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A scale of critical exponents for semilinear waves with time-dependent damping and mass terms

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Abstract

We consider the following Cauchy problem for a wave equation with time-dependent damping term $b(t)u_t$ and mass term $m(t)^2u$, and a power nonlinearity $|u|^p$:

$$\begin{cases} u_{tt} - \Delta u + b(t)u_t + m^2(t)u = |u|^p, & t \geq 0, \ x \in \mathbb{R}^n, \\ u(0,x) = f(x), \quad u_t(0,x) = g(x). \end{cases}$$

We discuss how the interplay between an effective time-dependent damping term and a time-dependent mass term influences the decay rate of the solution to the corresponding linear Cauchy problem, in the case in which the mass is dominated by the damping term, i.e. $m(t) = o(b(t))$ as $t \to \infty$.

Then we use the obtained estimates of solutions to linear Cauchy problems to prove the existence of global in-time energy solutions to the Cauchy problem with power nonlinearity $|u|^p$ at the right-hand side of the equation, in a supercritical range $p > \bar{p}$, assuming small data in the energy space $(f, g) \in H^1 \times L^2$. In particular, we find a threshold case for the behavior of $m(t)$ with respect to $b(t)$.

Below the threshold, the mass has no influence on the critical exponent, so that $\bar{p} = 1 + 4/n$, as in the case with $m = 0$. Above the threshold, $\bar{p} = 1$, global (in time) small data energy solutions exist for any $p > 1$, as it happens for the classical damped Klein-Gordon equation ($b = m = 1$). Along the threshold, varying a parameter $\beta \in [0, \infty]$ which depends on the behavior of $m(t)$ with respect to $b(t)$, we find a scale of critical exponents, namely, $\bar{p} = 1 + 4/(n + 4\beta)$. This scale of critical exponents is consistent with the diffusion phenomenon, that is, it is the same scale of critical exponents of the Cauchy problem for the corresponding diffusive equation

$$\begin{cases} -\Delta v + b(t)v_t + m^2(t)v = |v|^p, & t \geq 0, \ x \in \mathbb{R}^n, \\ v(0,x) = \varphi(x). \end{cases}$$

Keywords: critical exponent, wave equation, effective damping, damped Klein-Gordon equation
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1. Introduction

In this paper, we look for global (in time) small data energy solutions to the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + b(t)u_t + m^2(t)u = h(u), & t \geq 0, \ x \in \mathbb{R}^n, \\ u(0,x) = f(x), \quad u_t(0,x) = g(x), \end{cases} \tag{1}$$

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where $b(t)u_t$ and $m(t)^2u$, with $b(t), m(t) > 0$, respectively represent a damping and a mass term, and the power nonlinearity $h(u)$ verifies

$$h(0) = 0, \quad |h(u) - h(v)| \lesssim |u - v| \, (|u| + |v|)^{p-1} \tag{2}$$

for a given $p > 1$, for instance, $h(u) = |u|^p$. In order to do that, we derive suitable estimates for solutions to the corresponding linear Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + b(t)u_t + m^2(t)u = 0, & t \geq 0, \ x \in \mathbb{R}^n, \\ u(0, x) = f(x), \quad u_t(0, x) = g(x), \end{cases} \tag{3}$$

and we apply a contraction argument to construct the solution to (1).

In [1, 7], the model without mass has been considered,

$$\begin{cases} u_{tt} - \Delta u + b(t)u_t = |u|^p, & t \geq 0, \ x \in \mathbb{R}^n, \\ u(0, x) = f(x), \quad u_t(0, x) = g(x), \end{cases} \tag{4}$$

and it has been proved that the critical exponent for global (in time) small data energy solutions to (4) remains the same as for the Cauchy problem with $b = 1$ (see [13, 12, 15, 18, 23, 27]). Here the assumption of *effectiveness* of the damping term (see later, Hypothesis 1) was essential to derive suitable estimates for solutions to the corresponding linear Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + b(t)u_t = 0, & t \geq 0, \ x \in \mathbb{R}^n, \\ u(0, x) = f(x), \quad u_t(0, x) = g(x). \end{cases} \tag{5}$$

In particular, global existence holds for $p > 1 + 2/n$ if initial data are assumed to be small in exponentially weighted energy spaces. In the subcritical and critical range, $1 < p \leq 1 + 2/n$, no global in time small data Sobolev solutions exist, under a suitable sign assumption [5] for the data. If smallness of the data is assumed only in the standard energy space $H^1 \times L^2$ and in L^1, then the same result holds in space dimension $n = 1, 2$. If also the additional L^1 smallness is dropped, then the critical exponent becomes $1 + 4/n$.

In this paper, by effectiveness of the damping term we mean that for a suitable large class of damping coefficients $b(t)$, the estimates obtained for (5) are the same obtained for the solution to the corresponding Cauchy problem for the heat equation

$$\begin{cases} b(t)v_t - \Delta v = 0, \quad t \geq 0, \ x \in \mathbb{R}^n, \\ v(0, x) = \varphi(x), \end{cases} \tag{6}$$

for suitable initial datum $\varphi = \varphi(f, g, b)$ (see [26]). In the case of polynomial shape $b(t) = \mu(1 + t)^k$, the damping is effective if $k \in (-1, 1]$ (see [17, 19, 25] for the corresponding global existence result), and partially effective if $k = -1$.

In this latter case $b(t) = \mu(1 + t)^{-1}$, the critical exponent of global small data solutions to (4) remains $1 + 2/n$ if the positive coefficient μ is sufficiently large [2, 24], whereas it seems to increase to $\max\{p_S(n + \mu), 1 + 2/n\}$, as μ becomes smaller with respect to the space dimension, as conjectured in [6, 8] (see also [10, 16]), where p_S is the Strauss exponent for the semilinear undamped wave equation [9, 14, 22]. The overdamping case $b(t) = \mu(1 + t)^k$, with $k > 1$ has been studied in [11].

On the other hand, it is well-known that global small data solutions to the Cauchy problem for the damped Klein-Gordon equation

$$\begin{cases} u_{tt} - \Delta u + u_t + u = h(u), & t \geq 0, \ x \in \mathbb{R}^n, \\ u(0, x) = f(x), \quad u_t(0, x) = g(x), \end{cases} \tag{7}$$

exist for any $p > 1$ (see [3] for more results), that is, the interplay of a damping term and a mass term may lead to an improvement in the critical exponent of (1). This interaction has been recently studied in the scale invariant case, $b(t) = \mu_1(1 + t)^{-1}$ and $m(t) = \mu_2(1 + t)^{-1}$ in [20].

The main purpose of this paper is to show that under a simple condition on the interaction between $b(t)$ and $m(t)$, and assuming only small initial data in the energy space, one may find a scale of critical exponents, which continuously move from $1 + 4/n$ to 1, as the mass becomes more influent, with respect to the damping term. Our result applies for a large class of coefficients $b(t)$ and $m(t)$. In particular, it applies to the case in which b is constant (see Remark 2.5).

The scale of critical exponents is the same for solutions of the Cauchy problem for the corresponding diffusive equation

$$\begin{cases} b(t)v_t - \Delta v + m^2(t)v = v^p, & t \geq 0,\ x \in \mathbb{R}^n, \\ v(0,x) = \varphi(x) \geq 0, \end{cases} \tag{8}$$

which can be easily determined (see Section 5), and this hints to the optimality of the exponents obtained in this paper.

The scheme of the paper is the following:

- in Section 2 we present the main results for the Cauchy problems (1) and (3);
- in Section 3 we derive estimates for solutions of the associated linear Cauchy problem (Theorem 1);
- in Section 4 we prove the main results for the global in time existence of small data solutions (Theorems 2 and 3);
- in Section 5 we briefly discuss the Cauchy problem for the corresponding heat equation (8), having in mind Remark 2.2.

2. Main results

The fundamental assumption is that the damping term $b(t)u_t$ is *effective*, in particular, that the following assumptions are satisfied [7, 26] for $b = b(t)$.

Hypothesis 1. We assume that $b \in C^2$, with $b(t) > 0$, is monotone and it has controlled oscillations, this means

$$|b^{(k)}(t)| \leq C\, b(t)(1+t)^{-k}, \qquad k = 1, 2, \tag{9}$$

and that

$$tb(t) \to \infty, \qquad \text{as } t \to \infty, \tag{10}$$

$$\frac{1}{b(t)(1+t)^2} \in L^1(\mathbb{R}_+). \tag{11}$$

Moreover, we assume that b' verifies

$$tb'(t) \leq ab(t), \qquad \text{for some } a \in [0,1), \tag{12}$$

being this latter trivially satisfied if $b'(t) \leq 0$.

As a consequence of (9) and (10), we derive

$$|b'(t)| = o(b(t)^2), \qquad \text{as } t \to \infty. \tag{13}$$

By virtue of Hypothesis 1, the function

$$B(t,s) = \int_s^t \frac{1}{b(\tau)}\, d\tau$$

is positive, monotone increasing with respect to t and decreasing with respect to s and, for any fixed $s \geq 0$, it holds $B(t,s) \to \infty$, as $t \to \infty$. Indeed, $1/b$ is not in L^1 as a consequence of (10). For the sake of brevity, we fix

$$B(t) = B(t,0).$$

We also assume that the influence of the damping term dominates the influence of the mass term in the equation, so that the presence of the mass has a minor influence on the profile of the solution, which is mainly determined by the effective action of the damping term on the wave equation.

Hypothesis 2. We assume that $m \in C^2$, with $m(t) > 0$, has controlled oscillations, this means,

$$|m^{(k)}(t)| \leq C\, m(t)(1+t)^{-k}, \qquad k = 1, 2, \tag{14}$$

and that

$$m(t) = o(b(t)), \qquad \text{as } t \to \infty. \tag{15}$$

Due to the effectiveness of the damping, guaranteed by Hypotheses 1 and 2, the solution to problem (3) has the same decay properties of the solution to the corresponding diffusive equation (see later, Remark 2.2):

$$\begin{cases} b(t)v_t - \Delta v + m(t)^2 v = 0, & t \geq 0,\ x \in \mathbb{R}^n, \\ v(0,x) = \varphi(x). \end{cases} \tag{16}$$

In view of this, we may assume that the initial data from the energy space also belong to L^η, for some $\eta \in [1,2)$, to obtain an extra decay rate for the solution to (3). Indeed, this is a well-known property for heat equations, as (16). Having this in mind, we introduce the following notation.

Notation 1. For $\eta \in [1,2]$, we define the function spaces

$$\mathcal{A}_\eta = \left(L^\eta(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)\right) \times \left(L^\eta(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)\right),$$

with norms

$$\|(f,g)\|_{\mathcal{A}_\eta} = \|f\|_{L^\eta(\mathbb{R}^n)} + \|f\|_{H^1(\mathbb{R}^n)} + \|g\|_{L^\eta(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)}.$$

In particular, $\mathcal{A}_2 = H^1 \times L^2$.

Assuming initial data in $\mathcal{A}_\eta$ for some $\eta \in [1,2]$, we derive the following decay estimates for the solution to the linear Cauchy problem (3).

Theorem 1. *Let $n \geq 1$ and $\eta \in [1,2]$. Let us assume that $b = b(t)$ satisfies Hypothesis 1 and $m = m(t)$ satisfies Hypothesis 2. Let $(f,g) \in \mathcal{A}_\eta$. Then, the solution to Cauchy problem (3) satisfies the following decay estimates:*

$$\|u(t,\cdot)\|_{L^2} \leq C\,\gamma(t)\,(1+B(t))^{-\frac{n}{2}\left(\frac{1}{\eta}-\frac{1}{2}\right)}\,\|(f,g)\|_{\mathcal{A}_\eta}, \tag{17}$$

$$\|\nabla u(t,\cdot)\|_{L^2} \leq C\,\gamma(t)\,(1+B(t))^{-\frac{n}{2}\left(\frac{1}{\eta}-\frac{1}{2}\right)-\frac{1}{2}}\,\|(f,g)\|_{\mathcal{A}_\eta}, \tag{18}$$

$$\|u_t(t,\cdot)\|_{L^2} \leq C\,\gamma(t)\,\frac{m(t)^2 + (1+B(t))^{-1}}{b(t)}\,(1+B(t))^{-\frac{n}{2}\left(\frac{1}{\eta}-\frac{1}{2}\right)}\|(f,g)\|_{\mathcal{A}_\eta}, \tag{19}$$

where we define

$$\gamma(t) = \exp\left(-\int_0^t \frac{m^2(\tau)}{b(\tau)}\,d\tau\right), \tag{20}$$

and the constant $C > 0$ does not depend on the data.

