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$L^p - L^q$ estimates for a parameter-dependent multiplier with oscillatory and diffusive components

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Abstract

In this paper, we derive long time $L^p - L^q$ decay estimates, in the full range $1 \leq p \leq q \leq \infty$, for time-dependent multipliers in which an interplay between an oscillatory component and a diffusive component with different scaling appears. We estimate $\|m(t,\cdot)\|_{M_p^q}$ as $t \to \infty$ for multipliers of type

$$m(t,\xi) = e^{\pm i|\xi|^\sigma t - |\xi|^\theta t},$$

and suitable perturbations, under the assumption that the scaling of the diffusive component is worse, i.e., $\theta > \sigma$. These multipliers are, for instance, related to the fundamental solution to the Cauchy problem for the σ-evolution equation with structural damping:

$$u_{tt} + (-\Delta)^\sigma u + (-\Delta)^{\frac{\theta}{2}} u_t = 0, \quad t \geq 0,\ x \in \mathbb{R}^n,$$

in the so-called non-effective case $\sigma < \theta$.

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1. Introduction

We consider the forward Cauchy problem for a σ-evolution equation with a so-called structural damping $(-\Delta)^{\frac{\theta}{2}} u_t$:

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + (-\Delta)^{\frac{\theta}{2}} u_t = 0, & x \in \mathbb{R}^n,\ t \in \mathbb{R}_+, \\ u(0,x) = u_0(x), \\ u_t(0,x) = u_1(x). \end{cases} \tag{1}$$

The term $(-\Delta)^\sigma$, $\sigma > 0$, stands for possibly non-integer powers of the Laplace operator. In the non-integer case, $(-\Delta)^\sigma f = \mathscr{F}^{-1}(|\xi|^{2\sigma}\hat{f})$, for $f \in \mathcal{S}$ and its action is extended by density. Equations whose "principal part" is

$$w_{tt} + (-\Delta)^\sigma w = 0, \tag{2}$$

like the plate equation which is attained for $\sigma = 2$, are called σ-evolution equations in the sense of Petrowsky, since their symbols $\tau^2 + |\xi|^{2\sigma}$ have only pure imaginary, distinct, roots $\tau = \pm i|\xi|^\sigma$ for all $\xi \neq 0$. The set of

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1-evolution operators coincides with the set of strictly hyperbolic operators. However, several properties of the hyperbolic operators are missing when $\sigma \neq 1$.

The term $(-\Delta)^{\frac{\theta}{2}} u_t$ represents a damping, called structural or strong when $\theta > 0$, a term whose action may dissipate the energy

$$E(t) = \frac{1}{2} \|u_t(t,\cdot)\|_{L^2}^2 + \frac{1}{2} \|(-\Delta)^{\frac{\sigma}{2}} u(t,\cdot)\|_{L^2}^2. \tag{3}$$

Indeed,

$$E'(t) = -\|(-\Delta)^{\frac{\theta}{4}} u_t(t,\cdot)\|_{L^2}^2 \leq 0.$$

The case $\sigma = 2$ in (1) is an important model in the literature, known as beam operator and plate operator in the case of space dimension $n = 1$ and $n = 2$, respectively. Models to study the vibrations of thin plates ($n = 2$) given by the full von Kármán system have been studied by several authors, in particular, see [6, 17, 25]. We mention that some plate models include also a term $-\Delta u_{tt}$ called rotational inertia. Energy estimates for solutions, for which a regularity-loss type decay appears, have been investigated in [2, 3, 4, 31]. Strichartz estimates and estimates in modulations spaces for the plate equation are obtained in [7, 8].

Models in (1) have been studied in the abstract setting:

$$u''(t) + A^{\frac{\theta}{2}} u'(t) + Au = 0, \quad \theta = 0, 2,$$

where A is a nonnegative self-adjoint operator in a real Hilbert space H (see [5] and [16] and reference therein).

We consider the full symbol of the equation in (1):

$$\lambda^2 + |\xi|^\theta \lambda + |\xi|^{2\sigma}. \tag{4}$$

For any given compact subset of $\mathbb{R}^n \setminus \{0\}$, the real parts of the roots of (4) are bounded from above by a negative constant. On the other hand, for sufficiently small values of $|\xi|$, the symbol (4) admits two roots whose real parts vanish as $\xi \to 0$. The asymptotic profile of the solution to (1) is determined by its fundamental solution localized at low frequencies, if $\theta \in [0, 2\sigma]$. (We mention that the asymptotic profile and the decay rate structure may be very different if $\theta > 2\sigma$ and new effects appear: we will not investigate this case in details, but we address the interested reader to [13]).

These roots are real-valued in the "effective" case $\theta < \sigma$ and are conjugate complex-valued in the "noneffective" case $\theta > \sigma$ (see the classification in [9]). In the effective case, $L^p - L^q$ estimates for the solution may be obtained with techniques similar to the ones employed for diffusive equations [10]. In the non effective case, which we consider in this paper, $L^p - L^q$ estimates are more challenging due to the interplay of a diffusive component and an oscillatory component.

Let $\chi \in \mathcal{C}_c^\infty([0,\infty))$, with $\chi = 1$ for $\rho \in [0,1]$ and $\chi = 0$ for $\rho \geq 2$. The fundamental solution to (1) localized at low frequencies, say $|\xi| \leq \varepsilon_0$ for sufficiently small ε_0, is $K_0(t,\cdot)$, with

$$\hat{K}_0(t,\xi) = \chi(2|\xi|/\varepsilon_0)\, t\, e^{-\frac{t}{2}|\xi|^\theta} \operatorname{sinc}(\omega t), \tag{5}$$

where $\operatorname{sinc}\rho = \rho^{-1}\sin\rho$ is the cardinal sin function,

$$\omega = |\xi|^\sigma \sqrt{1 - |\xi|^{2\theta - 2\sigma}/4}. \tag{6}$$

The interplay between the diffusive part and the oscillatory part of $\hat{K}_0$ leads to a delicate equilibrium. In this manuscript, we discuss an optimal strategy to derive $L^p - L^q$ long-time estimates for the convolution operator $K_0(t,\cdot)*$. The novel idea in this paper consists in treating separately the two components of the solution to (1), the oscillatory one and the diffusive one, in a particular t-dependent zone of the frequencies space. This strategy allows us to treat an equation which is not scale-invariant by splitting it into two terms with different scaling properties. In particular, we use the scaling $t|\xi|^\sigma \mapsto |\xi|^\sigma$ of the oscillatory part as far as we may control the influence of the oscillations. When the oscillations are too strong, we rely on the scaling $t|\xi|^\theta \mapsto |\xi|^\theta$ of the diffusive part to produce some extra decay rate, tough slower due to the worse scaling (since $\theta > \sigma$).

To find the optimal (p,q) range in which we may take advantage of the better scaling of the oscillatory part of the multiplier, we derive a sharp estimate, with respect to a parameter τ, for the M_p^q norm (see Notation 1.3) of multipliers compactly supported in a ball with radius τ, with an oscillatory and a diffusive components (Section 5). The multipliers considered are perturbations of

$$m(\tau,\xi) = \chi(2\tau^{-1}|\xi|)(1-\chi(|\xi|))\,|\xi|^a\,e^{\pm i|\xi|^\sigma - \tau^{\sigma-\theta}|\xi|^\theta}.$$

In particular, we may estimate $\|m(\tau,\cdot)\|_{M_p^q}$ with a constant independent of τ when (p,q) is in the same range in which the multiplier $(1-\chi(|\xi|))\,|\xi|^a\,e^{\pm i|\xi|^\sigma}$ is in M_p^q, whereas in the remaining case we find the speed at which $\|m(\tau,\cdot)\|_{M_p^q}$ grows as $\tau \to \infty$. In this latter case, a benefit on the control of the M_p^q norm of $m(\tau,\cdot)$, comes from the presence of the diffusive component, since without it, the growth speed would be faster (see Section 7).

We investigate the general case of $\sigma > 0$, provided that $\sigma \neq 1$ (wave equation case). Indeed, as it is well-known, the condition for the multiplier $(1-\chi(|\xi|))\,|\xi|^a\,e^{\pm i|\xi|^\sigma}$ to be in M_p^q is different if $\sigma = 1$ or $\sigma \neq 1$ (see [19, 23, 28]). This peculiarity is also preserved for a perturbation ω of $|\xi|^\sigma$, as in (6), and our approach is not suitable to treat the special case $\sigma = 1$. In future, we plan to investigate the case $\sigma = 1$ with different strategies.

In a forthcoming paper, we will apply the optimal $L^p - L^q$ estimates obtained in this paper to study some nonlinear equations. In particular, in low space dimension we will show that the critical exponent for the problem (1) to which we add a right-hand side in the equation in the form $|u|^p$ or, respectively $|u_t|^p$, is $\bar{p} = 1 + 2\sigma/(n-\sigma)$ or, respectively, $\bar{p} = 1 + \sigma/n$. By critical exponent we mean that global small data solutions exist for $p > \bar{p}$, whereas no global solution exists for $1 < p \leq \bar{p}$, under suitable sign data assumption. This will complement the analysis initiated by the authors in [10], who found the critical exponents in the easier effective case $\theta \leq \sigma$ (see also [24]; see [12] for the case $\sigma = 1$).

However, the nonlinear problem mentioned above may only be solved using optimal decay $L^p - L^q$ estimates, since non optimal decay estimates only produce partial results for non-sharp ranges of existence. The need to find optimal decay estimates for problem (1) inspired this manuscript in a first moment.

Notation used trough the paper

In this paper, we use the following notation.

Notation 1.1. The notation $\mathcal{C}_c^k = \mathcal{C}_c^k(\mathbb{R}^n)$, $k \in \mathbb{N}$, denotes the space of compactly supported, k-times differentiable functions with continuous derivatives. The notation $\mathcal{C}_0^k = \mathcal{C}_0^k(\mathbb{R}^n)$, $k \in \mathbb{N}$, denotes the space of k-times differentiable functions with continuous derivatives, which vanish as $|x| \to \infty$. The notation $\mathcal{S}$ denotes the Schwartz space of functions with infinitely many rapidly decreasing derivatives, and $\mathcal{S}'$ denotes the space of tempered distributions, i.e. of the continuous linear functionals mapping $\mathcal{S}$, equipped with its standard convergence, into $\mathbb{C}$.

Notation 1.2. The notation $\hat{f} = \mathscr{F}f$ or $\hat{f}(t,\cdot) = \mathscr{F}f(t,\cdot)$ denotes the Fourier transform, with respect to the space variable x, of a tempered distribution or of a function, in the appropriate distributional or functional sense. The notation $\mathscr{F}^{-1}$ denotes the inverse Fourier transform, in the appropriate sense.

Notation 1.3 (see, for instance, [14]). The notation $L_p^q = L_p^q(\mathbb{R}^n)$ denotes the space of tempered distributions $T \in \mathcal{S}'$ such that $T * f \in L^q$ for any $f \in \mathcal{S}$, and

$$\|T * f\|_{L^q} \leq C\|f\|_{L^p}$$

for all $f \in \mathcal{S}$ with a constant C, which is independent of f. In this case, the operator $T*$ is extended by density from $\mathcal{S}$ to L^p.

The notation $M_p^q = M_p^q(\mathbb{R}^n)$ denotes the set of Fourier transforms $\hat{T}$ of distributions $T \in L_p^q$, equipped with the norm

$$\|m\|_{M_p^q} := \sup\big\{\|\mathscr{F}^{-1}(m\mathscr{F}(f))\|_{L^q} : f \in \mathcal{S}, \|f\|_{L^p} = 1\big\}.$$

and we set $M_p = M_p^p$. The elements in M_p^q are called multipliers of type (p,q).

We collect some multiplier theorems in Section 8.

In this paper we will also make us of a dyadic partition of unity and of the related notion of Besov space (see [32]).

Notation 1.4. We fix a nonnegative function $\psi \in \mathcal{C}^\infty$, having compact support in $\{\xi \in \mathbb{R}^n : 2^{-1} \le |\xi| \le 2\}$, such that:

$$\sum_{k=-\infty}^{+\infty} \psi_k(\xi) = 1, \qquad \text{where } \psi_k(\xi) := \psi(2^{-k}\xi). \tag{7}$$

(This property is easily obtained if $\psi(\xi) = \varphi(\xi/2) - \varphi(\xi)$, for some $\varphi \in \mathcal{C}^\infty$, with $\varphi(\xi) = 1$ for $|\xi| \le 1/2$ and $\varphi(\xi) = 0$ if $|\xi| \ge 1$). For any $p \in [1,\infty]$, we define the Besov space

$$B^0_{p,2} = \{f \in \mathcal{S}' : \ \forall k \in \mathbb{Z},\ \mathscr{F}^{-1}(\psi_k \hat{f}) \in L^p, \quad \|f\|_{B^0_{p,2}} < \infty\},$$

where

$$\|f\|_{B^0_{p,2}} = \|\mathscr{F}^{-1}(\psi_k \hat{f})\|_{\ell^2(L^p)} = \left(\sum_{k=-\infty}^{+\infty} \|\mathscr{F}^{-1}(\psi_k \hat{f})\|^2_{L^p} \right)^{\frac{1}{2}}.$$

2. Results for the evolution equation

If u solves (1), then $\hat{u}$ solves the Cauchy problem for the damped harmonic oscillator

$$\begin{cases} \hat{u}_{tt} + |\xi|^{2\sigma}\hat{u} + |\xi|^\theta \hat{u}_t = 0, & t \in \mathbb{R}_+, \\ \hat{u}(0,\xi) = \hat{u}_0(\xi), \\ \hat{u}_t(0,\xi) = \hat{u}_1(\xi), \end{cases} \tag{8}$$

for any $\xi \in \mathbb{R}^n$. If we write

$$u(t,\cdot) = K(t,\cdot) * u_1 + \big(\partial_t K + (-\Delta)^\theta K\big)(t,\cdot) * u_0,$$

where K is the fundamental solution to (1), then

$$\hat{K}(t,\xi) = t\, e^{-\frac{t}{2}|\xi|^\theta}\ \mathrm{sinc}\,(t\omega), \quad \omega = |\xi|^\sigma \sqrt{1 - |\xi|^{2\theta-2\sigma}/4},$$

for any ξ such that $|\xi|^{\theta-\sigma} < 2$, whereas

$$\hat{K}(t,\xi) = \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}, \quad \lambda_\pm(\xi) = -\frac{1}{2}|\xi|^\theta \big(1 \mp \sqrt{1 - 4|\xi|^{-2(\theta-\sigma)}}\big), \tag{9}$$

for any ξ such that $|\xi|^{\theta-\sigma} > 2$.

For a given $\varepsilon_0 \in (0,1)$ which we will fix later, we define $\varphi_0(\xi) = \chi(2|\xi|/\varepsilon_0)$, where $\chi \in \mathcal{C}^\infty_c([0,\infty))$, with $\chi = 1$ for $\rho \in [0,1]$ and $\chi = 0$ for $\rho \ge 2$. In particular, φ_0 is supported in $\{\xi : \ |\xi| \le \varepsilon_0\}$.

