

# Carleman estimates and null controllability for a degenerate population model

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## Abstract

We deal with a degenerate model describing the dynamics of a population depending on time, on age and on space. We assume that the degeneracy can occur at the boundary or in the interior of the space domain and we focus on null controllability problem. To this aim, we prove first Carleman estimates for the associated adjoint problem, then, via cut off functions, we prove the existence of a null control function localized in the interior of the space domain.

**Resumé** Nous traitons un modèle dégénéré décrivant la dynamique d'une population en fonction du temps, de l'âge et de l'espace. Nous supposons que la dégénérescence peut se produire à la limite ou à l'intérieur du domaine spatial et nous nous concentrons sur un problème de contrôlabilité à zéro. Dans ce but, nous démontrons les premières estimations de Carleman pour le problème adjoint associé, puis, via des fonctions de coupure, nous prouvons l'existence d'une fonction de contrôle à zéro localisée à l'intérieur du domaine spatial.

*Key words:* population equations, degenerate equations, Carleman estimates, observability inequalities.

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## 1. Introduction

We consider the following linear population model describing the dynamics of a single species:

$$\begin{cases} \frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} - k(x)y_{xx} + \mu(t, a, x)y = f(t, a, x)\chi_\omega & \text{in } Q, \\ y(t, a, 1) = y(t, a, 0) = 0 & \text{on } Q_{T,A}, \\ y(0, a, x) = y_0(a, x) & \text{in } Q_{A,1}, \\ y(t, 0, x) = \int_0^A \beta(a, x)y(t, a, x)da & \text{in } Q_{T,1}, \end{cases} \quad (1)$$

in the domain  $Q := (0, T) \times (0, A) \times (0, 1)$ . Moreover,  $Q_{T,A} := (0, T) \times (0, A)$ ,  
 5  $Q_{A,1} := (0, A) \times (0, 1)$  and  $Q_{T,1} := (0, T) \times (0, 1)$ . Here  $y(t, a, x)$  is the distribution of certain individuals of age  $a \in (0, A)$  at time  $t \in (0, T)$  and location  $x \in (0, 1)$ , while  $\chi_\omega$  is the characteristic function of  $\omega \subset (0, 1)$ , which is the region where the control  $f$  acts;  $A$  is the maximal age of life, and  $\beta$  and  $\mu$  are the natural fertility and the death rate, respectively. Thus, the formula  $\int_0^A \beta y da$   
 10 denotes the distribution of newborn individuals at time  $t$  and location  $x$ . The function  $k$  is the dispersion coefficient and we assume that it depends on the space variable  $x$  and can degenerate at the boundary or in the interior of the state space.

In the last centuries, population models have been widely investigated by  
 15 many authors from many points of view (see, for example, [9], [19], [21], [28], [29], [34], [36], [37], [41], [42], [44], [45]). In particular, one of the most studied problem has been the controllability of the system. Indeed,  $y$  can represent the distribution of a *demaging insect population* or of a *pest population* (see, for example, [35]), thus it is important to control it. For example in [35], where  
 20 models an insect growth, the control corresponds to a removal of individuals by using pesticides.

However, in the cited papers, the function  $k$  is either a constant or a strictly positive function depending on  $a$ . In such cases, it is well known from the general theory that all nontrivial solutions of the corresponding system (commonly  
 25 named Lotka-McKendrick systems) are asymptotically exponentially growing

or decaying, according to the size of a certain biological quantity (the so called net reproduction rate), see [6] and also [29] for related results concerning time-independent steady states.

In this paper we are not interested in large time controllability, i.e. asymptotic behavior of the solution of (1), but we want to address the problem of null controllability at each fixed time  $T > 0$ . More precisely, we will give sufficient conditions so that, for all initial data  $y_0$  in a suitable space, there exists a control  $f$  that brings the solution  $y$  of (1) at time  $T$  at zero, i.e.

$$y(T, a, x) = 0$$

for all  $x \in (0, 1)$  and all  $a$  in a suitable subdomain of  $(0, A)$ .

Our study has obviously many connections with related ones for the heat equation. Let us recall that the null controllability for linear parabolic equations has been extensively studied in the last years using Carleman inequalities and duality argument, not only when  $k$  is a constant (see, for example, [38], [39]), but also when  $k$  degenerates at the boundary of the space domain (see, for example, [5], [13]-[18], [23], [25]) or in the interior (see, for example, [8], [11], [26], [27], [30]-[32]). As far as we know, the first controllability result for an age population dynamics model is established in [4], where the authors proved that a set of profiles is approximately reachable. Later, in [1] a local exact controllability was proved. In particular, the authors showed that, if the initial distribution is small enough, one can find a control that leads the population to extinction (see also [3] and [7]). Null controllability is also studied for *nonlinear* population dynamics models, see [3] and [43]: in the first paper the authors studied the controllability of nonlinear diffusive dynamic populations when the fertility and the mortality rates depend on the total population; in the second one, the authors considered a nonlinear distribution of newborns of the form  $F(\int_0^A \beta(t, a, x)y(t, a, x)da)$ . However, in all the previous papers the dispersion coefficient  $k$  is a constant or a strictly positive function.

To our best knowledge, [2] is the first paper where the dispersion coefficient, which depends on the space variable  $x$ , can degenerate. In particular, the au-

55 thors assume that  $k$  degenerates at the boundary (for example  $k(x) = x^\alpha$ , being  
 $x \in (0, 1)$  and  $\alpha > 0$ ). Using Carleman estimates for the adjoint problem, the  
 authors prove null controllability for (1) under the condition  $T \geq A$ . However,  
 this assumption is not realistic when  $A$  is too large. To overcome this problem  
 in [20], the authors used Carleman estimates and a fixed point method via the  
 60 Leray - Schauder Theorem. However, while in [2] and in [20], the degenerating  
 operator is in *divergence form* - shortly (Df) -, i.e.  $(k(x)y_x)_x$ , in this paper we  
 consider the degenerating operator in *nondivergence form* - shortly (NDf) - and  
 we allow the function  $k$  to degenerate not only at the boundary, but also in the  
 interior of the state space. Observe that, in the case of a boundary degeneracy,  
 65 we cannot derive the null controllability for (1) by the one of the problems in  
 divergence form. Indeed, it is proved in [13] that in this situation, i.e. when  
 the degeneracy is at the boundary of the domain, and when the functions are  
 independent of  $a$  (i.e. if we have the degenerate heat equation), the equation of  
 (1) can be rewritten as

$$\frac{\partial y}{\partial t} - (k(x)y_x)_x + k_x(x)y_x + \mu(t, x)y = f(t, x)\chi_\omega \quad (2)$$

70 at the price of adding the drift term  $k_x(x)y_x$ . Such an addition has major  
 consequences: as described in [15], degenerate equations of the form (2) are well  
 posed in  $L^2(0, 1)$  under the structural assumption

$$k_x(x) \leq C\sqrt{k(x)},$$

for a strictly positive constant  $C$ . Imposing this condition on  $k_x$ , for  $k(x) = x^\alpha$ ,  
 gives  $\alpha \geq 2$ . This necessary condition that ensures the well posedness of (1)  
 75 makes it not null controllable (see [31] for the interior degeneracy). For this  
 reason, in this paper as in [13], [14], [26], [27] or [31], we prove null controllability  
 for (1) without deducing it by the previous results for the problem in divergence  
 form. Therefore, this paper complements [2]. Indeed, we do not require as in [2],  
 that  $T \geq A$ , but  $T < A$  (see Hypothesis 4.2). Clearly, this assumption is more  
 80 interesting, since it is reasonable to control the population in small times and  
 this is important if  $y$  represents, for example, a demaging insect population or a

pest population. Moreover, while in [20] the authors used Carleman estimates and a generalization of the Leray - Schauder fixed point Theorem and the multi-valued theory, here we use only Carleman estimates for the non degenerate and  
85 the degenerate problem, and a technique based on cut off functions, making the proof slimmer and easier to read. Last but not the least, we underline that in [2] and in [20] only the case of a boundary degeneracy is considered. If the function  $k$  in (1) degenerates in the interior of  $(0, 1)$  and the problem is in divergence form, related results can be founded in [10]. To our best knowledge,  
90 as written before, this is the first paper where the problem in *nondivergence form* is considered allowing the diffusion coefficient to degenerate at the boundary or in the interior of  $(0, 1)$  (when  $y$  is independent of  $a$  we refer, for example, to [31]). We underline that in [10] the authors assume that, if  $x_0 \in (0, 1)$  is the degenerate point, the function  $k \in C[0, 1] \cap C^1([0, 1] \setminus \{x_0\})$ ; moreover, they  
95 require the existence of a constant  $M \in [0, 1)$  such that  $(x - x_0)k' \leq Mk$  a.e. in  $[0, 1]$ . In this paper, we consider a less regular function  $k$  and we allow the constant  $M$  to approach 2, i.e.  $M \in [0, 2)$ , considering the so-called strongly degenerate case.

The paper is organized in the following way: in Section 2 we study the well  
100 posedness of the problem in the case that the dispersion coefficient  $k$  degenerates either at the boundary or in the interior of the state space. Section 3 is divided into three subsections: in the first one we deduce a Carleman estimate for the non degenerate problem in *nondivergence form* by a Carleman estimate for the non degenerate problem in *divergence form* (for the reader's convenience, we give  
105 its proof in the Appendix); the second and the third subsections are devoted to study Carleman estimates in the case that  $k$  degenerates at the boundary of the state space or in its interior, respectively. Finally, in Section 4 we prove null controllability via a null controllability result for an intermediate system, observability inequalities and cut off functions.

110 A final comment on the notation: by  $c$  or  $C$  we shall denote *universal* strictly positive constants, which are allowed to vary from line to line.

