This is the Pre-print of the paper
M.Berardi, F.Difonzo, M.Vurro, L.Lopez
The 1D Richards' equation in two layered soils: a Filippov approach to treat discontinuities
Advances in Water Resources 115 (2018), 264-272
DOI: https://doi.org/10.1016/j.advwatres.2017.09.027

The 1D Richards' equation in two layered soils: a Filippov approach to treat discontinuities

Marco Berardi[a,*], Fabio Difonzo[b], Michele Vurro[a], Luciano Lopez[c,a]

[a]*Istituto di Ricerca sulle Acque, Consiglio Nazionale delle Ricerche, via F. De Blasio 5, 70132, Bari, Italy*
[b]*Code Architects Automation, Via Stazione 164, 70029 Santeramo in Colle, Italy.*
[c]*Dipartimento di Matematica, Università degli Studi di Bari Aldo Moro, via E. Orabona 4, 70125 Bari, Italy*

Abstract

The infiltration process into the soil is generally modeled by the Richards' partial differential equation (PDE). In this paper a new approach for modeling the infiltration process through the interface of two different soils is proposed, where the interface is seen as a discontinuity surface defined by suitable state variables. Thus, the original 1D Richards' PDE, enriched by a particular choice of the boundary conditions, is first approximated by means of a time semidiscretization, that is by means of the *transversal* method of lines (TMOL). In such a way a sequence of discontinuous initial value problems, described by a sequence of second order differential systems in the space variable, is derived. Then, Filippov theory on discontinuous dynamical systems may be applied in order to study the relevant dynamics of the problem. The numerical integration of the semidiscretized differential system will be performed by using a one-step method, which employs an event driven procedure to locate the discontinuity surface and to adequately change the vector field.

Keywords: Richards' equation, event-driven numerical methods, layered soils, discontinuous differential systems, Filippov theory.

1. Introduction to the physical problem

The infiltration process into the soil is generally modeled by means of Richards' equation, an advection-diffusion equation derived by conservation of mass, generalizing Darcy's law for saturated flow in porous media.
Referring to a wide bibliography (see [1, 2, 3, 4]), we will consider just the infiltration along the vertical dimension z, since this process is mainly governed by gravity force. Nevertheless, the numerical solution for 2D or 3D problems is generally accomplished by Mixed Finite Element methods and Mixed Finite Volume methods, according to classical and recent papers,

*Corresponding author
Email addresses: marco.berardi@ba.irsa.cnr.it (Marco Berardi), fdifonzo@codearchitects.com (Fabio Difonzo), michele.vurro@ba.irsa.cnr.it (Michele Vurro), luciano.lopez@uniba.it (Luciano Lopez)

see for instance [5, 6, 7, 8, 9, 10].

It is well known that Richards' equation can be expressed in terms of *pressure-head* ψ or in terms of *water content* θ, where both formulations have advantages and drawbacks. Loosely speaking, we can think of *pressure-head* as the internal energy of the fluid due to the pressure exerted on its container, whereas *water content* is simply the volume of water per bulk volume of the medium (see [11]).
Here we will consider the following ψ-based form of the 1D Richards' equation:

$$C(\psi)\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial z}\left[K(\psi)\left(\frac{\partial \psi}{\partial z} - 1\right)\right], \ 0 \leq t \leq T, \text{ and } 0 \leq z \leq Z \ , \tag{1}$$

being $C(\psi)$, $K(\psi)$ and $\psi = \psi(t,z)$ real valued functions, with $C(\psi)$ the *hydraulic capacity* and $K(\psi)$ the *hydraulic conductivity*.

The function $C(\psi)$ is given in terms of the *water retention curve* $\theta(\psi)$, and is given by:

$$C(\psi) := \frac{\mathrm{d}\theta}{\mathrm{d}\psi}, \tag{2}$$

where $\theta(\psi)$ and $K(\psi)$ are suitable functions defined by the physical characteristics of the model (see for example [12]).

Equation (1) is usually enriched with the known value of the solution at initial time $t = 0$, that is:

$$\psi(0,z) := \psi^0(z), \quad \text{for every } z \in [0,Z], \tag{3}$$

and boundary conditions at the top and at the bottom of the vertical column $[0,Z]$, that is:

$$\psi(t,0) := \psi_0(t), \tag{4a}$$

$$\psi(t,Z) := \psi_Z(t), \qquad \text{for every } t \in [0,T]. \tag{4b}$$

The study of the infiltration process into a medium, constituted by two (or more) different layered soils, is an open problem. In particular, the wet front behavior on the interface between two layers is the most challenging situation, and in recent years some efforts have been made.

In [13] it is highlighted that water movement through the interface is possible just where the water accumulates on it and the pressure grows enough to overcome the capillary forces. In [13] an approach based on the method of lines is used for simulating water flow, and the continuity of normal flux and pressure is imposed at the interface. Assuming the threshold depending on the depth, the mesh is designed so that a node lies in the interface. A fictitious node is introduced at that point in order to assume the soil properties of the upper soil while the real node will assume the soil properties of the second one.
In [12] emphasis is posed on the infiltration into very dry soil, and it is highlighted that the dry condition is the most challenging physical case from the numerical point of view. Moreover, a water content based algorithm is proposed, and its efficiency is shown in very

dry soils.
From a mathematical point of view, the problem of handling the discontinuity consists in choosing conveniently the values for $C(\psi)$ and $K(\psi)$ at the interface, being these values not well-defined there; in fact, at the interface different hydraulic functions could be applied, according to the medium under consideration. Towards this goal, a *regularization* of hydraulic conductivity at the interface can be accomplished by means of a geometric average, as in [1]. It should be stressed that, also in one homogeneous soil, the problem of evaluating flow variables, i.e. the hydraulic conductivity function, is handled by using inter-nodal hydraulic conductivity, that is a weighted average of known hydraulic conductivities at two neighboring nodes of the spatial grid system. In [14] a refinement of this technique is studied, in the context of layered soils; this led to the concept of inter-layer hydraulic conductivity, and a comparison of different regularization techniques is presented therein.
Recently, in [15], a water-content formulation is used, both in a vertex-centered approach and in a cell-centered one. In particular, for the first approach, assuming that water retention curves are specified, a new composite curve is formed at the interface, expressing the relation between water content and pressure-head under composite effects of two different materials at the interface. An arithmetic average is used for computing the conductivity function at the interface and a correction term is introduced in order to take into account the soil heterogeneity.
Because of the continuity of flux and mass over the whole integration domain, and particularly across the surface, the continuity of the pressure head ψ is assumed over the whole domain.
Therefore, at the interface of two different soils (denoted briefly with 1 and 2), characterized by different hydraulic capacities, $C_i(\psi)$ for $i = 1, 2$, and different hydraulic conductivities, $K_i(\psi)$ for $i = 1, 2$, we have:

$$\theta_1(\overline{\psi}) \neq \theta_2(\overline{\psi}), \tag{5}$$

where water content functions θ_1 and θ_2 assume different values at the interface, $z = \overline{z}$, where the pressure value is $\overline{\psi} = \psi(\overline{t}, \overline{z})$. This justifies the discontinuity of the water content function at the interface, as claimed in [13, 16, 15], and supports our choice of the ψ-form of Richards' equation. Thus, we can assume that the state variable ψ is continuous on the interface while its derivative $\frac{\partial \psi}{\partial z}$ (the spatial derivative) is discontinuous.

Finally, from the analytical point of view, in [17], a closed-form solution is obtained under the assumption that the capillarity rise assumes the same value at each layer, while, in the recent work [18], an analytical solution is proposed in general cases of different values for hydraulic parameters, and for any variation of the saturated hydraulic conductivity varying with the depth: such an analytical solution is obtained just for Gardner' s exponential model, though. Nevertheless, both in [18] and in [19], the saturation condition is imposed at the bottom.

Thus, one way to model this discontinuous problem is considering the pair of Richards' equations:

$$C_1(\psi)\frac{\partial\psi}{\partial t} = \frac{\partial}{\partial z}\left[K_1(\psi)\left(\frac{\partial\psi}{\partial z} - 1\right)\right], \quad \text{if} \quad z < \overline{z}, \tag{6a}$$

$$C_2(\psi)\frac{\partial\psi}{\partial t} = \frac{\partial}{\partial z}\left[K_2(\psi)\left(\frac{\partial\psi}{\partial z} - 1\right)\right], \quad \text{if} \quad z > \overline{z}, \tag{6b}$$

according to the two different soils under investigation, and for a fixed threshold value $\overline{z}$.

