**This is the pre-print of the paper:**
**S. De Candia; M. Falcitelli,**
**Curvature of C_5 ⊕ C_12 -manifolds;**
**Mediterr. J. Math. 16 (2019), no. 4, Paper No. 105, 23 pp**

**DOI: https://dx.doi.org/10.1007/s00009-019-1382-2**

# CURVATURE OF $C_5 \oplus C_{12}$-MANIFOLDS

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Abstract. The Chinea-Gonzalez class $C_5 \oplus C_{12}$ consists of the almost contact metric manifolds that are locally described as double-twisted product manifolds $I \times_{(\lambda_1,\lambda_2)} \widehat{M}$, $I \subset \mathbb{R}$ being an open interval, $\widehat{M}$ a Kähler manifold and $\lambda_1, \lambda_2$ smooth positive functions. In this article we investigate the behavior of the curvature of $C_5 \oplus C_{12}$-manifolds. Particular attention to the $N(k)$-nullity condition is given and some local classification theorems in dimension $2n+1 \geq 5$ are stated. This allows us to classify $C_5 \oplus C_{12}$-manifolds that are generalized Sasakian space-forms. In addition, we provide explicit examples of these spaces.

## 1. Introduction

Double-twisted products play an interesting role in clarifying the interrelation between almost Hermitian (a.H.) and almost contact metric (a.c.m.) manifolds. In fact, the Chinea-Gonzalez class $C_{1-5} \oplus C_{12} = \bigoplus_{1 \leq i \leq 5} C_i \oplus C_{12}$ consists of the a.c.m. manifolds that are, locally, double-twisted products $]-\epsilon, \epsilon[\times_{(\lambda_1,\lambda_2)}\widehat{M} = (]-\epsilon, \epsilon[\times \widehat{M}, \varphi, \xi, \eta, g_{(\lambda_1,\lambda_2)})$, $\epsilon > 0$, $(\widehat{M}, \widehat{J}, \widehat{g})$ being an a.H. manifold, $\lambda_1, \lambda_2 \colon ]-\epsilon, \epsilon[\times \widehat{M} \to \mathbb{R}$ smooth positive functions and $(\varphi, \xi, \eta, g_{(\lambda_1,\lambda_2)})$ the structure defined in (2.1). The class $C_5 \oplus C_{12}$ is the subclass of $C_{1-5} \oplus C_{12}$ consisting of the a.c.m. manifolds that are locally realized as double-twisted products $]-\epsilon, \epsilon[\times_{(\lambda_1,\lambda_2)}\widehat{M}$, where $(\widehat{M}, \widehat{J}, \widehat{g})$ is a Kähler manifold [9]. This points out the interrelation between Kähler and $C_5 \oplus C_{12}$-manifolds.

Relevant results involving the behavior of the curvature of Kähler manifolds are well-known [13, 17].

In this article we develop a systematic study of the curvature of $C_5 \oplus C_{12}$-manifolds and obtain some classification theorems for those manifolds that satisfy suitable curvature conditions. We also recall that, considering an a.c.m. manifold $(M, \varphi, \xi, \eta, g)$ with fundamental 2-form $\Phi$ and Levi-Civita connection $\nabla$, the $C_5$, $C_{12}$ components of $\nabla\Phi$ are determined by the codifferential $\delta\eta$ and the 1-form $\nabla_\xi \eta$, respectively [6]. This allows to specify the defining conditions for the manifolds which fall in the class $C_5 \oplus C_{12}$ and in its proper subclasses $C_5$, $C_{12}$.

Let $(M, \varphi, \xi, \eta, g)$ be a $C_5 \oplus C_{12}$-manifold, with $\dim M = 2n+1$, and put $\alpha = -\frac{\delta\eta}{2n}$, $V = \nabla_\xi \xi$. For any vector fields $X, Y$, the "cosymplectic defect" $R(X,Y) \circ \varphi - \varphi \circ R(X,Y)$, $R$ denoting the curvature of $\nabla$, depends on $\alpha$, $d\alpha$, $V$ and $\nabla V$. In Section 3 we evaluate the cosymplectic defect and derive several consequences, involving the Ricci and the $*$-Ricci tensors, also.
We put our attention to the $(k, \mu)$-condition proving that, in the context of $C_5 \oplus C_{12}$-manifolds, it is equivalent to the $N(k)$-condition. Considering an $N(k)$-manifold of dimension $2n+1 \geq 5$, the function $k$ is expressed as a combination of $\alpha$, $\xi(\alpha)$ and $\operatorname{div} V$. Several properties of $N(k)$-manifolds are derived. In particular, we prove that a manifold with constant sectional curvature $k$ either is a $C_5$-manifold and $k < 0$ or it is flat and falls in the class $C_{12}$. Moreover, suitable $N(k)$-spaces are locally isometric to a warped product $N \times_\lambda N'$, $N$ being a 2-dimensional Riemannian manifold of Gaussian curvature $k$ and $N'$ is endowed with an $\overline{\alpha}$-Sasakian structure.
Section 6 deals with $C_5 \oplus C_{12}$-manifolds that are generalized Sasakian (g.S.) space-forms. These spaces are characterized as the $N(k)$-manifolds with pointwise constant $\varphi$-sectional curvature, say

2010 *Mathematics Subject Classification.* primary 53C25, 53C15, secondary 53B20, 53C21.
*Key words and phrases.* Double-twisted product manifold, Curvature, Cosymplectic defect, $N(k)$-manifold, Generalized Sasakian space-form.

$c$. Denoting by $M^{2n+1}(c,k)$, $n \geq 2$, a g.S. space-form, we prove that the function $c+\alpha^2$ satisfies a suitable differential equation. This allows us to state a classification theorem. More precisely, if $M^{2n+1}(c,k)$ is a g.S. space-form in the class $C_5 \oplus C_{12}$ and $\alpha = 0$, then either $M$ is cosymplectic or it falls in the class $C_{12}$ and $c = 0$. If $\alpha \neq 0$, then either $M$ is locally conformal to $C_{12}$-manifolds that are g.S. space-forms with zero $\varphi$-sectional curvature or $M$ is $\alpha$-Kenmotsu and globally conformal to a cosymplectic manifold with constant $\varphi$-sectional curvature.
Finally, in Section 7, for any $n \geq 2$, we construct a family of $C_{12}$-manifolds $M^{2n+1}(0,k)$.

Throughout this article, all manifolds are assumed smooth and connected.

## 2. Preliminaries

Given an almost Hermitian (a.H.) manifold $(\widehat{M}, \widehat{J}, \widehat{g})$, an open interval $I \subset \mathbb{R}$ and two smooth positive functions $\lambda_1, \lambda_2 \colon I \times \widehat{M} \to \mathbb{R}$, one considers the almost contact metric (a.c.m.) structure $(\varphi, \xi, \eta, g_{(\lambda_1,\lambda_2)})$ on the product manifold $I \times \widehat{M}$, acting as

$$\begin{aligned} &\varphi\Big(a\frac{\partial}{\partial t}, X\Big) = (0, \widehat{J}X), \quad \eta\Big(a\frac{\partial}{\partial t}, X\Big) = a\lambda_1, \\ &\xi = \frac{1}{\lambda_1}\Big(\frac{\partial}{\partial t}, 0\Big), \quad g_{(\lambda_1,\lambda_2)} = \lambda_1^2\pi_1^*(dt \otimes dt) + \lambda_2^2\pi_2^*(\widehat{g}), \end{aligned} \tag{2.1}$$

for any $a \in \mathfrak{F}(I \times \widehat{M}), X \in \Gamma(T\widehat{M})$, $\pi_1 \colon I \times \widehat{M} \to I$, $\pi_2 \colon I \times \widehat{M} \to \widehat{M}$ denoting the canonical projections. Note that $g_{(\lambda_1,\lambda_2)}$ is the double-twisted product of the Euclidean metric $g_0$ and $\widehat{g}$ [16]. The a.c.m. manifold $I \times_{(\lambda_1,\lambda_2)} \widehat{M} = (I \times \widehat{M}, \varphi, \xi, \eta, g_{(\lambda_1,\lambda_2)})$ is named the double-twisted product manifold of $(I, g_0)$ and $(\widehat{M}, \widehat{J}, \widehat{g})$ by $(\lambda_1, \lambda_2)$. If $\lambda_1 = 1$, $I \times_{(1,\lambda_2)} \widehat{M}$ is denoted by $I \times_{\lambda_2} \widehat{M}$ and is called the twisted product manifold of $(I, g_0)$ and $(\widehat{M}, \widehat{J}, \widehat{g})$ by $\lambda_2$. If $\lambda_2 = 1$, the manifold $I \times_{(\lambda_1,1)} \widehat{M}$ is denoted by ${}_{\lambda_1}I \times \widehat{M}$. In the case that $\lambda_1$ is independent of the Euclidean coordinate $t$ and $\lambda_2$ only depends on $t$, $I \times_{(\lambda_1,\lambda_2)} \widehat{M}$ is called a double-warped product manifold, the metric $g_{(\lambda_1,\lambda_2)}$ being just the double-warped product metric of $g_0$ and $\widehat{g}$ by $(\lambda_1, \lambda_2)$. If $\lambda_2$ only depends on $t$, $I \times_{\lambda_2} \widehat{M}$ is said to be a warped product manifold.

Applying the theory developed in [6], [9], we are able to specify the Chinea-Gonzalez class of the mentioned manifolds. In particular, if $\dim \widehat{M} = 2$, then $I \times_{(\lambda_1,\lambda_2)} \widehat{M}$ belongs to the class $C_5 \oplus C_{12}$. In the case that $\dim \widehat{M} = 2n \geq 4$, $(\widehat{J}, \widehat{g})$ is a Kähler structure and the function $\lambda_2$ is constant on $\widehat{M}$, then $I \times_{(\lambda_1,\lambda_2)} \widehat{M}$ is a $C_5 \oplus C_{12}$-manifold. Furthermore, if $\lambda_2 = 1$, ${}_{\lambda_1}I \times \widehat{M}$ falls in the class $C_{12}$. It is also known that any warped product manifold $I \times_{\lambda_2} \widehat{M}$, where $(\widehat{M}, \widehat{J}, \widehat{g})$ is a Kähler manifold, belongs to the class $C_5$ and is called an $\alpha$-Kenmotsu manifold, where $\alpha = \xi(\log \lambda_2)$. More generally, any double-warped product manifold $I \times_{(\lambda_1,\lambda_2)} \widehat{M}$, such that $(\widehat{M}, \widehat{J}, \widehat{g})$ is Kähler and both the functions $\lambda_1$, $\lambda_2$ are non constant, is in the class $C_5 \oplus C_{12} \setminus (C_5 \cup C_{12})$. This shows that $C_5$, $C_{12}$ are proper subclasses of $C_5 \oplus C_{12}$. Cosymplectic manifolds set up the class $C = C_5 \cap C_{12}$.

