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Identities and central polynomials with involution for the Grassmann algebra

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ABSTRACT

Let E be the infinite dimensional Grassmann algebra over an infinite field of characteristic p different from 2. Given an involution φ on E, denote by $Id(E,\varphi)$ and $C(E,\varphi)$ the set of all $*$-polynomial identities and $*$-central polynomials of (E,φ) respectively. In this paper we describe $Id(E,\varphi)$ and $C(E,\varphi)$. Moreover, we prove that $C(E,\varphi)$ is not finitely generated as a $T(*)$-space if $p > 2$.

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$*$-central polynomials
$*$-polynomial identities

1. Introduction

In what follows F will always denote a field and every algebra or vector space is intended to be over F.

The goal of this paper is describing the full set of $*$-identities and $*$-central polynomials for the infinite dimensional Grassmann algebra over an infinite field endowed with an involution.

The so called central polynomials play an important role as in PI-theory as in a general algebraic setting. We recall that given an associative algebra A and a polynomial f in some non-commutative variables, f is a *central polynomial* for the algebra A if it takes values in the center of A only.

In a celebrated work (see [20]), Kaplansky conjectured that $M_n(F)$, the matrix algebra of size n over the field F, has central polynomials which are not polynomial identities. This conjecture turned out to be true because of the two independent works by Formanek (see [16]) and Razmyslov (see [27]) in the first part of the decade of 70's.

The importance of central polynomials can be revealed, for instance, also in the paper [8] by Kanel-Belov, Malev and Rowen. More precisely, it was conjectured, again by Kaplansky, that if f is a multilinear polynomial in some non-commutative variables with coefficients in a quadratically closed field F of any characteristic, then the image of f evaluated on $M_n(F)$ is either the set of matrices of trace zero or all of $M_n(F)$, or the 0 matrix (f is a polynomial identity), or the set of scalar matrices (f is a central polynomial). In [8] the authors proved this conjecture in the case $n = 2$.

Some results concerning the construction of some classes of central polynomials for specific algebras can be found in the paper [26] by Razmyslov or in the paper [6] by Belov. We also refer to Drensky and Formanek's book [13] or the one by Giambruno and Zaicev (see [18]) for other results concerning the theory of central polynomials.

Although this amount of literature, the precise structure of the central polynomials of an algebra A, denoted by $C(A)$, is far from being understood. If F is a field of characteristic 0 we may cite the paper [25] in which the author described the generators of $C(M_2(F))$, and the paper [15] by Formanek where he exhibited the module structure of the center of the generic matrix algebra of order 2. If F is an infinite field of characteristic $\neq 2$ we have the work [11] by Colombo and Koshlukov in which the complete set of generators for $C(M_2(F))$ is given. With respect to infinite dimensional Grassmann algebra E, in [9] the authors showed up a finite set of polynomials generating $C(E)$ as a T-space (vector space that is invariant under the endomorphisms of the free algebra) when F is a field of characteristic 0, and an infinite list of polynomials generating $C(E)$ as a T-space when F is infinite of characteristic $p > 2$. We recall the latter result turned out to be the first example of an infinitely generated T-space which is the set of central polynomials of an algebra.

We would also like to highlight the relevance of the Grassmann algebra as in mathematics as in other areas of scientific research. Indeed the role played by the infinite dimensional Grassmann algebra E is crucial in PI-theory. In one of his famous works (see [23] or his monograph [22]) Kemer proved that any variety $\mathfrak{V}$ of associative algebras is finitely generated when F has characteristic 0. In particular the identities of $\mathfrak{V}$ are the same as the identities of the so called Grassmann envelope of a finite dimensional superalgebra. The Grassmann algebra plays also the same crucial role for other kinds of "weaker" identities such as graded identities (see [3]), or identities of algebra with a module algebra action of a Hopf algebra (see [21]) and last but not least identities of algebras with involution (see [2]).

The systematic use of involutions in an algebra structure dates back to a paper by Albert (see [1]) dealing with the genus of a certain surface and the main technical problem solved in that paper was to give necessary and sufficient conditions on a division algebra over the rational field to be a multiplication algebra. The use of polynomials associated to involutions only appeared some decades later the paper by Albert whereas nowadays is largely studied especially after the work [31] by Zelmanov.

We have exploited now the importance of the three main characters of this work, then we would like to make explicit the results obtained in.

Let $Y = \{y_1, y_2, \dots\}$ and $Z = \{z_1, z_2, \dots\}$ be two disjoint countable sets of indeterminates. Denote by $F\langle Y \cup Z\rangle$ the free unitary associative algebra freely generated by $Y \cup Z$ over F endowed with the involution $*$, where

$$y_i^* = y_i \text{ and } z_i^* = -z_i$$

for all i. Let E be the infinite dimensional Grassmann algebra over an infinite field F of characteristic p different from 2. Given an involution φ on E, denote by $Id(E, \varphi)$ and $C(E, \varphi)$ the subsets of $F\langle Y \cup Z\rangle$ formed by all $*$-polynomial identities and $*$-central polynomials of (E, φ) respectively. In this paper we exhibit a finite set of generators of $Id(E, \varphi)$ as a $T(*)$-ideal. Moreover, we exhibit a finite set of generators of $C(E, \varphi)$ as a $T(*)$-space when $p = 0$ and an infinite set of generators when $p > 2$. We remark that Anisimov described $Id(E, \varphi)$ when the ground field has characteristic 0 (see [4]) but here we give a new proof. We also prove that $C(E, \varphi)$ is not finitely generated as a $T(*)$-space when $p > 2$ and this is the first example of an algebra whose $*$-central polynomials are not finitely generated as a $T(*)$-space.

On this purpose we would like to recall that infinitely generated T-ideals firstly were found by Belov [7], Grishin [19] and Shchigolev [29]. Finally in [30] Shchigolev showed some examples of infinitely generated T-spaces constructed via the Grassmann algebra.

2. Preliminaries

Let $X = \{x_1, x_2, \dots\}$ be a countable set of indeterminates. Denote by $F\langle X\rangle$ the free unitary associative algebra freely generated by X over F. The elements of $F\langle X\rangle$ are called *polynomials*. Let A be an F-algebra, then a polynomial $f = f(x_1, \dots, x_n)$ is said to be a *polynomial identity* of A if $f(a_1, \dots, a_n) = 0$ for all $a_1, \dots, a_n \in A$.

We shall denote by $Id(A)$ the subset of $F\langle X\rangle$ formed by all polynomial identities of A. In particular $Id(A)$ is a *T-ideal* (transformation ideal) of $F\langle X\rangle$ that is an ideal invariant under all endomorphisms of $F\langle X\rangle$. Moreover, if $S \subseteq F\langle X\rangle$, then $\langle S\rangle^T$, the smallest T-ideal containing S is said to be the T-ideal of $F\langle X\rangle$ *generated by* S. If $Id(A) \neq \{0\}$ for a given algebra A, then A is said to be a *PI-algebra*. Analogously we can speak about *T-spaces*, i.e., linear subspaces of $F\langle X\rangle$ that are invariant under all endomorphisms of $F\langle X\rangle$. Again if $S \subseteq F\langle X\rangle$, then $\langle S\rangle^{TS}$, the smallest T-space containing S is said to be the T-space of $F\langle X\rangle$ generated by S.

