote the associated function $[0, \infty) \to \mathbb{R}$ of the radial variable $r = |x|$.

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2. Preliminaries

We first introduce some notation. We recall that $\mathcal{M}(\mathbb{R}^2)$ denotes the space of real-valued measurable functions $\mathbb{R}^2$, and we let $\mathcal{M}_+(\mathbb{R}^2)$ to denote the subset of nonnegative functions in $\mathcal{M}(\mathbb{R}^2)$. If $\Omega \subset \mathbb{R}^N$ is a measurable subset and u is a real-valued measurable function on Ω, we also regard u as a function in $\mathcal{M}(\mathbb{R}^2)$ by trivial extension. We then define the quadratic forms $b_\pm : \mathcal{M}_+(\mathbb{R}^2) \to [0, \infty]$ by

$$(v, w) \mapsto b_+(v, w) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln^+ \frac{1}{|x - y|} \, v(x) w(y) \, dx dy, \tag{2.1}$$

$$(v, w) \mapsto b_-(v, w) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln^+ |x - y| \, v(x) w(y) \, dx dy. \tag{2.2}$$

Moreover, we define

$$b_0(v, w) := b_+(v, w) - b_-(v, w) \in [-\infty, \infty)$$

for all functions $v, w \in \mathcal{M}_+(\mathbb{R}^2)$ for which $b_+(v, w) < \infty$. For the sake of brevity, we also set

$$b_\pm(v) := b_\pm(v, v) \qquad \text{and} \qquad b_0(v) := b_0(v, v) \quad \text{if } b_+(v) < \infty.$$

Next we define

$$|v|_* := \int_{\mathbb{R}^2} \ln(1 + |x|)|v| \, dx \quad \in [0, \infty] \qquad \text{for } v \in \mathcal{M}(\mathbb{R}^2),$$

and we consider the space

$$L^1_{ln}(\mathbb{R}^2) := \big\{ v \in L^1(\mathbb{R}^2), \quad |v|_* < \infty \big\}.$$

Taking into account that

$$\ln^+|x-y| \le \ln^+(|x|+|y|) \le \ln(1+|x|)(1+|y|) = \ln(1+|x|)+\ln(1+|y|) \text{ for } x,y \in \mathbb{R}^2, \tag{2.3}$$

we infer that

$$[\ln^+|\cdot| * v](x) \le \ln(1+|x|)|v|_1 + |v|_* \qquad \text{for } v \in L^1_{ln}(\mathbb{R}^2),\ x \in \mathbb{R}^2$$

and therefore

$$\ln^+|\cdot| * v \ \in \ L^\infty_{loc}(\mathbb{R}^2) \qquad \text{for } v \in L^1_{ln}(\mathbb{R}^2). \tag{2.4}$$

We also infer that

$$b_-(v,w) \le |v|_1|w|_* + |w|_1|v|_* < \infty \qquad \text{for } v,w \in L^1_{ln}(\mathbb{R}^2) \cap \mathcal{M}_+(\mathbb{R}^2). \tag{2.5}$$

This property admits a partial converse.

Lemma 2.1. *Let $v,w \in \mathcal{M}_+(\mathbb{R}^2)$ be given with $v \not\equiv 0$, $w \in L^1_{loc}(\mathbb{R}^2)$ and $b_-(v,w) < \infty$. Then $w \in L^1_{ln}(\mathbb{R}^2)$.*

Proof. Since $v \not\equiv 0$, there exist constants $R > 1, c > 0$ and a measurable subset $A \subset B_R$ of positive measure with $v \ge c$ on A. Then we have $|x-y| \ge \frac{|y|}{2} \ge R$ for $x \in A$, $y \in \mathbb{R}^2 \setminus B_{2R}$ and therefore

$$\begin{aligned} b_-(v,w) &\ge c \int_A \int_{\mathbb{R}^2} w(y) \ln^+|x-y|\, dydx \ge c|A| \int_{\mathbb{R}^2 \setminus B_{2R}} w(y) \ln^+ \frac{|y|}{|2|}\, dy \\ &\ge c_1 \int_{\mathbb{R}^2 \setminus B_{2R}} w(y) \ln^+(1+|y|)\, dy \ge c_1\big(|w|_* - \ln(1+2R)\|w\|_{L^1(B_{2R})}\big) \end{aligned}$$

with a constant $c_1 > 0$. Hence $|w|_* < \infty$, and therefore $w \in L^1_{ln}(\mathbb{R}^2)$. □

The following corollary is immediate.

Corollary 2.2. *If $v \in L^1_{loc}(\mathbb{R}^2)$ is nonnegative and satisfies $b_-(v) < \infty$, then $v \in L^1_{ln}(\mathbb{R}^2)$.*

We also note that by (2.5) we have

$$b_-(v,w) \le |v|_1|w|_* + |w|_1|v|_* \le \big(2\ln 2\big)|v|_1|w|_1 < \infty \tag{2.6}$$

for $v,w \in L^1(\mathbb{R}^2)$ with $v \equiv w \equiv 0$ on $\mathbb{R}^N \setminus B_1$.
Next, let v^* denote the Schwarz symmetrization of a function $v \in \mathcal{M}(\mathbb{R}^2)$. So $v^* \in \mathcal{M}_+(\mathbb{R}^2)$ is radial and nonincreasing in the radial variable. We note the following Riesz rearrangement type inequalities.

Lemma 2.3. *Let $v \in \mathcal{M}_+(\mathbb{R}^2)$. Then we have:*

- *(i) $b_+(v^*) \ge b_+(v)$;*
- *(ii) $b_-(v^*) \le b_-(v)$;*
- *(iii) If $b_+(v^*) < \infty$, then*

$$b_0(v^*) \ge b_0(v). \tag{2.7}$$

If, in addition, $v \in L^p(\mathbb{R}^2)$ for some $p \in [1,2]$ and $b_+(v) < \infty$ and $b_-(v) < \infty$, then equality holds in (2.7) if and only if $v = v^*(\cdot - x_0)$ for some $x_0 \in \mathbb{R}^2$.

Here we note that, under the assumptions of (iii), both $b_0(v^*)$ and $b_0(v)$ are well defined by the inequalities in (i) and (ii).

Proof. (i) follows from the Riesz's rearrangement inequality (see e.g. Theorem 3.7 in [18]), since the function $\ln^+\frac{1}{|\cdot|}$ is nonnegative, radial and nonincreasing in the radial variable.
(ii) follows by arguing as in Lemma 3.2 in [24].
To prove (iii), we first note that (2.7) is a consequence of (i) and (ii). Let $p \in [1,2]$, and let $v \in L^p(\mathbb{R}^2)$ be given with $b_\pm(v) < \infty$ and with the property that equality holds in (2.7). In particular,

this also implies that $b_-(v^*) \le b_-(v) < \infty$ and $b_+(v^*) = b_0(v^*) + b_-(v^*) = b_0(v) + b_-(v^*) < \infty$. We now define $\tilde{b}_\pm : \mathcal{M}_+(\mathbb{R}^2) \to [0,\infty]$ by

$$\tilde{b}_+(v) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\bigl(1 + \frac{1}{|x-y|}\bigr) v(x)v(y)\,dxdy,$$
$$\tilde{b}_-(v) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\bigl(1 + |x-y|\bigr) v(x)v(y)\,dxdy.$$

In the same way as in (i) and (ii), we see that

$$\tilde{b}_+(v^*) \ge \tilde{b}_+(v) \qquad \text{and} \qquad \tilde{b}_-(v^*) \le \tilde{b}_-(v). \tag{2.8}$$

Moreover, we claim that

$$\tilde{b}_\pm(v),\ \tilde{b}_\pm(v^*) < \infty \qquad \text{and} \qquad b_0(v) = \tilde{b}_+(v) - \tilde{b}_-(v), \quad b_0(v^*) = \tilde{b}_+(v^*) - \tilde{b}_-(v^*). \tag{2.9}$$

Once this is proved, it follows from the equality in (2.7) that equality must hold in each of the inequalities in (2.8). Since the function $r \mapsto \ln\bigl(1 + \frac{1}{r}\bigr)$ is positive and strictly decreasing on $(0,\infty)$, it then follows from the strict version of Riesz's rearrangement inequality (see e.g. Theorem 3.9 in [18]) that $v = v^*(\cdot - x_0)$ for some $x_0 \in \mathbb{R}^2$.
To see (2.9), we note that

$$\ln(1 + |\cdot|) \le 2\ln^+|\cdot| + h \qquad \text{in } \mathbb{R}^2$$

with the continuous and compactly supported function $h = \Bigl(\ln(1+|\cdot|) - 2\ln^+|\cdot|\Bigr)^+$. Consequently, by Young's inequality

$$\begin{aligned}\tilde{b}_-(v) &:= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(1 + |x-y|) v(x)v(y)\,dxdy \\ &\le 2b_-(v) + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} h(x-y)v(x)v(y)\,dxdy \\ &\le 2b_-(v) + |v|_p^2 |h|_s < \infty\end{aligned}$$

with $s = \bigl(2 - \frac{2}{p}\bigr)^{-1}$ if $p \in (1,2]$ and $s = \infty$ if $p = 1$. In the same way, we see that $\tilde{b}_-(v^*) < \infty$. Next we note that

$$\ln(1 + \frac{1}{|\cdot|}) \le c\bigl(\ln^+\frac{1}{|\cdot|} + \ln(1 + |\cdot|)\bigr) \qquad \text{in } \mathbb{R}^2$$

with a constant $c > 0$ and therefore

$$\tilde{b}_+(v) \le c\bigl(b_+(v) + \tilde{b}_-(v)\bigr) < \infty.$$

In the same way, we see that $\tilde{b}_+(v^*) < \infty$. Since $\tilde{b}_\pm(v) < \infty$ and $\tilde{b}_\pm(v^*) < \infty$, the equalities in (2.9) now merely follow from the fact that $\ln\frac{1}{r} = \ln\bigl(1 + \frac{1}{r}\bigr) - \ln\bigl(1 + r\bigr)$ for $r > 0$. The proof is thus finished. □

We also note the following

Lemma 2.4. *For any $u \in H^1(\mathbb{R}^2)$ we have $u^* \in H^1(\mathbb{R}^2)$ and*

$$|u^*|_p = |u|_p, \quad |\nabla u^*|_2 \le |\nabla u|_2 \qquad \text{for all } 2 \le p < \infty.$$

Consequently, we have $u^* \in \mathcal{B}_1$ if $u \in \mathcal{B}_1$ and $u^* \in \mathcal{B}_\infty$ if $u \in \mathcal{B}_\infty$.

Proof. See e.g. [18, Lemma 7.17]. □

Next we wish to collect some nonuniform nonlinear estimates for functions $u \in H_0^1(B_1)$ and $u \in H^1(\mathbb{R}^2)$, respectively.

Lemma 2.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying*

$$|f(t)| \le ce^{\alpha t^2} \qquad \text{for } t \in \mathbb{R} \text{ with constants } c, \alpha > 0. \tag{2.10}$$

(i) If $u \in H_0^1(B_1)$*, then we have* $f(u) \in L^s(B_1)$ *for* $1 \le s < \infty$ *and*

$$\ln^+ \frac{1}{|\cdot|} * \big(1_{B_1} f(u)\big), \quad \ln^+ |\cdot| * \big(1_{B_1} f(u)\big) \quad \in L^\infty(B_1).$$

(ii) If $u \in H^1(\mathbb{R}^2)$*, then* $f(u) \in L^s_{loc}(\mathbb{R}^2)$ *for* $1 \le s < \infty$*.*

If, in addition, $|f(t)| = O(|t|)$ *as* $t \to 0$*, then* $f(u) \in L^2(\mathbb{R}^2)$ *and*

$$\ln^+ \frac{1}{|\cdot|} * f(u) \quad \in L^s(\mathbb{R}^2) \qquad \textit{for } 2 \le s \le \infty.$$

Proof. We first recall that

$$\int_{B_1} e^{\alpha u^2} dx < \infty \qquad \text{for any } u \in H_0^1(B_1) \text{ and any } \alpha > 0 \tag{2.11}$$

as shown in [27] (see also [19, p.195]). To prove (i), we now let $u \in H_0^1(B_1)$ and define $v = 1_{B_1} f(u)$. By (2.10) and (2.11), we have $v \in L^s(\mathbb{R}^2)$ for $s \in [1, \infty)$. Moreover,

$$\ln^+ \frac{1}{|\cdot|} \in L^s(\mathbb{R}^2) \qquad \text{for } s \in [1, \infty). \tag{2.12}$$

Choosing $s = 2$, we deduce by Young's inequality that $\ln^+ \frac{1}{|\cdot|} * v \in L^\infty(\mathbb{R}^2)$. Moreover, we deduce that

$$[\ln^+ |\cdot| * v](x) = \int_{B_1} \ln^+ |x - y| v(y)\, dy < \ln 2 |v|_1 \qquad \text{for } x \in B_1$$

and therefore $\ln^+ |\cdot| * v \in L^\infty(B_1)$.

(ii) Let $u \in H^1(\mathbb{R}^2)$. For given $R > 0$ and $s \in [1, \infty)$, we wish to prove that $1_{B_R} f(u) \in L^s(\mathbb{R}^2)$. For this, we may, by (2.10), assume without loss of generality that $f(t) = e^{\alpha t^2}$ for $t \in \mathbb{R}$. Then we have $[1_{B_R} f(u)]^* \le 1_{B_R} [f(u)]^* = 1_{B_R} f(u^*)$ and therefore

$$\big|1_{B_R} f(u)\big|_s = \big|[1_{B_R} f(u)]^*\big|_s \le \big|1_{B_R} f(u^*)\big|_s,$$

so it suffices to consider the case where $u = u^*$ in the following. Let $v := f(u)$. Since u is locally bounded on $\mathbb{R}^2 \setminus \{0\}$, the same is true for v. It thus suffices to prove that $v \in L^s(B_1)$. For this we consider the function

$$U := \Big(1 + u^2(1)\Big)^{\frac{1}{2}} [u - u(1)]^+ \in H_0^1(B_1).$$

Since

$$u^2 = [u - u(1)]^2 + 2u(1)[u - u(1)] + u^2(1) \le U^2 + 1 + u^2(1) \quad \text{in } B_1$$

we have

$$\int_{B_1} |u|^s\, dx = \int_{B_1} e^{2s\alpha u^2}\, dx \le e^{s\alpha\left(1 + u^2(1)\right)} \int_{B_1} e^{s\alpha U^2}\, dx < \infty$$

by (2.11). It thus follows that $v \in L^s_{loc}(\mathbb{R}^2)$.