In Table 1 we list the defining conditions of any a.c.m. manifold $(M, \varphi, \xi, \eta, g)$ which falls in $C_5 \oplus C_{12}$ or in its subclasses. These conditions are formulated in terms of the covariant derivatives $\nabla\varphi$, $\nabla\eta$, $\nabla$ denoting the Levi-Civita connection of $M$. Note that, since $\nabla_\xi \xi$ is the vector field $g$-associated to the 1-form $\nabla_\xi \eta$, the vanishing of $\nabla_\xi \xi$ is equivalent to the condition that the considered manifold is in the class $C_5$, namely it is an $\alpha$-Kenmotsu manifold. Moreover, it is known that any $C_5 \oplus C_{12}$-manifold satisfies

$$\nabla_X \xi = \alpha(X - \eta(X)\xi) + \eta(X)\nabla_\xi \xi, \quad X \in \Gamma(TM) \tag{2.2}$$

$$d\eta = \eta \wedge \nabla_\xi \eta, \quad d(\nabla_\xi \eta) = -(\alpha \nabla_\xi \eta + \nabla_\xi(\nabla_\xi \eta)) \wedge \eta, \tag{2.3}$$

where $\dim M = 2n+1$ and $\alpha = -\frac{\delta\eta}{2n}$. Furthermore, if $\dim M \geq 5$, the Lee form of $M$ is $\omega = -\alpha\eta$ and it is closed. Applying (2.3), one has

$$d\alpha = \xi(\alpha)\eta + \alpha\nabla_\xi\eta. \tag{2.4}$$

**Table 1**

| Classes | Defining conditions |
|---|---|
| $C_5 \oplus C_{12}$ | $(\nabla_X\varphi)Y = \alpha(g(\varphi X, Y)\xi - \eta(Y)\varphi X) -\eta(X)((\nabla_\xi\eta)\varphi Y\xi + \eta(Y)\varphi(\nabla_\xi\xi))$ |
| $C_5$ | $(\nabla_X\varphi)Y = \alpha(g(\varphi X, Y)\xi - \eta(Y)\varphi X)$ |
| $C_{12}$ | $(\nabla_X\varphi)Y = -\eta(X)((\nabla_\xi\eta)\varphi Y\xi + \eta(Y)\varphi(\nabla_\xi\xi))$ |
| $C$ | $\nabla\varphi = 0$ |

In the sequel, given a $C_5 \oplus C_{12}$-manifold $(M, \varphi, \xi, \eta, g)$ we will denote by $D$, $D^\perp$ the mutually orthogonal distributions associated to the subbundles $\mathrm{Ker}\eta$ and $\mathrm{span}\{\xi\}$ of the tangent bundle $TM$, respectively. These distributions are both totally umbilical foliations. More precisely, $H = -\alpha\xi_{|N}$ is the mean curvature vector field of any leaf $(N, g')$ of $D$, $g'$ being the metric induced by $g$. Furthermore, $(J = \varphi_{|TN}, g')$ is a Kähler structure on $N$. For the sake of simplicity, we will denote by $V$ the vector field $\nabla_\xi\xi$, which represents the mean curvature vector field of any integral curve of $D^\perp$.

Applying the main results in [9], [16], one obtains a local description of a $C_5 \oplus C_{12}$-manifold $(M, \varphi, \xi, \eta, g)$. More precisely, for any point $x \in M$, there exist an open neighborhood $U$ of $x$, $\epsilon > 0$, a Riemannian manifold $(F, \widehat{g})$, two smooth positive functions $\lambda_1, \lambda_2\colon ]-\epsilon, \epsilon[\times F \to \mathbb{R}$ and an isometry $f\colon (]-\epsilon, \epsilon[\times F, g_{(\lambda_1,\lambda_2)}) \to (U, g_{|U})$ such that the canonical foliations of the product manifold correspond to the distributions $D$, $D^\perp$. It follows that $f_*(\frac{1}{\lambda_1}\frac{\partial}{\partial t}) = \xi_{|U}$ and, for any $t \in ]-\epsilon, \epsilon[$, $f_t(F)$ is a leaf of $D$, where $f_t = f(t, \cdot)$. Note that there exists $t_0 \in ]-\epsilon, \epsilon[$ such that $\widehat{g} = f_{t_0}^*(g_{|U})$. Furthermore, considering the Kähler structure $(\widehat{J} = (f_*^{-1} \circ \varphi \circ f_*)_{|TF}, \widehat{g})$ on $F$ and the corresponding a.c.m. manifold $]-\epsilon, \epsilon[\times_{(\lambda_1,\lambda_2)}F$ defined as in (2.1), then the map $f\colon ]-\epsilon, \epsilon[\times_{(\lambda_1,\lambda_2)}F \to (U, \varphi_{|U}, \xi_{|U}, \eta_{|U}, g_{|U})$ is an almost contact isometry.

Finally, if $(M, \varphi, \xi, \eta, g)$ is a $C_{12}$-manifold, then $D$ is a totally geodesic foliation. By [16], it follows that $\lambda_2 = 1$ so that $M$ is, locally, realized as the a.c.m. manifold $_\lambda]-\epsilon, \epsilon[\times F$, $F$ being a Kähler manifold.

## 3. Some curvature relations

In this section we focus on the main properties of the curvature $R$ of the Levi-Civita connection $\nabla$ of a $C_5 \oplus C_{12}$-manifold $(M, \varphi, \xi, \eta, g)$, $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. For the Riemannian curvature we adopt the convention $R(X, Y, Z, W) = g(R(Z, W, Y), X) = -g(R(X, Y, Z), W)$. This allows us to obtain an explicit expression of the cosymplectic defect, namely the $(0, 4)$-tensor field $\Lambda$ acting as

$$\Lambda(X, Y, Z, W) = R(X, Y, Z, W) - R(X, Y, \varphi Z, \varphi W).$$

We also state some properties of the Ricci tensor $\rho$ and the $*$-Ricci tensor $\rho^*$ and evaluate the mixed sectional curvature, denoted by $K(X, \xi)$, for any unit vector $X$ orthogonal to $\xi$.

**Proposition 3.1.** *Let* $(M,\varphi,\xi,\eta,g)$ *be a* $C_5\oplus C_{12}$*-manifold. For any vector fields* $X,Y,Z$ *on* $M$ *one has*

$$\begin{aligned} R(X,Y)\varphi Z =&\varphi(R(X,Y)Z)+\alpha(\alpha g(\varphi Y,Z)+\eta(Y)g(\varphi V,Z))X\\ &-\alpha(\alpha g(\varphi X,Z)+\eta(X)g(\varphi V,Z))Y\\ &+(Y(\alpha)\eta(Z)+\alpha^2 g(Y,Z)+\alpha\eta(Y)g(V,Z))\varphi X\\ &-(X(\alpha)\eta(Z)+\alpha^2 g(X,Z)+\alpha\eta(X)g(V,Z))\varphi Y\\ &+\alpha(\eta(X)g(\varphi Y,Z)-\eta(Y)g(\varphi X,Z))V\\ &+(\eta(X)(\alpha g(Y,Z)-\eta(Z)g(V,Y))-\eta(Y)(\alpha g(X,Z)-\eta(Z)g(V,X)))\varphi V\\ &+\eta(Z)(\eta(X)\nabla_Y\varphi V-\eta(Y)\nabla_X\varphi V)\\ &+(X(\alpha)g(\varphi Y,Z)-Y(\alpha)g(\varphi X,Z)+g(\varphi V,Z)(\eta(X)g(V,Y)-\eta(Y)g(V,X))\\ &\quad-\eta(X)g(\nabla_Y\varphi V,Z)+\eta(Y)g(\nabla_X\varphi V,Z))\xi. \end{aligned}$$

*Proof.* Since $M$ is a $C_5\oplus C_{12}$-manifold, for any $X,Y\in\Gamma(TM)$ one has

$$(\nabla_X\varphi)Y=\alpha(g(\varphi X,Y)\xi-\eta(Y)\varphi X)-\eta(X)(g(V,\varphi Y)\xi+\eta(Y)\varphi V). \tag{3.1}$$

Let $X,Y,Z$ be vector fields on $M$. By direct calculus, applying (2.2), (3.1), we have

$$\begin{aligned} R(X,Y)\varphi Z =&\varphi(R(X,Y)Z)+\nabla_X((\nabla_Y\varphi)Z)-\nabla_Y((\nabla_X\varphi)Z)-(\nabla_{[X,Y]}\varphi)Z\\ &+(\nabla_X\varphi)(\nabla_Y Z)-(\nabla_Y\varphi)(\nabla_X Z)\\ =&\varphi(R(X,Y)Z)-2d\eta(X,Y)(\eta(Z)\varphi V-g(\varphi V,Z)\xi)\\ &-\alpha\eta(Z)(\nabla_X\varphi Y-\nabla_Y\varphi X-\varphi[X,Y])\\ &+X(\alpha)(g(\varphi Y,Z)\xi-\eta(Z)\varphi Y)-Y(\alpha)(g(\varphi X,Z)\xi-\eta(Z)\varphi X)\\ &+\alpha^2(g(\varphi Y,Z)(X-\eta(X)\xi)-g(\varphi X,Z)(Y-\eta(Y)\xi))\\ &+\alpha(\eta(X)g(\varphi Y,Z)-\eta(Y)g(\varphi X,Z))V\\ &+\eta(Z)(\eta(X)\nabla_Y\varphi V-\eta(Y)\nabla_X\varphi V)\\ &+\alpha g(\varphi V,Z)(\eta(Y)X-\eta(X)Y)\\ &-(\nabla_X\eta)Z(\alpha\varphi Y+\eta(Y)\varphi V)+(\nabla_Y\eta)Z(\alpha\varphi X+\eta(X)\varphi V)\\ &+(\alpha(g(\nabla_X\varphi Y,Z)-g(\nabla_Y\varphi X,Z)-g(\varphi[X,Y],Z))\\ &\quad+\eta(Y)g(\nabla_X\varphi V,Z)-\eta(X)g(\nabla_Y\varphi V,Z))\xi. \end{aligned} \tag{3.2}$$

By (3.1) we also have

$$\begin{aligned} \nabla_X\varphi Y-\nabla_Y\varphi X =&\varphi[X,Y]+(\nabla_X\varphi)Y-(\nabla_Y\varphi)X\\ =&\varphi[X,Y]+\alpha(\eta(X)\varphi Y-\eta(Y)\varphi X)\\ &+(2\alpha g(\varphi X,Y)+\eta(X)g(\varphi V,Y)-\eta(Y)g(\varphi V,X))\xi. \end{aligned}$$

Then, substituting into (3.2) and applying (2.2), (2.3), one obtains the statement. □

**Corollary 3.1.** *Let* $(M,\varphi,\xi,\eta,g)$ *be a* $C_5\oplus C_{12}$*-manifold such that* $\dim M=2n+1$*. The following properties hold*

i) *For any* $X,Y\in\Gamma(TM)$ *we have*

$$\begin{aligned} R(X,Y)\xi =&X(\alpha)(Y-\eta(Y)\xi)-Y(\alpha)(X-\eta(X)\xi)+\alpha^2(\eta(X)Y-\eta(Y)X)\\ &+(\eta(X)g(V,Y)-\eta(Y)g(V,X))(V-\alpha\xi)-\eta(X)\nabla_Y V+\eta(Y)\nabla_X V\\ =&X(\alpha)(Y-\eta(Y)\xi)-Y(\alpha)(X-\eta(X)\xi)\\ &+\eta(X)(R(\xi,Y)\xi-\xi(\alpha)Y)-\eta(Y)(R(\xi,X)\xi-\xi(a)X). \end{aligned}$$

ii) *For any unit vector* $X$ *orthogonal to* $\xi$*, one has*

$$K(X,\xi)=-(\xi(\alpha)+\alpha^2)-g(V,X)^2+g(\nabla_X V,X).$$

iii) *The Ricci tensor satisfies*

$$\rho(\xi,\xi) = -2n(\xi(\alpha)+\alpha^2) - \operatorname{div} V,$$
$$\rho(X,\xi) = -(2n-1)(X-\eta(X)\xi)(\alpha) + \eta(X)\rho(\xi,\xi).$$