We consider $L^p - L^q$ estimates, $1 \le p \le q \le \infty$, for K_0 as in (5), that is, $K_0(t,\cdot) = \mathscr{F}^{-1}(\varphi_0 \hat{K}(t,\cdot))$. It is easy to see that, due to the compact support of $\hat{K}_0$ for any given $T > 0$, there exists $C(T)$ such that

$$\||\xi|^b \partial_t^j \hat{K}_0(t,\cdot)\|_{M_p^q} \le C(T), \quad t \in [0,T],$$

for any $1 \le p \le q \le \infty$, $b \ge 0$ and $j \in \mathbb{N}$, so that we dedicate ourselves to investigate the behavior of the norm above as $t \to \infty$. The estimates for $\hat{K}_0(t,\cdot)$ determine the decay rate profile for the solution to (1), since the remainder decays in time with exponential speed for $t \gg 1$, at least if $\theta < 2\sigma$ (see Section 9).

For a given couple (p,q), with $1 \le p \le q \le \infty$, we define

$$d(p,q) = n\left(\frac{1}{p} - \frac{1}{q}\right) + n\sigma \max\left\{ \left(\frac{1}{2} - \frac{1}{p}\right), \left(\frac{1}{q} - \frac{1}{2}\right) \right\}. \tag{10}$$

The quantity in (10) is related to multipliers of type $(1-\chi(|\xi|))|\xi|^a\, e^{i|\xi|^\sigma}$, in the sense that they are bounded in M_p^q (see Notation 1.3) when $\sigma\neq 1$ if, and only if, $d(p,q)+a\leq 0$ if $p>1$ and $q<\infty$, or if, and only if, $d(p,q)+a<0$ if $p=1$ or $q=\infty$ (see [20, Theorem 4.2]). For our multiplier (5), this quantity does not regulate whether $\hat{K}_0$ is in M_p^q or not, but it regulates the time-dependent size of its norm $\|\hat{K}_0(t,\cdot)\|_{M_p^q}$.

Before stating our main result, it is convenient to define another quantity, dependent on p,q, which comes into play as the power of a logarithmic loss of decay in a limit case:

$$\gamma(p,q)=\max\left\{\left(\frac{1}{2}-\frac{1}{p}\right)_+,\left(\frac{1}{q}-\frac{1}{2}\right)_+\right\}+\frac{\delta_1^p+\delta_q^\infty}{2}\,, \tag{11}$$

where $\delta_1^p=1$ if $p=1$ and $\delta_1^p=0$ if $p>1$, $\delta_q^\infty=1$ if $q=\infty$ and $\delta_q^\infty=0$ if $q<\infty$. It is clear that $\gamma(p,q)=\gamma(q',p')$ (duality argument). We stress that γ is discontinuous at $p=1$ and at $q=\infty$, but is otherwise continuous. This singular behavior is due to the lack of Besov embeddings for L^1 and L^∞ (see later, the proof of Lemma 5.2). In particular, $\gamma(p,q)=0$ if $1<p\leq 2\leq q<\infty$. For $1\leq p\leq q\leq p'$, $\gamma(p,q)$ may be written in the form

$$\gamma(p,q)=\begin{cases}1/\min\{2,q\}-1/2 & \text{if } p>1,\\ 1/\min\{2,q\} & \text{if } p=1 \text{ and } q<\infty,\\ 1 & \text{if } p=1 \text{ and } q=\infty,\end{cases}$$

and by duality we get $\gamma(p,q)=\gamma(q',p')$ when $q'\leq p\leq q\leq\infty$.

We are now ready to state our main result for (5).

Theorem 2.1. *Let $\theta>\sigma>0$ with $\sigma\neq 1$, and $b\geq 0$. Fix $1\leq p\leq q\leq\infty$, and let $d(p,q)$ be as in (10). Then*

$$\||\xi|^b\,\hat{K}_0(t,\cdot)\|_{M_p^q}\leq\begin{cases}C\,t^{1-\frac{n}{\sigma}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{b}{\sigma}} & \textit{if } d+b<\sigma,\\ C\,t^{n\max\left\{\frac{1}{2}-\frac{1}{p},\frac{1}{q}-\frac{1}{2}\right\}\left(1-\frac{\sigma}{\theta}\right)+\frac{\sigma-b}{\theta}-\frac{n}{\theta}\left(\frac{1}{p}-\frac{1}{q}\right)}\,(\log t)^{\gamma(p,q)} & \textit{if } d+b\geq\sigma,\end{cases} \tag{12}$$

for any $t\gg 1$, with $C>0$, independent of t. If $d+b>\sigma$, in the cases $p=1<q<\infty$ and $1<p<\infty=q$, the logarithmic loss in estimate (12) may be refined, replacing $(\log t)^{\gamma(p,q)}$ by $(\log t)^{\max\left\{\left(\frac{1}{2}-\frac{1}{p}\right)_+,\left(\frac{1}{q}-\frac{1}{2}\right)_+\right\}}$.

Moreover, $|\xi|^b\partial_t^j\hat{K}_0(t,\cdot)$ verifies the same estimates as $|\xi|^{b+j\sigma}\,\hat{K}_0(t,\cdot)$, for $j=1,2,\ldots$

Remark 2.1. We stress that the power of t in (12) when $d+b\geq\sigma$ may be equivalently written as

$$1-\frac{n}{\sigma}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{b}{\sigma}+(d+b-\sigma)\left(\frac{1}{\sigma}-\frac{1}{\theta}\right),$$

and so it is the same power of the case $d+b<\sigma$ when $d+b=\sigma$, but is worse in the case $d+b>\sigma$, due to $\theta>\sigma$ and $d+b-\sigma>0$.

Remark 2.2. We stress that the power $1-(n(1/p-1/q)+b)/\sigma$ of t, appearing in (12) when $d(p,q)+b\leq\sigma$, only depends on σ and is independent of θ. More precisely, it is the power related to the scaling $t^{\frac{1}{\sigma}}\xi\mapsto\xi$ for the M_p^q norm of multipliers of type $|\xi|^{b-\sigma}\,e^{\pm it|\xi|^\sigma}$.

The case $d(p,q)+b\leq\sigma$, indeed, corresponds to the necessary condition for $|\xi|^{b-\sigma}\,e^{\pm it|\xi|^\sigma}(1-\chi(|\xi|))$ to be a multiplier in M_p^q, where χ is smooth, compactly supported, and $\chi=1$ in a neighborhood of the origin (see [20, Theorem 4.1]). For our problem, assumption $d(p,q)+b\leq\sigma$ regulates the (p,q) range in which the norm of the multiplier $|\xi|^{b-\sigma}\,e^{\pm it|\xi|^\sigma}\chi(t^{\frac{1}{\sigma}}|\xi|)(1-\chi(|\xi|))$ may behave as $t^{1-\frac{n}{\sigma}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{b}{\sigma}}$. When $d(p,q)+b>\sigma$, a loss appears in the estimate for the M_p^q norm. The loss is, however, compensated by the presence of the diffusive component in the multiplier. We notice that the loss vanishes as $\theta\to\sigma$. The optimality of this loss is discussed in Section 5.

In Theorem 2.1, we focused our attention on the multiplier localized at low frequencies $\hat{K}_0$, since the multiplier localized at high frequencies shows an exponential decay. In particular, if $\theta < 2\sigma$, the smoothing effect of the multiplier at high frequencies guarantees that

$$\| |\xi|^b \, \partial_t^j (\hat{K} - \hat{K}_0)(t,\cdot) \|_{M_p^q} \leq C \, e^{-ct} \,, \tag{13}$$

for $t \gg 1$, where $C > 0$ and $c = c(\varepsilon) > 0$ are independent of t, for any $b \geq 0$, $j = 0, 1, 2, \ldots$, and for any $1 \leq p \leq q \leq \infty$. If $\theta = 2\sigma$, estimate (13) remains valid under additional assumptions, which we omit here, for brevity. We postpone the details about high frequencies estimates to Section 9.

As a consequence of these estimates and of Theorem 2.1, we obtain the following $L^p - L^q$ estimates for the solution to (1). We assume $u_0 \equiv 0$ for brevity.

Corollary 2.2. *Let $\sigma \neq 1$, $\sigma < \theta < 2\sigma$ and $b \geq 0$. Fix $1 \leq p \leq q \leq \infty$, and let $d(p,q)$ be as in (10). Let $u(t,\cdot)$ be the solution to (1) with $u_0 \equiv 0$ and $u_1 \in L^p$, for $t \gg 1$. Then there exists $C > 0$, independent of t, such that*

$$\|(-\Delta)^{\frac{b}{2}} \, \partial_t^j u(t,\cdot)\|_{L^q} \leq C \, t^{1 - \frac{n}{\sigma}\left(\frac{1}{p} - \frac{1}{q}\right) - \frac{b}{\sigma} - j} \, \|u_1\|_{L^p}, \tag{14}$$

if $d + b + (j-1)\sigma < 0$, and

$$\|(-\Delta)^{\frac{b}{2}} \, \partial_t^j u(t,\cdot)\|_{L^q} \leq C \, t^{n \max\left\{\frac{1}{2} - \frac{1}{p}, \frac{1}{q} - \frac{1}{2}\right\}\left(1 - \frac{\sigma}{\theta}\right) + \frac{\sigma(1-j) - b}{\theta} - \frac{n}{\theta}\left(\frac{1}{p} - \frac{1}{q}\right)} \, (\log t)^{\gamma(p,q)} \, \|u_1\|_{L^p} \tag{15}$$

if $d + b + (j-1)\sigma \geq 0$, where $\gamma(p,q)$ is as in (11). If $d + b + (j-1)\sigma > 0$, in the cases $p = 1 < q < \infty$ and $1 < p < \infty = q$, the logarithmic loss in estimate (15) may be refined, replacing $(\log t)^{\gamma(p,q)}$ by $(\log t)^{\max\left\{\left(\frac{1}{2} - \frac{1}{p}\right)_+, \left(\frac{1}{q} - \frac{1}{2}\right)_+\right\}}$.

Proof. The proof is an immediate consequence of Theorem 2.1 and estimate (13).

In the case $\theta > 2\sigma$, $L^p - L^q$ high frequencies may be treated with a different approach, due to the regularity-loss decay structure, and then combined with Theorem 2.1 to estimate the solution to (1).

3. Examples

Theorem 2.1 has a wide range of scenarios, due to the fact that $L^p - L^q$ estimates are considered in the full range $1 \leq p \leq q \leq \infty$, so we collect here some examples of special cases: energy estimates (Example 3.1), estimates for the solution with initial data in L^1 (Example 3.2), $L^p - L^p$ estimates (Example 3.3), $L^p - L^{p'}$ estimates (Example 3.4).

Example 3.1 (Energy estimates). Let us fix $q = 2$ and $b = \sigma$. We want to estimate the energy $E(t)$ in (3), as $t \to \infty$.

If $p = 2$, due to $d(2,2) = 0$ and $\gamma(2,2) = 0$, using (15), we have the classic energy estimate for the solution to (1): $E(t) \leq C \, \|u_1\|_{L^2}^2$. If $p \in [1,2)$, then $d(p,2) > 0$ and the energy decays as $t \to \infty$, as a consequence of the damping. Using (15), we find:

$$E(t) \leq C \, t^{-\frac{2n}{\theta}\left(\frac{1}{p} - \frac{1}{2}\right)} \, \|u_1\|_{L^p}^2 \,.$$

Example 3.2 (Estimates for the solution with L^1 data). Let us fix $p = 1$ and $b = 0$, and let $q \in [1,\infty]$ (by duality, this example also provides $L^{q'} - L^\infty$ estimates for the solution to (1), when $q' \in [1,\infty]$). If $d(1,q) < \sigma$, that is,

$$1 - \frac{n}{\sigma}\left(1 - \frac{1}{q}\right) > n\left(\frac{1}{q} - \frac{1}{2}\right),$$

then (14) gives the following estimate for the solution to (1):

$$\|u(t,\cdot)\|_{L^q} \leq C \, t^{1 - \frac{n}{\sigma}\left(1 - \frac{1}{q}\right)} \, \|u_1\|_{L^1}.$$

Otherwise, by (15), we find

$$\|u(t,\cdot)\|_{L^q} \le C\, t^{n\left(\frac{1}{q}-\frac{1}{2}\right)\left(1-\frac{\sigma}{\theta}\right)+\frac{\sigma}{\theta}-\frac{n}{\theta}\left(1-\frac{1}{q}\right)}\, (\log t)^{\gamma(1,q)}\, \|u_1\|_{L^1}.$$

If $d(1,q) > \sigma$ and $p = 1 < q < \infty$ or $1 < p < \infty = q$, we may replace the logarithmic loss above by $(\log t)^{\left(\frac{1}{q}-\frac{1}{2}\right)_+}$.

In the special case $q = 2$, we get

$$\|u(t,\cdot)\|_{L^2} \le \begin{cases} C\, t^{1-\frac{n}{2\sigma}}\, \|u_1\|_{L^1} & \text{if } n < 2\sigma, \\ C\, (\log t)^{\frac{1}{2}}\, \|u_1\|_{L^1} & \text{if } n = 2\sigma, \\ C\, t^{-\frac{n-2\sigma}{2\theta}}\, \|u_1\|_{L^1} & \text{if } n > 2\sigma. \end{cases}$$

The optimality of the estimate for this latter model with $\theta = 2\sigma$ is also discussed by direct computations in [15].

In the special case $q = \infty$, we get

$$\|u(t,\cdot)\|_{L^\infty} \le \begin{cases} C\, t^{1-\frac{n}{\sigma}}\, \|u_1\|_{L^1} & \text{if } n(2-\sigma) < 2\sigma, \\ C\, t^{-\frac{n}{2}\left(1-\frac{\sigma}{\theta}\right)+\frac{\sigma}{\theta}-\frac{n}{\theta}}\, (\log t)\, \|u_1\|_{L^1} & \text{if } n(2-\sigma) \ge 2\sigma. \end{cases}$$

We mention that the previous $L^1 - L^q$ estimates can be obtained interpolating the $L^1 - L^2$ estimate and the $L^1 - L^\infty$ estimate only if $n \max\{1, 2-\sigma\} < 2\sigma$. Otherwise, a straightforward interpolation gives a worse estimate.

In the special case $q = 1$, we get

$$\|u(t,\cdot)\|_{L^1} \le \begin{cases} C\, t\, \|u_1\|_{L^1} & \text{if } n = 1, \\ C\, t^{\frac{n}{2}\left(1-\frac{\sigma}{\theta}\right)+\frac{\sigma}{\theta}}\, (\log t)\, \|u_1\|_{L^1} & \text{if } n \ge 2. \end{cases}$$

These estimate have been obtained when $\theta = 2\sigma$, in space dimension $n \ge 4$, in [11]. We stress that if $\sigma = 1$, the $L^1 - L^1$ estimates are the same as above in even space dimension, but are better in odd space dimension, see [26] for the case $\theta = 2$ and [21] for the case $\theta \in (1,2)$.