It is time to introduce one of the main character of this paper. If $\mathrm{char}(F) \neq 2$, we consider the ideal I of $F\langle X\rangle$ generated by the set $\{x_i x_j + x_j x_i | 1 \leq i, j\}$ and we denote by E the quotient algebra $F\langle X\rangle/I$. The latter is an infinite dimensional unitary algebra called *Grassmann algebra* and is generated by the following set $\{\xi_1,\ \xi_2,\ \dots\}$ over F, where $\xi_i = x_i + I$. Indeed the Grassmann algebra E is a PI-algebra and the complete set of generators of $Id(E)$ as a T-ideal is known and we shall present it soon below (see for instance [24,17]) but first let us recall the definition of left-normed commutators. Let $[x_1, x_2] := x_1x_2 - x_2x_1$ being the Lie commutator, then let us define the *left-normed commutator* of degree n by

$$[x_1, \dots, x_n] := [[x_1, \dots, x_{n-1}], x_n],$$

where $[x_1, \dots, x_{n-1}]$ is defined by induction.

Theorem 2.1. *If F is an infinite field of characteristic different from 2, then*

$$Id(E) = \langle [x_1, x_2, x_3]\rangle^T.$$

An interesting and well known fact is that the next two polynomials

$$[x_1, x_2][x_2, x_3] \text{ and } [x_1, x_2][x_3, x_4] + [x_1, x_3][x_2, x_4]$$

belong to $Id(E)$. Then it turns out

$$[x_1, x_2]x_3[x_4, x_1] = [x_1, x_2, x_3][x_4, x_1] + x_3[x_1, x_2][x_4, x_1] \in Id(E).$$

Hence we get the next result which intent will be shown later on.

Lemma 2.2. *If F is an infinite field of characteristic different from 2, then*

$$[x_1, x_2]x_3[x_4, x_1] \in Id(E).$$

We will also need the following pretty obvious result.

Lemma 2.3. *Let F be a field of characteristic $p > 2$. For all $g_1,\ g_2 \in F\langle X\rangle$ we have*

$$(g_1g_2)^p + Id(E) = g_1^p g_2^p + Id(E).$$

We recall once again that a *central polynomial* for an algebra A is a polynomial $f(x_1, \dots, x_n) \in F\langle X\rangle$ such that

$$f(a_1, \dots, a_n) \in Z(A)$$

for all $a_1, \dots, a_n \in A$, where $Z(A)$ denotes the center of A. We denote by $C(A)$ the set of all central polynomials for A. Indeed $C(A)$ is a vector space carrying the property of being invariant under endomorphisms of $F\langle X\rangle$, i.e., it is a T-space. A complete set of generators of $C(E)$ as a T-space is also known if F is an infinite field. Here we have a sharp difference between the characteristic 0 case and the case F is a field of characteristic $p > 2$. Below we will show the content of two of the main results of [9].

Theorem 2.4. *If F is a field of characteristic 0, then*

$$C(E) = \langle 1,\ [x_1, x_2],\ x_1[x_2, x_3, x_4]\rangle^{TS}.$$

Let

$$q(x_1, x_2) := x_1^{p-1}[x_1, x_2]x_2^{p-1}$$

and for each $n \geq 1$

$$q_n = q_n(x_1, \dots, x_{2n}) := q(x_1, x_2)q(x_3, x_4)\cdots q(x_{2n-1}, x_{2n}).$$

Then we have the next.

Theorem 2.5. *If F is an infinite field of characteristic $p > 2$, then*

$$C(E) = \langle\{x_1[x_2, x_3, x_4], x_0^p, x_0^p q_n | n \geq 1\}\rangle^{TS}.$$

Moreover $C(E)$ is not finitely generated as a T-space.

Remark 2.6. Let F be an infinite field of $\mathrm{char}(F) = p > 2$. Denote by $\mathbb{I}$ the ideal of $F\langle X\rangle$ generated by the elements f^p where $f = f(x_1, \dots, x_n) \in F\langle X\rangle$ and $f(0, \dots, 0) = 0$. Let W_n be the T-space in $F\langle X\rangle$ generated by

$$x_0^p,\ x_0^p q_1,\ x_0^p q_2,\ \dots, x_0^p q_n.$$

By using Lemma 13 of [30], the authors showed in [9, Page 136] that

$$q_{n+1} \notin W_n + Id(E) + \mathbb{I}. \tag{1}$$

The aim of this paper is to study polynomial identities and central polynomials for E over an infinite field once E is endowed with an involution $*$ and Remark 2.6 will be useful in the proof of the analog of Theorem 2.5 for $*$-central polynomials.

3. $T(*)$-ideals

In order to study polynomial identities and central polynomials of algebras endowed with an involution, we need a weaker notion of polynomial. To do so we fix a field F of characteristic different from 2. Let $Y = \{y_1, y_2, \ldots\}$ and $Z = \{z_1, z_2, \ldots\}$ be two disjoint countable sets of indeterminates. Denote by $F\langle Y \cup Z\rangle$ the free unitary associative algebra freely generated by $Y \cup Z$ over F endowed with the involution $*$, where

$$y_i^* = y_i \text{ and } z_i^* = -z_i$$

for all i. The elements of $F\langle Y \cup Z\rangle$ are called *$*$-polynomials*.

Given an algebra A with involution $\circ$ and $S \subseteq A$, we write

$$S^{(+,\circ)} = \{a \in S \mid a^\circ = +a\} \text{ and } S^{(-,\circ)} = \{a \in S \mid a^\circ = -a\}. \tag{2}$$

Of course the elements of $S^{(+,\circ)}$ are called *symmetric* whereas the elements of $S^{(-,\circ)}$ are called *anti-symmetric*. As in the ordinary case, a $*$-polynomial $f = f(y_1, \ldots, y_m, z_1, \ldots, z_n)$ is said to be a *$*$-polynomial identity* for the involution algebra $(A, \circ)$ if

$$f(Y_1, \ldots, Y_m, Z_1, \ldots, Z_n) = 0$$

for all $Y_1, \ldots, Y_m \in A^{(+,\circ)}$ and $Z_1, \ldots, Z_n \in A^{(-,\circ)}$. We denote by $Id(A, \circ)$ the set of all $*$-polynomial identities for $(A, \circ)$. This set is a *$T(*)$-ideal*, that is, a $*$-ideal of $F\langle Y \cup Z\rangle$ closed under all $*$-endomorphisms of $F\langle Y \cup Z\rangle$.

Given a subset $S \subseteq F\langle Y \cup Z\rangle$ we denote by $\langle S\rangle^{T(*)}$ the $T(*)$-ideal generated by S, i.e., the smallest $T(*)$-ideal containing S. Note that $\langle S\rangle^{T(*)}$ is the vector subspace of $F\langle Y \cup Z\rangle$ spanned by all elements

$$uf(g_1, \ldots, g_m, h_1, \ldots, h_n)v$$

where $u, v \in F\langle Y \cup Z\rangle$; $g_1, \ldots, g_m \in F\langle Y \cup Z\rangle^{(+,*)}$; $h_1, \ldots, h_n \in F\langle Y \cup Z\rangle^{(-,*)}$; $f(y_1, \ldots, y_m, z_1, \ldots, z_n) \in S \cup S^*$, where $S^* = \{f^* : f \in S\}$.

In [4, Corollary 4.4] the author exhibits a complete set of generators of $*$-polynomial identities of the Grassmann algebra with involution in the case the ground field is of characteristic zero.

Theorem 3.1. *If $char(F) = 0$ and φ is an involution on E, then*

$$Id(E, \varphi) = \langle [y_1 + z_1, y_2 + z_2, y_3 + z_3\rangle^{T(*)}.$$

Throughout this paper we will also give a new proof of the previous result and we will also describe $Id(E,\varphi)$ when F is any infinite field. We start off with some well known definitions.