Estimates for solutions to (3) with partially effective damping and mass term have been studied in [21]. In particular, the case $b(t) = b_0(1+t)^{-1}$ and $m(t)^2 = m_0(1+t)^{-2}$ is considered under the assumption $b_0(b_0-2) \leq 4m_0$.

Remark 2.1. As a consequence of Hypothesis 1, it holds

$$B(t,s) \approx \frac{t}{b(t)} - \frac{s}{b(s)}, \tag{21}$$

(see Section 4.2 in [7]), so that $1 + B(t)$ in (17)-(18) may be conveniently replaced by $1 + t/b(t)$.

Remark 2.2. The decreasing function $\gamma = \gamma(t)$ in (20) represents the influence on the estimates of the mass term with respect to the damping term, under Hypotheses 1 and 2. In particular, there is no essential influence if $m(t)^2/b(t)$ is integrable. This influence is consistent with the effectiveness of the damping term, i.e. the analogy between the decay properties of solutions to problems (3) and (16). Indeed, the solution to (16) is given by

$$v(t,x) = \gamma(t)\,(4\pi B(t))^{-n/2}\, e^{-\frac{|x|^2}{4B(t)}} \ *_{(x)}\ \varphi(x)\,.$$

Our considerations hint to the opportunity to have the diffusion phenomenon, that is, the asymptotic profile of the solution to (3) may be expressed by the solution to (16), for a suitable choice of initial datum $\varphi = \varphi(f,g,b,m)$. However, we will not investigate this question in this paper. For a discussion about effectiveness of the damping term and diffusion phenomenon for a wide class of evolution equations with time-dependent damping, we address the interested reader to [4].

Remark 2.3. In estimate (19) there is a competition between the influence from $m(t)^2$ and from $B(t)^{-1}$ or, equivalently, from $b(t)/t$ (see (21)). In particular, if $tm^2(t) \lesssim b(t)$, then (19) reduces to

$$\|u_t(t,\cdot)\|_{L^2} \leq C\,\gamma(t)\,\frac{1}{b(t)}\,(1+B(t))^{-1-\frac{n}{2}\left(\frac{1}{\eta}-\frac{1}{2}\right)}\|(f,g)\|_{\mathcal{A}_\eta}.$$

On the other hand, if $b(t) \lesssim tm(t)^2$ (in particular, if $\beta > 0$ in (22)), then (19) reduces to

$$\|u_t(t,\cdot)\|_{L^2} \leq C\,\gamma(t)\,\frac{m(t)^2}{b(t)}\,(1+B(t))^{-\frac{n}{2}\left(\frac{1}{\eta}-\frac{1}{2}\right)}\|(f,g)\|_{\mathcal{A}_\eta}.$$

The competition between the two terms is consistent with the fact that if v is the solution to (16), then

$$v_t(t,x) = -\frac{m(t)^2}{b(t)}\,v(t,x) + \frac{1}{b(t)}\,\Delta v(t,x).$$

In order to study the influence of the term $\gamma(t)$ on the critical exponent for global in time small data energy solutions to (1), we will relate $m(t)$ and $B(t)$. Without making any further assumption on $m(t)$ and $B(t)$, we may define the parameter

$$\beta = \liminf_{t\to\infty} B(t)\,m(t)^2. \tag{22}$$

The critical exponent for (1) will only depend on this parameter $\beta \in [0,\infty]$.

Under the assumption of smallness of the data in the energy space $H^1 \times L^2$, we may state the following result.

Theorem 2. *Let $n \geq 1$, and assume that $b = b(t)$ satisfies Hypothesis 1 and $m = m(t)$ satisfies Hypothesis 2. Let β be as in (22). Moreover, assume $\beta > -1 + n/4$, if $n \geq 4$.*

Under these assumptions, for any $p > p_\beta(n)$, where

$$p_\beta(n) = \begin{cases} 1 + \frac{4}{n+4\beta} & \text{if } \beta \in [0,\infty), \\ 1 & \text{if } \beta = \infty, \end{cases} \tag{23}$$

and $p \leq 1 + 2/(n-2)$ if $n \geq 3$, there exists $\epsilon_0 > 0$ such that for any initial data $(f,g) \in H^1 \times L^2$ with

$$\|(f,g)\|_{H^1\times L^2} \leq \epsilon_0,$$

there exists a unique energy solution $u \in C([0,\infty), H^1) \cap C^1([0,\infty), L^2)$ to Cauchy problem (1). Moreover, it satisfies the following estimates:

$$\|u(t,\cdot)\|_{L^2} \leq C(1+B(t))^{-\alpha}\|(f,g)\|_{H^1\times L^2}, \tag{24}$$

$$\|\nabla u(t,\cdot)\|_{L^2} \leq C(1+B(t))^{-\alpha-\frac{1}{2}}\|(f,g)\|_{H^1\times L^2}, \tag{25}$$

$$\|u_t(t,\cdot)\|_{L^2} \leq C\,\frac{m(t)^2 + (1+B(t))^{-1}}{b(t)}\,(1+B(t))^{-\alpha}\|(f,g)\|_{H^1\times L^2}, \tag{26}$$

for any $\alpha < \beta$ if $\beta \in (0,\infty]$, and with $\alpha = 0$ if $\beta = 0$.

The estimates (24), (25), (26) are consistent with the estimates (17), (18), (19) for solutions to the linear Cauchy problem (3).

Remark 2.4. We mention that in the special case in which $\gamma(t) \approx B(t)^{-\beta}$ for some $\beta \in (0, \infty)$, that is,

$$\exists \lim_{t\to\infty} B(t)\, m(t)^2 = \beta,$$

$$\exists \int_0^\infty \Big(\frac{m(t)^2}{b(t)} - \frac{\beta}{b(t)B(t)}\Big)\, dt \in \mathbb{R},$$

the estimates (24), (25), (26) also hold for $\alpha = \beta$.

In particular, for a given coefficient $b(t)$ as in Hypothesis 1, the class of coefficients $m(t)$, for which

$$\exists \lim_{t\to\infty} B(t)\, m(t)^2 \in (0, \infty), \tag{27}$$

represents a threshold between two completely different situations. If $B(t)m(t)^2 \to 0$, then the mass term is too weak to modify the critical exponent for (1), which remains the same critical exponent for problem (4). If $B(t)m(t)^2 \to \infty$, i.e. $\beta = \infty$, then the mass term is so strong that global small data solutions to (1) exist for any $p > 1$. If (27) holds, then the critical exponent for (1) varies between the two limit situations, according to the value of β.

Remark 2.5. Even the case $b \equiv 1$ is of interest. So, we consider the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + u_t + m^2(t)u = h(u), & t \geq 0,\ x \in \mathbb{R}^n, \\ u(0, x) = f(x), \quad u_t(0, x) = g(x). \end{cases} \tag{28}$$

In this case, the interplay between the influence of the damping coefficient and the influence of the mass term clearly simplifies. In particular,

$$\beta = \liminf_{t\to\infty} t\, m(t)^2,$$

in (22). The estimates (24), (25), (26) read as follows:

$$\|u(t, \cdot)\|_{L^2} \leq C(1 + t)^{-\alpha}\|(f, g)\|_{H^1\times L^2}, \tag{29}$$

$$\|\nabla u(t, \cdot)\|_{L^2} \leq C(1 + t)^{-\alpha-\frac{1}{2}}\|(f, g)\|_{H^1\times L^2}, \tag{30}$$

$$\|u_t(t, \cdot)\|_{L^2} \leq C\, (m(t)^2 + (1 + t)^{-1})\, (1 + t)^{-\alpha}\|(f, g)\|_{H^1\times L^2}, \tag{31}$$

for any $\alpha < \beta$ if $\beta \in (0, \infty]$, and with $\alpha = 0$ if $\beta = 0$. In particular, the influence from the mass term brings an additional decay $(1 + t)^{-\alpha}$ in (29) and (30), with respect to the decay for the damped wave equation (see [18]). The same also happens in (31) if $tm(t)^2$ is bounded.

Remark 2.6. The condition $\beta > -1 + n/4$, if $n \geq 4$, in Theorem 2 is equivalent to have a nonempty range $(p_\beta(n), 1 + 2/(n-2)]$. However, the bound from above on p may be relaxed with several strategies (for instance, assuming higher data regularity), but this is beyond the scopes of this paper.

2.1. Examples

The easiest class of coefficients $b(t)$ and $m(t)$ which can be considered are of polynomial type.

Example 1. Let

$$b(t) = \mu(1 + t)^k, \qquad m(t) = \nu(1 + t)^\ell,$$

for some $\mu, \nu > 0$ and $k, \ell \in \mathbb{R}$. Then Hypothesis 1 holds if, and only if, $k \in (-1, 1)$. In this case,

$$B(t) = \frac{(1 + t)^{1-k} - 1}{\mu(1 - k)}\, .$$

In particular, $1 + B(t) \approx (1 + t)^{1-k}$. On the other hand, (15) in Hypothesis 2 holds if, and only if, $\ell < k$. Moreover,

- if $\ell < (k-1)/2$, then γ does not vanish as $t \to \infty$, so that $\beta = 0$ in Theorem 2;
- if $\ell = (k-1)/2$, then
$$\gamma(t) = \exp\left(-\int_0^t \frac{\nu^2}{\mu(1+\tau)}\, d\tau\right) = (1+t)^{-\frac{\nu^2}{\mu}},$$
that is, we are in the threshold case described by (27) with $\beta = \nu^2/(\mu(1-k))$ and estimates (24)-(25) also hold for $\alpha = \beta$;
- if $\ell \in ((k-1)/2, k)$, then Theorem 2 holds with $\beta = \infty$.

Example 2. Let
$$b(t) = \mu(1+t)^k\,(\log(e+t))^a, \qquad m(t) = \nu(1+t)^\ell\,(\log(e+t))^b,$$
for some $\mu, \ell > 0$, and $k, \ell, a, b \in \mathbb{R}$. Then Hypothesis 1 holds if, and only if, either $k \in (-1,1)$, or $k = -1$ and $a > 0$, and we get
$$B(t) = \frac{(1+t)^{1-k}(\log(e+t))^{-a}}{\mu(1-k)}(1+o(1))\,.$$
On the other hand, (15) in Hypothesis 2 holds if, and only if, either $\ell < k$, or $\ell = k$ and $b < a$. Although the explicit computation of γ is not hard, it is quicker to apply the definition in (22):

- if either $\ell < (k-1)/2$ or $\ell = (k-1)/2$ and $2b < a$, then $\beta = 0$ in (22);
- if $\ell = (k-1)/2$ and $a = 2b$, we are in the threshold case described by (27) with $\beta = \nu^2/(\mu(1-k))$;
- if either $\ell \in ((k-1)/2, k)$ or $\ell = (k-1)/2$ and $a < 2b$, then (22) holds with $\beta = \infty$.

In particular, if $\ell = (k-1)/2$ and $2b \in [a-1, a)$, then γ vanishes as $t \to \infty$, but the dissipation is too weak to modify the critical exponent for (1).

2.2. Using L^η smallness of the data

The use of L^1 smallness of the initial data should lead to replace the critical exponent in (23) by
$$p_{\beta,1}(n) = \begin{cases} 1 + \frac{2}{n+2\beta} & \text{if } \beta \in [0,\infty), \\ 1 & \text{if } \beta = \infty, \end{cases}$$
consistently with the corresponding result for $m = 0$ (see [7]). However, even in the case $b = 1$ and $m = 0$, the use of the smallness of initial data in the energy space and in L^1 leads to a technical limit, if no further assumption on the data are taken, namely, only power nonlinearities with $p \geq 2$ may be efficiently treated. To overcome such difficulty, different techniques need to be employed, as estimates in weighted energy spaces, or $L^p - L^q$ estimates.