Example 3.3 (**$L^p - L^p$ estimates for the solution).** Let us fix $p = q$ and $b = 0$. Then $d(p,p) = n\sigma|1/p - 1/2|$. If

$$n\left|\frac{1}{p} - \frac{1}{2}\right| < 1,$$

then (14) gives us

$$\|u(t,\cdot)\|_{L^p} \le C\, t\, \|u_1\|_{L^p};$$

otherwise, (15) gives us

$$\|u(t,\cdot)\|_{L^p} \le C\, t^{n\left|\frac{1}{p}-\frac{1}{2}\right|\left(1-\frac{\sigma}{\theta}\right)+\frac{\sigma}{\theta}}\, (\log t)^{\left|\frac{1}{p}-\frac{1}{2}\right|}\, \|u_1\|_{L^p},$$

if $p \ne 1, \infty$, with the logarithmic loss replaced by $\log t$ if $p = 1$ or $p = \infty$ (as in Example 3.2).

Example 3.4 (**$L^p - L^{p'}$ estimates for the solution).** Let us fix $p \in [1,2]$ and $b = 0$. Then

$$d(p,p') = n(2-\sigma)\left(\frac{1}{p} - \frac{1}{2}\right).$$

If $\sigma \ge 2$ or $d(p,p') < \sigma$, then (14) gives us

$$\|u(t,\cdot)\|_{L^{p'}} \le C\, t^{1-\frac{2n}{\sigma}\left(\frac{1}{p}-\frac{1}{2}\right)}\, \|u_1\|_{L^p};$$

if $\sigma < 2$ and $d(p,p') \ge \sigma$, then (15) gives us

$$\|u(t,\cdot)\|_{L^{p'}} \le C\, t^{-n\left(\frac{1}{p}-\frac{1}{2}\right)\left(1+\frac{2-\sigma}{\theta}\right)+\frac{\sigma}{\theta}}\, \|u_1\|_{L^p},$$

exception given for the case $p = 1$ and $n(2-\sigma) = 2\sigma$, in which a logarithmic loss $\log t$ appears (as in Example 3.2).

4. Preliminary analysis of the multiplier $\hat{K}_0$

To study the multiplier in Theorem 2.1, we will present a result for a more general class of parameter-dependent radial multipliers in Section 5. Before doing this, we will manipulate $\hat{K}_0$ by a change of variable and a second step of localization in "new" frequencies.

Even if $\hat{K}$ is not homogeneous at low frequencies, due to $\theta > \sigma$, it is convenient to perform a change of variable in $\hat{K}_0$, which takes advantage of the component with the better scaling, i.e., the oscillatory part of $\hat{K}_0$.

By the change of variable $t^{\frac{1}{\sigma}}\xi \mapsto \xi$, for any $1 \leq p \leq q \leq \infty$, it holds

$$\||\xi|^b\,\hat{K}_0(t,\cdot)\|_{M_p^q} = t^{1-\frac{n}{\sigma}(\frac{1}{p}-\frac{1}{q})-\frac{b}{\sigma}}\||\xi|^b\,\operatorname{sinc}(\tilde{\omega}(t,\cdot))\,e^{-\frac{1}{2}t^{1-\frac{\theta}{\sigma}}|\xi|^\theta}\,\tilde{\varphi}_0(t,\cdot)\|_{M_p^q}, \tag{16}$$

where we put

$$\tilde{\omega}(t,\xi) = |\xi|^\sigma\sqrt{1-t^{2-\frac{2\theta}{\sigma}}|\xi|^{2\theta-2\sigma}/4}, \quad \tilde{\varphi}_0(t,\xi) = \varphi_0(t^{-\frac{1}{\sigma}}\xi) = \chi(2t^{-\frac{1}{\sigma}}|\xi|/\varepsilon_0).$$

Formula (16) is a consequence of the scaling of multipliers in M_p^q:

$$\begin{aligned}\|m(\lambda\cdot)\|_{M_p^q} &= \sup_{0\neq f\in\mathcal{S}}\frac{\|\mathscr{F}^{-1}(m(\lambda\cdot)\hat{f})\|_{L^q}}{\|f\|_{L^p}} = \sup_{0\neq f\in\mathcal{S}}\frac{\lambda^{-n+\frac{n}{q}}\|\mathscr{F}^{-1}(m\hat{f}(\cdot/\lambda))\|_{L^q}}{\lambda^{\frac{n}{p}}\,\|f(\lambda\cdot)\|_{L^p}}\\ &= \sup_{0\neq f\in\mathcal{S}}\frac{\lambda^{\frac{n}{q}}\,\|\mathscr{F}^{-1}(m\mathscr{F}(f(\lambda\cdot)))\|_{L^q}}{\lambda^{\frac{n}{p}}\,\|f(\lambda\cdot)\|_{L^p}} = \lambda^{-n(\frac{1}{p}-\frac{1}{q})}\,\|m\|_{M_p^q}\,.\end{aligned}$$

We now divide our analysis in two cases, considering small and large values of the "new frequencies", and we define

$$\begin{aligned}\hat{K}_{0,0}(t,\xi) &= \chi(|\xi|)\,\operatorname{sinc}(\tilde{\omega}(t,\xi))\,e^{-\frac{1}{2}t^{1-\frac{\theta}{\sigma}}|\xi|^\theta},\\ \hat{K}_{0,1}(t,\xi) &= (1-\chi(|\xi|))\tilde{\varphi}_0(t,\xi)\,\operatorname{sinc}(\tilde{\omega}(t,|\xi|))\,e^{-\frac{1}{2}t^{1-\frac{\theta}{\sigma}}2|\xi|^\theta}.\end{aligned}$$

To write $K_{0,0}$, we used that $\tilde{\varphi}_0(t,\cdot) = 1$ in the support of χ, for sufficiently large t. It is easy to see that $\|\hat{K}_{0,0}(t,\cdot)\|_{M_p^q}$ is uniformly bounded, with respect to t, for any $1 \leq p \leq q \leq \infty$. Indeed, $\chi \in \mathcal{C}_c^\infty$, is independent of t, so that (see [14, Theorem 1.8]):

$$\||\xi|^b\,\hat{K}_{0,0}(t,\cdot)\|_{M_p^q} \leq C\min\{\||\xi|^b\,\hat{K}_{0,0}(t,\cdot)\|_{M_p^p},\ \||\xi|^b\,\hat{K}_{0,0}(t,\cdot)\|_{M_q^q}\}\,.$$

If $(p,q) \notin \{(1,1),(\infty,\infty),(1,\infty)\}$, by Mikhlin-Hörmander theorem (Theorem 8.1) we immediately obtain that the multiplier norm is bounded by a uniform constant, with respect to t. We stress that here (and in the following step) a key role is played by the fact that $b \geq 0$ and that sinc is smooth in zero, so that the derivative of order k of $\rho^b\operatorname{sinc}\rho$ may be estimated by $C\,\rho^{b-k}$ for $\rho \leq 2$.

Also, we notice that k derivatives of the exponential term $e^{-\frac{1}{2}t^{1-\frac{\theta}{\sigma}}|\xi|^\theta}$ are still estimated by $|\xi|^{-k}$, due to

$$\left|\partial_\rho^k e^{-\frac{1}{2}t^{1-\frac{\theta}{\sigma}}2\rho^\theta}\right| \leq C\,\rho^{-k}(1+t^{1-\frac{\theta}{\sigma}}\rho^\theta)^k\,e^{-\frac{1}{2}t^{1-\frac{\theta}{\sigma}}\rho^\theta} \leq C_1\,\rho^{-k}.$$

Now let $(p,q) \in \{(1,1),(\infty,\infty),(1,\infty)\}$. If $b > 0$ we apply the integration by parts method in Lemma 8.5 to obtain

$$\||\xi|^b\,\hat{K}_{0,0}(t,\cdot)\|_{M_1^1} = \||\xi|^b\,\hat{K}_{0,0}(t,\cdot)\|_{M_\infty^\infty} \leq \|(-\Delta)^{\frac{b}{2}}\,K_{0,0}(t,\cdot)\|_{L^1} \leq C\,,$$

with $C > 0$, independent of t. Indeed, if $b > 0$, we find $|\partial_\xi^\alpha(|\xi|^b\,\hat{K}_{0,0}(t,\cdot))| \leq C\,|\xi|^{b-|\alpha|}$ as discussed before, so that Remark 8.1 is applicable. If $b = 0$ then we apply Lemma 8.5 to $\hat{K}_{0,0}(t,\xi) - e^{-\frac{1}{2}t^{1-\frac{\theta}{\sigma}}2|\xi|^\theta}$. Indeed, this difference produces an additional power coming from the cancelation $|\operatorname{sinc}\tilde\omega(t,\xi)-1|\approx(\tilde\omega(t,\xi))^2\approx|\xi|^{2\sigma}$ as $\xi\to0$. Namely, for $|\xi|\le1$ we have

$$\begin{aligned}&\partial_\xi^\alpha(\hat K_{0,0}(t,\xi)-e^{-\frac12 t^{1-\frac{\theta}{\sigma}}2|\xi|^\theta})\\&=(\operatorname{sinc}\tilde\omega(t,\xi)-1)\partial_\xi^\alpha e^{-\frac12 t^{1-\frac{\theta}{\sigma}}2|\xi|^\theta}+\sum_{0\neq\beta\le\alpha}\partial_\xi^\beta(\operatorname{sinc}\tilde\omega(t,\xi))\,\partial_\xi^{\alpha-\beta}e^{-\frac12 t^{1-\frac{\theta}{\sigma}}2|\xi|^\theta}\,,\end{aligned}$$

so that

$$|\partial_\xi^\alpha(\hat K_{0,0}(t,\xi)-e^{-\frac12 t^{1-\frac{\theta}{\sigma}}2|\xi|^\theta})|\le C\,|\xi|^{2\sigma-|\alpha|}\,,$$

and Remark 8.1 is applicable. In turn,

$$\|\hat K_{0,0}(t,\cdot)\|_{M_1^1}\le\|\hat K_{0,0}(t,\cdot)-e^{-\frac12 t^{1-\frac{\theta}{\sigma}}2|\xi|^\theta}\|_{M_1^1}+\|e^{-\frac12 t^{1-\frac{\theta}{\sigma}}2|\xi|^\theta}\|_{M_1^1}\le C.$$

Therefore, we may focus our attention on the term $\hat K_{0,1}$. We fix the parameter $\tau=\varepsilon_0\,t^{\frac1\sigma}$, so that

$$\begin{aligned}\varphi_0(t,\xi)&=\chi(2\tau^{-1}|\xi|)\\\tilde\omega(t,\xi)&=\omega_0(\tau,|\xi|)=|\xi|^\sigma\sqrt{1-\varepsilon_0^{\frac{2\theta}{\sigma}-2}(|\xi|/\tau)^{2\theta-2\sigma}/4}\\e^{-\frac12 t^{1-\frac{\theta}{\sigma}}|\xi|^\theta}&=e^{-\eta\tau^{\sigma-\theta}\,|\xi|^\theta},\end{aligned}$$

with $\eta=\eta(\varepsilon_0)>0$.

We notice that $\omega_0(\tau,\rho)$ is a smooth perturbation of ρ^σ such that

$$\partial_\rho\omega_0(\tau,\rho)=\rho^{\sigma-1}\,\frac{2\sigma-(\theta/2)\varepsilon_0^{\frac{2\theta}{\sigma}-2}\,(\rho/\tau)^{2\theta-2\sigma}}{2\sqrt{1-\varepsilon_0\,(\rho/\tau)^{2\theta-2\sigma}/4}}\ge\rho^{\sigma-1}\,(\sigma-(\theta/4)\varepsilon_0^{\frac{2\theta}{\sigma}-2})\,,$$

for any $\sigma>0$, where we used $\rho\le\tau$. In particular, $\partial_\rho\omega_0(\tau,\rho)\ge c\,\rho^{\sigma-1}$ for a sufficiently small ε_0. Similarly,

$$|\partial_\rho^2\omega_0(\tau,\rho)|\ge c\,\rho^{\sigma-2},$$

for any $\sigma>0$, with $\sigma\neq1$, for a sufficiently small ε_0. Replacing $\operatorname{sinc}\omega_0=(2i\omega_0)^{-1}(e^{i\omega_0}-e^{-i\omega_0})$, we may then study radial multipliers of type

$$|\xi|^b\,\omega_0^{-1}\,\chi(2\tau^{-1}|\xi|)\,(1-\chi(|\xi|))\,e^{\pm i\omega_0-\eta\tau^{\sigma-\theta}\,|\xi|^\theta}.$$

This study will be carried on in a more general case, in Section 5.

5. Result for a class of parameter-dependent multipliers

We consider radial multipliers in the form

$$m(\tau,\xi)=\chi(2\tau^{-1}|\xi|)(1-\chi(|\xi|))\,f(\tau,|\xi|)\,e^{\pm i\omega_0(\tau,|\xi|)-\eta\tau^{\sigma-\theta}\,|\xi|^\theta}\,,\tag{17}$$

where $f(\tau,\cdot)$ and $\omega_0(\tau,\cdot)$ are in $\mathcal C^\infty((0,\infty))$ for any τ and verify the following assumptions.

Assumption 5.1. Let $a\in\mathbb R$ and $\sigma>0$, with $\sigma\neq1$. We assume that there exist $C>0$, $c>0$, independent of τ, such that

$$|\partial_\rho^\ell f(\tau,\rho)|\le C\rho^{a-\ell},\qquad \ell\in\mathbb N\,,\tag{18}$$

$$|\partial_\rho^\ell\omega_0(\tau,\rho)|\le C\rho^{\sigma-\ell},\qquad \ell\in\mathbb N\,,\tag{19}$$

$$|\partial_\rho^\ell\omega_0(\tau,\rho)|\ge c\rho^{\sigma-\ell},\qquad \ell=0,1,2,\tag{20}$$

for any $\rho\ge1$.

It is clear that $m(\tau,\cdot)$ is a smooth multiplier compactly supported at "intermediate frequencies", namely, in the annulus $\{\xi\,:\ 1\leq|\xi|\leq\tau\}$, but the size of the support depends on the parameter τ. Due to $m(\tau,\cdot)\in\mathcal{C}_c^{n+1}(\mathbb{R}^n)$, it follows that $m(\tau,\cdot)\in M_p^q$ for any $1\leq p\leq q\leq\infty$ (by Young inequality), but we are interested in obtain an estimate of $\|m(\tau,\cdot)\|_{M_p^q}$ with respect to τ. Assumptions (18) and (19) mean that f and ω_0 have very slow oscillations, comparable to those of a polynomial, away from the origin.