A polynomial $f = f(y_1, \ldots, y_m, z_1, \ldots, z_n) \in F\langle Y \cup Z\rangle$ is said to be *multihomogeneous* of *multidegree* $d = (d_{y_1}, \ldots, d_{y_m}, d_{z_1}, \ldots, d_{z_n})$ if $\deg_{y_i} w = d_{y_i}$ and $\deg_{z_i} w = d_{z_i}$ for all monomials w of f and all i. If $d = (1, \ldots, 1, 1, \ldots, 1)$, then f is said to be *multilinear*.

If $f \in F\langle Y \cup Z\rangle$ is any polynomial, we can write

$$f = \sum_d f^{(d)},$$

where $f^{(d)}$ is the multihomogeneous component of f with multidegree d. By using the same argument as in [12, Proposition 4.2.3] we can prove that: if F is infinite then

$$\langle f\rangle^{T(*)} = \langle f^{(d)}; d\rangle^{T(*)}. \tag{3}$$

Moreover, if $\mathrm{char}(F) = 0$ and f is multihomogeneous, then $\langle f\rangle^{T(*)} = \langle \widehat{f}\rangle^{T(*)}$ for some multilinear polynomial $\widehat{f}$. Thus, if J is a $T(*)$-ideal, then J is generated by its multihomogeneous elements if F is infinite, and J is generated by its multilinear elements if $\mathrm{char}(F) = 0$.

Lemma 3.2. *If F is an infinite field of characteristic different from 2, then*

$$\langle [y_1 + z_1, y_2 + z_2, y_3 + z_3]\rangle^{T(*)} = \langle [x_1, x_2, x_3] |\ x_1, x_2, x_3 \in Y \cup Z\rangle^{T(*)}.$$

Proof. It is a direct consequence of (3). □

A polynomial $f \in F\langle Y \cup Z\rangle$ is called *Y-proper* if it is a linear combination of polynomials

$$z_1^{a_1} z_2^{a_2} \cdots z_m^{a_m} c_1 c_2 \cdots c_t,$$

where $a_1, a_2, \ldots, a_m \geq 0$, $m \geq 0$, $t \geq 0$ and $c_1, c_2, \ldots, c_t$ are commutators. Let us denote by B_Y the set of all Y-proper polynomials.

The following lemma can be found in [14, Lemma 2.1].

Lemma 3.3. *If F is an infinite field of characteristic different from 2 and J is a $T(*)$-ideal, then J is generated, as a $T(*)$-ideal, by its multihomogeneous Y-proper elements.*

Consider the following order $<$ on $Y \cup Z$:

$$y_1 < y_2 < \ldots < z_1 < z_2 < \ldots.$$

Lemma 3.4. *Let F be an infinite field of characteristic different from 2 and let J be a $T(*)$-ideal such that*

$$[y_1 + z_1, y_2 + z_2, y_3 + z_3] \in J.$$

Then the quotient vector space $B_Y/(B_Y \cap J)$ is spanned by the elements

$$z_1^{a_1} z_2^{a_2} \cdots z_m^{a_m} [x_1, x_2][x_3, x_4] \cdots [x_{2n-1}, x_{2n}] + J,$$

where $x_1 < x_2 < x_3 < \ldots < x_{2n}$ are in $Y \cup Z$, $a_1, a_2, \ldots, a_m \geq 0$, $m \geq 0$ and $n \geq 0$.

Proof. The proof is similar to that of [12, Theorem 5.1.2] by using Lemma 3.2. □

4. Involutions on E

In this section, F will denote an infinite field of characteristic different from 2. Let E be the unitary Grassmann algebra generated by the infinite set $\{\xi_1, \ \xi_2, \ \ldots\}$. Then E has a linear basis $\mathcal{B}$ formed by 1 and all elements

$$\xi = \xi_{i_1} \xi_{i_2} \cdots \xi_{i_k},$$

where $1 \leq i_1 < i_2 < \ldots < i_k$. We say k is the *length* of ξ.

Let φ be an involution on E. Firstly, since φ is onto, we have $\varphi(1) = 1$. Denote by L the vector space spanned by $\{\xi_1, \ \xi_2, \ \ldots\}$. Let us write

$$\varphi(\xi_j) = \sum_{i=1}^{\infty} \alpha_{ij} \xi_i + c_j,$$

where c_j is a linear combination of elements in $\mathcal{B}$ with length ≥ 2 and $\alpha_{ij} \in F$. The linear map $\varphi_l : L \to L$ defined by

$$\varphi_l(\xi_j) = \sum_{i=1}^{\infty} \alpha_{ij} \xi_i$$

is a linear isomorphism. In particular we get a new involution on E simply extending φ_l to E. For the sake of notations we shall denote this new involution by φ_l too. See [4, Lemma 2.1] and [4, Remark on page 4216] for details. The next fact is well known.

Lemma 4.1. *The vector space L has a linear basis $\Omega = \{e_1, \ e_2, \ \ldots\}$ consisting of symmetric and skew-symmetric elements with respect to φ_l, that is,*

$$\varphi_l(e_j) = \pm e_j$$

for all j.

From now on, we fix a linear basis $\Omega = \{e_1,\ e_2,\ \dots\}$ for L as in Lemma 4.1. The unitary subalgebra of E generated by Ω is E and $e_ie_j = -e_je_i$ for all i, j. Let D be the set formed by 1 and all elements

$$v = e_{i_1}e_{i_2}\cdots e_{i_k}$$

where $1 \leq i_1 < \ldots < i_k$, $k \geq 1$. Then D is a basis for E and $\varphi_l(v) = \pm v$. Moreover,

$$\varphi(v) = \varphi_l(v) + d = \pm v + d, \tag{4}$$

where d is a linear combination of elements of D with length $\geq k+1$.

Lemma 4.2. *Let $v \in D$ and $e_i, e_j \in \Omega$ such that $ve_ie_j \neq 0$. Then we have:*
a) If $v \in D^{(+,\varphi_l)}$ and $e_i, e_j \in \Omega^{(+,\varphi_l)}$, then $ve_ie_j \in D^{(-,\varphi_l)}$.
b) If $v \in D^{(-,\varphi_l)}$ and $e_i, e_j \in \Omega^{(+,\varphi_l)}$, then $ve_ie_j \in D^{(+,\varphi_l)}$.
c) If $v \in D^{(+,\varphi_l)}$ and $e_i, e_j \in \Omega^{(-,\varphi_l)}$, then $ve_ie_j \in D^{(-,\varphi_l)}$.
d) If $v \in D^{(-,\varphi_l)}$ and $e_i, e_j \in \Omega^{(-,\varphi_l)}$, then $ve_ie_j \in D^{(+,\varphi_l)}$.

Proof. a) Since φ_l is an involution, we have that

$$\varphi_l(ve_ie_j) = \varphi_l(e_j)\varphi_l(e_i)\varphi_l(v) = e_je_iv = ve_je_i = -ve_ie_j.$$

Thus, $ve_ie_j \in D^{(-,\varphi_l)}$ and the proof follows.

Items b), c) and d) are analogous. □

Lemma 4.3. *Let $f(y_1, \dots, y_m, z_1, \dots, z_n) \in F\langle Y \cup Z\rangle$ be a multilinear polynomial. If F is a field of characteristic different from 2 then $f(y_1, \dots, y_m, z_1, \dots, z_n) \in Id(E, \varphi_l)$ if and only if $f(x_1, \dots, x_m, x_{m+1}, \dots, x_{m+n}) \in Id(E)$.*

Proof. We have to consider two cases. The first is when $\Omega^{(+,\varphi_l)}$ is infinite and the second when $\Omega^{(-,\varphi_l)}$ is infinite. We will study the first case only because the other one can be treated analogously.

