The qq -bit (III): Symmetric q -Jordan–Wigner embeddings

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Abstract. We prove that, replacing the left Jordan–Wigner q -embedding by the symmetric q -embedding described in section 2, the result of the corresponding central limit theorem changes drastically with respect to those obtained in [1]. In fact, in the latter case, for any $q \in \mathbb{C}$, the limit space is precisely the 1-mode Interacting Fock Space (IFS) that realizes the canonical quantum decomposition of the limit classical random variable. In the former case this happens if and only if $q = \pm 1$. Furthermore, as shown in section 4, the limit classical random variable turns out to coincide with the 1-mode version of the q -deformed quantum Brownian introduced by Parthasarathy [7], [8], and extended to the general context of bi-algebras by Schürman [10], [9]. The last section of the paper (Appendix) describes this continuous version in white noise language, leading to a simplification of the original proofs, based on quantum stochastic calculus.

1. Introduction

This is the third of a series of four papers devoted to central limits of Bernoulli-type quantum processes defined by different embeddings of copies of the algebra $M_2(\mathbb{C})$ of complex 2×2 matrices into the infinite tensor product $\bigotimes_{\mathbb{N}} M_2(\mathbb{C})$. In the paper [1] the embeddings are x -deformations ($x \in \mathbb{C}$) of the known Jordan–Wigner (JW) embedding used to represent the Fermion algebra into the infinite tensor product $\bigotimes_{\mathbb{N}} M_2(\mathbb{C})$. In the present paper we use a symmetric x -deformation of the JW embedding described in section 2 below.

We prove that, with this choice of the embeddings, for any $x \in \mathbb{C}$, the limit

space is precisely the 1-mode Interacting Fock Space (IFS) that realizes the canonical quantum decomposition of the limit classical random variable. With the embeddings used in [1], this happens if and only if $q = \pm 1$.

In [1] and [2] the limit space is a deformation of the monotone Fock space, for which to speak of 1-mode realization makes no sense, while in section 4 we show that the limit quantum random variable turns out to coincide with the 1-mode version of the q -deformed quantum Brownian introduced by Parthasarathy [7], [8], and Schürman [10], [9].

Contrarily to the JW- x -embeddings used in the papers [1] and [2], where the total order of $\mathbb{N}$ plays a crucial role, the symmetric x -embeddings used here can be defined when the index set is an arbitrary graph. This implies that, if the graph is a discrete approximation of a manifold, the functional central limit theorem associated to the present CLT will produce a quantum random field on this manifold. When the graph is a 1-dimensional lattice, the transition from a CLT to its functional version is obtained with a now standard procedure, therefore we do not include this adaptation. The limit space in the continuous case is described in section 4 where we give a new simplified proof, in the language of white noise, of the bosonization result obtained by Hudson and Parthasarathy [6] in the Fermion case and extended by Parthasarathy [7], [8] and Schürman [10], [9] to the q -deformed case ($q \neq 0$). The multi-dimensional case will be discussed in a forthcoming paper.

2. Statement of the central limit problem with symmetric x -embeddings

In the following $\mathbb{C}^2$ is considered as a Hilbert space with respect to the usual scalar product, i.e. the unique one making the pair

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad ; \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

an orthonormal basis. Finite tensor powers of $\mathbb{C}^2$ are meant in the Hilbert space sense unless explicitly stated. From [1], [2], [3] we recall the notations:

$$\sigma^{(+1)} := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad ; \quad \sigma^{(-1)} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Let $\{V_N\}_{N \in \mathbb{N}}$ be a family of finite sets such that

- $V_N \subset V_{N+1}$ for any N ;
- $|V_N| \rightarrow \infty$ as $N \rightarrow \infty$, where $|\cdot|$ denotes cardinality.

For any $N \in \mathbb{N}$, $\forall k \in V_N$ and $x \in \mathbb{C}$, define

$$\sigma_k^{(+1)}(x) := \bigotimes_{h \in V_N} b_h \quad ; \quad b_h := \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}, & \text{if } h \in V_N \setminus \{k\} \\ \sigma^{(+1)}, & \text{if } h = k \end{cases}$$

$$\sigma_k^{(-1)}(x) := \left[\sigma_k^{(+1)}(x) \right]^* = \bigotimes_{h \in V_N} b_h^* \quad ; \quad b_h^* := \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & \bar{x} \end{pmatrix}, & \text{if } h \in V_N \setminus \{k\} \\ \sigma^{(-1)}, & \text{if } h = k \end{cases}$$

The map

$$\sigma_k^{(\pm)} \in M_2(\mathbb{C}) \mapsto \sigma_k^{(\pm)}(x) \in \bigotimes_{h \in V_N} M_2(\mathbb{C}) \quad (2.1)$$

is called **the symmetric x -embedding into the k -factor of $\bigotimes_{h \in V_N} M_2(\mathbb{C})$** .

We use x (resp. q) to denote the deformation parameter before (resp. after) the limit. In general q is a function of x . In the present case $q = |x|^2$.

Remark. Clearly the embeddings (2.1) depend also on the *volume* V_N . The fact that we omit this dependence from the notation will cause no problem, because these embeddings will only appear inside **central limit sums** of the form

$$S_N^{(\varepsilon)}(x) := \frac{1}{\sqrt{|V_N|}} \sum_{k \in V_N} \sigma_k^{(\varepsilon)}(x) \quad ; \quad \forall \varepsilon \in \{\pm 1\} \quad (2.2)$$

and we make the convention that, if a symbol $\sigma_k^{(\pm)}(x)$ appears in a central limit sum with index V_N , the the embedding is also referred to V_N . Since products of sums referred to different V_N never appear, no ambiguity can arise from this convention. For each V_N we consider the vector

$$\Phi_N := \bigotimes_{h \in V_N} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \bigotimes_{h \in V_N} (\mathbb{C}^2)_h$$

For $n, N \in \mathbb{N}$ and $\varepsilon := (\varepsilon(1), \dots, \varepsilon(n)) \in \{-1, +1\}^n$, the expectation values

$$\left\langle \Phi_N, S_N^{(\varepsilon(1))}(x) \dots S_N^{(\varepsilon(n))}(x) \Phi_N \right\rangle = \quad (2.3)$$

$$= \frac{1}{\sqrt{|V_N|}^n} \sum_{\{k_1, \dots, k_n\} \subset V_N} \left\langle \Phi_N, \sigma_{k_1}^{(\varepsilon(1))}(x) \dots \sigma_{k_n}^{(\varepsilon(n))}(x) \Phi_N \right\rangle \quad (2.4)$$

are called the mixed Φ_N -moments of the pair of operator random variables $S_N^{(\pm 1)}(x) \in \bigotimes_{h \in V_N} M_2(\mathbb{C})$.