Assumption (20) on the lower bound of ω_0 and its derivatives up to the second order plays a key role in the use of stationary phase methods to estimate $\|m(\tau,\cdot)\|_{M_p^q}$ away from the diagonal line $p=q$. Indeed, it guarantees a nonsingular Hessian matrix, uniformly with respect to τ, which is a crucial ingredient to apply Lemma 8.4.

Proposition 5.1. *Assume that* (20) *holds for some* $\sigma>0$. *Then the determinant of the Hessian matrix verifies the following estimate from below:*

$$|\det H_{\omega_0(\tau,|\xi|)}|\geq\left(c|\xi|^{\sigma-2}\right)^n.$$

PROOF. By straightforward computation, we find

$$\partial_{\xi_k}\partial_{\xi_j}(\omega_0(\tau,|\xi|))=\Big(\rho^{-1}\partial_\rho\omega_0\,\delta_j^k+\rho^{-1}\partial_\rho\big(\rho^{-1}\partial_\rho\omega_0\big)\,\xi_j\,\xi_k\Big)_{\rho=|\xi|},$$

so that the Hessian matrix is

$$H_{\omega_0}=\alpha\,\mathrm{I_n}+\beta\xi\otimes\xi/|\xi|^2,\qquad\text{with}\quad\alpha=|\xi|^{-1}\partial_\rho\omega_0,\quad\beta=\partial_\rho^2\omega_0-\alpha,$$

where $\mathrm{I_n}$ is the identity matrix, and $\xi\otimes\xi$ is the matrix with entries $(\xi_k\xi_j)_{j,k}$. By the matrix determinant lemma,

$$\det H_{\omega_0}=(1+\beta\alpha^{-1})\,\alpha^n=(\partial_\rho^2\omega_0)\,\big(|\xi|^{-1}\partial_\rho\omega_0\big)^{n-1}.$$

This concludes the proof.

We are now in the position to state our main result.

Lemma 5.2. *Let* $\tau\gg1$, *and assume* (18), (19), (20) *for some* $a\in\mathbb{R}$, $\sigma>0$, *with* $\sigma\neq1$, *and* $\theta>\sigma$. *Let* $1\leq p\leq q\leq\infty$, *and define* $d(p,q)$ *as in* (10) *and* $\gamma(p,q)$ *as in* (11). *Then the multiplier in* (17) *verifies the following estimates*

$$\|m(\tau,\cdot)\|_{M_p^q}\leq\begin{cases}C & \text{if } d+a<0,\\ C\,(\log\tau)^{\gamma(p,q)} & \text{if } d+a=0,\\ C\,\tau^{\left(1-\frac{\sigma}{\theta}\right)(d+a)}\,(\log\tau)^{\gamma(p,q)} & \text{if } d+a>0,\end{cases}\tag{21}$$

where $C>0$ *is independent of* τ. *If* $d+a>0$, *in the cases* $p=1<q<\infty$ *and* $1<p<\infty=q$, *the logarithmic loss may be refined and* (21) *is replaced by*

$$\|m(\tau,\cdot)\|_{M_p^q}\leq C\,\tau^{\left(1-\frac{\sigma}{\theta}\right)(d+a)}\,(\log\tau)^{\max\left\{\left(\frac12-\frac1p\right)_+,\left(\frac1q-\frac12\right)_+\right\}}.\tag{22}$$

PROOF. Due to $\|\cdot\|_{M_p^q}=\|\cdot\|_{M_{q'}^{p'}}$, it is sufficient to prove Lemma 5.2 for $q\leq p'$.

We first consider the case $d+a<0$. In this case, we do not need to take advantage of the presence of the exponential component of the multiplier, and we estimate

$$\|m(\tau,\cdot)\|_{M_p^q}\leq\|m_0(\tau,\cdot)\|_{M_p^q}\,\|e^{-\eta\tau^{\sigma-\theta}|\xi|^\theta}\|_{M_q^q}=C\,\|m_0(\tau,\cdot)\|_{M_p^q},\tag{23}$$

where

$$m_0(\tau,\xi)=\chi(2\tau^{-1}|\xi|)(1-\chi(|\xi|))\,f(\tau,|\xi|)\,e^{\pm i\omega_0(\tau,|\xi|)}.\tag{24}$$

Indeed, the M_q^q norm is invariant by dilations, and $e^{-|\xi|^\theta}$ is in M_q^q, for any $q\in[1,\infty]$.

We consider a dyadic partition of unity $\{\psi_k\}_{k\in\mathbb{Z}}$ as in Notation 1.4. Due to $\operatorname{supp}\psi_k \subset \{\xi:\ 2^{k-1}\leq|\xi|\leq 2^{k+1}\}$, if we define

$$k_0 = k_0(\tau) = \max\{k\in\mathbb{Z}:\ 2^{k-1}\leq\tau\},$$

we now see that (for $\tau\geq 8$):

$$m_0(\tau,\xi)\psi_k(\xi) = \begin{cases} 0 & \text{if } k\leq -1, \\ (1-\chi(|\xi|))\,\psi_k\, f(\tau,|\xi|)\,e^{\pm i\omega_0} & \text{if } k=0,1 \\ \psi_k\, f(\tau,|\xi|)\,e^{\pm i\omega_0} & \text{if } 2\leq k\leq k_0-3, \\ \chi(2\tau^{-1}|\xi|)\psi_k\, f(\tau,|\xi|)\,e^{\pm i\omega_0} & \text{if } k=k_0-2,k_0-1,k_0, \\ 0 & \text{if } k\geq k_0+1. \end{cases}$$

In particular,

$$m_0(\tau,\xi) = \sum_{k=0}^{k_0(\tau)} \psi_k(\xi)\, m_0(\tau,\xi),$$

so that

$$\|m_0(\tau,\cdot)\|_{M_p^q} \leq \sum_{k=0}^{k_0(\tau)} \|\psi_k m_0(\tau,\cdot)\|_{M_p^q}.$$

Thanks to (18), we immediately obtain

$$\|\psi_k m_0(\tau,\cdot)\|_{M_2^2} = \|\psi_k m_0(\tau,\cdot)\|_{L^\infty} \leq C \max_{2^{k-1}\leq\rho\leq 2^{k+1}} \rho^a = C\,2^{ka}. \tag{25}$$

On the other hand, thanks to (19), by

$$\|\partial_\xi^\beta(\psi_k m_0(\tau,\cdot))\|_{L^2} \leq C\Big(\int_{2^{k-1}\leq|\xi|\leq 2^{k+1}} |\xi|^{2a+2(\sigma-1)|\beta|}\,d\xi\Big)^{\frac{1}{2}} \leq C_1\,2^{k(\frac{n}{2}+a+|\beta|(\sigma-1))},$$

we derive, choosing some $N>n/2$ (see Theorem 8.3), the estimate

$$\|\psi_k m_0(\tau,\cdot)\|_{M_1^1} \leq \|\psi_k m_0(\tau,\cdot)\|_{L_2}^{1-\frac{n}{2N}} \sum_{|\beta|=N} \|\partial_\xi^\beta(\psi_k m_0(\tau,\cdot))\|_{L^2}^{\frac{n}{2N}} \leq C_2\,2^{k\left(a+\frac{n}{2}\sigma\right)}. \tag{26}$$

Now let $k=2,\ldots,k_0-3$. Recalling that $m_0(\tau,\xi)\psi_k(\xi)=\psi_k\, f(\tau,|\xi|)\,e^{\pm i\omega_0}$, we replace ξ by $2^k\xi$ and define

$$\varphi(\tau,\rho) = 2^{-k\sigma}\,\omega_0(\tau,2^k\rho), \qquad g(\tau,\rho) = 2^{-ka}\, f(\tau,2^k\rho).$$

Thanks to (18) and (19), φ and g are uniformly bounded with respect to τ, together with their derivatives. Moreover, thanks to Proposition 5.1, assumption (20) gives us

$$|\det H_\varphi| = 2^{-k(\sigma-2)n}\,|\det H_{\omega_0(\tau,2^k\rho)}| \geq 2^{-k(\sigma-2)n+(k-1)(\sigma-2)n}\,c^n = (2^{2-\sigma}\,c)^n > 0, \tag{27}$$

uniformly with respect to τ. Applying Littman's lemma (Lemma 8.4) we conclude

$$\begin{aligned} \|m_0(\tau,\cdot)\psi_k\|_{M_1^\infty} &= \Big\|\mathscr{F}^{-1}_{\xi\to x}\Big(e^{\pm i\omega_0(\tau,|\xi|)}\psi_k(\xi)\,f(\tau,|\xi|)\Big)\Big\|_{L^\infty(\mathbb{R}^n)} \\ &= 2^{k(n+a)}\Big\|\mathscr{F}^{-1}_{\xi\to x}\Big(e^{\pm i2^{k\sigma}\varphi}\psi(\xi)\,g(\tau,|\xi|)\Big)\Big\|_{L^\infty(\mathbb{R}^n)} \\ &\leq C2^{k(n+a)}(1+2^{k\sigma})^{-\frac{n}{2}} \approx 2^{k\left(n+a-\frac{n}{2}\sigma\right)}, \end{aligned} \tag{28}$$

for all $k=2,\ldots,k_0-3$.

As a consequence of Riesz-Thorin interpolation theorem, by (25) and (28), we get the estimate on the conjugate line

$$\|m_0(\tau,\cdot)\psi_k\|_{M_{p_0}^{p_0'}} \leq C2^{k\left(a+n(2-\sigma)\left(\frac{1}{p_0}-\frac{1}{2}\right)\right)}\,. \tag{29}$$

Using (26), (29) and Riesz-Thorin interpolation theorem, we conclude that

$$\|m_0(\tau,\cdot)\psi_k\|_{M_p^q} \leq C2^{k\left(a+n\left(\frac{1}{p}+\frac{\sigma-1}{q}-\frac{\sigma}{2}\right)\right)} = C2^{k(d+a)},$$

where d is as in (10), so that

$$\|m_0(\tau,\cdot)\|_{M_p^q} \leq C\sum_{k=0}^{k_0(\tau)} 2^{k(d+a)}. \tag{30}$$

The latter sum is uniformly bounded with respect to τ due to $d+a<0$.

Now let $d+a=0$. In this case, using (30), we find

$$\|m_0(\tau,\cdot)\|_{M_p^q} \leq C\sum_{k=0}^{k_0(\tau)} 1 = C(k_0(\tau)+1) \approx \log\tau.$$

However, exception given for the cases $(p,q)=(1,1),(1,\infty),(\infty,\infty)$, we may reduce the power of $\log\tau$, using the embeddings for Besov spaces (see Notation 1.4).

Indeed (see, for instance, [27]), for any $p,q\in(1,\infty)$ it holds

$$L^p \hookrightarrow B^0_{p,\,\max\{p,2\}}, \qquad B^0_{q,\,\min\{2,q\}} \hookrightarrow L^q\,.$$

We distinguish three cases, recalling that we assumed $p\leq q'$. If $1<p\leq 2\leq q<\infty$, then

$$\|\mathscr{F}^{-1}(m_0(\tau,\cdot)\hat{f})\|_{L^q} \leq C_1\,\|\mathscr{F}^{-1}(m_0(\tau,\cdot)\hat{f})\|_{B^0_{q,2}} \leq C_2\,\|f\|_{B^0_{p,2}} \leq C_3\,\|f\|_{L^p}.$$

The inequality in the middle is a consequence of the fact the sum in (7) is finite for any given ξ, in particular, $\#\{k:\ \psi_k(\xi)\neq 0\}\leq 3$ (see also [1]), so that by

$$\|\mathscr{F}^{-1}(m_0(\tau,\cdot)\psi_k\hat{f})\|_{L^q}^2 \leq C\sum_{j=k-1}^{k+1}\|\mathscr{F}^{-1}(\psi_j\hat{f})\|_{L^p}^2,$$

we obtain

$$\|\mathscr{F}^{-1}(m_0(\tau,\cdot)\hat{f})\|_{B^0_{q,2}}^2 = \sum_{k=0}^{k_0(\tau)}\|\mathscr{F}^{-1}(m_0(\tau,\cdot)\psi_k\hat{f})\|_{L^q}^2 \leq (3C)^2\sum_{k=0}^{k_0(\tau)}\|\mathscr{F}^{-1}(\psi_k\hat{f})\|_{L^p}^2.$$

Now let $1<p\leq q<2$. In this case, we estimate

$$\begin{aligned}\|\mathscr{F}^{-1}(m_0(\tau,\cdot)\hat{f})\|_{L^q} &\leq C_1\,\|\mathscr{F}^{-1}(m_0(\tau,\cdot)\hat{f})\|_{B^0_{q,q}} \leq C_2\Big(3\sum_{k=0}^{k_0(\tau)}\|\mathscr{F}^{-1}(\psi_k\hat{f})\|_{L^p}^q\Big)^{\frac{1}{q}}\\ &\leq C_3\,(\log\tau)^{\frac{1}{q}-\frac{1}{2}}\Big(\sum_{k\in\mathbb{Z}}\|\mathscr{F}^{-1}(\psi_k\hat{f})\|_{L^p}^2\Big)^{\frac{1}{2}} \leq C_4\,(\log\tau)^{\frac{1}{q}-\frac{1}{2}}\,\|f\|_{L^p}\,.\end{aligned}$$

We stress that we used Hölder inequality with the sequence $\{\|\mathscr{F}^{-1}(\psi_k\hat{f})\|_{L^p}\}_{k\in\mathbb{Z}}\in\ell^2$ and the compactly supported sequence $\chi_{[0,k_0(\tau)]}$. In particular,

$$\|\chi_{[0,k_0(\tau)]}\|_{\ell^r} = (k_0(\tau)+1)^{\frac{1}{r}}, \quad \frac{1}{q}=\frac{1}{2}+\frac{1}{r}\,.$$

Finally, take $p=1$ and $q\in(1,\infty)$. Then, we can use only one of the Besov embeddings, obtaining

$$\begin{aligned}\|\mathscr{F}^{-1}(m_0(\tau,\cdot)\hat{f})\|_{L^q} &\leq C_1\,\|\mathscr{F}^{-1}(m_0(\tau,\cdot)\hat{f})\|_{B^0_{q,\min\{2,q\}}}\\ &\leq C_2\,\|f\|_{L^1}\,\Big(\sum_{k=0}^{k_0(\tau)}1\Big)^{\frac{1}{\min\{2,q\}}}\leq C_3\,(\log\tau)^{\frac{1}{\min\{2,q\}}}\,\|f\|_{L^1}\,.\end{aligned}$$

Recalling the definition of $\gamma(p,q)$, this concludes the proof when $d+a=0$.