We will only prove the necessary condition being the sufficient one trivial.

Suppose $f(x_1, \dots, x_m, x_{m+1}, \dots, x_{m+n})$ is not an identity for E. Since f is multilinear, we have

$$f(v_1, \dots, v_m, v_{m+1}, \dots, v_{m+n}) \neq 0$$

for some $v_1, \dots, v_{m+n} \in D$.

Given $v = e_{i_1}e_{i_2}\dots e_{i_k} \in D$, denote by $\delta(v)$ the set $\{e_{i_1}, e_{i_2},\ \dots, e_{i_k}\}$. Suppose

$$\delta(v_1) \cup \delta(v_2) \cup \ldots \cup \delta(v_{m+n}) \subseteq \{e_1, e_2, \dots, e_\theta\}.$$

Since $\Omega^{(+,\varphi_l)}$ is infinite, there exists

$$\{e_{i_1},\ e_{i_2},\ \ldots,\ e_{i_{2(m+n)-1}},\ e_{i_{2(m+n)}}\} \subset \Omega^{(+,\varphi_l)},$$

where $\theta < i_1 < i_2 < \ldots < i_{2(m+n)-1} < i_{2(m+n)}$. Define $\overline{v}_j = v_j u_j$ as follows:

(a) If $j = 1, \ldots, m$ and $v_j \in D^{(+,\varphi_l)}$, then $u_j = 1$.
(b) If $j = 1, \ldots, m$ and $v_j \in D^{(-,\varphi_l)}$, then $u_j = e_{i_{2j-1}} e_{i_{2j}}$.
(c) If $j = m+1, \ldots, m+n$ and $v_j \in D^{(+,\varphi_l)}$, then $u_j = e_{i_{2j-1}} e_{i_{2j}}$.
(d) If $j = m+1, \ldots, m+n$ and $v_j \in D^{(-,\varphi_l)}$, then $u_j = 1$.

By Lemma 4.2 we have

$$\overline{v_1}, \ldots, \overline{v_m} \in D^{(+,\varphi_l)} \quad \text{and} \quad \overline{v_{m+1}}, \ldots, \overline{v_{m+n}} \in D^{(-,\varphi_l)}.$$

Now,

$$0 = f(\overline{v_1}, \ldots, \overline{v_{m+n}}) = f(v_1, \ldots, v_{m+n}) u_1 \cdots u_{m+n} \neq 0,$$

which is an absurd. Hence $f(x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n}) \in Id(E)$ as claimed. □

Let $f(x_1, \ldots, x_t) \in F\langle X\rangle$ be a multihomogeneous polynomial. Given $a_1, \ldots, a_t \geq 1$ let $x_{ij} \in X$ be distinct variables, where $i = 1, \ldots, t$ and $j = 1, \ldots, a_i$.

A multihomogeneous component of

$$f\left(\sum_{j=1}^{a_1} x_{1j}, \sum_{j=1}^{a_2} x_{2j}, \ldots, \sum_{j=1}^{a_t} x_{tj}\right)$$

is called a *partial linearization* of $f(x_1, \ldots, x_t)$. Denote by $Lin(f)$ the set of all partial linearizations of f.

The following lemma is well known (see for instance [10, Lemma 2]).

Lemma 4.4. *Let F be an infinite field of characteristic different from 2 and let $f(x_1, \ldots, x_t) \in F\langle X\rangle$ be a multihomogeneous polynomial. Then $f(x_1, \ldots, x_t) \in Id(E)$ if and only if*

$$\widehat{f}(v_1, \ldots, v_l) = 0$$

for all $\widehat{f}(x_1, \ldots, x_l) \in Lin(f)$ and $v_1, \ldots, v_l \in D$.

Let $f(y_1, \ldots, y_m, z_1, \ldots, z_n) \in F\langle Y \cup Z\rangle$ be a multihomogeneous polynomial. Given $a_1, \ldots, a_m, c_1, \ldots, c_n \geq 1$, let $y_{ij} \in Y$ be distinct variables, where $i = 1, \ldots, m$ and

$j = 1, \ldots, a_i$. Moreover let $z_{ij} \in Z$ some distinct variables where $i = 1, \ldots, n$ and $j = 1, \ldots, c_i$. A multihomogeneous component of

$$f\left(\sum_{j=1}^{a_1} y_{1j}, \ldots, \sum_{j=1}^{a_m} y_{mj}, \sum_{j=1}^{c_1} z_{1j}, \ldots, \sum_{j=1}^{c_n} z_{nj}\right)$$

is called a *yz-partial linearization* of $f(y_1, \ldots, y_m, z_1, \ldots, z_n)$. Denote by $Lin_{yz}(f)$ the set of all yz-partial linearizations of f.

Lemma 4.5. *Let F be an infinite field of characteristic different from 2 and let $f(y_1, \ldots, y_m, z_1, \ldots, z_n) \in F\langle Y \cup Z\rangle$ be a multihomogeneous polynomial. Then $f(y_1, \ldots, y_m, z_1, \ldots, z_n) \in Id(E, \varphi_l)$ if and only if*

$$\widehat{f}(v_1, \ldots, v_l, w_1, \ldots, w_k) = 0$$

for all $\widehat{f}(y_1, \ldots, y_l, z_1, \ldots, z_k) \in Lin_{yz}(f)$; $v_1, \ldots, v_l \in D^{(+,\varphi_l)}$; $w_1, \ldots, w_k \in D^{(-,\varphi_l)}$.

Proof. The proof is similar to the proof of [10, Lemma 2]. □

Lemma 4.6. *If F is an infinite field of characteristic different from 2, then*

$$Id(E, \varphi) \subseteq Id(E, \varphi_l).$$

Proof. Let $f \in Id(E, \varphi)$ be a multihomogeneous Y-proper polynomial and $f \notin Id(E, \varphi_l)$. Denote by J the $T(*)$-ideal

$$J = \langle [y_1 + z_1, y_2 + z_2, y_3 + z_3] \rangle^{T(*)}.$$

Note that $J \subseteq Id(E, \varphi)$ and $J \subseteq Id(E, \varphi_l)$. By Lemma 3.4 it follows that $f = g + h$, where $h \in J$ and g is a linear combination of polynomials

$$z_1^{a_1} z_2^{a_2} \cdots z_m^{a_m} [x_1, x_2][x_3, x_4] \cdots [x_{2n-1}, x_{2n}],$$

where $x_1 < x_2 < x_3 < \cdots < x_{2n}$ are in $Y \cup Z$, $a_1, a_2, \ldots, a_m \geq 0$, $m \geq 0$ and $n \geq 0$. Note that $g \in Id(E, \varphi)$ and $g \notin Id(E, \varphi_l)$. Since F is infinite, we can suppose g multihomogeneous with the same multidegree of f.

By Lemma 4.5, there exist $\widehat{g}(y_1, \ldots, y_l, z_1, \ldots, z_k) \in Lin_{yz}(g)$, $v_1, \ldots, v_l \in D^{(+,\varphi_l)}$ and $w_1, \ldots, w_k \in D^{(-,\varphi_l)}$ such that

$$\widehat{g}(v_1, \ldots, v_l, w_1, \ldots, w_k) \neq 0. \tag{5}$$

Note that if $l \geq 1$, then $\deg_{y_i} \widehat{g} = 1$ for all $i = 1, \ldots, l$ and y_i is inside a commutator. Thus $\widehat{g}(y_1, \ldots, 1, \ldots, y_l, z_1, \ldots, z_k) = 0$. In particular, $v_1, \ldots, v_l \in D - \{1\}$. Since $1 \in D^{(+,\varphi_l)}$

we have $w_1, \dots, w_k \in D - \{1\}$ too. Since $\widehat{g}$ is multihomogeneous and $w_i^2 = 0$ it follows $\widehat{g}$ is multilinear.