In real applications, the difficulty often arises to assign a significant value to $\psi(t, Z)$; therefore, in the present paper, we suppose to replace (4b) with the following condition at $z = 0$:

$$\chi_0(t) := \left[\frac{\partial\psi(t,z)}{\partial z}\right]_{z=0}, \quad \text{for every} \quad t \in [0, T] \ , \tag{7}$$

namely, we assume that what happens at the bottom reflects (and is reflected by) the spatial derivative of ψ at the top.

We have to point out that the replacement of the boundary condition of $\psi(t, z)$ at $z = Z$ with (7) modifies the nature of our PDE and suggests to consider nonstandard methods for the semidiscretization of (1). In particular the Transversal Method of Lines (TMOL) will be used, instead of the more classical Method of Lines (MOL), in order to derive, for each value of t, an initial value problem described by a second order differential system in the space variable.

This way, Filippov theory (see [20]) may be applied to treat the discontinuity of (6) in the space variable (i.e. along $z = \bar{z}$). This approach represents the main novelty of the paper. We have to notice that Filippov theory has also been used in the numerical solution of geosciences and geochemical models but only to treat discontinuities in time (see [21], [22]).

The plan of the paper is the following: in Section 2 we sketch some basic concepts of Filippov theory; in Section 3 we semi-discretize Richards' PDE in time variable by using the transversal method of lines (TMOL), in order to derive a discontinuous system of second order differential equations in the space variable; in Section 4 we rewrite such a system in terms of Filippov system; in Section 5, for solving the Filippov discontinuous system, we employ an event driven method (see [23, 24]) based on a one-step scheme of low order. Particularly, such a method is a sort of predictor-corrector scheme applied to the semidiscretized system: the predictor is the Implicit Euler scheme applied to a Differential Algebraic Equation (DAE) associated to the semidiscretized system while the corrector performs a correction of the previous solution. Finally, in Section 6, some numerical simulations are reported in order to justify our results.

2. Background on Filippov theory

The Filippov theory is a well-known tool for dealing with discontinuous systems of ordinary differential equations (ODEs). The mathematical problem to which this theory applies

Please cite this article as: Marco Berardi, Fabio Difonzo, Michele Vurro, Luciano Lopez, The 1D Richards' equation in two layered soils: a Filippov approach to treat discontinuities, Advances in Water Resources (2017), **doi: 10.1016/j.advwatres.2017.09.027**

is generally expressed by a first order differential system in $\mathbb{R}^s$ of the following form:

$$x'(t) = f(x(t)) = \begin{cases} f_1(x(t)) & \text{when } h(x(t)) < 0 \\ f_2(x(t)) & \text{when } h(x(t)) > 0 \end{cases}, \qquad t \in [0, T] \tag{8}$$

with initial condition $x_0 = x(0)$, where $x : [0, T] \to \mathbb{R}^s$, $f : \mathbb{R}^s \to \mathbb{R}^s$ and $h : \mathbb{R}^s \to \mathbb{R}$ defines the discontinuity surface:

$$\Sigma = \{x \in \mathbb{R}^s |\ h(x) = 0\}\,. \tag{9}$$

The manifold Σ splits the phase space in such a way that $\mathbb{R}^s = \overline{R}_1 \cup \Sigma \cup \overline{R}_2$, where:

$$R_1 = \{x \in \mathbb{R}^s |\ h(x) < 0\}\,, \quad R_2 = \{x \in \mathbb{R}^s |\ h(x) > 0\}\,.$$

We also assume that Σ is smooth, that is the gradient $\nabla h(x) \neq 0$ for all $x \in \Sigma$.
The vector field f is discontinuous along Σ (that is $f_1(\overline{x}) \neq f_2(\overline{x})$, $\overline{x} \in \Sigma$), with f_1 and f_2 smooth in the regions $R_1 \cup \Sigma$ and $R_2 \cup \Sigma$, respectively.

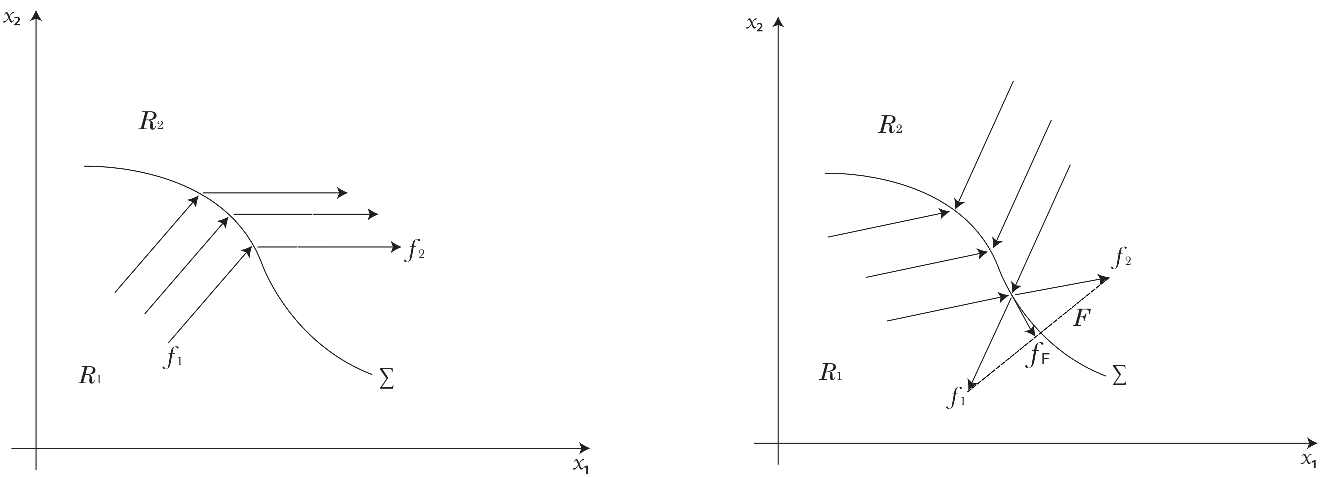


Figure 1: On the left, the crossing behavior; on the right, the sliding behavior. Both figures are depicting phase space plots.

The pioneering work on this subject is Filippov's book [20]. In the last decade a great effort has been devoted to study discontinuous ODEs, either from an theoretical point of view (see, for example, [25, 26, 27, 28]), or from a computational point of view (see [24]). Here we just sketch the most important aspects of such a theory; for example, the existence and uniqueness of solution to the initial value problem (8) is not guaranteed and particular attention has to be payed when the solution approaches the discontinuity.
We also assume that the trajectory $x(t)$ reaches the discontinuity surface Σ in a finite time $\bar{t}$ (called *event time*) and at a certain state $\overline{x} = x(\bar{t})$ (called *event point*). Thus, the solution

behavior can be determined according to some geometrical conditions. If we define:

$$w_1(\overline{x}) = \nabla h^\top(\overline{x}) \cdot f_1(\overline{x}), \tag{10a}$$

$$w_2(\overline{x}) = \nabla h^\top(\overline{x}) \cdot f_2(\overline{x}), \tag{10b}$$

the solution behavior follows one of these two situations, as represented in Figure 1:

- *Crossing.* The solution, coming from R_1 (resp. R_2), crosses Σ at some point $\overline{x} \in \Sigma$ and switches to the other vector field f_2 (resp. f_1). A sufficient and necessary condition for the crossing is

$$w_1(\overline{x})\, w_2(\overline{x}) > 0. \tag{11}$$

- *Sliding.* Roughly speaking, in this case the solution trajectory reaching $\overline{x} \in \Sigma$ will not leave Σ, and therefore will slide along Σ. A sufficient and necessary condition for attractive sliding is

$$w_1(\overline{x}) > 0 \quad \text{and} \quad w_2(\overline{x}) < 0. \tag{12}$$

  In this case, the solution satisfies a new initial value problem $x' = f_F(x)$, $x(0) = \overline{x}$, where the vector field f_F is the sliding Filippov vector field, computed as a suitable convex combination of f_1, f_2, that is

$$f_F(x) = (1 - \alpha(x)) f_1(x) + \alpha(x) f_2(x)\ , \tag{13}$$

  where $\alpha(x)$ is the value which guarantees $f_F(x)$ to lie on the tangent plane to Σ at x, that is the value for which $(\nabla h(x))^\top f_F(x) = 0$:

$$\alpha(x) = \frac{(\nabla h(x))^\top \cdot f_1(x)}{(\nabla h(x))^\top (f_1(x) - f_2(x))}. \tag{14}$$

From a numerical point of view, when a discontinuous system has to be simulated, the accuracy of the numerical method is lost crossing the discontinuity. To overcome this problem one could regularize the discontinuous vector field obtaining a continuous, but stiff, dynamical system (see [29]) whose solution requires expensive computational methods. Instead, if the event point $\overline{x}$ is located accurately, we can restart with the vector field f_2 (crossing case) or f_F (sliding attractive case) and no accuracy reduction will occur in the numerical solution obtained. These numerical procedures are known in literature as event-driven methods or event location techniques (see [30, 31, 32, 33, 34]).