*Proof.* Let $X, Y$ be vector fields on $M$. By Proposition 3.1, we get

$$\begin{aligned} R(X,Y)\xi = -\varphi^2(R(X,Y)\xi) =& (Y(\alpha)+\alpha^2\eta(Y))\varphi^2 X - (X(\alpha)+\alpha^2\eta(X))\varphi^2 Y \\ &- \eta(X)(\alpha\eta(Y) - g(V,Y))V + \eta(Y)(\alpha\eta(X) - g(V,X))V \\ &+ \eta(X)\varphi(\nabla_Y\varphi V) - \eta(Y)\varphi(\nabla_X\varphi V). \end{aligned}$$

Moreover, using (3.1), we have

$$\begin{aligned} \eta(X)\varphi(\nabla_Y\varphi V) - \eta(Y)\varphi(\nabla_X\varphi V) =& -\eta(X)(\nabla_Y\varphi)\varphi V + \eta(Y)(\nabla_X\varphi)\varphi V \\ &- \eta(X)\nabla_Y V + \eta(Y)\nabla_X V \\ =& -\alpha(\eta(X)g(V,Y) - \eta(Y)g(V,X))\xi \\ &- \eta(X)\nabla_Y V + \eta(Y)\nabla_X V. \end{aligned}$$

Thus, substituting into the previous formula, we obtain the first equality in i). The second relation follows by a direct calculus.

To prove property ii) it is enough to apply i) observing that, for any $X \in TM$, $X \perp \xi$, $||X|| = 1$, one has $K(\xi, X) = -g(R(\xi,X)\xi, X)$.

Let $\{e_1,\dots,e_{2n},e_{2n+1}=\xi\}$ be a local orthonormal frame on $M$. Since $V$ is orthogonal to $\xi$, applying ii) we have

$$\begin{aligned} \rho(\xi,\xi) = \sum_{i=1}^{2n} K(\xi,e_i) =& -2n(\xi(\alpha)+\alpha^2) - ||V||^2 + \sum_{i=1}^{2n} g(\nabla_{ei}V, e_i) \\ =& -2n(\xi(\alpha)+\alpha^2) + \sum_{i=1}^{2n+1} g(\nabla_{ei}V, e_i). \end{aligned}$$

Thus, the first formula in iii) is proved. Finally, by i) we obtain

$$\begin{aligned} \rho(X,\xi) &= \sum_{i=1}^{2n} R(X,e_i,\xi,e_i) \\ &= -2nX(\alpha) + \sum_{i=1}^{2n} e_i(\alpha)g(X-\eta(X)\xi, e_i) + \eta(X)\rho(\xi,\xi) + 2n\eta(X)\xi(\alpha) \\ &= -(2n-1)(X-\eta(X)\xi)(\alpha) + \eta(X)\rho(\xi,\xi). \end{aligned}$$

□

We recall that, given two (symmetric) $(0,2)$-tensor fields $P$, $Q$, the Kulkarni-Nomizu product $P \owedge Q$ acts as

$$\begin{aligned} (P \owedge Q)(X,Y,Z,W) =& P(X,Z)Q(Y,W) + P(Y,W)Q(X,Z) \\ &- P(X,W)Q(Y,Z) - P(Y,Z)Q(X,W). \end{aligned} \tag{3.3}$$

In particular, for the sake of simplicity, one puts $\pi_1 = \frac{1}{2} g \owedge g$.

**Proposition 3.2.** *Let* $(M,\varphi,\xi,\eta,g)$ *be a* $C_5 \oplus C_{12}$*-manifold such that* $\dim M = 2n+1$. *For any* $X, Y, Z, W \in \Gamma(TM)$ *one has*

$$\begin{aligned} \Lambda(X,Y,Z,W) =& -\alpha^2(\pi_1(X,Y,Z,W) - \pi_1(X,Y,\varphi Z,\varphi W)) \\ &- \alpha((g \owedge (\eta\otimes\nabla_\xi\eta))(X,Y,Z,W) - (g \owedge (\eta\otimes\nabla_\xi\eta))(X,Y,\varphi Z,\varphi W)) \\ &- (g \owedge (d\alpha\otimes\eta))(X,Y,Z,W) + ((\eta\otimes\eta) \owedge (\nabla(\nabla_\xi\eta) - \nabla_\xi\eta\otimes\nabla_\xi\eta))(X,Y,Z,W). \end{aligned}$$

*Proof.* We only outline the proof, which requires a quite long calculation.
Let $X, Y, Z, W$ be vector fields on $M$. Starting by the equality

$$\Lambda(X, Y, Z, W) = g(R(X, Y)\varphi Z - \varphi(R(X, Y)Z), \varphi W) + g(R(X, Y)\xi, Z)\eta(W),$$

one applies Proposition 3.1, Corollary 3.1 and adopts the notation

$$\nabla(\nabla_\xi \eta)(X, Y) = (\nabla_X(\nabla_\xi \eta))Y = g(\nabla_X V, Y).$$

Then the statement follows by direct calculation, also applying (3.3). □

*Remark* 3.1. In [9] the cosymplectic defect of a manifold that belongs to a class containing $C_5 \oplus C_{12}$ as a proper subclass was evaluated with respect to the minimal $U(n)$-connection. Considering a manifold in the class $C_5 \oplus C_{12}$, it is easy to verify that the formulas in Proposition 3.2 and in [9] are equivalent.

**Corollary 3.2.** *Let $(M, \varphi, \xi, \eta, g)$ be a $C_5 \oplus C_{12}$-manifold with* $\dim M = 2n + 1$. *The following properties hold*

i) *For any $X, Y \in \Gamma(D)$, we get*
$$\Lambda(X, Y, X, Y) = -\alpha^2(||X||^2||Y||^2 - g(X, Y)^2 - g(X, \varphi Y)^2).$$
ii) *For any $X, Y \in \Gamma(TM)$, we have*
$$\begin{aligned}(\rho - \rho^*)(X, Y) = &- ((2n - 1)\alpha^2 + \xi(\alpha))g(X, Y) - \alpha^2\eta(X)\eta(Y)\\ &- ((2n - 1)X(\alpha) + \operatorname{div} V\eta(X) - \alpha g(V, X))\eta(Y)\\ &- (2(n - 1)\alpha\eta(X) + g(V, X))g(V, Y) + g(\nabla_X V - \eta(X)\nabla_\xi V, Y).\end{aligned}$$
iii) *Denoting by $\tau$, $\tau^*$ the scalar and $*$-scalar curvatures, we get*
$$\tau - \tau^* = -2(2n^2\alpha^2 + 2n\xi(\alpha) + \operatorname{div} V).$$
iv) *The skew-symmetric component of $\rho^*$ is given by*
$$\begin{aligned}\rho^*(X, Y) - \rho^*(Y, X) =&(2n - 1)(X(\alpha)\eta(Y) - Y(\alpha)\eta(X))\\ &+ 2(n - 1)\alpha(g(V, Y)\eta(X) - g(V, X)\eta(Y)).\end{aligned}$$

*Proof.* Property i) is a direct consequence of Proposition 3.2.

Let $X, Y$ be vector fields on $M$. With respect to a local orthonormal frame $\{e_1, \ldots, e_{2n}, \xi\}$, we write $(\rho - \rho^*)(X, Y) = \sum_{i=1}^{2n} \Lambda(X, e_i, Y, e_i) - R(X, \xi, \xi, Y)$ and apply Proposition 3.2 and Corollary 3.1. So, we obtain ii) and then iii). Furthermore, since $\rho$ is symmetric, by ii) we have

$$\begin{aligned}\rho^*(X, Y) - \rho^*(Y, X) =&(2n - 1)(X(\alpha)\eta(Y) - Y(\alpha)\eta(X) - \alpha g(V, X)\eta(Y) + \alpha g(V, Y)\eta(X))\\ &- g(\nabla_X V - \eta(X)\nabla_\xi V, Y) + g(\nabla_Y V - \eta(Y)\nabla_\xi V, X).\end{aligned}$$

On the other hand, applying (2.3) we get

$$\begin{aligned}0 =&g(\nabla_X V, Y) - g(\nabla_Y V, X) + (\alpha g(V, X) + g(\nabla_\xi V, X))\eta(Y)\\ &- (\alpha g(V, Y) + g(\nabla_\xi V, Y))\eta(X).\end{aligned}$$

Hence, substituting into the previous formula, we obtain iv). □

**Proposition 3.3.** *Let $(M, \varphi, \xi, \eta, g)$ be an a.c.m. manifold with* $\dim M \geq 5$. *If $M$ is $\alpha$-Kenmotsu or a $C_{12}$-manifold, then $\rho^*$ is symmetric.*

*Proof.* Since $\dim M \geq 5$, by (2.4) and Corollary 3.2, for any $X, Y \in \Gamma(TM)$ we have

$$\rho^*(X, Y) - \rho^*(Y, X) = \alpha(g(V, X)\eta(Y) - g(V, Y)\eta(X)).$$

□

**Proposition 3.4.** *Let $(M, \varphi, \xi, \eta, g)$ be a $C_5 \oplus C_{12}$-manifold with* $\dim M \geq 5$. *The following properties are satisfied*

i) *For any $X, Y, Z, W \in \Gamma(TM)$, one has*

$$\begin{aligned} R(X,Y,Z,W) =& R(\varphi X, \varphi Y, \varphi Z, \varphi W) - \alpha^2 (g \owedge (\eta \otimes \eta))(X,Y,Z,W) \\ &+ (g \owedge (\eta \otimes (d\alpha - \alpha \nabla_\xi \eta)))(X,Y,Z,W) \\ &- (g \owedge (\eta \otimes (d\alpha - \alpha \nabla_\xi \eta)))(X,Y,\varphi Z,\varphi W) \\ &+ ((\eta \otimes \eta) \owedge (\nabla(\nabla_\xi \eta) - \nabla_\xi \eta \otimes \nabla_\xi \eta))(X,Y,Z,W). \end{aligned}$$

ii) *For any $X, Y \in \Gamma(TM)$, one has*

$$\begin{aligned} \rho(X,Y) =& \rho(\varphi X, \varphi Y) - (2n\alpha^2 + \operatorname{div} V)\eta(X)\eta(Y) \\ &- (2(n-1)\alpha(\nabla_\xi \eta)Y + (\nabla_\xi(\nabla_\xi \eta))Y + Y(\alpha))\eta(X) \\ &+ (\alpha(\nabla_\xi \eta)X - (2n-1)X(\alpha))\eta(Y) \\ &+ (\nabla_X(\nabla_\xi \eta))Y - (\nabla_\xi \eta)X(\nabla_\xi \eta)Y \\ &- (\nabla_{\varphi X}(\nabla_\xi \eta))\varphi Y + (\nabla_\xi \eta)\varphi X(\nabla_\xi \eta)\varphi Y. \end{aligned}$$

*Proof.* We observe that, for any $X, Y, Z, W \in \Gamma(TM)$, one has

$$R(X,Y,Z,W) - R(\varphi X, \varphi Y, \varphi Z, \varphi W) = \Lambda(X,Y,Z,W) + \Lambda(\varphi Z, \varphi W, X, Y).$$

Thus property i) follows by Proposition 3.2.