Since F is an infinite field and $g \in Id(E, \varphi)$ we get $Lin_{yz}(g) \subseteq Id(E, \varphi)$. Thus, $\widehat{g} \in Id(E, \varphi)$. Now,

$$\frac{v_i + \varphi(v_i)}{2} \in E^{(+,\varphi)} \quad \text{and} \quad \frac{w_j - \varphi(w_j)}{2} \in E^{(-,\varphi)}$$

for all i, j that is

$$\widehat{g}\left(\frac{v_1 + \varphi(v_1)}{2}, \dots, \frac{v_l + \varphi(v_l)}{2}, \frac{w_1 - \varphi(w_1)}{2}, \dots, \frac{w_k - \varphi(w_k)}{2}\right) = 0.$$

By (4) we have

$$\frac{v_i + \varphi(v_i)}{2} = v_i + v_i' \text{ and } \frac{w_j - \varphi(w_j)}{2} = w_j + w_j',$$

where v_i' is a linear combination of elements of D with length strictly greater than the length of v_i whereas w_j' is a linear combination of elements of D with length strictly greater than the length of w_j. Thus,

$$\begin{aligned} 0 &= \widehat{g}\left(\frac{v_1 + \varphi(v_1)}{2}, \dots, \frac{v_l + \varphi(v_l)}{2}, \frac{w_1 - \varphi(w_1)}{2}, \dots, \frac{w_k - \varphi(w_k)}{2}\right) \\ &= \widehat{g}(v_1 + v_1', \dots, v_l + v_l', w_1 + w_1', \dots, w_k + w_k') \\ &= \widehat{g}(v_1, \dots, v_l, w_1, \dots, w_k) + u, \end{aligned}$$

where u is a linear combination of elements of D with length strictly greater than the length of $\widehat{g}(v_1, \dots, v_l, w_1, \dots, w_k)$. It turns out $\widehat{g}(v_1, \dots, v_l, w_1, \dots, w_k) = 0$ and by (5) we have an absurd and we are done. □

5. $Id(E, \varphi)$ when $\mathrm{char}(F) = 0$

For the sake of completeness of the paper, in this section we give a new proof of [4, Corollary 4.4].

Theorem 5.1. *If $char(F) = 0$ and φ is an involution on E, then*

$$Id(E, \varphi) = \langle [y_1 + z_1, y_2 + z_2, y_3 + z_3] \rangle^{T(*)}.$$

Proof. Denote by

$$J = \langle [y_1 + z_1, y_2 + z_2, y_3 + z_3] \rangle^{T(*)}.$$

Since $[x_1, x_2, x_3] \in Id(E)$ we have $J \subseteq Id(E, \varphi)$. By Lemma 4.6 we have $Id(E, \varphi) \subseteq Id(E, \varphi_l)$. It remains to show that $Id(E, \varphi_l) \subseteq J$.

Let $f = f(y_1, \ldots, y_m, z_1, \ldots, z_n) \in Id(E, \varphi_l)$ be a multilinear polynomial. By Lemma 4.3 we have

$$f(x_1, \ldots, x_{m+n}) \in Id(E) = \langle [x_1, x_2, x_3] \rangle^T.$$

Thus we can write f as

$$f(x_1, \ldots, x_{m+n}) = \sum \alpha g_0 [g_1, g_2, g_3] g_4,$$

where $g_i = g_i(x_1, \ldots, x_{m+n}) \in F\langle X \rangle$ for all i. Therefore

$$f(y_1, \ldots, y_m, z_1, \ldots, z_n) = \sum \alpha \overline{g_0} [\overline{g_1}, \overline{g_2}, \overline{g_3}] \overline{g_4},$$

where $\overline{g_i} = g_i(y_1, \ldots, y_m, z_1, \ldots, z_n)$. It follows that

$$f(y_1, \ldots, y_m, z_1, \ldots, z_n) \in \langle [y_1 + z_1, y_2 + z_2, y_3 + z_3] \rangle^{T(*)}$$

as desired. □

6. $Id(E, \varphi)$ when $\mathrm{char}(F) > 2$

As announced in the introduction, we aim to give an explicit set of generators of the $T(*)$-ideal of $Id(E, \varphi)$ in the case F is an infinite field of characteristic $p > 2$. In this section F will denote an infinite field with $\mathrm{char}(F) = p > 2$.

The following fact holds. See, for example, [28, Lemma 1.2-b].

Lemma 6.1. *If E' is the Grassmann algebra without 1 generated by the infinite set $\{\xi_1, \xi_2, \ldots\}$ over F, then $x^p \in Id(E')$.*

The next is an easy result but we think it deserves an explicit proof.

Lemma 6.2. *If φ is an involution on E, then $E^{(-,\varphi)} \subseteq E'$.*

Proof. Let $u = \alpha + v \in E^{(-,\varphi)}$, where $\alpha \in F$ and $v \in E'$. By Lemma 6.1, we have

$$-(\alpha^p) = -(u^p) = (-u)^p = (\varphi(u))^p = \varphi(u^p) = \varphi(\alpha^p) = (\varphi(\alpha))^p = \alpha^p.$$

Thus $2\alpha^p = 0$, and therefore, $\alpha = 0$. □

The combined combination of Lemmas 6.1 and 6.2 gives us the following.

Proposition 6.3. *If φ is an involution on E, then $z_1^p \in Id(E, \varphi)$.*

From now on, we denote

$$I = \langle z_1^p,\ [y_1 + z_1, y_2 + z_2, y_3 + z_3]\rangle^{T(*)} \quad \text{and} \quad Id := Id(E, \varphi_l).$$

The next one is the main result of this section.

Theorem 6.4. *Let F be an infinite field of characteristic $p > 2$. If φ is an involution on E, then*

$$Id(E, \varphi) = \langle z_1^p, \quad [y_1 + z_1, y_2 + z_2, y_3 + z_3]\rangle^{T(*)}.$$

Notice that by Lemma 4.6 we have

$$I \subseteq Id(E, \varphi) \subseteq Id. \tag{6}$$

Then the only thing to be proved in the main theorem is the inclusion $Id \subseteq I$.

Let $M = (p^{c_1}, \dots, p^{c_m})$ and $N = (p^{b_1}, \dots, p^{b_n})$, where $c_i, b_j \geq 0$. Denote by B_{MN} the subspace of $F\langle Y, Z\rangle$ formed by all Y-proper multihomogeneous polynomials $f(y_1, \dots, y_m, z_1, \dots, z_n)$ with $\deg_{y_i} f = p^{c_i}$ and $\deg_{z_j} f = p^{b_j}$. By Lemma 3.3 and similar arguments as the ones used in Theorem 6 of [5] (page 101) we obtain that every $T(*)$-ideal is generated by its elements in B_{MN}. Thus $Id \subseteq I$ if and only if $Id \cap B_{MN} \subseteq I \cap B_{MN}$ for all M, N.