We now briefly describe an event driven procedure: let us suppose to start with $x_0 \in R_1$ (i.e. $h(x_0) < 0$). Let x^n be the numerical approximation of the solution $x(t_n)$ at the generic instant $t_n = t_{n-1} + \Delta t$ (Δt being the time step); what we do in practice is to check the quantity $h(x^n) \cdot h(x^{n+1})$: if this product is greater than zero, then x^{n+1} is in the same region as x^n, and the numerical integration continues using the same vector field (i.e. f_1) as in the previous step; otherwise, if the previous product is less than zero, then x^{n+1} belongs to a different region with respect to x^n, i.e. R_2, and so we need to compute the event point $x^{\bar{n}}$ on Σ and select the vector field f_2 (when the crossing condition is satisfied), or Filippov sliding vector field f_F (when the sliding condition is satisfied instead).

Please cite this article as: Marco Berardi, Fabio Difonzo, Michele Vurro, Luciano Lopez, The 1D Richards' equation in two layered soils: a Filippov approach to treat discontinuities, Advances in Water Resources (2017), **doi: 10.1016/j.advwatres.2017.09.027**

3. Semidiscretization in time

In this paper we wish to study what happens when an infiltrating water flow switches from one soil to another, according to different thresholds values of depth or head pressure gradient. Thus it seems natural to first discretize the time derivative operator (rather than the spatial derivative operator, as in MOL technique) using the transversal method of lines (TMOL). This approach is not usual when dealing with the Richards' equation, for which it is standard to first discretize the second order spatial derivative, and then integrate the resulting differential system with respect to the time variable (see, for example, [4, 13]). Let $\{t_0, \dots, t_{N_T}\}$ be a uniform partition of the time interval $[0, T]$ obtained by using the constant stepsize Δt. For any $n = 1, \dots, N_T$ consider the following set of equations:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[K\left(\psi^n\right)\left(\frac{\mathrm{d}\psi^n}{\mathrm{d}z} - 1\right)\right] = C\left(\psi^n\right) D_t(\psi^n), \qquad n = 1, \dots, N_T \ , \tag{15}$$

where $\psi^n = \psi^n(z)$ denotes the approximation of the exact solution at (t_n, z) for $0 \le z \le Z$, and

$$D_t(\psi^n) = \frac{\psi^n - \psi^{n-1}}{\Delta t}, \tag{16}$$

even if any discrete time differential operator could be used.

4. Discontinuous models: Filippov formulation

For every $n = 1, \dots, N_T$, let us define the following functions from $[0, Z]$ to $\mathbb{R}$:

$$x^n(z) = z, \qquad y^n(z) = \psi^n(z), \qquad s^n(z) = \left(y^n(z)\right)' \ ,$$

where the derivative of $y^n(z)$ is taken with respect to z.

From (15), it follows that

$$(s^n)' = \frac{C(y^n)}{K(y^n)} D_t(y^n) - \frac{K'(y^n)}{K(y^n)}(s^n - 1)s^n,$$

while $(x^n)' = 1$ and $(y^n)' = s^n$. Thus, we get the following differential system:

$$\begin{bmatrix} x^n \\ y^n \\ s^n \end{bmatrix}' = \begin{bmatrix} 1 \\ s^n \\ \frac{C(y^n)}{K(y^n)} D_t(y^n) - \frac{K'(y^n)}{K(y^n)}(s^n - 1)s^n \end{bmatrix}, \tag{17}$$

for each $n = 1, \dots, N_T$, and on the spatial interval $[0, Z]$.

Next, we will analyze (17) when the dynamics faces a discontinuity surface.

More specifically, two different scenarios will be considered, according to the choice of the function h which defines the discontinuity surface:

- if the discontinuity occurs along x^n, then we take

$$h(x^n, y^n, s^n) := x^n - \tilde{z}, \tag{18}$$

Please cite this article as: Marco Berardi, Fabio Difonzo, Michele Vurro, Luciano Lopez, The 1D Richards' equation in two layered soils: a Filippov approach to treat discontinuities, Advances in Water Resources (2017), **doi: 10.1016/j.advwatres.2017.09.027**

- if the discontinuity occurs along s^n, then we take

$$h(x^n, y^n, s^n) := s^n - \tilde{\psi}', \tag{19}$$

where $\tilde{z}$ and $\tilde{\psi}'$ are reference real values for the height and the pressure gradient, respectively.

For instance, the first choice - i.e., the threshold depending on the depth - refers to the classical papers on this subject (e.g., see [13, 16]), whereas the second choice links the trespass into the second soil with a threshold value of the pressure gradient.

Let the discontinuity surface

$$\Sigma^n := \{(x^n, y^n, s^n) \in \mathbb{R}^3 \, : \, h(x^n, y^n, s^n) = 0\},$$

which splits $\mathbb{R}^3$ into two regions:

$$R_1 := \{(x^n, y^n, s^n) \in \mathbb{R}^3 \, : \, h(x^n, y^n, s^n) < 0\}, \; R_2 := \{(x^n, y^n, s^n) \in \mathbb{R}^3 \, : \, h(x^n, y^n, s^n) > 0\}.$$

We now define the vector fields f_1 and f_2 in the smooth regions R_1 and R_2 of the discontinuous system, that is:

$$f_i^n(x^n, y^n, s^n) := \begin{bmatrix} 1 \\ s^n \\ \dfrac{C_i(y^n)}{K_i(y^n)} D_t(y^n) - \dfrac{K_i'(y^n)}{K_i(y^n)}(s^n - 1)s^n \end{bmatrix}, \qquad i = 1, 2, \tag{20}$$

where the functions C_i, K_i for $i = 1, 2$, define the Richards' equation in the two different soils.

Thus, (17) may be written in the Filippov form as:

$$\begin{bmatrix} x^n \\ y^n \\ s^n \end{bmatrix}' = \begin{cases} f_1^n(x^n, y^n, s^n), & x \in R_1, \\ f_2^n(x^n, y^n, s^n), & x \in R_2, \end{cases} \tag{21}$$

with initial conditions given by

$$x^n(0) = 0, \quad y^n(0) = \psi_0(t_n), \quad s^n(0) = \chi_0(t_n) \; . \tag{22}$$

4.1. Discontinuity in z

This setting provides a contact of two layered soils at the line $h(x^n) = x^n - \tilde{z} = 0$, so that (21) becomes a Filippov differential system with a discontinuity surface Σ^n defined by (18). Since $\nabla h(x^n) = [1 \; 0 \; 0]^\top$, we have

$$w_1 = \nabla h(x^n)^\top f_1^n(x^n, y^n, s^n) = 1,$$
$$w_2 = \nabla h(x^n)^\top f_2^n(x^n, y^n, s^n) = 1,$$

thus, from (11) the state variable x^n can only undergo a crossing phenomenon along the spatial direction.