The main result of the present paper (see Theorem 1 below) is that the limits, as $N \rightarrow \infty$, of the Φ_N -moments (2.3) exist and are equal to the corresponding vacuum moments of the creation and annihilation operators in the unique 1-MIFS with principal Jacobi sequence

$$\omega_m := m |x|^{2(m-1)} = \frac{1}{|x|^2} m |x|^{2m}$$

The meaning of this principal Jacobi sequence is explained in section 3.

2.1. Properties of pair partitions

Let us recall, from [3], section 3 and [1], section 2, the following notations. For $n, m, N \in \mathbb{N}$,

$$\{-1, +1\}^m := \{\text{functions from } \{1, \dots, m\} \rightarrow \{-1, +1\}\}$$

$$\begin{aligned} \{-1, 1\}_+^{2n} &:= \left\{ \varepsilon \in \{-1, 1\}^{2n} : \sum_{h=1}^{2n} \varepsilon_h = 0 \text{ and } \forall m \in \{1, \dots, 2n\}, \sum_{h=m}^{2n} \varepsilon_h \geq 0 \right\} \\ &:= \left\{ \varepsilon \in \{-1, 1\}^{2n} : \sum_{h=1}^{2n} \varepsilon_h = 0 \text{ and } \forall m \in \{1, \dots, 2n\}, \sum_{h=1}^m \varepsilon_h \leq 0 \right\} \end{aligned} \quad (2.5)$$

$$\{-1, +1\}_-^{2n} := \{-1, +1\}^{2n} \setminus \{-1, +1\}_+^{2n}$$

$$F(N, n) := \text{the set of functions } \{1, \dots, 2n\} \rightarrow V_N$$

$$\begin{aligned} PPF(N, n) &:= \text{the set of \textbf{pair-partition functions} } \{1, \dots, 2n\} \rightarrow \{1, \dots, N\} \\ &:= \{\mathbf{k} \in F(N, n) : \forall r \in \{1, \dots, 2n\}, |\{\mathbf{k}^{-1}(\mathbf{k}(r))\}| = 2\} \end{aligned}$$

Moreover, for $\varepsilon \in \{-1, 1\}^{2n}$:

$$PP(2n, \varepsilon) := \{(l_h, r_h)_{h=1}^n \in PP(2n) : l_1 < \dots < l_n, l_h < r_h, \quad (2.6)$$

$$\varepsilon(l_h) = -1, \varepsilon(r_h) = 1, \forall h \in \{1, 2, \dots, n\} \quad (2.7)$$

$$NCPP(2n, \varepsilon) := PP(2n, \varepsilon) \cap NCPP(2n) \quad (2.8)$$

and for $\pi = (l_h, r_h)_{h=1}^n, 2n < N$,

$$PF(N, n, \pi) := \{\mathbf{k} \in PF(N, n) : \forall h \in \{1, \dots, 2n\}, \mathbf{k}(l_h) = \mathbf{k}(r_h)\}$$

It is known, see [4] (we give a short direct proof of this in Lemma 1 below), that for any $n \in \mathbb{N}$ and $\varepsilon \in \{-1, +1\}_+^{2n}$, there is a unique element of $NCPP(2n)$, denoted by $\{(l_h^\varepsilon, r_h^\varepsilon)\}_{h=1}^n$, such that

$$\varepsilon(l_h^\varepsilon) = -1 \text{ and } \varepsilon(r_h^\varepsilon) = 1 \quad ; \quad \forall h \in \{1, 2, \dots, n\} \quad (2.9)$$

Lemma 1. For any $n \in \mathbb{N}$, $\varepsilon \in \{-1, +1\}_+^{2n}$, and $\{(l_h, r_h)\}_{h=1}^n \in PP(2n, \varepsilon)$, and for $\{(l_h^\varepsilon, r_h^\varepsilon)\}_{h=1}^n$ given by (2.9), one has:

$$l_h = l_h^\varepsilon \quad ; \quad \forall h \in \{1, \dots, n\} \quad (2.10)$$

$$\{r_h : h \in \{1, \dots, n\}\} = \{r_h^\varepsilon : h \in \{1, \dots, n\}\} \text{ (as un-ordered sets)} \quad (2.11)$$

$$|PP(2n, \varepsilon)| = \prod_{h=1}^n (2h - l_h^\varepsilon) = \prod_{h=1}^n (2h - l_h) \quad (2.12)$$

Remark. (2.10) and (2.11) mean that the class $PP(2n, \varepsilon)$ is characterized by the left elements of its unique non-crossing pair partition $(l_h^\varepsilon, r_h^\varepsilon)_{h=1}^n$ in the sense that two elements of the class have the same l_h and the corresponding r_h differ by a permutation.

Proof. Let $\varepsilon \in \{-1, +1\}_+^{2n}$. We first construct a non-crossing pair partition satisfying (2.7). From (2.5) we know that $2n \in \varepsilon^{-1}(\{+1\})$ and that $|\varepsilon^{-1}(\{-1\})| = |\varepsilon^{-1}(\{+1\})|$. Therefore there must exist $k \in \{1, \dots, n\}$ such that $\varepsilon(k-1) = -1$ and $\varepsilon(k) = +1$. Denote r_n^ε the largest among these k and $l_n^\varepsilon := r_n - 1$. Use the unique order preserving identification of $\{1, \dots, 2n\} \setminus \{l_n^\varepsilon, r_n^\varepsilon\}$ with $\{1, \dots, 2(n-1)\}$ to identify $\varepsilon[\{1, \dots, 2n\} \setminus \{l_n^\varepsilon, r_n^\varepsilon\}]$ with a function $\varepsilon' : \{1, \dots, 2(n-1)\} \rightarrow \{-1, +1\}$. The fact that $(l_n^\varepsilon, r_n^\varepsilon)$ is a consecutive pair implies that $\varepsilon' \in \{-1, +1\}_+^{2(n-1)}$ and that $(l_h, r_h)_{h=1}^{n-1}$ is mapped by this

identification into a non-crossing pair partition of $\{1, \dots, 2(n-1)\}$. Therefore one can iterate this procedure and the construction gives a non-crossing pair partition $(l_h^\varepsilon, r_h^\varepsilon)_{h=1}^n$ of $\{1, \dots, 2n\}$ satisfying (2.7) and such that

$$l_1^\varepsilon < \dots < l_n^\varepsilon \quad ; \quad l_h^\varepsilon < r_h^\varepsilon, \quad h \in \{1, \dots, 2n\}$$

To prove uniqueness, we show that, if $(l_h, r_h)_{h=1}^n$ of $\{1, \dots, 2n\}$ is a non-crossing pair partition satisfying (2.7), then it must coincide with $(l_h^\varepsilon, r_h^\varepsilon)_{h=1}^n$. Since one can always suppose that the numeration of the $\{l_1, \dots, l_n\}$ has been chosen so that

$$l_1 < \dots < l_n$$

and since both partitions satisfy (2.7), it follows that:

$$l_h = l_h^\varepsilon, \quad h \in \{1, \dots, n\} \quad ; \quad \{r_1, \dots, r_n\} = \{r_1^\varepsilon, \dots, r_n^\varepsilon\} \text{ as un-ordered sets}$$

