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Local behavior of solutions to subelliptic problems with Hardy potential on Carnot groups

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Abstract

We determine the exact behavior at the singularity of solutions to semilinear subelliptic problems of the type $-\Delta_{\mathbb{G}} u - \mu \frac{\psi^2}{d^2} u = f(\xi, u)$ in Ω, $u = 0$ on $\partial\Omega$, where $\Delta_{\mathbb{G}}$ is a sub-Laplacian on a Carnot group $\mathbb{G}$ of homogeneous dimension Q, Ω is an open subset of $\mathbb{G}$, $0 \in \Omega$, d is the gauge norm on $\mathbb{G}$, $\psi := |\nabla_{\mathbb{G}} d|$, where $\nabla_{\mathbb{G}}$ is the horizontal gradient associated to $\Delta_{\mathbb{G}}$, f has at most critical growth and $0 \le \mu < \overline{\mu}$, where $\overline{\mu} = \left(\frac{Q-2}{2}\right)^2$ is the best Hardy constant on $\mathbb{G}$.

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1 Introduction

Let $\mathbb{G}$ be a Carnot group of homogeneous dimension $Q \ge 3$ and let Ω be an arbitrary open subset of $\mathbb{G}$, $0 \in \Omega$. We consider semilinear subelliptic problems of the type

$$\begin{cases} -\Delta_{\mathbb{G}} u - \mu \dfrac{\psi^2}{d^2} u = f(\xi, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\Delta_{\mathbb{G}}$ is a sub-Laplacian operator on $\mathbb{G}$, d is the natural gauge associated with the fundamental solution of $-\Delta_{\mathbb{G}}$ on $\mathbb{G}$, $\psi := |\nabla_{\mathbb{G}} d|$, where $\nabla_{\mathbb{G}}$ is the horizontal gradient associated to $\Delta_{\mathbb{G}}$ and $0 \le \mu < \overline{\mu}$, where $\overline{\mu} = \left(\frac{Q-2}{2}\right)^2$ is the best constant in the Hardy inequality on Carnot groups

$$\int_\Omega |\nabla_{\mathbb{G}} u|^2 \, \mathrm{d}\xi \ge \overline{\mu} \int_\Omega \psi^2 \frac{|u|^2}{d(\xi)^2} \, \mathrm{d}\xi, \quad \forall u \in C_0^\infty(\Omega),$$

and it is never attained (see Garofalo and Lanconelli [18], D'Ambrosio [8]). Here the function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

$$|f(\xi, t)| \le C(|t| + |t|^{2^*-1}), \quad \forall t \in \mathbb{R}, \tag{1.2}$$

where $2^* = \frac{2Q}{Q-2}$ is the critical Sobolev exponent in the stratified Lie context.

We are interested in qualitative properties of weak solutions to pb. (1.1), i.e. solutions in the Folland-Stein space $S_0^1(\Omega)$, defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{S_0^1(\Omega)} := \left(\int_\Omega |\nabla_{\mathbb{G}} u|^2 \, \mathrm{d}\xi\right)^{1/2}$. We recall that $u \in S_0^1(\Omega)$ is called a weak solution of (1.1) if it satisfies

$$\int_\Omega \nabla_{\mathbb{G}} u \cdot \nabla_{\mathbb{G}} \varphi \, \mathrm{d}\xi - \mu \int_\Omega \psi^2 \frac{u\varphi}{d^2} \, \mathrm{d}\xi = \int_\Omega f(\xi, u)\varphi \, \mathrm{d}\xi, \qquad \forall \varphi \in S_0^1(\Omega).$$

In the Euclidean elliptic setting, i.e. when $\mathbb{G}$ is the ordinary Euclidean space $(\mathbb{R}^n, +)$, the semilinear model problem (1.1) with $f(x,u) = |u|^{2^*-2}u + \lambda u$ on bounded domains of $\mathbb{R}^n$ has been widely studied by many authors (see e.g. Terracini [31], Jannelli [22], Ferrero-Gazzola [15], Chen [6], Felli-Terracini [14], but the list cannot be exhaustive). In particular, Jannelli in [22] proved that for $0 < \mu \le \overline{\mu} - 1$, such problem admits a positive solution for all $\lambda \in (0, \lambda_1(\mu))$, $\lambda_1(\mu)$ being the first eigenvalue of the positive operator $-\Delta - \frac{\mu}{|x|^2}$ $(0 \le \mu < \overline{\mu})$ with Dirichlet boundary conditions; if, instead, $\overline{\mu} - 1 < \mu < \overline{\mu}$, there exists $\lambda^* \in (0, \lambda_1(\mu))$ such that pb (1.1) admits solutions if and only if $\lambda \in (\lambda^*, \lambda_1(\mu))$.

The qualitative properties of solutions to semilinear problems with inverse square potential on $\mathbb{R}^n$ have been studied in a series of papers (see e.g. [30], [13], [5], [21], [14]). In particular, Han [21] obtained the exact asymptotic behavior of solutions at the singularity for problem (1.1) under the growth assumption (1.2) on f. Such behavior reflects the one of positive entire solutions to the critical problem $-\Delta u - \mu\frac{u}{|x|^2} = u^{2^*-1}$ in the whole $\mathbb{R}^n$, whose solutions are explicitly known and take the form

$$U_\varepsilon(x) = \frac{C_\varepsilon}{(\varepsilon|x|^{\frac{\gamma'}{\sqrt{\overline{\mu}}}} + |x|^{\frac{\gamma}{\sqrt{\overline{\mu}}}})^{\sqrt{\overline{\mu}}}}, \quad \varepsilon > 0 \tag{1.3}$$

where $\gamma' = \sqrt{\overline{\mu}} - \sqrt{\overline{\mu} - \mu}$ and $\gamma = \sqrt{\overline{\mu}} + \sqrt{\overline{\mu} - \mu}$ (see [22], [31]).

For what concerns the abstract framework of Carnot groups, few results are known about nonlinear problems of the type (1.1). We recall that the study of the perturbed operator $\mathcal{L} = -\Delta_{\mathbb{G}} - \mu\dfrac{\psi^2}{d^2}$ in the case of the Heisenberg group $\mathbb{G} = \mathbb{H}^n$ was introduced by Garofalo and Lanconelli in the seminal paper [18], where the Hardy type inequality on $\mathbb{H}^n$ was established and unique continuation results were obtained. Concerning problems of type (1.1) with a power-type nonlinearity, a first existence result on $\mathbb{H}^n$ in the case of a *subcritical* nonlinearity is due to Mokrani [29]. Recently, Liouville theorems for general quasilinear operators with Hardy perturbation have been proved in [9] in the setting of Carnot groups.

In this paper, as a first step in the study of problem (1.1) under the general growth assumption (1.2), we establish the exact behavior of solutions at the singularity. The proof is based on the L^p-regularity of solutions and the application of Moser's iteration technique in the subelliptic framework, and it extends to the Carnot setting Euclidean results in [21], [5], [13]. Denoted by $B(\xi, R)$ the ball with center at ξ and radius R with respect to the gauge d, our main result is the following:

Theorem 1.1. *Let $u \in S^1_0(\Omega)$ be a solution to problem* (1.1) *under the assumption* (1.2). *Then, there exists a constant $C > 0$ such that*

$$|u(\xi)| \leq C d(\xi)^{-\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu}}, \quad \forall \xi \in B(0,r) \setminus \{0\}, \tag{1.4}$$

for $r > 0$ sufficiently small.

If, moreover, u is positive and $f(\xi,t) \geq 0$ for $t > 0$, there exists a constant $C' > 0$ such that

$$u(\xi) \geq C' d(\xi)^{-\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu}}, \quad \forall \xi \in B(0,r) \setminus \{0\}, \tag{1.5}$$

for $r > 0$ sufficiently small.

So, from the above theorem, we get that any positive solution of the semilinear problem (1.1) has stronger and stronger singularity as $\mu \to \overline{\mu}$, where the singularity is completely determined by the operator $-\Delta_{\mathbb{G}} - \mu \dfrac{\psi^2}{d^2}$ and does not depend on the explicit form of f. We recall that in the case $\mu = 0$, instead, weak solutions of (1.1) are proved to be bounded (see e.g. [24], [25], [2]).