Next we assume that $|f(t)| = O(|t|)$ as $t \to 0$. To prove that $v = f(u) \in L^2(\mathbb{R}^2)$, it suffices, by (2.10), to consider the case where $f(t) = e^{\alpha t^2} - 1$. It then follows that $|f(u)|_2 = |[f(u)]^*|_2 = |f(u^*)|_2$, so we may assume again that $u = u^*$. Then u is bounded on $\mathbb{R}^N \setminus B_1$, and therefore it follows $|v| \le C|u|$ on $\mathbb{R}^N \setminus B_1$, so $v 1_{\mathbb{R}^N \setminus B_1} \in L^2(\mathbb{R}^2)$ since $u \in L^2(\mathbb{R}^2)$. Since we already proved that $v 1_{B_1} \in L^2(\mathbb{R}^2)$, we infer that $v \in L^2(\mathbb{R}^2)$. By (2.12) and Young's inequality, it now follows that $\ln^+ \frac{1}{|\cdot|} * v \in L^s(\mathbb{R}^2)$ for $2 \le s \le \infty$. □

Corollary 2.6.

(i) If F *satisfies assumption* (A) *and* $u \in H_0^1(B_1)$*, then* $\Phi_\pm(u) < \infty$*.*

(ii) If F *satisfies assumption* (A_1) *and* $u \in H^1(\mathbb{R}^2)$*, then* $\Psi_+(u) < \infty$*.*

Proof. This is a rather direct consequence of Lemma 2.5, applied to $f = F$. □

Lemma 2.7. *Let $v \in L^1_{ln}(\mathbb{R}^2)$ be a nonnegative radial function. Then the convolution $[(\ln|\cdot|) * v](x) \in \mathbb{R}$ is well defined for $x \in \mathbb{R}^N \setminus \{0\}$ and given by*

$$[(\ln|\cdot|) * v](x) = \ln|x| \int_{B_{|x|}} v(y)dy + \int_{\mathbb{R}^N \setminus B_{|x|}} \ln|y| v(y)dy \tag{2.13}$$

Proof. Let $x \in \mathbb{R}^N \setminus \{0\}$. For nonnegative radial functions $v \in L^\infty(\mathbb{R}^N)$ with bounded support, both the LHS and RHS of (2.13) are well-defined, and the formula holds by Newton's theorem (see for instance [18, Theorem 9.7]).

Next we let $v \in L^1_{ln}(\mathbb{R}^2)$ be a nonnegative radial function. By (2.3), we have

$$[(\ln^+|\cdot| * v](x) \le \ln(1+|x|)|v|_1 + |v|_* < \infty.$$

We consider an increasing sequence of nonnegative radial functions $v_n \in L^\infty(\mathbb{R}^N)$ with bounded support and such that $v_n \to v$ pointwisely. Then

$$[(\ln^+|\cdot|) * v_n](x) \le [(\ln^+|\cdot| * v](x) \quad \text{for all } n \qquad \text{and} \qquad \lim_{n\to\infty} [(\ln^+|\cdot|) * v_n](x) = [(\ln^+|\cdot| * v](x).$$

The latter property is a consequence of the monotone convergence theorem. Moreover, also by monotone convergence,

$$\begin{aligned}
[(\ln^+\frac{1}{|\cdot|}) * v](x) &= \lim_{n\to\infty}[(\ln^+\frac{1}{|\cdot|}) * v_n](x) = \lim_{n\to\infty}\Big([(\ln^+|\cdot|) * v_n](x) - [(\ln|\cdot|) * v_n](x)\Big) \\
&= [(\ln^+|\cdot| * v](x) - \lim_{n\to\infty}\Big(\ln|x| \int_{B_{|x|}} v_n(y)dy + \int_{\mathbb{R}^N \setminus B_{|x|}} \ln|y| v_n(y)dy\Big) \\
&= [(\ln^+|\cdot| * v](x) - \Big(\ln|x| \int_{B_{|x|}} v(y)dy + \int_{\mathbb{R}^N \setminus B_{|x|}} \ln|y| v(y)dy\Big).
\end{aligned}$$

We thus conclude that

$$[(\ln|\cdot|) * v](x) = [(\ln^+|\cdot| * v](x) - [(\ln^+\frac{1}{|\cdot|}) * v](x) = \ln|x| \int_{B_{|x|}} v(y)dy + \int_{\mathbb{R}^N \setminus B_{|x|}} \ln|y| v(y)dy,$$

as claimed. □

Corollary 2.8. *Let $v, w \in L^1_{loc}(\mathbb{R}^N)$ be nonnegative radial functions.*

(i) If $b_\pm(v,w) < \infty$ and $v \not\equiv 0$, $w \not\equiv 0$, then $v, w \in L^1_{ln}(\mathbb{R}^2)$ and

$$\frac{b_0(v,w)}{(2\pi)^2} = \int_0^\infty rv(r)\Big(\ln\frac{1}{r}\int_0^r \rho w(\rho)d\rho + \int_r^\infty \rho(\ln\frac{1}{\rho})w(\rho)d\rho\Big)dr \tag{2.14}$$

$$= \int_0^1 rw(r)\Big(\ln\frac{1}{r}\int_0^r \rho v(\rho)d\rho + \int_r^\infty \rho(\ln\frac{1}{\rho})v(\rho)d\rho\Big)dr. \tag{2.15}$$

(ii) If $b_\pm(v) < \infty$, then $v \in L^1_{ln}(\mathbb{R}^2)$ and

$$\frac{b_0(v)}{(2\pi)^2} = 2\int_0^\infty rv(r)\ln\frac{1}{r}\int_0^r \rho v(\rho)d\rho dr.$$

Proof. (i) This follows from Lemma 2.1, Lemma 2.7 and Fubini's theorem.
(ii) The claim is obvious if $v \equiv 0$. If $v \not\equiv 0$, we have, by (i),

$$\begin{aligned}
\frac{b_0(v)}{(2\pi)^2} &= \int_0^\infty rv(r)\ln\frac{1}{r}\int_0^r \rho v(\rho)d\rho dr + \int_0^\infty rv(r)\int_r^\infty \rho(\ln\frac{1}{\rho})v(\rho)d\rho dr \\
&= \int_0^\infty rv(r)\ln\frac{1}{r}\int_0^r \rho v(\rho)d\rho dr + \int_0^\infty \rho(\ln\frac{1}{\rho})v(\rho)\int_0^\rho rv(r)drd\rho \\
&= 2\int_0^\infty rv(r)\ln\frac{1}{r}\int_0^r \rho v(\rho)d\rho dr.
\end{aligned}$$

□

Remark 2.9. *(i) If F satisfies (A), it follows from Corollary 2.6(i) and Corollary 2.8(ii) that*

$$\Phi(u) = 2(2\pi)^2 \int_0^1 rF(u(r)) \ln\frac{1}{r} \int_0^r \rho F(u(\rho))d\rho dr > 2(2\pi)^2 \int_0^1 rF(u(0)) \ln\frac{1}{r} \int_0^r \rho F(u(0))d\rho dr = \Phi(0)$$

for every $u \in H_0^1(B_1) \setminus \{0\}$. Hence $m_1(F) \in \bigl(\Phi(0),\infty\bigr]$, and therefore $m_1(F)$ is not attained at $u = 0$.
(ii) If F satisfies (A_1), we have

$$\Psi(u) = \Phi(u) > \Phi(0) = \Psi(0) = 0 \quad \text{for every function } u \in H_0^1(B_1) \subset H^1(\mathbb{R}^2) \text{ with } u \not\equiv 0.$$

In particular, we have $m_\infty(F) \in \bigl(0,\infty\bigr]$, and therefore $m_\infty(F)$ is not attained at $u = 0$.

We finally prove a key estimate which will be used in the following sections.

Lemma 2.10. *Let $\beta_1, \beta_2 \le 0$ satisfy $\beta_1 + \beta_2 \le -2$, and let $\mathcal{B}_{1,rad} := \{u \in \mathcal{B}_1 \,:\, u \text{ radial}\}$. Then*

$$\Phi_{\beta_1,\beta_2}(u_1,u_2) := \int_0^1 r(1+|u_1(r)|)^{\beta_1} e^{4\pi u_1^2(r)} \ln\frac{1}{r} \int_0^r \rho(1+|u_1(\rho)|)^{\beta_2} e^{4\pi u_1^2(\rho)} d\rho\, dr \tag{2.16}$$

defines a bounded functional $\Phi_{\alpha,\beta} : \mathcal{B}_{1,rad} \times \mathcal{B}_{1,rad} \to [0,\infty)$.

Proof. We fix $0 < \varepsilon < 1/4\pi$. Moreover, we let $u_1, u_2 \in \mathcal{B}_{1,rad}$, and we define

$$A_i^+ := \{r \in (0,1] \,:\, u_i(r) \ge \sqrt{\varepsilon(-\ln r)}\}, \qquad A_i^- := \{r \in (0,1] \,:\, u_i(r) < \sqrt{\varepsilon(-\ln r)}\}$$

for $i = 1,2$. With $v_i := (1+|u_i|)^{\beta_i} e^{4\pi u_i^2}$ we then have

$$v_i(r) \le e^{4\pi u_i^2(r)} \le r^{-4\pi\varepsilon} \qquad \text{for } r \in A_i^-, \tag{2.17}$$

and

$$v_i(r) \le \bigl(1+\sqrt{\varepsilon(-\ln r)}\bigr)^{\beta_i} e^{4\pi u_i^2(r)} \qquad \text{for } r \in A_i^+. \tag{2.18}$$

Moreover,

$$\begin{aligned}\Phi_{\beta_1,\beta_2}(u_1,u_2) &= \int_{A_1^+} rv_1(r)\ln\frac{1}{r}\int_{A_2^+\cap[0,r]} \rho v_2(\rho)d\rho dr + \int_{A_1^-} rv_1(r)\ln\frac{1}{r}\int_{A_2^-\cap[0,r]} \rho v_2(\rho)d\rho dr \\ &+ \int_{A_1^+} rv_1(r)\ln\frac{1}{r}\int_{A_2^-\cap[0,r]} \rho v_2(\rho)d\rho dr + \int_{A_1^-} rv_1(r)\ln\frac{1}{r}\int_{A_2^+\cap[0,r]} \rho v_2(\rho)d\rho dr,\end{aligned}$$

where

$$\begin{aligned}\int_{A_1^-} rv_1(r)\ln\frac{1}{r}\int_{A_2^-\cap[0,r]} \rho v_2(\rho)d\rho dr &\le \int_{A_1^-} r^{1-4\pi\varepsilon}\ln\frac{1}{r}\int_{A_2^-\cap[0,r]} \rho^{1-4\pi\varepsilon} d\rho dr \\ &\le \frac{1}{2-4\pi\varepsilon}\int_0^1 r^{3-8\pi\varepsilon}\ln\frac{1}{r}dr = C_1 < \infty\end{aligned}$$

and

$$\begin{aligned}&\int_{A_1^+} rv_1(r)\ln\frac{1}{r}\int_{A_2^+\cap[0,r]} \rho v_2(\rho)d\rho dr \\ &\le \int_{A_1^+} r\bigl(1+\sqrt{\varepsilon(-\ln r)}\bigr)^{\beta_1} e^{4\pi u_1^2(r)}\ln\frac{1}{r}\int_{A_2^+\cap[0,r]} \rho\bigl(1+\sqrt{\varepsilon(-\ln \rho)}\bigr)^{\beta_2} e^{4\pi u_2^2(\rho)} d\rho dr \\ &\le \int_0^1 (-\ln r)(1+\sqrt{\varepsilon(-\ln r)}\bigr)^{\beta_1+\beta_2} re^{4\pi u_1^2(r)}\int_0^r \rho e^{4\pi u_2^2(\rho)}d\rho dr \\ &\le \frac{1}{\varepsilon}\Bigl(\int_0^1 re^{4\pi u_1^2(r)}dr\Bigr)\Bigl(\int_0^1 \rho e^{4\pi u_2^2(\rho)}d\rho\Bigr) \le C_2.\end{aligned}$$

Here we used the assumption $\beta_1+\beta_2\le -2$ and the Trudinger-Moser inequality (1.1). Moreover, $C_1, C_2, \dots$ are positive constants independent of $u_1, u_2 \in \mathcal{B}_{1,rad}$. We also have

$$\int_{A_1^+} rv_1(r)\ln\frac{1}{r}\int_{A_2^-\cap[0,r]}\rho v_2(\rho)d\rho dr \le \int_0^1 r\big(1+\sqrt{\varepsilon(-\ln r)}\big)^{\beta_1}e^{4\pi u_1^2(r)}\ln\frac{1}{r}\int_0^r \rho^{1-4\pi\varepsilon}d\rho dr$$
$$\le \frac{1}{2-4\pi\varepsilon}\int_0^1 r^{3-4\pi\varepsilon}\Big(\ln\frac{1}{r}\Big)e^{4\pi u_1^2(r)}\,dr \le C_3\int_0^1 re^{4\pi u_1^2(r)}\,dr \le C_4$$

and

$$\int_{A_1^-} rv_1(r)\ln\frac{1}{r}\int_{A_2^+\cap[0,r]}\rho v_2(\rho)d\rho dr \le \int_0^1 r^{1-4\pi\varepsilon}\ln\frac{1}{r}\int_0^r \rho e^{4\pi u_2^2(\rho)}d\rho dr$$
$$\le \int_0^1 r^{1-4\pi\varepsilon}\ln\frac{1}{r}dr\int_0^1 \rho e^{4\pi u_2^2(\rho)}d\rho \le C_5\int_0^1 r^{1-4\pi\varepsilon}\ln\frac{1}{r}\,dr \le C_6$$

again by the Trudinger-Moser inequality (1.1). The proof is thus finished. □

3. Maximization problem on the ball

In this section we complete the proof of Theorem 1.2. We recall that $\mathcal{B}_1 := \{u \in H_0^1(B_1) : |\nabla u|_2 \le 1\}$, and we let

$$\mathcal{B}_1^* := \{u^* \,:\, u \in \mathcal{B}_1\}$$

denote the corresponding Schwarz symmetrized set. By Lemma 2.4, we then have $\mathcal{B}_1^* \subset \mathcal{B}_1$.