## 2. Well posedness result

To study well posedness we assume that the dispersion coefficient  $k$  satisfies one of the following assumptions:

115 **Hypotheses 2.1. Boundary degenerate case (BD):**

$$k \in C([0, 1]) \quad k > 0 \text{ in } (0, 1) \text{ and } k(0) = 0 \text{ or } k(1) = 0.$$

**Hypotheses 2.2. Interior weakly degenerate case (IWD):** *There exists  $x_0 \in (0, 1)$  such that  $k(x_0) = 0$ ,  $k > 0$  on  $[0, 1] \setminus \{x_0\}$ ,  $k \in W^{1,1}(0, 1)$  and there exists  $M \in (0, 1)$  such that  $(x - x_0)k' \leq Mk$  a.e. in  $[0, 1]$ .*

**Hypotheses 2.3. Interior strongly degenerate case (ISD):** *There exists*  
 120  *$x_0 \in (0, 1)$  such that  $k(x_0) = 0$ ,  $k > 0$  on  $[0, 1] \setminus \{x_0\}$ ,  $k \in W^{1,\infty}(0, 1)$  and there exists  $M \in [1, 2)$  such that  $(x - x_0)k' \leq Mk$  a.e. in  $[0, 1]$ .*

Thus, we assume that the function  $k$  can degenerate at the boundary of the domain or at an interior point; for example, as  $k$  one can consider  $k(x) = x^\alpha$ ,  $k(x) = (1 - x)^\alpha$  or  $k(x) = |x - x_0|^\alpha$ ,  $\alpha > 0$ .

125 On the rates  $\mu$  and  $\beta$  we assume:

**Hypotheses 2.4.** *The functions  $\mu$  and  $\beta$  are such that*

$$\begin{aligned} &\bullet \beta \in C(\bar{Q}_{A,1}) \text{ and } \beta \geq 0 \text{ in } Q_{A,1}, \\ &\bullet \mu \in C(\bar{Q}) \text{ and } \mu \geq 0 \text{ in } Q. \end{aligned} \tag{3}$$

To prove well posedness of (1), we introduce, as in [13] or in [14], the following weighted Lebesgue and Hilbert spaces

$$L_{\frac{1}{k}}^2(0, 1) := \left\{ u \in L^2(0, 1) \mid \int_0^1 u^2 \frac{1}{k} dx < \infty \right\},$$

$$H_{\frac{1}{k}}^1(0, 1) := L_{\frac{1}{k}}^2(0, 1) \cap H_0^1(0, 1) \tag{4}$$

130 and

$$H_{\frac{1}{k}}^2(0, 1) := \left\{ u \in H_{\frac{1}{k}}^1(0, 1) \mid ku_{xx} \in L_{\frac{1}{k}}^2(0, 1) \right\}, \tag{5}$$

in the boundary degenerate case; while in the interior degenerate case, as in [31], we consider, in place of  $H_{\frac{1}{k}}^2(0, 1)$ , the space

$$H_{\frac{1}{k}, x_0}^2(0, 1) := \left\{ u \in H_{\frac{1}{k}}^1(0, 1) \mid u' \in H^1(0, 1) \right\},$$

that can be written in a more appealing way as

$$H_{\frac{1}{k}, x_0}^2(0, 1) := \left\{ u \in H_{\frac{1}{k}}^1(0, 1) \mid u' \in H^1(0, 1) \text{ and } ku'' \in L_{\frac{1}{k}}^2(0, 1) \right\}.$$

In every case, we consider the following norms

$$\|u\|_{\frac{1}{k}}^2 := \int_0^1 u^2 \frac{1}{k} dx,$$

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$$\|u\|_{1, \frac{1}{k}}^2 := \int_0^1 u^2 \frac{1}{k} dx + \int_0^1 u_x^2 dx$$

and

$$\|u\|_{2, \frac{1}{k}}^2 := \|u\|_{1, \frac{1}{k}}^2 + \int_0^1 ku_{xx}^2 dx.$$

Observe that, if  $k$  is nondegenerate, the spaces  $L_{\frac{1}{k}}^2(0, 1)$ ,  $H_{\frac{1}{k}}^1(0, 1)$  and  $H_{\frac{1}{k}}^2(0, 1)$  (or  $H_{\frac{1}{k}, x_0}^2(0, 1)$ ) coincide, respectively, with  $L^2(0, 1)$ ,  $H_0^1(0, 1)$  and  $H^2(0, 1) \cap H_0^1(0, 1)$ .

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Denoting by  $\mathcal{H}_{\frac{1}{k}}^2(0, 1)$  the space  $H_{\frac{1}{k}}^2(0, 1)$  or  $H_{\frac{1}{k}, x_0}^2(0, 1)$ , we have, as in [13], [14] or [31], that the operator

$$\mathcal{A}_0 u := ku_{xx}, \quad D(\mathcal{A}_0) := \mathcal{H}_{\frac{1}{k}}^2(0, 1)$$

is self-adjoint, nonpositive and generates an analytic contraction semigroup of angle  $\pi/2$  on the space  $L_{\frac{1}{k}}^2(0, 1)$ .

Now, setting  $\mathcal{A}_a u := \frac{\partial u}{\partial a}$ , we have that

$$\mathcal{A}u := \mathcal{A}_a u - \mathcal{A}_0 u,$$

145 for

$$u \in D(\mathcal{A}) = \left\{ u \in L^2(0, A; D(\mathcal{A}_0)) : \frac{\partial u}{\partial a} \in L^2(0, A; H_{\frac{1}{k}}^1(0, 1)), u(0, x) = \int_0^A \beta(a, x) u(a, x) da \right\},$$

generates a strongly continuous semigroup on  $L^2(0, A) \times L^2_{\frac{1}{k}}(0, 1)$  (see also [7]).

Moreover, the operator  $B(t)$  defined as

$$B(t)u := -\mu(t, a, x)u,$$

for  $u \in D(\mathcal{A})$ , can be seen as a bounded perturbation of  $\mathcal{A}$  (see, for example, [5]); thus also  $(\mathcal{A} + B(t), D(\mathcal{A}))$  generates a strongly continuous semigroup.

150 Setting  $L^2_{\frac{1}{k}}(Q) := L^2(Q_{T,A}) \times L^2_{\frac{1}{k}}(0, 1)$  and  $L^2_{\frac{1}{k}}(Q_{A,1}) := L^2(0, A) \times L^2_{\frac{1}{k}}(0, 1)$ , the following well posedness result holds (see [22], [40]):

**Theorem 2.1.** *Assume that Hypotheses 2.4 and one among Hypothesis 2.1 - 2.3 are satisfied. For all  $f \in L^2_{\frac{1}{k}}(Q)$  and  $y_0 \in L^2_{\frac{1}{k}}(Q_{A,1})$ , the system (1) admits a unique solution  $y \in \mathcal{U} := C([0, T]; L^2_{\frac{1}{k}}(Q_{A,1})) \cap L^2(0, T; H^1(0, A) \times H^1_{\frac{1}{k}}(0, 1))$ .*

155 *In addition, if  $f \equiv 0$ ,  $u \in C^1([0, T]; L^2_{\frac{1}{k}}(Q_{A,1}))$ .*

### 3. Carleman estimates

In this section we show Carleman estimates for the following system:

$$\begin{cases} \frac{\partial z}{\partial t} + \frac{\partial z}{\partial a} + k(x)z_{xx} - \mu(t, a, x)z = f, & (t, a, x) \in Q, \\ z(t, a, 0) = z(t, a, 1) = 0, & (t, a) \in Q_{T,A}, \\ z(t, A, x) = 0, & (t, x) \in Q_{T,1}, \end{cases} \quad (6)$$

where the function  $k$  is non degenerate (this will be crucial for the following) or satisfies one of Hypothesis 2.1, 2.2 or 2.3.

160 *Carleman inequalities in the non degenerate case.* First of all assume that  $k$  is non degenerate. Then, the following estimate holds:

**Theorem 3.1.** *Let  $z \in \mathcal{V} := L^2(Q_{T,A}; H^2(0, 1) \cap H^1_0(0, 1)) \cap H^1(Q_{T,A}; H^1_0(0, 1))$  be the solution of (6) where  $f \in L^2(Q)$  and  $k \in C^1([0, 1])$  is a strictly positive function. Then, there exist two strictly positive constants  $C$  and  $s_0$ , such that,*

165 *for any  $s \geq s_0$ ,  $z$  satisfies the estimate*

$$\int_Q (s^3 \phi^3 z^2 + s \phi z_x^2) e^{2s\Phi} dx da dt \leq C \left( \int_Q f^2 e^{2s\Phi} dx da dt - s\kappa \int_0^T \int_0^A [ke^{2s\Phi} \phi(z_x)^2]_{x=0}^{x=1} da dt \right). \quad (7)$$



Here the functions  $\phi$  and  $\Phi$  are defined as follows

$$\begin{aligned}\phi(t, a, x) &= \Theta(t, a)e^{\kappa\sigma(x)}, \quad \Theta(t, a) = \frac{1}{t^4(T-t)^4a^4}, \\ \Phi(a, t, x) &= \Theta(t, a)\Psi(x), \quad \Psi(x) = e^{\kappa\sigma(x)} - e^{2\kappa\|\sigma\|_\infty},\end{aligned}\tag{8}$$

$(t, a, x) \in Q$ ,  $\kappa > 0$  and  $\sigma(x) := \mathfrak{d} \int_x^1 \frac{1}{k(t)} dt$ , where  $\mathfrak{d} = \|k'\|_{L^\infty(0,1)}$ .

The proof of the previous result is based on the next Carleman estimate which is proved in the Appendix.

**Theorem 3.2.** *Let  $z \in \mathcal{V}$  be the solution of*

$$\begin{cases} \frac{\partial z}{\partial t} + \frac{\partial z}{\partial a} + (k(x)z_x)_x - \mu(t, a, x)z = f, & (t, x, a) \in Q, \\ z(t, a, 0) = z(t, a, 1) = 0, & (t, a) \in Q_{T,A}, \\ z(t, A, x) = 0, & (t, x) \in Q_{T,1}, \end{cases}\tag{9}$$

where  $f$  and  $k$  are as in the previous theorem. Then, there exist two strictly positive constants  $C$  and  $s_0$ , such that, for any  $s \geq s_0$ ,  $z$  satisfies the estimate

$$\int_Q (s^3\phi^3z^2 + s\phi z_x^2)e^{2s\Phi} dx dadt \leq C \left( \int_Q f^2 e^{2s\Phi} dx dadt - s\kappa \int_0^T \int_0^A [ke^{2s\Phi}\phi(z_x)^2]_{x=0}^{x=1} dadt \right),\tag{10}$$

with  $\phi$  and  $\Phi$  defined as in (8).