Remark 1. *We notice that no particular difficulty arises if the value $\tilde{z}$ depends on time t. In fact, in this case $h(x^n) := x^n - \tilde{z}^n$, and again $w_1 = w_2 = 1$.*

Please cite this article as: Marco Berardi, Fabio Difonzo, Michele Vurro, Luciano Lopez, The 1D Richards' equation in two layered soils: a Filippov approach to treat discontinuities, Advances in Water Resources (2017), **doi: 10.1016/j.advwatres.2017.09.027**

4.2. Discontinuity in $\frac{\partial\psi}{\partial z}$

We now suppose that h is defined as in (19), that is $h(s^n) := s^n - \tilde{\psi}$. This means that the presence of a discontinuity is detected by some rapid variation of the state variable from one point of the depth mesh to the following one. We point out that, in the context of Filippov systems, this would be equivalent to a *time-stepping* approach (see, for example, [35]) versus a *event-driven* one. As a matter of fact, in the classical *time-stepping* approaches the occurrence of a discontinuity is detected just by examining the behavior of the step-size control procedure. Similarly, the occurrence of a different soil in the spatial integration can be investigated just by rapid variation – beyond a certain threshold – in $\frac{\partial\psi}{\partial z}$.
Since $\nabla h(s^n)^\top = [0\ 0\ 1]$, on the discontinuity surface $s^n - \tilde{\psi}' = 0$ we have:

$$w_1(y^n) = \nabla h(s^n)^\top f_1^n(x^n, y^n, s^n) = \frac{C_1(y^n)}{K_1(y^n)}\left[D_t(y^n) - \Gamma_1(y^n)(\tilde{\psi}' - 1)\tilde{\psi}'\right], \tag{23a}$$

$$w_2(y^n) = \nabla h(s^n)^\top f_2^n(x^n, y^n, s^n) = \frac{C_2(y^n)}{K_2(y^n)}\left[D_t(y^n) - \Gamma_2(y^n)(\tilde{\psi}' - 1)\tilde{\psi}'\right], \tag{23b}$$

where

$$\Gamma_i(y^n) := \frac{K_i'(y^n)}{C_i(y^n)}, \quad i = 1, 2. \tag{24}$$

The crossing case is similar to the one where the discontinuity refers to x^n, while the sliding case apparently has never been treated so far.
Then, from (12), if the pressure gradient increases as z decreases, a sliding phenomenon occurs if and only if

$$w_1(y^n) > 0 \text{ and } w_2(y^n) < 0. \tag{25}$$

Proposition 1. *Let us assume that the dynamics is governed by* (21)*. Then a sliding motion on* $s^n - \tilde{\psi} = 0$ *occurs, if and only if*

$$\Gamma_1(y^n)(\tilde{\psi}' - 1)\tilde{\psi}' < D_t(y^n) < \Gamma_2(y^n)(\tilde{\psi}' - 1)\tilde{\psi}'. \tag{26}$$

Proof. If sliding motion occurs, then (25) holds, thus the claim will follow as we prove that $C_i(y^n) > 0$, $K_i(y^n) > 0$, for $i = 1, 2$, on $s^n - \tilde{\psi} = 0$. Since C_i is defined as in (2) while the usual form of θ will be the one in (40a), then $C_i(y^n) > 0$, for $i = 1, 2$, on $s^n - \tilde{\psi} = 0$. Further, since the form of K is the one in (40b), then $K_i(y^n) > 0$, $i = 1, 2$, on $s^n - \tilde{\psi} = 0$. Viceversa, if (26) holds, then

$$D_t(y^n) - \Gamma_1(y^n)(\tilde{\psi}' - 1)\tilde{\psi}' > 0,$$
$$D_t(y^n) - \Gamma_2(y^n)(\tilde{\psi}' - 1)\tilde{\psi}' < 0.$$

Hence, by using positivity of C_i, K_i, for $i = 1, 2$, on $s^n - \tilde{\psi} = 0$, then (25) follows. □

From (26), it follows that during the sliding motion on $s^n - \tilde{\psi} = 0$ it is necessary that

$$\frac{K_1'(y^n)}{C_1(y^n)}(\tilde{\psi}' - 1)\tilde{\psi}' < \frac{K_2'(y^n)}{C_2(y^n)}(\tilde{\psi}' - 1)\tilde{\psi}', \tag{27}$$

which represents a condition depending just on the characteristic properties of hydraulic conductivity and hydraulic capacity of the two soils and on the threshold value for s.

Please cite this article as: Marco Berardi, Fabio Difonzo, Michele Vurro, Luciano Lopez, The 1D Richards' equation in two layered soils: a Filippov approach to treat discontinuities, Advances in Water Resources (2017), **doi: 10.1016/j.advwatres.2017.09.027**

5. Discretization in space

The full discretization of the Richards' equation is usually obtained by a discretization of the spatial domain by using finite difference or finite element methods, followed by the discretization of the time variable by implicit methods as the Implicit Euler (IE) scheme (see [36, 37, 38]). Due to the stiff and nonlinear nature of the Richards' equation, computational difficulties arise in the nonlinear systems we need to solve at each time step. In literature, solving such nonlinear systems is often performed by using the Picard iteration, or more effective techniques as the modified Picard iteration or the L-scheme, in order to have more rapid convergence to the solutions (see [36, 37, 38]).

We have to notice that the substitution of boundary condition (4), at $z = Z$, with the initial condition (7), at $z = 0$, leads to significant changes in the numerical methods for the full discretization of our problem.

In fact, here we will first discretize with respect to the time variable and then with respect to the space variable by using an implicit scheme. Because of the occurrence of the discontinuities, the space discretization of the ODE system will be obtained by means of an event driven procedure of low order of accuracy; particularly, starting with the set of initial conditions in (22) we will integrate (21) in region R_1 (where $z \leq \tilde{z}$, $C = C_1$ and $K = K_1$) until reaching the discontinuity surface; then, depending on the crossing or sliding condition, we will adequately change the vector field and continue the numerical integration.

In this section, we will consider just the case of discontinuity in z, that is the crossing case, because the sliding case may be treated in a similar way by using Filippov vector field (13) during the sliding mode.

Let $z_{k+1} = z_k + \Delta z$, for $k = 0, \ldots, K_Z - 1$, be a uniform partition of the spatial interval $[0, Z]$ with spatial step Δz. Let $y_k^n := y^n(z_k)$ and $s_k^n := s^n(z_k)$: then the Implicit Euler (IE) scheme reads as:

$$y_{k+1}^n = y_k^n + \Delta z \; s_{k+1}^n; \tag{28a}$$

$$s_{k+1}^n = s_k^n + \Delta z \left[\frac{C_1(y_{k+1}^n)}{K_1(y_{k+1}^n)} D_t(y_{k+1}^n) - \frac{K_1'(y_{k+1}^n)}{K_1(y_{k+1}^n)} (s_{k+1}^n - 1) s_{k+1}^n \right]; \tag{28b}$$

for $k = 0, \ldots, \tilde{K}_Z - 1$, and $z_{\tilde{K}_Z} = \tilde{z}$; from (28b) we get

$$K_1(y_{k+1}^n) s_{k+1}^n = K_1(y_{k+1}^n) s_k^n + \Delta z \left[C_1(y_{k+1}^n) D_t(y_{k+1}^n) - K_1'(y_{k+1}^n)(s_{k+1}^n - 1) s_{k+1}^n \right]. \tag{29}$$

We point out that is not necessary to integrate x^n, being equal to z. Once the value $\tilde{z}$ is reached, we restart with $(y_{\tilde{K}_Z}^n, \; s_{\tilde{K}_Z}^n)$ by integrating (21) in region R_2, where C_1 and K_1 are replaced by C_2 and K_2 respectively, for any $k = \tilde{K}_Z, \ldots, K_Z - 1$.

However, we have to notice that in real situations the functions C_1 and K_1 assume positive and very small values; in particular K_1 is much smaller that C_1 and the function $\sigma(y) = \frac{C_1(y)}{K_1(y)}$ in (29) is often unbounded (see for instance (40)). Thus the numerical solution of the nonlinear system (28) by the standard or modified Picard method is not effective (as our numerical simulations have shown) and an alternative procedure is necessary.

Here, using the fact that K_1 is positive and small, we suggest a sort of predictor-corrector method based on the IE scheme, where the predictor is the scheme derived by IE setting $K_1 = 0$ in (29). The numerical solution obtained this way will be corrected by the IE scheme. That is, we first solve the nonlinear system:

$$0 = y_{k+1}^n - y_k^n - \Delta z \; s_{k+1}^n; \tag{30a}$$

$$0 = \left[C_1(y_{k+1}^n) D_t(y_{k+1}^n) - K_1'(y_{k+1}^n)(s_{k+1}^n - 1)s_{k+1}^n\right]; \tag{30b}$$

in order to determine a solution (y_{k+1}^n, s_{k+1}^n), which is the predictor. We notice that the previous nonlinear system does not suffer of the possible unboundedness of $\sigma(y)$.

The nonlinear system in (30) may be solved by a Newton or Picard type method within a tolerance TOL. Define the following function from $\mathbb{R}^2$ to $\mathbb{R}$:

$$F(y_{k+1}^n, s_{k+1}^n) := C_1(y_{k+1}^n) D_t(y_{k+1}^n) - K_1'(y_{k+1}^n)(s_{k+1}^n - 1)s_{k+1}^n \; , \tag{31}$$

which is L_f-Lipschitz because of the smoothness of C_1 and K_1. Thus we can suppose that when we stop the iterations of the numerical procedure to solve (30) its right hand side is of order TOL.