Considering an adapted local orthonormal frame $\{e_1, \ldots, e_n, e_{n+1} = \varphi e_1, \ldots, e_{2n} = \varphi e_n, \xi\}$ on $M$, for any $X, Y \in \Gamma(TM)$, we write

$$\begin{aligned} \rho(X,Y) - \rho(\varphi X, \varphi Y) =& \sum_{i=1}^{2n} (R(X,e_i,Y,e_i) - R(\varphi X, \varphi e_i, \varphi Y, \varphi e_i)) \\ &+ g(R(X,\xi)\xi, Y) - g(R(\varphi X, \xi)\xi, \varphi Y). \end{aligned}$$

Then, applying i) and Corollary 3.1, one proves ii). □

*Remark* 3.2. We point out that, being $\rho$ symmetric, the tensor field considered at the right side of formula ii) in Proposition 3.4 has to be symmetric. This is equivalent to the condition

$$\begin{aligned} 0 =& 2(n-1)((X(\alpha) - \alpha(\nabla_\xi \eta)X)\eta(Y) - (Y(\alpha) - \alpha(\nabla_\xi \eta)Y)\eta(X)) \\ &+ Q(X,Y) - Q(Y,X) - Q(\varphi X, \varphi Y) + Q(\varphi Y, \varphi X), \end{aligned}$$

for any $X, Y \in \Gamma(TM)$, where $Q = \nabla(\nabla_\xi \eta) + (\nabla_\xi(\nabla_\xi \eta) + \alpha \nabla_\xi \eta) \otimes \eta$. In fact, by (2.3) we know that $Q$ is symmetric. Thus, if $\dim M = 3$, the above equality reduces to an identity. If $\dim M \geq 5$, by (2.4) we obtain that $(d\alpha - \alpha \nabla_\xi \eta) \otimes \eta$ is symmetric, also.

## 4. The $k$-nullity condition

In Contact Geometry the behavior of the tensor field $h = \frac{1}{2} L_\xi \varphi$, $L_\xi$ denoting the Lie derivative with respect to $\xi$, plays an important role for the classification of manifolds satisfying suitable curvature conditions [2, 3].

The following result shows that the vector field $V$ of any $C_5 \oplus C_{12}$-manifold specifies $h$.

**Lemma 4.1.** *Let $(M, \varphi, \xi, \eta, g)$ be a $C_5 \oplus C_{12}$-manifold. For any $X \in \Gamma(TM)$ one has $h(X) = -\frac{1}{2} g(V, \varphi X)\xi$. Therefore, $h$ vanishes if and only if $M$ falls in the class $C_5$.*

*Proof.* By direct calculation, for any $X \in \Gamma(TM)$ one has

$$2h(X) = (\nabla_\xi \varphi)X - \nabla_{\varphi X}\xi + \varphi(\nabla_X \xi) = -(\nabla_\xi \eta)\varphi X \xi = -g(V, \varphi X)\xi.$$

Since $V$ is orthogonal to $\xi$, we obtain $h = 0$ if and only if $V = 0$. □

**Lemma 4.2.** *Let $(M,\varphi,\xi,\eta,g)$ be a $C_5\oplus C_{12}$-manifold. Assume the existence of smooth functions $k,\mu$ on $M$ such that*

$$R(X,Y)\xi = k(\eta(Y)X-\eta(X)Y)+\mu(\eta(Y)h(X)-\eta(X)h(Y)), \tag{4.1}$$

*for any $X,Y\in\Gamma(TM)$. Then, one has $\mu h=0$ and $d\alpha=\xi(\alpha)\eta$.*

*Proof.* By Corollary 3.1 and the hypothesis, for any $X,Y\in\Gamma(D)$, we have

$$X(\alpha)Y-Y(\alpha)X=R(X,Y)\xi=0.$$

It follows that $X(\alpha)=0$ so that $d\alpha=\xi(\alpha)\eta$.

Given $X$ orthogonal to $\xi$, by Corollary 3.1 and Lemma 4.1, we obtain

$$\begin{aligned} -kX+\frac{1}{2}\mu g(V,\varphi X)\xi &= -R(X,\xi)\xi \\ &= (\xi(\alpha)+\alpha^2)X+g(V,X)(V-\alpha\xi)-\nabla_X V. \end{aligned}$$

Taking the inner product by $\xi$, we get $-\alpha g(V,X)-g(\nabla_X V,\xi)=\frac{1}{2}\mu g(V,\varphi X)$. Moreover, applying (2.2) one has $g(\nabla_X V,\xi)=-g(\nabla_X\xi,V)=-\alpha g(V,X)$. It follows that $\mu g(V,\varphi X)=0$, for any $X\in\Gamma(TM)$. □

Condition (4.1) was firstly considered in [4] in the context of contact manifolds, $k,\mu$ being suitable real numbers. Contact manifolds satisfying (4.1), also named $(k,\mu)$-manifolds, have been deeply studied ([3] and References therein). We call $N(k)$-space an a.c.m. manifold $(M,\varphi,\xi,\eta,g)$ admitting a smooth function $k$ such that

$$R(X,Y)\xi=k(\eta(Y)X-\eta(X)Y), \quad X,Y\in\Gamma(TM). \tag{4.2}$$

Lemma 4.2 clarifies that conditions (4.1), (4.2) are equivalent in the case of a $C_5\oplus C_{12}$-manifold. In [15] the authors proved that the curvature of an $\alpha$-Kenmotsu manifold always satisfies (4.2), where $k=-(\xi(\alpha)+\alpha^2)$. The next results show that this property does not extend to $C_5\oplus C_{12}$-manifolds.

**Proposition 4.1.** *Let $(M,\varphi,\xi,\eta,g)$ be a $C_5\oplus C_{12}$-manifold such that $\dim M=2n+1\geq 5$. If $M$ is an $N(k)$-manifold, the following properties hold*

i) $d\alpha=\xi(\alpha)\eta$, $\alpha V=0$.
ii) $k=-(\xi(\alpha)+\alpha^2)-\frac{1}{2n}\operatorname{div} V$.
iii) $\alpha\operatorname{div} V=0$.
iv) *For any $X\in\Gamma(TM)$, one has*

$$\nabla_X V=-\frac{1}{2n}\operatorname{div} V(X-\eta(X)\xi)+g(V,X)V+\eta(X)\nabla_\xi V.$$

*Proof.* By Lemma 4.2, we have $d\alpha=\xi(\alpha)\eta$ and comparing with (2.4) we obtain $\alpha V=0$. Then, also applying Corollary 3.1, for any $X\in\Gamma(D)$ one gets

$$kX=R(X,\xi)\xi=-(\xi(\alpha)+\alpha^2)X-g(V,X)V+\nabla_X V. \tag{4.3}$$

Let $\{e_1,\dots,e_{2n},e_{2n+1}=\xi\}$ be a local orthonormal frame on $M$. By (4.3) we have

$$\begin{aligned} 2nk=\sum_{i=1}^{2n} g(R(e_i,\xi)\xi,e_i) &= -2n(\xi(\alpha)+\alpha^2)-||V||^2+\sum_{i=1}^{2n} g(\nabla_{e_i}V,e_i) \\ &= -2n(\xi(\alpha)+\alpha^2)-\operatorname{div} V. \end{aligned}$$

Then, ii) follows. Moreover, since $\alpha V=0$, we get $0=\sum_{i=1}^{2n} g(\nabla_{e_i}(\alpha V),e_i)=d\alpha(V)-\alpha\operatorname{div} V=-\alpha\operatorname{div} V$. This proves iii). Finally, using (4.3), for any $X$ orthogonal to $\xi$ we have

$$\nabla_X V=\Big(-\frac{1}{2n}\operatorname{div} V\Big)X+g(V,X)V.$$

This relation entails iv). □

We point out that the distribution $D$ on any manifold as in Proposition 4.1 is spherical. In fact, the equation $d\alpha = \xi(\alpha)\eta$ means that the leaves of $D$ are extrinsic spheres.

**Proposition 4.2.** *Let $(M, \varphi, \xi, \eta, g)$ be a $C_5 \oplus C_{12}$-manifold such that $\dim M = 2n + 1 \geq 5$. Assume that $M$ is an $N(k)$-manifold. Then, for any $U, X \in \Gamma(D)$, one has*

i) $R(U, X)V = (U(k) - kg(V, U))X - (X(k) - kg(V, X))U$.
ii) $U(k + \frac{1}{2n} \operatorname{div} V) = (k + \frac{1}{2n} \operatorname{div} V)(\nabla_\xi \eta)U$.

*Proof.* Let $U, X, Y$ be vector fields on $M$. By direct calculation, applying (2.2) and (4.2), one has

$$\begin{aligned}(\nabla_U R)(X, Y)\xi =& U(k)(\eta(Y)X - \eta(X)Y) + k\eta(U)(g(V, Y)X - g(V, X)Y) \\ &+ \alpha k(g(U, Y)X - g(U, X)Y) - \alpha R(X, Y)U - \eta(U)R(X, Y)V.\end{aligned} \tag{4.4}$$

Now we consider $U, X$ orthogonal to $\xi$ and apply the second Bianchi identity, namely

$$(\nabla_U R)(X, \xi)\xi + (\nabla_X R)(\xi, U)\xi + (\nabla_\xi R)(U, X)\xi = 0.$$

By(4.4) we get

$$U(k)X - X(k)U + k(g(V, X)U - g(V, U)X) - R(U, X)V = 0.$$

Hence, i) follows. Furthermore, applying Proposition 4.1, we have

$$\begin{aligned}R(U, X)V &= \nabla_U(\nabla_X V) - \nabla_X(\nabla_U V) - \nabla_{[U,X]}V \\ &= -\frac{1}{2n} \operatorname{div} V(g(V, X)U - g(V, U)X) - \frac{1}{2n}(U(\operatorname{div} V)X - X(\operatorname{div} V)U).\end{aligned}$$

Thus, comparing with i), one has

$$U\Big(k + \frac{1}{2n} \operatorname{div} V\Big)X - X\Big(k + \frac{1}{2n} \operatorname{div} V\Big)U = \Big(k + \frac{1}{2n} \operatorname{div} V\Big)(g(V, U)X - g(V, X)U).$$