The development of these estimates beyond the purpose of this paper, but the restriction $p \geq 2$ is consistent with the critical exponent above only if $n = 1$ and $\beta \leq 1/2$, or $n = 2$ and $\beta = 0$.

For the sake of completeness, we state the global existence result in space dimension $n \geq 1$, assuming small data in $\mathcal{A}_\eta$, with $\eta \in [1,2)$, with the additional restriction $p \geq 2$. The theoretical critical exponent, corresponding to L^η smallness of data, is
$$p_{\beta,\eta}(n) = \begin{cases} 1 + \frac{2\eta}{n+2\eta\beta} & \text{if } \beta \in [0,\infty), \\ 1 & \text{if } \beta = \infty. \end{cases} \tag{32}$$
We notice that $p_{\beta,\eta}(n) = p_\beta(2n/\eta)$. In other words, the influence on the critical exponent coming from the use of additional regularity L^η corresponds to multiply the space dimension by a factor $2/\eta$ (see [23]).

Theorem 3. *Let $n \geq 1$, $\eta \in [1,2)$, and assume that $b = b(t)$ satisfies Hypothesis* 1 *and $m = m(t)$ satisfies Hypothesis* 2. *Let β be as in* (22).

Under these assumptions, for any $p \geq 2/\eta$ such that $p > p_{\beta,\eta}(n)$, and $p \leq 1 + 2/(n-2)$ if $n \geq 3$, there exists $\epsilon_0 > 0$ such that if

$$\|(f,g)\|_{\mathcal{A}_\eta} \leq \epsilon_0,$$

then there exists a unique energy solution $u \in C([0,\infty), H^1) \cap C^1([0,\infty), L^2)$ to Cauchy problem (1). *Moreover, it satisfies the following estimates:*

$$\|u(t,\cdot)\|_{L^2} \leq C(1+B(t))^{-\frac{n}{2}\left(\frac{1}{\eta}-\frac{1}{2}\right)-\alpha}\|(f,g)\|_{\mathcal{A}_\eta}, \tag{33}$$

$$\|\nabla u(t,\cdot)\|_{L^2} \leq C(1+B(t))^{-\frac{n}{2}\left(\frac{1}{\eta}-\frac{1}{2}\right)-\alpha-\frac{1}{2}}\|(f,g)\|_{\mathcal{A}_\eta}, \tag{34}$$

$$\|u_t(t,\cdot)\|_{L^2} \leq C\,\frac{m(t)^2 + (1+B(t))^{-1}}{b(t)}\,(1+B(t))^{-\frac{n}{2}\left(\frac{1}{\eta}-\frac{1}{2}\right)-\alpha}\|(f,g)\|_{\mathcal{A}_\eta}, \tag{35}$$

for any $\alpha < \beta$ if $\beta \in (0,\infty]$, and with $\alpha = 0$ if $\beta = 0$.

Remark 2.7. For any fixed β and n, setting $\eta = 1$ in Theorem 3 gives the best possible critical exponent $p_{\beta,\eta}(n)$. On the other hand, the limitation $p \geq 2/\eta$ gives the most restrictive range. In particular, if we set $\eta = 1$, Theorem 3 gives no information in space dimension $n \geq 5$. On the other hand, as $\eta \to 2$, condition $p \geq 2/\eta$ becomes less restrictive, and it allows more freedom in higher space dimension, provided that β is sufficiently large. In particular, at $\beta = \infty$ there is no benefit at all from taking $\eta \in [1,2)$, and one may just rely on Theorem 2. In other words, the influence of the mass term in (1) allows to overcome the bound of space dimension $n \leq 6$ which appears in [1], and in [13, Theorem 1.2] for $b \equiv 1$.

3. Decay estimates for solutions to linear Cauchy problems

In order to prove Theorems 2 and 3, we plan to apply Duhamel's principle. However, due to the presence of time-dependent coefficients, the equation in (3) is not invariant by time translations. Having this in mind, we derive decay estimates for the solution to a family of parameter-dependent Cauchy problems

$$\begin{cases} u_{tt} - \Delta u + b(t)u_t + m^2(t)u = 0, & t \geq s, \\ u(s,x) = f(s,x), \quad u_t(s,x) = g(s,x), \end{cases} \tag{36}$$

where $s \geq 0$, obtaining decay rates which depend on both t and s. At $s = 0$, we will obtain the estimates for solutions of the linear Cauchy problem, stated in Theorem 1. On the other hand, for any $s > 0$ we may assume $f(s,\cdot) = 0$, and we prove the following result.

Lemma 3.1. *Let $b = b(t)$ satisfy Hypothesis* 1 *and $m = m(t)$ satisfy Hypothesis* 2. *Let $f(s,\cdot) = 0$ and $g(s,\cdot) \in L^\eta \cap L^2$ for some $\eta \in [1,2]$.*

Then, the solution to Cauchy problem (36) *satisfies the following estimates:*

$$\|u(t,\cdot)\|_{L^2} \leq C\frac{1}{b(s)}(1+B(t,s))^{-\frac{n}{2}\left(\frac{1}{\eta}-\frac{1}{2}\right)}\,\frac{\gamma(t)}{\gamma(s)}\,\|g(s,\cdot)\|_{L^\eta\cap L^2}, \tag{37}$$

$$\|\nabla u(t,\cdot)\|_{L^2} \leq C\frac{1}{b(s)}(1+B(t,s))^{-\frac{n}{2}\left(\frac{1}{\eta}-\frac{1}{2}\right)-\frac{1}{2}}\,\frac{\gamma(t)}{\gamma(s)}\,\|g(s,\cdot)\|_{L^\eta\cap L^2}, \tag{38}$$

$$\|u_t(t,\cdot)\|_{L^2} \leq C\frac{1}{b(s)}\left(\frac{m(t)^2}{b(t)} + b(t)(1+b(t)^2B(t,s))^{-1}\right)(1+B(t,s))^{-\frac{n}{2}\left(\frac{1}{\eta}-\frac{1}{2}\right)}\,\frac{\gamma(t)}{\gamma(s)}\,\|g(s,\cdot)\|_{L^\eta\cap L^2}, \tag{39}$$

where the constant C is independent of s.

Remark 3.1. The structure of estimate (39) is more complicated than the structure of (37) and (38), since it arises from the competition of three factors. In particular, as $s \to t$ the term $b(t)(1+b(t)^2B(t,s))^{-1}$ behaves as $b(t)$, whereas, for a fixed s, it behaves as $(b(t)B(t))^{-1}$ as $t \to \infty$. In the first case, it also holds $m(t)^2/b(t) \lesssim b(t)$, whereas in the second one, one has to compare $m(t)^2$ and $B(t)^{-1}$ (see Remark 2.3). The more complicated structure of estimate (39) is related to a refinement process of the estimate, which is only helpful for large values of $B(t,s)$ (see later, Lemma 3.9 and the proof of Theorem 2).

We notice that

$$\frac{\gamma(t)}{\gamma(s)} = \exp\Big(-\int_s^t \frac{m^2(\tau)}{b(\tau)}\,d\tau\Big).$$

In order to prove Theorems 2 and 3, we will also make use of the following easy statement (which motivates the choice of the use of the parameter β in (22)).

Lemma 3.2. *Let $\beta > 0$ be as in* (22)*, and fix $\alpha \in (0,\beta)$. Then there exists a constant $C = C(\alpha) > 0$ such that, for any $0 \le s \le t$, it holds*

$$\exp\left(-\int_s^t \frac{m^2(\tau)}{b(\tau)}\,d\tau\right) \le C\,B(t)^{-\alpha}\,B(s)^{\alpha}.$$

We notice that $B(t) \ge B(s)$, so that the right-hand term is well-defined.

Proof. For any $\alpha \in (0,\beta)$ there exists $t_0 = t_0(\alpha)$ such that

$$\frac{m(\tau)^2}{b(\tau)} \ge \frac{\alpha}{b(\tau)B(\tau)}, \qquad \forall \tau \ge t_0.$$

If $s \le t \le t_0$, the statement trivially follows by the compactness of $[0,t_0]$, for $C \ge C_1(t_0)$. If $t \ge s \ge t_0$, the statement follows for $C \ge 1$ by integrating the previous inequality, so that

$$\exp\left(-\int_s^t \frac{m^2(\tau)}{b(\tau)}\,d\tau\right) \le \exp\left(-\alpha\int_s^t \frac{1}{b(\tau)B(\tau)}\,d\tau\right) = B(t)^{-\alpha}\,B(s)^{\alpha}.$$

If $s < t_0 < t$ the statement follows by combining the previous arguments.

The rest of this section is devoted to prove Theorem 1 and Lemma 3.1.

Following the approach in [26], we transform our Cauchy problem (3) with dissipation and mass terms in a Cauchy problem with time dependent mass. By applying the Fourier transform, we obtain

$$\hat{u}_{tt} + |\xi|^2\hat{u} + b(t)\hat{u}_t + m^2(t)\hat{u} = 0.$$

Let

$$\hat{v}(t,\xi) := \lambda(t)\hat{u}(t,\xi), \qquad \lambda(t) := \exp\Big(\frac{1}{2}\int_0^t b(\tau)d\tau\Big).$$

The function $\hat{v}$ solves

$$\hat{v}_{tt} + M(t,\xi)\hat{v} = 0,$$

where

$$M(t,\xi) = |\xi|^2 - \frac{1}{4}b^2(t) - \frac{1}{2}b'(t) + m^2(t). \tag{40}$$

We notice that, in particular,

$$\hat{v}(0,\xi) = \hat{f}(\xi), \quad \hat{v}_t(0,\xi) = \frac{b(0)}{2}\hat{f}(\xi) + \hat{g}(\xi).$$

We divide the extended phase space $\mathbb{R}_+ \times \mathbb{R}^n$ in the hyperbolic region Π_{hyp} and the elliptic region Π_{ell}, as follows:

$$\Pi_{\text{hyp}} = \Big\{(t,\xi) \in \mathbb{R}_+ \times \mathbb{R}^n : |\xi| > \frac{b(t)}{2}\Big\}, \qquad \Pi_{\text{ell}} = \Big\{(t,\xi) \in \mathbb{R}_+ \times \mathbb{R}^n : |\xi| < \frac{b(t)}{2}\Big\}.$$

Let us define the auxiliary symbol

$$\langle\xi\rangle_{b(t)} = \sqrt{\left||\xi|^2 - \frac{b(t)^2}{4}\right|}.$$

We introduce the hyperbolic zone, the elliptic zone and the reduced zone as follows:

$$\begin{aligned} Z_{\text{hyp}} &= \left\{(t,\xi) \in \{t \geq s\} \times \mathbb{R}^n : \langle\xi\rangle_{b(t)} \geq \epsilon\frac{b(t)}{2}\right\} \cap \Pi_{\text{hyp}}, \\ Z_{\text{ell}} &= \left\{(t,\xi) \in \{t \geq s\} \times \mathbb{R}^n : \langle\xi\rangle_{b(t)} \geq \epsilon\frac{b(t)}{2}\right\} \cap \Pi_{\text{ell}}, \\ Z_{\text{red}} &= \left\{(t,\xi) \in \{t \geq s\} \times \mathbb{R}^n : \langle\xi\rangle_{b(t)} \leq \epsilon\frac{b(t)}{2}\right\}, \end{aligned}$$

where $\epsilon > 0$ will be a sufficiently small constant. Then it holds

$$\langle\xi\rangle_{b(t)} \sim \begin{cases} |\xi| & \text{in } Z_{\text{hyp}}, \\ b(t) & \text{in } Z_{\text{ell}}. \end{cases}$$

3.1. Estimates in the hyperbolic zone

In the hyperbolic zone we introduce the micro-energy

$$V = (\langle\xi\rangle_{b(t)}\hat{v}, D_t\hat{v})^T.$$

Then V solves the system

$$D_tV = A(t,\xi)V = \left[\begin{pmatrix} 0 & \langle\xi\rangle_{b(t)} \\ \langle\xi\rangle_{b(t)} & 0 \end{pmatrix} + \begin{pmatrix} \frac{D_t\langle\xi\rangle_{b(t)}}{\langle\xi\rangle_{b(t)}} & 0 \\ \frac{-b'(t)+2m^2(t)}{2\langle\xi\rangle_{b(t)}} & 0 \end{pmatrix}\right] V, \tag{41}$$

in the hyperbolic zone. Here and in the following, we use the notation $D_t = -i\partial_t$.