Now let $d+a>0$. In this case, we take advantage of the presence of the exponential term to mitigate the growth of $\|m_0(\tau,\cdot)\|_{M_p^q}$ as $\tau\to\infty$ (see Remark 5.2). Replacing (23) by

$$\|m(\tau,\cdot)\|_{M_p^q}\leq\|m_0(\tau,\cdot)\,|\xi|^{-d-a}\|_{M_p^q}\,\||\xi|^{d+a}\,e^{-\eta\tau^{\sigma-\theta}\,|\xi|^\theta}\|_{M_q^q},\tag{31}$$

we may now apply the previous step to $m_0(\tau,\xi)\,|\xi|^{-d-a}$, obtaining

$$\|m_0(\tau,\cdot)\,|\xi|^{-d-a}\|_{M_p^q}\leq C\,(\log\tau)^{\gamma(p,q)},$$

whereas

$$\||\xi|^{d+a}\,e^{-\eta\tau^{\sigma-\theta}\,|\xi|^\theta}\|_{M_q^q}=\tau^{\left(1-\frac{\sigma}{\theta}\right)(d+a)}\,\||\xi|^{d+a}\,e^{-\eta\,|\xi|^\theta}\|_{M_q^q}=C\,\tau^{\left(1-\frac{\sigma}{\theta}\right)(d+a)}\,.$$

This concludes the proof of (21). Now let $p=1<q<\infty$ and $d(1,q)+a>0$. To prove (22), we replace (31) by

$$\|m(\tau,\cdot)\|_{M_1^q}\leq\|m_0(\tau,\cdot)\,|\xi|^{-d(1+\delta,q)-a}\|_{M_{1+\delta}^q}\,\||\xi|^{d(1+\delta,q)+a}\,e^{-\eta\tau^{\sigma-\theta}\,|\xi|^\theta}\|_{M_1^{1+\delta}},$$

for a sufficiently small $\delta>0$, such that $d(1+\delta,q)+a>0$ and $1+\delta\leq q'$. Therefore,

$$\|m_0(\tau,\cdot)\,|\xi|^{-d(1+\delta,q)-a}\|_{M_{1+\delta}^q}\leq C\,(\log\tau)^{\gamma(1+\delta,q)}=C(\log\tau)^{(1/q-1/2)_+},$$

and

$$\||\xi|^{d(1+\delta,q)+a}\,e^{-\eta\tau^{\sigma-\theta}\,|\xi|^\theta}\|_{M_1^{1+\delta}}=C\,\tau^{\left(1-\frac{\sigma}{\theta}\right)\left(n\left(1-\frac{1}{1+\delta}\right)+d(1+\delta,q)+a\right)}.$$

The proof of (22) follows noticing that

$$n\left(1-\frac{1}{1+\delta}\right)+d(1+\delta,q)+a=d(1,q)+a.$$

Remark 5.2. If we do not take advantage of the exponential term when $d+a>0$, that is, we still rely on (23) instead of using (31), then, by (30), we find the estimate

$$\|m_0(\tau,\cdot)\|_{M_p^q}\leq C\sum_{k=0}^{k_0(\tau)}2^{k(d+a)}\approx 2^{k_0(\tau)(d+a)}\approx\tau^{d+a}.$$

The above estimate is, indeed, sharp for multipliers without the diffusive component, as m_0 is. We postpone the discussion of this optimality to Section 7, where we will show that

$$\|m_0(\tau,\cdot)\|_{M_p^q}\approx\tau^{d(p,q)+a},\qquad\text{if } d(p,q)+a>0.$$

Then it is clear that the presence of the diffusive component of the multiplier is crucial to improve the estimate of the M_p^q norm of $m(\tau,\xi)$ from τ^{d+a} to $\tau^{\left(1-\frac{\sigma}{\theta}\right)(d+a)}$. As $\theta\to\infty$, the gain vanishes.

Remark 5.3. It is clear that the estimate in Lemma 5.2 is optimal when $d(p,q)+a<0$, since it does not depend on τ, and is also optimal when $d(p,q)+a=0$ with $1<p\leq 2\leq q<\infty$, so that $\gamma(p,q)=0$. Apart from the logarithmic term, the estimate in Lemma 5.2 is also optimal when $d(p,q)+a>0$, see Proposition 7.2. Indeed, it is obtained by the better scaling of the oscillating component as far as it is possible, and then it is based on the scaling of the diffusive component only when it becomes necessary (since without the use of that scaling, the estimate would be worse, see Proposition 7.1).

In two special cases, we may prove the optimality of Lemma 5.2 when $d+a\geq 0$, with an easier approach than the one employed in Section 7.

Example 5.4. Let $p=q=2$ and $a>0$. Moreover, assume that $|f(\rho,\tau)|\geq c_1\,\rho^a$, uniformly with respect to τ. Then

$$\|m(\tau,\cdot)\|_{M_2^2}=\|m(\tau,\cdot)\|_{L^\infty}=\sup_{1\leq\rho\leq\tau}\chi(2\tau^{-1}\rho)(1-\chi(\rho))\,|f(\rho,\tau)|\,e^{-\eta\tau^{\sigma-\theta}\,\rho^\theta}\geq c_1\sup_{2\leq\rho\leq\tau/2}\rho^a\,e^{-\eta\tau^{\sigma-\theta}\,\rho^\theta}.$$

Letting $\rho=\tau^{1-\frac{\sigma}{\theta}}$, so that $\eta\tau^{\sigma-\theta}\,\rho^\theta=\eta$, we find

$$\|m(\tau,\cdot)\|_{M_2^2}\geq c_1\,e^{-\eta}\,\tau^{\left(1-\frac{\sigma}{\theta}\right)a}.$$

Noticing that $d(2,2)=0$ in (10), we derive the optimality of the estimates in Lemma 5.2 in this case.

Example 5.5. Let $(p,q)=(1,2)$ (or, by duality, $(p,q)=(2,\infty)$) and $a\geq -n/2$. Moreover, assume that $|f(\rho,\tau)|\geq c_1\,\rho^a$, uniformly with respect to τ. Then it holds (see [14, Theorem 1.4])

$$\begin{aligned}\|m(\tau,\cdot)\|_{M_1^2}&=\|m(\tau,\cdot)\|_{M_2^\infty}=\|\mathscr{F}^{-1}m(\tau,\cdot)\|_{L^2}=c_n\,\|m(\tau,\cdot)\|_{L^2}\\&\geq c_2\Big(\int_2^{\tau/2}\rho^{n-1+2a}\,e^{-2\eta\tau^{\sigma-\theta}\,\rho^\theta}\,d\rho\Big)^{\frac12}\\&\geq c_2\,e^{-\eta}\Big(\int_2^{\tau^{1-\frac{\sigma}{\theta}}}\rho^{n-1+2a}\,d\rho\Big)^{\frac12}\approx\begin{cases}(\log\tau)^{\frac12} & \text{if } a=-n/2,\\ \tau^{\left(1-\frac{\sigma}{\theta}\right)\left(\frac{n}{2}+a\right)} & \text{if } a>-n/2.\end{cases}\end{aligned}$$

Noticing that $d(1,2)=n/2$ according to (10), and that $\gamma(1,2)=1/2$, we derive the optimality of the estimates in Lemma 5.2 in both the cases $d+a=0$ and $d+a>0$.

6. Proof of Theorem 2.1

The application of Lemma 5.2 is crucial to conclude the proof of Theorem 2.1.

PROOF (PROOF OF THEOREM 2.1). We first consider $|\xi|^b\hat{K}_0$, for any $b\geq 0$. As discussed in Section 4, due to (16), and to the fact that $\|\hat{K}_{0,0}(t,\cdot)\|_{M_p^q}\leq C$, with C independent of t, it remains to estimate

$$\hat{K}_{0,1}(t,\xi)=m(\tau,\xi),$$

with m as in (17), where

$$\tau=\varepsilon_0\,t^{\frac{1}{\sigma}},\quad f=\omega^{-1}\,|\xi|^b,\quad a=b-\sigma.$$

Applying Lemma 5.2, we then obtain:

$$\|\hat{K}_{0,1}(t,\cdot)\|_{M_p^q}\leq\begin{cases}C & \text{if } d(p,q)+b<\sigma,\\ C\,(\log t)^{\gamma(p,q)} & \text{if } d(p,q)+b=\sigma,\\ C\,t^{\left(1-\frac{\sigma}{\theta}\right)\left(\frac{d(p,q)+b}{\sigma}-1\right)}\,(\log t)^{\gamma(p,q)} & \text{if } d(p,q)+b>\sigma.\end{cases}$$

In particular, we notice that when $d+b=\sigma$ then

$$1-\frac{n}{\sigma}\left(\frac1p-\frac1q\right)-\frac{b}{\sigma}=n\max\left\{\left(\frac12-\frac1p\right),\left(\frac1q-\frac12\right)\right\}.$$

If $d+b>\sigma$, then we may compute

$$\begin{aligned}1-\frac{n}{\sigma}\left(\frac1p-\frac1q\right)-\frac{b}{\sigma}+\left(1-\frac{\sigma}{\theta}\right)\left(\frac{d(p,q)+b}{\sigma}-1\right)\\=n\max\left\{\frac12-\frac1p,\frac1q-\frac12\right\}\left(1-\frac{\sigma}{\theta}\right)+\frac{\sigma-b}{\theta}-\frac{n}{\theta}\left(\frac1p-\frac1q\right),\end{aligned}$$

and this concludes the proof.

We may now turn our attention to $\partial_t^j \hat{K}_0$, for $j = 1, 2, \ldots$ We notice that

$$\partial_t \hat{K}_0 = -\frac{1}{2}|\xi|^\theta \hat{K}_0 \varphi_0 + e^{-\frac{t}{2}|\xi|^\theta} \cos t\omega \varphi_0 = \frac{1}{2}\varphi_0 \left(-|\xi|^\theta \hat{K}_0 + e^{-\frac{t}{2}|\xi|^\theta} (e^{it\omega} + e^{-it\omega})\right).$$

Iterating for higher order time derivatives, and proceeding as we did for $\hat{K}_0$, we find the desired estimate.

7. Optimality of Lemma 5.2

As discussed in Remark 5.2, following the proof of Lemma 5.2, we find that

$$\|m_0(\tau,\cdot)\|_{M_p^q} \leq C\,\tau^{d(p,q)+a},$$

when $d(p,q) + a > 0$. In this section, we prove the optimality of this estimate. We assume for simplicity that $\omega_0 = \rho^\sigma$ and $f = \rho^a$ in (24), but more general assumptions are possible.

Proposition 7.1. *Let $\tau \gg 1$, and assume that $\omega_0 = \rho^\sigma$ and $f = \rho^a$ in (24), where $a \in \mathbb{R}$ and $\sigma > 0$, with $\sigma \neq 1$. Let $1 \leq p \leq q \leq \infty$, and define $d(p,q)$ as in (10). If $d(p,q) + a > 0$, then there exists $c > 0$ such that*

$$\|m_0(\tau,\cdot)\|_{M_p^q} \geq c\,\tau^{d+a}, \tag{32}$$

where $c > 0$ is independent of τ.

Proof. The proof is inspired by [18, Lemma 8]. By duality, we may assume $1 \leq p \leq q \leq p'$. We fix $g \in \mathcal{C}_c^\infty$, radial. For any $\tau \gg 1$, we define $g_\tau(x) = \tau^{\frac{n}{p}} g(\tau x)$, so that $\|g_\tau\|_{L^p} = \|g\|_{L^p}$. Then $\hat{g}_\tau(\xi) = \tau^{-\frac{n}{p'}} \hat{g}(\tau^{-1}\xi)$.

Let $\rho = |\xi|$. Abusing the notation for radial functions, we may write

$$Tg_\tau(|x|) = \int_1^\tau m_0(\rho)\,\hat{g}_\tau(\rho)\,J_{\frac{n-2}{2}}(|x|\rho)\,|x|^{1-\frac{n}{2}}\,\rho^{\frac{n}{2}}\,d\rho,$$

where $J_{\frac{n-2}{2}}$ denotes the Bessel function of first kind. Due to $\rho \geq 1$, for $|x| \gg 1$ we may use the asymptotic expansion of the Bessel function

$$J_{\frac{n-2}{2}}(|x|\rho) = (|x|\rho)^{-\frac{1}{2}} \left(p_+ e^{i|x|\rho} + p_- e^{-i|x|\rho} + O(|x|^{-N}\rho^{-N})\right),$$

where $p_\pm = c_\pm + \sum_{j=1}^{N-1} c_{j,\pm}\,(|x|\rho)^{-j}$. Therefore,

$$\begin{aligned} Tg_\tau(|x|) = |x|^{-\frac{n-1}{2}}\,\tau^{-\frac{n}{p'}} \int_1^\tau m_0(\rho)\,\hat{g}(\tau^{-1}\rho)\left(p_+ e^{i|x|\rho} + p_- e^{-i|x|\rho}\right)\rho^{\frac{n-1}{2}}\,d\rho \\ + O\Big(|x|^{-\frac{n-1}{2}-N}\,\tau^{-\frac{n}{p'}} \int_1^\tau \rho^{a+\frac{n-1}{2}-N}\,d\rho\Big). \end{aligned}$$

We now look for an interval I, such that $|x| \approx \tau^{\sigma-1}$ for $|x| \in I$, I has length $\approx \tau^{\sigma-1}$, and

$$|Tg_\tau(|x|)| \geq c\,\tau^{-\frac{n}{2}\sigma+\frac{n}{p}+a}, \qquad \text{for any } |x| \in I. \tag{33}$$

If we find such interval as in (33), then it follows that

$$\begin{aligned} \|Tg_\tau\|_{L^q} \geq \|Tg_\tau\|_{L^q(I)} \geq c\,\tau^{-\frac{n}{2}\sigma+\frac{n}{p}+a} \Big(\int_{|x|\in I} 1\,dx\Big)^{\frac{1}{q}} \\ \approx \tau^{-\frac{n}{2}\sigma+\frac{n}{p}+a+\frac{n}{q}(\sigma-1)} = \tau^{n\left(\frac{1}{p}-\frac{1}{q}\right)+n\sigma\left(\frac{1}{q}-\frac{1}{2}\right)+a} = \tau^{d(p,q)+a}, \end{aligned}$$

so the proof is concluded. Hence, the rest of the proof is devoted to prove (33).

It is clear that if $|x| \approx \tau^{\sigma-1}$, then

$$|x|^{-\frac{n-1}{2}-N}\,\tau^{-\frac{n}{p'}}\int_1^\tau \rho^{a+\frac{n-1}{2}-N}\,d\rho = o\big(\tau^{-\frac{n}{2}\sigma+\frac{n}{p}+a}\big),$$

for a sufficiently large N. We now consider the leading terms $c_\pm$ of the polynomials $p_\pm$ in Tg_τ, being the treatment of the other terms easier. That is, we consider

$$\begin{aligned}\int_1^\tau &\chi(2\tau^{-1}\rho)(1-\chi(\rho))\,\rho^{a+\frac{n-1}{2}}\,\big(c_+e^{i(\pm\rho^\sigma+|x|\rho)}+c_-e^{i(\pm\rho^\sigma-|x|\rho)}\big)\,\hat{g}(\tau^{-1}\rho)\,d\rho\\ &= c_+\,\tau^{a+\frac{n+1}{2}}\int_{1/\tau}^1 \chi(2\rho)(1-\chi(\tau\rho))\,\rho^{a+\frac{n-1}{2}}\,e^{i\tau^\sigma\,(\pm\rho^\sigma+\tau^{1-\sigma}\rho|x|)}\,\hat{g}(\rho)d\rho\\ &\quad + c_-\,\tau^{a+\frac{n+1}{2}}\int_{1/\tau}^1 \chi(2\rho)(1-\chi(\tau\rho))\,\rho^{a+\frac{n-1}{2}}\,e^{i\tau^\sigma\,(\pm\rho^\sigma-\tau^{1-\sigma}\rho|x|)}\,\hat{g}(\rho)d\rho\end{aligned}$$

where we performed the change of variable $\rho \mapsto \tau\rho$.