Being K_1 very small, we are going to choose TOL of the same order (or less) of K_1, so to have $\left|\frac{F(y_{k+1}^n, s_{k+1}^n)}{K_1(y_{k+1})}\right|$ bounded. Then, once (y_{k+1}^n, s_{k+1}^n) has been found, we correct just s_{k+1}^n by using the value of $F(y_{k+1}^n, s_{k+1}^n)$, that is we will assume:

$$s_{k+1}^n = s_k^n + \Delta z \, \frac{F(y_{k+1}^n, s_{k+1}^n)}{K_1(y_{k+1})} \; . \tag{32}$$

5.1. How to solve the nonlinear system (30).

Now, let us define the Courant-Fischer-Lewy number:

$$\gamma := \frac{\Delta t}{\Delta z^2} \; , \tag{33}$$

recall that $D_t(y_{k+1}^n) = \frac{y_{k+1}^n - y_{k+1}^{n-1}}{\Delta t}$ and set $w_{k+1}^n = y_{k+1}^n - y_k^n$.

Thus, replacing (30a) in (30b), we derive:

$$0 = \left[C_1(y_{k+1}^n) - \gamma \;\; K_1'(y_{k+1}^n)(w_{k+1}^n - \Delta z)\right] w_{k+1}^n + C_1(y_{k+1}^n)(y_k^n - y_{k+1}^{n-1}), \tag{34}$$

which is a nonlinear algebraic equation that can be solved by an iterative procedure.

Set $w_{k+1,m}^n := y_{k+1,m}^n - y_k^n$, with $y_{k+1,0}^n$ known, and define the following modified Picard iteration:

$$0 = \left(C_1(y_{k+1,m}^n) - \gamma \; K_1'(y_{k+1,m}^n)(w_{k+1,m}^n - \Delta z)\right) w_{k+1,m+1}^n + C_1(y_{k+1,m}^n)(y_k^n - y_{k+1}^{n-1}),$$

from which we derive

$$w_{k+1,m+1}^n = \frac{y_{k+1}^{n-1} - y_k^n}{\left[1 - \gamma \frac{K_1'(y_{k+1,m}^n)}{C_1(y_{k+1,m}^n)}(w_{k+1,m}^n - \Delta z)\right]} \; , \quad m \geq 0 \; . \tag{35}$$

Thus, if the sequence $\left\{w_{k+1,m}^n\right\}$ converges to w_{k+1}^n for $m \to \infty$, then we assume $y_{k+1}^n = w_{k+1}^n + y_k^n$.

Please cite this article as: Marco Berardi, Fabio Difonzo, Michele Vurro, Luciano Lopez, The 1D Richards' equation in two layered soils: a Filippov approach to treat discontinuities, Advances in Water Resources (2017), **doi: 10.1016/j.advwatres.2017.09.027**

5.2. Convergence of iterations

Proposition 2. *Let us consider the scheme* (30)*. Let us suppose that* $|\Delta y_0^n|$ *and* $|\Delta s_0^n|$ *are bounded quantities for each* n*. Then, there exists a constant* $M > 0$ *independent on* k, n *such that*

$$|\Delta y_k^n| \leq M,$$

where $\Delta y_k^n := y_{k-1}^{n-1} - y_k^n$*, for any* $n = 1, \ldots, N_T$*,* $k = 0, \ldots, K_Z$*.*

Proof. Instead of (30) we consider:

$$y_{k+1}^n = y_k^n + \Delta z\ s_{k+1}^n; \tag{36a}$$

$$\varepsilon s_{k+1}^n = \varepsilon s_k^n + \Delta z\ F(y_{k+1}^n, s_{k+1}^n); \tag{36b}$$

where $F(y_{k+1}^n, s_{k+1}^n)$ is the smooth function in (31) and where ε is positive and small so that the second equation of (36) is a perturbation of order ε of (30b).

Now, by using $\Delta y_k^n := y_{k-1}^{n-1} - y_k^n$, and $\Delta s_k^n := s_{k-1}^{n-1} - s_k^n$, we derive:

$$\begin{cases} \Delta y_k^n = \Delta y_{k-1}^n + \Delta z \cdot \Delta s_k^n. \\ \varepsilon \Delta s_k^n = \varepsilon \Delta s_{k-1}^n + \Delta z \cdot (F(y_{k+1}^{n-1}, s_{k+1}^{n-1}) - F(y_k^n, s_k^n)) \ , \end{cases} \tag{37}$$

and from the second equation of (37) it follows that

$$|\Delta s_k^n| \leq \frac{|\Delta s_{k-1}^n| + \Delta z(\varepsilon)\ L_f |\Delta y_k^n|}{1 - \Delta z(\varepsilon)\ L_f} \tag{38}$$

where $\Delta z(\varepsilon) = \Delta z / \varepsilon$, $1 - \Delta z(\varepsilon) L_f > 0$ and Δz is sufficiently small.

Using (38) into the first of (37) we derive

$$|\Delta y_k^n| \leq |\Delta y_{k-1}^n| + \Delta z \frac{|\Delta s_{k-1}^n| + \Delta z(\varepsilon)\ L_f |\Delta y_k^n|}{1 - \Delta z(\varepsilon)\ L_f}$$

from which

$$|\Delta y_k^n| \leq \frac{1}{1 - \frac{\Delta z\ \Delta z(\varepsilon)}{1 - \Delta z(\varepsilon) L_f}} \left[|\Delta y_{k-1}^n| + \frac{\Delta z}{1 - \Delta z(\varepsilon) L_f} |\Delta s_{k-1}^n| \right]$$

By recursion on k, we finally get:

$$|\Delta y_k^n| \leq c_y(\Delta z, L_f, k, \varepsilon) |\Delta y_0^n| + c_s(\Delta z, L_f, k, \varepsilon) |\Delta s_0^n|,$$

for a fixed n, where c_y, c_s are constants depending on Δz, L_f, k and ε that can be uniformly bounded with respect to k by some values $\overline{c}_y$, $\overline{c}_s$, in order to have:

$$|\Delta y_k^n| \leq \overline{c}_y |\Delta y_0^n| + \overline{c}_s |\Delta s_0^n|.$$

Finally, from the boundedness of $|\Delta y_0^n|$ and $|\Delta s_0^n|$ and using induction on n , we get $|\Delta y_k^n| \leq M$, $\forall k, n$. □

We then prove the following.

Theorem 1. *There exists $c > 0$ such that, if γ in (33) belongs to $(0, c)$, then the sequence $\{w^n_{k+1,m}\}$ converges, as $m \to \infty$, to a unique limit w^n_{k+1}.*

Proof. In order to prove the claim, we set

$$G(\xi) := \frac{\Delta y^n_k}{1 - \gamma f(\xi)},$$

where

$$f(\xi) := \frac{K'_1(\xi + y^n_k)}{C_1(\xi + y^n_k)}(\xi - \Delta z),$$

and we prove that

$$|G'(\xi)| < 1, \quad \xi \in \mathbb{R}.$$

We observe that:

$$|G'(\xi)| = \gamma \left| \frac{\Delta y^n_k}{(1 - \gamma f(\xi))^2} f'(\xi) \right|,$$

and then, from Proposition 2 it follows that:

$$|G'(\xi)| \leq \gamma \frac{M}{(1 - \gamma f(\xi))^2} \|f\|,$$

where, for any function $g \in \mathscr{C}^1$, we defined $\|g\| := \|g\|_2 + \|g'\|_2$. Therefore, if the right-hand side of the equation above is strictly less than one, we obtain that $\{w^n_{k+1,m}\}$ converges to w^n_{k+1}, and also that this limit is unique. Then, we have convergence if:

$$\gamma \frac{M}{(1 - \gamma f(\xi))^2} \|f\| < 1,$$

that is, if and only if

$$\|f\|^2 \gamma^2 - (M + 2)\|f\|\gamma + 1 > 0 .$$

We observe that $\frac{M+2-\sqrt{M^2+4M}}{2\|f\|} > 0$, thus the last inequality holds if and only if

$$0 < \gamma < \frac{M + 2 - \sqrt{M^2 + 4M}}{2\|f\|} \quad \text{or} \quad \gamma > \frac{M + 2 + \sqrt{M^2 + 4M}}{2\|f\|}. \tag{39}$$

The second inequality imposes a lower bound for γ which has no numerical or physical meaning. Hence, setting $c := \frac{M+2-\sqrt{M^2+4M}}{2\|f\|}$ concludes the proof. □

Please cite this article as: Marco Berardi, Fabio Difonzo, Michele Vurro, Luciano Lopez, The 1D Richards' equation in two layered soils: a Filippov approach to treat discontinuities, Advances in Water Resources (2017), **doi: 10.1016/j.advwatres.2017.09.027**