In particular (2.10) and (2.11) hold. Since the partition $\{(l_h, r_h)\}_{h=1}^n$ is non-crossing and l_n is the largest among the left elements, the interval $[l_n, r_n] \cap \{1, \dots, 2n\}$ cannot contain more than 2 points, that is:

$$r_n = l_n + 1 = l_n^\varepsilon + 1 = r_n^\varepsilon \iff (l_n, r_n) = (l_n^\varepsilon, r_n^\varepsilon)$$

Identifying $\{1, \dots, 2n\} \setminus \{l_n, r_n\}$ with $\{1, \dots, 2(n-1)\}$ and arguing as in the first part of the proof, one concludes that $(l_h, r_h) = (l_h^\varepsilon, r_h^\varepsilon)$ for each $h \in \{1, \dots, n\}$. This proves uniqueness.

To prove (2.12), notice that, since the l_h are common to all pairs in $PP(2n, \varepsilon)$, the pairs in this class are parametrized by the possible choices of the r_h .

Since $l_n < r_n$, as $\{(l_h, r_h)\}_{h=1}^n$ runs over $PP(2n, \varepsilon)$,

– r_n runs over $\{l_n + 1, \dots, 2n\}$;

– r_{n-1} runs over $\{l_{n-1} + 1, \dots, 2n\} \setminus \{l_n, r_n\}$;

– $\dots$;

– r_m runs over $\{l_m + 1, \dots, 2n\} \setminus \{l_h, r_h\}_{h=m+1}^n$ ($m \in \{1, \dots, n\}$) (where $\{l_h, r_h\}_{h=n+1}^n := \emptyset$).

Therefore

$$\begin{aligned} |PP(2n, \varepsilon)| &= (2n - l_n)(2n - 2 - l_{n-1})(2n - 4 - l_{n-2}) \dots (2n - 2(n-1) - l_1) \\ &= \prod_{m=1}^n (2(n - m + 1) - l_{n-m+1}) = \prod_{h=1}^n (2h - l_h^\varepsilon) \end{aligned}$$

and this proves the first equality in (2.12). The second one follows from (2.10). $\square$

2.2. Existence and structure of the limit moments

Lemma 2. As N tends to ∞ , the mixed moment (2.4) tends to zero if either of the following happens:

- n is odd;
- $n = 2m$ but $\varepsilon \in \{-1, +1\}_-^{2m}$.

For any $\varepsilon \in \{-1, +1\}_+^{2n}$, the scalar product

$$\left\langle \Phi_N, S_N^{(\varepsilon(1))}(x) \dots S_N^{(\varepsilon(2n))}(x) \Phi_N \right\rangle$$

has the form

$$o(1) + \frac{1}{|V_N|^n} \sum_{\pi=\{(l_h, r_h)\}_{h=1}^{2n} \in PP(2n, \varepsilon)} \sum_{\mathbf{k} \in PF(N, n, \pi)} \left\langle \Phi_N, \sigma_{\mathbf{k}(1)}^{(\varepsilon(1))}(x) \dots \sigma_{\mathbf{k}(2n)}^{(\varepsilon(2n))}(x) \Phi_N \right\rangle$$

Proof. This follows from von Waldenfels Lemma (see [5] for a general formulation). $\square$

Lemma 3. For any $\varepsilon \in \{-1, 1\}_+$, $\pi = \{(l_h, r_h)\}_{h=1}^{2n} \in PP(2n, \varepsilon)$ and $\mathbf{k} \in PF(N, n, \pi)$, one has

$$\begin{aligned} & \sigma_{\mathbf{k}(1)}^{(\varepsilon(1))}(x) \dots \sigma_{\mathbf{k}(2n)}^{(\varepsilon(2n))}(x) \Phi_N = \\ & = x^{\sum_{m=1}^n |\{h \in \{1, \dots, n\} : l_m < r_h < r_m\}|} \bar{x}^{\sum_{m=1}^n |\{h \in \{1, \dots, n\} : l_m < l_h < r_m\}|} \Phi_N \end{aligned} \quad (2.13)$$

Proof. The definition of symmetric embedding implies that, for any $j \in V_N$, the j -th tensor factor of $\sigma_{\mathbf{k}(1)}^{(\varepsilon(1))}(x) \dots \sigma_{\mathbf{k}(2n)}^{(\varepsilon(2n))}(x)$ has the form

$$\begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & x_1 \end{pmatrix} \sigma^{(-1)} \begin{pmatrix} 1 & 0 \\ 0 & x_2 \end{pmatrix} \sigma^{(+1)} \begin{pmatrix} 1 & 0 \\ 0 & x_3 \end{pmatrix}, & \text{if } j \in \text{Range}(\mathbf{k}) \\ \begin{pmatrix} 1 & 0 \\ 0 & |x|^{2n} \end{pmatrix}, & \text{if } j \notin \text{Range}(\mathbf{k}) \end{cases}$$

Moreover, if for some $m \in \{1, \dots, n\}$, either $j = \mathbf{k}(l_m)$ or $j = \mathbf{k}(r_m)$)

$$x_2 := x^{|\{h \in \{1, \dots, n\} : l_m < r_h < r_m\}|} \bar{x}^{|\{h \in \{1, \dots, n\} : l_m < l_h < r_m\}|}$$

Since the range of $\mathbf{k}$ consists of n -points, one gets

$$\sigma_{\mathbf{k}(1)}^{(\varepsilon(1))}(x) \dots \sigma_{\mathbf{k}(2n)}^{(\varepsilon(2n))}(x) \Phi_N = x^{\sum_{m=1}^n |\{h \in \{1, \dots, n\} : l_m < r_h < r_m\}|} \bar{x}^{\sum_{m=1}^n |\{h \in \{1, \dots, n\} : l_m < l_h < r_m\}|}$$

and this is (2.13). $\square$

The following Lemma gives the explicit expression of the sums on the right hand side of (2.13).