Let us make some comments on some particular cases of problem (1.1). First of all, let us consider the linear case, i.e. the case when $f(x,u) = \lambda u$, with $\lambda \in \mathbb{R}$. In this case, the asymptotic estimates given in Theorem 1.1 apply, in particular, to the eigenfunctions of the operator $\mathcal{L} = -\Delta_{\mathbb{G}} - \mu \dfrac{\psi^2}{d^2}$, $0 \leq \mu < \overline{\mu}$, with Dirichlet boundary conditions on bounded domains $\Omega \subset \mathbb{G}$. We recall that the spectrum of $\mathcal{L}$ is discrete and positive and denoted by λ_i, $i = 1, 2 \ldots$, the eigenvalues of $\mathcal{L}$, it holds that $\lambda_i \to \infty$, as $i \to \infty$ (see e.g. [29] for a proof in the Heisenberg case $\mathbb{H}^n$). Let e_i, $i = 1, 2, \ldots$, denote the L^2-normalized eigenfunctions of the operator $\mathcal{L}$ on $S^1_0(\Omega)$ corresponding to the eigenvalues λ_i. Then, e_i satisfies

$$\begin{cases} -\Delta_{\mathbb{G}} e_i - \mu \dfrac{\psi^2}{d^2} e_i = \lambda_i e_i & \text{in } \Omega, \\ e_i = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence, by Theorem 1.1, we get that

$$|e_i(\xi)| \leq C d(\xi)^{-\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu}}, \quad \forall \xi \in \Omega \setminus \{0\},$$

generalizing the Euclidean result in [4].

Another interesting limit case of problem (1.1) is the case when f is a purely critical nonlinearity, i.e.

$$-\Delta_{\mathbb{G}} u - \mu \frac{\psi^2}{d^2} u = |u|^{2^*-2} u \quad \text{in } \Omega \subset \mathbb{G}. \tag{1.6}$$

Let us consider the above equation when Ω is an unbounded domain.

In the Euclidean case, due to the conformal invariance of the critical equation (1.6), the knowledge of the exact behavior of solutions at 0 immediately gives the exact decay of solutions at ∞. This result is achieved by means of the classical Kelvin transform on $\mathbb{R}^n$, as one can see e.g. in [12], [14].

In the Carnot context, an analogous technique is not available for an arbitrary Carnot group, since a suitable inversion with good conformal properties is defined only in a special subclass of Stratified groups, the so-called Iwasawa-type groups (see Garofalo-Vassilev [20, Section 8]; see also the monograph [1], Chapter 18). At least for these groups, which include the well-known model case of the Heisenberg group $\mathbb{G} = \mathbb{H}^n$, we are able to infer from Theorem 1.1 the exact behavior of solutions at infinity for critical growth equations. The result is the following.

Theorem 1.2. *Let $\mathbb{H}$ be a Iwasawa-type group and let $\Omega \subset \mathbb{H}$ be a neighborhood of ∞. If $u \in S_0^1(\Omega)$ is a solution to*

$$-\Delta_{\mathbb{H}} u - \mu \frac{\psi^2}{d^2} u = |u|^{2^*-2} u \text{ in } \Omega, \tag{1.7}$$

there exists $C > 0$ such that

$$|u(\xi)| \leq C d(\xi)^{-\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}}, \quad \text{for } d(\xi) \text{ large.} \tag{1.8}$$

If, moreover, u is positive, there exists $C' > 0$ such that

$$u(\xi) \geq C' d(\xi)^{-\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}}, \quad \text{for } d(\xi) \text{ large.} \tag{1.9}$$

Note that the above asymptotic decay reflects the one of the extremal functions (1.3) in $\mathbb{R}^n$. The last part of the paper will be devoted to the proof of the above theorem. In an arbitrary Carnot group, instead, the study of analogous decay properties shall require the implementation of different techniques which will constitute one of the future developments of this research. We conclude by pointing out that the knowledge of the behavior at infinity of entire solutions to critical equations has a fundamental rôle in the study of the associated Brezis-Nirenberg type problems, as shown in [26], [23], [28].

The plan of the paper is the following: in Section 2 we briefly introduce the main definitions and notations about Carnot groups; in Section 3 we study the L^p-regularity of solutions to problem (1.1) under the general growth assumption (1.2) on f and prove the main Theorem 1.1 on the behavior of solutions at the singularity; in Section 4 we indicate some generalizations to subelliptic problems with multi-singular Hardy potentials. Finally, in Section 5, we focus on the critical case, i.e. when $f(\xi, u) = |u|^{2^*-1}u$ and Ω is an unbounded domain, and we derive the exact decay of solutions at ∞ from the asymptotic behavior at 0 in the case when $\mathbb{G}$ is an H-type group of Iwasawa type.

2 The functional setting

A *Carnot group* (or *Stratified group*) $(\mathbb{G}, \circ)$ is a connected, simply connected nilpotent Lie group, whose Lie algebra $\mathfrak{g}$ admits a stratification, i.e. $\mathfrak{g} = \bigoplus_{j=1}^r \mathfrak{G}_j$, such that $[\mathfrak{G}_1, \mathfrak{G}_j] = \mathfrak{G}_{j+1}$ for $1 \leq j < r$, and $[\mathfrak{G}_1, \mathfrak{G}_r] = \{0\}$. The number r is called the *step* of the group $\mathbb{G}$. The integer $Q = \sum_{i=1}^r i \dim(\mathfrak{G}_i)$ is called the *homogeneous dimension* of $\mathbb{G}$. We shall assume throughout that $Q \geq 3$.

By means of the natural identification of $\mathbb{G}$ with its Lie algebra via the exponential map (which we shall assume throughout), it is not restrictive to suppose that $\mathbb{G}$ be a homogeneous Lie group on $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \ldots \times \mathbb{R}^{N_r}$, with $N_i = \dim(\mathfrak{G}_i)$, equipped with a family of group-automorphisms (called *dilations*) δ_λ of the form

$$\delta_\lambda(\xi) = (\lambda\, \xi^{(1)}, \lambda^2\, \xi^{(2)}, \cdots, \lambda^r\, \xi^{(r)}),$$

where $\xi^{(j)} \in \mathbb{R}^{N_j}$ for $j = 1, \ldots, r$. Let $m := N_1$ and let $X_1, \ldots, X_m$ be the set of left invariant vector fields of $\mathfrak{G}_1$ that coincide at the origin with the first m partial derivatives. The second order differential operator

$$\Delta_{\mathbb{G}} = \sum_{i=1}^{m} X_i^2$$

is called the canonical *sub-Laplacian* on $\mathbb{G}$. We shall denote by

$$\nabla_{\mathbb{G}} = (X_1, \ldots, X_m)$$

the related subelliptic gradient. Moreover, for any C^1 vector field $h = (h_1, h_2, \ldots, h_m)$, we shall indicate by

$$\mathrm{div}_{\mathbb{G}} h = \sum_{i=1}^{m} X_i h_i,$$

the divergence with respect to the vector fields X_j's. Note that $\Delta_{\mathbb{G}}$ is left-translation invariant w.r.t. the group action and δ_λ-homogeneous of degree two. In other words, $\Delta_{\mathbb{G}}(u \circ \tau_\xi) = \Delta_{\mathbb{G}} u \circ \tau_\xi$, $\Delta_{\mathbb{G}}(u \circ \delta_\lambda) = \lambda^2 \Delta_{\mathbb{G}} u \circ \delta_\lambda$. Moreover, due to the stratification condition, the Lie algebra generated by $X_1, \ldots, X_m$ is the whole $\mathfrak{g}$, and therefore it is everywhere of rank N; therefore, the sub-Laplacian operator $\Delta_{\mathbb{G}}$ satisfies the well-known Hörmander's hypoellipticity condition.

In the last part of the paper, a special subclass of Carnot groups will be considered, namely the H-type groups of Iwasawa-type, which include the well-known Heisenberg group $\mathbb{H}^n$. For the definition and properties about these groups, we refer to [1, Chapter 18] and the references therein.

When $Q \geq 3$, Carnot groups possess the following property: there exists a suitable homogeneous norm d on $\mathbb{G}$ such that

$$\Gamma(\xi) = \frac{C}{d(\xi)^{Q-2}} \tag{2.1}$$

is a fundamental solution of $-\Delta_{\mathbb{G}}$ with pole at 0, for a suitable constant $C > 0$ (see [16]). By definition, a *homogeneous norm* on $\mathbb{G}$ is a continuous function $d : \mathbb{G} \to [0, +\infty)$, smooth away from the origin, such that $d(\delta_\lambda(\xi)) = \lambda\, d(\xi)$, for every $\lambda > 0$ and $\xi \in \mathbb{G}$, $d(\xi^{-1}) = d(\xi)$ and $d(\xi) = 0$ iff $\xi = 0$. Moreover, if we define $d(\xi, \eta) := d(\eta^{-1} \circ \xi)$, then d is a pseudo-distance on $\mathbb{G}$. In particular, d satisfies the pseudo-triangular inequality

$$d(\xi, \eta) \leq \beta(d(\xi, \zeta) + d(\zeta, \eta)), \quad \xi, \eta, \zeta \in \mathbb{G},$$

for a suitable constant $\beta > 0$. Throughout the paper, we shall denote by d the homogeneous norm associated to the fundamental solution of the sub-Laplacian by (2.1). We shall indicate by $B(\xi, r)$ the d-ball with center at ξ and radius r.