**PROOF (PROOF OF THEOREM 3.1).** Rewrite the equation of (6) as  $\frac{\partial z}{\partial t} + \frac{\partial z}{\partial a} + (k(x)z_x)_x - \mu(t, a, x)z = \bar{f}$ , where  $\bar{f} := f + k'z_x$ . Then, applying Theorem 3.2, there exist two strictly positive constants  $C$  and  $s_0 > 0$ , such that, for all  $s \geq s_0$ ,

$$\int_Q (s^3\phi^3z^2 + s\phi z_x^2)e^{2s\Phi} dx dadt \leq C \left( \int_Q \bar{f}^2 e^{2s\Phi} dx dadt - s\kappa \int_0^T \int_0^A [ke^{2s\Phi}\phi(z_x)^2]_{x=0}^{x=1} dadt \right).\tag{11}$$

Using the definition of  $\bar{f}$ , the term  $\int_Q \bar{f}^2 e^{2s\Phi(t,x)} dx dadt$  can be estimated in the following way

$$\begin{aligned}\int_Q \bar{f}^2 e^{2s\Phi} dx dadt &\leq 2 \int_Q f^2 e^{2s\Phi} dx dadt + 2\|k'\|_{L^\infty(0,1)}^2 \int_Q e^{2s\Phi} (z_x)^2 dx dadt \\ &\leq 2 \int_Q f^2 e^{2s\Phi} dx dadt + 2\|k'\|_{L^\infty(0,1)}^c \int_Q \Theta e^{\kappa\sigma} e^{2s\Phi} (z_x)^2 dx dadt,\end{aligned}\tag{12}$$

180 where  $c := A^4 \max_{[0,T]}(t(T-t))^4 = A^4 \left(\frac{T}{2}\right)^8$ . Thus, by (11) and (12), one has

$$\begin{aligned} & \int_Q \left( s^3 \phi^3 z^2 + s \phi z_x^2 - 2 \|k'\|_{L^\infty(0,1)}^2 c \phi z_x^2 \right) e^{2s\Phi} dx dadt \\ & \leq C \left( \int_Q f^2 e^{2s\Phi} dx dadt - s \kappa \int_0^T \int_0^A [k e^{2s\Phi} \phi(z_x)^2]_{x=0}^{x=1} dadt \right). \end{aligned}$$

Now, let  $s_1 > 0$  be such that  $\frac{s_1}{2} \geq 2 \|k'\|_{L^\infty(0,1)}^2 c$ . Then, for all  $s \geq s_1$

$$\int_Q \left( s \phi z_x^2 - 2 \|k'\|_{L^\infty(0,1)}^2 c \phi z_x^2 \right) e^{2s\Phi} dx dadt \geq \frac{s}{2} \int_Q \phi z_x^2 e^{2s\Phi} dx dadt.$$

Hence the claim follows for all  $s \geq \max\{s_0, s_1\}$ .

Actually we can prove Theorem 3.1 directly, but we have to assume on  $k$  more regularity, for example  $k \in C^2[0, 1]$  or, at least,  $k \in W^{2,\infty}(0, 1)$ . Indeed, in this

185 case, we have to estimate an integral containing the term  $(k\Phi_{xx})_x$ .

**Remark 3.1.** *The previous Theorems still hold under the weaker assumption  $k \in W^{1,\infty}(0, 1)$  without any additional assumption.*

*On the other hand, if we require  $k \in W^{1,1}(0, 1)$  then we have to add the following hypothesis: there exist two functions  $\mathfrak{g} \in L^1(0, 1)$ ,  $\mathfrak{h} \in W^{1,\infty}(0, 1)$  and two*

190 *strictly positive constants  $\mathfrak{g}_0, \mathfrak{h}_0$  such that  $\mathfrak{g}(x) \geq \mathfrak{g}_0$  and*

$$-\frac{k'(x)}{2\sqrt{k(x)}} \left( \int_x^1 \mathfrak{g}(t) dt + \mathfrak{h}_0 \right) + \sqrt{k(x)} \mathfrak{g}(x) = \mathfrak{h}(x) \quad \text{for a.e. } x \in [0, 1],$$

*in the divergence case,*

$$\frac{k'(x)}{2\sqrt{k(x)}} \left( \int_x^1 \mathfrak{g}(t) dt + \mathfrak{h}_0 \right) + \sqrt{k(x)} \mathfrak{g}(x) = \mathfrak{h}(x) \quad \text{for a.e. } x \in [0, 1],$$

*in the nondivergence one.*

*In this case, i.e. if  $k \in W^{1,1}(0, 1)$ , the function  $\Psi$  in (8) becomes*

$$\Psi(x) := -r \left[ \int_0^x \frac{1}{\sqrt{k(t)}} \int_t^1 \mathfrak{g}(s) ds dt + \int_0^x \frac{\mathfrak{h}_0}{\sqrt{k(t)}} dt \right] - \mathfrak{c}, \quad (13)$$

*where  $r$  and  $\mathfrak{c}$  are suitable strictly positive functions.*

195 Thus we have the next theorem

### Hypotheses 3.1.

(a<sub>1</sub>)  $k \in W^{1,1}(0,1)$ , and there exist two functions  $\mathfrak{g} \in L^1(0,1)$ ,  $\mathfrak{h} \in W^{1,\infty}(0,1)$  and two strictly positive constants  $\mathfrak{g}_0, \mathfrak{h}_0$  such that  $\mathfrak{g}(x) \geq \mathfrak{g}_0$  and

$$-\frac{k'(x)}{2\sqrt{k(x)}} \left( \int_x^1 \mathfrak{g}(t)dt + \mathfrak{h}_0 \right) + \sqrt{k(x)}\mathfrak{g}(x) = \mathfrak{h}(x) \quad \text{for a.e. } x \in [0,1],$$

200 in the (Df) case,

$$\frac{k'(x)}{2\sqrt{k(x)}} \left( \int_x^1 \mathfrak{g}(t)dt + \mathfrak{h}_0 \right) + \sqrt{k(x)}\mathfrak{g}(x) = \mathfrak{h}(x) \quad \text{for a.e. } x \in [0,1],$$

in the (Ndf) one, or

(a<sub>2</sub>)  $k \in W^{1,\infty}(0,1)$ .

Define  $\Phi(t, a, x)$ ,  $\phi(t, a, x)$ ,  $\Theta(t, a)$  and  $\sigma$  as in (8) and

$$\Psi(x) := \begin{cases} -r \left[ \int_0^x \frac{1}{\sqrt{k(t)}} \int_t^1 \mathfrak{g}(s)dsdt + \int_0^x \frac{\mathfrak{h}_0}{\sqrt{k(t)}}dt \right] - \mathfrak{c}, & \text{if } (a_1) \text{ holds,} \\ e^{r\sigma(x)} - \mathfrak{c}, & \text{if } (a_2) \text{ holds,} \end{cases} \quad (14)$$

205 where  $r > 0$  and  $\mathfrak{c} > 0$  is chosen in the second case in such a way that  $\max_{[0,1]} \Psi < 0$ .

**Theorem 3.3.** Assume that Hypothesis 3.1 is satisfied. Let  $z \in \mathcal{V}$  be the solution of (6) or of (9) where  $f \in L^2(Q)$ . Then, there exist two strictly positive constants  $C$  and  $s_0$ , such that, for any  $s \geq s_0$ ,  $z$  satisfies the estimate

$$\int_Q (s\Theta(z_x)^2 + s^3\Theta^3 z^2) e^{2s\Phi} dx dadt \leq C \left( \int_Q f^2 e^{2s\Phi} dx dadt - (B.T.) \right), \quad (15)$$

210 where

$$(B.T.) := \begin{cases} sr \int_0^T \int_0^A \Theta(t) \left[ \sqrt{k} \left( \int_x^1 \mathfrak{g}(\tau)d\tau + \mathfrak{h}_0 \right) (z_x)^2 e^{2s\Phi} \right]_{x=0}^{x=1} dadt, & \text{in the (Ndf),} \\ sr \int_0^T \int_0^A \left[ k^{3/2} e^{2s\Phi} \Theta \left( \int_x^1 \mathfrak{g}(\tau)d\tau + \mathfrak{h}_0 \right) (z_x)^2 \right]_{x=0}^{x=1} dadt, & \text{in the (Df),} \end{cases}$$

if  $(a_1)$  holds and

$$\int_Q (s\Theta e^{r\sigma}(z_x)^2 + s^3\Theta^3 e^{3r\sigma} z^2) e^{2s\Phi} dx dadt \leq C \left( \int_Q f^2 e^{2s\Phi} dx dadt - (B.T.) \right), \quad (16)$$

where  $(B.T.) := sr \int_0^T \int_0^A [ke^{2s\Phi} \Theta e^{r\sigma}(z_x)^2]_{x=0}^{x=1} dadt$ , if  $(a_2)$  is in force.

(See the Appendix for the proof.)

*Carleman inequalities when the degeneracy is at the boundary.* In this subsection we will consider the case when  $k(0) = 0$  or  $k(1) = 0$ . In both cases we assume that  $\mu$  satisfies (3). On the other hand, on  $k$  we make different assumptions:

**Hypotheses 3.2.** The function  $k \in C^0[0, 1] \cap C^2(0, 1]$  is such that  $k(0) = 0$ ,  $k > 0$  on  $(0, 1]$  and there exist  $\varepsilon \in (0, 1]$  and  $M \in (0, 2)$  such that the function  $\frac{xk_x}{k(x)} \in L^\infty(0, \varepsilon)$ ,  $\frac{xk_x(x)}{k(x)} \leq M$  and  $\left( \frac{xk_x(x)}{k(x)} \right)_x \in L^\infty(0, \varepsilon)$ .

**Hypotheses 3.3.** The function  $k \in C^0[0, 1] \cap C^2[0, 1)$  is such that  $k(1) = 0$ ,  $k > 0$  on  $(0, 1)$  and there exist  $\varepsilon \in (0, 1]$  and  $M \in (0, 2)$  such that the function  $\frac{(x-1)k_x}{k(x)} \in L^\infty(1-\varepsilon, 1)$ ,  $\frac{(x-1)k_x(x)}{k(x)} \leq M$  and  $\left( \frac{(x-1)k_x(x)}{k(x)} \right)_x \in L^\infty(1-\varepsilon, 1)$ .