Lemma 4. For any $\{(l_h, r_h)\}_{h=1}^{2n} \in PP(2n)$,

$$\begin{aligned} \sum_{m=1}^n |\{h \in \{1, \dots, n\} : l_m < r_h < r_m\}| &= \sum_{m=1}^n |\{h \in \{1, \dots, n\} : l_m < l_h < r_m\}| \\ &= \frac{1}{2} \sum_{h=1}^n (r_h - l_h - 1) \end{aligned} \quad (2.14)$$

Proof. The identity (2.14) is trivial for $n = 1$. Suppose by induction that it is true for n , and let $\{(l_h, r_h)\}_{h=1}^{n+1} \in PP(2n+2)$. We know that $l_{n+1} < r_{n+1} \leq 2n+2$ and that l_{n+1} is the maximum among the left elements of the pair partition. Denote by

$$p := r_{n+1} - 1 - l_{n+1} = |\{l_{n+1} + 1, \dots, r_{n+1} - 1\}|$$

$$s := 2n + 2 - r_{n+1} = |\{r_{n+1} + 1, \dots, 2n + 2\}|$$

$$\{l_{n+1} + 1, \dots, r_{n+1} - 1\} =: \{r_{k_1}, \dots, r_{k_p}\} ; \{r_{n+1} + 1, \dots, 2n + 2\} =: \{r_{h_1}, \dots, r_{h_s}\}$$

Then

$$\begin{aligned} & \sum_{m=1}^{n+1} |\{h \in \{1, \dots, n+1\} : l_m < r_h < r_m\}| \\ &= \sum_{m=1}^n |\{h \in \{1, \dots, n+1\} : l_m < r_h < r_m\}| + \\ & \quad + |\{h \in \{1, \dots, n+1\} : l_{n+1} < r_h < r_{n+1}\}| \end{aligned}$$

For any $m \in \{h_1, \dots, h_s\}$, one has $l_m < r_{n+1} < r_m$, therefore

$$\begin{aligned} & |\{h \in \{1, \dots, n+1\} : l_m < r_h < r_m\}| = |\{h \in \{1, \dots, n\} : l_m < r_h < r_m\}| + 1 \\ & \sum_{m \in \{h_1, \dots, h_s\}} |\{h \in \{1, \dots, n+1\} : l_m < r_h < r_m\}| \\ &= \sum_{m \in \{h_1, \dots, h_s\}} |\{h \in \{1, \dots, n\} : l_m < r_h < r_m\}| + s \end{aligned}$$

If either $r_m < l_{n+1}$, or $m \in \{k_1, \dots, k_p\}$, one has $r_m < r_{n+1}$, therefore

$$\begin{aligned} & |\{h \in \{1, \dots, n+1\} : l_m < r_h < r_m\}| = |\{h \in \{1, \dots, n\} : l_m < r_h < r_m\}| \\ & \sum_{m \in \{k_1, \dots, k_p\} \cup \{m \in \{1, \dots, n+1\} : r_m < l_{n+1}\}} |\{h \in \{1, \dots, n+1\} : l_m < r_h < r_m\}| \\ &= \sum_{m \in \{k_1, \dots, k_p\}} |\{h \in \{1, \dots, n\} : l_m < r_h < r_m\}| \end{aligned}$$

Therefore

$$\begin{aligned} & \sum_{m \in \{k_1, \dots, k_p\} \cup \{m \in \{1, \dots, n+1\} : r_m < l_{n+1}\} \cup \{h_1, \dots, h_s\}} |\{h \in \{1, \dots, n+1\} : l_m < r_h < r_m\}| \\ &= \sum_{m \in \{k_1, \dots, k_p\} \cup \{m \in \{1, \dots, n+1\} : r_m < l_{n+1}\} \cup \{h_1, \dots, h_s\}} |\{h \in \{1, \dots, n\} : l_m < r_h < r_m\}| + s \end{aligned}$$

Since the index set of the summations can be identified to $\{1, \dots, n\}$, we conclude that

$$\begin{aligned} & \sum_{m \in \{1, \dots, n\}} |\{h \in \{1, \dots, n+1\} : l_m < r_h < r_m\}| \\ &= \sum_{\{1, \dots, n\}} |\{h \in \{1, \dots, n\} : l_m < r_h < r_m\}| + s \end{aligned}$$

This implies

$$\begin{aligned} & \sum_{m=1}^{n+1} |\{h \in \{1, \dots, n+1\} : l_m < r_h < r_m\}| \\ &= \sum_{m=1}^n |\{h \in \{1, \dots, n+1\} : l_m < r_h < r_m\}| + |\{h \in \{1, \dots, n+1\} : l_{n+1} < r_h < r_{n+1}\}| \\ &= \sum_{m=1}^n |\{h \in \{1, \dots, n+1\} : l_m < r_h < r_m\}| + s + p \end{aligned}$$

Now consider the inequalities of the form $l_m < l_h < r_m$. In the above notations, if $m \in \{k_1, \dots, k_p\} \cup \{h_1, \dots, h_s\}$, then $l_m < l_{n+1} < r_m$, so that

$$|\{h \in \{1, \dots, n+1\} : l_m < l_h < r_m\}| = |\{h \in \{1, \dots, n\} : l_m < l_h < r_m\}| + 1$$

Hence

$$\begin{aligned} & \sum_{m \in \{k_1, \dots, k_p\} \cup \{h_1, \dots, h_s\}} |\{h \in \{1, \dots, n+1\} : l_m < l_h < r_m\}| \\ &= \sum_{m \in \{k_1, \dots, k_p\} \cup \{h_1, \dots, h_s\}} |\{h \in \{1, \dots, n\} : l_m < l_h < r_m\}| + p + s \end{aligned}$$

From this, since we know that, if $r_m < l_{n+1}$, then clearly

$$|\{h \in \{1, \dots, n+1\} : l_m < l_h < r_m\}| = |\{h \in \{1, \dots, n\} : l_m < l_h < r_m\}|$$

we deduce that

$$\begin{aligned} & \left(\sum_{m \in \{k_1, \dots, k_p\} \cup \{h_1, \dots, h_s\}} + \sum_{m \in \{m \in \{1, \dots, n+1\} : r_m < l_{n+1}\}} \right) \\ & \quad |\{h \in \{1, \dots, n+1\} : l_m < r_h < r_m\}| \\ &= \left(\sum_{m \in \{k_1, \dots, k_p\} \cup \{h_1, \dots, h_s\}} + \sum_{m \in \{m \in \{1, \dots, n+1\} : r_m < l_{n+1}\}} \right) \\ & \quad |\{h \in \{1, \dots, n\} : l_m < r_h < r_m\}| + p + s \end{aligned}$$

This implies, using again the identification of $\{1, \dots, n\}$ with $\{k_1, \dots, k_p\} \cup \{m \in \{1, \dots, n+1\} : r_m < l_{n+1}\} \cup \{h_1, \dots, h_s\}$, that

$$\begin{aligned} & \sum_{m \in \{1, \dots, n\}} |\{h \in \{1, \dots, n+1\} : l_m < r_h < r_m\}| \\ &= \sum_{m \in \{1, \dots, n\}} |\{h \in \{1, \dots, n\} : l_m < r_h < r_m\}| + p + s \end{aligned}$$

In conclusion

$$\begin{aligned} & \sum_{m=1}^{n+1} |\{h \in \{1, \dots, n+1\} : l_m < l_h < r_m\}| \\ &= \sum_{m=1}^n |\{h \in \{1, \dots, n+1\} : l_m < l_h < r_m\}| \\ & \quad + |\{h \in \{1, \dots, n+1\} : l_{n+1} < l_h < r_{n+1}\}| \\ &= \sum_{m=1}^n |\{h \in \{1, \dots, n+1\} : l_m < l_h < r_m\}| \\ &= \sum_{m \in \{1, \dots, n\}} |\{h \in \{1, \dots, n\} : l_m < l_h < r_m\}| + p + s \end{aligned}$$

and the identity (2.14) follows from the induction assumption.