If $\mathbb{G}$ is a Carnot group of dimension $Q \geq 3$, the following Sobolev-type inequality due to Folland and Stein [16] holds on $\mathbb{G}$: there exists a positive constant $S = S(\mathbb{G})$ such that

$$\int_{\mathbb{G}} |\nabla_{\mathbb{G}} u|^2 \, \mathrm{d}\xi \geq S \left(\int_{\mathbb{G}} |u|^{2^*} \, \mathrm{d}\xi \right)^{2/2^*} \qquad \forall \, u \in C_0^\infty(\mathbb{G}), \tag{2.2}$$

where $2^* = 2Q/(Q-2)$ is the critical exponent in this context. Moreover, the following Hardy-type inequality holds

$$\int_{\mathbb{G}} |\nabla_{\mathbb{G}} u|^2 \, \mathrm{d}\xi \geq \left(\frac{Q-2}{2} \right)^2 \int_{\mathbb{G}} \psi^2 \frac{|u|^2}{d^2} \, \mathrm{d}\xi \quad \forall u \in C_0^\infty(\mathbb{G}), \tag{2.3}$$

where d is the gauge associated with the fundamental solution of $\Delta_{\mathbb{G}}$ on $\mathbb{G}$ and $\psi = |\nabla_{\mathbb{G}} d|$. The preceding inequality was firstly proved by Garofalo and Lanconelli in [18] for the Heisenberg group. Then, it has been extended to all Carnot groups in [8]. We recall that the constant in the r.h.s. of the formula (2.3) is sharp and it is never attained (see [8]).

We explicitly note that the weight function ψ appearing in the r.h.s. of (2.3) is constant if and only if $\mathbb{G}$ is the Euclidean group (see [1, Prop. 9.8.9]). Moreover, ψ is δ_λ-homogeneous of degree 0, hence ψ is bounded.

For a complete treatment of Carnot groups, we refer to the book [1] and to the classical papers [16], [17].

In what follows, we shall use the following spaces. For an open set $\Omega \subset \mathbb{G}$, we shall denote by $S_0^1(\Omega)$ the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{S_0^1(\Omega)} := \left(\int_\Omega |\nabla_{\mathbb{G}} u|^2 \, \mathrm{d}\xi \right)^{1/2}$. Moreover, we shall indicate by $\Gamma^2(\Omega)$ the Folland-Stein space of all continuous functions $u \in C(\Omega)$ such that $X_j u$, $X_i X_j u \in C(\Omega)$, for $i, j = 1, \ldots, m$.

3 Proof of the main results

We begin by introducing the following general Lemma, which provides the L^p-regularity of solutions at the origin, by means of a Brezis-Kato type argument [3]. We recall that analogous regularity theorems in the case $\mu = 0$ were obtained in [19], [25], [2]. In the present singular case, we shall adapt the Euclidean proof due to Smets [30].

Lemma 3.1. *Let Ω be a bounded neighborhood of 0 in $\mathbb{G}$, $0 < \mu < \overline{\mu}$. Assume that $V \in L^{Q/2}(\Omega)$ and $g \in L^q(\Omega)$, $q \geq 2$. If $u \in S_0^1(\Omega)$ is a weak solution of*

$$-\Delta_{\mathbb{G}} u - \mu \frac{\psi^2}{d^2} u - V u + \nu u = g \quad \textit{in } \Omega, \tag{3.1}$$

where ν is such that the linear operator on the l.h.s. is positive, then

$$u \in \bigcap_{p < p_{lim}} L^p(\Omega), \quad \textit{where } p_{lim} = 2^* \min \left\{ \frac{q}{2}, \frac{\sqrt{\overline{\mu}}}{\sqrt{\overline{\mu}} - \sqrt{\overline{\mu} - \mu}} \right\}.$$

Proof. First note that, under the assumptions $\mu < \overline{\mu}$ and $V \in L^{Q/2}$, for sufficiently large ν the operator $-\Delta_{\mathbb{G}} - (\mu\psi^2/d^2 + V)I + \nu I$ is positive. Let us assume that g is positive, otherwise we can decompose $g = g^+ - g^-$.

Let $W_k := \min\{k, \mu\psi^2/d^2 + V\}$ and let u_k be the unique weak solution to

$$\begin{cases} -\Delta_{\mathbb{G}} u_k - W_k u_k + \nu u_k &= g \quad \text{in } \Omega, \\ \qquad\qquad\qquad u_k &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Note that u_k is positive. Assume that $u_k \in L^p(\Omega)$ for some $p \le q$. Let $u_k^n := \min\{n, u_k\}$. By testing problem (3.2) with $(u_k^n)^{p-1}$ we get

$$\begin{aligned} (p-1)\int_\Omega (u_k^n)^{p-2} |\nabla_{\mathbb{G}} u_k^n|^2 \,\mathrm{d}\xi \le& \int_\Omega W_k^+ (u_k^n)^p \,\mathrm{d}\xi \\ &+ \int_{u_k>n} k n^{p-1} u_k \,\mathrm{d}\xi + \int_\Omega g (u_k^n)^{p-1} \,\mathrm{d}\xi. \end{aligned} \tag{3.3}$$

By Hardy's inequality (2.3) and using that, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\int_\Omega |V(\xi)| v^2 \,\mathrm{d}\xi \le \varepsilon \|\nabla_{\mathbb{G}} v\|_2^2 + C_\varepsilon \|v\|_2^2, \tag{3.4}$$

from (3.3) we get

$$\begin{aligned} \frac{4(p-1)}{p^2} \|\nabla_{\mathbb{G}} \left((u_k^n)^{p/2}\right)\|_2^2 \le& C\|g\|_q \|u_k^n\|_p^{p-1} + \left(\frac{\mu}{\overline{\mu}} + \varepsilon\right) \|\nabla_{\mathbb{G}} \left((u_k^n)^{p/2}\right)\|_2^2 \\ &+ C_\varepsilon \|u_k^n\|_p^p + k\int_{u_k>n} (u_k)^p \,\mathrm{d}\xi. \end{aligned} \tag{3.5}$$

Now, a direct calculation shows that if $p < 2p_{lim}/2^*$, then

$$\frac{\mu}{\overline{\mu}} < \frac{4(p-1)}{p^2}.$$

Hence, choosing ε sufficiently small in (3.5) and using Sobolev inequality (2.2), we get that

$$\|u_k^n\|_{2^*p/2}^p \le C_p \left(\|g\|_q \|u_k^n\|_p^{p-1} + \|u_k^n\|_p^p + k\int_{u_k>n} (u_k)^p \mathrm{d}\xi \right).$$

Since by assumption $u_k \in L^p$, taking the limit as $n \to \infty$ we obtain

$$\|u_k\|_{2^*p/2}^p \le C_p \left(\|g\|_q \|u_k\|_p^{p-1} + \|u_k\|_p^p \right).$$

Therefore, $u_k \in L^{2^*p/2}$, and being the above estimate uniform with respect to k, we get that $u \in L^{2^*p/2}$. Now, starting from $p = 2$ and arguing recursively, we can improve the integrability order of u, as long as $p < 2p_{lim}/2^*$. □

By Lemma 3.1, the following regularity result follows for the solutions of (1.1).