Now, let us introduce the weight functions

$$\varphi(t, a, x) := \Theta(t, a)(p(x) - 2\|p\|_{L^\infty(0,1)}), \quad (17)$$

and

$$\bar{\varphi}(t, a, x) := \Theta(t, a)(\bar{p}(x) - 2\|\bar{p}\|_{L^\infty(0,1)}), \quad (18)$$

where  $\Theta$  is as in (8),  $p(x) := \int_0^x \frac{y}{k(y)} e^{Ry^2} dy$  and  $\bar{p}(x) := \int_0^x \frac{y-1}{k(y)} e^{R(y-1)^2} dy$ , with  $R > 0$ , if  $k$  satisfies Hypothesis 3.2 or Hypothesis 3.3, respectively. Observe that  $\varphi(t, a, x), \bar{\varphi}(t, a, x) < 0$  for all  $(t, x) \in Q$  and  $\varphi(t, a, x), \bar{\varphi}(t, a, x) \rightarrow -\infty$  as  $t \rightarrow 0^+, T^-$  or  $a \rightarrow 0^+$ . The following estimates hold:

**Theorem 3.4.** Assume that Hypothesis 3.2 is satisfied for some  $\varepsilon \in (0, 1]$ . Then, there exist two strictly positive constants  $C$  and  $s_0$  such that every solution  $v$  of (6) in

$$\mathcal{V}_1 := L^2(Q_{T,A}; H_{\frac{1}{k}}^2(0, 1)) \cap H^1(Q_{T,A}; H_{\frac{1}{k}}^1(0, 1))$$

satisfies, for all  $s \geq s_0$ ,

$$\begin{aligned} \int_Q \left( s\Theta v_x^2 + s^3\Theta^3 \left( \frac{x}{k} \right)^2 v^2 \right) e^{2s\varphi} dx dadt &\leq C \int_Q f^2 \frac{e^{2s\varphi}}{k} dx dadt \\ &+ sC \int_0^T \int_0^A \Theta(t, a) \left[ x v_x^2 e^{2s\varphi} \right] (t, a, 1) dadt. \end{aligned}$$

**Theorem 3.5.** Assume that Hypothesis 3.3 is satisfied for some  $\varepsilon \in (0, 1]$ .

235 Then, there exist two strictly positive constants  $C$  and  $s_0$  such that every solution  $v$  of (6) in  $\mathcal{V}_1$  satisfies, for all  $s \geq s_0$ ,

$$\begin{aligned} \int_Q \left( s\Theta v_x^2 + s^3\Theta^3 \left( \frac{x-1}{k} \right)^2 v^2 \right) e^{2s\bar{\varphi}} dx dadt &\leq C \int_Q f^2 \frac{e^{2s\bar{\varphi}}}{k} dx dadt \\ &+ sC \int_0^T \int_0^A \Theta(t, a) \left[ (1-x) v_x^2 e^{2s\bar{\varphi}} \right] (t, a, 0) dadt. \end{aligned}$$

Clearly the previous Carleman estimates hold for every function  $v$  that satisfies (6) in  $(0, T) \times (0, A) \times (0, B)$  or  $(0, T) \times (0, A) \times (B, 1)$  as long as  $(0, 1)$  is substituted by  $(0, B)$  or  $(B, 1)$  and  $k$  satisfies Hypothesis 3.2 in  $(0, B)$  or Hypothesis 240 3.3 in  $(B, 1)$ , respectively.

**Remark 3.2.** Observe that Theorems 3.4 and 3.5 improve [13, Theorems 3.3. and 3.4] and [14, Theorem 3 and 4]. Indeed, here we assume that  $k$  is of class  $C^2(0, 1]$  (or  $C^2[0, 1)$ ) and not  $C^3(0, 1]$  (or  $C^3[0, 1)$ ) as therein, where  $y$  was independent of  $a$ .

245 In the following, we will prove only Theorem 3.4 since the proof of Theorem 3.5 is analogous.

*Proof of Theorem 3.4.* As a first step assume that  $\mu \equiv 0$ .

In order to prove Theorem 3.4, we define, for  $s > 0$ , the function

$$w(t, a, x) := e^{s\varphi(t, a, x)} v(t, a, x)$$

where  $v$  is the solution of (6) in  $\mathcal{V}_1$ ; observe that, since  $v \in \mathcal{V}_1$ ,  $w \in \mathcal{V}_1$ . Clearly,

250 one has that  $w$  satisfies

$$\begin{cases} (e^{-s\varphi}w)_t + (e^{-s\varphi}w)_a + k(x)(e^{-s\varphi}w)_{xx} = f(t, a, x), & (t, x) \in Q, \\ w(0, a, x) = w(T, a, x) = 0, & (a, x) \in Q_{A,1}, \\ w(t, A, x) = w(t, 0, x) = 0, & (t, x) \in Q_{T,1}, \\ w(t, a, 0) = w(t, a, 1) = 0, & (t, a) \in Q_{T,A}. \end{cases} \quad (19)$$

Defining  $Lw := w_t + w_a + kw_{xx}$  and  $L_s w := e^{s\varphi}L(e^{-s\varphi}w)$ , the equation of (19) can be recast as follows

$$L_s w = L_s^+ w + L_s^- w = e^{s\varphi} f,$$

where

$$\begin{cases} L_s^+ w := kw_{xx} - s(\varphi_t + \varphi_a)w + s^2 k \varphi_x^2 w, \\ L_s^- w := w_t + w_a - 2sk\varphi_x w_x - sk\varphi_{xx}w. \end{cases}$$

255 Moreover, set  $\langle u, v \rangle_{L^2_{\frac{1}{k}}(Q)} := \int_Q uv \frac{1}{k} dx dadt$ , one has

$$\|L_s^+ w\|_{L^2_{\frac{1}{k}}(Q)}^2 + \|L_s^- w\|_{L^2_{\frac{1}{k}}(Q)}^2 + 2\langle L_s^+ w, L_s^- w \rangle_{L^2_{\frac{1}{k}}(Q)} = \|fe^{s\varphi}\|_{L^2_{\frac{1}{k}}(Q)}^2. \quad (20)$$

Now, we compute the inner product  $\langle L_s^+ w, L_s^- w \rangle_{L^2_{\frac{1}{k}}(Q)}$  whose first expression is given in the following lemma

**Lemma 3.1.** *Assume Hypothesis 3.2. The following identity holds*

$$\begin{aligned} \langle L_s^+ w, L_s^- w \rangle_{L^2_{\frac{1}{k}}(Q)} &= s \int_Q (k\varphi_{xx} + (k\varphi_x)_x) w_x^2 dx dadt \\ &\quad + s^3 \int_Q \varphi_x^2 (k\varphi_{xx} + (k\varphi_x)_x) w^2 dx dadt \\ &\quad - 2s^2 \int_Q \varphi_x \varphi_{xt} w^2 dx dadt + \frac{s}{2} \int_Q \frac{\varphi_{tt} + \varphi_{aa}}{k} w^2 dx dadt \\ &\quad + s \int_{Q_T} (k\varphi_{xx})_x w w_x dx dadt \\ &\quad + s \int_Q \frac{\varphi_{ta}}{k} w^2 dx dadt - 2s^2 \int_Q \varphi_x \varphi_{xa} w^2 dx dadt \end{aligned} \left. \vphantom{\int_Q} \right\} \{D.T.\} \quad (21)$$

$$\{B.T.\} \left\{ \begin{array}{l} -\frac{1}{2} \int_0^A \int_0^1 [w_x^2]_0^T dx da + \int_0^T \int_0^A [w_x(w_t + w_a)]_0^1 dadt \\ -s \int_0^T \int_0^A [k\varphi_x w_x^2]_0^1 dadt \\ -s \int_0^T \int_0^A [k\varphi_{xx} w w_x]_0^1 dadt + \frac{1}{2} \int_0^A \int_0^1 \left[ (s^2 \varphi_x^2 - s \frac{\varphi_t + \varphi_a}{k}) w^2 \right]_0^T dx da \\ -s \int_0^T \int_0^A \left[ (s^2 k \varphi_x^3 - s \varphi_x \varphi_t - s \varphi_x \varphi_a) w^2 \right]_0^1 dadt \\ -\frac{1}{2} \int_0^T \int_0^1 [w_x^2]_0^A dx dt + \frac{1}{2} \int_0^T \int_0^1 \left[ (s^2 \varphi_x^2 - s \frac{\varphi_t + \varphi_a}{k}) w^2 \right]_0^A dx dt. \end{array} \right.$$

260 PROOF. It results, integrating by parts,

$$\langle L_s^+ w, L_s^- w \rangle_{L^2_{\frac{1}{k}}(Q)} = I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 = \int_Q w_{xx}(w_t - 2sk\varphi_x w_x - sk\varphi_{xx} w) dx dadt,$$

$$I_2 = \int_Q \frac{1}{k} (-s\varphi_t w + s^2 k \varphi_x^2 w) (w_t - 2sk\varphi_x w_x - sk\varphi_{xx} w) dx dadt,$$

$$I_3 = \int_Q (w_{xx} - s \frac{(\varphi_t + \varphi_a)}{k} w + s^2 \varphi_x^2 w) w_a dx dadt$$

and

$$I_4 = -s \int_Q \frac{\varphi_a w}{k} (w_t - 2sk\varphi_x w_x - sk\varphi_{xx} w) dx dadt.$$

265 By several integrations by parts in space and in time (see [13] or [14]), we get

$$\begin{aligned}
I_1 + I_2 = & s \int_Q (k\varphi_{xx} + (k\varphi_x)_x) w_x^2 dx dadt \\
& + s^3 \int_Q \varphi_x^2 (k\varphi_{xx} + (k\varphi_x)_x) w^2 dx dadt \\
& - 2s^2 \int_Q \varphi_x \varphi_{xt} w^2 dx dadt + \frac{s}{2} \int_Q \frac{\varphi_{tt}}{k} w^2 dx dadt \\
& + s \int_{Q_T} (k\varphi_{xx})_x w w_x dx dadt \\
& - \frac{1}{2} \int_0^A \int_0^1 [w_x^2]_0^T dx da + \int_0^T \int_0^A [w_x w_t]_0^1 dadt \\
& - s \int_0^T \int_0^A [a\varphi_x w_x^2]_0^1 dadt \\
& - s \int_0^T \int_0^A [a\varphi_{xx} w w_x]_0^1 dadt + \frac{1}{2} \int_0^A \int_0^1 \left[ (s^2 \varphi_x^2 - s \frac{\varphi_t}{a}) w^2 \right]_0^T dx da \\
& - s \int_0^T \int_0^A \left[ (s^2 a \varphi_x^3 - s \varphi_x \varphi_t) w^2 \right]_0^1 dadt.
\end{aligned} \tag{22}$$