Lemma 3.1. *Suppose that F satisfies (A) and has at most 0-critical growth. Then the functional Φ_- is uniformly bounded on $\mathcal{B}_1$.*

Proof. Let $u \in \mathcal{B}_1$. By definition and (2.6) we have

$$\Phi_-(u) = b_-(1_{B_1}F(u), 1_{B_1}F(u)) \le \big(2\ln 2\big)|1_{B_1}F(u)|_1^2$$

where

$$|1_{B_1}F(u)|_1 \le c_1\int_{B_1} e^{4\pi u^2}\,dx \le c_2$$

with constants $c_1, c_2 > 0$ independent of u by assumption and the Trudinger-Moser inequality (1.1). □

Proposition 3.2. *Suppose that F satisfies (A) and has at most β-critical growth for some $\beta \le -1$. Then we have*

$$m_1(F) \le m_1^+(F) < \infty, \qquad \textit{where} \qquad m_1(F) = \sup_{\mathcal{B}_1}\Phi \quad \textit{and} \quad m_1^+(F) = \sup_{\mathcal{B}_1}\Phi^+$$

Proof. By Lemma 2.3, we have

$$\Phi^+(u) = b_+\big(1_{B_1}F(u), 1_{B_1}F(u)\big) \le b_+\big([1_{B_1}F(u)]^*, [1_{B_1}F(u)]^*\big) = b_+\big(1_{B_1}F(u^*), 1_{B_1}F(u^*)\big) = \Phi(u^*)$$

for every $u \in \mathcal{B}_1$ and therefore

$$m_1^+(F) = \sup_{u\in\mathcal{B}_1^*}\Phi^+(u). \tag{3.1}$$

By assumption, we have

$$F(s) \le c_1\frac{e^{4\pi s^2}}{1+|s|} \qquad \text{for every } s \in \mathbb{R} \tag{3.2}$$

with a constant $c_1 > 0$. Let $u \in \mathcal{B}_1^*$ and $v := 1_{B_1}F(u)$. By Corollary 2.6, we then have $b_\pm(v) < \infty$. Therefore we have, by Corollary 2.8(ii) and (3.2),

$$\frac{\Phi(u)}{2(2\pi)^2} = \frac{b_0(v)}{2(2\pi)^2} = \int_0^1 rv(r)\ln\frac{1}{r}\int_0^r \rho v(\rho)d\rho dr \le c_2$$

with

$$c_2 := c_1^2 \sup_{u \in \mathcal{B}_{1,rad}} \Phi_{\beta_1,\beta_2}(u,u) < \infty,$$

where $\mathcal{B}_{1,rad}$ and Φ_{β_1,β_2} are defined in Lemma 2.10 for $\beta_1 = \beta_2 = -1$. Combining this inequality with Lemma 3.1, we thus deduce that

$$\Phi_+(u) = \Phi(u) + \Phi_-(u) \le c_3 \quad \text{for } u \in \mathcal{B}_1^* \text{ with } \quad c_3 := 2(2\pi)^2 c_2 + \sup_{\mathcal{B}_1} \Phi_- < \infty. \tag{3.3}$$

Hence $m_1^+(F) \le c_3 < \infty$, and therefore also $m_1(F) \le m_1^+(F) < \infty$, as claimed. $\square$

The following Proposition shows the optimality of the growth exponent $\beta = -1$ for the finiteness of $m_1(F)$.

Proposition 3.3. *Suppose that F satisfies (A) and has at least β-critical growth for some $\beta > -1$. Then there exists a sequence of functions $u_n \in \mathcal{B}_1 \cap L^\infty(B_1)$ with $\Phi(u_n) \to \infty$ as $n \to \infty$.*

Proof. The assumption implies the existence of a constant $c_1 > 0$ with

$$F(s) \ge \frac{s^\beta e^{4\pi s^2}}{c_1} \qquad \text{for } s \ge c_1. \tag{3.4}$$

For $n \in \mathbb{N}$, $n \ge 2$, we now define $u_n = m_n \in H_0^1(B_1) \cap L^\infty(B_1)$ as in [23, Eq. (2.12)], namely

$$m_n(x) = \frac{1}{\sqrt{2\pi}} \frac{\ln(|x|)}{(\ln n)^{1/2}} \big(1 - \frac{1}{4 \ln n}\big)^{1/2}, \qquad \frac{1}{n} \le |x| \le 1$$

and

$$m_n(x) = \frac{1}{\sqrt{2\pi}} (\ln n)^{1/2} \big(1 - \frac{1}{4 \ln n}\big)^{1/2} \qquad 0 \le |x| \le \frac{1}{n}.$$

As noted in [23, p. 346], we then have $|\nabla u_n|_2 \le 1$ for n large and thus $u_n \in \mathcal{B}_1^*$. Moreover, $v_n := F(u_n) \in L^\infty(B_1)$ and therefore

$$\Phi_\pm(u_n) = b_\pm(v_n, v_n) < \infty \qquad \text{for } n \in \mathbb{N}.$$

By (3.4) we have, for n large,

$$v_n \;\ge\; c_1^{-1} \Big(\frac{\ln n}{2\pi} - \frac{1}{8\pi}\Big)^{\frac{\beta}{2}} e^{4\pi\left(\frac{\ln n}{2\pi} - \frac{1}{8\pi}\right)} \;\ge\; c_2 (\ln n)^{\frac{\beta}{2}} n^2 \qquad \text{on } B_{\frac{1}{n}}(0)$$

with a constant $c_2 > 0$. We derive that

$$\begin{aligned} \frac{b_0(v_n, v_n)}{(2\pi)^2} &\ge 2 \int_0^{\frac{1}{n}} r v_n(r) \ln \frac{1}{r} \int_0^r \rho v_n(\rho) d\rho dr \ge 2 \Big(c_2 (\ln n)^{\frac{\beta}{2}} n^2\Big)^2 \int_0^{\frac{1}{n}} r \ln \frac{1}{r} \int_0^r \rho d\rho dr \\ &= c_2^2 (\ln n)^\beta n^4 \int_0^{\frac{1}{n}} r^3 \ln \frac{1}{r} dr = -c_2^2 (\ln n)^\beta n^4 \Big(\frac{r^4}{4} \ln r \Big|_0^{\frac{1}{n}} - \int_0^{\frac{1}{n}} \frac{r^3}{4} dr\Big) \\ &= -c_2^2 (\ln n)^\beta n^4 \Big(\frac{r^4}{4} \ln r - \frac{r^4}{16}\Big)\Big|_0^{\frac{1}{n}} \ge \frac{1}{4} c_2^2 (\ln n)^{1+\beta}, \end{aligned}$$

so that $b_0(v_n, v_n) \to \infty$ as $n \to \infty$ since $\beta > -1$. This shows that $\Phi(u_n) = b_0(v_n, v_n) \to \infty$ as $n \to \infty$, as required. $\square$

Next we wish to prove the existence of a maximizer $u \in \mathcal{B}_1^*$ for $m_1(F)$ under the assumptions of Proposition 3.2. We first prove the following convergence result.

Proposition 3.4. *Suppose that F satisfies (A) and has at most β-critical growth with $\beta < -1$, and let $(u_n)_n$ be a sequence in $\mathcal{B}_1^*$ with $u_n \rightharpoonup 0$ in $H_0^1(B_1)$. Then*

$$\Phi(u_n) \to \Phi(0) \qquad \textit{as } n \to \infty. \tag{3.5}$$

Proof. Since $u_n \rightharpoonup 0$ in $H_0^1(B_1)$ and $H_0^1(B_1)$ is compactly embedded into $L^p(B_1)$ for $2 < p < \infty$, we have

$$u_n \to 0 \qquad \text{in } L^p(B_1) \text{ for } 2 < p < \infty. \tag{3.6}$$

Since $u_n \in \mathcal{B}_1^*$ for every $n \in \mathbb{N}$, (3.6) implies that

$$u_n \to 0 \qquad \text{uniformly in } [\delta, 1] \text{ for every } \delta \in (0,1). \tag{3.7}$$

We now write $F = \kappa_0 + \tilde{F}$ with $\kappa_0 = F(0)$, where the function $\tilde{F} = F - \kappa_0$ also satisfies (A), $\tilde{F}(0) = 0$ and has at most β-critical growth. Consequently,

$$\tilde{F}(t) \le c_1(1+|t|)^\beta e^{4\pi t^2} \le c_1 e^{4\pi t^2} \qquad \text{for } t \in \mathbb{R} \text{ with a constant } c_1 > 0. \tag{3.8}$$

With

$$v_n := 1_{B_1}\tilde{F}(u_n) \qquad \text{for } n \in \mathbb{N},$$

we then have

$$\Phi(u_n) = b_0(v_n) + 2b_0\big(1_{B_1}\kappa_0, v_n\big) + b_0(\kappa_0 1_{B_1}) = b_0(v_n) + 2b_0\big(1_{B_1}\kappa_0, v_n\big) + \Phi(0).$$

By Corollary 2.8(i), we have

$$b_0\big(1_{B_1}\kappa_0, v_n\big) = (2\pi)^2\kappa_0 \int_0^1 r v_n(r) g(r)dr \qquad \text{with} \quad g(r) = \ln\frac{1}{r}\int_0^r \rho d\rho + \int_r^1 \rho(\ln\frac{1}{\rho})d\rho.$$

Moreover, for any $\delta \in (0,1)$ we have, by (1.1) and (3.8),

$$\Big|\int_0^\delta r v_n g(r)dr\Big| \le \frac{c_1}{2\pi}\|g\|_{L^\infty(0,\delta)}\int_{B_1} e^{4\pi u_n^2}dx \le \frac{c_1 c(B_1)}{2\pi}\|g\|_{L^\infty(0,\delta)}. \tag{3.9}$$

By (3.7) and since $\tilde{F}(0) = 0$, we also have

$$v_n \to 0 \qquad \text{uniformly in } [\delta, 1] \text{ for every } \delta \in (0,1). \tag{3.10}$$

Combining (3.9), (3.10) and the fact that $g(r) \to 0$ as $r \to 0$, we see that

$$b_0\big(1_{B_1}\kappa_0, v_n\big) \to 0 \qquad \text{as } n \to \infty.$$

To prove (3.5), it thus remains to show that

$$b_0(v_n) \to 0 \qquad \text{as } n \to \infty. \tag{3.11}$$

To see (3.11), we note that for every $\delta \in (0,1)$ we have

$$\frac{b_0(v_n)}{2(2\pi)^2} = \int_0^1 r v_n(r)\ln\frac{1}{r}\int_0^r \rho v_n(\rho)d\rho dr = M_n^\delta + N_n^\delta,$$

where, by (1.1), (3.8) and (3.10)

$$\begin{aligned} M_n^\delta &:= \int_\delta^1 r v_n(r)\ln\frac{1}{r}\int_0^r \rho v_n(\rho)d\rho dr \le c_1\int_\delta^1 r v_n(r)\ln\frac{1}{r}\int_0^1 \rho e^{4\pi u_n^2(\rho)}d\rho dr \\ &\le \frac{c_1 c(B_1)}{2\pi}\int_\delta^1 r v_n(r)\ln\frac{1}{r}dr \to 0 \qquad \text{as } n \to \infty. \end{aligned} \tag{3.12}$$

To estimate

$$N_n^\delta := \int_0^\delta r v_n(r)\ln\frac{1}{r}\int_0^r \rho v_n(\rho)d\rho dr$$

we fix $\varepsilon \in (0, \frac{1}{4\pi})$ and define, for any $n \in \mathbb{N}$,

$$A_n^+ = \{r \in (0,1] \,:\, u_n(r) \ge \sqrt{\varepsilon(-\ln r)}\}, \qquad A_n^- := \{r \in (0,1] \,:\, u_n(r) < \sqrt{\varepsilon(-\ln r)}\}.$$

By (3.8), we have

$$v_n(r) \le c_1 e^{4\pi u_n^2(r)} \le \frac{c_1}{r^{4\pi\varepsilon}} \qquad \text{for } r \in A_n^-, \text{ and} \tag{3.13}$$

$$v_n(r) \le c_1\big(1+\sqrt{\varepsilon(-\ln r)}\big)^{\beta} e^{4\pi u_n^2(r)} \qquad \text{for } r \in A_n^+. \tag{3.14}$$