Let

$$B_{MN}(I) := \frac{B_{MN}}{I \cap B_{MN}}.$$

By Lemma 3.4 the vector space $B_{MN}(I)$ is spanned by the elements

$$z_1^{a_1} z_2^{a_2} \cdots z_n^{a_n} [x_1, x_2][x_3, x_4] \cdots [x_{2k-1}, x_{2k}] + I \cap B_{MN},$$

where $x_1 < x_2 < x_3 < \cdots < x_{2k}$ are in $Y \cup Z$.

We shall present now a list of technical results which are in some sense the steps of the proof of Theorem 6.4.

Lemma 6.5. *If $c_i \geq 1$ for some i, then $Id \cap B_{MN} \subseteq I \cap B_{MN}$.*

Proof. Since $y_i = x_j$ for some j and $x_1 < x_2 < x_3 < \dots < x_{2k}$ we obtain $B_{MN}(I) = 0$. Thus $Id \cap B_{MN} \subseteq B_{MN} = I \cap B_{MN}$. □

Lemma 6.6. *If $M = (1, \dots, 1)$ and $b_j \geq 2$ for some j, then $Id \cap B_{MN} \subseteq I \cap B_{MN}$.*

Proof. If $b_j \geq 2$, then $a_j \geq p^2 - 1 > p$. Then we get

$$z_1^{a_1} \cdots z_j^{a_j} \cdots z_n^{a_n} [x_1, x_2] \cdots [x_{2k-1}, x_{2k}] = z_1^{a_1} \cdots z_j^{a_j - p} z_j^p \cdots z_n^{a_n} [x_1, x_2] \cdots [x_{2k-1}, x_{2k}] \in I,$$

$B_{MN}(I) = 0$ and $Id \cap B_{MN} \subseteq B_{MN} = I \cap B_{MN}$. □

Lemma 6.7. *If $M = (1, \dots, 1)$ and $b_j = 0$ for all j, then $Id \cap B_{MN} \subseteq I \cap B_{MN}$.*

Proof. Let $f(y_1, \dots, y_m, z_1, \dots, z_n) \in Id \cap B_{MN}$. Since f is a multilinear polynomial, we can use the same argument as in Theorem 5.1 to obtain $f \in I \cap B_{MN}$. □

Lemma 6.8. *Let $M = (1, \dots, 1)$ and $0 \leq b_j \leq 1$ for all j. If $b_j = 1$ for some j, then $Id \cap B_{MN} \subseteq I \cap B_{MN}$.*

Proof. Let $f(y_1, \dots, y_m, z_1, \dots, z_n) \in Id \cap B_{MN}$. Denote by d the cardinality of the set

$$\{j \in \{1, \dots, n\} | b_j = 1\}.$$

We shall prove, by induction on d, that $f(y_1, \dots, y_m, z_1, \dots, z_n) \in I \cap B_{MN}$.

By Lemma 6.7 the case $d = 0$ is already proved.

Suppose $d \geq 1$. Since $z_i z_j = z_j z_i + [z_i, z_j]$, by Lemma 2.2 we can take $b_1 = 1$, then $B_{MN}(I)$ is spanned by the elements

$$z_1^{p-1} z_2^{a_2} \cdots z_n^{a_n} [z_1, x_2][x_3, x_4] \cdots [x_{2k-1}, x_{2k}] + I \cap B_{MN},$$

where $x_2 < x_3 < \dots < x_{2k}$ are in $Y \cup Z$. Hence there exist $\widehat{f} \in B_{M\widehat{N}}$ and $\overline{f} \in I$ such that

$$f = z_1^{p-1} \widehat{f} + \overline{f},$$

where, $\widehat{N} = (1, p^{b_2}, \dots, p^{b_n})$ and $\widehat{f}$ is a linear combination of polynomials

$$z_2^{a_2} \cdots z_n^{a_n} [z_1, x_2][x_3, x_4] \cdots [x_{2k-1}, x_{2k}].$$

Claim. $\widehat{f} \in Id$.

Proof of the Claim: Suppose $\widehat{f} \notin Id$.

Since $I \subseteq Id$ we have $\overline{f} \in Id$, then $z_1^{p-1} \widehat{f} \in Id$ too.

Again we have to consider the case $\Omega^{(+,\varphi_l)}$ is infinite or the case $\Omega^{(-,\varphi_l)}$ is infinite. We will study only the first case because the other one is to be handled similarly.

Given $w = e_{i_1} e_{i_2} \cdots e_{i_k} \in D$, let us denote by $\delta(w)$ the set $\{e_{i_1}, e_{i_2}, \dots, e_{i_k}\}$. Moreover, if $u = \alpha_1 w_1 + \cdots + \alpha_t w_t$ where $\alpha_i \in F$, $w_i \in D$, we denote by $\delta(u)$ the set $\delta(w_1) \cup \cdots \cup \delta(w_t)$.

Since $\widehat{f} \notin Id$, there exist $v_1, \dots, v_m \in E^{(+,\varphi_l)}$ and $w_1, \dots, w_n \in E^{(-,\varphi_l)}$ such that

$$\widehat{f}(v_1, \dots, v_m, w_1, \dots, w_n) \neq 0$$

and $w_1 \in D^{(-,\varphi_l)}$. Remark that the length of w_1 is odd.

Let $s \geq 1$ such that

$$\delta(v_1) \cup \cdots \cup \delta(v_m) \cup \delta(w_1) \cup \cdots \cup \delta(w_n) \subseteq \{e_1, e_2, \ldots, e_s\}.$$

Since $\Omega^{(+,\varphi_l)}$ is infinite, there exists $\{e_{i_1}, e_{i_2}, \ldots, e_{i_{2p-3}}, e_{i_{2p-2}}\} \subseteq \Omega^{(+,\varphi_l)}$, with $s < i_1 < i_2 < \ldots < i_{2p-3} < i_{2p-2}$. Let

$$w'_1 = e_{i_1} e_{i_2}, \ \ldots \ , \ w'_{p-1} = e_{i_{2p-3}} e_{i_{2p-2}}.$$

By Lemma 4.2 $w'_i \in E^{(-,\varphi_l)}$ for all i.

Since $z_1^{p-1}\widehat{f} \in Id$, we have $u = 0$, where

$$u = \left(w_1 + \sum_{i=1}^{p-1} w'_i\right)^{p-1} \widehat{f}(v_1, \ldots, v_m, w_1 + \sum_{i=1}^{p-1} w'_i, w_2, \ldots, w_n).$$

On the other hand

$$u = (p-1)! \left(w'_1 \cdots w'_{p-1}\right) \widehat{f}(v_1, \ldots, v_m, w_1, \ldots, w_n) \neq 0$$

that is an absurd. It turns out $\widehat{f} \in Id$ and the claim is proved.

By induction hypothesis, $\widehat{f} \in I$. In particular,

$$f = z_1^{p-1}\widehat{f} + \overline{f} \in I$$

as desired. □

Indeed the proof of Theorem 6.4 follows directly from (6), Lemmas 6.5, 6.6, 6.7 and 6.8.

7. $*$-central polynomials of E

Throughout this section F will denote an infinite field of characteristic different from 2. We will introduce $*$-central polynomials for a given algebra and we will study the $T(*)$-space of the $*$-central polynomials of the Grassmann algebra.

A polynomial $f = f(y_1, \ldots, y_m, z_1, \ldots, z_n) \in F\langle Y \cup Z\rangle$ is a $*$-*central polynomial* for an algebra with involution $(A, \circ)$ if

$$f(Y_1, \ldots, Y_m, Z_1, \ldots, Z_n) \in Z(A)$$

for all $Y_1, \ldots, Y_m \in A^{(+,\circ)}$ and $Z_1, \ldots, Z_n \in A^{(-,\circ)}$. We denote by $C(A, \circ)$ the set of all $*$-central polynomials for $(A, \circ)$. This set is a $T(*)$-space, i.e., a vector subspace of $F\langle Y \cup Z\rangle$ closed under all $*$-endomorphisms of $F\langle Y \cup Z\rangle$.