To prove the identity (2.14), notice that, for any $m \in \{1, \dots, n\}$

$$\begin{aligned} & |\{h \in \{1, \dots, n\} : l_m < r_h < r_m\}| + |\{h \in \{1, \dots, n\} : l_m < l_h < r_m\}| \\ &= \text{the cardinality of } \{\text{right indexes in } \{l_m + 1, \dots, r_m - 1\}\} + \\ &+ \text{the cardinality of } \{\text{left indexes in } \{l_m + 1, \dots, r_m - 1\}\} \\ &= r_m - 1 - l_m \end{aligned}$$

Summing over $m \in \{1, \dots, n\}$ and using (2.14), one gets

$$\begin{aligned} \sum_{m=1}^n (r_m - l_m - 1) &= \sum_{m \in \{1, \dots, n\}} |\{h \in \{1, \dots, n\} : l_m < r_h < r_m\}| + \\ &+ \sum_{m \in \{1, \dots, n\}} |\{h \in \{1, \dots, n\} : l_m < l_h < r_m\}| \\ &= 2 \sum_{m \in \{1, \dots, n\}} |\{h \in \{1, \dots, n\} : l_m < l_h < r_m\}| \\ &= 2 \sum_{m \in \{1, \dots, n\}} |\{h \in \{1, \dots, n\} : l_m < r_h < r_m\}| \end{aligned}$$

which is equivalent to (2.14). $\square$

Lemma 5. For any $x \in \mathbb{C}$, $\varepsilon \in \{-1, 1\}_+^{2n}$, $\pi = \{(l_h, r_h)\}_{h=1}^{2n} \in PP(2n, \varepsilon)$ and for any

$\mathbf{k} \in PF(N, n, \pi)$, in the notation (2.9) one has

$$\begin{aligned} \lim_{N \rightarrow \infty} \left\langle \Phi_N, S_N^{(\varepsilon(1))}(x) \dots S_N^{(\varepsilon(2n))}(x) \Phi_N \right\rangle &= |PP(2n, \varepsilon)| \cdot |x|^{\sum_{h=1}^{2n} (r_h^\varepsilon - l_h^\varepsilon - 1)} \\ &= \prod_{h=1}^{2n} (2h - l_h^\varepsilon) |x|^{(r_h^\varepsilon - l_h^\varepsilon - 1)} \quad (2.15) \end{aligned}$$

Proof. From Lemma 2 we know that for any $\varepsilon \in \{-1, +1\}_+^{2n}$,

$$\begin{aligned} & \left\langle \Phi_N, S_N^{(\varepsilon(1))}(x) \dots S_N^{(\varepsilon(2n))}(x) \Phi_N \right\rangle \\ &= o(1) + \frac{1}{|V_N|^n} \sum_{\pi = \{(l_h, r_h)\}_{h=1}^{2n} \in PP(2n, \varepsilon)} \sum_{\mathbf{k} \in PF(N, n, \pi)} \left\langle \Phi_N, \sigma_{\mathbf{k}(1)}^{(\varepsilon(1))}(x) \dots \sigma_{\mathbf{k}(2n)}^{(\varepsilon(2n))}(x) \Phi_N \right\rangle \end{aligned}$$

Using (2.13), (2.14), (2.14), (2.12) (in the order), this becomes

$$\begin{aligned} & o(1) + \frac{1}{|V_N|^n} \sum_{\pi = \{(l_h, r_h)\}_{h=1}^{2n} \in PP(2n, \varepsilon)} \sum_{\mathbf{k} \in PF(N, n, \pi)} \\ & x^{\sum_{m=1}^n |\{h \in \{1, \dots, n\} : l_m < r_h < r_m\}|} \bar{x}^{\sum_{m=1}^n |\{h \in \{1, \dots, n\} : l_m < l_h < r_m\}|} = \\ &= o(1) + \frac{1}{|V_N|^n} \sum_{\pi = \{(l_h, r_h)\}_{h=1}^{2n} \in PP(2n, \varepsilon)} \sum_{\mathbf{k} \in PF(N, n, \pi)} x^{\frac{1}{2} \sum_{h=1}^{2n} (r_h - l_h - 1)} \bar{x}^{\frac{1}{2} \sum_{h=1}^{2n} (r_h - l_h - 1)} = \\ &= o(1) + |PP(2n, \varepsilon)| \frac{|PF(N, n, \pi)|}{|V_N|^n} |x|^{\sum_{h=1}^{2n} (r_h - l_h - 1)} \end{aligned}$$

$$\begin{aligned}
&= o(1) + |PP(2n, \varepsilon)| |x|^{\sum_{h=1}^n (r_h - l_h - 1)} \\
&= o(1) + (2h - l_h^\varepsilon) |x|^{\sum_{h=1}^n (r_h - l_h - 1)}
\end{aligned}$$

and this is equivalent to (2.15). $\square$

Definition 1. A sequence $\omega := (\omega_n)$ of positive real numbers is called a *principal Jacobi sequence* if

$$\omega_n = 0 \Rightarrow \omega_{n+k} = 0 \quad ; \quad \forall n, k \in \mathbb{N} \quad (2.16)$$

The **1-mode interacting Fock space (1-MIFS) with principal Jacobi sequence** (ω_n) is the (unique up to isomorphism) pre-Hilbert space, denoted $(\Gamma_\omega, a_\omega^+, \Phi_\omega)$ characterized by the property of containing a linear orthogonal basis (Φ_n) satisfying

$$\|\Phi_n\|^2 = \omega_n! := \omega_n \omega_{n-1} \cdots \omega_1 \quad (2.17)$$

The **creation operator** a_ω^+ is defined by $a_\omega^+ \Phi_n := \Phi_{n+1}$. Its adjoint, called **annihilation operator** and denoted a_ω , satisfies $a_\omega \Phi_0 = 0$.

Remark. Condition (2.16) guarantees the existence of the adjoint.