Next, we compute  $I_3$  and  $I_4$

$$\begin{aligned}
I_3 = & - \int_Q w_{xa} w_x dx dadt + \int_0^T \int_0^A [w_x w_a]_0^1 dadt \\
& + \int_Q (s^2 \varphi_x^2 - s \frac{\varphi_t + \varphi_a}{k}) w w_a dx dadt \\
= & - \frac{1}{2} \int_0^T \int_0^1 [w_x^2]_0^A dx dt + \int_0^T \int_0^A [w_x w_a]_0^1 dadt + \frac{1}{2} \int_0^T \int_0^1 [(s^2 \varphi_x^2 - s \frac{\varphi_t + \varphi_a}{a}) w^2]_0^A dx dt \\
& + \frac{1}{2} \int_Q \left( -s^2 \varphi_x^2 + s \frac{\varphi_t + \varphi_a}{k} \right)_a w^2 dx dadt \\
= & - \frac{1}{2} \int_0^T \int_0^1 [w_x^2]_0^A dx dt + \int_0^T \int_0^A [w_x w_a]_0^1 dadt + \frac{1}{2} \int_0^T \int_0^1 [(s^2 \varphi_x^2 - s \frac{\varphi_t + \varphi_a}{a}) w^2]_0^A dx dt \\
& + \frac{s}{2} \int_Q \frac{\varphi_{aa}}{k} w^2 dx dadt + \frac{s}{2} \int_Q \frac{\varphi_{ta}}{k} w^2 dx dadt - s^2 \int_Q \varphi_x \varphi_{xa} w^2 dx dadt.
\end{aligned} \tag{23}$$



On the other hand

$$\begin{aligned}
I_4 &= -s \int_Q \frac{\varphi_a w w_t}{k} dx dadt + 2s^2 \int_Q \varphi_x \varphi_a w w_x dx dadt + s^2 \int_Q \varphi_a \varphi_{xx} w^2 dx dadt \\
&= -\frac{s}{2} \int_Q \frac{\varphi_a}{k} (w^2)_t dx dadt + s^2 \int_Q \varphi_x \varphi_a (w^2)_x dx dadt + s^2 \int_Q \varphi_a \varphi_{xx} w^2 dx dadt \\
&= \frac{s}{2} \int_Q \frac{\varphi_{at}}{k} w^2 dx dadt - s^2 \int_Q (\varphi_x \varphi_a)_x w^2 dx dadt + s^2 \int_Q \varphi_a \varphi_{xx} w^2 dx dadt \\
&\quad - \frac{s}{2} \int_0^A \int_0^1 \left[ \frac{\varphi_a}{k} w^2 \right]_0^T dx da + s^2 \int_0^T \int_0^A [\varphi_x \varphi_a w^2]_0^1 dadt \\
&= \frac{s}{2} \int_Q \frac{\varphi_{at}}{k} w^2 dx dadt - s^2 \int_Q \varphi_x \varphi_{ax} w^2 dx dadt \\
&\quad - \frac{s}{2} \int_0^A \int_0^1 \left[ \frac{\varphi_a}{k} w^2 \right]_0^T dx da + s^2 \int_0^T \int_0^A [\varphi_x \varphi_a w^2]_0^1 dadt.
\end{aligned} \tag{24}$$

Adding (22) - (24), (21) follows immediately.

The next lemma holds.

270 **Lemma 3.2.** *Assume Hypothesis 3.2. The boundary terms in (21) become*

$$\{B.T.\} = -s e^R \int_0^T \int_0^A \Theta(t, a) w_x^2(t, a, 1) dadt. \tag{25}$$

The proof is based on the next result:

**Lemma 3.3.** *For all  $\gamma \geq M$  the map  $x \mapsto \frac{x^\gamma}{k}$  is nondecreasing in  $(0, 1]$  so that  $\lim_{x \rightarrow 0} \frac{x^\gamma}{k} = 0$  for all  $\gamma > M$ .*

PROOF (PROOF OF LEMMA 3.2). Using the definition of  $\varphi$  and [13, Lemma

275 3.9], the boundary terms of  $\langle L_s^+ w, L_s^- w \rangle_{L^2_{\frac{1}{k}}(Q)}$  become

$$\begin{aligned}
\{B.T.\} &= -se^R \int_0^T \int_0^A \Theta(t) w_x^2(t, a, 1) dadt + \int_0^T \int_0^A [w_x w_a]_0^1 dadt \\
&\quad - \frac{s}{2} \int_0^A \int_0^1 \left[ \frac{\varphi_a}{k} w^2 \right]_0^T dx da + s^2 \int_0^T \int_0^A [\varphi_x \varphi_a w^2]_0^1 dadt \\
&\quad - \frac{1}{2} \int_0^T \int_0^1 [w_x^2]_0^A dx dt - \frac{s}{2} \int_0^T \int_0^1 \left[ \frac{\varphi_a}{k} w^2 \right]_0^A dx dt \\
&= -se^R \int_0^T \int_0^A \Theta(t) w_x^2(t, a, 1) dadt - \frac{1}{2} \int_0^T \int_0^1 [w_x^2]_0^A dx dt + \int_0^T \int_0^A [w_x w_a]_0^1 dadt \\
&\quad - \frac{s}{2} \int_0^A \int_0^1 \left[ \Theta_a \frac{p(x) - 2\|p\|_{L^\infty(0,1)}}{k} w^2 \right]_0^T dx da - \frac{s}{2} \int_0^T \int_0^1 \left[ \Theta_a \frac{p(x) - 2\|p\|_{L^\infty(0,1)}}{k} w^2 \right]_0^A dx dt \\
&\quad + s^2 \int_0^T \int_0^A \left[ \Theta \Theta_a \frac{x}{k} (p(x) - 2\|p\|_{L^\infty(0,1)}) e^{Rx^2} w^2 \right]_0^1 dadt.
\end{aligned}$$

Since  $w \in \mathcal{V}_1$ ,  $w(0, a, x)$ ,  $w(T, a, x)$ ,  $w_x(0, a, x)$ ,  $w_x(T, a, x)$ ,  $w(t, 0, x)$ ,  $w(t, A, x)$  and  $\int_0^T \int_0^1 [w_x^2]_0^A dx dt$  are well defined; thus, using the boundary conditions and the definition of  $w$  itself, we get

$$\begin{aligned}
\int_0^T \int_0^1 [w_x^2]_0^A dx dt &= \int_0^A \int_0^1 \left[ \Theta_a \frac{p(x) - 2\|p\|_{L^\infty(0,1)}}{k} w^2 \right]_0^T dx da \\
&= \int_0^T \int_0^1 \left[ \Theta_a \frac{p(x) - 2\|p\|_{L^\infty(0,1)}}{k} w^2 \right]_0^A dx dt = 0.
\end{aligned}$$

Moreover, since  $w \in \mathcal{V}_1$ , we have that  $w_a(t, a, 0)$  and  $w_a(t, a, 1)$  make sense.

280 Moreover, also  $w_x(t, a, 0)$  and  $w_x(t, a, 1)$  are well defined, since  $w(t, a, \cdot) \in H^2_{\frac{1}{k}}(0, 1)$ . Thus  $\int_0^T \int_0^A [w_x w_a]_{x=0}^{x=1} dadt$  is well defined and actually equals 0. Indeed, by the boundary conditions, we find

$$|w_a(t, a, x)| \leq \int_0^x |w_{ax}(t, a, y)| dy \leq \sqrt{x} \left( \int_0^x |w_{ax}(t, a, y)|^2 dy \right)^{1/2} \rightarrow 0$$

as  $x \rightarrow 0$ , the integral being finite. Now, we consider the term

$$\int_0^T \int_0^A \left[ \Theta \Theta_a \frac{x}{k} (p(x) - 2\|p\|_{L^\infty(0,1)}) e^{Rx^2} w^2 \right]_0^1 dadt.$$

Since  $w(t, a, 1) = 0$ ,

$$\int_0^T \int_0^A \left[ \Theta \Theta_a \frac{x}{k} (p(x) - 2\|p\|_{L^\infty(0,1)}) e^{Rx^2} w^2 \right] (t, a, 1) dadt = 0.$$

285 Moreover, by Hölder inequality,  $w^2(t, a, x) \leq x \int_0^x w_x^2(t, a, y) dy$ ; hence, by Lemma 3.3, one has

$$\left| \Theta(t, a) \Theta_a(t, a) \frac{x}{k} w^2(t, a, x) \right| \leq \Theta(t, a) |\Theta_a(t, a)| \frac{x^2}{k(x)} \int_0^x w_x^2(t, a, y) dy \rightarrow 0,$$

as  $x \rightarrow 0^+$ . Thus

$$\begin{aligned} & 2s^2 \|p\|_{L^\infty(0,1)} \int_0^T \int_0^A \Theta(t, a) \Theta_a(t, a) \left[ e^{Rx^2} \frac{x}{k} w^2 \right] (t, a, 0) dadt \\ &= \lim_{\epsilon \rightarrow 0} 2s^2 \|p\|_{L^\infty(0,1)} \int_0^T \int_0^A \Theta(t, a) \Theta_a(t, a) \left[ e^{Rx^2} \frac{x}{k} w^2 \right] (t, a, \epsilon) dadt = 0. \end{aligned}$$

Finally, using the fact that the function  $x \mapsto \frac{x^M}{k}$  is nondecreasing, one has that

$$\int_0^T \int_0^A \Theta(t, a) \Theta_a(t, a) \left[ e^{Rx^2} \frac{x}{k} p(x) w^2 \right] (t, a, 0) dadt = 0.$$

Indeed, if  $M \leq 1$ ,

$$\left| \Theta(t, a) \Theta_a(t, a) e^{Rx^2} \frac{x}{k} p(x) w^2(t, a, x) \right| \leq \Theta(t, a) |\Theta_a(t, a)| e^R \left( \frac{x^2}{k(x)} \right)^2 \int_0^x w_x^2(t, a, y) dy \rightarrow 0,$$

290 as  $x \rightarrow 0$ . If  $M > 1$ ,

$$\begin{aligned} \left| \Theta(t, a) \Theta_a(t, a) e^{Rx^2} \frac{x}{k} p(x) w^2(t, a, x) \right| &\leq \Theta(t, a) |\Theta_a(t, a)| e^R \frac{x^{M+1}}{k^2(x)} w^2(t, a, x) \int_0^x \frac{1}{y^{M-1}} dy \\ &= \Theta(t, a) |\Theta_a(t, a)| e^R \frac{x^3}{k^2(x)} w^2(t, a, x). \\ &\leq \Theta(t, a) |\Theta_a(t, a)| e^R \frac{x^4}{k^2(x)} \int_0^x w_x^2(t, a, y) dy \rightarrow 0, \end{aligned}$$

as  $x \rightarrow 0$ . Hence the thesis.