We now write

$$N_n^\delta = \int_{A_n^- \cap (0,\delta)} r v_n(r) \ln\frac{1}{r} \int_0^r \rho v_n(\rho) d\rho dr + \int_{A_n^+ \cap (0,\delta)} r v_n(r) \ln\frac{1}{r} \int_0^r \rho v_n(\rho) d\rho dr,$$

where, by (1.1) and (3.13),

$$\int_{A_n^- \cap (0,\delta)} r v_n(r) \ln\frac{1}{r} \int_0^r \rho v_n(\rho) d\rho dr \le c_1^2 \int_0^\delta r^{1-4\pi\varepsilon} \ln\frac{1}{r} \int_0^1 \rho e^{4\pi u_n^2(\rho)} d\rho dr$$

$$\le -\frac{c_1^2 c(B_1)}{2\pi} \int_0^\delta r^{1-4\pi\varepsilon} \ln r dr = \frac{c_1^2 c(B_1)}{2\pi}\Big(\frac{\delta^{1-4\pi\varepsilon}}{1-4\pi\varepsilon} - \frac{\delta^{2-4\pi\varepsilon}\ln\delta}{2-4\pi\varepsilon}\Big) \tag{3.15}$$

for all $n \in \mathbb{N}$. Moreover, we have

$$\int_{A_n^+ \cap (0,\delta)} r v_n(r) \ln\frac{1}{r} \int_0^r \rho v_n(\rho) d\rho dr$$

$$= \int_{A_n^+ \cap (0,\delta)} r v_n(r) \ln\frac{1}{r} \int_{A_n^+ \cap (0,r)} \rho v_n(\rho) d\rho dr + \int_{A_n^+ \cap (0,\delta)} r v_n(r) \ln\frac{1}{r} \int_{A_n^- \cap (0,r)} \rho v_n(\rho) d\rho dr,$$

where

$$\int_{A_n^+ \cap (0,\delta]} r v_n(r) \ln\frac{1}{r} \int_{A_n^+ \cap [0,r]} \rho v_n(\rho) d\rho dr$$

$$\le c_1^2 \int_{A_n^+ \cap (0,\delta]} r\big(1+\sqrt{\varepsilon(-\ln r)}\big)^{\beta} e^{4\pi u_n^2(r)} \ln\frac{1}{r} \int_{A_n^+ \cap [0,r]} \rho\big(1+\sqrt{\varepsilon(-\ln \rho)}\big)^{\beta} e^{4\pi u_n^2(\rho)} d\rho dr$$

$$\le c_1^2 \int_{A_n^+ \cap (0,\delta]} (-\ln r)\big(1+\sqrt{\varepsilon(-\ln r)}\big)^{2\beta} r e^{4\pi u_n^2(r)} \int_0^r \rho e^{4\pi u_n^2(\rho)} d\rho dr$$

$$\le \frac{c_1^2}{\varepsilon}\big(1+\sqrt{\varepsilon(-\ln\delta)}\big)^{2+2\beta}\Big(\int_0^1 r e^{4\pi u_n^2(r)} dr\Big)^2 \le \frac{\big(c_1 c(B_1)\big)^2}{\varepsilon} \sqrt{\varepsilon(-\ln\delta)}\big)^{2+2\beta} \tag{3.16}$$

again by (1.1). Here we used the assumption $\beta < -1$. Finally, we have

$$\int_{A_n^+ \cap [0,\delta]} r v_n(r) \ln\frac{1}{r} \int_{A_n^- \cap [0,r]} \rho v_n(\rho) d\rho dr \le c_1^2 \int_{A_n^+ \cap [0,\delta]} r e^{4\pi u_n^2(r)} \ln\frac{1}{r} \int_0^r \rho^{1-4\pi\varepsilon} d\rho dr$$

$$\le \frac{c_1^2}{2-4\pi\varepsilon} \int_0^\delta r^{3-4\pi\varepsilon} e^{4\pi u_n^2(r)} \ln\frac{1}{r}\, dr \le c_1^2 C_\delta \int_0^1 r e^{4\pi u_n^2(r)}\, dr \le \frac{c_1^2 C_\delta c(B_1)}{2\pi} \tag{3.17}$$

by (1.1) with $C_\delta = \sup_{r\in[0,\delta]} r^{2-4\pi\varepsilon} \ln\frac{1}{r}$. Since, as $\varepsilon \in (0,\frac{1}{4\pi})$ and $\beta < -1$, the RHS of (3.15), (3.16) and (3.17) tend to zero as $\delta \to 0^+$, we infer that

$$\lim_{\delta\to 0} \sup_{n\in\mathbb{N}} N_n^\delta = 0.$$

Combining this with (3.12), we infer (3.11), as claimed. □

Proposition 3.5. *Suppose that F satisfies (A) and has at most β-critical growth with $\beta < -1$. Then the value $m_1(F) < \infty$ is attained by a function $u \in \mathcal{B}_1^*$.*

Proof. Let $(u_n)_n$ be a maximizing sequence in $\mathcal{B}_1$ for m_F. By Lemma 2.3, we may assume that $u_n \in \mathcal{B}_1^*$ for $n \in \mathbb{N}$. Since $\mathcal{B}_1$ is bounded in $H_0^1(B_1)$, we may also assume that

$$u_n \rightharpoonup u \in H_0^1(B_1) \qquad \text{with } u \in \mathcal{B}_1^*. \tag{3.18}$$

By [19, Theorem 1.6], we have two possibilities. Either

i) $u = 0$, or

ii) $u \neq 0$, and $\int_{B_1} e^{(4\pi+t)u_n^2}\,dx$ is bounded for some $t > 0$ and thus

$$e^{4\pi u_n^2} \to e^{4\pi u^2} \quad \text{in} \quad L^1(B_1). \tag{3.19}$$

Assume first that *i*) holds. In this case Proposition 3.4 implies that

$$m_1(F) = \lim_{n\to\infty} \Phi(u_n) = \Phi(0),$$

which is impossible by Remark 2.9. So this case does not occur.
It remains to consider the case where $(u_n)_n$ satisfies *ii*). Set $v_n := 1_{B_1}F(u_n)$ for $n \in \mathbb{N}$ and $v := 1_{B_1}F(u)$. By *ii*), $\int_{B_1} e^{(4\pi+t)u_n^2}\,dx$ is bounded for some $t > 0$ and thus v_n is bounded in $L^{s_0}(\mathbb{R}^2)$ with $s_0 = 1 + \frac{t}{4\pi} > 1$. Moreover, since

$$v_n \to v \qquad \text{in } L^1(\mathbb{R}^2),$$

interpolation yields that

$$v_n \to v \qquad \text{in } L^s(\mathbb{R}^2) \qquad \text{for } 1 \le s < s_0.$$

Moreover,

$$\begin{aligned}
\frac{\Phi(u_n)-\Phi(u)}{2(2\pi)^2} &= \int_0^1 r v_n(r)\ln\frac{1}{r}\int_0^r \rho v_n(r)\,d\rho dr - \int_0^1 r v(r)\ln\frac{1}{r}\int_0^r \rho v(\rho)\,d\rho dr\\
&= \int_0^1 r v_n(r)\ln\frac{1}{r}\int_0^r \rho[v_n(\rho)-v(\rho)]\,d\rho dr + \int_0^1 r[v_n(r)-v(r)]\ln\frac{1}{r}\int_0^r \rho v(\rho)\,d\rho dr
\end{aligned}$$

where, for fixed $s \in (1, s_0)$,

$$\begin{aligned}
\Big|\int_0^1 r v_n(r)\ln\frac{1}{r}\int_0^r \rho[v_n(\rho)-v(\rho)]\,d\rho dr\Big| &\le \frac{|v_n-v|_s}{2\pi}\int_0^1 r|B_r|^{\frac{1}{s'}} v_n(r)\ln\frac{1}{r}dr\\
\le \frac{|v_n-v|_s}{2\pi^{1-\frac{1}{s'}}}\int_0^1 r^{1+\frac{2}{s'}} v_n(r)\ln\frac{1}{r}dr &\le C|v_n-v|_s|v_n|_1 \to 0 \qquad \text{as } n\to\infty
\end{aligned}$$

with $C := \frac{\pi^{\frac{1}{s'}-2}}{4}\sup_{r\in(0,1]} r^{\frac{2}{s'}}\ln\frac{1}{r}$ and also

$$\begin{aligned}
\Big|\int_0^1 r[v_n(r)-v(r)]\ln\frac{1}{r}\int_0^r \rho v(\rho)\,d\rho dr\Big| &\le \frac{|v|_s}{2\pi}\int_0^1 r|B_r|^{\frac{1}{s'}}[v_n(r)-v(r)]\ln\frac{1}{r}dr\\
\le \frac{|v|_s}{2\pi^{1-\frac{1}{s'}}}\int_0^1 r^{1+\frac{2}{s'}}[v_n(r)-v(r)]\ln\frac{1}{r}dr &\le C|v|_s|v_n-v|_1 \to 0 \qquad \text{as } n\to\infty.
\end{aligned}$$

We thus conclude that

$$m_1(F) = \lim_{n\to\infty}\Phi(u_n) = \Phi(u),$$

so $m_1(F)$ is attained at $u \in \mathcal{B}_1^*$. □

The proof of Theorem 1.2 is now completed by the following lemma.

Lemma 3.6. *Let F satisfy (A), and let $u \in \mathcal{B}_1$ be a maximizer for $\Phi\big|_{\mathcal{B}_1}$. Then, up to a change of sign, we have $u \in \mathcal{B}_1^*$, and u is strictly positive in B_1.*

Proof. We already know that $u \not\equiv 0$. We first assume that $u \in \mathcal{B}_1^*$, and we suppose by contradiction that there exists $\tau \in (0,1)$ with $u(r) > 0$ for $r \in (0,\tau)$ and $u(r) = 0$ for $r \in [\tau, 1)$. Let then $\tilde{u} \in H_0^1(B_1)$ be defined by $\tilde{u}(r) = u(\tau r)$. Then we have $|\nabla\tilde{u}|_2 = |\nabla u|_2$ and therefore $\tilde{u} \in \mathcal{B}_1^*$. Moreover, with $v := F(u)$ we have

$$\Phi(\tilde{u}) = 2(2\pi)^2\int_0^1 r v(\tau r)\ln\frac{1}{r}\int_0^r \rho v(\tau\rho)\,d\rho dr,$$

where

$$v(\tau r) \geq v(r) \quad \text{for } r \in (0,\tau) \qquad \text{and} \qquad v(\tau r) = F(u(\tau r)) > F(0) = v(r) \quad \text{for } r \in [\tau, 1),$$

since F is strictly increasing on $[0,\infty)$ by assumption (A). It thus follows that

$$\Phi(\tilde{u}) > 2(2\pi)^2 \int_0^1 r v(r) \ln\frac{1}{r} \int_0^r \rho v(\rho) d\rho dr = \Phi(u),$$

contrary to the maximizing property of u. We thus conclude that u is strictly positive in B_1. Next, we let $u \in \mathcal{B}_1$ be a general maximizer of $\Phi\big|_{\mathcal{B}_1}$. By Lemma 2.3(iii), it then follows that $u^* \in \mathcal{B}_1^*$ is also a maximizer of $\Phi\big|_{\mathcal{B}_1}$. Moreover, since $F(u) \in L^2(\mathbb{R}^2)$ by Lemma 2.5 and $b_\pm(F(u),F(u)) < \infty$, it also follows from Lemma 2.3(iii) that $F(u)$ equals $F(u)^*$ up to translation. Since F is even and strictly increasing on $[0,\infty)$ by assumption (A), this implies that u equals u^* up to sign and translation. Since $u \in H_0^1(B_1)$ and, as we have proved above, $u^* > 0$ in B_1, it follows that $u \equiv u^* \in \mathcal{B}_1^*$ or $-u \equiv u^* \in \mathcal{B}_1$. The proof is thus finished. □

4. MAXIMIZATION PROBLEM IN THE ENTIRE SPACE $\mathbb{R}^2$

In this section we complete the proof of Theorem 1.4. We recall that $\mathcal{B}_\infty := \{u \in H^1(\mathbb{R}^2) : \|u\| \leq 1\}$, and we let

$$\mathcal{B}_\infty^* := \{u^* \ : \ u \in \mathcal{B}_\infty\}$$

denote the corresponding Schwarz symmetrized set. By Lemma 2.4, $\mathcal{B}_\infty^*$ is a subset of $\mathcal{B}_\infty$. Throughout this section, we assume that $F : \mathbb{R} \to \mathbb{R}$ satisfies assumption (A_1).

Lemma 4.1. *Suppose that F has at most β-critical growth for some $\beta \leq -1$. Then*

$$m_\infty^+(F) = \sup_{\mathcal{B}_\infty} \Psi_+ < \infty.$$

Proof. By assumption, there exists a constant $c_1 > 0$ with

$$F(s) \leq \widetilde{F}(s) := c_1 \frac{e^{4\pi s^2}}{\sqrt{1+s^2}} \qquad \text{for } s \in \mathbb{R}. \tag{4.1}$$

We note that $\widetilde{F}$ satisfies assumption (A). Moreover, we note the property that

$$\widetilde{F}(\sqrt{t+s}) = c_1 \frac{e^{4\pi(t+s)}}{\sqrt{1+(s+t)}} \leq c_1 e^{4\pi s} \frac{e^{4\pi t}}{\sqrt{1+t}} = e^{4\pi s}\widetilde{F}(\sqrt{t}) \quad \text{for } t,s \geq 0. \tag{4.2}$$

Since $\Psi_+(u^*) \geq \Psi_+(u)$ for every $u \in H^1(\mathbb{R}^2)$ by Lemma 2.3(i), it now suffices to consider functions $u \in \mathcal{B}_\infty^*$. We first note, by the radial lemma (see [6, Lemma A.IV]), we have

$$u(r) \leq \frac{|u|_2}{\sqrt{\pi}} r^{-\frac{1}{2}} \leq \frac{\|u\|}{\sqrt{\pi}} r^{-\frac{1}{2}} \leq \frac{r^{-\frac{1}{2}}}{\sqrt{\pi}}. \tag{4.3}$$

Let $v = F(u)$. We write

$$v = F(u) = F(u 1_{\mathbb{R}^N \setminus B_1}) + F(u 1_{B_1}) =: v_1 + v_2.$$

Since F is nonnegative and $F(t) = O(|t|)$ as $t \to 0$, it follows that

$$0 \leq v(r) \leq c_2 |u(r)| \qquad \text{for } r \geq 1$$

with a constant $c_2 > 0$ and therefore

$$|v_1|_2 \leq c_2 |u|_2 \leq c_2 \|u\| \leq c_2. \tag{4.4}$$

To estimate v_2, we now consider the radial function

$$U := \Big(1 + \frac{|u|_2^2}{\pi}\Big)^{\frac{1}{2}} [u - u(1)]^+ \ \in \ H_0^1(B_1).$$