Proposition 3.2. *Let Ω be an arbitrary open subset of $\mathbb{G}$, $0 \in \Omega$. Let $u \in S^1_0(\Omega)$ be a solution of problem* (1.1) *under the assumption* (1.2). *Then*

$$u \in L^p_{loc}(\Omega), \quad \forall\, p < p_{lim} = 2^* \frac{\sqrt{\bar{\mu}}}{\sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu}}. \tag{3.6}$$

Proof. By the subelliptic regularity theory (see Folland [16]), we can infer that $u \in \Gamma^2_{loc}(\Omega \setminus \{0\})$, henceforth it is locally bounded out of the origin. To study the summability at the origin, let $R > 0$ be such that $B_R(0) \subset\subset \Omega$ and let $\eta \in C^\infty_0(B_R(0))$ be a cut-off function such that $\eta \equiv 1$ on $B_{R/2}(0)$, $0 \le \eta \le 1$. Then, the function $w := \eta u$ satisfies the equation

$$-\Delta_{\mathbb{G}} w - \mu \frac{\psi^2}{d^2} w - V w + \nu w = g, \qquad w \in S^1_0(B_R(0)),$$

where $V := f(\xi, u)/u \in L^{Q/2}(B_R)$ and $g := -2\nabla_{\mathbb{G}} u \cdot \nabla_{\mathbb{G}} \eta - u \Delta_{\mathbb{G}} \eta + \nu w$. Since $\nabla_{\mathbb{G}} \eta \equiv 0$ on $B_{R/2}(0)$ and $u \in \Gamma^2_{loc}(B_R \setminus \{0\})$, it follows that the L^p regularity of g is given by that of w. Therefore, starting from $g \in L^{2^*}$ and arguing recursively, by using Lemma 3.1 we can improve the regularity of w, until we reach the threshold imposed by μ. □

Proof of Theorem 1.1 *Estimate from above.* We shall adapt the proof in [21]. Let u be a solution of (1.1) and define

$$v := d^{\sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu}} u\,. \tag{3.7}$$

First observe that $v \in S^1_0(\Omega, d^{-2(\sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu})}\, \mathrm{d}\xi)$. Indeed, by Hardy's inequality (2.3)

$$\begin{aligned}
& \int_\Omega d^{-2\left(\sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu}\right)} |\nabla_{\mathbb{G}} v|^2 \, \mathrm{d}\xi \\
= \; & \int_\Omega d^{-2\left(\sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu}\right)} \left| d^{\sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu}} \nabla_{\mathbb{G}} u + \left(\sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu}\right) d^{\sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu}} \frac{u \nabla_{\mathbb{G}} d}{d} \right|^2 \mathrm{d}\xi \\
\le \; & 2 \int_\Omega \left(|\nabla_{\mathbb{G}} u|^2 + \left(\sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu}\right)^2 \frac{|u|^2 \psi^2}{d^2} \right) \mathrm{d}\xi \\
\le \; & \frac{2\left(\bar{\mu} + (\sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu})^2\right)}{\bar{\mu}} \int_\Omega |\nabla_{\mathbb{G}} u|^2 \, \mathrm{d}\xi \\
\le \; & C.
\end{aligned}$$

Now, by using that the sub-Laplacian acts on smooth radial functions $h = h(d)$ as follows (see [1, Prop. 5.4.3])

$$\Delta_{\mathbb{G}} h = \psi^2 \left(h'' + \frac{Q-1}{d} h' \right), \tag{3.8}$$

we can see by direct calculation that, for any $\xi \in \Omega \setminus \{0\}$, v satisfies

$$-\mathrm{div}_{\mathbb{G}} \left(d^{-2(\sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu})} \nabla_{\mathbb{G}} v \right) = d^{-(\sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu})} f(\xi, d^{-(\sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu})} v). \tag{3.9}$$

Indeed, if we let $\alpha := \sqrt{\bar\mu} - \sqrt{\bar\mu - \mu}$ to simplify notation, then $v = d^\alpha u$ satisfies

$$-(\Delta_{\mathbb{G}} d^{-\alpha})v - d^{-\alpha}\Delta_{\mathbb{G}} v - 2\nabla_{\mathbb{G}} d^{-\alpha}\cdot\nabla_{\mathbb{G}} v - \mu\frac{\psi^2}{d^2} v d^{-\alpha} = f(\xi, d^{-\alpha}v) \tag{3.10}$$

that is, taking into account (3.8)

$$-d^{-\alpha}\Delta_{\mathbb{G}} v - 2\nabla_{\mathbb{G}} d^{-\alpha}\cdot\nabla_{\mathbb{G}} v - \psi^2\left(\alpha^2 - \alpha(Q-2) + \mu\right) d^{-\alpha-2} = f(\xi, d^{-\alpha}v)$$

which reduces to

$$-d^{-\alpha}\Delta_{\mathbb{G}} v - 2\nabla_{\mathbb{G}} d^{-\alpha}\cdot\nabla_{\mathbb{G}} v = f(\xi, d^{-\alpha}v) \tag{3.11}$$

since $\alpha^2 - \alpha(Q-2) + \mu = 0$ iff $\alpha = \sqrt{\bar\mu} \pm \sqrt{\bar\mu - \mu}$.

Then, multiplying equation (3.11) by $d^{-\alpha}$, we finally get the following equation for v

$$-\mathrm{div}_{\mathbb{G}}(d^{-2\alpha}\nabla_{\mathbb{G}} v) = d^{-\alpha} f(\xi, d^{-\alpha}v), \quad \xi\in\Omega\setminus\{0\},$$

that is (3.9).

By the subelliptic regularity theory (see Folland [16]), it follows that $v \in \Gamma^2_{loc}(\Omega\setminus\{0\})$. Let $\rho > 0$ be small enough such that $B_\rho(0) \subset\subset \Omega$. Set

$$\varphi = \eta^2 v v_t^{2(s-1)} \in S_0^1\left(\Omega, d^{-2(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\,\mathrm{d}\xi\right),$$
$$\text{where } s, t > 1, \quad v_t = \min\{|v|, t\}, \quad \eta \in C_0^\infty\left(B_\rho(0)\right),$$

where $0 \le \eta \le 1$, $\eta \equiv 1$ in $B_r(0)$, where $0 < r < \rho$ and $|\nabla_{\mathbb{G}}\eta| \le \frac{4}{\rho - r}$. By using φ as a test function in (3.9) we get

$$\int_\Omega d^{-2\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)}\nabla_{\mathbb{G}} v\cdot\nabla_{\mathbb{G}}\varphi\,\mathrm{d}\xi = \int_\Omega d^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}} f\left(\xi, d^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}}v\right)\varphi\,\mathrm{d}\xi,$$

that can be rewritten as

$$\begin{aligned}
&\int_\Omega d^{-2\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)} \\
&\times\left(2\eta v v_t^{2(s-1)}\nabla_{\mathbb{G}}\eta\cdot\nabla_{\mathbb{G}} v + \eta^2 v_t^{2(s-1)}\left|\nabla_{\mathbb{G}} v\right|^2 + 2\,(s-1)\,\eta^2 v_t^{2(s-1)}\left|\nabla_{\mathbb{G}} v_t\right|^2\right)\mathrm{d}\xi \\
= &\int_\Omega d^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}} f\left(\xi, d^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}}v\right)\eta^2 v v_t^{2(s-1)}\,\mathrm{d}\xi.
\end{aligned} \tag{3.12}$$

Observe that for any $\varepsilon > 0$ small

$$\begin{aligned}
&\left|2\int_\Omega d^{-2\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)}\eta v v_t^{2(s-1)}\nabla_{\mathbb{G}}\eta\cdot\nabla_{\mathbb{G}} v\,\mathrm{d}\xi\right| \\
\le\ &\varepsilon\int_\Omega d^{-2\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)}\eta^2 v_t^{2(s-1)}\left|\nabla_{\mathbb{G}} v\right|^2\,\mathrm{d}\xi \\
&+C(\varepsilon)\int_\Omega d^{-2\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)}\left|v\right|^2 v_t^{2(s-1)}\left|\nabla_{\mathbb{G}}\eta\right|^2\,\mathrm{d}\xi,
\end{aligned} \tag{3.13}$$

Moreover, by assumption (1.2), the r.h.s. of (3.12) can be estimated as follows

$$
\begin{aligned}
&\left|\int_\Omega d^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}} f\left(\xi, d^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}} v\right)\eta^2 v v_t^{2(s-1)}\,\mathrm{d}\xi\right| \\
\leq\ & C\int_\Omega d^{-2\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)}\eta^2 |v|^2 v_t^{2(s-1)}\,\mathrm{d}\xi \\
&+C\int_\Omega d^{-2^*\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)}\eta^2 |v|^{2^*} v_t^{2(s-1)}\,\mathrm{d}\xi.
\end{aligned}
\tag{3.14}
$$

Taking $\varepsilon=\frac{1}{2}$ and inserting (3.13) and (3.14) into (3.12), we obtain

$$
\begin{aligned}
&\int_\Omega d^{-2\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)}\left(\eta^2 v_t^{2(s-1)}|\nabla_{\mathbb{G}} v|^2+2(s-1)\eta^2 v_t^{2(s-1)}|\nabla_{\mathbb{G}} v_t|^2\right)\mathrm{d}\xi \\
\leq\ & C\int_\Omega d^{-2\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)}\left(\eta^2+|\nabla_{\mathbb{G}}\eta|^2\right)|v|^2 v_t^{2(s-1)}\,\mathrm{d}\xi \\
&+C\int_\Omega d^{-2^*\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)}\eta^2|v|^{2^*} v_t^{2(s-1)}\,\mathrm{d}\xi.
\end{aligned}
\tag{3.15}
$$