Lemma 3.3. *The fundamental solution $\mathcal{E}_{\text{hyp}} = \mathcal{E}_{\text{hyp}}(t,s,\xi)$ to the system* (41) *satisfies the following estimate:*

$$\|\mathcal{E}_{\text{hyp}}(t,s,\xi)\| \lesssim \left(\frac{\lambda(t)}{\lambda(s)}\right)^{2\epsilon},$$

for $(t,\xi),\ (s,\xi) \in Z_{\text{hyp}}$.

Proof. We consider the matrices

$$P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

and we set $V^{(0)} := P^{-1}V$.
By diagonalizing the principal part of the system we reduce to estimate the fundamental solution to

$$D_tV^{(0)} = [D(t,\xi) + R(t,\xi)]V^{(0)}, \tag{42}$$

where

$$\begin{aligned} D(t,\xi) &= \begin{pmatrix} \langle\xi\rangle_{b(t)} & 0 \\ 0 & -\langle\xi\rangle_{b(t)} \end{pmatrix}, \\ R(t,\xi) &= \frac{D_t\langle\xi\rangle_{b(t)}}{2\langle\xi\rangle_{b(t)}}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{-b'(t)+2m^2(t)}{4\langle\xi\rangle_{b(t)}}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}. \end{aligned}$$

For a sufficiently large $t_0 > 0$ (depending on ϵ), due to Hypotheses 1 and 2, it holds

$$\frac{m^2(\tau) + |b'(\tau)|/2}{b^2(\tau)} \leq \epsilon^2,$$

for any $\tau \geq t_0$. Therefore, using $\langle \xi \rangle_{b(\tau)} \geq \epsilon\, b(\tau)/2$, we obtain

$$\int_s^t \frac{m^2(\tau)}{2\langle \xi \rangle_{b(\tau)}} d\tau + \int_s^t \frac{|b'(\tau)|}{4\langle \xi \rangle_{b(\tau)}} d\tau \leq \epsilon \int_s^t b(\tau) d\tau,$$

for any $s \geq t_0$. Moreover, we have

$$\int_s^t \frac{D_t \langle \xi \rangle_{b(\tau)}}{2\langle \xi \rangle_{b(\tau)}} d\tau = \frac{1}{2} \log\left(\frac{\langle \xi \rangle_{b(t)}}{\langle \xi \rangle_{b(s)}}\right),$$

so that

$$\exp\Big(\int_s^t \frac{\left|D_t \langle \xi \rangle_{b(\tau)}\right|}{2\langle \xi \rangle_{b(\tau)}} d\tau\Big) \leq \left(\frac{\langle \xi \rangle_{b(t)}}{\langle \xi \rangle_{b(s)}}\right)^{\pm\frac{1}{2}} \sim \frac{|\xi|^{\frac{1}{2}}}{|\xi|^{\frac{1}{2}}} = 1,$$

where $\pm$ stays for $+$ or, respectively, for $-$ if $b(t)$ is increasing, or, respectively, decreasing.

This concludes the proof for $s \geq t_0$. For $t \leq t_0$, it is clear that $\mathcal{E}_{\text{hyp}}(t, s, \xi)$ is bounded, so that the proof of the lemma follows by combining the two cases.

3.2. Estimates in the reduced zone

Let us define in the reduced zone the micro-energy

$$V = (\epsilon b(t)\hat{v}, D_t \hat{v})^T.$$

Then V solves the system

$$D_t V = \begin{pmatrix} \frac{D_t b(t)}{b(t)} & \epsilon b(t) \\ \frac{|\xi|^2 - \frac{1}{4}b^2(t) - \frac{1}{2}b'(t) + m^2(t)}{\epsilon b(t)} & 0 \end{pmatrix} V, \tag{43}$$

in the reduced zone.

Lemma 3.4. *The fundamental solution $\mathcal{E}_{\text{red}} = \mathcal{E}_{\text{red}}(t, s, \xi)$ to* (43) *satisfies the following estimate in the reduced zone:*

$$\|\mathcal{E}_{\text{red}}(t, s, \xi)\| \lesssim \left(\frac{\lambda(t)}{\lambda(s)}\right)^{4\epsilon},$$

for $t \geq s \geq t_0$ with a sufficiently large t_0 and (t, ξ), $(s, \xi) \in Z_{\text{red}}(\epsilon)$.

Proof. By Hypothesis 1 on $b = b(t)$ we have that for $t > t_0$ sufficiently large $b'(t) \leq \epsilon b^2(t)$. Thus, it holds

$$\frac{|b'(t)|}{b(t)} \leq \epsilon b(t).$$

Moreover, by Hypotheses 1 and 2, for $t > t_0$ sufficiently large, we have

$$\frac{m^2(t)}{\epsilon b(t)} + \frac{|b'(t)|}{\epsilon b(t)} \leq \epsilon b(t).$$

Hence, in the reduced zone we have the following estimates for $t > t_0$:

$$\frac{|\xi|^2 - \frac{1}{4}b^2(t) - \frac{1}{2}b'(t) + m^2(t)}{\epsilon b(t)} = \frac{\langle \xi \rangle_{b(t)}^2}{\epsilon b(t)} - \frac{b'(t)}{2\epsilon b(t)} + \frac{m^2(t)}{\epsilon b(t)} \leq 2\epsilon b(t),$$

since in the reduced zone it holds $\langle \xi \rangle_{b(t)} \leq \epsilon b(t)$. This concludes the proof.

3.3. Estimates in the elliptic zone

In this zone it is more convenient to introduce the micro-energy

$$V = (\sqrt{-M(t,\xi)}\hat{v}, D_t\hat{v})^T,$$

with $M(t,\xi)$ as in (40). Indeed,

$$-M(t,\xi) = \frac{b(t)^2}{4} + \frac{b'(t)}{2} - m^2(t) - |\xi|^2,$$

is positive for sufficiently large t, due to

$$-M(t,\xi) \geq \epsilon^2\frac{b(t)^2}{4} + \frac{b'(t)}{2} - m^2(t),$$

and $m(t)^2 = o(b(t)^2)$, $b'(t) = o(b(t)^2)$. For the same reason, $\sqrt{-M(t,\xi)} \approx b(t)$. Then V solves the system

$$D_tV = A(t,\xi)V, \qquad A(t,\xi) = \sqrt{-M(t,\xi)}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \frac{D_t\sqrt{-M}}{\sqrt{-M}}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{44}$$

in the elliptic zone.

Lemma 3.5. *The fundamental solution $\mathcal{E}_{\mathrm{ell}} = \mathcal{E}_{\mathrm{ell}}(t,s,\xi)$ to* (44) *satisfies the following estimate in the elliptic zone:*

$$\|\mathcal{E}_{\mathrm{ell}}(t,s,\xi)\| \lesssim \left(\frac{b(t)}{b(s)}\right)^{\frac{1}{2}} \exp\left\{\int_s^t \sqrt{-M(\tau,\xi)}\,d\tau\right\}.$$

We introduce the matrices

$$Q = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}, \quad Q^{-1} = \frac{1}{2}\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix},$$

and we set $V^{(0)} := Q^{-1}V$.

By applying the first step of diagonalization we get that the fundamental solution of (44) satisfies the same estimates as the fundamental solution to the system

$$D_tV^{(0)} = [D(t,\xi) + R(t,\xi)]V^{(0)}, \tag{45}$$

where we have

$$D(t,\xi) = i\sqrt{-M(t,\xi)}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$R(t,\xi) = \frac{D_t\sqrt{-M(t,\xi)}}{2\sqrt{-M(t,\xi)}}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Now we carry out the second step of diagonalization.

We denote $N^{(0)} = \mathrm{I}_2$, $B^{(0)} = R$, $F^{(0)} = \mathrm{diag}(B^{(0)})$, and we set

$$N^{(1)} = \frac{1}{2\sqrt{-M(t,\xi)}}\begin{pmatrix} 0 & -iB^{(0)}_{12} \\ iB^{(0)}_{21} & 0 \end{pmatrix} = \frac{D_t\sqrt{-M(t,\xi)}}{-4M(t,\xi)}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

$$B^{(1)} = (D_t - R + F^{(0)})N^{(1)},$$

$$N_1(t,\xi) = \mathrm{I}_2 + N^{(1)}(t,\xi).$$

We have that N_1 is invertible for $t > t_0$ with t_0 sufficiently large and its inverse N_1^{-1} is bounded. In fact, it holds

$$\|N^{(1)}\| \leq \frac{C}{(1+t)b(t)},$$

due to the control of the oscillations, and due to $|m(t)| \lesssim b(t)$, after using once again $\sqrt{-M(t,\xi)} \approx b(t)$. Due to $tb(t) \to \infty$, we obtain that $\det N_1(t,\xi) \geq c > 0$, for sufficiently large t.

Thus, we can define

$$R_1(t,\xi) = -N_1^{-1}(t,\xi)B^{(1)}(t,\xi).$$

We have the operator identity

$$(D_t - D(t,\xi) - R(t,\xi))(N_1(t,\xi)W) = N_1(t,\xi)(D_t - D(t,\xi) - F^{(0)}(t,\xi) - R_1(t,\xi))W,$$

for all matrices W. Now we may prove Lemma 3.5 as a direct consequence of the following statement.

Lemma 3.6. *Assume Hypothesis* 1 *for* $b = b(t)$ *and Hypothesis* 2 *for* $m = m(t)$*. Then the fundamental solution* $\mathcal{E}_{\mathrm{ell},1} = \mathcal{E}_{\mathrm{ell},1}(t,s,\xi)$ *to the transformed equation*

$$(D_t - D(t,\xi) - F_0(t,\xi) - R_1(t,\xi))\mathcal{E}_{\mathrm{ell},1} = 0$$

can be represented as

$$\mathcal{E}_{\mathrm{ell},1}(t,s,\xi) = \Big(\frac{b(t)}{b(s)}\Big)^{\frac{1}{2}} \exp\Big\{\int_s^t \sqrt{-M(\tau,\xi)}\,d\tau\Big\}Q_{\mathrm{ell},1}(t,s,\xi),$$

for $(t,\xi),(s,\xi) \in Z_{\mathrm{ell}}$*, where* $Q_{\mathrm{ell},1} = Q_{\mathrm{ell},1}(t,s,\xi)$ *is a uniformly bounded matrix in* Z_{ell}*.*

Proof. If we differentiate the term

$$\exp\Big\{-i\int_s^t (D(\tau,\xi) + F^{(0)}(\tau,\xi))d\tau\Big\}\mathcal{E}_{\mathrm{ell},1}(t,s,\xi),$$

then we obtain

$$\begin{aligned}
&D_t\Big(\exp\Big\{-i\int_s^t (D(\tau,\xi) + F^{(0)}(\tau,\xi))d\tau\Big\}\mathcal{E}_{\mathrm{ell},1}(t,s,\xi)\Big)\\
&= -(D(t,\xi) + F^{(0)}(t,\xi))\exp\Big\{-i\int_s^t (D(\tau,\xi) + F^{(0)}(\tau,\xi))d\tau\Big\}\mathcal{E}_{\mathrm{ell},1}(t,s,\xi)\\
&+ \exp\Big\{-i\int_s^t (D(\tau,\xi) + F^{(0)}(\tau,\xi))d\tau\Big\}(D(t,\xi) + F^{(0)}(t,\xi) + R_1(t,\xi))\mathcal{E}_{\mathrm{ell},1}(t,s,\xi)\\
&= \exp\Big\{-i\int_s^t (D(\tau,\xi) + F^{(0)}(\tau,\xi))d\tau\Big\}R_1(t,\xi)\mathcal{E}_{\mathrm{ell},1}(t,s,\xi).
\end{aligned}$$