Definition 2. If $\theta = \{(l_h, r_h)\}_{h=1}^n \in PP(2n)$, and $k \in \{1, \dots, n\}$, the **depth of the pair** (l_k, r_k) in θ is defined by

$$\begin{aligned}
d_\theta(l_k, r_k) &:= 1 + |\{h \in \{1, \dots, n\} : l_h \leq l_k < r_k \leq r_h\}| \\
&= |\{\text{pairs} \in \theta \text{ containing } (l_k, r_k)\}|
\end{aligned} \quad (2.18)$$

Theorem 1. For any $m \in \mathbb{N}$ and $\varepsilon \in \{-1, +1\}^m$, one has:

$$\begin{aligned}
&\lim_{N \rightarrow \infty} \left\langle \Phi_N, S_N^{(\varepsilon(1))}(x) \dots S_N^{(\varepsilon(m))}(x) \Phi_N \right\rangle = \\
&= \begin{cases} \prod_{h=1}^n (2h - l_h^\varepsilon) |x|^{(r_h^\varepsilon - l_h^\varepsilon - 1)} & , \text{ if } m = 2n \text{ and } \varepsilon \in \{-1, +1\}_+^{2n} \\ 0, & \text{ otherwise} \end{cases}
\end{aligned} \quad (2.19)$$

Moreover, denoting by $(\mathcal{H}_L, a_L^+, \Phi_L)$ the cyclic space of the limit (2.19) and $(\Gamma_\omega, a_\omega^+, \Phi_\omega)$ the 1-MIFS with principal Jacobi sequence

$$\omega_m := m |x|^{2(m-1)} \quad ; \quad \forall m \in \mathbb{N} \quad (2.20)$$

then the map

$$a_L^{(\varepsilon(1))} \dots a_L^{(\varepsilon(m))} \Phi_L \mapsto a_\omega^{(\varepsilon(1))} \dots a_\omega^{(\varepsilon(m))} \Phi_\omega \quad ; \quad m \in \mathbb{N}, \varepsilon \in \{-1, +1\}^m \quad (2.21)$$

uniquely extends to a unitary pre-Hilbert space isomorphism $U : \mathcal{H}_L \rightarrow \Gamma_\omega$ satisfying

$$U a_L^{(\pm 1)} U^* = a_\omega^{(\pm 1)} \quad ; \quad U \Phi_L = \Phi_\omega \quad (2.22)$$

Proof. (2.19) follows from Lemma 5. Moreover

$$\prod_{h=1}^n (2h - l_h^\varepsilon) |x|^{(r_h^\varepsilon - l_h^\varepsilon - 1)} = \prod_{h=1}^n (2h - l_h^\varepsilon) |x|^{\sum_{h=1}^n (r_h^\varepsilon - l_h^\varepsilon - 1)}$$

and from Lemma 3 and (2.18) we know that

$$\sum_{m=1}^n |\{h \in \{1, \dots, n\} : l_m < r_h < r_m\}| \quad (2.23)$$

$$\begin{aligned} &= \sum_{m=1}^n |\{h \in \{1, \dots, n\} : l_m < l_h < r_m\}| = \\ &= \sum_{h=1}^n (d_\theta(l_h, r_h) - 1) = \frac{1}{2} \sum_{h=1}^n (r_h - l_h - 1) \end{aligned} \quad (2.24)$$

On the other hand, from Lemma 3.1 of [3] we know that,

$$d_\theta(l_h, r_k) = 2h - l_h \quad (2.25)$$

Therefore

$$\sum_{h=1}^n (r_h - l_h - 1) = 2 \sum_{h=1}^n (2h - l_h - 1)$$

so that

$$\prod_{h=1}^n (2h - l_h^\varepsilon) |x|^{(r_h^\varepsilon - l_h^\varepsilon - 1)} = \prod_{h=1}^n (2h - l_h^\varepsilon) |x|^{2 \sum_{k=1}^n (2k - l_k^\varepsilon - 1)} = \prod_{h=1}^n (2h - l_h^\varepsilon) |x|^{2(2h - l_h^\varepsilon) - 1}$$

From Corollary (3.1) of [3], one knows that, for general ω :

$$\langle \Phi, a_\omega^{\varepsilon(1)} \dots a_\omega^{\varepsilon(2n)} \Phi \rangle = \prod_{k=1}^n \omega_{2k - l_k} \quad ; \quad \forall \varepsilon \in \{-1, 1\}_+^{2n}, \quad \forall n \quad (2.26)$$

where $\{(l_h^\varepsilon, r_h^\varepsilon)\}_{h=1}^n$ denotes the unique non-crossing pair partition of $\{1, \dots, 2n\}$ corresponding to ε . But in our case, from (2.20) we know that $\omega_m := m |x|^{2(m-1)}$. Therefore, with this choice of ω , the limit (2.19) becomes equal to

$$\prod_{h=1}^n (2h - l_h^\varepsilon) |x|^{2(2h - l_h^\varepsilon) - 1} = \prod_{h=1}^n (2k - l_k^\varepsilon) |x|^{2(2k - l_k^\varepsilon - 1)} = \prod_{k=1}^n \omega_{2k - l_k}$$

which coincides with the quantum moment

$$\langle \Phi_\omega, a_\omega^{(\varepsilon(1))} \dots a_\omega^{(\varepsilon(m))} \Phi_\omega \rangle$$

In particular the map (2.21) preserves scalar products. Since Φ_L (resp. Φ_ω) is cyclic for the polynomial algebra generated by the $a_L^{(\pm 1)}$ (resp. $a_\omega^{(\pm 1)}$), it follows that this map extends to a unitary pre-Hilbert space isomorphism. The intertwining properties (2.22) are easy consequences of (2.21). $\square$

3. Description of the limit 1–Mode Interacting Fock Space

Using the commutation relations for the 1–Mode Interacting Fock Space associated to the principal Jacobi sequence emerging from Theorem 1, i.e.

$$\omega_m := m |x|^{2(m-1)} \quad ; \quad \forall m \in \mathbb{N} \quad (3.1)$$

one finds

$$\begin{aligned} aa^+ - |x|^2 a^+ a &= \omega_{\Lambda+1} - |x|^2 \omega_\Lambda = (\Lambda + 1) |x|^{2\Lambda} - \Lambda |x|^2 |x|^{2(\Lambda-1)} \\ &= (\Lambda + 1) |x|^{2\Lambda} - \Lambda |x|^{2\Lambda} = |x|^{2\Lambda} \end{aligned} \quad (3.2)$$

Denoting $q := |x|^2$, the identity (3.2) shows that the random variable corresponding to the principal Jacobi sequence (3.1) is 1–mode version of the q^Λ –deformation of the Boson Fock Brownian motion, introduced by Parthasarathy in his quantum decomposition of the Azema martingale [7].