Now we apply the following weighted Sobolev inequality on $\mathbb{G}$

$$
\left(\int_\Omega d^{-2^* a}|w|^{2^*}\,\mathrm{d}\xi\right)^{\frac{2}{2^*}}\leq C\int_\Omega d^{-2a}|\nabla_{\mathbb{G}} w|^2\,\mathrm{d}\xi,\quad \forall w\in S_0^1\left(\Omega, d^{-2a}\,\mathrm{d}\xi\right),
\tag{3.16}
$$

where $-\infty<a<\frac{Q-2}{2}$ and C is a positive constant depending on a. Note that the general quasilinear version of (3.16) has been proved in [11] for the subclass of H-type groups, but the same proof works for general Carnot groups in the semilinear case under consideration (see Theorem 6.1 in the Appendix). In the sequel, we shall take $a=\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}<\frac{Q-2}{2}$. Choosing $w=\eta v v_t^{s-1}$ in (3.16) and using (3.15), we get

$$
\begin{aligned}
&\left(\int_\Omega d^{-2^*\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)}|\eta v v_t^{s-1}|^{2^*}\,\mathrm{d}\xi\right)^{\frac{2}{2^*}} \\
&\leq C\int_\Omega d^{-2\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)}\left|\nabla_{\mathbb{G}}\left(\eta v v_t^{s-1}\right)\right|^2\mathrm{d}\xi \\
&\leq C\int_\Omega d^{-2\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)} \\
&\times\left(|\nabla_{\mathbb{G}}\eta|^2|v|^2 v_t^{2(s-1)}+\eta^2 v_t^{2(s-1)}|\nabla_{\mathbb{G}} v|^2+(s-1)^2\eta^2 v_t^{2(s-1)}|\nabla_{\mathbb{G}} v_t|^2\right)\mathrm{d}\xi \\
&\leq Cs\int_\Omega d^{-2\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)}\left(\eta^2+|\nabla_{\mathbb{G}}\eta|^2\right)|v|^2 v_t^{2(s-1)}\,\mathrm{d}\xi \\
&+Cs\int_\Omega d^{-2^*\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)}\eta^2|v|^{2^*} v_t^{2(s-1)}\,\mathrm{d}\xi.
\end{aligned}
\tag{3.17}
$$

Let us fix $\frac{Q}{2}<q<\frac{Q(Q-2)}{2(Q-2-2\sqrt{\bar\mu-\mu})}$. Then

$$
(2^*-2)q<\frac{2Q}{Q-2-2\left(\sqrt{\bar\mu-\mu}\right)}\quad\text{and}\quad 2<\frac{2q}{q-1}<2^*.
$$

By Proposition 3.2, $u \in L^{(2^*-2)q}$ locally in Ω. Therefore, the last integral in (3.17) can be estimated as follows, for any $\varepsilon > 0$

$$\begin{aligned}
&\int_\Omega d^{-2^*\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)}\eta^2 \left|v\right|^{2^*} v_t^{2(s-1)}\,\mathrm{d}\xi \\
&= \int_\Omega d^{-2\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)} \left|u\right|^{2^*-2} \left|\eta v v_t^{s-1}\right|^2 \mathrm{d}\xi \qquad (3.18)\\
&\le \|u\|^{2^*-2}_{L^{(2^*-2)q}(supp\,\eta)} \|d^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}}\eta v v_t^{s-1}\|^2_{L^{\frac{2q}{q-1}}(\Omega)} \\
&\le \|u\|^{2^*-2}_{L^{(2^*-2)q}(supp\,\eta)} \left(\varepsilon\|d^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}}\eta v v_t^{s-1}\|_{L^{2^*}} + C_q\varepsilon^{-\frac{Q}{2q-Q}}\|d^{-\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}}\eta v v_t^{s-1}\|_{L^2}\right)^2 \\
&\le C\varepsilon^2\left(\int_\Omega d^{-2^*\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)} \left|\eta v v_t^{s-1}\right|^{2^*} \mathrm{d}\xi\right)^{\frac{2}{2^*}} \\
&+ C\varepsilon^{-\frac{2Q}{2q-Q}}\int_\Omega d^{-2\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)}|\eta v v_t^{s-1}|^2\,\mathrm{d}\xi.
\end{aligned}$$

Inserting (3.18) into (3.17), we obtain

$$\begin{aligned}
&\left(\int_\Omega d^{-2^*\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)} \left|\eta v v_t^{s-1}\right|^{2^*} \mathrm{d}\xi\right)^{\frac{2}{2^*}} \\
&\le Cs\varepsilon^2\left(\int_\Omega d^{-2^*\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)} \left|\eta v v_t^{s-1}\right|^{2^*} \mathrm{d}\xi\right)^{\frac{2}{2^*}} \\
&+ Cs\int_\Omega d^{-2\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)} \left(\eta^2 + |\nabla_{\mathbb{G}}\eta|^2\right) |v|^2 v_t^{2(s-1)}\,\mathrm{d}\xi \qquad (3.19)\\
&+ Cs\varepsilon^{-\frac{2Q}{2q-Q}}\int_\Omega d^{-2\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)} \left|\eta v v_t^{s-1}\right|^2 \mathrm{d}\xi.
\end{aligned}$$

Taking $\varepsilon = \frac{1}{\sqrt{2Cs}}$ in (3.19), we finally get

$$\begin{aligned}
&\left(\int_\Omega d^{-2^*\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)} \left|\eta v v_t^{s-1}\right|^{2^*} \mathrm{d}\xi\right)^{\frac{2}{2^*}} \\
&\le Cs^\alpha\int_\Omega d^{-2\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)} \left(\eta^2 + |\nabla_{\mathbb{G}}\eta|^2\right) |v|^2 v_t^{2(s-1)}\,\mathrm{d}\xi, \qquad (3.20)
\end{aligned}$$

where $\alpha = \frac{2q}{2q-Q} > 0$. Note that

$$\int_\Omega d^{-2^*\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)}\eta^{2^*} |v|^2 v_t^{2^*s-2}\,\mathrm{d}\xi \le \int_\Omega d^{-2^*\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)} \left|\eta v v_t^{s-1}\right|^{2^*} \mathrm{d}\xi. \qquad (3.21)$$

Hence from (3.20) and (3.21) we have

$$\begin{aligned}
&\left(\int_\Omega d^{-2^*\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)}\eta^{2^*} |v|^2 v_t^{2^*s-2}\,\mathrm{d}\xi\right)^{\frac{2}{2^*}} \\
&\le \; Cs^\alpha\int_\Omega d^{-2\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)} \left(\eta^2 + |\nabla_{\mathbb{G}}\eta|^2\right) |v|^2 v_t^{2(s-1)}\,\mathrm{d}\xi \\
&\le \; Cs^\alpha\int_\Omega d^{-2^*\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)} \left(\eta^2 + |\nabla_{\mathbb{G}}\eta|^2\right) |v|^2 v_t^{2(s-1)}\,\mathrm{d}\xi.
\end{aligned}$$

By the properties of the cut-off function η, we then obtain

$$\left(\int_{B_r(0)} d^{-2^*\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)}\,|v|^2\,v_t^{2^*s-2}\,\mathrm{d}\xi\right)^{\frac{2}{2^*}} \le \frac{Cs^\alpha}{(\rho-r)^2}\int_{B_\rho(0)} d^{-2^*\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)}\,|v|^2\,v_t^{2s-2}\,\mathrm{d}\xi. \tag{3.22}$$

Choosing s^* such that

$$\frac{Q}{Q-2}<s^*<\frac{Q}{Q-2-2\sqrt{\bar\mu-\mu}}\,,$$

we define the sequence

$$s_j=s^*\left(\frac{2^*}{2}\right)^j,\quad j=0,1,2,\dots.$$

Let ρ_0 be sufficiently small that $B_{2\rho_0}(0)\subset\subset\Omega$ and let $r_j=\rho_0(1+\rho_0^j)$, $j=0,1,2,\dots$. Taking $\rho=r_j$, $r=r_{j+1}$ and $s=s_j$ in (3.22), by Moser's iteration we get

$$\begin{aligned}
&\left(\int_{B_{r_{j+1}}(0)} d^{-2^*\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)}|v|^2 v_t^{2s_{j+1}-2}\,\mathrm{d}\xi\right)^{\frac{1}{2s_{j+1}}}\\
&\le\left(\frac{Cs_j^\alpha}{(\rho_0-\rho_0^2)\rho_0^j}\right)^{1/2s_j}\left(\int_{B_{r_j}(0)} d^{-2^*\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)}|v|^2 v_t^{2s_j-2}\,\mathrm{d}\xi\right)^{\frac{1}{2s_j}}\\
&\le\cdots\\
&\le\left(\frac{C}{(1-\rho_0)\rho_0}\right)^{\sum_{k=0}^j\frac{1}{2s_k}}\rho_0^{-\sum_{k=0}^j\frac{k}{2s_k}}\\
&\times\prod_{k=0}^j s_k^{\frac{\alpha}{2s_k}}\left(\int_{B_{r_0}(0)} d^{-2^*\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)}|v|^2 v_t^{2s^*-2}\,\mathrm{d}\xi\right)^{\frac{1}{2s^*}}.
\end{aligned}\tag{3.23}$$