Hence, by integration on the interval $[s,t]$, we obtain that $\mathcal{E}_{\mathrm{ell},1}(t,s,\xi)$ satisfies the following integral equation:

$$\begin{aligned}
\mathcal{E}_{\mathrm{ell},1}(t,s,\xi) &= \exp\Big\{i\int_s^t (D(\tau,\xi) + F^{(0)}(\tau,\xi))d\tau\Big\}\mathcal{E}_{\mathrm{ell},1}(s,s,\xi)\\
&\quad + i\int_s^t \exp\Big\{i\int_\theta^t (D(\tau,\xi) + F^{(0)}(\tau,\xi))d\tau\Big\}R_1(\theta,\xi)\mathcal{E}_{\mathrm{ell},1}(\theta,s,\xi)d\theta.
\end{aligned}$$

We may write

$$iD(\tau,\xi) + iF^{(0)}(\tau,\xi) = \omega(\tau,\xi)\,\mathrm{I}_2 + \sqrt{-M(\tau,\xi)}\begin{pmatrix} 0 & 0\\ 0 & -2\end{pmatrix},$$

where

$$\omega(t,\xi) = \sqrt{-M(t,\xi)} + \frac{\partial_t \sqrt{-M(t,\xi)}}{2\sqrt{-M(t,\xi)}}.$$

Then

$$Q_{\mathrm{ell},1}(t,s,\xi) = \exp\Big\{-\int_s^t \omega(\tau,\xi)d\tau\Big\}\mathcal{E}_{\mathrm{ell},1}(t,s,\xi)$$

satisfies the new integral equation

$$\begin{aligned}\mathcal{Q}_{\mathrm{ell},1}(t,s,\xi) &= \exp\Big\{\int_s^t (iD(\tau,\xi)+iF^{(0)}(\tau,\xi)-\omega(\tau,\xi)\,\mathrm{I}_2\,)d\tau\Big\} \\ &+ \int_s^t \exp\Big\{\int_\theta^t (iD(\tau,\xi)+iF^{(0)}(\tau,\xi)-\omega(\tau,\xi)\,\mathrm{I}_2\,)d\tau\Big\} R_1(\theta,\xi)\mathcal{Q}_{\mathrm{ell},1}(\theta,s,\xi)d\theta.\end{aligned} \tag{46}$$

The function $R_1 = R_1(\theta,\xi)$ is uniformly integrable in Z_{ell}. In fact, by using Hypotheses 1 and 2, it is easy to prove that

$$\|R_1\| \lesssim \|B^{(1)}\| \lesssim \frac{1}{b(t)}\|B^{(0)}\|^2 + \frac{1}{b(t)}\|D_tB^{(0)}\| \lesssim \frac{1}{(1+t)^2 b(t)},$$

that is, R_1 is uniformly integrable thanks to Hypothesis 1.

By estimating

$$\exp\Big\{\int_s^t \frac{\partial_\tau\sqrt{-M(\tau,\xi)}}{2\sqrt{-M(\tau,\xi)}}d\tau\Big\} = \Big(\frac{\sqrt{-M(t,\xi)}}{\sqrt{-M(s,\xi)}}\Big)^{\frac{1}{2}} \approx \Big(\frac{b(t)}{b(s)}\Big)^{\frac{1}{2}},$$

the proof is completed.

3.4. *Proof of Theorem* 1 *and Lemma* 3.1

In order to derive the estimates in Theorem 1 and Lemma 3.1, it is convenient to consider the function

$$a(t,s,\xi) = \frac{\lambda(s)}{\lambda(t)}\frac{h(s,\xi)}{h(t,\xi)}\|\mathcal{E}(t,s,\xi)\|,$$

where $h(t,\xi)$ behaves as the weight employed in the three different zones of the extended phase space, in particular,

$$h(t,\xi) \approx \begin{cases} |\xi| & (t,\xi)\notin Z_{\mathrm{ell}}, \\ b(t) & (t,\xi)\in Z_{\mathrm{ell}}. \end{cases}$$

We use the notation $a_j(t,s,\xi)$, when (t,ξ) and (s,ξ) belong to the same zone Z_j, with $j = \mathrm{hyp}, \mathrm{red}, \mathrm{ell}$, as we did for $\mathcal{E}(t,s,\xi)$.

Thanks to Lemmas 3.3 and 3.4, we know that

$$a_{\mathrm{hyp}}(t,s,\xi) \lesssim \left(\frac{\lambda(s)}{\lambda(t)}\right)^{1-2\epsilon},$$
$$a_{\mathrm{red}}(t,s,\xi) \lesssim \left(\frac{\lambda(s)}{\lambda(t)}\right)^{1-4\epsilon},$$

and it is straight-forward to prove the following statement.

Lemma 3.7. *In* Z_{ell} *it holds*

$$a_{\mathrm{ell}}(t,s,\xi) \lesssim \exp\Big(-|\xi|^2 B(t,s) - \int_s^t \frac{m^2(\tau)}{b(\tau)}d\tau\Big).$$

Proof. It is sufficient to prove that

$$\frac{\lambda(s)}{\lambda(t)}\exp\Big\{\int_s^t \sqrt{-M(\tau,\xi)}\,d\tau\Big\} \lesssim \Big(\frac{b(t)}{b(s)}\Big)^{\frac{1}{2}} \exp\Big(-|\xi|^2 B(t,s) - \int_s^t \frac{m^2(\tau)}{b(\tau)}d\tau\Big).$$

This latter is true if and only if it holds

$$-M(t,\xi) \le \left(\frac{b(t)}{2} + \frac{b'(t)}{2b(t)} - \frac{|\xi|^2}{b(t)} - \frac{m^2(t)}{b(t)}\right)^2.$$

But this inequality is true after taking account that the right-hand side is equal to

$$-M(t,\xi) + \left(\frac{b'(t)}{2b(t)} - \frac{|\xi|^2}{b(t)} - \frac{m^2(t)}{b(t)}\right)^2.$$

This concludes the proof.

To prove Theorem 1 and Lemma 3.1, we first notice that the solution to (36) verifies the pointwise estimates

$$\begin{aligned} |\hat{u}(t,\xi)| &\lesssim a(t,s,\xi)\left(|\hat{f}(s,\xi)| + \frac{1}{h(s,\xi)}|\hat{g}(s,\xi)|\right), \\ |\hat{u}_t(t,\xi)| &\lesssim (h(t,\xi)+b(t))\,a(t,s,\xi)\left(|\hat{f}(s,\xi)| + \frac{1}{h(s,\xi)}|\hat{g}(s,\xi)|\right). \end{aligned}$$

In the second estimate, we used $\lambda(t)\hat{u}_t = \hat{v}_t + (b(t)/2)\,\hat{v}$.

Due to the fact that the estimates in $Z_{\rm red}$ and in $Z_{\rm hyp}$ are essentially the same, for any $t \geq 0$, we define

$$\Theta(t) = \frac{\sqrt{1-\epsilon^2}}{2}\,b(t),$$

so that $(t,\xi) \in Z_{\rm ell}$ if, and only if, $|\xi| \leq \Theta(t)$.

We first derive an estimate for low frequencies. For any fixed $t \geq s \geq 0$, let us introduce the function

$$\Theta_0(t,s) = \min\{\Theta(t), \Theta(s)\}.$$

Then we may prove the following statement.

Lemma 3.8. *Let $j = 0, 1$. For any $w(s,\cdot) \in L^\eta \cap H^j$, with $\eta \in [1,2]$, it holds*

$$\||\xi|^j e^{-|\xi|^2 B(t,s)}\hat{w}(s,\cdot)\|_{L^2(|\xi|\leq\Theta_0(t,s))} \lesssim (1+B(t,s))^{-\frac{n}{2}\left(\frac{1}{\eta}-\frac{1}{2}\right)-\frac{j}{2}}\left(\|w(s,\cdot)\|_{L^\eta} + \|w(s,\cdot)\|_{H^j}\right). \tag{47}$$

Proof. By Hölder's inequality, we get

$$\||\xi|^j\, \exp\{-|\xi|^2 B(t,s)\}\,\hat{w}(s,\cdot)\|_{L^2} \lesssim \||\xi|^j\, \exp\{-|\xi|^2 B(t,s)\}\|_{L^r}\,\|\hat{w}(s,\cdot)\|_{L^{\eta'}},$$

where $\eta' = \eta/(\eta-1)$ is the Hölder's conjugate of η, and

$$\frac{1}{r} = \frac{1}{2} - \frac{1}{\eta'} = \frac{1}{\eta} - \frac{1}{2}.$$

By the change of variable $\mu = \sqrt{B(t,s)}\xi$, we get

$$\||\xi|^j\, \exp\{-|\xi|^2 B(t,s)\}\|_{L^r} \leq B(t,s)^{-\frac{n}{2r}-\frac{j}{2}}.$$

Recalling that

$$\|\hat{w}(s,\cdot)\|_{L^{\eta'}} \lesssim \|w(s,\cdot)\|_{L^\eta},$$

being $\eta \in [1,2]$, we derive

$$\||\xi|^j e^{-|\xi|^2 B(t,s)}\hat{w}(s,\cdot)\|_{L^2(|\xi|\leq\Theta_0(t,s))} \lesssim B(t,s)^{-\frac{n}{2}\left(\frac{1}{\eta}-\frac{1}{2}\right)-\frac{j}{2}}\,\|w(s,\cdot)\|_{L^\eta}.$$

The above estimate becomes singular as $s \to t$. In this case, we just use the previous estimate with $\eta = 2$ and $j = 0$, replacing $\hat{w}$ by $|\xi|^j\hat{w}$, so that

$$\||\xi|^j e^{-|\xi|^2 B(t,s)}\hat{w}(s,\cdot)\|_{L^2(|\xi|\leq\Theta_0(t,s))} \lesssim \||\xi|^j\hat{w}(s,\cdot)\|_{L^2}.$$

Combining the two cases, we conclude the proof of (47).

As a consequence of Lemma 3.8, we obtain

$$\||\xi|^j a(t,s,\xi)\hat{w}(s,\cdot)\|_{L^2(|\xi|\leq\Theta_0(t,s))} \lesssim (1+B(t,s))^{-\frac{n}{2}\left(\frac{1}{\eta}-\frac{1}{2}\right)-\frac{j}{2}}\,\frac{\gamma(t)}{\gamma(s)}\,(\|w(s,\cdot)\|_{L^\eta}+\|w(s,\cdot)\|_{H^j}),$$

for $j=0,1$. We now derive an estimate for high frequencies. For this reason we introduce for any fixed $t\geq s\geq 0$ the function

$$\Theta_\infty(t,s)=\max\{\Theta(t),\Theta(s)\}.$$

Then we immediately obtain

$$\||\xi|^j a(t,s,\cdot)\hat{w}(s,\cdot)\|_{L^2(|\xi|\geq\Theta_\infty(t,s))} \lesssim \Big(\frac{\lambda(s)}{\lambda(t)}\Big)^{1-4\epsilon}\|w(s,\cdot)\|_{H^j}. \tag{48}$$

We shall now turn to the case of intermediate frequencies.

- If b is increasing, intermediate frequencies are described by $\Theta(s)\leq|\xi|\leq\Theta(t)$.
- If b is decreasing, intermediate frequencies are described by $\Theta(t)\leq|\xi|\leq\Theta(s)$.