It follows that ii) holds. □

*Remark* 4.1. By Proposition 4.1, it is easy to verify that property ii) of Proposition 4.2 is equivalent to the condition

$$U(\xi(\alpha)) = \xi(\alpha)g(V, U), \quad U \in \Gamma(D).$$

**Proposition 4.3.** *Let $(M, \varphi, \xi, \eta, g)$ be a $C_5 \oplus C_{12}$-manifold such that $\dim M = 2n + 1 \geq 5$. Assume that $M$ is an $N(k)$-manifold. For any $X, Y, Z, W \in \Gamma(TM)$ one has*

i) $$\begin{aligned}R(X, Y)\varphi Z =& \varphi(R(X, Y)Z) + (k + \alpha^2)\eta(X)(\eta(Z)\varphi Y - g(\varphi Y, Z)\xi) \\ &- (k + \alpha^2)\eta(Y)(\eta(Z)\varphi X - g(\varphi X, Z)\xi) \\ &+ \alpha^2(g(\varphi Y, Z)X - g(\varphi X, Z)Y + g(Y, Z)\varphi X - g(X, Z)\varphi Y).\end{aligned}$$

ii) $$\begin{aligned}\Lambda(X, Y, Z, W) =& -\alpha^2(\pi_1(X, Y, Z, W) - \pi_1(X, Y, \varphi Z, \varphi W)) \\ &+ (k + \alpha^2)(g \owedge (\eta \otimes \eta))(X, Y, Z, W).\end{aligned}$$

*Proof.* Let $X, Y, Z, W$ be vector fields on $M$. By Propositions 3.1, 4.1 we have

$$\begin{aligned}R(X, Y)\varphi Z =& \varphi(R(X, Y)Z) + \xi(\alpha)\eta(Z)(\eta(Y)\varphi X - \eta(X)\varphi Y) \\ &+ \alpha^2(g(\varphi Y, Z)X - g(\varphi X, Z)Y + g(Y, Z)\varphi X - g(X, Z)\varphi Y) \\ &+ \eta(Z)(g(V, X)\eta(Y) - g(V, Y)\eta(X))\varphi V \\ &+ \eta(Z)(\eta(X)\nabla_Y \varphi V - \eta(Y)\nabla_X \varphi V) \\ &+ (\xi(\alpha)(\eta(X)g(\varphi Y, Z) - \eta(Y)g(\varphi X, Z)) \\ &\quad + g(\varphi V, Z)(\eta(X)g(V, Y) - \eta(Y)g(V, X)) \\ &\quad - \eta(X)g(\nabla_Y \varphi V, Z) + \eta(Y)g(\nabla_X \varphi V, Z))\xi.\end{aligned}$$

Moreover, applying (3.1) and Proposition 4.1, we get

$$\nabla_X \varphi V = (\nabla_X \varphi)V + \varphi(\nabla_X V) = g(V, X)\varphi V - \Big(\frac{1}{2n} \operatorname{div} V\Big)\varphi X + \eta(X)\varphi(\nabla_\xi V).$$

Substituting into the previous formula and using property ii) of Proposition 4.1, i) follows.

Finally, property ii) is obtained by i) and the relation

$$\Lambda(X,Y,Z,W) = g(R(X,Y)\varphi Z - \varphi(R(X,Y)Z), \varphi W) + k\eta(W)(\eta(Y)g(X,Z) - \eta(X)g(Y,Z)).$$

□

**Theorem 4.1.** *Let $(M,\varphi,\xi,\eta,g)$ be a $C_5 \oplus C_{12}$-manifold such that $\dim M \geq 5$. Assume that $M$ has constant sectional curvature $k$. Then, either $M$ is an $\alpha$-Kenmotsu manifold and $k = -\alpha^2$, or $M$ is flat and falls in the class $C_{12}$.*

*Proof.* Let $x$ be a point of $M$ and consider unit vectors $X, Y \in T_xM$ such that $g_x(X,Y) = g_x(X,\varphi Y) = \eta_x(X) = \eta_x(Y) = 0$. Since $M$ has constant sectional curvature, we have $R = k\pi_1$, so that

$$R_x(X,Y)\varphi_x Y - \varphi_x(R_x(X,Y)Y) = -k\varphi_x X.$$

On the other hand, by Proposition 4.3, one obtains

$$R_x(X,Y)\varphi_x Y - \varphi_x(R_x(X,Y)Y) = \alpha(x)^2\varphi_x X.$$

It follows $k + \alpha(x)^2 = 0$. Thus, $\alpha$ is a constant function . Since $\alpha V = 0$, one of the following two cases occurs

i) $\alpha \neq 0$, $V = 0$, $k = -\alpha^2$,
ii) $\alpha = 0$, $k = 0$.

In case i), $M$ falls in the class $C_5$, namely it is $\alpha$-Kenmotsu, $\alpha$ = constant and $k = -\alpha^2 < 0$. In case ii), $M$ is flat and falls in $C_{12}$. □

We remark that, for any $\alpha \in \mathbb{R}$, $\alpha \neq 0$, an $\alpha$-Kenmotsu manifold with constant sectional curvature $k = -\alpha^2$ is locally a warped product $]-\epsilon,\epsilon[\times_\lambda F$, where $F$ is a flat Kähler manifold and $\lambda(t) = a\exp(-|\alpha|t)$, $a = \text{const} > 0$. On the other hand, a flat $C_{12}$-manifold is locally realized as a product $_\lambda]-\epsilon,\epsilon[\times F$, $F$ being a flat Kähler manifold and $\lambda\colon ]-\epsilon,\epsilon[\times F \to \mathbb{R}$ a smooth positive function. The action of $\lambda$ will be specified in Section 7.

**Proposition 4.4.** *Let $(M,\varphi,\xi,\eta,g)$ be a $C_5 \oplus C_{12}$-manifold such that $\dim M \geq 5$. If $M$ is an $N(k)$-manifold, the curvature satisfies the following identities*

$$\begin{aligned}R(X,Y,Z,W) =& R(X,Y,\varphi Z,\varphi W) + R(\varphi X,Y,Z,\varphi W) + R(X,\varphi Y,Z,\varphi W)\\ &+ k\eta(W)(\eta(Y)g(X,Z) - \eta(X)g(Y,Z)),\end{aligned} \tag{4.5}$$

$$R(X,Y,Z,W) = R(\varphi X,\varphi Y,\varphi Z,\varphi W) + k(g \owedge (\eta\otimes\eta))(X,Y,Z,W), \tag{4.6}$$

*for any $X,Y,Z,W \in \Gamma(TM)$.*

*Proof.* The statement follows by Proposition 4.3 observing that, for any vector fields $X,Y,Z,W$ on $M$, one has

$$\begin{aligned}R(X,Y,Z,W) =& R(X,Y,\varphi Z,\varphi W) + R(\varphi X,Y,Z,\varphi W) + R(X,\varphi Y,Z,\varphi W)\\ &+ \Lambda(X,Y,Z,W) - \Lambda(Z,\varphi W,X,\varphi Y) + \eta(Y)R(Z,\varphi W,\xi,\varphi X)\end{aligned}$$

and

$$R(X,Y,Z,W) = R(\varphi X,\varphi Y,\varphi Z,\varphi W) + \Lambda(X,Y,Z.W) + \Lambda(\varphi Z,\varphi W,X,Y).$$

□

*Remark* 4.2. If $k = \text{const} = 1$, properties (4.5) and (4.6) correspond to the identities, called $G_2$, $G_3$ identities, introduced and studied in [14]. Obviously, the curvature of any $\alpha$-Kenmotsu manifold satisfies (4.5), (4.6), being $k = -(\xi(\alpha) + \alpha^2)$.

## 5. Local description of $N(k)$-manifolds

We are going to provide some local descriptions of a $C_5 \oplus C_{12}$-manifold $(M, \varphi, \xi, \eta, g)$ satisfying the $N(k)$-condition, examining suitable distributions on $M$. Assuming that $V$ is nowhere zero, we can consider the rank 2 distribution $D_1 = \mathrm{span}\{\xi, V\}$ and its orthogonal complement $D_1^\perp = \ker\eta \cap \ker\nabla_\xi\eta$. By (2.3), one gets that $D_1^\perp$ is integrable. Moreover, Proposition 4.1 entails that $M$ falls in the class $C_{12}$. It follows that, if $D^\perp = \mathrm{span}\{\xi\}$ is spherical, equivalently $\nabla_\xi V = -||V||^2\xi$, $M$ is, locally, the a.c.m. manifold ${}_\lambda] - \epsilon, \epsilon[\times F$, $F$ being a Kähler manifold and $\lambda \colon F \to \mathbb{R}^*_+$ a smooth function [9].

We recall that a Riemannian submanifold $N$ of an a.c.m. manifold $(M, \varphi, \xi, \eta, g)$ is said to be a semi-invariant $\xi^\perp$-submanifold if the vector field $\xi \in \Gamma(T^\perp N)$ and there exist two orthogonal distributions, $\overline{D}$ and $\overline{D}^\perp$, on $N$ such that $TN = \overline{D} \oplus \overline{D}^\perp$, $\varphi(\overline{D}) = \overline{D}$ and $\varphi\overline{D}^\perp \subseteq T^\perp N$ [5].

In the sequel, for the sake of simplicity, by $V \neq 0$ we mean that $V$ is nowhere zero on $M$.

**Proposition 5.1.** *Let $(M, \varphi, \xi, \eta, g)$ be a $C_{12}$-manifold such that $\dim M = 2n+1 \geq 5$, $V \neq 0$ and $\nabla_\xi V = -||V||^2\xi$. If $M$ is an $N(k)$-manifold, then the distribution $D_1$ is totally geodesic and $D_1^\perp$ is spherical. Furthermore, each leaf of $D_1$ is an anti-invariant submanifold of $M$ with Gaussian curvature $k$ and each leaf of $D_1^\perp$ is a semi-invariant $\xi^\perp$-submanifold of $M$ admitting a $C_6$-structure.*

*Proof.* By hypotheses and Proposition 4.1, we have that $k = -\frac{1}{2n}\,\mathrm{div}\, V$ and

$$\nabla_X V = kX + g(V, X)V - \eta(X)(||V||^2 + k)\xi, \quad X \in \Gamma(TM). \tag{5.1}$$

It follows that

$$d(||V||^2) = 2(||V||^2 + k)\nabla_\xi\eta. \tag{5.2}$$

By (2.3) and (5.2), we get $0 = d(||V||^2 + k) \wedge \nabla_\xi\eta - (||V||^2 + k)\nabla_\xi(\nabla_\xi\eta) \wedge \eta$. Since $\nabla_\xi V = -||V||^2\xi$, it follows that $\nabla_\xi(\nabla_\xi\eta) \wedge \eta = 0$ and thus

$$dk = \frac{1}{||V||^2} V(k)\nabla_\xi\eta. \tag{5.3}$$

Applying (2.2) and (5.1), it is easy to verify that the distribution $D_1$ is totally geodesic. Moreover, considering a leaf $N$ of $D_1$, we have $\varphi(TN) \subseteq T^\perp N$, namely $N$ is anti-invariant, and the Gauss curvature of $N$ is given by $k(x) = \frac{R_x(\xi, V, \xi, V)}{||V||^2}$, $x \in N$.