Now, observe that the integral in the r.h.s. of (3.23) if finite, since

$$\begin{aligned}
&\int_{B_{r_0}(0)} d^{-2^*\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)}|v|^2 v_t^{2s^*-2}\,\mathrm{d}\xi\\
&\le\int_{B_{r_0}(0)} d^{(2s^*-2^*)\left(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}\right)}|u|^{2s^*}\,\mathrm{d}\xi\\
&\le r_0^{(2s^*-2^*)(\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu})}\int_{B_{r_0}(0)}|u|^{2s^*}\,\mathrm{d}\xi\\
&\le C
\end{aligned}\tag{3.24}$$

by the choice of s^*. Moreover, it is easily seen that

$$\sum_{k=0}^\infty\frac{1}{2s_k}<\infty,\quad\sum_{k=0}^\infty\frac{k}{2s_k}<\infty,\quad\prod_{k=0}^\infty s_k^{\frac{\alpha}{2s_k}}<\infty. \tag{3.25}$$

Hence, by letting $j \to \infty$ in (3.23) and taking into account (3.24) and (3.25), we obtain that

$$\|v_t\|_{L^\infty(B_{\rho_0}(0))} \leq C$$

where C does not depend on t. Therefore, by taking $t \to +\infty$, we get that v is bounded on $B_{\rho_0}(0)$, which is equivalent to (1.4).

Estimate from below. Let u be a positive solution of (1.1) and suppose that $f(\xi, t) \geq 0$ for $t > 0$. Reasoning as in [4, Theorem 1.1], let $0 < t_1 < t_2$ (such that $B(0, t_2) \subset\subset \Omega$) and let $\phi(t) := \min_{d(\xi)=t} v(\xi)$, $t_1 < t < t_2$, where v is the auxiliary function defined by (3.7). Consider the function

$$g(\xi) := Ad(\xi)^{-2\sqrt{\bar{\mu}-\mu}} + B,$$

where A and B are chosen so that $g(\xi) = \phi(t_i)$ for $d(\xi) = t_i$, $i = 1, 2$. It follows that

$$A = \frac{\phi(t_2) - \phi(t_1)}{t_2^{-2\sqrt{\bar{\mu}-\mu}} - t_1^{-2\sqrt{\bar{\mu}-\mu}}}, \quad B = \frac{\phi(t_1)t_2^{-2\sqrt{\bar{\mu}-\mu}} - \phi(t_2)t_1^{-2\sqrt{\bar{\mu}-\mu}}}{t_2^{-2\sqrt{\bar{\mu}-\mu}} - t_1^{-2\sqrt{\bar{\mu}-\mu}}}.$$

It is easy to verify that

$$\operatorname{div}_{\mathbb{G}}\left(d^{-2(\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu})}\nabla_{\mathbb{G}} g\right) = 0, \quad \forall\, \xi \neq 0.$$

Moreover, by (3.9) and the sign hypothesis on f, it holds that

$$-\operatorname{div}_{\mathbb{G}}\left(d^{-2(\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu})}\nabla_{\mathbb{G}} v\right) \geq 0 \quad \text{in } \Omega \setminus \{0\}.$$

Henceforth

$$-\operatorname{div}_{\mathbb{G}}\left(d^{-2(\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu})}\nabla_{\mathbb{G}}(v - g)\right) \geq 0 \quad \text{in } \Omega \setminus \{0\}.$$

By the definition of g, we know that $v \geq g$ on $\partial(B(0, t_2) \setminus B(0, t_1))$. Therefore, by the weak maximum principle, we obtain that

$$v \geq g \quad \text{in } B(0, t_2) \setminus B(0, t_1). \tag{3.26}$$

So, taking into account the explicit form of g, from (3.26) we get

$$\begin{aligned} v(\xi) &\geq \frac{\phi(t_2) - \phi(t_1)}{t_2^{-2\sqrt{\bar{\mu}-\mu}} - t_1^{-2\sqrt{\bar{\mu}-\mu}}}\, d(\xi)^{-2\sqrt{\bar{\mu}-\mu}} + \frac{\phi(t_1)t_2^{-2\sqrt{\bar{\mu}-\mu}} - \phi(t_2)t_1^{-2\sqrt{\bar{\mu}-\mu}}}{t_2^{-2\sqrt{\bar{\mu}-\mu}} - t_1^{-2\sqrt{\bar{\mu}-\mu}}} \\ &= \frac{t_2^{-2\sqrt{\bar{\mu}-\mu}} - d(\xi)^{-2\sqrt{\bar{\mu}-\mu}}}{t_2^{-2\sqrt{\bar{\mu}-\mu}} - t_1^{-2\sqrt{\bar{\mu}-\mu}}}\phi(t_1) + \frac{d(\xi)^{-2\sqrt{\bar{\mu}-\mu}} - t_1^{-2\sqrt{\bar{\mu}-\mu}}}{t_2^{-2\sqrt{\bar{\mu}-\mu}} - t_1^{-2\sqrt{\bar{\mu}-\mu}}}\phi(t_2) \\ &\geq \frac{d(\xi)^{2\sqrt{\bar{\mu}-\mu}} - t_1^{2\sqrt{\bar{\mu}-\mu}}}{d(\xi)^{2\sqrt{\bar{\mu}-\mu}}\left(1 - t_1^{2\sqrt{\bar{\mu}-\mu}}t_2^{-2\sqrt{\bar{\mu}-\mu}}\right)}\phi(t_2), \end{aligned}$$

for every $\xi \in B(0, t_2) \setminus B(0, t_1)$.

By letting $t_1 \to 0$, we conclude that $v(\xi) \geq \phi(t_2) = \min_{d(\eta)=t_2} v(\eta) > 0$, for all $\xi \in B(0, t_2) \setminus \{0\}$, that is equivalent to (1.5). $\square$

4 Generalization to multi-singular inverse square potentials

It is not difficult to prove that the methods used in Section 3 also apply to nonlinear problems with multi-singular inverse square potentials, i.e. for problems of the type

$$\begin{cases} -\Delta_{\mathbb{G}} u - \displaystyle\sum_{i=1}^{k} \mu_i \frac{\psi^2}{d(\xi, a_i)^2} u &= f(\xi, u) \quad \text{in } \Omega, \\ u &= 0 \qquad\quad\ \text{on } \partial\Omega \end{cases} \tag{4.1}$$

where Ω is an open subset of $\mathbb{G}$, $a_i \in \Omega$, for $1 \le i \le k$, where $k \in \mathbb{N} \setminus \{0\}$, the numbers μ_i are positive real numbers such that

$$\sum_{i=1}^{k} \mu_i < \overline{\mu} = \left(\frac{Q-2}{2} \right)^2$$

and f satisfies assumption (1.2). In this general multi-singular case, the behavior of weak solutions at the singularities a_i can be described as follows, extending analogous results in [4] (see also [14]).

Theorem 4.1. *Under the assumption* (1.2)*, any solution* $u \in S_0^1(\Omega)$ *of problem* (4.1) *satisfies*

$$|u(\xi)| \le C d(\xi, a_i)^{-\sqrt{\overline{\mu}} + \sqrt{\overline{\mu} - \mu_i}}, \quad \forall \xi \in B(a_i, r) \setminus \{a_i\}, \quad 1 \le i \le k,$$

for $r > 0$ *sufficiently small.*

Moreover, if u *is positive and* $f(\xi, t) \ge 0$ *for* $t > 0$*, then*

$$u(\xi) \ge C' d(\xi, a_i)^{-\sqrt{\overline{\mu}} + \sqrt{\overline{\mu} - \mu_i}}, \quad \forall \xi \in B(a_i, r) \setminus \{a_i\}, \quad 1 \le i \le k,$$

for $r > 0$ *sufficiently small. In the above estimates,* r *is small enough so that* $B(a_i, r) \subset\subset \Omega$ *for* $1 \le i \le k$ *and* $a_j \notin B(a_i, r)$ *for any* $i \ne j$.

Proof. We omit the details, referring to the Euclidean outline in [4]. □

5 Asymptotic decay results for critical equations

In this final section, we focus on the case when f is a critical nonlinearity in problem (1.1), i.e. $f(\xi, u) = |u|^{2^*-2}u$, and Ω is an unbounded domain.