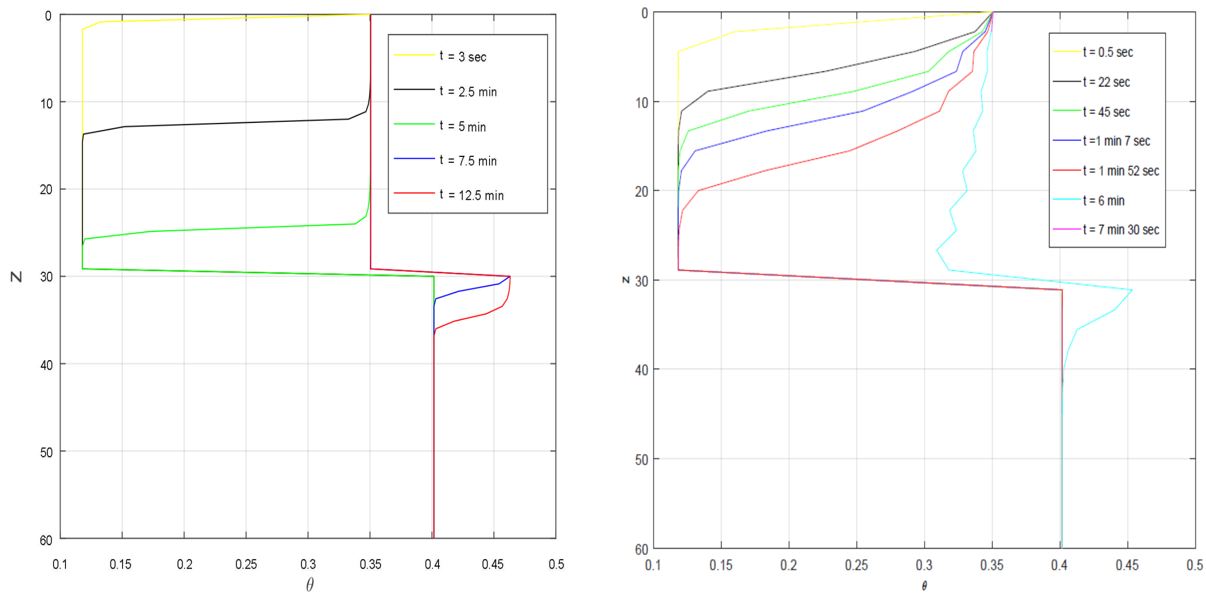
6. Results

For the simulations reported in this section, the water retention curve and the unsaturated hydraulic conductivity are defined here by means of Van Genutchen model reported in [12], i.e.

$$\theta(\psi) = \theta_r + \frac{\theta_S - \theta_r}{(1 + |\alpha\psi|^n)^m}, \quad \text{being} \quad m = 1 - \frac{1}{n}; \tag{40a}$$

$$K(\psi) = K_S \left[\frac{1}{1 + |\alpha\psi|^n}\right]^{\frac{m}{2}} \left[1 - \left(1 - \frac{1}{1 + |\alpha\psi|^n}\right)^m\right]^2, \tag{40b}$$

with the usual meaning for the parameters θ_S, θ_r, K_S, that are -respectively- the saturated water content, the residual water content, the saturated hydraulic conductivity, whereas α, n are fitting parameters.



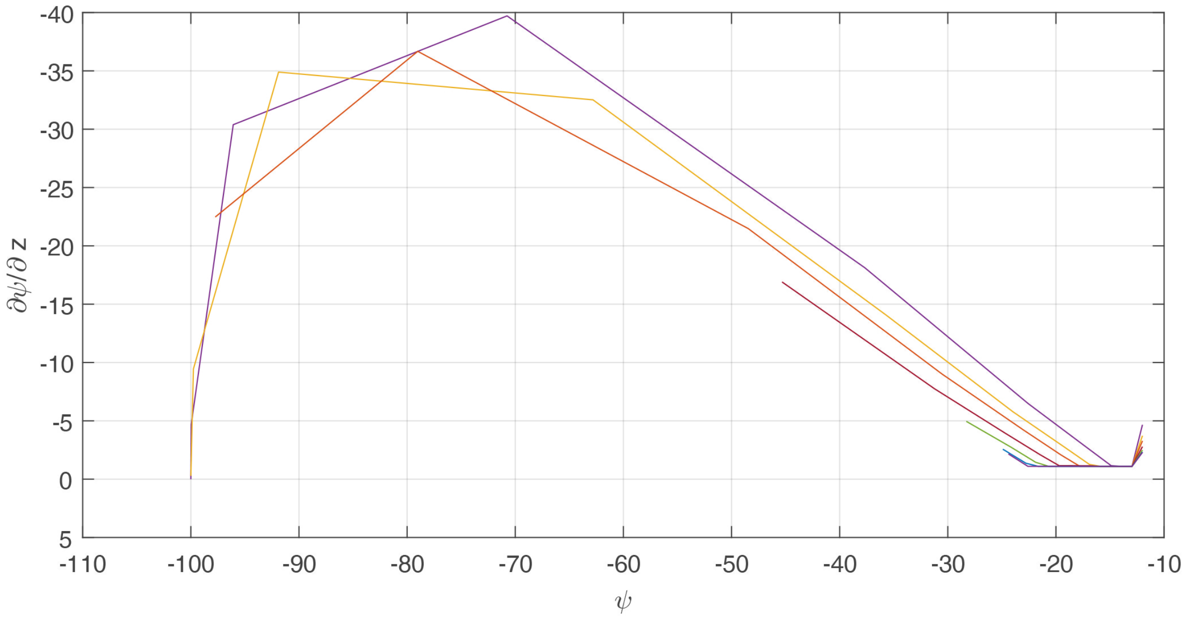
(a) (b)

Figure 2: Water content profiles at different time points: the discontinuous behavior of θ is evident at the threshold. The dynamics is faster in the upper soil, according to the greater permeability – corresponding to a greater velocity of water– in fine sands than in clay loams. The left plot is obtained by the new TMOL approach. In the right plot: the profile obtained by MOL

The different soil characteristics will be defined according to a different choice of parameters in (40). For both simulations, let us assume that for the upper soil the value of ψ at the top is identically equal to -12, and the starting profile corresponding to (3) on the vertical column is the vector

$$\left[\psi^0(z_0) \quad \psi^0(z_1) \quad \ldots \psi^0(z_{K_Z})\right]^\top = \left[-12 \quad -100 \quad \ldots \quad -100\right]^\top. \tag{41}$$

Please cite this article as: Marco Berardi, Fabio Difonzo, Michele Vurro, Luciano Lopez, The 1D Richards' equation in two layered soils: a Filippov approach to treat discontinuities, Advances in Water Resources (2017), **doi: 10.1016/j.advwatres.2017.09.027**



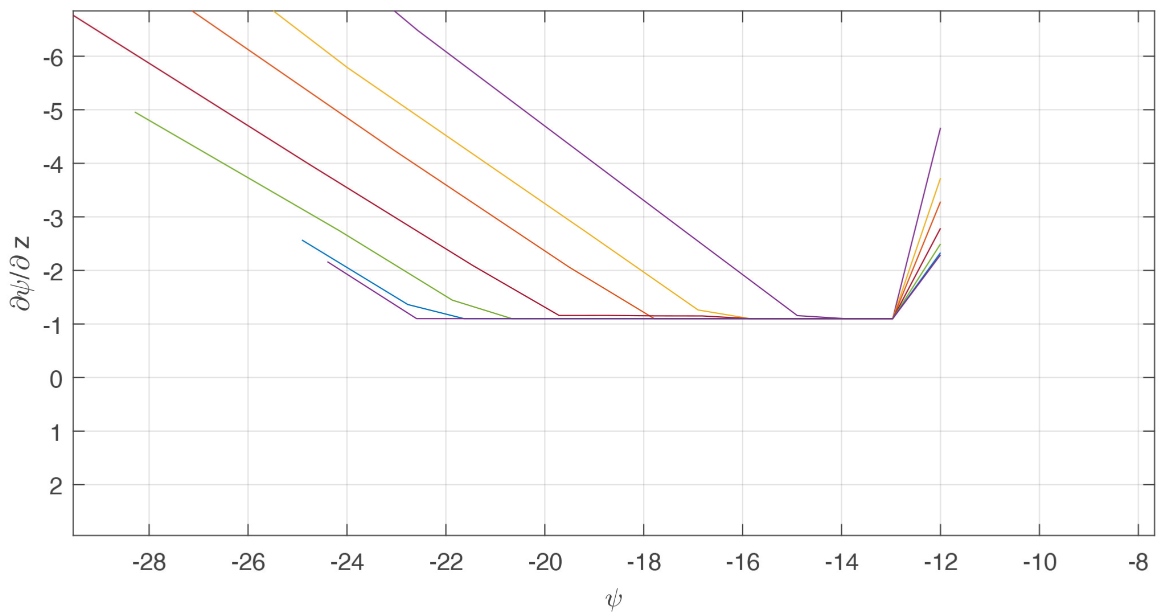


Figure 3: The sliding behavior is here evident, over the threshold $\tilde{\psi}' = -1.1$, between $\psi \approx -22.59$ and $\psi \approx -12.96$. In the lower plot, a magnification of this phenomenon. Each colored line represents one time instant, in which sliding occurs.