Recalling that $|x| \approx \tau^{\sigma-1}$, for any $N \ge 1$ we may estimate (see, for instance, Proposition 1 at page 331 in [29])

$$\Big|\int_{1/\tau}^1 \chi(2\rho)(1-\chi(\tau\rho))\,\rho^{a+\frac{n-1}{2}}\,e^{\pm i\tau^\sigma(\rho^\sigma+\tau^{1-\sigma}|x|\rho)}\,\hat{g}(\rho)d\rho\Big| \le C_N\,\tau^{-N\sigma},$$

due to the lack of stationary points of the phase function $\rho^\sigma + \tau^{1-\sigma}|x|\rho$.

To obtain the desired estimate from below, we shall now consider the stationary point of the other phase function $h(\rho) = \rho^\sigma - \tau^{1-\sigma}|x|\rho$. We find

$$h'(\rho) = \sigma\rho^{\sigma-1} - \tau^{1-\sigma}|x|, \qquad h''(\rho) = \sigma\,(\sigma-1)\,\rho^{\sigma-2},$$

so that, for any $\sigma \neq 1$, there is a unique stationary point $\rho = \rho_0(|x|) = \tau^{-1}\,(|x|/\sigma)^{\frac{1}{\sigma-1}}$.

We now fix the desired interval $I = [r_1, r_2]$ such that $\rho_0(r_1) = 1/6$ and $\rho_0(r_2) = 1/3$. In particular, $|x| \approx \tau^{\sigma-1}$ for any $|x| \in I$ and I has length $\approx \tau^{\sigma-1}$. Also, possibly restricting the interval, we may assume that $\inf_{r\in I_1} |\hat{g}(\rho_0)| > 0$.

Now, let $\delta > 0$, be such that $\delta < \rho_0/2$, so that $\chi(2\rho) = 1$ for $\rho \le \rho_0 + \delta$, and $\chi(\tau\rho) = 0$ for $\rho \ge \rho_0 - \delta$, for $\tau \ge 24$. For any $|x| \in I_1$, away from the stationary point $\rho_0(|x|)$, for any $N \ge 1$, we may estimate

$$\Big|\int_{1/\tau}^{\rho_0-\delta}(1-\chi(\tau\rho))\,\rho^{a+\frac{n-1}{2}}\,e^{\pm i\tau^\sigma h(\rho)}\,\hat{g}(\rho)d\rho\Big| \le C_{\delta,N}\,\tau^{-N\sigma},$$

$$\Big|\int_{\rho_0+\delta}^{1}\chi(2\rho)\,\rho^{a+\frac{n-1}{2}}\,e^{\pm i\tau^\sigma h(\rho)}\,\hat{g}(\rho)d\rho\Big| \le C_{\delta,N}\,\tau^{-N\sigma}.$$

Indeed, $|h'(\rho)| \ge c_\delta$ for any $\rho \notin (\rho_0 - \delta, \rho_0 + \delta)$.

We may now study the integral near the stationary point:

$$J_\delta = \int_{\rho_0-\delta}^{\rho_0+\delta}\rho^{a+\frac{n-1}{2}}\,e^{\pm i\tau^\sigma h(\rho)}\,\hat{g}(\rho)d\rho.$$

By Taylor's formula,

$$h(\rho) - h(\rho_0) = \frac{\sigma(\sigma-1)}{2}\,\rho_0^{\sigma-2}\,(\rho-\rho_0)^2 + O(\rho-\rho_0)^3,$$

so that, for a sufficiently small $\delta > 0$, we may perform the change of variable

$$h(\rho) - h(\rho_0) = \begin{cases} r^2 & \text{if } \sigma > 1,\\ -r^2 & \text{if } \sigma < 1,\end{cases}$$

and

$$J_\delta = e^{\pm i h(\rho_0)} \int_{a_-}^{a_+} \psi(r)\, e^{\pm i\tau^\sigma r^2} dr, \qquad a_\pm = \sqrt{|h(\rho_0 \pm \delta) - h(\rho_0)|}\,,$$

where $\psi(r) \approx \rho_0^{a+\frac{n-1}{2}}\, \hat g(\rho_0)$. Therefore, $|J_\delta| \geq c\,\tau^{-\frac{\sigma}{2}}$ (see, for instance, Proposition 3 at page 334 in [29], or the proof of Lemma 8 in [18]). Due to

$$|x|^{-\frac{n-1}{2}}\, \tau^{-\frac{n}{p'}+a+\frac{n+1-\sigma}{2}} \approx \tau^{-\frac{n-1}{2}(\sigma-1)-n+\frac{n}{p}+a+\frac{n+1-\sigma}{2}}$$

for $|x| \in I$, we conclude the proof of (33).

In a similar way, it is possible to prove that the estimate for $\|m(\tau,\cdot)\|_{M_p^q}$ in Lemma 5.2, when $d+a>0$, is also optimal, apart from the logarithmic terms.

Proposition 7.2. *Let $\tau \gg 1$, and assume that $\omega_0 = \rho^\sigma$ and $f = \rho^a$ in (17), where $a \in \mathbb{R}$ and $\theta > \sigma > 0$, with $\sigma \neq 1$. Let $1 \leq p \leq q \leq \infty$, and define $d(p,q)$ as in (10). If $d(p,q)+a>0$, then there exists $c>0$ such that*

$$\|m(\tau,\cdot)\|_{M_p^q} \geq c\,\tau^{(d+a)\left(1-\frac{\sigma}{\theta}\right)}, \tag{34}$$

where $c>0$ is independent of τ.

Proof. We follow the proof of Proposition 7.1, but with $g_\tau(x) = \tau^{\frac{n}{p}\left(1-\frac{\sigma}{\theta}\right)}\, g(\tau^{1-\frac{\sigma}{\theta}}x)$, so that

$$\hat g_\tau(\xi) = \tau^{-\frac{n}{p'}\left(1-\frac{\sigma}{\theta}\right)}\, \hat g(\tau^{-\left(1-\frac{\sigma}{\theta}\right)}\xi),$$

and we look for an interval I such that $|x| \approx \tau^{(\sigma-1)\left(1-\frac{\sigma}{\theta}\right)}$ for $|x| \in I$, I has length $\approx \tau^{(\sigma-1)\left(1-\frac{\sigma}{\theta}\right)}$, and

$$|Tg_\tau(|x|)| \geq c\,\tau^{\left(-\frac{n}{2}\sigma+\frac{n}{p}+a\right)\left(1-\frac{\sigma}{\theta}\right)}, \qquad \text{for any } |x| \in I. \tag{35}$$

If we find such interval as in (35), then it follows that

$$\begin{aligned}\|Tg_\tau\|_{L^q} \geq \|Tg_\tau\|_{L^q(I)} &\geq c\,\tau^{\left(-\frac{n}{2}\sigma+\frac{n}{p}+a\right)\left(1-\frac{\sigma}{\theta}\right)} \left(\int_{|x|\in I} 1\,dx\right)^{\frac{1}{q}}\\ &\approx \tau^{\left(-\frac{n}{2}\sigma+\frac{n}{p}+a+\frac{n}{q}(\sigma-1)\right)\left(1-\frac{\sigma}{\theta}\right)} = \tau^{(d(p,q)+a)\left(1-\frac{\sigma}{\theta}\right)},\end{aligned}$$

so the proof of is concluded. Hence, the rest of the proof is devoted to prove (35).

We now shall consider:

$$\begin{aligned}&\int_1^\tau \chi(2\tau^{-1}\rho)(1-\chi(\rho))\,\rho^{a+\frac{n-1}{2}}\, e^{-\eta\tau^{\sigma-\theta}\rho^\theta}\left(c_+ e^{i(\pm\rho^\sigma+|x|\rho)} + c_- e^{i(\pm\rho^\sigma-|x|\rho)}\right) \hat g(\tau^{-\left(1-\frac{\sigma}{\theta}\right)}\rho)\,d\rho\\ &\quad= c_+\,\tau^{\left(a+\frac{n+1}{2}\right)\left(1-\frac{\sigma}{\theta}\right)} \int_{\tau^{\frac{\sigma}{\theta}-1}}^{\tau^{\frac{\sigma}{\theta}}} \chi(2\tau^{-\frac{\sigma}{\theta}}\rho)(1-\chi(\tau^{1-\frac{\sigma}{\theta}}\rho))\\ &\qquad\qquad \times \rho^{a+\frac{n-1}{2}}\, e^{-\eta\rho^\theta + i\tau^{\sigma\left(1-\frac{\sigma}{\theta}\right)}\left(\pm\rho^\sigma+|x|\tau^{(1-\sigma)\left(1-\frac{\sigma}{\theta}\right)}\rho\right)}\, \hat g(\rho)\,d\rho\\ &\qquad + c_-\,\tau^{\left(a+\frac{n+1}{2}\right)\left(1-\frac{\sigma}{\theta}\right)} \int_{\tau^{\frac{\sigma}{\theta}-1}}^{\tau^{\frac{\sigma}{\theta}}} \chi(2\tau^{-\frac{\sigma}{\theta}}\rho)(1-\chi(\tau^{1-\frac{\sigma}{\theta}}\rho))\\ &\qquad\qquad \times \rho^{a+\frac{n-1}{2}}\, e^{-\eta\rho^\theta + i\tau^{\sigma\left(1-\frac{\sigma}{\theta}\right)}\left(\pm\rho^\sigma-|x|\tau^{(1-\sigma)\left(1-\frac{\sigma}{\theta}\right)}\rho\right)}\, \hat g(\rho)\,d\rho\,,\end{aligned}$$

where we used the change of variable $\rho \mapsto \tau^{1-\frac{\sigma}{\theta}}\rho$.

As in the proof of Proposition 7.1, for any $N \geq 1$ we may estimate

$$\left|\int_{\tau^{\frac{\sigma}{\theta}-1}}^{\tau^{\frac{\sigma}{\theta}}} \chi(2\tau^{-\frac{\sigma}{\theta}}\rho)(1-\chi(\tau^{1-\frac{\sigma}{\theta}}\rho))\,\rho^{a+\frac{n-1}{2}}\, e^{-\eta\rho^\theta \pm i\tau^{\sigma\left(1-\frac{\sigma}{\theta}\right)}\left(\rho^\sigma+|x|\tau^{(1-\sigma)\left(1-\frac{\sigma}{\theta}\right)}\rho\right)}\, \hat g(\rho)\,d\rho\right| \leq C_N\,\tau^{-N\sigma\left(1-\frac{\sigma}{\theta}\right)},$$

due to the lack of stationary points of the phase function $\rho^\sigma + |x|\tau^{(1-\sigma)(1-\frac{\sigma}{\theta})}\rho$. We stress that we implicitly used the boundedness of the function $\rho^{a+\frac{n-1}{2}}\, e^{-\eta\rho^\theta}$ and its derivatives to treat the integral at large values of ρ.

We proceed similarly for $e^{-\eta\rho^\theta \pm i\tau^{\sigma(1-\frac{\sigma}{\theta})}h(\rho)}$ out of the interval $(\rho_0-\delta, \rho_0+\delta)$, where now ρ_0 is the stationary point of the function

$$h(\rho) = \rho^\sigma - |x|\tau^{(1-\sigma)(1-\frac{\sigma}{\theta})}\rho,$$

so that $\rho_0 = \tau^{-\left(1-\frac{\sigma}{\theta}\right)}(|x|/\sigma)^{\frac{1}{\sigma-1}} \approx 1$.

We may now study the integral near the stationary point and we apply Taylor's formula for a sufficiently small $\delta > 0$, as in the proof of Proposition 7.1, to get:

$$J_\delta = \int_{\rho_0-\delta}^{\rho_0+\delta} \rho^{a+\frac{n-1}{2}}\, e^{-\eta\rho^\theta \pm i\tau^{\sigma(1-\frac{\sigma}{\theta})}h(\rho)}\, \hat{g}(\rho)d\rho = e^{\pm i h(\rho_0)} \int_{a_-}^{a_+} \psi(r)\, e^{\pm i\tau^{\sigma\left(1-\frac{\sigma}{\theta}\right)}r^2} dr,$$

where $\psi(r) \approx \rho_0^{a+\frac{n-1}{2}}\, e^{-\eta\rho_0^\theta}\, \hat{g}(\rho_0)$. Therefore, $|J_\delta| \geq c\,\tau^{-\frac{\sigma}{2}\left(1-\frac{\sigma}{\theta}\right)}$. Due to

$$|x|^{-\frac{n-1}{2}}\, \tau^{\left(-\frac{n}{p'}+a+\frac{n+1-\sigma}{2}\right)\left(1-\frac{\sigma}{\theta}\right)} \approx \tau^{\left(-\frac{n-1}{2}(\sigma-1)-n+\frac{n}{p}+a+\frac{n+1-\sigma}{2}\right)\left(1-\frac{\sigma}{\theta}\right)}$$

for $|x| \in I$, we conclude the proof of (35).

7.1. Remarks about the case $\sigma = 1$

In the case $\sigma = 1$, the proof of Proposition 7.1 does not work. However, we may prove the following result.

Proposition 7.3. *Let $\tau \gg 1$, and assume that $\omega_0 = \rho$ and $f = \rho^a$ in (24), where $a \in \mathbb{R}$. Let $1 \leq p \leq q \leq \infty$, and define*

$$\tilde{d}(p,q) = n\left(\frac{1}{p} - \frac{1}{q}\right) + (n-1)\max\left\{\frac{1}{2} - \frac{1}{p}, \frac{1}{q} - \frac{1}{2}\right\}.$$

If $\tilde{d}(p,q) + a > 0$, then there exists $c > 0$ such that

$$\|m_0(\tau,\cdot)\|_{M_p^q} \geq c\,\tau^{\tilde{d}+a}, \tag{36}$$

where $c > 0$ is independent of τ.