Since

$$|\nabla U|_2^2 \le \Big(1+\frac{|u|_2^2}{\pi}\Big)|\nabla u_2|_2^2 \le \Big(1+\frac{|u|_2^2}{\pi}\Big)|\nabla u|_2^2 \le |u|_2^2 + |\nabla u|_2^2 \le 1, \tag{4.5}$$

we have $U \in \mathcal{B}_1^*$, which implies in particular that

$$|v_2|_1 \le c_1 \int_{B_1} e^{4\pi U^2}\,dx \le c_1 c(B_1) =: c_3 \tag{4.6}$$

by the Trudinger-Moser inequality (1.1). Since $u(1) \le \frac{|u|_2}{\sqrt{\pi}} \le \frac{1}{\sqrt{\pi}}$ by (4.3), we also have

$$u^2 = [u-u(1)]^2 + 2u(1)[u-u(1)]^+ + u^2(1) \le (1+u^2(1))[u-u(1)]^2 + 1 + u^2(1) \le U^2 + 1 + \frac{1}{\pi}. \tag{4.7}$$

Consequently, by (4.2) we have

$$v_2 \le \widetilde{F}(\sqrt{u^2}) \le \widetilde{F}\Big(\sqrt{U^2 + (1+\frac{1}{\pi})}\Big) \le c_1 e^{4\pi(1+\frac{1}{\pi})}\widetilde{F}(U) \qquad \text{in } B_1.$$

We now have

$$\Psi_+(u) = b_+(v) = b_+(v_1) + b_+(v_2) + 2b_+(v_1, v_2) \tag{4.8}$$

where

$$b_+(v_2) \le c_1^2 e^{8\pi(1+\frac{1}{\pi})} b_+(\widetilde{F}(U), \widetilde{F}(U)) \le c_1^2 e^{8\pi(1+\frac{1}{\pi})} m_1^+(\widetilde{F}) =: c_4 < \infty$$

by Proposition 3.2. Moreover, by Young's inequality, (4.4) and (4.6) we have

$$b_+(v_1, v_2) \le |v_1|_2 \Big|\ln^+ \frac{1}{|\cdot|}\Big|_2 |v_2|_1 \le c_2 \Big|\ln^+ \frac{1}{|\cdot|}\Big|_2 c_3 =: c_5$$

Moreover, again by Young's inequality,

$$b_+(v_1) \le |v_1|_2^2 \Big|\ln^+ \frac{1}{|\cdot|}\Big|_1 \le c_2^2 \Big|\ln^+ \frac{1}{|\cdot|}\Big|_1 =: c_6.$$

Inserting these estimates in (4.8), we deduce that $\Psi_+(u) \le c_4 + c_6 + 2c_5 < \infty$. Thus the claim is proved. ☐

Remark 4.2. *From Lemma 4.1 it follows that*

$$m_\infty(F) = \sup_{\mathcal{B}_\infty} \Psi \le m_\infty^+(F) < \infty.$$

Moreover, it follows from Lemma 2.3 that

$$m_\infty(F) := \sup_{u \in \mathcal{B}_\infty^*} \Psi(u) \tag{4.9}$$

where $\mathcal{B}_\infty^* := \{u \in \mathcal{B}_\infty \,:\, u = u^*\}$ *is the Schwarz symmetrized set.*

Our next aim is to prove the existence of maximizers for $m_\infty(F)$.

Proposition 4.3. *Suppose that* F *satisfies* (A), $F(s) = O(|s|)$, *and that* F *has at most* β*-critical growth for some* $\beta < -1$. *Then the value* $m_\infty(F) < \infty$ *is attained by a function* $u \in \mathcal{B}_\infty^*$.

Proof. Let $(u_n)_n$ be a maximizing sequence in $\mathcal{B}_\infty$ for $m_\infty(F)$. By Lemma 2.3, we may assume that $u_n \in \mathcal{B}_\infty^*$ for $n \in \mathbb{N}$. Moreover, we have $b_-(F(u_n)) < \infty$ for every $n \in \mathbb{N}$. Hence, by Corollary 2.8, we have

$$\Psi(u_n) = b_0(F(u), F(u)) = 8\pi^2\Big(\Psi_2(u_n) - \Psi_1(u_n)\Big) \qquad \text{for } n \in \mathbb{N},$$

with the nonnegative functionals $\Psi_1, \Psi_2 : \mathcal{B}_\infty^* \to [0, \infty)$ given by

$$\Psi_1(u) = \int_1^\infty rF(u(r)) \ln r \int_0^r \rho F(u(\rho))d\rho dr, \qquad \Psi_2(u) = \int_0^1 rF(u(r)) \ln\frac{1}{r} \int_0^r \rho F(u(\rho))d\rho dr.$$

Since $\mathcal{B}_\infty$ is bounded in $H^1(\mathbb{R}^2)$, we may assume that

$$u_n \rightharpoonup u \in H^1(\mathbb{R}^2) \qquad \text{with } u \in \mathcal{B}_\infty^*. \tag{4.10}$$

We first claim that

$$\Psi_2(u) \ge \limsup_{n\to\infty} \Psi_2(u_n). \tag{4.11}$$

By [19, Theorem 1.6], we have two possibilities. Either

i) $u = 0$, or
ii) $u \neq 0$, and $\int_{B_1} e^{(4\pi+t)u_n^2}\,dx$ is bounded for some $t > 0$ and thus

$$e^{4\pi u_n^2} \to e^{4\pi u^2} \quad \text{in} \quad L^1(B_1). \tag{4.12}$$

We first assume that i) holds. As in the proof of Proposition 3.4, we then deduce that

$$u_n(r) \to 0 \qquad \text{as } n \to \infty \text{ for all } r > 0. \tag{4.13}$$

Moreover, there exists a constant $c > 0$ with

$$F(s) \le \widetilde{F}(s) := c(1+s^2)^{\frac{\beta}{2}} e^{4\pi s^2} \qquad \text{for } s \in \mathbb{R}. \tag{4.14}$$

We may assume that $-8\pi < \beta < 1$ from now on. Then

$$\widetilde{F}'(s) = ce^{4\pi s^2}(1+s^2)^{\frac{\beta}{2}-1}\big(8\pi s(1+s^2) + \beta s\big) > 0 \qquad \text{for } s > 0$$

and therefore $\widetilde{F}$ satisfies assumption (A). Let

$$U_n := \Big(1 + \frac{|u_n|_2^2}{\pi}\Big)^{\frac{1}{2}}\Big(u_n - u_n(1)\Big)^+ \ \in\ H_0^1(B_1).$$

As in the proof of Lemma 4.1, we see that, for all $n \in \mathbb{N}$,

$$\begin{aligned} &u_n(1) \le \frac{|u_n|_2}{\sqrt{\pi}} \le \frac{\|u_n\|}{\sqrt{\pi}} \le \frac{1}{\sqrt{\pi}} \\ &u_n^2 \le U_n^2 + 1 + \frac{1}{\pi}, \quad \text{and} \\ &|\nabla U_n|_2 \le 1, \quad \text{i.e.,} \quad U_n \in \mathcal{B}_1^*. \end{aligned}$$

Setting $v_n := F(u_n)$, we thus deduce that

$$v_n = F(\sqrt{u_n^2}) \le F(\sqrt{U_n^2 + 1 + \frac{1}{\pi}}) \le \tilde{F}(\sqrt{U_n^2 + 1 + \frac{1}{\pi}}) \le ce^{4\pi(1+\frac{1}{\pi})}\widetilde{F}(U_n) \qquad \text{for } n \in \mathbb{N}.$$

For given $\varepsilon > 0$, we also have, by (4.13 and since F is increasing on $[0,\infty)$,

$$v_n = F(u_n) = F\Big(\big(1 + \frac{|u_n|_2^2}{\pi}\big)^{-\frac{1}{2}}U_n + u_n(1)\Big) \le F(U_n + \varepsilon) \quad \text{in } B_1 \text{ for } n \text{ sufficiently large}$$

by (4.13). We now fix $\varepsilon > 0$ and define $F_\varepsilon : \mathbb{R} \to \mathbb{R}$ by $F_\varepsilon(t) = \min\{F(|t|+\varepsilon), ce^{4\pi(1+\frac{1}{\pi})}\widetilde{F}(t)\}$. Then F_ε has at most β-critical growth and satisfies (A). Moreover, we have $v_n \le F_\varepsilon(U_n)$ and therefore

$$\begin{aligned} \Psi_2(u_n) \le \int_0^1 r v_n(r) \ln\frac{1}{r} \int_0^r \rho v_n(\rho) d\rho dr &\le \int_0^1 r F_\varepsilon(U_n(r)) \ln\frac{1}{r}\int_0^r \rho F_\varepsilon(U_n(\rho)) d\rho dr \\ &= b_0(1_{B_1}F_\varepsilon(U_n), 1_{B_1}F_\varepsilon(U_n)) \end{aligned}$$

for n sufficiently large. We now define

$$\Phi_\varepsilon : \mathcal{B}_1 \to \mathbb{R}, \qquad \Phi_\varepsilon(U) = b_0(1_{B_1}F_\varepsilon(U), 1_{B_1}F_\varepsilon(U))$$

Then Proposition 3.4 applies to Φ_ε and yields that

$$\Psi_2(u_n) \le \Phi_\varepsilon(U_n) \to \Phi_\varepsilon(0) \qquad \text{as } n \to \infty,$$

since $U_n \rightharpoonup 0$ in $H_0^1(B_1)$. Moreover, since $F(0) = 0$ by assumption, for $\varepsilon > 0$ sufficiently small we have

$$\Phi_\varepsilon(0) = b_0(1_{B_1}F_\varepsilon(0), 1_{B_1}F_\varepsilon(0)) = b_0(1_{B_1}F(\varepsilon), 1_{B_1}F(\varepsilon))$$

whereas

$$b_0(1_{B_1}F(\varepsilon), 1_{B_1}F(\varepsilon)) \to b_0(1_{B_1}F(0), 1_{B_1}F(0)) = b_0(0,0) = 0 \qquad \text{as } \varepsilon \to 0^+.$$

It thus follows that

$$\limsup_{n\to\infty} \Psi_2(u_n) \le 0 = \Psi_2(0) = \Psi_2(u)$$

and therefore (4.11) holds in this case.

We now assume that alternative (ii) holds. With $v_n = F(u_n)$ and $v = F(u)$, we then deduce, as in the proof of Proposition 3.5, that v_n is bounded in $L^{s_0}(\mathbb{R}^2)$ with $s_0 = 1 + \frac{t}{4\pi} > 1$, and that

$$v_n \to v \qquad \text{in } L^s(\mathbb{R}^2) \qquad \text{for } 1 \le s < s_0.$$

Moreover, since

$$\Psi_2(u_n) = \int_0^1 r v_n(r) \ln\frac{1}{r} \int_0^r \rho v_n(r) d\rho dr$$

and therefore

$$\begin{aligned}\Psi_2(u_n) - \Psi_2(u) &= \int_0^1 r v_n(r) \ln\frac{1}{r} \int_0^r \rho v_n(r) d\rho dr - \int_0^1 r v(r) \ln\frac{1}{r} \int_0^r \rho v(\rho) d\rho dr \\ &= \int_0^1 r v_n(r) \ln\frac{1}{r} \int_0^r \rho[v_n(\rho) - v(\rho)] d\rho dr + \int_0^1 r[v_n(r) - v(r)] \ln\frac{1}{r} \int_0^r \rho v(r) d\rho dr\end{aligned}$$

we may argue exactly as in the proof of Proposition 3.5 to see that $\Psi_2(u_n) - \Psi_2(u) \to 0$ as $n \to \infty$. Hence (4.11) also holds in this case.

Finally, by Fatou's lemma, we also have $\Psi_1(u) \le \liminf_{n\to\infty} \Psi_1(u_n)$. Combining this with (4.11), we conclude that $\Psi(u) \ge \limsup_{n\to\infty} \Psi(u_n)$. Hence u is a maximizer of Ψ in $\mathcal{B}_\infty$. ☐

Proposition 4.4. *Suppose that F satisfies (A) and has at least β-critical growth for some $\beta > -1$. Then there exists a sequence of functions $u_n \in \mathcal{B}_\infty \cap L^\infty(\mathbb{R}^2)$ with $\Psi(u_n) \to \infty$ as $n \to \infty$.*

Proof. It suffices to take the sequence $(u_n)_n$ considered in the proof of Proposition 3.3, since $\Psi(u_n) = \Phi(u_n)$ for every $n \in \mathbb{N}$ and $\|u_n\| \le 1$ for n sufficiently large. ☐

The following lemma completes the proof of Theorem 1.4.

Lemma 4.5. *Let F satisfy (A_1), and let $u \in \mathcal{B}_\infty$ be a maximizer for $\Psi\big|_{\mathcal{B}_\infty}$. Then $u = u^*$ up to a change of sign and translation.*

Proof. Let $u \in \mathcal{B}_\infty$ be a general maximizer of $\Psi\big|_{\mathcal{B}_\infty}$. By Lemma 2.3(iii), it then follows that $u^* \in \mathcal{B}_\infty^*$ is also a maximizer of $\Psi\big|_{\mathcal{B}_\infty}$. Moreover, since $F(u) \in L^2(\mathbb{R}^2)$ by Lemma 2.5 and $b_\pm(F(u), F(u)) < \infty$, it also follows from Lemma 2.3(iii) that $F(u)$ equals $F(u)^*$ up to translation. Since F is even and strictly increasing on $[0,\infty)$ by assumption (A), this implies that u equals u^* up to sign and translation. ☐

5. The Euler-Lagrange equation

In this section we discuss whether or not a maximizer u of Φ resp. Ψ on $\mathcal{B}_1$, $\mathcal{B}_\infty$, respectively satisfies corresponding Euler-Lagrange equations at least in a weak sense. For this it will be convenient to extend the definition of the quadratic forms $b_\pm, b_0$ to possibly sign changing functions. Let $v, w \in \mathcal{M}(\mathbb{R}^2)$.

If $b_+(|v|,|w|) < \infty$, then $b_+(v,w)$ is well-defined as the double integral in (2.1) in Lebesgue sense. By Young's inequality and since $\ln^+\frac{1}{|\cdot|} \in L^s$ for $1 \le s < \infty$, this is true, in particular, if $v, w \in L^p(\mathbb{R}^2)$

for some $p \in (1,2]$.
Moreover, if $b_-(|v|,|w|) < \infty$, then $b_-(v,w)$ is well-defined as the double integral in (2.2) in Lebesgue sense. By (2.5), this is true in particular if $v,w \in L^1_{ln}(\mathbb{R}^2)$.
If both $b_\pm(v,w)$ are well-defined, then also $b_0(v,w) = b_+(v,w) - b_-(v,w)$ is well defined.