Let $N'$ be a leaf of $D_1^\perp$. For any $X, Y \in \Gamma(TN')$, by (2.2), (5.1), we obtain $g(\nabla_X Y, \xi) = 0$ and $g(\nabla_X Y, V) = -kg(X, Y)$. By the Gauss formula, it follows that $N'$ is totally umbilical with mean curvature vector field $H = -\frac{k}{||V||^2}V$. Moreover, denoting by $\nabla^\perp$ the normal connection of $N'$, we have

$$\nabla^\perp_X H = -\left(X\left(\frac{k}{||V||^2}\right)V + \frac{k}{||V||^2}\nabla^\perp_X V\right), \quad X \in \Gamma(TN').$$

On the other hand, by (5.2), (5.3), we get $X(\frac{k}{||V||^2}) = 0$. Moreover, using (5.1), we have $\nabla^\perp_X V = 0$. Substituting into the above equation, it follows that $N'$ is an extrinsic sphere.
Now, we consider the distribution $\mathrm{span}\{\varphi V\}$ on $N'$ and denote by $\overline{D}$ its orthogonal complement on $N'$. Since $\varphi^2 V = -V \in \Gamma(T^\perp N')$, we have $\varphi(\mathrm{span}\{\varphi V\}) \subseteq T^\perp N'$. Moreover, for any $X \in \Gamma(\overline{D})$ one has $g(\varphi X, \varphi V) = 0$, namely $\varphi(\overline{D}) = \overline{D}$. This means that $N'$ is a semi-invariant $\xi^\perp$-submanifold of $M$.
Finally, putting $g' = g_{|_{TN' \times TN'}}$, $\xi' = \frac{1}{||V||}\varphi V$, $\eta' = \xi'^b$, we consider the (1,1)-tensor field $\varphi'$on $N'$ such that $\varphi'(\xi') = 0$ and $\varphi'(X) = \varphi X$, for any $X \perp \xi'$. In particular, for any $X \in \Gamma(TN')$ one has

$$\varphi'(X) = \varphi X + \frac{1}{||V||^2} g(\varphi V, X)V. \tag{5.4}$$

It is easy to check that $(\varphi', \xi', \eta', g')$ is an a.c.m. structure on $N'$. Furthermore, we denote by $\nabla'$ the Levi-Civita connection of $(N', g')$, apply the Gauss formula and obtain

$$\nabla_X Y = \nabla'_X Y - \frac{k}{||V||^2} g(X,Y)V, \quad X, Y \in \Gamma(TN').$$

Then, by direct calculation, also applying (5.1), (5.4), one has

$$(\nabla'_X \varphi')Y = -\frac{k}{||V||}(g'(X,Y)\xi' - \eta'(Y)X), \quad X, Y \in \Gamma(TN').$$

It follows that $(N', \varphi', \xi', \eta', g')$ is an $\overline{\alpha}$-Sasakian manifold, with $\overline{\alpha} = -\frac{k}{||V||}$, and it falls in the class $C_6$ [3, 6]. □

Applying Proposition 5.1 and the decomposition theorem of Hiepko, we are able to state the following classification theorem.

**Theorem 5.1.** *Let $(M, \varphi, \xi, \eta, g)$ be a $C_{12}$-manifold such that $\dim M = 2n+1 \geq 5$, $V \neq 0$ and $\nabla_\xi V = -||V||^2\xi$. If $M$ is an $N(k)$-manifold, then $(M, g)$ is locally isometric to a warped product $N \times_\lambda N'$, where $\dim N = 2$, $N$ has Gaussian curvature $k$ and $N'$ is an $\overline{\alpha}$-Sasakian manifold, $\overline{\alpha} = -\frac{k}{||V||}$.*

**Corollary 5.1.** *Let $(M, \varphi, \xi, \eta, g)$ be a $C_{12}$-manifold such that $\dim M = 2n+1 \geq 5$, $V \neq 0$ and $\nabla_\xi V = -||V||^2\xi$. If $M$ is flat, then $(M, g)$ is locally isometric to a Riemannian product $N \times N'$, $\dim N = 2$ and $N$, $N'$ are flat manifolds. Furthermore, $N'$ admits a cosymplectic structure.*

*Proof.* Since $M$ is flat, $M$ is an $N(0)$-manifold. Hence, using Proposition 5.1, both the distributions $D_1$ and $D_1^\perp$ are totally geodesic. In fact, for any $X \in \Gamma(D_1^\perp)$ one has $\nabla_X V = 0 = \nabla_X \xi$. By Theorem 5.1, $(M, g)$ is locally isometric to a Riemannian product $N \times N'$, where $N$ is a flat 2-dimensional manifold and $N'$ admits an $\overline{\alpha}$-Sasakian structure, with $\overline{\alpha} = 0$. □

We end this section considering the distribution $D' = \operatorname{span}\{\xi, V, \varphi V\}$ on $M$. As in the previous case, we assume $V \neq 0$ and $D^\perp$ spherical.

**Proposition 5.2.** *Let $(M, \varphi, \xi, \eta, g)$ be a $C_{12}$-manifold such that $\dim M = 2n+1 \geq 5$, $V \neq 0$ and $\nabla_\xi V = -||V||^2\xi$. If $M$ is an $N(k)$-manifold, the distribution $D'$ is totally geodesic and each leaf of $D'$ is an $N(k)$-manifold belonging to the class $C_{12}$.*

*Proof.* By Proposition 4.1, we get $k = -\frac{1}{2n} \operatorname{div} V$. Moreover, applying (2.2), (5.1) and the defining condition of the class $C_{12}$ (see Table 1), an easy calculus entails

$$\nabla_V \xi = 0 = \nabla_{\varphi V} \xi = \nabla_\xi \varphi V,$$
$$\nabla_V V = (||V||^2 + k)V, \quad \nabla_V \varphi V = (||V||^2 + k)\varphi V, \quad \nabla_{\varphi V} V = k\varphi V, \quad \nabla_{\varphi V} \varphi V = -kV.$$

The above formulas, together with the hypothesis $\nabla_\xi V = -||V||^2\xi$, imply that the distribution $D'$ is totally geodesic.

Let $N'$ be a leaf of $D'$. It is easy to verify that $(\varphi' = \varphi_{|TN'}, \xi' = \xi_{|TN'}, \eta' = \eta_{|TN'}, g' = g_{|TN' \times TN'})$ is an a.c.m. structure on $N'$. Since $N'$ is totally geodesic, $(N', \varphi', \xi', \eta', g')$ is an $N(k)$-manifold and falls in the class $C_{12}$. □

**Theorem 5.2.** *Let $(M, \varphi, \xi, \eta, g)$ be a $C_{12}$-manifold such that $\dim M = 2n+1 \geq 5$, $V \neq 0$ and $\nabla_\xi V = -||V||^2\xi$. If $M$ is flat, then $(M, g)$ is locally isometric to a Riemannian product $N' \times N''$, where $N'$ is a 3-dimensional $C_{12}$-manifold, $N''$ is a Kähler manifold and $N'$, $N''$ are both flat.*

*Proof.* Since $M$ is flat, $M$ is an $N(0)$-manifold. Let $D'^\perp$ be the orthogonal complement of $D'$. By (2.2), (5.1), for any $X, Y \in \Gamma(D'^\perp)$ we get $g(\nabla_X Y, \xi) = 0 = g(\nabla_X Y, V) = g(\nabla_X Y, \varphi V)$. Hence, the distribution $D'^\perp$ is totally geodesic and each leaf $N''$ of $D'^\perp$ is totally geodesic and flat. Moreover, $(J'' = \varphi_{|TN''}, g'' = g_{|TN'' \times TN''})$ is a Kähler structure on $N''$. Then, also applying Proposition 5.2, we get the statement. □

## 6. The case of generalized Sasakian space-forms

In this section we consider a $C_5 \oplus C_{12}$-manifold $(M, \varphi, \xi, \eta, g)$ which is a generalized Sasakian space-form (g.S. space-form), namely $M$ admits three smooth functions $f_1$, $f_2$, $f_3$ such that the curvature tensor satisfies

$$R = f_1\pi_1 + f_2 S + f_3 T, \tag{6.1}$$

where $\pi_1$, $S$, $T$ are the tensor fields acting as

$$\begin{aligned}
\pi_1(X,Y,Z) &= g(Y,Z)X - g(X,Z)Y,\\
S(X,Y,Z) &= g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z,\\
T(X,Y,Z) &= \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi.
\end{aligned}$$

This class of a.c.m. manifolds was introduced in [1] and subsequently studied by a number of mathematicians from several points of view. In particular, in [8] it was proved that $M$ is a g.S. space-form if and only if $M$ is an $N(k)$-manifold with pointwise constant $\varphi$-sectional curvature $c$ and, for any $X, Y \in \Gamma(D)$, the cosymplectic defect satisfies $\Lambda(X,Y,X,Y) = l(||X||^2||Y||^2 - g(X,Y)^2 - g(X,\varphi Y)^2)$, $l$ being a smooth function on $M$.

Now, also applying Corollary 3.2 and Proposition 4.1, it is easy to obtain the following result.

**Proposition 6.1.** *Let $(M, \varphi, \xi, \eta, g)$ be a $C_5 \oplus C_{12}$-manifold with $\dim M = 2n+1 \geq 5$. The following conditions are equivalent*

i) *$M$ is a g.S. space-form.*
ii) *$M$ is an $N(k)$-manifold with pointwise constant $\varphi$-sectional curvature $c$.*

*Moreover, if one of the previous conditions holds, one has $k = -(\xi(\alpha)+\alpha^2) - \frac{1}{2n}\operatorname{div} V$, $f_1 = \frac{c-3\alpha^2}{4}$, $f_2 = \frac{c+\alpha^2}{4}$, $f_3 = f_1 - k = \frac{c+\alpha^2}{4} + \xi(\alpha) + \frac{1}{2n}\operatorname{div} V$.*

Taking into account Proposition 6.1, we denote by $M^{2n+1}(c,k)$ a g.S. space-form with pointwise constant $\varphi$-sectional curvature $c$ and satisfying the $k$-nullity condition.

**Proposition 6.2.** *Let $(M, \varphi, \xi, \eta, g)$ be a $C_5 \oplus C_{12}$-manifold. If $M^{2n+1}(c,k)$, $n \geq 2$, is a g.S. space-form, the following properties hold*

i) *For any point $x_0 \in M$, the leaf $(N, J, g')$ of $D$ through $x_0$ is a Kähler manifold with constant holomorphic sectional curvature $(c+\alpha^2)(x_0)$.*
ii) *$dc = \xi(c)\eta$.*
iii) *For any $X \in \Gamma(D)$, one has $X(\xi(c)) = \xi(c)g(V,X)$.*
iv) *$cV = 0$.*
v) *$dk = \xi(k)\eta + k\nabla_\xi\eta$.*

*Proof.* Let $x_0 \in M$ and $(N, J = \varphi_{|TN}, g' = g_{|TN\times TN})$ be the leaf of the distribution $D$ through $x_0$. Since $M$ is a $C_5 \oplus C_{12}$-manifold, we know that $(J, g')$ is a Kähler structure on $N$ and $N$ is totally umbilical with mean curvature vector field $H = -\alpha\xi_{|N}$. Denoting by $R'$ the Riemannian curvature of $N$ and applying the Gauss equation, for any $x \in N$ and any unit vector $X \in T_xN$, we get

$$R'_x(X, J_xX, X, J_xX) = R_x(X, \varphi_xX, X, \varphi_xX) + \alpha(x)^2 = (c+\alpha^2)(x).$$

Since $\dim N \geq 4$, it follows that $N$ has constant holomorphic sectional curvature $(c+\alpha^2)_{|N}$. So, we obtain i). On the other hand, by Proposition 6.1, $M$ is an $N(k)$-manifold. Hence, applying Proposition 4.1, $\alpha$ is constant on $N$. This implies that $c$ is constant on $N$. It follows that the function $c$ is constant on any leaf of $D$, that is ii) holds.