In the Euclidean elliptic setting, we know that, due to the conformal invariance of such equation, by means of the Kelvin transform we can immediately derive the behavior at infinity of solutions by means of their behavior at 0 (see e.g. [12], [14]).

In the framework of Carnot groups, a well-behaved analogue of the classical Kelvin transform is defined only in a special class of groups, namely the H-type groups of Iwasawa-type. Indeed, in any H-type group, a suitable inversion map can be explicitly written; however, the related Kelvin transform possesses suitable conformal properties (namely, it preserves $\Delta_{\mathbb{G}}$-harmonicity) only for H-type groups of Iwasawa type, as proved in [7, Theorem 4.2].

For a treatment about this topics and its applications, we refer to [7], [20] and [1, Chapter 18]. See also [27], where a Kelvin-type transform is used to obtain asymptotic estimates for critical problems involving Grushin-type operators.

In what follows, after briefly introducing the notations and properties about Kelvin transform on Iwasawa type groups, we shall derive for this class of groups the asymptotic decay of solutions for critical equations with Hardy potential, as announced in the statement of Theorem 1.2. Our result will cover the model case of the Heisenberg group $\mathbb{H}^n$.

Following the notations in [1], let $\mathbb{H} = (\mathbb{R}^{m+n}, \circ)$ be an H-type group and let us denote by $\xi = (x, t)$ the coordinates of points in $\mathbb{H}$, where $x = (x_1, \dots, x_m) \in \mathbb{R}^m$ and $t = (t_1, \dots, t_n) \in \mathbb{R}^n$. We recall that the composition law on $\mathbb{H}$ takes the form

$$(x,t) \circ (\xi, \tau) = \left(x + \xi, t_1 + \tau_1 + \frac{1}{2} < U^{(1)}x, \xi >, \dots, t_n + \tau_n + \frac{1}{2} < U^{(n)}x, \xi > \right)$$

where $U^{(1)}, \dots, U^{(n)}$ are fixed $m \times m$ skew-symmetric and orthogonal matrices and $U^{(r)}U^{(s)} = -U^{(s)}U^{(r)}$ for every $r, s \in \{1, \dots n\}$, $r \neq s$. Denoted by $\Delta_{\mathbb{H}}$ the canonical sub-Laplacian on $\mathbb{H}$, we recall that the fundamental solution of $\Delta_{\mathbb{H}}$ is given by $\Gamma = Cd^{2-Q}$, $Q = m + 2n$, where C is a suitable positive constant and d is the following gauge on $\mathbb{H}$

$$d(x,t) = (|x|^4 + 16|t|^2)^{1/4}.$$

Definition 5.1 (*H-inversion and H-Kelvin transform*)**.** *Let $\mathbb{H}$ be an H-type group. The H-inversion map $\sigma : \mathbb{H} \setminus \{0\} \to \mathbb{H} \setminus \{0\}$ is defined as*

$$\sigma(x,t) = \left(-\frac{|x|^2 x - 4\sum_{k=1}^{n} t_k U^{(k)} x}{|x|^4 + 16|t|^2}, -\frac{t}{|x|^4 + 16|t|^2} \right).$$

If σ is as above, the H-Kelvin transform u^* of a function $u : \mathbb{H} \to \mathbb{R}$ is the function $u^* : \mathbb{H} \setminus \{0\} \to \mathbb{R}$ defined as

$$u^*(x,t) = d(x,t)^{2-Q} u(\sigma(x,t)).$$

Hereafter, we recall some properties of σ (see e.g. [1, Prop. 18.5.2]). Clearly σ is involutive, i.e. $\sigma(\sigma(x,t)) = (x,t)$. Moreover,

$$d(\sigma(x,t)) = \frac{1}{d(x,t)}. \tag{5.1}$$

If we denote $\sigma(x,t) = (\sigma^1(x,t), \sigma^2(x,t))$, then it holds

$$|\sigma^1(x,t)| = \frac{|x|}{d^2(x,t)}, \qquad |\sigma^2(x,t)| = \frac{|t|}{d^4(x,t)}. \tag{5.2}$$

The following theorem summarizes the main fundamental properties of the H-inversion and the H-Kelvin transform on groups of Iwasawa type.

Theorem 5.2 ([7]). *Let $\mathbb{H}$ be a group of Iwasawa type. Then, the Jacobian of the inversion σ is given by*

$$|J(\sigma)(x,t)| = d(x,t)^{-2Q}, \qquad \forall (x,t) \in \mathbb{H} \setminus \{0\}.$$

Moreover, the H-Kelvin transform u^* of a function $u \in C^2(\mathbb{H} \setminus \{0\})$ satisfies

$$\Delta_{\mathbb{H}}(u^*(x,t)) = d^{-(Q+2)}(x,t)(\Delta_{\mathbb{H}} u)(\sigma(x,t)), \qquad \forall (x,t) \in \mathbb{H} \setminus \{0\}.$$

In what follows, we shall denote by Ω^* the image of a generic domain Ω under the inversion σ. We observe that, if Ω is a neighborhood of ∞, by which we mean that there exists a ball $B(0,R)$ such that $\overline{B(0,R)}^C \subset \Omega$, then Ω^* is a punctured neighborhood of 0, i.e. $\Omega^* = D \setminus \{0\}$, where D is an open set, $0 \in D$.

Now, let us consider the following norm on $S_0^1(\Omega)$:

$$\|u\|_\mu := \left(\int_\Omega |\nabla_{\mathbb{H}} u|^2 \, \mathrm{d}\xi - \mu \int_\Omega \psi^2 \frac{|u|^2}{d^2} \, \mathrm{d}\xi \right)^{1/2}. \tag{5.3}$$

This is a norm, which is equivalent to $\|u\|_{S_0^1} = \left(\int_\Omega |\nabla_{\mathbb{H}} u|^2 \, \mathrm{d}\xi \right)^{1/2}$, due to Hardy inequality. The following property holds.

Theorem 5.3. *The H-Kelvin transform is an isometry between $S_0^1(\Omega)$ and $S_0^1(\Omega^*)$ with respect to the norm* (5.3)*.*

Proof. Let $u, v \in S_0^1(\Omega)$ and let u^*, v^* their H-Kelvin transform. Reasoning as in [20, Theorem 8.6], we get

$$\int_\Omega \nabla_{\mathbb{H}} u(\xi) \cdot \nabla_{\mathbb{H}} v(\xi) \, \mathrm{d}\xi = \int_{\Omega^*} \nabla_{\mathbb{H}} u^*(\xi') \cdot \nabla_{\mathbb{H}} v^*(\xi') \, \mathrm{d}\xi'.$$

Indeed, for $u, v \in C_0^\infty(\Omega)$ (and then using a density argument), by the properties recalled in Theorem 5.2, we have

$$\begin{aligned}
&\int_{\Omega^*} \nabla_{\mathbb{H}} u^*(\xi') \cdot \nabla_{\mathbb{H}} v^*(\xi') \, \mathrm{d}\xi' \\
&\qquad = - \int_{\Omega^*} u^*(\xi') \Delta_{\mathbb{H}} v^*(\xi') \, \mathrm{d}\xi' \\
&\qquad = - \int_{\Omega^*} d(\xi')^{-2Q} u(\sigma(\xi')) \Delta_{\mathbb{H}} v^*(\sigma(\xi')) \, \mathrm{d}\xi' \\
&\qquad = - \int_\Omega u(\xi) \Delta_{\mathbb{H}} v(\xi) \, \mathrm{d}\xi \\
&\qquad = \int_\Omega \nabla_{\mathbb{H}} u(\xi) \cdot \nabla_{\mathbb{H}} v(\xi) \, \mathrm{d}\xi.
\end{aligned}$$

Moreover

$$
\begin{aligned}
&\int_{\Omega^*} \psi^2(\xi') \frac{u^*(\xi')v^*(\xi')}{d^2(\xi')}\, \mathrm{d}\xi' \\
&\qquad = \int_{\Omega^*} \psi^2(\xi') d(\xi')^{2(2-Q)} \frac{u(\sigma(\xi'))v(\sigma(\xi'))}{d^2(\xi')}\, \mathrm{d}\xi' \\
&\qquad = \int_{\Omega} \psi^2(\xi) \frac{u(\xi)v(\xi)}{d^2(\xi)}\, \mathrm{d}\xi,
\end{aligned}
$$

where we have used that, since $\psi(x,t) = \dfrac{|x|}{d(x,t)}$ in any H-type group, by (5.2) and (5.1) we get that $\psi(\sigma(x,t)) = \psi(x,t)$, for all $(x,t) \in \mathbb{H} \setminus \{0\}$. The proof is therefore complete. $\square$