PROOF. The proof is inspired by [18, Lemma 7]. By duality, we may assume $1 \leq p \leq q \leq p'$. We fix g and g_τ, and we write Tg_τ, as in the proof of Proposition 7.1. By the change of variable $\rho \to \rho\tau$, we may deal with the leading terms $c_\pm$ of the asymptotic expansion of $J_{\frac{n-2}{2}}(|x|\rho)$. In particular,

$$\begin{aligned}
&|x|^{-\frac{n-1}{2}}\, \tau^{-\frac{n}{p'}} \int_1^\tau \chi(2\tau^{-1}\rho)(1-\chi(\rho))\, \rho^{a+\frac{n-1}{2}}\, e^{i(1\pm|x|)\rho}\, \hat{g}(\rho/\tau)\, d\rho \\
&= |x|^{-\frac{n-1}{2}}\, \tau^{-\frac{n}{p'}+a+\frac{n+1}{2}} \int_{1/\tau}^1 \chi(2\rho)(1-\chi(\tau\rho))\, \rho^{a+\frac{n-1}{2}}\, e^{i(1\pm|x|)\tau\rho}\, \hat{g}(\rho)\, d\rho \\
&= |x|^{-\frac{n-1}{2}}\, \tau^{-\frac{n}{p'}+a+\frac{n+1}{2}}\, h((1\pm|x|)\tau),
\end{aligned}$$

where

$$\hat{h}(\rho) = \chi(2\rho)(1-\chi(\tau\rho))\, \rho^{a+\frac{n-1}{2}}\, \hat{g}(\rho)\,.$$

Due to $\hat{h} \in L^1$, it follows that $h(|y|)$ vanishes as $|y| \to \infty$. In particular, $h((1+|x|)\tau) \to 0$ for any x, as $\tau \to \infty$, and $h((1-|x|)\tau) \to 0$ for any fixed $|x| \neq 1$ as $\tau \to \infty$. We know that h is nonzero in some point (otherwise $\hat{h} = 0$). Assume for brevity that $h(0) \neq 0$. There exists $\delta > 0$ such that $|h(|y|)| > c$ for $|y| \leq \delta$, so that

$$|x|^{-\frac{n-1}{2}}\, \tau^{-\frac{n}{p'}+a+\frac{n+1}{2}}\, |h((1\pm|x|)\tau)| \geq c\,\tau^{-\frac{n}{p'}+a+\frac{n+1}{2}}$$

for any $|x| \in I = [1-\delta/\tau, 1+\delta/\tau]$. Then

$$\|Tg_\tau\|_{L^q} \geq \|Tg_\tau\|_{L^q(I)} \approx \tau^{-\frac{n}{p'}+a+\frac{n+1}{2}} \Big(\int_{1-\frac{\delta}{\tau}\leq|x|\leq 1+\frac{\delta}{\tau}} 1\,dx\Big)^{\frac{1}{q}}$$
$$\approx \tau^{-n+\frac{n}{p}+a+\frac{n+1}{2}-\frac{1}{q}} = \tau^{n\left(\frac{1}{p}-\frac{1}{q}\right)+(n-1)\left(\frac{1}{q}-\frac{1}{2}\right)+a} = \tau^{\tilde{d}(p,q)+a}.$$

8. Multiplier theorems

In this section, we collect some results employed in the paper to prove that a function is a multiplier in M_p^q, basing on suitable estimates for the function and its derivatives.

A key result for multipliers in M_p^p with $p \in (1,\infty)$ is the Mikhlin-Hörmander multiplier theorem.

Theorem 8.1 (see Theorem 2.5 in [14]). *Let $1 < p < \infty$ and $k = [n/2]+1$. Suppose that $m \in \mathcal{C}^k(\mathbb{R}^n \setminus \{0\})$ and*

$$\left|\partial_\xi^\gamma m(\xi)\right| \leq C\,|\xi|^{-|\gamma|}, \quad |\gamma| \leq k.$$

Then $m \in M_p^p$.

Mikhlin-Hörmander multiplier theorem may be used together with Hardy-Littlewood-Sobolev theorem for the Riesz potential I_a, whose action may be defined by $I_a f = \mathscr{F}^{-1}(|\xi|^{-a}\hat{f})$, for $f \in \mathcal{S}$ and then extended by density.

Theorem 8.2. *Let $a \in (0,n)$ and $p \in (1, n/a)$. Then $|\xi|^{-a} \in M_p^q(\mathbb{R}^n)$, that is, $I_a \in L_p^q(\mathbb{R}^n)$, with q obtained by*

$$\frac{1}{q} = \frac{1}{p} - \frac{a}{n}.$$

A function m is a multiplier in M_1^1 if $\mathscr{F}^{-1}m \in L^1$. In particular, this is true if $m \in H^N$, for some $N > n/2$. The following inequality also provides an estimate for $\|\mathscr{F}^{-1}m\|_{L^1}$.

Theorem 8.3 (see, for instance, Theorem 1.2 in [28]). *Let $n \geq 1$ and $N > n/2$. Assume that $m \in H^N$, then $\mathscr{F}^{-1}m \in L^1$ and there exists a constant $C > 0$ such that*

$$\|\mathscr{F}^{-1}m\|_{L^1} \leq C\,\|m\|_{L^2}^{1-\frac{n}{2N}}\,\|D^N m\|_{L^2}^{\frac{n}{2N}}.$$

The estimates provided by Theorem 8.3 will be used together with the estimates for $\|\mathscr{F}^{-1}m\|_{L^\infty}$, provided by the following application of Littman's lemma, based on stationary phase methods.

Lemma 8.4 (see, for instance, [22]). *Let us consider for $\tau \geq \tau_0$, τ_0 is a large positive number, the oscillatory integral*

$$F^{-1}_{\xi\to x}\big(e^{-i\tau\omega(\xi)}v(\xi)\big).$$

The amplitude function $v = v(\xi)$ is supposed to belong to $\mathcal{C}_c^\infty(\mathbb{R}^n)$ with support in $\{\xi \in \mathbb{R}^n : |\xi| \in [\frac{1}{2}, 2]\}$. The function $\omega = \omega(\xi)$ is C^∞ in a neighborhood of the support of v. Moreover, the Hessian $H_\omega(\xi)$ is nonsingular, i.e., $\det H_\omega(\xi) \neq 0$, on the support of v. Then the following $L^\infty - L^\infty$ estimate holds:

$$\big\|F^{-1}_{\xi\to x}\big(e^{-i\tau\omega(\xi)}v(\xi)\big)\big\|_{L^\infty(\mathbb{R}^n_x)} \leq C(1+\tau)^{-\frac{n}{2}} \sum_{|\beta|\leq n+1} \|D^\beta v(\xi)\|_{L^\infty(\mathbb{R}^n_\xi)},$$

where the constant C depends on C^{n+2} norm of the phase function ω, on the lower bound for $|\det H_\omega(\xi)|$ and on the diameter of the support of v.

In [30, Proposition 2.5, Chapter 8], one can find a simple proof of Lemma 8.4, from which it is easy to check that the statement remains valid whenever ω and v depend on some parameter, with a constant C, uniform with respect to the parameter, provided that the derivatives of ω and v, as well as $|\det H_\omega(t,\xi)|$ can be estimated uniformly with respect to the parameter. This property for the Hessian appears in (27), in this manuscript.

By Young inequality, $\|m\|_{M_p^q} \leq \|\mathscr{F}^{-1}m\|_{L^r}$, with $1-1/r = 1/p-1/q$ (see [14, Corollary 1.2]), in particular the equality holds if $p=1<q$ or $p<q=\infty$, whereas $M_1^1 = M_\infty^\infty$ is the set of bounded measures. The following lemma provides an easy way to determine whether $\mathscr{F}^{-1}m \in L^r$, in particular, when $r=1$, for smooth compactly supported multipliers m.

Lemma 8.5. *Assume that $m \in \mathcal{C}_c^\kappa(\mathbb{R}^n)$ for some $\kappa \geq 0$, and that it verifies the estimates*

$$\forall |\alpha| \leq \kappa: \qquad |\partial_\xi^\alpha m(\xi)| \leq C\,|\xi|^{-a}, \quad \text{for some } a<n.$$

Then $g = \mathscr{F}^{-1}m$ satisfies the estimate $|g(x)| \leq C'\,(1+|x|)^{-\kappa}$. Moreover, if $m \in \mathcal{C}_c^{\kappa+1}$ and

$$\forall |\alpha| = \kappa+1: \qquad |\partial_\xi^\alpha m(\xi)| \leq C\,|\xi|^{-(n+\delta)}, \quad \text{for some } \delta \in [0,1),$$

then

$$|g(x)| \leq \begin{cases} C'\,(1+|x|)^{-(n-a+\kappa)} & \text{if } a > n-1+\delta, \\ C'\,(1+|x|)^{-(\kappa+1-\delta)} & \text{if } a \leq n-1+\delta \text{ and } \delta \in (0,1), \\ C'\,(1+|x|)^{-(\kappa+1)}\,\log(e+|x|) & \text{if } a \leq n-1 \text{ and } \delta = 0. \end{cases} \tag{37}$$

The proof is very straightforward and likely standard, but we include it for the ease of reading.

Proof. Due to $a<n$, by the compact support of m, we obtain $\partial_\xi^\alpha m \in L^1$ for $|\alpha| \leq \kappa$, so that $(1+|x|)^\kappa g \in \mathcal{C}_0$. This proves the first part of the statement.

Thanks to

$$e^{ix\xi} = -\sum_{j=1}^n \frac{ix_j}{|x|^2}\,\partial_{\xi_j} e^{ix\xi}, \tag{38}$$

after integrating by parts κ times, we may write

$$g(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\xi} m(\xi)\,d\xi = (2\pi)^{-n}|x|^{-\kappa} \sum_{|\gamma|=\kappa} c_\gamma \int_{\mathbb{R}^n} e^{ix\xi}\partial_\xi^\gamma m(\xi)\,d\xi,$$

where we used that m is compactly supported. We now split the integral in two parts and we apply one extra step of integration by parts in the latter integral:

$$\int_{\mathbb{R}^n} e^{ix\xi}\partial_\xi^\gamma m(\xi)\,d\xi = \int_{|\xi|\leq|x|^{-1}} e^{ix\xi}\partial_\xi^\gamma m(\xi)\,d\xi - \sum_{j=1}^n \frac{ix_j}{|x|^2} \int_{|\xi|=|x|^{-1}} e^{ix\xi}\partial_\xi^\gamma m(\xi)\,d\xi$$
$$+ \sum_{j=1}^n \frac{ix_j}{|x|^2} \int_{|\xi|\geq|x|^{-1}} e^{ix\xi}\partial_{\xi_j}\partial_\xi^\gamma m(\xi)\,d\xi\,.$$

Let $M > |x|^{-1}$ be such that $\operatorname{supp} m \subset \{|\xi| < M\}$. Then we may estimate

$$\int_{|\xi|\leq|x|^{-1}} |\partial_\xi^\gamma m(\xi)|\,d\xi \leq C \int_{|\xi|\leq|x|^{-1}} |\xi|^{-a}\,d\xi = C_1\,|x|^{-(n-a)},$$
$$|x|^{-1} \int_{|\xi|=|x|^{-1}} |\partial_\xi^\gamma m(\xi)|\,d\xi = |x|^{-1} \int_{|\xi|=|x|^{-1}} |\xi|^{-a}\,d\xi = C_2\,|x|^{-(n-a)},$$
$$|x|^{-1} \int_{|\xi|\geq|x|^{-1}} |\partial_{\xi_j}\partial_\xi^\gamma m(\xi)|\,d\xi = |x|^{-1} \int_{|x|^{-1}\leq|\xi|\leq M} |\xi|^{-(n+\delta)}\,d\xi = \begin{cases} C_3\,|x|^{-(1-\delta)} & \text{if } \delta>0, \\ C_3\,|x|^{-1}\log(M|x|) & \text{if } \delta=0. \end{cases}$$

This concludes the proof.

Remark 8.1. Assume that $|g(x)| \le C'(1+|x|)^{-n+\kappa}$ for some $\kappa < n$. Then, $g \in L^r$ for any $r \in [1,\infty)$ if $\kappa < 0$ and for $r > n/(n-\kappa)$ otherwise. By Young inequality, we get $m \in M_p^q$ for any $1 \le p \le q \le \infty$ if $\kappa < 0$ and for $n(1/p - 1/q) > \kappa$ otherwise. However, by Hardy-Littlewood-Sobolev theorem (see [14, Theorem 2.4]), we obtain $m \in M_p^q$, if $1 < p \le q < \infty$ and $n(1/p - 1/q) = \kappa$.

In particular, if

$$\forall \alpha: \qquad |\partial_\xi^\alpha m(\xi)| \le C_\alpha (1 + |\xi|^{b-|\alpha|}),$$

for some $b > -n$, then we may apply Lemma 8.5 with $\kappa = n - 1 + \lceil b \rceil$, i.e., κ is the biggest integer verifying $\kappa < n + b$. Setting $a = \kappa - b$ (we notice that $a \in [n-1, n)$) and $\delta = a + 1 - n$, we get

$$|g(x)| \le \begin{cases} C'(1+|x|)^{-n-b} & \text{if } b \text{ is not integer,} \\ C'(1+|x|)^{-n-b} \log(e+|x|) & \text{if } b \text{ is integer.} \end{cases}$$

As a consequence:

- if $b > 0$, then $m \in M_p^q$ for any $1 \le p \le q \le \infty$;
- if $b = 0$, then $m \in M_p^q$ for any $1 < p \le q < \infty$;
- if $b \in (-n, 0)$, then $m \in M_p^q$ if $n(1/p - 1/q) > -b$; moreover, if b is not an integer, and $1 < p \le q < \infty$, then $m \in M_p^q$ if $n(1/p - 1/q) \ge -b$.

9. High frequencies estimates

In this section, we briefly collect the $L^p - L^q$ estimates for the fundamental solution localized at high frequencies, namely, for $(K - K_0)(t,\cdot)$, where K_0 is as in (5). Due to the fact that $\hat{K}(t,\xi)$ is smooth in the support of $1-\chi(2|\xi|/\varepsilon_0)$, and that for any given $N \gg \varepsilon_0$, the real parts of the roots of (4) are strictly negative in the compact annulus $\{\xi : \varepsilon_0/2 \le |\xi| \le 2N\}$, we find that $\hat{K}_1(t,\xi) = (1 - \chi(2|\xi|/\varepsilon_0))\chi(|\xi|/N)\hat{K}(t,\xi)$ trivially verifies the multiplier estimate

$$\| |\xi|^b \partial_t^j \hat{K}_1(t,\cdot) \|_{M_p^q} \le C\, e^{-ct}, \quad t \ge 0,$$

for any $1 \le p \le q \le \infty$, for some $C > 0$ and $c > 0$. Therefore, we consider in the following $\hat{K}_\infty(t,\xi) = \varphi_\infty(\xi)\,\hat{K}(t,\xi)$, with $\varphi_\infty(\xi) = 1 - \chi(|\xi|/N)$. For sufficiently large N, namely, such that $N^{\theta-\sigma} > 2$, it holds

$$\hat{K}_\infty(t,\xi) = \varphi_\infty(\xi)\, \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}, \quad \lambda_\pm(\xi) = -\frac{1}{2}\,|\xi|^\theta \big(1 \mp \sqrt{1 - 4|\xi|^{-2(\theta-\sigma)}}\big), \tag{39}$$

as in (9). In particular,

$$\lambda_- \sim -|\xi|^\theta, \quad \lambda_+ \sim |\xi|^{2\sigma-\theta}. \tag{40}$$

If $\theta < 2\sigma$, a smoothing effect appears which, in particular, allow us to deal with higher derivatives of the solution and to get a $L^p - L^q$ estimate for any $1 \le p \le q \le \infty$, but the estimate is possibly singular at $t = 0$. The singularity at $t = 0$ is related to the fact that the smoothing effect requires some positive time to produces its effect. This phenomenon is analogous to what happens in the heat equation and other diffusive equations. In the limit case $\theta = 2\sigma$, the smoothing effect only influences the time derivatives.