Given a subset $S \subseteq F\langle Y \cup Z\rangle$ denote by $\langle S\rangle^{TS(*)}$ the $T(*)$-space generated by S. Note that it is the vector subspace of $F\langle Y \cup Z\rangle$ spanned by all elements

$$f(g_1, \ldots, g_m, h_1, \ldots, h_n)$$

where $g_1, \ldots, g_m \in F\langle Y \cup Z\rangle^{(+,*)}$; $h_1, \ldots, h_n \in F\langle Y \cup Z\rangle^{(-,*)}$; $f(y_1, \ldots, y_m, z_1, \ldots, z_n) \in S$.

If $f \in F\langle Y \cup Z\rangle$ is any polynomial, then

$$f = \sum_d f^{(d)},$$

where $f^{(d)}$ is the multihomogeneous component of f with multidegree d. By using the same argument as in Proposition 4.2.3 of [12] we can prove that if F is infinite, then

$$\langle f\rangle^{TS(*)} = \langle f^{(d)}; d\rangle^{TS(*)}. \tag{7}$$

Moreover, if $\mathrm{char}(F) = 0$ and f is multihomogeneous, then $\langle f\rangle^{TS(*)} = \langle \widehat{f}\rangle^{TS(*)}$ for some multilinear polynomial $\widehat{f}$. Thus, if J is a $T(*)$-space, then J is generated by its multihomogeneous elements if F is infinite, and J is generated by its multilinear elements if $\mathrm{char}(F) = 0$.

We start off now our investigation toward the $*$-central polynomials of E. Let us denote by W the following $T(*)$-space

$$W := F + \langle [y_1 + z_1, y_2 + z_2]\rangle^{TS(*)} + Id(E, \varphi).$$

Note that $W \subseteq C(E, \varphi)$ if φ is an involution on E. A rather obvious consequence of Theorem 5.1 and Theorem 6.4 is that $Id(E, \varphi) = Id(E, \varphi_l)$, then it follows that $C(E, \varphi) = C(E, \varphi_l)$.

Lemma 7.1. *Let F be an infinite field of characteristic different from 2. Let φ be an involution on E. If $f \in C(E, \varphi)$ and there exists $x \in Y \cup Z$ such that f is homogeneous in x with $\deg_x(f) = 1$, then $f \in W$.*

Proof. Without loss of generality we can suppose $\varphi = \varphi_l$. The polynomial f is a linear combination of monomials $m = m_1 x m_2$. Since $m = x m_2 m_1 + [m_1, x m_2]$, we can write

$$f = x\widehat{f} + \overline{f},$$

where $\deg_x(\widehat{f}) = 0$ and $\overline{f} \in \langle [y_1 + z_1, y_2 + z_2]\rangle^{TS(*)}$. In particular, $x\widehat{f} \in C(E, \varphi_l)$.

Claim. $\widehat{f} \in Id(E, \varphi_l)$.

Let us set $\widehat{f} = \widehat{f}(y_1, \ldots, y_m, z_1, \ldots, z_n)$ and suppose $\widehat{f} \notin Id(E, \varphi_l)$. Then there exist $v_1, \ldots, v_m \in E^{(+,\varphi_l)}$ and $w_1, \ldots, w_n \in E^{(-,\varphi_l)}$ such that

$$\widehat{f}(v_1,\dots,v_m,w_1,\dots,w_n)=g_0+g_1\neq 0,$$

where g_0 and g_1 are linear combination of elements in D whose lengths are even and odd respectively. We will use the notation of δ introduced in the proof of Lemma 6.8. Let $s\geq 1$ such that

$$\delta(v_1)\cup\cdots\cup\delta(v_m)\cup\delta(w_1)\cup\cdots\cup\delta(w_n)\subseteq\{e_1,e_2,\dots,e_s\}.$$

There exist $s<i_1<i_2<i_3$ such that:
a1) $e_{i_1},e_{i_2},e_{i_3}\in\Omega^{(+,\varphi_l)}$ if $\Omega^{(+,\varphi_l)}$ is an infinite set,
b1) $e_{i_1},e_{i_2},e_{i_3}\in\Omega^{(-,\varphi_l)}$ if $\Omega^{(-,\varphi_l)}$ is an infinite set.
By Lemma 4.2 we get
a2) $1,e_{i_1}\in D^{(+,\varphi_l)}$ and $e_{i_1}e_{i_2},e_{i_1}e_{i_2}e_{i_3}\in D^{(-,\varphi_l)}$ if $\Omega^{(+,\varphi_l)}$ is an infinite set,
b2) $1,e_{i_1}e_{i_2}e_{i_3}\in D^{(+,\varphi_l)}$ and $e_{i_1},e_{i_1}e_{i_2}\in D^{(-,\varphi_l)}$ if $\Omega^{(-,\varphi_l)}$ is an infinite set.

Thus, if $g_0\neq 0$ then there exist $x^+\in D^{(+,\varphi_l)}$ and $x^-\in D^{(-,\varphi_l)}$ of odd lengths such that

$$x^{\pm}\widehat{f}(v_1,\dots,v_m,w_1,\dots,w_n)=x^{\pm}g_0+x^{\pm}g_1\notin Z(E).$$

If $g_1\neq 0$, then there exist $x^+\in D^{(+,\varphi_l)}$ and $x^-\in D^{(-,\varphi_l)}$ of length even such that

$$x^{\pm}\widehat{f}(v_1,\dots,v_m,w_1,\dots,w_n)=x^{\pm}g_0+x^{\pm}g_1\notin Z(E)$$

that is an absurd because $x\widehat{f}\in C(E,\varphi_l)$.

Therefore, $\widehat{f}\in Id(E,\varphi_l)$ and

$$f=x\widehat{f}+\overline{f}\in W,$$

as desired. □

We are in position to state and prove the main result of the section.

Theorem 7.2. *If F is a field of characteristic 0 and φ is an involution on E, then*

$$C(E,\varphi)=\langle 1,[y_1+z_1,y_2+z_2],(y_1+z_1)[y_2+z_2,y_3+z_3,y_4+z_4]\rangle^{TS(*)}.$$

Proof. By Theorem 5.1 we obtain

$$W=\langle 1,[y_1+z_1,y_2+z_2],(y_1+z_1)[y_2+z_2,y_3+z_3,y_4+z_4]\rangle^{TS(*)}$$

and $W\subseteq C(E,\varphi)$.

Since $char(F)=0$, every $T(*)$-space is generated by its multilinear elements. Thus, by Lemma 7.1 we have $W\supseteq C(E,\varphi)$ as desired. □

8. $C(E, \varphi)$ when char$(F) > 2$

In this section we shall describe $C(E, \varphi)$ and prove that it is not finitely generated over an infinite field of characteristic $p > 2$ as a $T(*)$-space.