Recalling the definition of $\lambda(t)$, if we define t_ξ as the unique value in $[s,t]$ such that $\Theta(t_\xi)=|\xi|$, then we have the following estimates:

$$\begin{aligned}&\||\xi|^j a(t,s,\cdot)\hat{w}(s,\cdot)\|_{L^2(\Theta(s)\leq|\xi|\leq\Theta(t))}\\ &\quad\lesssim (1+B(t,t_\xi))^{-\frac{n}{2}\left(\frac{1}{\eta}-\frac{1}{2}\right)-\frac{j}{2}}\,\exp\Big(-(1/2-2\epsilon)\int_s^{t_\xi}b(\tau)d\tau-\int_{t_\xi}^t\frac{m^2(\tau)}{b(\tau)}d\tau\Big)(\|w(s,\cdot)\|_{L^\eta}+\|w(s,\cdot)\|_{H^j}),\end{aligned} \tag{49}$$

if b is increasing, and

$$\begin{aligned}&\||\xi|^j a(t,s,\cdot)\hat{w}(s,\cdot)\|_{L^2(\Theta(t)\leq|\xi|\leq\Theta(s))}\\ &\quad\lesssim (1+B(t_\xi,s))^{-\frac{n}{2}\left(\frac{1}{\eta}-\frac{1}{2}\right)-\frac{j}{2}}\,\exp\Big(-\int_s^{t_\xi}\frac{m^2(\tau)}{b(\tau)}d\tau-(1/2-2\epsilon)\int_{t_\xi}^t b(\tau)d\tau\Big)(\|w(s,\cdot)\|_{L^\eta}+\|w(s,\cdot)\|_{H^j}),\end{aligned} \tag{50}$$

if b is decreasing.

We now compare (47), (48), (49) and (50). We first recall that $m=o(b(t))$, and we notice that, for any $\epsilon>0$ it holds

$$\exp\Big(-\int_s^t\frac{m^2(\tau)}{b(\tau)}d\tau\Big)\geq\exp\Big(-\epsilon\int_s^t b(\tau)\,d\tau\Big)=\Big(\frac{\lambda(s)}{\lambda(t)}\Big)^{2\epsilon},$$

for sufficiently large s. On the other hand, for any $\epsilon>0$ it holds

$$B(t,s)^{-1}=\exp\Big(-\int_s^t\frac{1}{B(\tau)b(\tau)}\,d\tau\Big)\geq\exp\Big(-\epsilon\int_s^t b(\tau)\,d\tau\Big)=\Big(\frac{\lambda(s)}{\lambda(t)}\Big)^{2\epsilon}$$

for sufficiently large s, due to $B(t)b(t)^2\approx tb(t)\to 0$ as $t\to\infty$. Similarly, using $b'(t)=o(b(t)^2)$, one may derive $\lambda(t)^\epsilon\lesssim b(t)$, which will be used later, to derive (51).

Therefore, it is clear that the decay rate in (47) is worse than the decay rates in the other cases.

Having in mind that $h(t,\xi)\approx|\xi|$ if $(t,\xi)\notin Z_{\mathrm{ell}}$, and $h(t,\xi)\approx b(t)$ if $(t,\xi)\in Z_{\mathrm{ell}}$, we may estimate the solution to the Cauchy problem (3) (or (36) at $s=0$) with $g=0$ by

$$\|\nabla^j\partial_t^k u(t,\cdot)\|_{L^2}\lesssim b(t)^k\,(1+B(t))^{-\frac{n}{2}\left(\frac{1}{\eta}-\frac{1}{2}\right)-\frac{j}{2}}\,\exp\Big(-\int_0^t\frac{m^2(\tau)}{b(\tau)}d\tau\Big)(\|f\|_{L^\eta}+\|f\|_{H^{j+k}}), \tag{51}$$

for $j+k=0,1$. On the other hand, recalling that $h(s,\xi)\approx b(s)$ if $(s,\xi)\in Z_{\mathrm{ell}}$ and that $h(s,\xi)^{-1}\approx|\xi|^{-1}\lesssim b(s)^{-1}$ if $(s,\xi)\notin Z_{\mathrm{ell}}$, we may estimate the solution to (36) with $f=0$ by

$$\|\nabla^j\partial_t^k u(t,\cdot)\|_{L^2}\lesssim b(t)^k\,b(s)^{-1}\,(1+B(t,s))^{-\frac{n}{2}\left(\frac{1}{\eta}-\frac{1}{2}\right)-\frac{j}{2}}\,\exp\Big(-\int_s^t\frac{m^2(\tau)}{b(\tau)}d\tau\Big)\|g(s,\cdot)\|_{L^\eta\cap L^2}, \tag{52}$$

for $j+k=0,1$. In this way the proofs of estimates (17) and (18), in Theorem 1, and of estimates (37) and (38), in Lemma 3.1, are completed.

However, the estimate provided for $\|\partial_t u(t,\cdot)\|_{L^2}$ can be refined for solutions to the Cauchy problem (3) as $t\to\infty$, and for the Cauchy problem (36) when t is away from s.

We proceed as in Lemma 20 in [26]. Let

$$u(t,x)=\varphi_0(t,s,x)*_{(x)}f(s,x)+\varphi_1(t,s,x)*_{(x)}g(s,x).$$

By construction, the multiplier $\hat\varphi_i(t,s,\xi)$ satisfies the Cauchy problem

$$\hat\varphi_i''+\left(|\xi|^2+m^2(t)\right)\hat\varphi_i+b(t)\hat\varphi_i'=0,\quad \hat\varphi_i(s,s,\xi)=\delta_i^0,\quad \hat\varphi_i'(s,s,\xi)=\delta_i^1.$$

After setting $\Psi_i(t,s,\xi)=\hat\varphi_i'(t,s,\xi)$ we get

$$\Psi_i'+b(t)\Psi_i=-\left(|\xi|^2+m^2(t)\right)\hat\varphi_i,\quad \Psi_i(s,s,\xi)=\delta_i^1.$$

Therefore,

$$\Psi_i(t,s,\xi)=\frac{\lambda(s)^2}{\lambda(t)^2}\delta_i^1-\int_s^t\frac{\lambda^2(\tau)}{\lambda^2(t)}\left(|\xi|^2+m^2(\tau)\right)\hat\varphi_i(\tau,s,\xi)d\tau. \tag{53}$$

Assume $(t,\xi),(s,\xi)\in Z_{\rm ell}$. By the previous estimate obtained for $\hat\varphi_i$, we derive

$$|\Psi_i(t,s,\xi)|\lesssim\frac{\lambda(s)^2}{\lambda(t)^2}\delta_i^1+I,$$

where

$$I=\int_s^t\frac{\lambda^2(\tau)}{\lambda^2(t)}\Big(\frac{|\xi|^2+m^2(\tau)}{b(s)^i}\Big)\exp\Big\{-\int_s^\tau\frac{|\xi|^2+m^2(\theta)}{b(\theta)}d\theta\Big\}\,d\tau.$$

After integration by parts we get

$$\begin{aligned}
I&=\int_s^t\Big(\partial_\tau\frac{\lambda^2(\tau)}{\lambda^2(t)}\Big)\Big(\frac{|\xi|^2+m^2(\tau)}{b(\tau)b(s)^i}\Big)\exp\Big\{-\int_s^\tau\frac{|\xi|^2+m^2(\theta)}{b(\theta)}d\theta\Big\}\,d\tau\\
&=\Big[\frac{\lambda(\tau)^2}{\lambda(t)^2}\Big(\frac{|\xi|^2+m^2(\tau)}{b(\tau)b(s)^i}\Big)\exp\Big\{-\int_s^\tau\frac{|\xi|^2+m^2(\theta)}{b(\theta)}d\theta\Big\}\Big]_s^t+J\\
J&=-\int_s^t\frac{\lambda^2(\tau)}{\lambda^2(t)}\partial_\tau\Big(\Big(\frac{|\xi|^2+m^2(\tau)}{b(\tau)b(s)^i}\Big)\exp\Big\{-\int_s^\tau\frac{|\xi|^2+m^2(\theta)}{b(\theta)}d\theta\Big\}\Big)d\tau.
\end{aligned}$$

We may estimate $J\le I/2$. Indeed, by Hypotheses 1 and 2, using that $|\xi|^2<b(t)^2/4$ in $Z_{\rm ell}$, we get

$$\Big(-\frac{2m(\tau)m'(\tau)}{b(\tau)}+\frac{(|\xi|^2+m^2(\tau))b'(\tau)}{b(\tau)^2}+\frac{(|\xi|^2+m^2(\tau))^2}{b(\tau)^2}\Big)<\frac{1}{2}(|\xi|^2+m^2(\tau)),$$

for $t>s>t_0$, for t_0 sufficiently large. Therefore, we have

$$|\Psi_i(t,s,\xi)|\lesssim\frac{\lambda(s)^2}{\lambda(t)^2}\delta_i^1+\frac{|\xi|^2+m^2(t)}{b(t)b(s)^i}\exp\Big\{-|\xi|^2B(t,s)-\int_s^t\frac{m^2(\tau)}{b(\tau)}d\tau\Big\}. \tag{54}$$

Due to $m(t)\lesssim b(t)$ and $|\xi|\lesssim b(t)$, estimate (54) is a refinement of the previous estimate for the time derivative of the solution (see (51)-(52) for $j=0$ and $k=1$).

We may now proceed as we did to estimate $\|\nabla^j u(t,\cdot)\|_{L^2}$, using that

$$|\hat u_t(t,\xi)|\lesssim|\Psi_0(t,s,\xi)|\,|\hat f(s,\xi)|+|\Psi_1(t,s,\xi)|\,|\hat g(s,\xi)|,$$

for $(t,\xi),(s,\xi)\in Z_{\rm ell}$, but relying on the following modification of Lemma 3.8.

Lemma 3.9. *For any $w(s,\cdot)\in L^\eta\cap L^2$, with $\eta\in[1,2]$, it holds*

$$\left\| \frac{|\xi|^2+m^2(t)}{b(t)b(s)^i}\, e^{-|\xi|^2 B(t,s)}\hat{w}(t,\cdot)\right\|_{L^2(|\xi|\leq\Theta_0(t,s))} \lesssim \frac{m^2(t)+B(t,s)^{-1}}{b(t)b(s)^i}\,(1+B(t,s))^{-\frac{n}{2}\left(\frac{1}{\eta}-\frac{1}{2}\right)}\,(\|w(s,\cdot)\|_{L^\eta}+\|w(s,\cdot)\|_{L^2}), \qquad (55)$$

for any $s\geq 0$ and $t>s$.

We notice that estimate (55) is singular at $t=s$, but we plan to use it only for $B(t,s)\geq 1$.

Proof. Proceeding as in the proof of Lemma 3.8, we derive

$$\||\xi|^{2k}\, e^{-|\xi|^2 B(t,s)}\hat{w}(s,\cdot)\|_{L^2(|\xi|\leq\Theta_0(t,s))} \lesssim B(t,s)^{-\frac{n}{2}\left(\frac{1}{\eta}-\frac{1}{2}\right)-k}\,\|w(s,\cdot)\|_{L^\eta},$$

for $k=0,1$. In particular,

$$\||\xi|^{2k}\, e^{-|\xi|^2 B(t,s)}\hat{w}(s,\cdot)\|_{L^2(|\xi|\leq\Theta_0(t,s))} \lesssim B(t,s)^{-k}\,\|w(s,\cdot)\|_{L^2}.$$

Using the L^η regularity for $B(t,s)\geq 1$ and the L^2 regularity for $B(t,s)\leq 1$, we conclude the proof.

Combining estimate (52) (when $B(t,s)\leq 1$) with estimates (54) and (55) (when $B(t,s)\geq 1$), and taking into account that the decay in

$$\frac{\lambda(s)^2}{\lambda(t)^2}\delta_i^1$$

is faster than the decay in (55), we derive (39). In particular, to derive the term

$$\frac{m(t)^2}{b(t)}+b(t)(1+b(t)^2B(t,s))^{-1}$$

in (39), valid for all possible choices of s,t (see Remark 3.1), we use the property that:

$$0<h(t)\leq\min\{g_1(t),g_2(t)+g_3(t)\}\Rightarrow h(t)\leq g_2(t)+2\left(\frac{1}{g_1(t)}+\frac{1}{g_3(t)}\right)^{-1},$$

with $g_1(t)=C_1b(t)$, $g_2(t)=C_2m(t)^2/b(t)$ and $g_3(t)=C_3(b(t)B(t,s))^{-1}$.