Lemma 5.4. *Let $u \in S_0^1(\Omega)$ be a solution of*

$$
\begin{cases} -\Delta_{\mathbb{H}} u - \mu \dfrac{\psi^2}{d^2} u = |u|^{2^*-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.4}
$$

Then, its Kelvin transform u^* satisfies (5.4) *in Ω^*.*

Proof. Let u be a solution to (5.4). We know by Theorem 5.3 that $u^* \in S_0^1(\Omega^*)$. Let $\varphi \in C_0^\infty(\Omega^*)$; we can write $\varphi = \phi^*$, for some $\varphi \in C_0^\infty(\Omega)$. Applying Theorem 5.2 and 5.3 we have

$$
\begin{aligned}
&\int_{\Omega^*} \nabla_{\mathbb{H}} u^*(\xi') \cdot \nabla_{\mathbb{H}} \varphi(\xi')\, \mathrm{d}\xi' - \mu \int_{\Omega^*} \psi^2(\xi') \frac{u^*(\xi')\varphi(\xi')}{d^2(\xi')}\, \mathrm{d}\xi' \\
&\quad = \int_{\Omega^*} \nabla_{\mathbb{H}} u^*(\xi') \cdot \nabla_{\mathbb{H}} \phi^*(\xi')\, \mathrm{d}\xi' - \mu \int_{\Omega^*} \psi^2(\xi') \frac{u^*(\xi')\phi^*(\xi')}{d^2(\xi')}\, \mathrm{d}\xi' \\
&\quad = \int_{\Omega} \nabla_{\mathbb{H}} u(\xi) \cdot \nabla_{\mathbb{H}} \phi(\xi)\, \mathrm{d}\xi - \mu \int_{\Omega} \psi^2(\xi) \frac{u(\xi)\phi(\xi)}{d^2(\xi)}\, \mathrm{d}\xi \\
&\quad = \int_{\Omega} |u(\xi)|^{2^*-2} u(\xi)\phi(\xi)\, \mathrm{d}\xi \\
&\quad = \int_{\Omega^*} |u(\sigma(\xi'))|^{2^*-2} u(\sigma(\xi'))\phi(\sigma(\xi')) d(\xi')^{-2Q}\, \mathrm{d}\xi' \\
&\quad = \int_{\Omega^*} |u^*(\xi')|^{2^*-2} u^*(\xi')\phi^*(\xi')\, \mathrm{d}\xi' \\
&\quad = \int_{\Omega^*} |u^*(\xi')|^{2^*-2} u^*(\xi')\varphi(\xi')\, \mathrm{d}\xi'.
\end{aligned}
$$

By the arbitrariness of $\varphi \in C_0^\infty(\Omega^*)$, the thesis follows. $\square$

Proof of Theorem 1.2 Let u be a solution of (1.7). Then, by Lemma 5.4, u^* satisfies (1.7) in a punctured neighborhood of the origin. Therefore, by Theorem 1.1, $|u^*(\xi)| \le Cd(\xi)^{-\sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu}}$ for $d(\xi)$ small. So, taking into account that $u(\xi) = d(\xi)^{2-Q} u^*(\sigma(\xi))$, the asymptotic decay estimate (1.8) follows.

The estimate from below (1.9) for positive solutions of (1.7) can be proved analogously. Indeed, if u is positive, then also u^* is positive and so, by Theorem 1.1, u^* satisfies the estimate from below $|u^*(\xi)| \geq C' d(\xi)^{-\sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu}}$, for $d(\xi)$ small. The thesis, then, follows as before. □

6 Appendix. Some weighted Sobolev inequalities

In this appendix, we prove the weighted Sobolev inequality (3.16), which is used as a tool in the proof of Theorem 1.1. This kind of inequalities have been proved in [11, Theorem 1.2] for the general L^p-weighted norm of the horizontal gradient, with $1 < p < Q$, in the subclass of H-type groups. The result can be easily extended to arbitrary Carnot groups in the semilinear case $p = 2$ under consideration, as we show below.

Theorem 6.1. *Let $\mathbb{G}$ be a Carnot group of dimension $Q > 2$ and let $0 \leq s \leq 2$, $\alpha > \frac{2-Q}{2}$, $2^*(s) = \frac{2(Q-s)}{Q-2}$. Then there exists $C = C(s, \alpha, Q)$ such that for every $u \in C_0^\infty(\mathbb{G})$*

$$\int_{\mathbb{G}} \psi^s \frac{|d^\alpha u|^{2^*(s)}}{d^s} \,\mathrm{d}\xi \leq C \left(\int_{\mathbb{G}} |d^\alpha \nabla_{\mathbb{G}} u|^2 \,\mathrm{d}\xi \right)^{\frac{Q-s}{Q-2}}, \tag{6.1}$$

where $\psi = |\nabla_{\mathbb{G}} d|$.

Proof. Observe that the condition $\alpha > \frac{2-Q}{2}$ ensures that the integrals in (6.1) are finite, since it implies that $\alpha 2^*(s) - s + Q > 0$ and $2\alpha + Q > 0$. For any $u \in C_0^\infty(\mathbb{G})$, let us set $w = d^\alpha u$. Then

$$\begin{aligned} \int_{\mathbb{G}} |d^\alpha \nabla_{\mathbb{G}} u|^2 \,\mathrm{d}\xi &= \int_{\mathbb{G}} |\nabla_{\mathbb{G}} w - \frac{\alpha}{d} w \nabla_{\mathbb{G}} d|^2 \,\mathrm{d}\xi \\ &= \int_{\mathbb{G}} \left(|\nabla_{\mathbb{G}} w|^2 + \alpha^2 \psi^2 \frac{|w|^2}{d^2} - 2\alpha \frac{w}{d} \nabla_{\mathbb{G}} d \cdot \nabla_{\mathbb{G}} w \right) \mathrm{d}\xi. \end{aligned} \tag{6.2}$$

Moreover

$$\begin{aligned} 2\alpha \int_{\mathbb{G}} \frac{w}{d} \nabla_{\mathbb{G}} d \cdot \nabla_{\mathbb{G}} w \,\mathrm{d}\xi &= \alpha \int_{\mathbb{G}} d^{-1} \nabla_{\mathbb{G}} d \cdot \nabla_{\mathbb{G}} w^2 \,\mathrm{d}\xi \\ &= -\alpha \int_{\mathbb{G}} w^2 \mathrm{div}_{\mathbb{G}}(d^{-1} \nabla_{\mathbb{G}} d) \,\mathrm{d}\xi \\ &= -\alpha(Q-2) \int_{\mathbb{G}} \psi^2 \frac{w^2}{d^2} \,\mathrm{d}\xi. \end{aligned} \tag{6.3}$$

By (6.2) and (6.3), it follows that

$$\int_{\mathbb{G}} |d^\alpha \nabla_{\mathbb{G}} u|^2 \,\mathrm{d}\xi = \int_{\mathbb{G}} |\nabla_{\mathbb{G}} w|^2 \,\mathrm{d}\xi + \alpha(Q-2+\alpha) \int_{\mathbb{G}} \psi^2 \frac{w^2}{d^2} \,\mathrm{d}\xi. \tag{6.4}$$

Since $\alpha > \frac{2-Q}{2}$, it turns out that $Q - 2 + \alpha > 0$. Now, if $\alpha < 0$, by the Hardy inequality (2.3), we get from (6.4) that

$$\int_{\mathbb{G}} |d^\alpha \nabla_{\mathbb{G}} u|^2 \,\mathrm{d}\xi \geq C_1 \int_{\mathbb{G}} |\nabla_{\mathbb{G}} w|^2 \,\mathrm{d}\xi, \tag{6.5}$$

where

$$C_1 = 1 + \alpha(Q-2+\alpha)\left(\frac{2}{Q-2}\right)^2 = \left(1 + \frac{2\alpha}{Q-2}\right)^2 > 0.$$

If $\alpha \geq 0$, (6.5) obviously holds with $C_1 = 1$. Finally, by (6.5) and using Hardy-Sobolev inequality on $\mathbb{G}$ (see e.g. [10]) we obtain

$$\begin{aligned}\int_{\mathbb{G}} |d^\alpha \nabla_{\mathbb{G}} u|^2 \, \mathrm{d}\xi &\geq C_1 \int_{\mathbb{G}} \frac{|w|^{2^*(s)}}{d^s} \psi^s \, \mathrm{d}\xi \\ &= C_1 \int_{\mathbb{G}} \frac{|d^\alpha u|^{2^*(s)}}{d^s} \psi^s \, \mathrm{d}\xi.\end{aligned}$$

So, the proof is complete. □

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