By ii), we obtain $d(\xi(c)\eta) = 0$. So, applying (2.3), one has $(d\xi(c) - \xi(c)\nabla_\xi\eta) \wedge \eta = 0$ and iii) follows.

Finally, using the second Bianchi identity, one has $f_2V = 0$ and $dk = \xi(k)\eta - f_3\nabla_\xi\eta$ (cf. [7], Section 4). Applying Propositions 4.1, 6.1, we easily obtain iv) and v). □

*Remark* 6.1. In the same hypotheses of Proposition 6.2, applying the main results in [9], we have that $M$ is locally almost contact isometric to a double-twisted product manifold $]-\epsilon,\epsilon[\times_{(\lambda_1,\lambda_2)}F$, where $\epsilon>0$, $(F,\widehat{J},\widehat{g})$ is a Kähler manifold with constant holomorphic sectional curvature $(c+\alpha^2)_{|F}$ and $\lambda_1,\lambda_2\colon ]-\epsilon,\epsilon[\times F\to\mathbb{R}$ are smooth positive functions.

**Proposition 6.3.** *Let $(M,\varphi,\xi,\eta,g)$ be a $C_5\oplus C_{12}$-manifold. If $M^{2n+1}(c,k)$, $n\geq 2$, is a g.S. space-form, then the following differential equation holds*

$$d(c+\alpha^2)=-2(c+\alpha^2)\alpha\eta. \tag{6.2}$$

*Proof.* Let $U,X,Y$ be vector fields on $M$ and $Z\in\Gamma(D)$. By (6.1), we have

$$\begin{aligned}(\nabla_U R)(X,Y,Z)=&U(f_1)\pi_1(X,Y,Z)+U(f_2)S(X,Y,Z)+U(f_3)T(X,Y,Z)\\&+f_2(\nabla_U S)(X,Y,Z)+f_3(\nabla_U T)(X,Y,Z),\end{aligned}\tag{6.3}$$

where $f_1$, $f_2$, $f_3$ are related to $c$, $k$ as in Proposition 6.1. Furthermore, it is easy to verify the following relations

$$\begin{aligned}(\nabla_U S)(X,Y,Z)=&g(\varphi Y,Z)(\nabla_U\varphi)X-g((\nabla_U\varphi)Z,Y)\varphi X\\&-g(\varphi X,Z)(\nabla_U\varphi)Y+g((\nabla_U\varphi)Z,X)\varphi Y\\&+2g(\varphi Y,X)(\nabla_U\varphi)Z+2g((\nabla_U\varphi)Y,X)\varphi Z,\\(\nabla_U T)(X,Y,Z)=&(\eta(X)Y-\eta(Y)X)(\nabla_U\eta)Z\\&+(g(X,Z)(\nabla_U\eta)Y-g(Y,Z)(\nabla_U\eta)X)\xi\\&+(g(X,Z)\eta(Y)-g(Y,Z)\eta(X))\nabla_U\xi.\end{aligned}$$

In order to apply the second Bianchi identity, using the above formulas, Propositions 4.1, 6.1, 6.2 and (2.2), (3.1), a direct calculus entails

$$U(f_1)=\frac{1}{4}\xi(c-3\alpha^2)\eta(U),\quad U(f_2)=\frac{1}{4}\xi(c+\alpha^2)\eta(U),\tag{6.4}$$

$$\begin{aligned}f_2\underset{(U,X,Y)}{\sigma}(\nabla_U S)(X,Y,Z)=2\alpha f_2\Big(&\underset{(U,X,Y)}{\sigma}(g(\varphi X,Z)\eta(Y)-g(\varphi Y,Z)\eta(X))\varphi U\\&+2\underset{(U,X,Y)}{\sigma}g(\varphi Y,X)\eta(U)\varphi Z\Big),\end{aligned}\tag{6.5}$$

$$\begin{aligned}f_3\underset{(U,X,Y)}{\sigma}(\nabla_U T)(X,Y,Z)=f_3\Big(2\alpha&\underset{(U,X,Y)}{\sigma}(g(X,Z)\eta(Y)-g(Y,Z)\eta(X))U\\&+\underset{(U,X,Y)}{\sigma}\eta(U)(\eta(X)Y-\eta(Y)X)g(V,Z)\\&+\underset{(U,X,Y)}{\sigma}(g(X,Z)g(V,Y)-g(Y,Z)g(V,X))\eta(U)\xi\Big),\end{aligned}\tag{6.6}$$

where $\sigma$ represents the cyclic sum over $U,X,Y$.

Now, choosing $U=\xi$, $Y=Z$, $X\perp U,Y,\varphi Y$, and substituting into (6.3)-(6.6), the second Bianchi identity gives

$$\Big(\frac{1}{4}\xi(c-3\alpha^2)+2\alpha f_3\Big)||Z||^2X+(X(f_3)-f_3g(V,X))||Z||^2\xi=0.$$

This implies $\xi(c-3\alpha^2)+8\alpha f_3=0$. Using iii) in Proposition 4.1 and Proposition 6.1, it follows that $0=\xi(c-3\alpha^2)+2\alpha(c+\alpha^2)+8\alpha\xi(\alpha)=\xi(c+\alpha^2)+2\alpha(c+\alpha^2)$. On the other hand, by Propositions 4.1, 6.2, we know that $d(c+\alpha^2)=\xi(c+\alpha^2)\eta$. Hence, the statement holds. □

Now, we are able to classify g.S. space-forms belonging to the class $C_5\oplus C_{12}$.

**Theorem 6.1.** *Let $(M,\varphi,\xi,\eta,g)$ be a $C_5\oplus C_{12}$-manifold. If $M^{2n+1}(c,k)$, $n\geq 2$, is a g.S. space-form, then exactly one of the following cases occurs*

i) *$M$ is cosymplectic and $c$ is constant.*
ii) *$M$ falls in the class $C_{12}\smallsetminus C$ and $c=0$.*

iii) *$\alpha \neq 0$ and $c + \alpha^2 = 0$. Moreover, there exist an open covering $\{U_i\}_{i\in I}$ of $M$ and, for any $i \in I$, a smooth function $\sigma_i\colon U_i \to \mathbb{R}$ such that $(U_i, \varphi_i = \varphi_{|U_i}, \xi_i = \exp(-\sigma_i)\xi_{|U_i}, \eta_i = \exp(\sigma_i)\eta_{|U_i}, g_i = \exp(2\sigma_i)g_{|U_i})$ is a g.S. space-form with zero $\varphi$-sectional curvature, which falls in the class $C_{12}$.*

iv) *$M$ is $\alpha$-Kenmotsu and the function $c + \alpha^2$, which is nowhere zero, has constant sign. Moreover, $M$ is globally conformal to a cosymplectic manifold with constant $\varphi$-sectional curvature* $\mathrm{sign}(c + \alpha^2)$.

*Proof.* If $\alpha = 0$, by Proposition 6.3, we get that $c$ is a constant function. If $c \neq 0$, applying Proposition 6.2, it follows that the vector field $V$ vanishes, so that $M$ is a cosymplectic manifold. If $c = 0$, by Proposition 6.1 and (6.1), the curvature tensor of $M$ is given by $R = \left(\frac{1}{2n}\operatorname{div} V\right)T$. In this case, if $\operatorname{div} V \neq 0$, then $M$ is a $C_{12}$-manifold but it is not cosymplectic. If $\operatorname{div} V = 0$, $M$ is flat and either $M$ is cosymplectic or $M$ falls in the class $C_{12} \setminus C$. We conclude that, if $\alpha = 0$, one of the cases i), ii) occurs.

Now, we suppose that $\alpha \neq 0$. Since the Lee form $\omega = -\alpha\eta$ is closed, by Proposition 4.4 in [9], $M$ is a locally conformal $C_{12}$-manifold, namely there exist an open covering $\{U_i\}_{i\in I}$ of $M$ and, for any $i \in I$, a smooth function $\sigma_i\colon U_i \to \mathbb{R}$ such that $U_i$ is endowed with the $C_{12}$-structure $(U_i, \varphi_i = \varphi_{|U_i}, \xi_i = \exp(-\sigma_i)\xi_{|U_i}, \eta_i = \exp(\sigma_i)\eta_{|U_i}, g_i = \exp(2\sigma_i)g_{|U_i})$ and $d\sigma_i = \omega_{|U_i}$.
The Levi-Civita connections of the local metrics $g_i$ fit up to the Weyl connection $\overline{\nabla}$ acting as

$$\overline{\nabla}_X Y = \nabla_X Y - \alpha\eta(X)Y - \alpha\eta(Y)X + \alpha g(X,Y)\xi, \quad X, Y \in \Gamma(TM).$$

Furthermore, fixed $i \in I$ and denoting by $\overline{R}$ the $(0,4)$-curvature tensor of $\overline{\nabla}$, it is well-known that in $U_i$ one has

$$\exp(-2\sigma_i)\overline{R} = R - P \owedge g, \tag{6.7}$$

where $P = \nabla\omega - \omega \otimes \omega + \frac{1}{2}||\omega||^2 g$. Applying Proposition 4.1 and (2.2), it is easy to verify the following relations

$$P = -\xi(\alpha)\eta \otimes \eta - \frac{1}{2}\alpha^2 g,$$
$$(P \owedge g)(X,Y,Z,W) = \alpha^2 g(\pi_1(X,Y,Z),W) - \xi(\alpha)g(T(X,Y,Z),W).$$

Substituting into (6.7) and applying (6.1), Proposition 6.1, it follows that

$$\overline{R} = \frac{c+\alpha^2}{4}(\pi_1 + S) + \left(\frac{c+\alpha^2}{4} + \frac{1}{2n}\operatorname{div} V\right)T. \tag{6.8}$$

Since $\omega$ is closed, by (6.2) and the connectedness of $M$, one of the following two cases occurs

a) $c + \alpha^2 = 0$,
b) $c + \alpha^2 \neq 0$ everywhere.