In section 2.3 of the paper [3] it was proved that $q \neq 0$, (3.2) can be reformulated as the following **generalized q –commutation relation**:

$$aa^+ - Q_\Lambda a^+ a = 1 \quad (3.3)$$

where

$$Q_\Lambda \xi := \begin{cases} \xi, & \text{if } \xi = \Phi \\ \left(\Lambda |x|^{2(\Lambda-1)} \right)^{-1} \left((\Lambda + 1) |x|^{2\Lambda} - 1 \right) \xi, & \text{if } \xi \in \Gamma_1 \end{cases}$$

and

$$\Gamma_1 := \{\Phi\}^\perp := \{\mathbb{C}, \langle \cdot, \cdot \rangle_1\} \oplus \{\mathbb{C}, \langle \cdot, \cdot \rangle_2\} \oplus \{\mathbb{C}, \langle \cdot, \cdot \rangle_3\} \oplus \dots$$

i.e., in the decomposition $\Gamma = \mathbb{C} \cdot \Phi \oplus \Gamma_1$:

$$Q_\Lambda = 1 \oplus \left(\Lambda |x|^{2(\Lambda-1)} \right)^{-1} \left((\Lambda + 1) |x|^{2\Lambda} - 1 \right)$$

Remark. One easily verifies that, as x runs over $\mathbb{C}$,

- $\{a, a^+\}$ verifies the CCR if and only if $|x| = 1$ (so $Q_\Lambda = 1$);
- With the exception of the case $|x| = 1$, there is no choice of x for which the right hand side of (3.2) is constant;
- The case $x = 0$ is dealt with separately. In this case the space is isomorphic to $\mathbb{C}^2$ and the relations are the fermionic ones which, at the level of 1–mode space, cannot be distinguished from the Boolean ones.

4. White noise approach to the $q^{\Lambda_{[0,t]}}$ –deformation

In this section we describe the continuous version of the 1–mode IFS described in the previous sections and we show that it coincides with the $|q|^2$ –deformed quantum Brownian introduced by Parthasarathy and Schürman [7], [8], [10], [9]. The use of white noise language allows to drastically simplify the proofs in [6], [7] which are based on stochastic calculus.

Denote $a_s^\pm$ the standard Boson white noise over $\mathbb{R}$:

$$[a_s, a_r^+] = \delta(s - r) a_s \quad ; \quad [a_s, a_r] = 0 \quad (4.1)$$

Order will be important: in this section, we cannot replace $\mathbb{R}$ by $\mathbb{R}^d$.

4.1. Some known formulas

Lemma 6. With the notation

$$\int_0^t a_s^+ a_s ds = \Lambda_{[0,t]}(1) =: \Lambda_{[0,t]} \quad (4.2)$$

to be meant in the sense

$$\lim_{\varepsilon \rightarrow 0} \int_0^{t+\varepsilon} a_s^+ a_s ds = \Lambda_{[0,t]}(1) =: \Lambda_{[0,t]} \quad (4.3)$$

where the integral is a weak Bochner integral on the number vectors, one has

$$[\Lambda_{[0,t]}(1), a_r] = \begin{cases} -a_r, & \text{if } r \in [0, t] \\ 0, & \text{if } r \in (t, +\infty) \end{cases} \quad (4.4)$$

Moreover, for any $z \in \mathbb{C}$ and any $t \in \mathbb{R}$,

$$e^{z[\Lambda_{[0,t]}, \cdot]} a_t = e^{-z} a_t \quad (4.5)$$

Proof. We start from

$$[a_s^+ a_s, a_r] = [a_s^+, a_r] a_s = -\delta(r - s) a_s \quad (4.6)$$

(4.6) implies that

$$\begin{aligned} [\Lambda_{[0,t]}(1), a_r] &= \left[\lim_{\varepsilon \rightarrow 0} \int_0^{t+\varepsilon} a_s^+ a_s ds, a_r \right] = \lim_{\varepsilon \rightarrow 0} \int_0^{t+\varepsilon} ds [a_s^+ a_s, a_r] \\ &= - \lim_{\varepsilon \rightarrow 0} \int_0^{t+\varepsilon} ds \delta(s - r) a_r \end{aligned}$$

and this is equivalent to (4.4). Consequently

$$[\Lambda_{[0,t]}(1), \cdot]^n a_t = [\Lambda_{[0,t]}(1), \cdot]^{n-1} [\Lambda_{[0,t]}(1), a_t] = (-1) [\Lambda_{[0,t]}(1), \cdot]^{n-1} a_t = \cdots = (-1)^n a_t$$

and (4.5) follows from

$$e^{z[\Lambda_{[0,t]}(1), \cdot]} a_t = \sum_n \frac{z^n}{n!} [\Lambda_{[0,t]}(1), \cdot]^n a_t = \sum_n \frac{(-z)^n}{n!} a_t = e^{-z} a_t$$

Lemma 7. For general operators T and A on the Fock space, one has:

$$e^{z[T, \cdot]} A = e^{zT} A e^{-zT} \quad ; \quad \forall z \in \mathbb{C} \quad (4.7)$$

Proof. We only sketch the proof because this is a well known fact. Denoting,

$$F(z, A) := e^{z[T, \cdot]} A \quad ; \quad G(z, A) := e^{zT} A e^{-zT}$$

one has

$$\partial_z F(z, A) := e^{z[T, \cdot]} [T, A] = F(z, [T, A]) \quad ; \quad F(0, A) = A$$

$$\partial_z G(z, A) := e^{zT} T A e^{-zT} - e^{zT} A T e^{-zT} = e^{zT} [T, A] e^{-zT} = G(z, [T, A]) \quad ; \quad G(0, A) = A$$

Therefore

$$F(z, A) = G(z, A) \iff e^{z[T, \cdot]} A = e^{zT} A e^{-zT} \quad ; \quad \forall z \in \mathbb{C}$$

and this proves the statement.

4.2. Appendix: the $q^{\Lambda_{[0,t]}}$ -commutation relations

Lemma 8. For any $q \in \mathbb{C} \setminus \{0\}$ and any $t \in \mathbb{R}$, one has

$$a_t q^{\Lambda_{[0,t]}} = q q^{\Lambda_{[0,t]}} a_t \quad ; \quad q^{\Lambda_{[0,t]}} a_t^+ = q a_t^+ q^{\Lambda_{[0,t]}} \quad (4.8)$$

$$s < t \Rightarrow a_t q^{\Lambda_{[0,s]}} = q^{\Lambda_{[0,s]}} a_t \quad ; \quad q^{\Lambda_{(s,t]}} a_s = a_s q^{\Lambda_{(s,t]}} \quad (4.9)$$

Moreover the operators $a_{q,t}^{\pm}$, defined by

$$a_{q,t} := q^{\Lambda_{[0,t]}} a_t \quad (4.10)$$

for $q \neq 0$, satisfy the following q -commutation relations:

$$a_{q,s} a_{q,t}^+ - \bar{q} a_{q,t}^+ a_{q,s} = |q|^{2\Lambda_{[0,t]}} \delta(t-s) \quad , \text{ for } s < t \quad (4.11)$$

$$a_{q,s} a_{q,t}^+ - q a_{q,t}^+ a_{q,s} = |q|^{2\Lambda_{[0,t]}} \delta(t-s) \quad , \text{ for } s > t \quad (4.12)$$