We shall define the following polynomials inside $F\langle Y \cup Z\rangle$. For every $n \geq 1$ let

$$\widehat{q}_1 = \widehat{q}(y_1, y_2, z_1, z_2) := (y_1 + z_1)^{p-1}[y_1 + z_1, y_2 + z_2](y_2 + z_2)^{p-1}$$

and for each $n \geq 1$, let

$$\widehat{q}_n := \widehat{q}(y_1, y_2, z_1, z_2) \cdots \widehat{q}(y_{2n-1}, y_{2n}, z_{2n-1}, z_{2n}).$$

Theorem 8.1. *Let F be an infinite field of characteristic $p > 2$. Then the set $C(E, \varphi)$ of the $*$-central polynomials of E endowed with the involution φ is generated, as a $T(*)$-space, by the polynomials:*

(1) $(y_1 + z_1)z_2^p(y_3 + z_3)$,
(2) $(y_1 + z_1)[y_2 + z_2, y_3 + z_3, y_4 + z_4]$,
(3) y_0^p, $y_0^p\widehat{q}_1$, …, $y_0^p\widehat{q}_n$, …

Proof. Let U be the $T(*)$-space generated by the polynomials (1), (2) and (3). By Theorem 6.4 we have

$$U = I + \langle y_0^p\widehat{q}_n |\ n \geq 0\rangle^{TS(*)},$$

where $I = Id(E, \varphi)$. Since $y_0^p\widehat{q}_1 \in U$ we obtain

$$g = (1)^p(1 + y_1 + z_1)^{p-1}[1 + y_1 + z_1, 1 + y_2 + z_2](1 + y_2 + z_2)^{p-1} \in U$$

and each multihomogeneous component of g belongs to U too. Thus

$$[y_1, y_2], [y_1, z_2], [z_1, y_2], [z_1, z_2] \in U$$

and $W \subseteq U$.

By Theorem 2.5 we have $U \subseteq C(E, \varphi)$. We shall prove that $U = C(E, \varphi)$.

By using an argument similar to that of Theorem 6 of [5] (page 101) we obtain that every $T(*)$-space is generated by its multihomogeneous elements $f(y_1, \dots, y_m, z_1, \dots, z_n)$ with multidegree $(p^{b_1}, \dots, p^{b_m}, p^{c_1}, \dots, p^{c_n})$, where $c_i, b_j \geq 0$, $\deg_{y_i} f = p^{b_i}$ and $\deg_{z_j} f = p^{c_j}$. Let $f(y_1, \dots, y_m, z_1, \dots, z_n) \in C(E, \varphi)$ such polynomial.

By Lemma 7.1, if either $b_i = 0$ or $c_i = 0$ for some i, then $f \in W \subseteq U$.

Suppose $b_i \geq 1$ and $c_i \geq 1$ for all i. By Lemma 3.4, the polynomial $f + I$ in $F\langle Y \cup Z\rangle/I$ is a linear combination of elements $h + I$ where

$$h = y_1^{d_1} \cdots y_m^{d_m} z_1^{a_1} \cdots z_n^{a_n} [x_1, x_2][x_3, x_4] \cdots [x_{2k-1}, x_{2k}]$$

and $x_1 < x_2 < x_3 < \cdots < x_{2k}$ are in $Y \cup Z$. Since F is infinite, we can suppose h with the same multidegree of f. Thus, if $c_i \geq 2$ for some i, then $h \in I$ and $f \in I \subseteq U$ as desired.

Suppose $b_i \geq 1$ and $c_i = 1$ for all i. In this case, note that $a_i = p - 1$ for all i and so

$$h = y_1^{d_1} \cdots y_m^{d_m} z_1^{p-1} \cdots z_n^{p-1} [x_1, x_2][x_3, x_4] \cdots [x_{2k-1}, x_{2k}].$$

We know that if $u \in C(E, \varphi)$, then $uv + I = vu + I$ for every polynomial v. Since

$$[x_{2i-1}, x_{2i}],\ [x_{2i-1}, x_{2i}]x_{2i}^{p-1},\ x_{2i-1}^{p-1}[x_{2i-1}, x_{2i}]x_{2i}^{p-1} \in C(E, \varphi)$$

we can suppose

$$h = y_{i_1}^{p^{b_{i_1}}} \ldots y_{i_r}^{p^{b_{i_r}}} y_{j_1}^{p^{b_{j_1}-p}} \ldots y_{j_s}^{p^{b_{j_s}-p}} x_1^{p-1}[x_1, x_2]x_2^{p-1} \cdots x_{2n-1}^{p-1}[x_{2n-1}, x_{2n}]x_{2n}^{p-1}.$$

Since $(uv)^p + I = u^p v^p + I$ (see Lemma 2.3) we obtain

$$h \in \langle y_0^p \widehat{q}_n \rangle^{TS(*)} + I.$$

Thus $f \in U$ as desired. $\square$

A $T(*)$-space $\mathbb{V}$ of $F\langle Y \cup Z \rangle$ is said to be *finitely generated as a $T(*)$-space* if there exists a finite set S such that $\mathbb{V} = \langle S \rangle^{TS(*)}$.

Corollary 8.2. *Let F be an infinite field of characteristic $p > 2$. If φ is an involution on E, then $C(E, \varphi)$ is not finitely generated as a $T(*)$-space.*

Proof. Let C_n be the $T(*)$-space of $F\langle Y \cup Z \rangle$ generated by the polynomials

$$(y_1 + z_1)z_2^p(y_3 + z_3),\ (y_1 + z_1)[y_2 + z_2, y_3 + z_3, y_4 + z_4],\ y_0^p,\ y_0^p \widehat{q}_1,\ \ldots,\ y_0^p \widehat{q}_n.$$

Suppose that $C(E, \varphi)$ is finitely generated as a $T(*)$-space. Thus there exists a finite set S such that $C(E, \varphi) = \langle S \rangle^{TS(*)}$. Since

$$C(E, \varphi) = \bigcup_n C_n,$$

for each $f \in S$ there exists n_f such that $f \in C_{n_f}$. Thus, if $N = max\{n_f |\ f \in S\}$ then $C(E, \varphi) = C_N$. In particular,

$$f = f(y_1, \ldots, y_{2(N+1)}) = 1^p \widehat{q}_{N+1}(y_1, \ldots, y_{2(N+1)}, 0, \ldots, 0) \in C_N.$$

Thus, f is a linear combination of polynomials

$$f_1 f_2^p f_3, \quad g_1[g_2, g_3, g_4],$$
$$h_0^p, \quad h_1^p \widehat{q}_1(h_{1,1}, h_{1,2}, h_{1,3}, h_{1,4}), \quad \ldots, \quad h_N^p \widehat{q}_N(h_{N,1}, \ldots, h_{N,4N}),$$

for some $f_i, g_i, h_i, h_{i,j} \in F\langle Y \cup Z\rangle$. Since f_2 is a skew-symmetric polynomial and 1 is a symmetric polynomial we have $f_2(0, \ldots, 0) = 0$, i.e., f_2 does not have scalar term. Moreover, by replacing the variables $z_1, z_2, \ldots$ by 0 we can suppose $f_i, g_i, h_i, h_{i,j} \in F\langle Y\rangle$.

Now, we replace the variables y_j by x_j. Thus, $f_1 f_2^p f_3 \in \mathbb{I}$, $g_1[g_2, g_3, g_4] \in Id(E)$, $h_0^p \in W_N$,

$$h_i^p \widehat{q}_i(h_{i,1}, \ldots, h_{i,4i}) = h_i^p q_i(h_{i,1} + h_{i,2i+1}, \ldots, h_{i,2i} + h_{i,4i}) \in W_N$$

and

$$q_{N+1} = f(x_1, \ldots, x_{2(N+1)}) \in W_N + Id(E) + \mathbb{I}$$

that is an absurd by Remark 2.6.

Therefore $C(E, \varphi)$ is not finitely generated as a $T(*)$-space. □

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