At $s=0$, the calculations are simpler. Combining estimates (51) and (52) for $t\leq 1$ with estimates (54) and (55) for $t\geq 1$, we obtain (19).

This concludes the proofs of Theorem 1 and Lemma 3.1.

4. Proofs of Theorems 2 and 3

A function $u=u(t,x)$ solves the Cauchy problem (1) in a suitable space X of Sobolev solutions if and only if

$$u(t,x)=u^{\mathrm{lin}}(t,x)+(Nu)(t,x),$$

where u^{lin} is the solution to the Cauchy problem (3), and

$$Nu(t,x)=\int_0^t \Phi(t,s,\cdot)*_{(x)}h(u(s,\cdot))(x)ds,$$

in X, where by $\Phi(t,s,\cdot)*_{(x)}h(u(s,\cdot))(x)$ we denote the solution to the Cauchy problem (36) with $f=0$ and $g(s,\cdot)=h(u(s,\cdot))$.

To prove Theorems 2 and 3, we will rely on a standard contraction argument in the solution space $C^1([0,\infty),H^1)\times C([0,\infty),L^2)$, equipped with a suitable norm, defined accordingly to the decay estimates for the solutions to the corresponding linear Cauchy problems with vanishing right-hand side obtained in Theorem 1.

For any $T>0$, we define the Banach spaces

$$X_0(T)=C([0,T],H^1),\qquad X(T)=X_0(T)\cap C^1([0,T],L^2).$$

We will fix a suitable norm on $X(T)$ such that

$$\|u^{\mathrm{lin}}\|_{X(T)} \le C\|(f,g)\|_{\mathcal{A}_\eta}. \tag{56}$$

Then we will prove that

$$\|Nu\|_{X(T)} \le C\|u\|^p_{X_0(T)}, \tag{57}$$

$$\|Nu - Nv\|_{X(T)} \le C\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X_0(T)} + \|v\|^{p-1}_{X_0(T)}\big), \tag{58}$$

where $\eta = 2$ in Theorem 2 and $\eta \in [1,2)$ in Theorem 3, with a constant $C > 0$, independent of T. From condition (57) it follows that N maps $X_0(T)$ into $X(T)$.

Due to the inequalities (57) and (58), we can apply the Banach's fixed point theorem to prove that for all $T > 0$ there exists a uniquely determined solution to Cauchy problem (1), in $X(T)$, provided that $\|(f,g)\|_{\mathcal{A}_\eta}$ in (56) is sufficiently small. Since the constants in (56), (57), (58) do not depend on T, the solution is globally defined (in time).

We first prove Theorem 2.

Proof. If $\beta > 0$ in (22), due to $p > p_\beta(n)$, we may consider any $\alpha \in (0,\beta)$ such that $p > p_\alpha(n)$. If $\beta = 0$, we fix $\alpha = 0$.

To prove Theorem 2, we equip $X_0(T)$ and $X(T)$ with the norms

$$\|u\|_{X_0(T)} = \sup_{0\le t\le T}\big((1+B(t))^\alpha\|u(t,\cdot)\|_{L^2} + (1+B(t))^{\alpha+\frac{1}{2}}\|\nabla u(t,\cdot)\|_{L^2}\big)$$

$$\|u\|_{X(T)} = \|u\|_{X_0(T)} + \sup_{0\le t\le T}\big((1+B(t))^\alpha b(t)(m(t)^2 + (1+B(t))^{-1})^{-1}\|u_t(t,\cdot)\|_{L^2}\big).$$

By applying Theorem 1 with $\eta = 2$ to u^{lin} and using $(1+B(t))^\alpha \lesssim \gamma(t)^{-1}$, we derive (56).

As a consequence of Gagliardo-Nirenberg inequality, any function $u \in X_0(T)$ verifies the inequality

$$\|u(\tau,\cdot)\|_{L^q} \le C\,(1+B(\tau))^{-\frac{n}{2}\left(\frac{1}{2}-\frac{1}{q}\right)-\alpha}\,\|u\|_{X_0(T)}, \tag{59}$$

for any $\tau \in [0,T]$, for any $q \in [2,\infty)$ if $n = 1,2$ and for any $q \in [2, 2n/(n-2)]$ if $n \ge 3$.

Let $j = 0,1$. Then

$$\|\nabla^j Nu(t,\cdot)\|_{L^2} \le \int_0^t \|\nabla^j\Phi(t,s,\cdot) *_{(x)} h(u(s,\cdot))\|_{L^2}ds.$$

By Lemma 3.1 with $\eta = 2$, and using Lemma 3.2, we get

$$\|\nabla^j Nu(t,\cdot)\|_{L^2} \le C\int_0^t \frac{1}{b(s)}(1+B(t,s))^{-\frac{j}{2}}\,B(t)^{-\alpha}\,B(s)^\alpha\|h(u(s,\cdot))\|_{L^2}ds.$$

Using $|h(u)| \lesssim |u|^p$ and (59), noticing that $2p \le 2n/(n-2)$ if $n \ge 3$, we may estimate

$$\|h(u(s,\cdot))\|_{L^2} \lesssim \|u(s,\cdot)\|^p_{L^{2p}} \lesssim (1+B(s))^{-\frac{n}{4}(p-1)-p\alpha}\|u\|^p_{X_0(T)}. \tag{60}$$

For short time, say for $t \le 2$, it is clear that

$$\|\nabla^j Nu(t,\cdot)\|_{L^2} \lesssim \|u\|^p_{X_0(T)}.$$

Let $t \ge 2$. To estimate the integral terms we use the following properties for $B(t,s)$ (see [7]):

- $B(t,s) \approx B(t)$ in $[0,\frac{t}{2}]$,
- $B(s,0) \approx B(t)$ in $[\frac{t}{2},t]$,

as well as $B(t) \approx 1 + B(t)$. So, we get

$$\|\nabla^j Nu(t,\cdot)\|_{L^2} \lesssim \|u\|^p_{X_0(T)}\,(1+B(t))^{-\frac{j}{2}-\alpha} \int_0^{t/2} \frac{1}{b(s)}\,(1+B(s))^{-\left(\frac{n}{4}+\alpha\right)(p-1)}ds$$
$$+ \|u\|^p_{X_0(T)}\,B(t)^{-\left(\frac{n}{4}+\alpha\right)(p-1)-\alpha} \int_{t/2}^t \frac{1}{b(s)}(1+B(t,s))^{-\frac{j}{2}}\,ds.$$

By using the change of variable $\rho = B(s)$ in the first integral and $\rho = B(t,s)$ in the second one, we obtain

$$\int_0^{t/2} \frac{1}{b(s)}\,(1+B(s))^{-\left(\frac{n}{4}+\alpha\right)(p-1)}ds \leq C,$$
$$\int_{t/2}^t \frac{1}{b(s)}(1+B(t,s))^{-\frac{j}{2}}\,ds \lesssim B(t/2)^{1-\frac{j}{2}},$$

where we used $p > p_\alpha(n)$ in the first integral, and $j/2 < 1$ in the second one. Using $B(t/2) \sim 1 + B(t)$ for $t \geq 2$, and $p > p_\alpha(n)$ once again, we derive

$$\|\nabla^j Nu(t,\cdot)\|_{L^2} \lesssim \|u\|^p_{X_0(T)}\,(1+B(t))^{-\frac{j}{2}-\alpha}.$$

Now we estimate the term with the time derivative. So, we proceed as follows:

$$\|\partial_t Nu(t,\cdot)\|_{L^2} \leq \int_0^t \|\partial_t \Phi(t,s,\cdot) * h(u(s,\cdot))\|_{L^2}ds.$$

By Lemma 3.1 with $\eta = 2$, and using Lemma 3.2, we get

$$\|\partial_t Nu(t,\cdot)\|_{L^2} \leq C \int_0^t \frac{1}{b(s)}\left(\left(\frac{m(t)^2}{b(t)} + b(t)(1+b(t)^2 B(t,s))^{-1}\right)\right) B(t)^{-\alpha}\,B(s)^\alpha \|h(u(s,\cdot))\|_{L^2}ds.$$

Clearly, we may estimate

$$\frac{m(t)^2}{b(t)} \int_0^t \frac{1}{b(s)}\,B(t)^{-\alpha}\,B(s)^\alpha \|h(u(s,\cdot))\|_{L^2}ds \lesssim \frac{m(t)^2}{b(t)}\,(1+B(t))^{-\alpha},$$

as we did in the previous step for $j = 0$. Therefore we restrict ourselves to estimate

$$I(t) = \int_0^t \frac{1}{b(s)} b(t)(1+b(t)^2 B(t,s))^{-1}\,B(t)^{-\alpha}\,B(s)^\alpha \|h(u(s,\cdot))\|_{L^2}ds.$$

For short time, say for $t \leq 2$, it is clear that $I(t) \lesssim \|u\|^p_{X_0(T)}$. Let $t \geq 2$. Now we get

$$\|\partial_t Nu(t,\cdot)\|_{L^2} \lesssim \|u\|^p_{X_0(T)}\,b(t)\,(1+b(t)^2B(t))^{-1}\,(1+B(t))^{-\alpha} \int_0^{t/2} \frac{1}{b(s)}\,(1+B(s))^{-\left(\frac{n}{4}+\alpha\right)(p-1)}ds$$
$$+ \|u\|^p_{X_0(T)}\,b(t)\,B(t)^{-\left(\frac{n}{4}+\alpha\right)(p-1)-\alpha} \int_{t/2}^t \frac{1}{b(s)}(1+b(t)^2 B(t,s))^{-1}\,ds$$
$$\lesssim \|u\|^p_{X_0(T)}\,b(t)\,(1+b(t)^2B(t))^{-1}\,(1+B(t))^{-\alpha}$$
$$+ \|u\|^p_{X_0(T)}\,b(t)\,B(t)^{-\left(\frac{n}{4}+\alpha\right)(p-1)-\alpha}\,\frac{1}{b(t)^2}\,\log(1+b(t)^2B(t)).$$

In the second integral, we used the change of variable $\rho = b(t)^2 B(t,s)$. By using $1 + b(t)^2B(t) \approx b(t)^2B(t)$ (due to $b(t)^2B(t) \approx tb(t) \to \infty$), and controlling the logarithmic term with an arbitrarily small power (we also use (12)), this leads to

$$I(t) \lesssim \|u\|^p_{X_0(T)}\,\frac{1}{b(t)}\,(1+B(t))^{-1-\alpha}.$$

Summarizing, we proved that

$$\|Nu\|_{X(T)} \leq C\|u\|_{X_0(T)}^p,$$

with C independent of T. This concludes the proof of (57). We proceed similarly to prove (58). In particular, we replace (60) by

$$\begin{aligned}
\|h(u(s,\cdot)) - h(v(s,\cdot))\|_{L^2} &\lesssim \||u(s,\cdot) - v(s,\cdot)|\,(|u(s,\cdot)|^{p-1} + |v(s,\cdot)|^{p-1}\|_{L^2} \\
&\lesssim \|u(s,\cdot) - v(s,\cdot)\|_{L^{2p}}\,\||u(s,\cdot)|^{p-1} + |v(s,\cdot)|^{p-1}\|_{L^{2p'}} \\
&\lesssim (1+B(s))^{-\frac{n}{4}(p-1)-p\alpha}\|u-v\|_{X_0(T)}\,\big(\|u\|_{X_0(T)}^{p-1} + \|v\|_{X_0(T)}^{p-1}\big).
\end{aligned}$$

This concludes the proof.

Finally, we prove Theorem 3.