Theorem 9.1. *Let $1 \le p \le q \le \infty$, $\sigma < \theta \le 2\sigma$, $j \in \mathbb{N}$, and $b \in \mathbb{R}$. Define*

$$a = n\left(\frac{1}{p} - \frac{1}{q}\right) + b. \tag{41}$$

Then we have the following estimate

$$\| |\xi|^b \partial_t^j \hat{K}_\infty(t,\cdot) \|_{M_p^q} \le C\, t^{-\delta}\, e^{-ct}, \qquad \forall t > 0, \tag{42}$$

where:

- *if $\theta < 2\sigma$ and $a \geq \theta$, then*
$$\delta = j + \frac{a-\theta}{2\sigma-\theta},$$
if $p, q \in (1, \infty)$, whereas δ may be any positive number verifying
$$\delta > j + \frac{a-\theta}{2\sigma-\theta},$$
if $p = 1$ or $q = \infty$;
- *if $\theta \leq 2\sigma$ and $a \leq \theta$, then*
$$\delta = \Big(j - 1 + \frac{a}{\theta}\Big)_+,$$
if $p, q \in (1, \infty)$, whereas δ may be any nonnegative number verifying
$$\delta > j - 1 + \frac{a}{\theta},$$
if $p = 1$ or $q = \infty$, provided that we assume $a < \theta$ and $1/p - 1/q \geq 1/2$ if $\theta = 2\sigma$.

If we are interested in non-singular estimates, it is sufficient to take $\delta = 0$ in Theorem 9.1.

PROOF. It is not restrictive to assume that $1 \leq p \leq 2$ and $p \leq q \leq p'$, since $M_{q'}^{p'} = M_p^q$. Assume first that $p > 1$ (so that $q < \infty$ as well). Then
$$\||\xi|^b \partial_t^j \hat{K}_\infty(t,\cdot)\|_{M_p^q} \leq C\, \||\xi|^a \partial_t^j \hat{K}_\infty(t,\cdot)\|_{M_q^q},$$
by Hardy-Littlewood-Sobolev theorem (Theorem 8.2). Noticing that
$$\partial_t^j \hat{K}_\infty(t,\cdot) = \varphi_\infty \left(\frac{\lambda_+^j}{\lambda_+ - \lambda_-}\, e^{\lambda_+ t} - \frac{\lambda_-^j}{\lambda_+ - \lambda_-}\, e^{\lambda_- t} \right),$$
we employ (40) to obtain
$$\big|\partial_\xi^\gamma \big(|\xi|^a\, \partial_t^j \hat{K}_\infty(t,\xi)\big)\big| \leq C\, |\xi|^{a-\theta-|\gamma|} \Big(|\xi|^{(2\sigma-\theta)j}\, e^{-ct|\xi|^{2\sigma-\theta}} + |\xi|^{\theta j}\, e^{-ct|\xi|^\theta} \Big), \tag{43}$$
for any $|\xi| \geq N$, for some $C, c > 0$. Let $\kappa = 2\sigma - \theta, \theta$. We multiply and divide by t^δ with δ as in the statement, and we use
$$t^\delta |\xi|^{\delta\kappa}\, e^{-\frac{c}{2}t|\xi|^\kappa} \leq C, \quad e^{-\frac{c}{2}t|\xi|^\kappa} \leq e^{-c_1 t},$$
where the latter one is a consequence of $|\xi| \geq N$. In turn, we obtain
$$\begin{aligned} \big|\partial_\xi^\gamma \big(|\xi|^a\, \partial_t^j \hat{K}_\infty(t,\xi)\big)\big| &\leq C\, |\xi|^{a-\theta-|\gamma|} \Big(|\xi|^{(2\sigma-\theta)(j-\delta)} + |\xi|^{\theta(j-\delta)} \Big)\, t^{-\delta}\, e^{-c_1 t} \\ &\leq C\, |\xi|^{-|\gamma|}\, t^{-\delta}\, e^{-c_1 t}, \end{aligned} \tag{44}$$
and the proof follows by Mikhlin-Hörmander multiplier theorem (Theorem 8.1). If $p = 1$, then $\|\cdot\|_{M_1^q} = \|\cdot\|_{L^q}$ for $q > 1$ (see [14, Theorem 1.4]) and $\|\cdot\|_{M_1^1} \leq \|\cdot\|_{L^1}$ by the properties of the convolution product. The proof follows estimating
$$\|(-\Delta)^{\frac{b}{2}} \partial_t^j K_\infty(t,\cdot)\|_{L^q} \leq C\, t^{-\delta}\, e^{-c_1 t}.$$
For $q \geq 2$, the estimate above is an immediate consequence of Haussdorff-Young inequality:
$$\begin{aligned} \|(-\Delta)^{\frac{b}{2}} \partial_t^j K_\infty(t,\cdot)\|_{L^q} &\leq C\, \||\xi|^b \partial_t^j \hat{K}_\infty(t,\cdot)\|_{L^{q'}} \\ &\leq C\, t^{-\delta}\, e^{-c_1 t} \left(\int_{|\xi| \geq N} \Big| |\xi|^{b-\theta} \Big(|\xi|^{(2\sigma-\theta)(j-\delta)} + |\xi|^{\theta(j-\delta)} \Big) \Big|^{q'} \right)^{\frac{1}{q'}} \leq C_1\, t^{-\delta}\, e^{-c_1 t}. \end{aligned}$$

For $q \in [1,2)$ the desired estimate may be easily obtained integrating by parts, if $\theta < 2\sigma$. Indeed, we notice that $|\xi|^b \hat{K}_\infty(t,\cdot) \in \mathcal{S} \subset L^1$, for any $t > 0$, if $\theta < 2\sigma$, thanks to the exponentially decaying terms in (43). Therefore, the Fourier inversion formula is valid and we may use an integration by parts argument analogous to Lemma 8.5.

We give the proof for $q = 1$, being the proof for $q \in (1,2)$ analogous.

Let $g = t^\delta\, e^{c_1 t}\, (-\Delta)^{\frac{b}{2}} \partial_t^j K_\infty(t,\cdot)$ and $m = \hat{g}$. Setting $|\gamma| = n+1$ in (43) and integrating by parts, we find that

$$|g(t,x)| \lesssim |x|^{-(n+1)} \sum_{|\gamma|=n+1} \int_{|\xi|\geq N} |\partial_\xi^\gamma m(t,\xi)|\, d\xi \leq C\, |x|^{-(n+1)},$$

with $C > 0$ independent of t. On the other hand, integrating first $n-1$ times, splitting in two integrals and integrating by parts one more time in one of the two integrals (similarly to what we do in the proof of Lemma 8.5), we obtain

$$\begin{aligned}
|g(t,x)| &\lesssim |x|^{-(n-1)} \sum_{|\gamma|=n-1} \int_{N\leq|\xi|\leq|x|^{-1}} |\partial_\xi^\gamma m(t,\xi)|\, d\xi \\
&\quad + |x|^{-n} \sum_{j=1}^{n} \sum_{|\gamma|=n-1} \int_{|x|^{-1}\leq|\xi|} |\partial_{\xi_j}\partial_\xi^\gamma m(t,\xi)|\, d\xi \\
&\quad + |x|^{-n} \sum_{|\gamma|=n-1} \int_{|\xi|=|x|^{-1}} |\partial_\xi^\gamma m(t,\xi)|\, d\xi \\
&\lesssim |x|^{-(n-1)} \int_{N\leq|\xi|\leq|x|^{-1}} |\xi|^{b-\theta-(n-1)} \left(|\xi|^{(2\sigma-\theta)(j-\delta)} + |\xi|^{\theta(j-\delta)}\right) d\xi \\
&\quad + |x|^{-n} \int_{|x|^{-1}\leq|\xi|} |\xi|^{b-\theta-n} \left(|\xi|^{(2\sigma-\theta)(j-\delta)} + |\xi|^{\theta(j-\delta)}\right) d\xi \\
&\quad + |x|^{-n} \int_{|\xi|=|x|^{-1}} |\xi|^{b-\theta-(n-1)} \left(|\xi|^{(2\sigma-\theta)(j-\delta)} + |\xi|^{\theta(j-\delta)}\right) d\xi \\
&\lesssim |x|^{\varepsilon-n},
\end{aligned}$$

for some $\varepsilon > 0$ where we used that $b - \theta + (2\sigma-\theta)(j-\delta) < 0$ and $b - \theta + 2\theta(j-\delta) < 0$. Therefore, due to $|g(t,x)| \leq C\, \min\{|x|^{\varepsilon-n}, |x|^{-(n+1)}\}$, we find $\|g(t,\cdot)\|_{L^1} \leq C$, with $C > 0$, independent of t.

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References

[1] P. Brenner, On $L_p - L_{p'}$ estimates for the wave equation. Math. Z. 145 (1975), 251–254.

[2] R.C. Charão, C.R. da Luz, *Asymptotic properties for a semilinear plate equation in unbounded domains*, J. Hyperbolic Differ. Equ. **6** (2009) 269–294.

[3] R.C. Charão, C.R. da Luz, R. Ikehata, *New decay rates for a problem of plate dynamics with fractional damping*, J. Hyperbolic Differ. Equ. **10**, 3 (2013), 563–575.

[4] R.C. Charão, J.L. Horbach, R. Ikehata, *Optimal decay rates and asymptotic profile for the plate equation with structural damping*, J. Math. Anal. Appl. **440** (2016), 529–560.

[5] R. Chill and A. Haraux, An optimal estimate for the difference of solutions of two abstract evolution equations, J. Differential Equations, 193 (2003), 385–395.

[6] P. G. Ciarlet, *A justification of the von Kármán equations*, Arch. Rational Mech. Anal. **73** (1980), 349–389.
[7] E. Cordero, D. Zucco, *Strichartz estimates for the vibrating plate equation*, J. Evol. Equ. **11** (2011), 827–845.
[8] E. Cordero, D. Zucco, *The Cauchy problem for the vibrating plate equation in modulation spaces*, J. Pseudo-Differ. Oper. Appl. **2** (2011), 343-354.
[9] M. D'Abbicco, M.R. Ebert, *A classification of structural dissipations for evolution operators*, Math. Meth. Appl. Sci. **39** (2016), 2558-2582, http://dx.doi.org/10.1002/mma.3713.
[10] M. D'Abbicco, M.R. Ebert, A new phenomenon in the critical exponent for structurally damped semi-linear evolution equations. Nonlinear Analysis, Theory, Methods and Applications, 149 (2017), 1–40.
[11] M. D'Abbicco, G. Girardi, J. Liang, $L^1 - L^1$ *estimates for the strongly damped plate equation*, J. Math. Anal. Appl. **478** (2019), 2, 476–498, https://doi.org/10.1016/j.jmaa.2019.05.039.
[12] M. D'Abbicco, M. Reissig, *Semilinear structural damped waves*, Math. Meth. Appl. Sci. **37** (2014), 1570–1592, http://dx.doi.org/10.1002/mma.2913.
[13] M. Ghisi, M. Gobbino, A. Haraux, *Local and global smoothing effects for some linear hyperbolic equations with a strong dissipation*, Trans. Amer. Math. Soc. **368**, 3 (2016), 2039–2079.
[14] L. Hörmander, Estimates for translation invariant operators in L^p spaces. Acta Mathematica **104** (1960), 93–140.
[15] R. Ikehata, *A note on optimal* L^2*-estimates of solutions to some strongly damped* σ*-evolution equations*, Asymptotic Analysis **121** (2021) 59–74.
[16] R. Ikehata, G. Todorova, B. Yordanov, *Wave equations with strong damping in Hilbert spaces*, J. Differential Equations **254** (2013), 3352–3368.
[17] I. Lasiecka, *Uniform stability of a full von Kármán system with nonlinear boundary feedback*, SIAM J. Control Optim. **36** (1998), 1376–1422.
[18] B. Marshall, W. Strauss, S. Wainger, $L^p - L^q$ *estimates for the Klein-Gordon equation,* J. Math Pure Appl. **59** (1980), 417–440.
[19] A. Miyachi, *On some estimates for the wave equation in* L^p *and* H^p, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), 331–354.
[20] A. Miyachi, *On some singular Fourier multiplier*, Journal of the Faculty of Science, the University of Tokyo. Sect. 1 A, Mathematics **28**, 2 (1981), 267–315.
[21] T. Narazaki, M. Reissig, L^1 *estimates for oscillating integrals related to structural damped wave models*, in Studies in Phase Space Analysis with Applications to PDEs, Cicognani M, Colombini F, Del Santo D (eds), Progress in Nonlinear Differential Equations and Their Applications. Birkhäuser, 2013; 215–258.
[22] H. Pecher, L^p-Abschätzungen und klassische Lösungen für nichtlineare Wellengleichungen. I. Math. Z. 150 (1976), 159–183.
[23] J. Peral, L^p estimates for the Wave Equation, J. Funct. Anal. 36, 114–145 (1980).
[24] Pham Trieu Duong, M. Kainane and M. Reissig, *Global existence for semi-linear structurally damped* σ*-evolution models*, J. Math. Anal. Appl. **431** (2015), 569–596.
[25] J. P. Puel and M. Tucsnak, *Global existence for full von Kármán system*, Appl. Math. Optim. **34** (1996), 139–160.
[26] Y. Shibata, *On the rate of decay of solutions to linear viscoelastic equation*, Math. Meth. Appl. Sci. **23** (2000), 203–226.
[27] W. Sickel, H. Triebel, Hölder inequalities and sharp embeddings in function spaces of $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ type. Z. Anal. Anwendungen 14 (1995), 105–140.
[28] S. Sjöstrand, *On the Riesz means of the solution of the Shrödinger equation*, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze, 3^e série, **24** (1970), 2, 331–348.
[29] E.M. Stein, Harmonic Analysis: Real-variable methods, orthogonality, and oscillatory integrals. Princeton Univ. press, 1993.
[30] E. M. Stein and Rami Shakarchi, Functional analysis, Introduction to further topics in analysis.volume 4 of Princeton Lectures in Analysis. Princeton University Press, Princeton, NJ, 2011.
[31] Y. Sugitani and S. Kawashima, *Decay estimates of solutions to a semi-linear dissipative plate equation*, J. Hyperbolic Differ. Equ. **7** (2010) 471–501.
[32] H. Triebel. Theory of function spaces. Basel, Birkhäuser, 1983.