Theorem 5.1. *Suppose that $F \in C^1(\mathbb{R})$ satisfies (A), and suppose that $f := F'$ satisfies*

$$f(t) \le ce^{\alpha t^2} \qquad \text{for } t \in \mathbb{R} \text{ with constants } c, \alpha > 0. \tag{5.1}$$

Suppose furthermore that $u \in \mathcal{B}_1$ is a maximizer of $\Phi\big|_{\mathcal{B}_1}$. Then there exists $\theta > 0$ such that u satisfies the Euler-Lagrange equation in weak sense, i.e.,

$$\begin{aligned}\int_{B_1} \nabla u \nabla \varphi \, dx &= \theta \, b_0(1_{B_1}F(u), 1_{B_1}f(u)\varphi) \\ &= \theta \int_{B_1} (\ln \frac{1}{|\cdot|} * \big(1_{B_1}F(u)\big) f(u)\varphi \, dx \qquad \text{for all } \varphi \in H^1_0(B_1).\end{aligned} \tag{5.2}$$

Here we note that the term on the RHS of (5.2) is well-defined since $1_{B_1}F(u), 1_{B_1}f(u)\varphi \in L^1_{ln}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ by Lemma 2.5 and since these functions vanish on $\mathbb{R}^N \setminus B_1$.

Proof. We first consider $\varphi \in C^1_0(B_1)$ satisfying

$$\int_{B_1} \nabla u \nabla \varphi \, dx < 0. \tag{5.3}$$

We begin to prove that there exists

$$\lim_{t \to 0^+} \frac{\Phi(u+t\varphi) - \Phi(u)}{t} = 2b_0(1_{B_1}F(u), 1_{B_1}f(u)\varphi) = 2\int_{B_1} \big[\ln \frac{1}{|\cdot|} * 1_{B_1}F(u)\big] f(u)\varphi \, dx. \tag{5.4}$$

By (5.3) there exists $t_0 > 0$ with

$$u + t\varphi \in \mathcal{B}_1 \qquad \text{for } 0 \le t \le t_0. \tag{5.5}$$

Let $v_t := 1_{B_1}F(u+t\varphi)$ for $0 \le t \le t_0$. Since $u + t\varphi \in H^1_0(B_1)$, Lemma 2.5 implies that $v_t \in L^s(\mathbb{R}^2)$ for any $s \in [1,\infty)$. We therefore see that, for $t \in (0,t_0)$,

$$\begin{aligned}\Phi(u+t\varphi) - \Phi(u) &= b_0(v_t) - b_0(v_0) = b_0(v_t - v_0, v_t) + b_0(v_0, v_t - v_0) \\ &= t\Big(b_0(1_{B_1}f(u+s_t\varphi)\varphi, v_t) + b_0(v_0, 1_{B_1}f(u+s_t\varphi)\varphi)\Big)\end{aligned} \tag{5.6}$$

where $s_t = s_t(x)$ belongs to the interval $(0,t)$. Here we note that, for fixed $s \in [0,\infty)$, the functions v_t and $w_t := 1_{B_1}f(u+s_t\varphi)$ are uniformly bounded in $L^s(\mathbb{R}^2)$ for $0 \le t \le t_0$. To see this, we remark that $\psi := |u| + t_0|\varphi| \in H^1_0(B_1)$ and that

$$\max\{|u+s_t\varphi|, |u+t\varphi|\} \le \psi \qquad \text{in } B_1.$$

Consequently,

$$\max\big\{e^{\alpha(u+s_t\varphi)^2}, e^{\alpha(u+t\varphi)^2}\big\} \le e^{\alpha\psi^2} \qquad \text{for } 0 \le t \le t_0,$$

and $e^{\alpha\psi^2} \in L^s(B_1)$ for every $s \in [0,\infty)$ by Lemma 2.5. The claim thus follows from (1.4) and (5.1). Moreover, since $u \ne 0$ by Remark 2.9 and

$$u + t\varphi \to u \quad \text{and} \quad u + s_t\varphi \to u \in H^1_0(B_1) \qquad \text{as } t \to 0^+,$$

[19, Theorem 1.6] implies that

$$v_t \to v_0 \quad \text{and} \quad w_t \to 1_{B_1}f(u) \qquad \text{in } L^1(\mathbb{R}^2) \tag{5.7}$$

Together with the uniform boundedness property we just proved, this implies, by interpolation, that

$$v_t \to v_0 \quad \text{and} \quad w_t \to 1_{B_1}f(u) \qquad \text{in } L^s(\mathbb{R}^2) \text{ for } 1 \le s < \infty. \tag{5.8}$$

Since also $\ln^+ \frac{1}{|\cdot|} \in L^s(\mathbb{R}^2)$ for $1 \le s < \infty$, Young's inequality implies that

$$b_+(w_t\varphi, v_t) \to b_+(1_{B_1} f(u)\varphi, v_0) \quad \text{and} \quad b_+(v_0, w_t\varphi) \to b_+(v_0, 1_{B_1} f(u)\varphi) \tag{5.9}$$

as $t \to 0^+$. Moreover, using (2.6) and (5.7), we easily see that

$$b_-(w_t\varphi, v_t) \to b_-(1_{B_1} f(u)\varphi, v_0) \quad \text{and} \quad b_-(v_0, w_t\varphi) \to b_-(v_0, 1_{B_1} f(u)\varphi) \tag{5.10}$$

Combining (5.6), (5.9) and (5.10), we obtain (5.4). Since u is a maximizer of Φ on $\mathcal{B}_1$, it follows from (5.4) and (5.5) that

$$b_0(1_{B_1}F(u), 1_{B_1} f(u)\varphi) \le 0 \qquad \text{for all } \varphi \in C_0^1(B_1) \text{ satisfying (5.3).} \tag{5.11}$$

Next we let $\varphi \in C_0^1(B_1)$ satisfy

$$\int_{B_1} \nabla u \nabla \varphi \, dx = 0. \tag{5.12}$$

Approximating φ and $-\varphi$ with functions satisfying (5.3), we then deduce from (5.11) that

$$0 = b_0(1_{B_1}F(u), 1_{B_1} f(u)\varphi) = \int_{B_1} (\ln \frac{1}{|\cdot|} * \big(1_{B_1}F(u)\big) f(u)\varphi \, dx \tag{5.13}$$

for all $\varphi \in C_0^1(B_1)$ satisfying (5.12). Since $f(u) \in L^s(B_1)$ for $s \in [1,\infty)$, it follows from Lemma 2.5(i) that

$$(\ln \frac{1}{|\cdot|} * \big(1_{B_1}F(u)\big) f(u) \qquad \in L^s(B_1) \qquad \text{for } s \in [1,\infty).$$

Consequently, the RHS of (5.13) defines a continuous linear form on $H_0^1(B_1)$. By density, it thus follows that (5.13) holds for all $\varphi \in H_0^1(B_1)$ satisfying (5.12). Hence there exists $\Theta \in \mathbb{R}$ with the property that

$$\Theta \int_{B_1} \nabla u \nabla \varphi \, dx = \int_{B_1} (\ln \frac{1}{|\cdot|} * \big(1_{B_1}F(u)\big) f(u)\varphi \, dx \qquad \text{for all } \varphi \in H_0^1(B_1). \tag{5.14}$$

Next we claim that $\Theta > 0$. For this we recall that $u \in \mathcal{B}_1^*$ by Lemma 3.6. In particular, u is a radial function. Applying (5.14) to $\varphi = u$ and using Corollary 2.8, we deduce that

$$\Theta |\nabla u|_2^2 = \int_{B_1} (\ln \frac{1}{|\cdot|} * \big(1_{B_1}F(u)\big) f(u) u \, dx = 2(2\pi)^2 \int_0^1 g_u f(u) u \, dr \tag{5.15}$$

with the function

$$r \mapsto g_u(r) = \ln \frac{1}{r} \int_0^r \rho F(u(\rho)) d\rho + \int_r^1 \rho \big(\ln \frac{1}{\rho}\big) F(u(\rho)) d\rho.$$

By Lemma 3.6, we have $u > 0$ on $(0,1)$ and therefore also $g_u > 0$ on $(0,1)$. Moreover, we have $f > 0$ a.e. on $(0,\infty)$ by assumption (A), which implies in particular that $f(u) \ge 0$. Consequently, (5.15) implies that $\Theta \ge 0$. Moreover, if we suppose by contradiction that $\Theta = 0$, then we have

$$\int_0^1 g_u f(u) u \, dr = 0$$

and therefore $f(u) \equiv 0$. Since u is continuous on $(0,1]$ with $u(1) = 0$ and $f > 0$ a.e. on $(0,\infty)$, it then follows that $u \equiv 0$, a contradiction. We thus obtain $\Theta > 0$, as claimed. We conclude that (5.2) holds with $\theta = \frac{1}{\Theta} > 0$. The proof is finished. □

Remark 5.2. *Since, under the assumptions of Theorem 5.1, the function* $\ln \frac{1}{|\cdot|} * \big(1_{B_1}F(u)\big) f(u)$ *is bounded in* B_1 *by Lemma 2.5, standard elliptic regularity shows that solutions of (5.2) are contained in* $W^{2,p}(B_1) \cap C^{1,\sigma}(B_1)$ *for* $p < \infty$; $\sigma \in (0,1)$.

Next we derive the following result on the entire space.

Theorem 5.3. *Suppose that $F \in C^1(\mathbb{R})$ satisfies (A) and $F(0) = 0$. Moreover, suppose that $f := F'$ satisfies (5.1). If $u \in \mathcal{B}_\infty$ is a maximizer of $\Phi\big|_{\mathcal{B}_\infty}$, then there exists $\theta > 0$ such that u satisfies the Euler-Lagrange equation in weak sense, i.e.,*

$$\int_{\mathbb{R}^2} \big(\nabla u \nabla \varphi + u\varphi\big)\, dx = \theta b_0(F(u), f(u)\varphi) = \theta \int_{\mathbb{R}^2} \big(\ln \frac{1}{|\cdot|} * F(u)\big) f(u)\varphi \, dx \tag{5.16}$$

for all $\varphi \in H^1(\mathbb{R}^2)$ with bounded support.

Here we note that $u = u^*$ up to translation by Lemma 4.5, so $F(u) \in L^2(\mathbb{R}^2)$ and $f(u)\varphi \in L^2(\mathbb{R}^2)$ by Lemma 2.5. Since φ has compact support, we also see that $f(u)\varphi \in L^1_{ln}(\mathbb{R}^2)$. Moreover, $F(u) \in L^1_{ln}(\mathbb{R}^2)$ by Lemma 2.2 since $\Psi_-(u) = b_-(F(u), F(u)) < \infty$. Hence the term on the RHS of (5.16) is well-defined.

Proof. First we recall that $u = u^*$ up to sign and translation, so we may assume that $u \in \mathcal{B}^*_\infty$ in the following. Let first $\varphi \in C^1_0(\mathbb{R}^2)$, φ radial, and assume that

$$\int_{B_1} \big(\nabla u \nabla \varphi + u\varphi\big)\, dx < 0. \tag{5.17}$$

Then there exists $t_0 > 0$ with

$$u + t\varphi \in \mathcal{B}_\infty \qquad \text{for } 0 \le t \le t_0. \tag{5.18}$$

We shall prove first that there exists

$$\lim_{t \to 0^+} \frac{\Psi(u + t\varphi) - \Psi(u)}{t} = 2b_0(F(u), f(u)\varphi) = 2\int_{\mathbb{R}^2} \big(\ln \frac{1}{|\cdot|}\big) * F(u) f(u)\varphi \, dx \tag{5.19}$$

For this, we first note that $v_t := F(u + t\varphi) \in L^s_{loc}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ for $1 \le s < \infty$ by Lemma 2.5. Moreover, $v_t \in L^1_{ln}(\mathbb{R}^2)$ since $F(u) \in L^1_{ln}(\mathbb{R}^2)$ and φ has bounded support. We therefore see that, for $t \in (0, t_0)$,

$$\begin{aligned} \Phi(u + t\varphi) - \Phi(u) &= b_0(v_t) - b_0(v_0) = 2b_0(v_t - v_0, v_0) + b_0(v_t - v_0) \\ &= t\Big(b_0\big(f(u + s_t\varphi)\varphi, v_0\big) + t b_0\big(f(u + s_t\varphi)\varphi\big)\Big) \end{aligned} \tag{5.20}$$

where $s_t = s_t(x)$ belongs to the interval $(0, t)$. Here we note that, for fixed $s \in [0, \infty)$, the functions v_t and $w_t := 1_{B_1} f(u + s_t\varphi)$ are uniformly bounded in $L^s_{loc}(\mathbb{R}^2)$ for $0 \le t \le t_0$. To see this, we remark that $\psi := |u| + t_0|\varphi| \in H^1(\mathbb{R}^2)$ and that

$$\max\{|u + s_t\varphi|, |u + t\varphi|\} \le \psi \qquad \text{in } \mathbb{R}^2.$$

Consequently,

$$\max\big\{e^{\alpha(u+s_t\varphi)^2}, e^{\alpha(u+t\varphi)^2}\big\} \le e^{\alpha\psi^2} \qquad \text{for } 0 \le t \le t_0,$$

and $e^{\alpha\psi^2} \in L^s_{loc}(B_1)$ for every $s \in [0, \infty)$ by Lemma 2.5. The claim thus follows from (1.4) and (5.1). Moreover, since $u \ne 0$ by Remark 2.9 and

$$u + t\varphi \to u \quad \text{and} \quad u + s_t\varphi \to u \in H^1(\mathbb{R}^2) \qquad \text{as } t \to 0^+,$$

[19, Theorem 1.6] implies that

$$v_t \to v_0 \quad \text{and} \quad w_t, \tilde{w}_t \to f(u) \qquad \text{in } L^1_{loc}(\mathbb{R}^2). \tag{5.21}$$