Remark. (4.11) and (4.12) are understood in the sense that, for any pair of test functions f, g

$$\begin{aligned} [a_{q,f}, a_{q,g}^+]_q &:= \int \int_{s < t} ds dt \bar{f}(s) g(t) (a_{q,s} a_{q,t}^+ - \bar{q} a_{q,t}^+ a_{q,s}) \\ &\quad + \int \int_{s > t} ds dt \bar{f}(s) g(t) (a_{q,s} a_{q,t}^+ - q a_{q,t}^+ a_{q,s}) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} ds dt |q|^{2\Lambda_{[0,t]}} \bar{f}(s) g(t) \delta(t-s) = \int_{\mathbb{R}} dt |q|^{2\Lambda_{[0,t]}} \bar{f}(t) g(t) \end{aligned}$$

Proof. From (4.5) and (4.7) one has

$$e^{-z} a_t = e^{z[\Lambda_{[0,t]}(1), \cdot]} a_t = e^{z\Lambda_{[0,t]}(1)} a_t e^{-z\Lambda_{[0,t]}(1)} \iff e^{-z} a_t e^{z\Lambda_{[0,t]}(1)} = e^{z\Lambda_{[0,t]}} a_t$$

If $z = \log q$, then

$$e^{z\Lambda_{[0,t]}} = e^{\log q \Lambda_{[0,t]}} = e^{\log q^{\Lambda_{[0,t]}}} = q^{\Lambda_{[0,t]}} \quad (4.13)$$

and one has

$$q^{\Lambda_{[0,t]}} a_t = e^{z\Lambda_{[0,t]}} a_t = e^{-z} a_t e^{z\Lambda_{[0,t]}(1)} = \frac{1}{q} a_t q^{\Lambda_{[0,t]}} \iff a_t q^{\Lambda_{[0,t]}} = q q^{\Lambda_{[0,t]}} a_t$$

that is the first identity in (4.8). Taking adjoints of both sides of this one finds

$$\bar{q}^{\Lambda_{[0,t]}} a_t^+ = \bar{q} a_t^+ \bar{q}^{\Lambda_{[0,t]}}$$

and using the arbitrariness of q , one obtains the second identity in (4.8). Now let $a_{q,t}$ be defined by (4.10). Then

$$a_{q,t}^+ := a_t^+ \bar{q}^{\Lambda_{[0,t]}}$$

so that

$$a_{q,s} a_{q,t}^+ = q^{\Lambda_{[0,s]}} a_s a_t^+ \bar{q}^{\Lambda_{[0,t]}} \quad (4.14)$$

Using (4.8) with $q \neq 0$, one obtains, if $s < t$

$$\begin{aligned} a_{q,t}^+ a_{q,s} &= a_t^+ \bar{q}^{\Lambda_{[0,t]}} q^{\Lambda_{[0,s]}} a_s = a_t^+ q^{\Lambda_{[0,s]}} \bar{q}^{\Lambda_{[0,t]}} a_s = a_t^+ q^{\Lambda_{[0,s]}} \bar{q}^{\Lambda_{[0,s]}} \bar{q}^{\Lambda_{(s,t]}} a_s \\ &= \bar{q}^{-1} q^{\Lambda_{[0,s]}} a_t^+ a_s \bar{q}^{\Lambda_{[0,t]}} \end{aligned}$$

Therefore, from (4.14),

$$\begin{aligned} a_{q,s} a_{q,t}^+ - \bar{q} a_{q,t}^+ a_{q,s} &= q^{\Lambda_{[0,s]}} a_s a_t^+ \bar{q}^{\Lambda_{[0,t]}} - \bar{q} \bar{q}^{-1} q^{\Lambda_{[0,s]}} a_t^+ a_s \bar{q}^{\Lambda_{[0,t]}} = q^{\Lambda_{[0,s]}} [a_s, a_t^+] \bar{q}^{\Lambda_{[0,t]}} \\ &= q^{\Lambda_{[0,s]}} \delta(s-t) \bar{q}^{\Lambda_{[0,t]}} = \delta(s-t) |q|^{2\Lambda_{[0,t]}} \end{aligned}$$

which is (4.11). Similarly, if $s > t$

$$\begin{aligned} a_{q,t}^+ a_{q,s} &= a_t^+ \bar{q}^{\Lambda_{[0,t]}} q^{\Lambda_{[0,s]}} a_s = a_t^+ q^{\Lambda_{[0,s]}} \bar{q}^{\Lambda_{[0,t]}} a_s = a_t^+ q^{\Lambda_{[0,t]}} q^{\Lambda_{(t,s]}} \bar{q}^{\Lambda_{[0,t]}} a_s \\ &= q^{\Lambda_{(t,s]}} a_t^+ q^{\Lambda_{[0,t]}} a_s \bar{q}^{\Lambda_{[0,t]}} = q^{-1} q^{\Lambda_{(t,s]}} q^{\Lambda_{[0,t]}} a_t^+ a_s \bar{q}^{\Lambda_{[0,t]}} = q^{-1} q^{\Lambda_{[0,s]}} a_t^+ a_s q^{\Lambda_{[0,t]}} \end{aligned}$$

Therefore, using again (4.14),

$$\begin{aligned} a_{q,s} a_{q,t}^+ - q a_{q,t}^+ a_{q,s} &= q^{\Lambda_{[0,s]}} a_s a_t^+ \bar{q}^{\Lambda_{[0,t]}} - q q^{-1} q^{\Lambda_{[0,s]}} a_t^+ a_s q^{\Lambda_{[0,t]}} = q^{\Lambda_{[0,s]}} [a_s, a_t^+] \bar{q}^{\Lambda_{[0,t]}} \\ &= q^{\Lambda_{[0,s]}} \delta(s-t) \bar{q}^{\Lambda_{[0,t]}} = |q|^{2\Lambda_{[0,t]}} \delta(s-t) \end{aligned}$$

which is (4.12).

Remark. In (4.11), (4.12):

- choosing $q = +1$ one finds the standard Boson white noise,
- choosing $q = -1$ one finds the standard Fermi white noise,
- choosing $q \in \mathbb{R}$ one finds the Parthasarathy–Schürmann white noise.
- choosing $q = 0$ and with the convention

$$0^0 := 1 \tag{4.15}$$

we see that, for $q = 0$, the operator $a_{q,t}$ can be non-zero only on the 1-particle q -sub-space

$$a_{q,t}^+ \Phi := a_t^+ 0^{\Lambda_{[0,t]}} \Phi = a_t^+ \Phi \tag{4.16}$$

and that, for $n \geq 2$, all the n -particle sectors are zero. Therefore, in this case, the q -Fock space is 2-dimensional (in agreement with the canonical decomposition of a Bernoulli random variable) discussed in [1].

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