These conditions correspond to a situation quite close to saturation at the top of the column, and initially pretty dry in the subsoil.

In the first numerical simulation, the threshold is assumed to depend on depth, and a pair of Richards' equations is considered, as in (6), where C_i, K_i, for $i = 1, 2$ depend on

different soil characteristics, according to the choice of the hydraulic functions and to the choice of the parameters therein appearing.
In order to perform this simulation, we settled realistic initial and boundary conditions defined in (3), (4a) and (7).

Since the ψ-based form has been solved, a matrix of states has been produced for ψ, and therefore it is transformed into a matrix of states θ, according to the definition of the water retention curve; of course, the state θ is discontinuous on Σ, according to (5).

In particular, as in [12], let us set the following choice, respectively for the upper and the lower soil:

Berino loamy fine sand $\theta_r = 0.0286, \theta_S = 0.3658, \alpha = 0.0280, n = 2.2390, K_S = 541.0$ cm/d

Glendale clay loam $\theta_r = 0.1060, \theta_S = 0.4686, \alpha = 0.0104, n = 1.3954, K_S = 13.1$ cm/d.

The ψ-based form of Richards' equation has been used in all of the simulations; nevertheless, in Figure 2 the water content dynamics for the first minutes is represented. In spite of the continuity of the pressure head, by plotting the water content profiles, the discontinuous behavior of θ at $\overline{z} = 30$ cm is evident in Figure 2. The numerical method described in (30) and (32) has been performed to obtain the plot in Figure 2(a). For this method, the value of γ in (24) is settled to 0.4, with a time step worth 0.7. In Figure 2(b), a similar behavior is obtained by a classical MOL approach by integrating the same system forward in time by a semi-implicit method with low order of accuracy, with $\gamma = 0.048$, and with time-step equal to 0.045. Such a fine numerical approximation is needed for obtaining results comparable with the TMOL approach, and in spite of this, a crooked profile is showed mostly in the last times. Moreover, a harmonic average is used for choosing the hydraulic function at the interface.

In the second simulation (see Figure 3), a sliding behavior is observed, with a threshold depending on the spatial derivative of the pressure-head. In order to obtain the sliding mode, the upper soil is chosen as the Glendale clay loam described in [12]; for the lower soil, we have chosen an artificial soil, starting from the aforementioned Berino fine sand, but with a saturated hydraulic conductivity 100 times greater than the corresponding reference value in [12]. A threshold is defined for the spatial derivative of $\frac{\partial \psi}{\partial z}$: in our case, it is $\tilde{\psi}' = -1.1$, as can be seen in Figure 3. The initial state profile is the same as (41), and the states at the top are equally spaced in time between -120 and -12. The occurrence of sliding is detected by the fulfillment of condition (12). When sliding occurs, the Filippov vector field (13) is integrated by an explicit method, with a spatial integration much finer than the one used in the remaining numerical simulation. Once the sliding behavior is detected, a projection is needed onto the discontinuity surface (as, for example, in [24]), so that the numerical solution reflects the predicted sliding behavior.

7. Conclusions and future work

This paper deals with a new mathematical interpretation for the physical problem of the water infiltration into two layered soils, based on the Filippov theory for the treatment of discontinuities in ordinary differential systems. This work needs a peculiar discretization for the time derivative and the numerical solution of the corresponding ODEs system according to the spatial derivative. Here, the accurate detection of the discontinuity contributes to the stability of the numerical method.

In a future work, a challenge would be to study the Filippov approach for solving the 2D Richards' equation in multi-layered soils, because of the difficulty to express the discontinuity surface as depending on the state variable. Moreover, the study of mass balance errors of the proposed technique, as well as a hydro-geological interpretation of the sliding phenomenon, will be object of future research.

Future studies can follow very different branches. For instance, a time-lapse test can be accomplished, and a data assimilation approach (e.g, see [39, 40]) can be used for dynamically assimilating the data with the model (Richards' equation). Moreover, this type of approach could be of interest for modeling also fracture problems (see, for example, [41, 42]), or any type of preferential flows, occurring, for instance, in heterogeneous and structured soils, where a dual continuum approach is used, as in [43, 44].

References

[1] N. Romano, B. Brunone, A. Santini, Numerical analysis of one-dimensional unsaturated flow in layered soils, Advances in Water Resources 21 (4) (1998) 315–324, doi:http://dx.doi.org/10.1016/S0309-1708(96)00059-0.

[2] J. Bear, A. Verruijt, Modeling Groundwater Flow and Pollution, D. Reidel Publishing Company, 1987.

[3] K. Rathfelder, L. M. Abriola, Mass conservative numerical solutions of the head-based Richards equation, Water Resources Research 30 (1994) 2579–2586.

[4] M. D. Tocci, C. Kelley, C. T. Miller, Accurate and economical solution of the pressure-head form of Richards' equation by the method of lines, Advances in Water Resources 20 (1) (1997) 1 – 14, ISSN 0309-1708, doi:http://dx.doi.org/10.1016/S0309-1708(96)00008-5.

[5] T. Arbogast, M. Obeyesekere, M. Wheeler, Numerical methods for the simulation of flow in root-soil systems, SIAM Journal on Numerical Analysis 33 (6) (1993) 1677–1702, doi: https://doi.org/10.1137/0730086.

[6] T. Arbogast, M. Wheeler, N. Zhang, A Nonlinear Mixed Finite Element Method for a Degenerate Parabolic Equation Arising in Flow in Porous Media, SIAM Journal on Numerical Analysis 33 (4) (1996) 1669–1687, doi:10.1137/S0036142994266728.

[7] F. Radu, I. Pop, P. Knabner, Order of convergence estimates for an Euler implicit, mixed finite element discretization of Richards' equation, SIAM Journal on Numerical Analysis 42 (2004) 14521478.

[8] F. Radu, I. Pop, P. Knabner, Error estimates for a mixed nite element discretization of some degenerate parabolic equations, Numerische Mathematik 109 (2008) 285311.

[9] R. A. Klausen, F. A. Radu, G. T. Eigestad, Convergence of MPFA on triangulations and for Richards' equation, International Journal for Numerical Methods in Fluids 58 (12) (2008) 1327–1351, doi: 10.1002/fld.1787.

[10] R. Eymard, M. Gutnic, D. Hilhorst, The finite volume method for Richards equation, Computational Geosciences 3 (3-4) (1999) 259–294, doi:10.1023/A:1011547513583.

Please cite this article as: Marco Berardi, Fabio Difonzo, Michele Vurro, Luciano Lopez, The 1D Richards' equation in two layered soils: a Filippov approach to treat discontinuities, Advances in Water Resources (2017), **doi: 10.1016/j.advwatres.2017.09.027**

[11] J. R. Nimmo, Unsaturated Zone Flow Processes, John Wiley & Sons, Ltd, ISBN 9780470848944, doi:10.1002/0470848944.hsa161, URL `http://dx.doi.org/10.1002/0470848944.hsa161`, 2006.

[12] R. G. Hills, I. Porro, D. B. Hudson, P. J. Wierenga, Modeling one-dimensional infiltration into very dry soils: 1. Model development and evaluation, Water Resources Research 25 (6) (1989) 1259–1269, doi:10.1029/WR025i006p01259.

[13] C. Matthews, R. Braddock, G. Sander, Modeling Flow Through a One-Dimensional Multi-Layered Soil Profile Using the Method of Lines, Environmental Modeling & Assessment 9 (2) (2004) 103–113, doi:10.1023/B:ENMO.0000032092.10546.c6.