Together with the uniform boundedness property we just proved, this implies, by interpolation, that

$$w_t \to f(u) \qquad \text{in } L^s_{loc}(\mathbb{R}^2) \text{ for } 1 \le s < \infty. \tag{5.22}$$

Consequently, since φ has compact suppport,

$$w_t\varphi \to f(u)\varphi \qquad \text{in } L^s(\mathbb{R}^2) \text{ for } 1 \le s < \infty \text{ and in } L^1_{ln}(\mathbb{R}^2). \tag{5.23}$$

Since, as noted earlier, $v_0 \in L^2(\mathbb{R}^2)$ and $\ln^+ \frac{1}{|\cdot|} \in L^s(\mathbb{R}^2)$ for $1 \le s < \infty$, Young's inequality implies that

$$b_+(w_t\varphi, v_0) \to b_+(f(u)\varphi, v_0) \quad \text{and} \quad b_+(w_t\varphi) \to b_+(f(u)\varphi) \tag{5.24}$$

as $t \to 0^+$. Moreover, since also $v_0 \in L^1_{ln}(\mathbb{R}^2)$, we deduce from (2.6) and (5.23) that

$$b_-(w_t\varphi, v_0) \to b_-(f(u)\varphi, v_0) \quad \text{and} \quad b_-(w_t\varphi) \to b_-(f(u)\varphi). \tag{5.25}$$

Combining (5.20), (5.24) and (5.25), we infer (5.19). Now since u is a maximizer of Φ on $\mathcal{B}_\infty$, it follows that the RHS in (5.19) is less than or equal to zero for all $\varphi \in C^1_0(\mathbb{R}^2)$ satisfying (5.17). Precisely as in the proof of Theorem 5.1, we then deduce that

$$b_0(F(u), f(u)\varphi) = 0 \tag{5.26}$$

for all functions $\varphi \in C^1_0(\mathbb{R}^2)$ satisfying

$$\int_{\mathbb{R}^2} \big(\nabla u \nabla \varphi + u\varphi\big)\, dx = 0. \tag{5.27}$$

Moreover, by (1.4), (5.1) and Lemma 2.5, we have

$$\big(\ln \frac{1}{|\cdot|} * F(u)\big) f(u) \qquad \in L^s_{loc}(\mathbb{R}^2) \qquad \text{for } s \in [1,\infty).$$

Consequently, for arbitrary fixed $R > 0$, the RHS of (5.19) defines a continuous linear form on $H^1_0(B_R)$. By density, it thus follows that (5.27) holds for all $\varphi \in H^1_0(B_R)$ satisfying (5.12). Hence there exists $\Theta \in \mathbb{R}$ with the property that

$$\Theta \int_{\mathbb{R}^2} \big(\nabla u \nabla \varphi + u\varphi\big)\, dx = \int_{\mathbb{R}^2} \big(\ln \frac{1}{|\cdot|} * F(u)\big) f(u)\varphi\, dx \qquad \text{for all } \varphi \in H^1_0(B_R). \tag{5.28}$$

Since $H^1_0(B_R) \subset H^1_0(B_{R'})$ if $R < R'$, it follows that Θ does not depend on R and therefore (5.28) holds for all $\varphi \in H^1(\mathbb{R}^2)$ with bounded support.
Next we claim that $\Theta \neq 0$, and we suppose by contradiction that $\Theta = 0$. Then we have

$$(\ln \frac{1}{|\cdot|} * \big(1_{B_1} F(u)\big) f(u) = 0 \qquad \text{a.e. in } \mathbb{R}^2.$$

By Lemma 2.7 and since u is a radial function, this implies that

$$g_u(r) f(u(r)) = 0 \quad \text{for a.e. } r \in (0,\infty) \tag{5.29}$$

with

$$g_u(r) = \ln \frac{1}{r} \int_0^r \rho F(u(\rho))d\rho + \int_r^\infty \rho\big(\ln \frac{1}{\rho}\big) F(u(\rho))d\rho$$

Since $u \not\equiv 0$ by Remark 2.9, we have $F(u) \not\equiv 0$ and therefore $g_u(r) < 0$ for $r > 1$. Hence (5.29) implies that $f(u) \equiv 0$ on $(1,\infty)$. Noting again that $f(s) \neq 0$ a.e. on $\mathbb{R}$ by assumption (A), we deduce again that u is constant on $(1,\infty)$. Since $u \in H^1(\mathbb{R}^2)$, this implies that $u \equiv 0$ on $(1,\infty)$ and therefore $F(u) \equiv F(0) = 0$ on $(1,\infty)$. As a consequence, $F(u) \not\equiv 0$ on $(0,1)$ and therefore

$$g_u(r) = \ln \frac{1}{r} \int_0^r \rho F(u(\rho))d\rho + \int_r^1 \rho\big(\ln \frac{1}{\rho}\big) F(u(\rho))d\rho > 0 \qquad \text{for } r \in (0,1).$$

Then (5.29) implies that $f(u) \equiv 0$ on $(0,1)$. As above, we then deduce that u is constant on $(0,1)$. Since we already know that u is continuous on $(0,\infty)$ and satisfies $u \equiv 0$ on $[1,\infty)$, we conclude that $u \equiv 0$ on $(0,\infty)$, contradicting Remark 2.9. Hence $\Theta \neq 0$.
Next we show that $\Theta > 0$. For this, we first note that the radial function

$$r \mapsto h_u(r) = \Big(\ln \frac{1}{|\cdot|} * F(u)\Big)(r) = -2\pi\Big(\ln r \int_0^r \rho F(u(\rho))d\rho + \int_r^\infty \rho \ln \rho F(u(\rho))d\rho\Big)$$

is decreasing in r. Indeed we have

$$\begin{aligned} h_u'(r) &= -2\pi\Big(\frac{1}{r}\int_0^r F(u(\rho))\rho d\rho + r\ln r F(u(r)) - r\ln r F(u(r))\Big) \\ &= -\frac{2\pi}{r}\int_0^r F(u(\rho))\rho d\rho \le 0 \qquad \text{for } r \in (0,1). \end{aligned}$$

If $h_u(r) \le 0$ for all $r \in (0,1)$, then

$$\Phi(u) = \int_{\mathbb{R}^2} F(u) h_u dx \le 0 \le \Phi(0)$$

which is wrong for maximizers by Remark 2.9. Hence there exists $r_0 \in (0,1)$ with $h_u(r) > 0$ for $r \in (0, r_0)$. Making r_0 smaller if necessary, we may also assume that $u(r_0) > 0$. Applying (5.28) to the function $\varphi = [u - u(r_0)]^+ \in H_0^1(B_1)$, we see that

$$\Theta \int_{B_{r_0}} \big(|\nabla u|^2 + u[u - u(r_0)]\big)\, dx = \int_{B_{r_0}} f(u) h_u [u - u(r_0)]\, dx \tag{5.30}$$

We distinguish two cases.

Case 1: $u \equiv u_{r_0}$ on B_{r_0}. In this case it follows from (5.28) that

$$\Theta u(r_0) \int_{B_{r_0}} \varphi\, dx = f(u(r_0)) \int_{B_{r_0}} h_u \varphi\, dx \qquad \text{for all } \varphi \in H_0^1(B_{r_0}).$$

and therefore

$$\Theta u(r_0) = f(u(r_0)) h_u \qquad \text{a.e. on } B_{r_0}. \tag{5.31}$$

Since $\Theta \ne 0$, $u(r_0) > 0$, $f(u(r_0)) \ge 0$ and $h_u > 0$ on B_{r_0}, we deduce that $\Theta > 0$ in this case.

Case 2: $u \not\equiv u_{r_0}$ on B_{r_0}. In this case it follows from (5.30) that

$$\Theta = \Big(\int_{B_{r_0}} \big(|\nabla u|^2 + u[u - u(r_0)]\big)\, dx\Big)^{-1} \int_{B_{r_0}} f(u) h_u [u - u(r_0)]\, dx \ge 0.$$

Since we have already proved that $\Theta \ne 0$, we conclude again that $\Theta > 0$.

So in either case we have $\Theta > 0$, and then (5.16) holds with $\theta = \frac{1}{\Theta} > 0$. The proof is finished. □

Remark 5.4. *Since, under the assumptions of Theorem 5.3, the function* $\ln \frac{1}{|\cdot|} * \big(F(u)\big) f(u)$ *is contained in* $L^\infty_{loc}(\mathbb{R}^2)$ *by (2.4) and by Lemma 2.5, standard elliptic regularity shows that solutions of (5.2) are contained in* $W^{2,p}_{loc}(\mathbb{R}^2) \cap C^{1,\sigma}_{loc}(\mathbb{R}^2)$ *for* $p < \infty$; $\sigma \in (0,1)$.

6. SYMMETRY OF THE POSITIVE SOLUTIONS

Using the strict form of the Riesz rearrangement inequality, we have shown in Lemma 4.5 that, if F satisfies assumption (A_1), then all maximizers for Ψ on $\mathcal{B}_\infty$ are radial up to sign and translation. In this section we prove Theorem 1.6, thereby giving a partial answer to the question whether positive solutions of the corresponding Euler-Lagrange equation

$$-\Delta u + u = \theta\Big(\ln \frac{1}{|\cdot|} * F(u)\Big) F'(u) \qquad \text{in } \mathbb{R}^2, \tag{6.1}$$

are radial. Here $\theta > 0$ is a fixed Lagrangian multiplier. We recall that solutions of (6.1) correspond to of the nonlinear system

$$\begin{cases} -\Delta u + u = \theta\, w f(u) & \text{in } \mathbb{R}^2, \\ -\Delta w = 2\pi F(u) & \text{in } \mathbb{R}^2, \end{cases} \tag{6.2}$$

subject to the conditions

$$u \in L^\infty(\mathbb{R}^2) \quad \text{and} \quad w(x) \to -\infty \quad \text{as } |x| \to \infty. \tag{6.3}$$

Next we prove Theorem 1.6. For matters of convenience, we recall the statement.

Theorem 6.1. *Suppose that* $F \in C^1(\mathbb{R})$ *with* $f = F'$, *suppose that* $F(0) = f(0) = 0$, *and suppose that* f *is a nondecreasing Lipschitz function on* $[0, \infty)$. *Then every classical solution* (u, w) *of (6.2), (6.3) with* $u > 0$ *in* $\mathbb{R}^2$ *is radially symmetric up to translation and strictly decreasing in the distance from the symmetry center.*

The remainder of this section is devoted to the proof of this theorem. So from now on we suppose that F and $f := F'$ satisfy the assumptions above, and we let (u, w) be a classical solution of (6.2), (6.3) with $u > 0$ in $\mathbb{R}^2$.
The proof will be given by a variant of the moving plane method. We start with a preliminary lemma.

Lemma 6.2. *We have* $u(x) = O(e^{-|x|})$ *as* $|x| \to \infty$.

Proof. We first claim that

$$u(x) \to 0 \qquad \text{as } |x| \to \infty. \tag{6.4}$$

Arguing by contradiction, we suppose that $\kappa_\infty := \limsup\limits_{|x|\to\infty} u(x) > 0$, and we let $(x_n)_n$ be a sequence in $\mathbb{R}^2$ with $|x_n| \to \infty$ and $u(x_n) \to \kappa_\infty$ as $n \to \infty$. Then there exists $n_0 \in \mathbb{N}$ with

$$\kappa_n := \sup_{x \in \partial B_1(x_n)} u_n < e^{\frac{1}{5}} \kappa_\infty < e^{\frac{1}{4}} u(x_n) \qquad \text{for } n \geq n_0. \tag{6.5}$$

Moreover, by making n_0 larger if necessary, we may assume by (6.3) that $-\Delta u + u = \theta w f(u) \leq 0$ in $B_1(x_n)$ for $n \geq n_0$. Moreover, it is easy to see that the function $x \mapsto \tau(x) = e^{\frac{|x|^2}{4}}$ satisfies $-\Delta\tau + \tau \geq 0$ in B_1. We now consider the functions

$$\tau_n : B_1(x_n) \to \mathbb{R}, \qquad \tau_n(x) = \kappa_n e^{-\frac{1}{4}} \tau(x - x_n).$$

By construction, we then have

$$-\Delta(\tau_n - u) + (\tau_n - u) \geq 0 \qquad \text{in } B_1(x_n) \text{ for } n \geq n_0.$$

Consequently, the function $\tau_n - u$ cannot attain a negative minimum in $B_1(x_n)$ for $n \geq n_0$. Since, be definition of κ_n, we also have $\tau_n - u \geq 0$ on $\partial B_1(x_n)$, it follows that $\tau_n - u \geq 0$ in $B_1(x_n)$ and therefore $u(x_n) \leq \tau_n(x_n) = \kappa_n e^{-\frac{1}{4}}$ for $n \geq n_0$. This contradicts (6.5), and hence (6.4) holds.
Next, we fix $R > 0$ sufficiently large so that, by (6.3) we have $-\Delta u + u = \theta w f(u) \leq 0$ in $\mathbb{R}^2 \setminus \overline{B_R}$. Moreover, we note that the function

$$x \mapsto \sigma(x) = \|u\|_{L^\infty(\mathbb{R}^2)}\, e^{R-|x|}$$

satisfies $-\Delta\sigma + \sigma \geq 0$ in $\mathbb{R}^2 \setminus \overline{B_R}$. Consequently, the function $\tilde{u} := \sigma - u$ satisfies $-\Delta\tilde{u} + \tilde{u} \geq 0$ in $\mathbb{R}^2 \setminus \overline{B_R}$ and can therefore not attain a negative minimum in this set. Since, moreover, $\tilde{u} \geq 0$ on ∂B_R and $\tilde{u}(x) \to 0$ as $|x| \to \infty$ by (6.4), we conclude that $u \leq \sigma$ in $\mathbb{R}^2 \setminus \overline{B_R}$. We conclude that $u(x) = O(e^{-|x|})$ as $|x| \to \infty$, as claimed. ☐