The crucial step is to prove now the following estimate.

**Lemma 3.4.** *Assume Hypothesis 3.2. There exist two strictly positive constants  $C$  and  $s_0$  such that, for all  $s \geq s_0$ , all solutions  $w$  of (19) satisfy the following*

295 *estimate*

$$sC \int_Q \Theta w_x^2 dx dadt + s^3 C \int_Q \Theta^3 \left( \frac{x}{k} \right)^2 w^2 dx dadt \leq \{D.T.\}.$$

PROOF. The distributed terms of  $\langle L_s^+ w, L_s^- w \rangle_{L^2_{\frac{1}{k}}(Q)}$ , using the definition of  $\varphi$ , take the form

$$\begin{aligned}
\{D.T.\} = & s \int_Q \Theta \left( 2 - \frac{xk_x}{k} + 4Rx^2 \right) e^{Rx^2} w_x^2 dx dadt \\
& + s^3 \int_Q \Theta^3 \left( \frac{x}{k} \right)^2 \left( 2 - \frac{xk_x}{k} + 4Rx^2 \right) e^{3Rx^2} w^2 dx dadt \\
& - 2s^2 \int_Q \Theta \Theta_t \left( \frac{x}{k} \right)^2 e^{2Rx^2} w^2 dx dadt + \frac{s}{2} \int_Q \frac{\Theta_{tt}}{k} \left( p - 2\|p\|_{L^\infty(0,1)} \right) w^2 dx dadt \\
& + s \int_Q \Theta \left( e^{Rx^2} \left[ 1 + 2Rx - x \frac{k'}{k} \right] \right)_x w w_x dx dadt \\
& + \frac{s}{2} \int_Q \frac{\Theta_{aa}}{k} \left( p - 2\|p\|_{L^\infty(0,1)} \right) w^2 dx dadt \\
& + s \int_Q \frac{\Theta_{ta}}{k} \left( p - 2\|p\|_{L^\infty(0,1)} \right) w^2 dx dadt - 2s^2 \int_Q \Theta \Theta_a \left( \frac{x}{k} \right)^2 e^{2Rx^2} w^2 dx dadt.
\end{aligned} \tag{26}$$

Now, observe that there exists  $c > 0$  such that

$$\begin{aligned}
\Theta^\mu &\leq c\Theta^\nu \text{ if } 0 < \mu < \nu \\
|\Theta\Theta_t| &\leq c\Theta^3, |\Theta\Theta_a| \leq c\Theta^3, \\
|\Theta_{aa}| &\leq c\Theta^{\frac{3}{2}}, |\Theta_{tt}| \leq c\Theta^{\frac{3}{2}} \text{ and } |\Theta_{ta}| \leq c\Theta^{\frac{3}{2}}.
\end{aligned} \tag{27}$$

Hence, proceeding as in the proof of [13, Lemma 3.8] or of [31, Lemma 4.3], one

300 can deduce

$$\begin{aligned}
& s \int_Q \Theta \left( 2 - \frac{xk_x}{k} + 4Rx^2 \right) e^{Rx^2} w_x^2 dx dadt \\
& + s^3 \int_Q \Theta^3 \left( \frac{x}{k} \right)^2 \left( 2 - \frac{xk_x}{k} + 4Rx^2 \right) e^{3Rx^2} w^2 dx dadt \\
& - 2s^2 \int_Q \Theta \Theta_t \left( \frac{x}{k} \right)^2 e^{2Rx^2} w^2 dx dadt + \frac{s}{2} \int_Q \frac{\Theta_{tt}}{k} \left( p - 2\|p\|_{L^\infty(0,1)} \right) w^2 dx dadt \\
& + \frac{s}{2} \int_Q \frac{\Theta_{aa}}{k} \left( p - 2\|p\|_{L^\infty(0,1)} \right) w^2 dx dadt \\
& + s \int_Q \frac{\Theta_{ta}}{k} \left( p - 2\|p\|_{L^\infty(0,1)} \right) w^2 dx dadt - 2s^2 \int_Q \Theta \Theta_a \left( \frac{x}{k} \right)^2 e^{2Rx^2} w^2 dx dadt \\
& \geq sC \int_Q \Theta w_x^2 dx dadt + s^3 C \int_Q \Theta^3 \left( \frac{x}{k} \right)^2 w^2 dx dadt \\
& - s^2 \frac{C}{4} \int_Q \Theta^3 \left( \frac{x}{k} \right)^2 w^2 dx dadt - s \frac{C}{4} \int_Q \frac{\Theta^{\frac{3}{2}}}{k} w^2 dx dadt,
\end{aligned} \tag{28}$$

where  $C > 0$  denotes some universal strictly positive constant which may vary from line to line.

Now, consider the term  $\int_Q \Theta \left( e^{Rx^2} \left[ 1 + 2Rx - x \frac{k'}{k} \right] \right)_x w w_x dx dadt$ . Setting

$$\mathfrak{h} := e^{Rx^2} \left[ 1 + 2Rx - x \frac{k'}{k} \right]$$

305 and for  $\epsilon > 0$ , it results

$$\begin{aligned}
\left| s \int_Q \Theta \mathfrak{h}' w w_x dx dadt \right| & \leq \frac{1}{\epsilon} s \int_Q \Theta |\mathfrak{h}'|^2 w^2 dx dadt + \epsilon s \int_Q \Theta (w_x)^2 dx dadt \\
& \leq \frac{1}{\epsilon} sC \|\mathfrak{h}'\|_{L^\infty(0,1)}^2 \|k\|_{L^\infty(0,1)} \int_Q \Theta^{\frac{3}{2}} \frac{w^2}{k} dx dadt + \epsilon s \int_Q \Theta (w_x)^2 dx dadt.
\end{aligned} \tag{29}$$

As in [13], one has, for  $\gamma > 0$ ,

$$\begin{aligned}
\int_Q \frac{\Theta^{\frac{3}{2}}}{k} w^2 dx dadt & = \int_Q \left( \frac{1}{\gamma} \Theta^2 \left( \frac{x}{k} \right)^2 w^2 \right)^{\frac{1}{2}} \left( \gamma \frac{\Theta}{x^2} w^2 \right)^{\frac{1}{2}} dx dadt \\
& \leq \frac{1}{\gamma} \int_Q \Theta^2 \left( \frac{x}{k} \right)^2 w^2 dx dadt + \gamma \int_Q \frac{\Theta}{x^2} w^2 dx dadt.
\end{aligned}$$

By Hardy's inequality one has

$$\int_Q \frac{\Theta^{\frac{3}{2}}}{k} w^2 dx dadt \leq \frac{1}{\gamma} \int_Q \Theta^2 \left( \frac{x}{k} \right)^2 w^2 dx dadt + \gamma C \int_Q \Theta w_x^2 dx dadt, \tag{30}$$

for a strictly positive constant  $C$ .

Thus, for  $s_0$  large enough and  $\gamma$  small enough, by (28), (29) and (30), the  
 310 thesis follows.

As a consequence of Lemmas 3.2 and 3.4, we have

**Proposition 3.1.** *Assume Hypothesis 3.2. There exist two strictly positive constants  $C$  and  $s_0$  such that, for all  $s \geq s_0$ , all solutions  $w$  of (19) in  $\mathcal{V}_1$  satisfy*

$$\int_Q s \Theta w_x^2 + s^3 \Theta^3 \left( \frac{x}{k} \right)^2 w^2 dx dadt \leq C \left( \int_Q f^2 \frac{e^{2s\varphi}}{k} dx dadt + s \int_0^T \int_0^A \Theta(t, a) w_x^2(t, a, 1) dadt \right).$$

Recalling the definition of  $w$ , we have  $v = e^{-s\varphi} w$  and  $v_x = (w_x - s\varphi_x w) e^{-s\varphi}$ .

315 Thus, Theorem 3.4 follows immediately by Proposition 3.1 when  $\mu \equiv 0$ .

Now, we assume that  $\mu \neq 0$ .