In case a), the equation (6.8) reduces to $\overline{R} = \left(\frac{1}{2n}\operatorname{div} V\right)T$. In order to rewrite this equation with respect to the metrics $g_i$, $i \in I$, we put $V_i = \overline{\nabla}_{\xi_i}\xi_i$ and denote by $T_i$ the tensor field on $U_i$ defined as $T$. An easy calculation entails

$$V_i = \exp(-2\sigma_i)V_{|U_i}, \quad \operatorname{div} V_i = \exp(-2\sigma_i)\operatorname{div} V_{|U_i}, \quad T_{|U_i} = \exp(-2\sigma_i)T_i.$$

It follows that

$$\overline{R}_{|U_i} = \left(\frac{1}{2n}\operatorname{div} V_i\right)T_i, \quad i \in I.$$

Combining the above formula with Proposition 6.1, we get that the $C_{12}$-manifolds $(U_i, \varphi_i, \xi_i, \eta_i, g_i)$ are g.S. space-forms with zero $\varphi$-sectional curvature. Hence, iii) holds.
Finally, we examine case b). Since $M$ is connected, the function $c+\alpha^2$ has constant sign. Moreover, by Propositions 4.1, 6.2, we have $(c + \alpha^2)V = 0$. This implies that $V = 0$, namely $M$ is an $\alpha$-Kenmotsu manifold. On the other hand, solving (6.2), we get $\omega = d\log\sqrt{|c + \alpha^2|}$. Since $\omega$ is

exact, $M$ is globally conformal to the a.c.m. manifold $(M, \varphi, \frac{1}{\sqrt{|c+\alpha^2|}}\xi, \sqrt{|c+\alpha^2|}\eta, |c+\alpha^2|g)$, which is cosymplectic [15]. Furthermore, with respect to the metric $\overline{g} = |c+\alpha^2|g$, (6.8) becomes

$$\overline{R} = \frac{1}{4}\frac{c+\alpha^2}{|c+\alpha^2|}(\overline{\pi_1} + \overline{S} + \overline{T}) = \frac{1}{4}\operatorname{sign}(c+\alpha^2)(\overline{\pi_1} + \overline{S} + \overline{T}).$$

The above equation means that $(M, \varphi, \frac{1}{\sqrt{|c+\alpha^2|}}\xi, \sqrt{|c+\alpha^2|}\eta, |c+\alpha^2|g)$ has constant $\varphi$-sectional curvature $\operatorname{sign}(c+\alpha^2)$. Hence iv) occurs. □

*Remark* 6.2. In [7] the authors gave a local classification of g.S. space-forms $M^{2n+1}(f_1, f_2, f_3)$, $n \geq 2$, assuming that for any $i = 1, 2, 3$, if the function $f_i$ does not vanish, then $f_i \neq 0$ everywhere. The authors proved that nine cases can occur and these cases are not mutually exclusive. Obviously, a restriction on the Chinea-Gonzalez class of the g.S. space-form entails that some of the mentioned cases have to be excluded. Comparing the result stated in Theorem 6.1 with the main Theorem 1.3 in [7], we get that a $C_5 \oplus C_{12}$-manifold $M^{2n+1}(c, k)$ has to satisfy one of four cases listed in [7], namely the ones denoted by (a), (e), (f), (g). We also remark that in our context the hypothesis $f_i = 0$ or $f_i \neq 0$ everywhere is needless.

## 7. Examples

In Theorem 4.1 we have shown that a $C_5 \oplus C_{12}$-manifold $(M, \varphi, \xi, \eta, g)$ with $\dim M = 2n+1 \geq 5$ and constant sectional curvature is either an $\alpha$-Kenmotsu manifold or a flat $C_{12}$-manifold. Note that, as remarked in Section 4, in the first case it is known that $M$ is locally described as a warped product. Furthermore, the hyperbolic space $\mathbb{H}^{2n+1}(-\alpha^2)$ is the local model of space-forms carrying a non-cosymplectic $\alpha$-Kenmotsu structure.
More generally, in Theorem 6.1 we have classified g.S. space-forms $M^{2n+1}(c, k)$. Taking into account case ii), we are going to provide a method for constructing a whole family of g.S. space-forms $M^{2n+1}(0, k)$ falling in the class $C_{12} \setminus C$.

Let $(J_0, g_0)$ be the canonical Kähler structure on $\mathbb{R}^{2n}$, $n \geq 2$, $I \subset \mathbb{R}$ an open interval and $\lambda \colon I \times \mathbb{R}^{2n} \to \mathbb{R}$ a smooth positive function. We know that the a.c.m. manifold $M = {}_{\lambda}I \times \mathbb{R}^{2n}$, defined as in (2.1), falls in the class $C_{12} \setminus C$. According to Proposition 6.1, Theorem 6.1 and formula (6.1), the condition that $M$ is a g.S. space-form $M^{2n+1}(0, k)$ is equivalent to require that its curvature tensor satisfies

$$R = \Big(\frac{1}{2n}\operatorname{div} V\Big)T = -kT. \tag{7.1}$$

Using the curvature formulas in [16], we have

$$R(X, \xi)Z = (g(\nabla_X(\operatorname{grad}\log\lambda), Z) + X(\log\lambda)Z(\log\lambda))\xi, \quad X, Z \in \Gamma(D),$$

where $\nabla$ is the Levi-Civita connection on $(M, g = g_{(\lambda,1)})$ and grad is evaluated with respect to $g$. By an easy calculation, also considering Corollary 3.1 and Proposition 4.1, one can check that (7.1) is equivalent to the condition

$$g(\nabla_X(\operatorname{grad}\log\lambda), Z) + X(\log\lambda)Z(\log\lambda) = -kg(X, Z), \quad X, Z \in \Gamma(D).$$

Considering the orthonormal frame $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{2n}}, \xi\}$ on $M$, the above equation corresponds to the following PDE's system

$$\frac{\partial^2\lambda}{\partial x^i \partial x^j} + k\lambda\delta_{ij} = 0, \quad i, j = 1, \ldots, 2n. \tag{7.2}$$

Hence, for any $i \neq j$, one has $\frac{\partial^2\lambda}{\partial x^i \partial x^j} = 0$. It follows that $\lambda(t, x^1, \ldots, x^{2n}) = \sum_{k=1}^{2n} a_k(t, x^k)$, where $a_k$ is a function only depending on $t$ and $x^k$. Substituting into (7.2) and assuming $i = j$, we get $\frac{\partial^2 a_i}{\partial (x^i)^2} = -k\lambda$. This implies that the function $k\lambda$ only depends on $t$. Putting $-k\lambda = 2C(t)$, it follows that $a_i(t, x^i) = C(t)(x^i)^2 + B_i(t)x^i + E_i(t)$, for any $i = 1, \ldots, 2n$. We can conclude that (7.1) is satisfied if and only if

$$\lambda(t, x^1, \ldots, x^{2n}) = \sum_{i=1}^{2n} (C(t)(x^i)^2 + B_i(t)x^i) + E(t), \tag{7.3}$$

where $E(t) = \sum_{i=1}^{2n} E_i(t)$ and $C(t) = -\frac{1}{2}k\lambda$.

We observe that for $\lambda$ to be a positive function we have to narrow its domain. Supposing $0 \in I$, we can assume $C(0) \geq 0$, $E(0) > 0$ and $B_i(0) > 0$, $i = 1, \ldots, 2n$. Thus, there exists an open interval $J$, $0 \in J \subset I$, such that $C(t) \geq 0$, $E(t) > 0$ and $B_i(t) > 0$, for any $i = 1, \ldots, 2n$, $t \in J$. Putting $U = \mathbb{R}^*_+ \times \cdots \times \mathbb{R}^*_+$, the function $\lambda\colon J \times U \to \mathbb{R}$, defined as in (7.3), is smooth and positive.
We conclude that the a.c.m. manifolds $M = {}_\lambda J \times U$ are g.S. space-forms $M^{2n+1}(0, k)$ belonging to the class $C_{12} \smallsetminus C$.

*Remark* 7.1. The condition $k = 0$ is equivalent to require that the a.c.m. manifolds $M = {}_\lambda J \times U$ are flat and $\lambda(t, x^1, \ldots, x^{2n}) = \sum_{i=1}^{2n} B_i(t)x^i + E(t)$. Note that the method above described is similar to the procedure used in Theorem 5.2 in [7]. In our case the hypothesis that $f_3 = -k$ is nowhere zero is needless.

Finally, we provide an explicit example of a $C_{12}$-manifold satisfying the hypotheses of Theorem 5.2.

**Example 7.1.** Given three non negative real numbers $B_1, B_{n+1}, E$ such that $(B_1, B_{n+1}) \neq (0, 0)$, one considers the open set $W = \{(x^1, \ldots, x^{2n}) \in \mathbb{R}^{2n} | x^1 > 0, x^{n+1} > 0\}$ and the smooth positive function $\lambda\colon \mathbb{R} \times W \to \mathbb{R}$ acting as

$$\lambda(t, x^1, \ldots, x^{2n}) = B_1 x^1 + B_{n+1}x^{n+1} + E.$$

By Remark 7.1, we know that the a.c.m. manifold $M = {}_\lambda\mathbb{R} \times W = (\mathbb{R} \times W, \varphi, \xi = \frac{1}{\lambda}\frac{\partial}{\partial t}, \eta = \lambda dt, g = \lambda^2 dt \otimes dt + g_0)$ is flat and falls in the class $C_{12} \smallsetminus C$. Note that, for any $i = 1, \ldots, n$, $\varphi(\frac{\partial}{\partial x^i}) = J_0(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^{n+i}}$. Using the formulas in [16], it is easy to verify that the tensor field $V = \nabla_\xi \xi = -\frac{1}{\lambda}(B_1 \frac{\partial}{\partial x^1} + B_{n+1}\frac{\partial}{\partial x^{n+1}})$ satisfies the condition $\nabla_\xi V = -||V||^2\xi$. Moreover, considering the distribution $D' = \operatorname{span}\{\xi, V, \varphi V\}$ on $M$ and putting $U_1 = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^{n+1}}$, $U_2 = \varphi U_1$, we have $D' = \operatorname{span}\{\lambda\xi, U_1, U_2\}$.

Given the open set $N' = \{(t, y, z) \in \mathbb{R}^3 | y > 0, -y < z < y\}$, $(t_0, x_0) = (t_0, x_0^1, \ldots, x_0^{2n}) \in M$, we define the map $f\colon N' \to \mathbb{R} \times W$ acting as

$$f(t, y, z) = \Big(t, \frac{1}{\sqrt{2}}(y - z), x_0^2, \ldots, x_0^n, \frac{1}{\sqrt{2}}(y + z), \ldots, x_0^{2n}\Big).$$

Putting $\lambda' = \lambda \circ f$ and $g' = \lambda'^2 dt \otimes dt + dy \otimes dy + dz \otimes dz$, it is easy to check that $f$ is an isometric immersion with respect to the metrics $g'$ and $g$. Note that $(N', g')$ is the leaf of $D'$ through $(t_0, x_0)$ and, applying Proposition 5.2, $(N', \varphi' = -\frac{\partial}{\partial y} \otimes dz + \frac{\partial}{\partial z} \otimes dy, \xi = \frac{1}{\lambda'}\frac{\partial}{\partial t}, \eta' = \lambda' dt, g')$ is a flat $C_{12}$-manifold. Moreover, up to an isometry, the leaf of $D'^\perp$ through $(t_0, x_0)$ is $\mathbb{R}^{2n-2}$ endowed with its canonical Kähler structure. Thus, applying Theorem 5.2, $M$ is locally isometric to the Riemannian product $N' \times \mathbb{R}^{2n-2}$.

**Work partially supported by INDAM.**

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