Proof. We proceed as in the proof of Theorem 2. But, now we fix $\alpha \in (0,\beta)$ such that $p > p_{\alpha,\eta}(n)$ if $\beta > 0$, and we equip $X_0(T)$ and $X(T)$ with the norms

$$\begin{aligned}
\|u\|_{X_0(T)} &= \sup_{0\leq t\leq T}(1+B(t))^{\frac{n}{2}\left(\frac{1}{\eta}-\frac{1}{2}\right)+\alpha}(\|u(t,\cdot)\|_{L^2} + (1+B(t))^{\frac{1}{2}}\|\nabla u(t,\cdot)\|_{L^2}), \\
\|u\|_{X(T)} &= \|u\|_{X_0(T)} + \sup_{0\leq t\leq T}\big((1+B(t))^{\frac{n}{2}\left(\frac{1}{\eta}-\frac{1}{2}\right)+\alpha}b(t)(m(t)^2 + (1+B(t))^{-1})^{-1}\|u_t(t,\cdot)\|_{L^2}\big).
\end{aligned}$$

By Theorem 1, we derive (56). As a consequence of Gagliardo-Nirenberg inequality, any function $u \in X_0(T)$ verifies the inequality

$$\|u(\tau,\cdot)\|_{L^q} \leq C\,(1+B(\tau))^{-\frac{n}{2}\left(\frac{1}{\eta}-\frac{1}{q}\right)-\alpha}\,\|u\|_{X_0(T)}, \tag{61}$$

for any $\tau \in [0,T]$, for any $q \in [2,\infty)$ if $n = 1,2$ and for any $q \in [2, 2n/(n-2)]$ if $n \geq 3$.

Using $|h(u)| \lesssim |u|^p$ and (61), noticing that $\eta p \geq 2$ and $2p \leq 2n/(n-2)$ if $n \geq 3$, we may replace (60) by the two estimates

$$\|h(u(s,\cdot))\|_{L^\eta} \lesssim \|u(s,\cdot)\|_{L^{\eta p}}^p \lesssim (1+B(s))^{-\frac{n}{2\eta}(p-1)-p\alpha}\|u\|_{X_0(T)}^p, \tag{62}$$

$$\|h(u(s,\cdot))\|_{L^2} \lesssim \|u(s,\cdot)\|_{L^{2p}}^p \lesssim (1+B(s))^{-\frac{n}{2\eta}(p-1)-\frac{n}{2}\left(\frac{1}{\eta}-\frac{1}{2}\right)-p\alpha}\|u\|_{X_0(T)}^p. \tag{63}$$

Now, we use different estimates for $s \in [0,t/2]$ and $s \in [t/2,t]$, by using the L^η norm of $h(u(s,\cdot))$ only for $s \in [0,t/2]$. By applying Lemma 3.1, we obtain

$$\begin{aligned}
&\|\nabla^j Nu(t,\cdot)\|_{L^2} \\
&\quad\leq C\int_0^{t/2}\frac{1}{b(s)}(1+B(t,s))^{-\frac{n}{2}\left(\frac{1}{\eta}-\frac{1}{2}\right)-\frac{j}{2}}\,B(t)^{-\alpha}\,B(s)^{\alpha}(\|h(u(s,\cdot))\|_{L^\eta} + \|h(u(s,\cdot))\|_{L^2})ds \\
&\quad\quad + C\int_{t/2}^{t}\frac{1}{b(s)}(1+B(t,s))^{-\frac{j}{2}}\,B(t)^{-\alpha}\,B(s)^{\alpha}\|h(u(s,\cdot))\|_{L^2}ds,
\end{aligned}$$

for $j = 0,1$. For short time, say for $t \leq 2$, it is clear that

$$\|\nabla^j Nu(t,\cdot)\|_{L^2} \lesssim \|u\|_{X_0(T)}^p.$$

Let $t \geq 2$. Proceeding as in the proof of Theorem 2, using (62) and (63), we get

$$\begin{aligned}
\|\nabla^j Nu(t,\cdot)\|_{L^2} &\lesssim \|u\|_{X_0(T)}^p\,(1+B(t))^{-\frac{n}{2}\left(\frac{1}{\eta}-\frac{1}{2}\right)-\frac{j}{2}-\alpha}\int_0^{t/2}\frac{1}{b(s)}\,(1+B(s))^{-\left(\frac{n}{2\eta}+\alpha\right)(p-1)}ds \\
&\quad + \|u\|_{X_0(T)}^p\,B(t)^{-\left(\frac{n}{2\eta}+\alpha\right)(p-1)-\frac{n}{2}\left(\frac{1}{\eta}-\frac{1}{2}\right)-\alpha}\int_{t/2}^{t}\frac{1}{b(s)}(1+B(t,s))^{-\frac{j}{2}}\,ds.
\end{aligned}$$

Using $p > p_{\alpha,\eta}(n)$ it follows that the first integral is bounded. Hence, we arrive at

$$\|\nabla^j Nu(t,\cdot)\|_{L^2} \lesssim \|u\|^p_{X_0(T)} (1+B(t))^{-\frac{n}{2}\left(\frac{1}{\eta}-\frac{1}{2}\right)-\frac{j}{2}-\alpha}.$$

We estimate the term with the time derivative as we did in the proof of Theorem 2, obtaining

$$\|Nu\|_{X(T)} \leq C\|u\|^p_{X_0(T)},$$

with C independent of T. Once again, we proceed similarly to prove (58), and this concludes the proof.

5. The corresponding diffusive Cauchy problem

In order to motivate the critical exponents in Theorems 2 and 3, we consider the corresponding nonlinear Cauchy problem for the heat equation (8).

First we prove the existence of global (in time) small data Sobolev solutions for $p > p_{\beta,\eta}(n)$, where β is as in (22). The solution to the Cauchy problem depending on a parameter $s \geq 0$, that is, to

$$\begin{cases} b(t)v_t - \Delta v + m(t)^2 v = 0, & t \geq s,\ x \in \mathbb{R}^n, \\ v(s,x) = \varphi(s,x) \end{cases} \tag{64}$$

is given by

$$v(t,x) = \frac{\gamma(t)}{\gamma(s)} (4\pi B(t,s))^{-n/2} e^{-\frac{|x|^2}{4B(t,s)}} *_{(x)} \varphi(s,x).$$

By Young's inequality, $v = v(t,x)$ verifies

$$\|v(t,\cdot)\|_{L^{\eta p}} \lesssim \frac{\gamma(t)}{\gamma(s)} B(t,s)^{-\frac{n}{2\eta}\left(1-\frac{1}{p}\right)} \|\varphi(s,\cdot)\|_{L^\eta},$$

for $p, \eta \geq 1$. By following the proofs of Theorems 2 and 3 with minor modifications, this leads to obtain the existence of global (in time) small data Sobolev solutions to (8) for $p > p_{\beta,\eta}(n)$.

To prove nonexistence of global (in time) solutions to (8), we rely on a modified test function method argument (see Theorem 2.2 in [5]). Let us consider a suitable nonnegative test function $\Phi = \Phi(t,x)$. In particular, we assume that

- $\Phi(t,x)$ is supported on $[0,1] \times \{|x| \leq 1\}$,
- $\Phi(t,x)$ is constant with respect to t for any $t \in [0,1/2]$,
- Φ is the ℓ-th power of a test function, for a sufficiently large integer ℓ.

Multiplying the semilinear heat equation in (8) by $(\gamma(t)b(t))^{-1}\,\Phi(t,x)$, integrating and applying integration by parts, we derive

$$\begin{aligned} I = \int_0^\infty \int_{\mathbb{R}^n} v(t,x)^p\, (\gamma(t)b(t))^{-1}\Phi(t,x)dxdt &= -\int_{\mathbb{R}^n} \varphi(x)\,\Phi(0,x)dx \\ &+ \int_0^\infty \int_{\mathbb{R}^n} v(t,x)\,(\gamma(t)b(t))^{-1}\,(-b(t)\partial_t\Phi(t,x) - \Delta\Phi(t,x))dxdt. \end{aligned}$$

We notice that we used that

$$\gamma'(t) = -\frac{m^2(t)}{b(t)}\,\gamma(t).$$

Now, for the sake of simplicity, we fix $b(t) = \mu(1+t)^k$, with $k \in (-1,1)$, and $m(t) = \nu(1+t)^\ell$, with $\ell = (k-1)/2$, as in Example 1, so that

$$B(t) = \frac{(1+t)^{1-k}-1}{(1-k)\mu}, \qquad \gamma(t) = (1+t)^{-\beta(1-k)}, \quad \beta = \frac{\nu^2}{\mu(1-k)}.$$

We also assume $\beta > -1$, so that

$$\frac{1}{\gamma(t)b(t)} \approx (1+t)^{\beta(1-k)-k} = (1+t)^{(1+\beta)(1-k)-1},$$

is not integrable over $(0,\infty)$. We set $\Phi(t,x) = \Phi_R(t,x)$, where

$$\Phi_R = \Phi_R(t,x) = \Phi_1(R^{-1}t, R^{-\frac{1-k}{2}}x),$$

is the re-scaled function of a fixed test function $\Phi_1 = \Phi_1(t,x)$. Then

$$\partial_t\Phi_R(t,x) = R^{-1}(\partial_t\Phi_1)(t/R, R^{-\frac{1-k}{2}}x), \qquad \Delta\Phi_R(t,x) = R^{-(1-k)}(\Delta\Phi_1)(t/R, R^{-\frac{1-k}{2}}x).$$

By Young's inequality, for small positive δ, we may estimate

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{R}^n} v(t,x)\,(\gamma(t)b(t))^{-1}\,b(t)\,|\partial_t\Phi_R(t,x)|dxdt &\lesssim \delta I + C_\delta R^{-p'+\frac{n}{2}(1-k)} \int_{R/2}^R b(t)^{p'}(\gamma(t)b(t))^{-1}\,dt \\
&\lesssim \delta I + C_\delta R^{(-p'+1+\beta+n/2)(1-k)}, \\
\int_0^\infty \int_{\mathbb{R}^n} v(t,x)\,(\gamma(t)b(t))^{-1}\,|\Delta\Phi_R(t,x)|dxdt &\lesssim \delta I + C_\delta R^{-(1-k)p'+\frac{n}{2}(1-k)} \int_0^R (\gamma(t)b(t))^{-1}\,dt \\
&\lesssim \delta I + C_\delta R^{(-p'+1+\beta+n/2)(1-k)}.
\end{aligned}$$

Since $\varphi \geq 0$, it follows that v vanishes identically for any $p > 1$ such that

$$-p' + 1 + \beta + n/2 < 0,$$

that is, $p < p_{\beta,1}(n)$. On the other hand, let $\bar\eta \in (1,2]$ and $p < p_{\beta,\bar\eta}(n)$. We want to refine the previous argument to show that no global (in time) Sobolev solutions exist for suitable initial data in $L^{\bar\eta}$. Let $\eta \in [1,\bar\eta)$ be such that $p < p_{\beta,\eta}(n)$, as well. Then we may assume that

$$\varphi(x) \geq \epsilon\,(1+|x|)^{-n/\eta},$$

since this assumption is consistent with $\varphi \in L^{\bar\eta}$. But now a contradiction follows from

$$0 \leq -\epsilon\, R^{\frac{n(1-k)}{2}\left(1-\frac{1}{\eta}\right)} + CR^{(-p'+1+\beta+n/2)(1-k)},$$

since

$$-p' + 1 + \beta + \frac{n}{2\eta} < 0,$$

due to $p < p_{\beta,\eta}(n)$.

Summarizing, we sketched the proof that $p_{\beta,\eta}(n)$ is the critical exponent for (8), in the sense that, in general, the following two statements are true:

- global (in time) small data Sobolev solutions exist for small data in L^η for $p > p_{\beta,\eta}(n)$;
- no global (in time) Sobolev solutions exist under a suitable sign assumption on the data in L^η for $p < p_{\beta,\eta}(n)$.

For this reason, we expect that under a suitable sign assumption, global (in time) Sobolev solutions with L^η data may also be excluded for (1), when $p < p_{\beta,\eta}(n)$, verifying the optimality of the exponent found for the global (in time) existence of small data solutions in this paper.

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