To complete the proof of Theorem 3.2 we consider the function  $\bar{f} = f + \mu v$ . Hence, there are two strictly positive constants  $C$  and  $s_0$  such that, for all  $s \geq s_0$ , the following inequality holds

$$\begin{aligned} \int_Q \left( s \Theta v_x^2 + s^3 \Theta^3 \left( \frac{x}{k} \right)^2 v^2 \right) e^{2s\varphi} dx dadt &\leq C \int_Q \bar{f}^2 \frac{e^{2s\varphi}}{k} dx dadt \\ &+ s C \int_0^T \int_0^A \Theta(t, a) [x v_x^2 e^{2s\varphi}] (t, a, 1) dadt. \end{aligned} \quad (31)$$

320 On the other hand, we have

$$\int_Q |\bar{f}|^2 \frac{e^{2s\varphi}}{k} dx dadt \leq 2 \left( \int_Q |f|^2 \frac{e^{2s\varphi}}{k} dx dadt + \|\mu\|_{L^\infty(Q)}^2 \int_Q |v|^2 \frac{e^{2s\varphi}}{k} dx dadt \right). \quad (32)$$

Now, applying Hardy-Poincaré inequality to the function  $\nu := e^{s\varphi} v$ , we obtain

$$\begin{aligned} \int_Q |v|^2 \frac{e^{2s\varphi}}{k} dx dadt &= \int_Q \frac{\nu^2}{k} dx dadt = \int_Q \frac{x^2}{k} \frac{\nu^2}{x^2} dx dadt \leq C \int_Q \frac{\nu^2}{x^2} dx dadt \\ &\leq C \int_Q (e^{s\varphi} v)_x^2 dx dadt \leq C \int_Q e^{2s\varphi} v_x^2 dx dadt + C s^2 \int_Q \Theta^2 e^{2s\varphi} \left( \frac{x}{k} \right)^2 v^2 dx dadt. \end{aligned}$$

Using this last inequality in (32), it follows

$$\begin{aligned} \int_Q |\bar{f}|^2 \frac{e^{2s\varphi}}{k} dx dadt &\leq 2 \int_Q |f|^2 \frac{e^{2s\varphi}}{k} dx dadt + C \int_Q e^{2s\varphi} v_x^2 dx dadt \\ &+ C s^2 \int_Q \Theta^2 e^{2s\varphi} \left( \frac{x}{k} \right)^2 v^2 dx dadt. \end{aligned} \quad (33)$$

Substituting in (31), one can conclude

$$\begin{aligned} & \int_Q \left( s\Theta v_x^2 + s^3\Theta^3 \left( \frac{x}{k} \right)^2 v^2 \right) e^{2s\varphi} dx d\alpha dt \leq C \left( \int_Q |f|^2 \frac{e^{2s\varphi}}{k} dx d\alpha dt \right. \\ & \left. + \int_Q e^{2s\varphi} v_x^2 dx d\alpha dt + s^2 \int_Q \Theta^2 e^{2s\varphi} \left( \frac{x}{k} \right)^2 v^2 dx d\alpha dt + s \int_0^T \int_0^A \Theta(t, a) [x v_x^2 e^{2s\varphi}] (t, a, 1) d\alpha dt \right). \end{aligned}$$

This completes the proof of Theorem 3.4.

325 *Carleman inequalities when the degeneracy is in the interior.* Now, we prove Carleman inequalities for (6) when  $k$  has an interior degeneracy point. In particular, on  $k$  we assume

**Hypotheses 3.4.** *The function  $k$  satisfies Hypothesis 2.2 or Hypothesis 2.3. Moreover,*

$$\frac{(x - x_0)k'(x)}{k(x)} \in W^{1,\infty}(0, 1),$$

330 *and, if  $M \geq 1$ , there exists a constant  $\vartheta \in (0, M]$  such that the function*

$$x \mapsto \frac{k(x)}{|x - x_0|^\vartheta} \begin{cases} \text{is nonincreasing on the left of } x = x_0, \\ \text{is nondecreasing on the right of } x = x_0. \end{cases} \quad (34)$$

As before, we introduce the function  $\Gamma(t, a, x) := \Theta(t, a)\gamma(x)$ , where  $\Theta$  is defined as in (8) and

$$\gamma(x) := d_1 \left( \int_{x_0}^x \frac{y - x_0}{k(y)} e^{R(y-x_0)^2} dy - d_2 \right), \quad (35)$$

with  $R > 0$ ,  $d_2 > \max \left\{ \frac{(1 - x_0)^2 e^{R(1-x_0)^2}}{(2 - K)k(1)}, \frac{x_0^2 e^{Rx_0^2}}{(2 - K)k(0)} \right\}$  and  $d_1 > 0$ . Also in this case we have

$$-d_1 d_2 \leq \gamma(x) < 0 \quad \text{for every } x \in [0, 1].$$

335

**Theorem 3.6.** *Assume Hypothesis 3.4. Then, there exist two strictly positive constants  $C$  and  $s_0$  such that every solution  $v$  of (6) in*

$$\mathcal{V}_2 := L^2(Q_{T,A}; H_{\frac{1}{k}, x_0}^2(0, 1)) \cap H^1(Q_{T,A}; H_{\frac{1}{k}}^1(0, 1)) \quad (36)$$

satisfies

$$\begin{aligned}
& \int_Q \left( s\Theta(v_x)^2 + s^3\Theta^3 \left( \frac{x-x_0}{k} \right)^2 v^2 \right) e^{2s\Gamma} dx dadt \\
& \leq C \left( \int_Q f^2 \frac{e^{2s\Gamma}}{k} dx dadt + sd_1 \int_0^T \int_0^A [\Theta e^{2s\Gamma} (x-x_0)(v_x)^2 dadt]_{x=0}^{x=1} dadt \right)
\end{aligned} \tag{37}$$

for all  $s \geq s_0$ , where  $d_1$  is the constant of (35).

**Remark 3.3.** Observe that Theorem 3.6 is the same as [31, Theorem 4.2]. However, here we assume that  $k$  satisfies (34) only if  $M \geq 1$ , while in [31] condition (34) is required if  $M \geq \frac{1}{2}$ . Thus, also in this situation, we improve [31, Theorem 4.2] when  $y$  is independent of  $a$ .

*Proof of Theorem 3.6.* The proof of Theorem 3.6 follows the ideas of the one of [31, Theorem 4.2] or Theorem 3.4. As before, we consider, first of all, the case when  $\mu \equiv 0$ : for every  $s > 0$  consider the function

$$w(t, a, x) := e^{s\Gamma(t, a, x)} v(t, a, x),$$

where  $v$  is any solution of (6) in  $\mathcal{V}_2$ , so that also  $w \in \mathcal{V}_2$ , since  $\Gamma < 0$ . Moreover,  $w$  satisfies (19) and Lemma 3.1 still holds. We underline the fact that all integrals and integrations by parts are justified by the definition of  $D(\mathcal{A})$  and the choice of  $\Gamma$ , while before they were guaranteed by the choice of Dirichlet conditions at  $x = 0$  or  $x = 1$ , i.e. where the operator degenerates. Thus we start with the analogue of Lemma 3.4 in the weakly and in the strongly degenerate cases, which now gives the following estimate:

**Lemma 3.5.** Assume Hypothesis 3.4. Then there exists a strictly positive con-



stant  $s_0$  such that for all  $s \geq s_0$  the distributed terms of (21) satisfy the estimate

$$\begin{aligned}
& s \int_Q (k\Gamma_{xx} + (k\Gamma_x)_x)(w_x)^2 dx dadt + s^3 \int_Q (\Gamma_x)^2 (k\Gamma_{xx} + (k\Gamma_x)_x) w^2 dx dadt \\
& - 2s^2 \int_Q \Gamma_x \Gamma_{xt} w^2 dx dadt + \frac{s}{2} \int_Q \frac{\Gamma_{tt}}{k} w^2 dx dadt + s \int_Q (k\Gamma_{xx})_x w w_x dx dadt \\
& + \frac{s}{2} \int_Q \frac{\Gamma_{aa}}{k} w^2 dx dadt + s \int_Q \frac{\Gamma_{ta}}{k} w^2 dx dadt - 2s^2 \int_Q \Gamma_x \Gamma_{xa} w^2 dx dadt \\
& \geq Cs \int_Q \Theta(w_x)^2 dx dadt + Cs^3 \int_Q \Theta^3 \left( \frac{x-x_0}{k} \right)^2 w^2 dx dadt,
\end{aligned}$$

for a universal strictly positive constant  $C$ .

PROOF. Using the definition of  $\Gamma$ , the distributed terms of  $\int_Q \frac{1}{k} L_s^+ w L_s^- w dx dadt$  take the form

$$\{D.T.\}_1 \begin{cases} \frac{s}{2} \int_Q \frac{\Theta_{tt} + \Theta_{aa}}{k} \gamma w^2 dx dadt - 2s^2 \int_Q \Theta \Theta_t (\gamma')^2 w^2 dx dadt - 2s^2 \int_Q \Theta \Theta_a (\gamma')^2 w^2 dx dadt \\ + s \int_Q \Theta (2k\gamma'' + k'\gamma') (w_x)^2 dx dadt + s^3 \int_Q \Theta^3 (2k\gamma'' + k'\gamma') (\gamma')^2 w^2 dx dadt \\ + s \int_Q \Theta (k\gamma'')' w w_x dx dadt. \end{cases}$$

Because of the choice of  $\gamma(x)$ , one has

$$2k(x)\gamma''(x) + k'(x)\gamma'(x) = d_1 e^{R(x-x_0)^2} \frac{2k(x) - k'(x)(x-x_0) + 4R(x-x_0)^2 k(x)}{k(x)}.$$

As in [31], by Hypothesis 2.2 or 2.3, we immediately find

$$2 - \frac{(x-x_0)k'}{k} + 4R(x-x_0)^2 \geq 2 - M \quad \text{a.e. } x \in [0, 1],$$

for every  $R > 0$ . Thus, using the fact that  $e^{R(x-x_0)^2}$  is bounded and bounded away from 0 in  $[0, 1]$ , the distributed terms satisfy the estimate

$$\begin{aligned}
\{D.T.\}_1 & \geq \frac{s}{2} \int_Q \frac{\Theta_{tt} + \Theta_{aa}}{k} \gamma w^2 dx dadt - s^2 C \int_Q |\Theta \Theta_t| \left( \frac{x-x_0}{k} \right)^2 w^2 dx dadt \\
& - s^2 C \int_Q |\Theta \Theta_a| \left( \frac{x-x_0}{k} \right)^2 w^2 dx dadt \\
& + sC \int_Q \Theta(w_x)^2 dx dadt + s^3 C \int_Q \Theta^3 \left( \frac{x-x_0}{k} \right)^2 w^2 dx dadt \\
& + s \int_Q \Theta (k\gamma'')' w w_x dx dadt.
\end{aligned} \tag{38}$$

By (27), we conclude that, for  $s$  large enough,

$$\begin{aligned} s^2 C \int_Q (|\Theta\Theta_t| + |\Theta\Theta_a|) \left( \frac{x-x_0}{k} \right)^2 w^2 dx dadt &\leq c C s^2 \int_Q \Theta^3 \left( \frac{x-x_0}{k} \right)^2 w^2 dx dadt \\ &\leq \frac{C^3}{8} s^3 \int_Q \Theta^3 \left( \frac{x-x_0}{k} \right)^2 w^2 dx dadt. \end{aligned}$$