We note that it follows from Lemma 6.2 that also $F(u(x)) = O(e^{-|x|})$ as $|x| \to \infty$. Moreover, since every semibounded harmonic function $\mathbb{R}^2 \to \mathbb{R}$ is constant, we have

$$w(x) = c - \int_{\mathbb{R}^2} \ln|x - y| F(u(y))\, dy \qquad \text{for } x \in \mathbb{R}^2 \text{ with a constant } c \in \mathbb{R}. \tag{6.6}$$

Next, we fix some notation related to the moving plane method. For $\lambda \in \mathbb{R}$, we put

$$H_\lambda := \{x \in \mathbb{R}^2 \,:\, x_1 > \lambda\}, \qquad T_\lambda = \partial H_\lambda = \{x \in \mathbb{R}^2 \,:\, x_1 = \lambda\}.$$

Moreover, we let $\mathbb{R}^2 \to \mathbb{R}^2$, $x \mapsto x^\lambda$ denote the reflection of x at T_λ. From now on, we consider a fixed solution of (6.2), (6.3) with $u > 0$, and we set

$$u^\lambda(x) = u(x^\lambda), \qquad w^\lambda(x) = w(x^\lambda) \qquad \text{for } x \in \mathbb{R}^2,\ \lambda \in \mathbb{R}.$$

On H_λ we define the difference functions

$$u_\lambda = u^\lambda - u, \qquad\qquad w_\lambda = w^\lambda - w$$

which satisfy the system of equations

$$\begin{cases} -\Delta u_\lambda + u_\lambda = \theta[w_\lambda f(u^\lambda) + h_\lambda(x) w u_\lambda] \\ \qquad -\Delta w_\lambda = 2\pi K_\lambda(x) u_\lambda \end{cases} \quad \text{in } \mathcal{H}_\lambda \tag{6.7}$$

with

$$h_\lambda(x) := \begin{cases} \dfrac{f(u^\lambda(x)) - f(u(x))}{u_\lambda(x)}, & \text{if } u_\lambda(x) \neq 0, \\ 0, & \text{if } u_\lambda(x) = 0 \end{cases}$$

and

$$K_\lambda(x) := \begin{cases} \dfrac{F(u^\lambda(x)) - F(u(x))}{u_\lambda(x)}, & \text{if } u_\lambda(x) \neq 0, \\ 0, & \text{if } u_\lambda(x) = 0. \end{cases}$$

Since f is Lipschitz continuous on $[0, \|u\|_{L^\infty(\mathbb{R}^2)}]$, there exists a constant $C = C(u) > 0$ such that

$$\|h_\lambda\|_{L^\infty(H_\lambda)} \le C \qquad \text{for every } \lambda \in \mathbb{R} \tag{6.8}$$

It follows from (6.6) that

$$w_\lambda(x) = \int_{H_\lambda} \ln \frac{|x - y^\lambda|}{|x-y|} K_\lambda(y) u_\lambda(y)\, dy \qquad \text{for } x \in H_\lambda. \tag{6.9}$$

Since $\ln \frac{|x-y^\lambda|}{|x-y|} > 0$ for every $x, y \in H_\lambda$, and F is a nondecreasing function, we have the implication

$$u_\lambda \ge 0 \text{ in } H_\lambda \qquad \Longrightarrow \qquad w_\lambda \ge 0 \text{ in } H_\lambda \tag{6.10}$$

for every $\lambda \in \mathbb{R}$. In the following, we let $v^- := \min\{v, 0\}$ denote the negative part of a function defined on a subset of $\mathbb{R}^2$. Note that v^- is a nonpositive function with this convention. We need the following estimate:

Lemma 6.3. *There exists a constant $\kappa > 0$ such that*

$$\|w_\lambda^-\|_{L^2(H_\lambda)} \le \kappa\, c_\lambda \|u_\lambda^-\|_{L^2(H_\lambda)} \qquad \textit{for every } \lambda \in \mathbb{R},$$

where

$$c_\lambda = \Big(\int_{M_\lambda} (y_1 - \lambda)^2 [f(\xi_\lambda(y))]^2\, dy\Big)^{\frac{1}{2}} \qquad \textit{and} \qquad M_\lambda := \{x \in H_\lambda \ :\ u_\lambda(x) < 0\}$$

where $\xi_\lambda(x) = u(x) + \gamma_\lambda(x) u^\lambda(x)$ for some $\gamma_\lambda(x) \in]0,1[$.

Proof. We note that

$$0 \le \ln \frac{|x - y^\lambda|}{|x-y|} \le \ln(1 + \frac{|y - y^\lambda|}{|x-y|}) \le \frac{|y-y^\lambda|}{|x-y|} = \frac{2(y_1 - \lambda)}{|x-y|} \qquad \text{for } x, y \in H_\lambda.$$

Since also $u_\lambda(y) < 0$ implies that $0 \le u^\lambda(y) \le u(y)$, we may use the integral representation (6.9) to conclude that

$$w_\lambda^-(x) \ge \int_{M_\lambda} \frac{2(y_1 - \lambda)}{|x-y|} K_\lambda(y) u_\lambda^-(y)\, dy = 2\int_{M_\lambda} \frac{(y_1 - \lambda)}{|x-y|} f(\xi_\lambda(y)) u_\lambda^-(y)\, dy$$

where $\xi_\lambda(x) = u(x) + \gamma_\lambda(x)u^\lambda(x)$ with $\gamma_\lambda(x) \in]0,1[$, for any $x \in H_\lambda$. Then it follows by the Hardy-Littlewood-Sobolev inequality that

$$\|w_\lambda^-\|_{L^2(H_\lambda)} \le \kappa\, c_\lambda \|u_\lambda^-\|_{L^2(H_\lambda)}$$

with a constant $\kappa > 0$ independent of λ, as claimed. $\square$

Lemma 6.4. *There exists* $\bar\lambda > 0$ *such that* $u_\lambda \ge 0$ *in* H_λ *for* $\lambda \ge \bar\lambda$.

Proof. By (6.3) and (6.8), we may choose $\lambda_1 > 0$ such that $w \le 0$ in H_λ for $\lambda \ge \lambda_1$. Multiplying the first equation in (6.7) by u_λ^- and taking into account that f is nondecreasing function, we may estimate with the help of Lemma 6.3

$$\begin{aligned}\|u_\lambda^-\|^2_{L^2(H_\lambda)} \le &\|u_\lambda^-\|^2_{H^1(H_\lambda)} = \int_{H_\lambda} \theta[w_\lambda f(u^\lambda)u_\lambda^- + h_\lambda(x)w(u_\lambda^-)^2]\,dx \\ &\le \int_{H_\lambda} \theta w_\lambda f(u^\lambda)u_\lambda^-\,dx \le \\ &\le \theta\|w_\lambda^-\|_{L^2(H_\lambda)}\|f(u^\lambda)\|_{L^\infty(H_\lambda)}\|u_\lambda^-\|_{L^2(H_\lambda)} \le \kappa\, c_\lambda \|f(u)\|_{L^\infty(\mathbb{R}^2)}\|u_\lambda^-\|^2_{L^2(H_\lambda)}.\end{aligned}$$

Since $c_\lambda \to 0$ as $\lambda \to \infty$ by Lemma 6.2 and $f(0) = 0$, there exists $\bar\lambda \ge \lambda_1$ such that

$$\kappa\, c_\lambda\, \|f(u)\|_{L^\infty(\mathbb{R}^2)} < 1 \qquad \text{for } \lambda \ge \bar\lambda,$$

so that $u_\lambda^- \equiv 0$ on H_λ for $\lambda \ge \bar\lambda$, as claimed. $\square$

Lemma 6.5. *If* $\lambda \in \mathbb{R}$ *is such that* $u_\lambda \ge 0$ *in* H_λ, *then also* $w_\lambda \ge 0$ *on* H_λ. *Moreover, either* $u_\lambda \equiv 0 \equiv w_\lambda$ *or*

$$u_\lambda > 0, \quad w_\lambda > 0 \qquad \text{on } H_\lambda \tag{6.11}$$

and

$$\frac{\partial u}{\partial x_1} < 0, \quad \frac{\partial w}{\partial x_1} < 0 \qquad \text{on } T_\lambda. \tag{6.12}$$

Proof. We already noted in (6.10) that $u_\lambda \ge 0$ in H_λ implies $w_\lambda \ge 0$ in H_λ. Moreover, if $u_\lambda \not\equiv 0$, then w_λ is strictly positive in H_λ and

$$\frac{\partial w_\lambda}{\partial x_1} = -2\frac{\partial w}{\partial x_1} > 0 \qquad \text{on } T_\lambda$$

by the Hopf lemma. Conversely, if $w_\lambda \not\equiv 0$, then also $u_\lambda \not\equiv 0$ by (6.9), and u_λ satisfies

$$-\Delta u_\lambda + (1 - \theta h_\lambda w)u_\lambda = \theta w_\lambda\, f(u^\lambda) \ge 0 \qquad \text{in } H_\lambda.$$

Hence $u_\lambda > 0$ in H_λ by the maximum principle, and

$$\frac{\partial u_\lambda}{\partial x_1} = -2\frac{\partial u}{\partial x_1} > 0 \qquad \text{on } T_\lambda$$

by the Hopf lemma. $\square$

Lemma 6.6. *Let* $\lambda \in \mathbb{R}$. *If* $u_\lambda > 0$ *in* H_λ, *then there exists* $\varepsilon > 0$ *such that* $u_\mu \ge 0$ *in* H_μ *for* $\mu \in (\lambda - \varepsilon, \lambda)$.

Proof. Let $B_R := B_R(0)$ for $R > 0$. By (6.3), Lemma 6.2 and (6.8), we may fix $R > 1$ large enough such that

$$w \le 0 \qquad \text{in } H_\mu \setminus B_R \text{ for every } \mu \in \mathbb{R} \tag{6.13}$$

and

$$\Big(\int_{\mathbb{R}^2\setminus B_R} (y_1 - \mu)^2[f(u(y) + \xi u^\mu(y))]^2\,dy\Big)^{\frac{1}{2}} < \frac{1}{\kappa\|f(u)\|_{L^\infty(\mathbb{R}^2)}} \qquad \text{for every } \xi \in]0,1[,\ \mu \in [\lambda - 1, \lambda], \tag{6.14}$$

where κ is as in Lemma 6.3. Moreover, by (6.11), (6.12) and the continuity of u, $\frac{\partial u}{\partial x_1}$, there exists $0 < \varepsilon < 1$ such that

$$u_\mu > 0 \qquad \text{in } H_\mu \cap B_R \qquad \text{for } \mu \in (\lambda - \varepsilon, \lambda]. \tag{6.15}$$

Arguing similarly as in the proof of Lemma 6.3 with $\mu \in (\lambda - \varepsilon, \lambda]$ in place of λ, we may multiply the first equation in (6.7) – with μ in place of λ – with u_μ^- and integrate, obtaining the estimate

$$\begin{aligned}\|u_\mu^-\|^2_{L^2(H_\mu)} \le & \|u_\mu^-\|^2_{H^1(H_\mu)} = \int_{H_\mu \setminus B_R} \theta[w_\mu f(u^\mu)u_\mu^- + h_\mu(x)w(u_\mu^-)^2]\,dx \\ & \le \int_{H_\mu \setminus B_R} \theta w_\mu f(u^\mu) u_\mu^-\,dx \\ & \le \theta \|w_\mu^-\|_{L^2(H_\mu)} \|f(u^\mu)\|_{L^\infty(H_\mu)} \|u_\mu^-\|_{L^2(H_\lambda)} \le \kappa\, c_\mu \|f(u)\|_{L^\infty(\mathbb{R}^2)} \|u_\mu^-\|^2_{L^2(H_\mu)}. \end{aligned} \tag{6.16}$$

Here we used (6.15) in the second step, (6.13) in the third step and Lemma 6.3 in the last step. We note that (6.14) and (6.15) also imply that

$$c_\mu \le \Big(\int_{\mathbb{R}^2 \setminus B_R} (y_1 - \mu)^2 [f(u + \xi_\mu u^\mu)(y)]^2\,dy\Big)^{\frac{1}{2}} < \frac{1}{\kappa \|f(u)\|_{L^\infty(\mathbb{R}^2)}} \qquad \text{for } \mu \in (\lambda - \varepsilon, \lambda].$$

Consequently, (6.16) implies that $u_\mu^- \equiv 0$ for $\mu \in (\lambda - \varepsilon, \lambda]$. □

Proof of Theorem 6.1(completed). Put

$$\lambda_1 := \inf\{\lambda \in \mathbb{R} \,:\, u_\lambda \ge 0 \text{ in } H_\lambda\}$$

By Lemma 6.4 we have $\lambda_1 < \infty$, whereas the positivity of u and Lemma 6.2 imply that $\lambda_1 > -\infty$. Moreover, as a consequence of Lemmas 6.5 and 6.6, we have that $u_{\lambda_1} \equiv 0$ and $w_{\lambda_1} \equiv 0$. Repeating the same argument with x_1 replaced by the second coordinate direction x_2, we also find $\lambda_2 \in \mathbb{R}$ such that u and w are symmetric with respect to the hyperplane $\{x \in \mathbb{R}^2 \,:\, x_2 = \lambda_2\}$. Now consider $\lambda := (\lambda_1, \lambda_2) \in \mathbb{R}^2$ and the translated functions

$$\tilde{u}, \tilde{w} : \mathbb{R}^2 \to \mathbb{R}, \qquad \tilde{u}(x) = u(x - \lambda),\ \tilde{w}(x) = w(x - \lambda),$$

These functions also solve (6.2), (6.3) and satisfy

$$\tilde{u}(x) = \tilde{u}(-x), \qquad \tilde{w}(x) = \tilde{w}(-x) \qquad \text{for } x \in \mathbb{R}^2.$$

It is then easy to see that every symmetry hyperplane of $\tilde{u}$ and $\tilde{w}$ must contain the origin. Consequently, repeating the moving plane procedure in an arbitrary direction in place of the x_1-coordinate direction, we obtain that $\tilde{u}$ and $\tilde{w}$ are symmetric with respect to any hyperplane containing zero, hence radially symmetric. □

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