[14] B. Brunone, M. Ferrante, N. Romano, A. Santini, Numerical Simulations of One-Dimensional Infiltration into Layered Soils with the Richards Equation Using Different Estimates of the Interlayer Conductivity, Vadose Zone Journal 2 (2003) 193–200, doi:10.2136/vzj2003.1930.

[15] Y. Zha, J. Yang, L. Shi, X. Song, Simulating One-Dimensional Unsaturated Flow in Heterogeneous Soils with Water Content-Based Richards Equation, Vadose Zone Journal 12 (2), doi:10.2136/vzj2012.0109.

[16] C. Matthews, F. Cook, R. Knight, J.H. Braddock, Handling the water content discontinuity at the interface between layered soils within a numerical scheme, Australian Journal of Soil Research 43 (8) (2006) 945–955.

[17] S. Barontini, R. Ranzi, B. Bacchi, Water dynamics in a gradually nonhomogeneous soil described by the linearized Richards equation, Water Resources Research 43, doi:10.1029/2006WR005126.

[18] D. L. De Luca, J. M. Cepeda, Procedure to Obtain Analytical Solutions of One-Dimensional Richards Equation for Infiltration in Two-Layered Soils, Journal of Hydrologic Engineering 21 (7) (2016) 04016018, doi:10.1061/(ASCE)HE.1943-5584.0001356.

[19] M. Hayek, An analytical model for steady vertical flux through unsaturated soils with special hydraulic properties, Journal of Hydrology 527 (2015) 1153 – 1160, doi: http://dx.doi.org/10.1016/j.jhydrol.2015.06.010.

[20] A. Filippov, Differential Equations with Discontinuous Right-Hand Sides, Mathematics and Its Applications, Kluwer Academic, Dordrecht, 1988.

[21] A. Agosti, B. Giovanardi, L. Formaggia, A. Scotti, A numerical procedure for geochemical compaction in the presence of discontinuous reactions, Advances in Water Resources 94 (2016) 332 – 344, doi: http://dx.doi.org/10.1016/j.advwatres.2016.06.001.

[22] A. Agosti, L. Formaggia, A. Scotti, Analysis of a model for precipitation and dissolution coupled with a Darcy flux, Journal of Mathematical Analysis and Applications 431 (2) (2015) 752 – 781, doi: http://dx.doi.org/10.1016/j.jmaa.2015.06.003.

[23] N. Del Buono, L. Lopez, Direct event location techniques based on Adams multistep methods for discontinuous ODEs, Applied Mathematics Letters 49 (2015) 152 – 158, doi: http://dx.doi.org/10.1016/j.aml.2015.05.012.

[24] L. Dieci, L. Lopez, A survey of numerical methods for IVPs of ODEs with discontinuous right-hand side, Journal of Computational and Applied Mathematics 236 (16) (2012) 3967–3991, doi: https://doi.org/10.1016/j.cam.2012.02.011.

[25] M. di Bernardo, C. Budd, A. Champneys, P. Kowalczyk, Piecewise-smooth Dynamical Systems. Theory and Applications, Applied Mathematical Sciences 163. Springer-Verlag, Berlin, 2008.

[26] R. Casey, H. de Jong, G. J.-L., Piecewise-linear Models of Genetics Regulatory Networks: Equilibria and their Stability, Journal of Mathematical Biology 52 (2006) 27–56, doi:10.1007/s00285-005-0338-2.

[27] N. Del Buono, C. Elia, L. Lopez, On the Equivalence between the Sigmoidal Approach and Utkin's Approach for Piecewise-Linear Models of Gene Regulatory, SIAM Journal on Applied Dynamical Systems 13 (3) (2014) 1270–1292, doi:10.1137/130950483.

[28] L. Dieci, L. Lopez, Fundamental matrix solutions of piecewise smooth differential systems, Mathematics and Computers in Simulation 81 (5) (2011) 932–953, doi: http://dx.doi.org/10.1016/j.matcom.2010.10.012.

[29] M. Teixeira, P. da Silva, Regularization and singular perturbation techniques for non-smooth systems, Physica D: Nonlinear Phenomena 241 (22) (2012) 1948–1955, doi: http://dx.doi.org/10.1016/j.physd.2011.06.022.

Please cite this article as: Marco Berardi, Fabio Difonzo, Michele Vurro, Luciano Lopez, The 1D Richards' equation in two layered soils: a Filippov approach to treat discontinuities, Advances in Water Resources (2017), **doi: 10.1016/j.advwatres.2017.09.027**

[30] P. Piiroinen, Y. Kuznetsov, An event-driven method to simulate Filippov systems with accurate computing of sliding motions, ACM Transactions on Mathematical Software 34 (1–24, May, 2008) 13, doi:10.1145/1356052.1356054.

[31] M. Berardi, L. Lopez, On the continuous extension of Adams-Bashforth methods and the event location in discontinuous ODEs, Applied Mathematics Letters 25 (2012) 995–999, doi: https://doi.org/10.1016/j.aml.2011.11.014.

[32] M. Berardi, Rosenbrock-type methods applied to discontinuous differential systems, Mathematics and Computers in Simulation 95 (0) (2014) 229 – 243, doi:http://dx.doi.org/10.1016/j.matcom.2013.05.006.

[33] L. Dieci, L. Lopez, One-sided direct event location techniques in the numerical solution of discontinuous differential systems, BIT Numerical Mathematics 55 (4) (2015) 987–1003, doi:10.1007/s10543-014-0538-5.

[34] L. Lopez, S. Maset, Time-transformations for the event location in discontinuous ODEs, Mathematics of Computation doi:https://doi.org/10.1090/mcom/3305.

[35] C. Gear, O. Osterby, Solving Ordinary differential equations with discontinuities, ACM Transactions on Mathematical Software 10 (1984) 23–24, doi:10.1145/356068.356071.

[36] M. A. Celia, E. T. Bouloutas, R. L. Zarba, A general mass-conservative numerical solution for the unsaturated flow equation, Water Resources Research 26 (7) (1990) 1483–1496, ISSN 1944-7973, doi: 10.1029/WR026i007p01483, URL `http://dx.doi.org/10.1029/WR026i007p01483`.

[37] I. Pop, F. Radu, P. Knabner, Mixed finite elements for the Richards equation: linearization procedure, Journal of Computational and Applied Mathematics 168 (12) (2004) 365 – 373, doi: https://doi.org/10.1016/j.cam.2003.04.008.

[38] F. List, F. A. Radu, A study on iterative methods for solving Richards' equation, Computational Geosciences 20 (2) (2016) 341–353, doi:10.1007/s10596-016-9566-3.

[39] M. Berardi, A. Andrisani, L. Lopez, M. Vurro, A new data assimilation technique based on ensemble Kalman filter and Brownian bridges: An application to Richards equation, Computer Physics Communications 208 (2016) 43 – 53, doi:http://dx.doi.org/10.1016/j.cpc.2016.07.025.

[40] C. Paniconi, M. Marrocu, M. Putti, M. Verbunt, Newtonian nudging for a Richards equation-based distributed hydrological model, Advances in Water Resources 26 (2) (2003) 161 – 178, ISSN 0309-1708, doi:http://dx.doi.org/10.1016/S0309-1708(02)00099-4.

[41] C. Masciopinto, D. Palmiotta, Relevance of solutions to the Navier-Stokes equations for explaining groundwater flow in fractured karst aquifers, Water Resources Research 49 (2013) 1–17, doi: 10.1002/wrcr.20279.

[42] M. Caputo, L. De Carlo, C. Masciopinto, J. Nimmo, Measurement of field-saturated hydraulic conductivity on fractured rock outcrops near Altamura (Southern Italy) with an adjustable large ring infiltrometer, Environmental Earth Sciences 60 (2010) 583–590, doi:10.1007/s12665-009-0198-y.

[43] J. Dusek, T. Vogel, M. Dohnal, H. H. Gerke, Combining dual-continuum approach with diffusion wave model to include a preferential flow component in hillslope scale modeling of shallow subsurface runoff, Advances in Water Resources 44 (2012) 113 – 125, doi: http://dx.doi.org/10.1016/j.advwatres.2012.05.006.

[44] A. Coppola, H. H. Gerke, A. Comegna, A. Basile, V. Comegna, Dual-permeability model for flow in shrinking soil with dominant horizontal deformation, Water Resources Research 48 (8) (2012) 71 – 90, doi:10.1029/2011WR011376.

Please cite this article as: Marco Berardi, Fabio Difonzo, Michele Vurro, Luciano Lopez, The 1D Richards' equation in two layered soils: a Filippov approach to treat discontinuities, Advances in Water Resources (2017), **doi: 10.1016/j.advwatres.2017.09.027**