Again as in [31], by (27) we get

$$\begin{aligned} \left| \frac{s}{2} \int_Q \frac{\Theta_{tt} + \Theta_{aa}}{k} \gamma w^2 dx dadt \right| &\leq s \frac{d_1 d_2}{2} c \int_Q \Theta^{3/2} \frac{w^2}{k} dx dadt \\ &\leq \frac{C}{4} s \int_Q \Theta(w_x)^2 dx dadt \\ &\quad + \frac{C^3}{4} s^3 \int_Q \Theta^3 \left( \frac{x-x_0}{k} \right)^2 w^2 dx dadt. \end{aligned} \tag{39}$$

Now, we consider the last term in (38), i.e.  $s \int_Q \Theta(k\gamma'')' w w_x dx dadt$ . By Hypothesis 3.4 and using the definition of  $\gamma$ , as in [31], we get

$$\left| s \int_Q \Theta(k\gamma'')' w w_x dx dadt \right| \leq \frac{C}{4} s \int_Q \Theta(w_x)^2 dx dadt + s^3 \frac{C^3}{8} \int_Q \Theta^3 \left( \frac{x-x_0}{k} \right)^2 w^2 dx dadt.$$

Summing up, we obtain

$$\begin{aligned} \{D.T.\}_1 &\geq -\frac{C}{4} s \int_Q \Theta(w_x)^2 dx dadt - \frac{C^3}{4} s^3 \int_Q \Theta^3 \left( \frac{x-x_0}{k} \right)^2 w^2 dx dadt \\ &\quad - \frac{C^3}{8} s^3 \int_Q \Theta^3 \left( \frac{x-x_0}{k} \right)^2 w^2 dx dadt \\ &\quad + s C \int_Q \Theta(w_x)^2 dx dadt + s^3 C \int_Q \Theta^3 \left( \frac{x-x_0}{k} \right)^2 w^2 dx dadt \\ &\quad - \frac{C}{4} s \int_Q \Theta(w_x)^2 dx dadt - \frac{C^3}{8} s^3 \int_Q \Theta^3(w_x)^2 dx dadt \\ &= \frac{C}{2} s \int_Q \Theta(w_x)^2 dx dadt + \frac{C^3}{2} s^3 \int_Q \Theta^3 \left( \frac{x-x_0}{k} \right)^2 w^2 dx dadt. \end{aligned}$$

As for the boundary terms, similarly to Lemma 3.2, we have the following result, whose proof parallels the one of Lemma 3.2 and is thus omitted (see also [31, Lemma 4.4]).

**Lemma 3.6.** *Assume Hypothesis 3.4. Then the boundary terms in (21) reduce to*

$$-s d_1 \int_0^T \int_0^A \Theta(t) a \left[ (x-x_0) e^{R(x-x_0)^2} (w_x)^2 \right]_{x=0}^{x=1} dadt.$$

By Lemmas 3.5 and 3.6, there exist  $C > 0$  and  $s_0 > 0$  such that all solutions  $w$  of (19) satisfy, for all  $s \geq s_0$ ,

$$\begin{aligned} \int_Q \frac{1}{k} L_s^+ w L_s^- w dx dadt &\geq C s \int_Q \Theta(w_x)^2 dx dadt \\ &\quad + C s^3 \int_Q \Theta^3 \left( \frac{x - x_0}{k} \right)^2 w^2 dx dadt \\ &\quad - s d_1 \int_0^T \int_0^A \Theta(t, a) \left[ (x - x_0) e^{R(x - x_0)^2} (w_x)^2 \right]_{x=0}^{x=1} dadt. \end{aligned} \quad (40)$$

Thus, for all  $s \geq s_0$ , we obtain the next Carleman inequality for  $w$ :

$$\begin{aligned} s \int_Q \Theta(w_x)^2 dx dadt + s^3 \int_Q \Theta^3 \left( \frac{x - x_0}{k} \right)^2 w^2 dx dadt \\ \leq C \left( \int_Q f^2 \frac{e^{2s\Gamma}}{k} dx dadt + s d_1 \int_0^T \int_0^A \left[ \Theta(x - x_0) e^{R(x - x_0)^2} (w_x)^2 \right]_{x=0}^{x=1} dadt \right). \end{aligned}$$

Theorem 3.6 follows recalling the definition of  $w$ .

If  $\mu \neq 0$ , we can proceed as in the proof of Theorem 3.4, obtaining the thesis.

#### 4. Observability and controllability of linear equations

In this section we will prove, as a consequence of the Carleman estimates established in Section 3, observability inequalities for the associated adjoint problem of (1). To this aim, we assume that the control set  $\omega$  is such that

$$\omega = (\alpha, \rho) \subset\subset (0, 1), \quad (41)$$

if  $k$  degenerates at the boundary of  $(0, 1)$ . When  $k$  degenerates at  $x_0 \in (0, 1)$ ,  $\omega$  is such that

$$x_0 \in \omega = (\alpha, \rho) \subset\subset (0, 1), \quad (42)$$

or

$$\omega = \omega_1 \cup \omega_2, \quad (43)$$

where

$$\omega_i = (\lambda_i, \rho_i) \subset (0, 1), \quad i = 1, 2, \quad \text{and} \quad \rho_1 < x_0 < \lambda_2. \quad (44)$$

**Remark 4.1.** Observe that, if (42) holds, we can find two subintervals  $\omega_1 = (\lambda_1, \rho_1) \subset \subset (\alpha, x_0)$ ,  $\omega_2 = (\lambda_2, \rho_2) \subset \subset (x_0, \rho)$ .

Moreover,  $k$  and  $\beta$  satisfy the following assumptions:

**Hypotheses 4.1.** The function  $k$  is s.t. Hypothesis 3.2, 3.3 or 3.4 is satisfied.

390 Moreover, if Hypothesis 2.2 holds, there exist two functions  $\mathfrak{g} \in L_{\text{loc}}^\infty([-\rho_1, 1] \setminus \{x_0\})$ ,  $\mathfrak{h} \in W_{\text{loc}}^{1,\infty}([-\rho_1, 1] \setminus \{x_0\}, L^\infty(0, 1))$  and two strictly positive constants  $\mathfrak{g}_0, \mathfrak{h}_0$  such that  $\mathfrak{g}(x) \geq \mathfrak{g}_0$

$$\frac{\tilde{k}'(x)}{2\sqrt{\tilde{k}(x)}} \left( \int_x^B \mathfrak{g}(t) dt + \mathfrak{h}_0 \right) + \sqrt{\tilde{k}(x)} \mathfrak{g}(x) = \mathfrak{h}(x, B) \quad \text{for a.e. } x \in [-\rho_1, 1], B \in [0, 1] \quad (45)$$

with  $x < B < x_0$  or  $x_0 < x < B$ , where

$$\tilde{k}(x) := \begin{cases} k(x), & x \in [0, 1], \\ k(-x), & x \in [-1, 0]. \end{cases} \quad (46)$$

Observe that (45) implies the fact that  $\frac{k'}{\sqrt{k}} \in L_{\text{loc}}^\infty([0, 1] \setminus \{x_0\})$ .

395 **Hypotheses 4.2.** Assume  $T < A$  and suppose that there exists  $\bar{a} \leq T$  such that

$$\beta(a, x) = 0 \text{ for all } (a, x) \in [0, \bar{a}] \times [0, 1]. \quad (47)$$

Observe that Hypothesis 4.2 is the biological meaningful one. Indeed,  $\bar{a}$  is the minimal age in which the female of the population become fertile, thus it is natural that before  $\bar{a}$  there are no newborns. Obviously, if  $T < A$  and  $T = \bar{a}$ ,  
 400 then  $y(t, 0, x) = \int_T^A \beta(a, x) y(t, a, x) da$ . In this case, if  $(t, a) \in (0, T) \times (0, T)$ , only the mortality rate acts on the equation; hence it is natural to expect that the population is 0 at  $T$ . However, we will prove the observability inequalities also in this case, since they are independently interesting. Finally, we underline that, since  $T$  is strictly less than  $A$ , we are able to control the population also  
 405 in small times, thus complementing [2].

Under the previous hypotheses, the following observability inequality holds:

**Proposition 4.1.** *Suppose that Hypotheses 4.1 and 4.2 hold and assume that  $\omega$  satisfies (41), (42) or (43). Then, there exists a strictly positive constant  $C$  such that, for every  $\delta \in (T, A)$ , every solution  $v \in \mathcal{U}$  of*

$$\begin{cases} \frac{\partial v}{\partial t} + \frac{\partial v}{\partial a} + k(x)v_{xx} - \mu(t, a, x)v + \beta(a, x)v(t, 0, x) = 0, & (t, x, a) \in Q, \\ v(t, a, 0) = v(t, a, 1) = 0, & (t, a) \in Q_{T,A}, \\ v(T, a, x) = v_T(a, x) \in L^2(Q_{A,1}), & (a, x) \in Q_{A,1} \\ v(t, A, x) = 0, & (t, x) \in Q_{T,1}, \end{cases} \quad (48)$$

410 satisfies

$$\int_0^A \int_0^1 \frac{1}{k} v^2(T-\bar{a}, a, x) dx da \leq C \left( \int_0^\delta \int_0^1 \frac{v_T^2(a, x)}{k} dx da + \int_0^T \int_0^A \int_\omega \frac{v^2}{k} dx da dt \right). \quad (49)$$

Here  $v_T(a, x)$  is such that  $v_T(A, x) = 0$  in  $(0, 1)$ .

**Remark 4.2.** 1. If  $T = \bar{a}$ , the observability inequality given in the previous proposition is the corresponding of [2, Proposition 3.1], where the authors proved it for the divergence case under different assumptions and with  
415  $T \geq A$ .

2. Moreover, observe that in (49) the presence of the integral  $\int_0^\delta \int_0^1 \frac{v_T^2(a, x)}{k} dx da$  is related to the presence of the term  $\beta(a, x)v(t, 0, x)$  in the equation of (48). In fact, estimating such a term using the method of characteristic lines, we obtain the previous integral. Obviously, if  $v_T(a, x) = 0$  a.e. in  
420  $(0, \delta) \times (0, 1)$ , we obtain the classical observability inequality.

Before proving Proposition 4.1 we will give some results that will be very helpful. As a first step we introduce the following class of functions

$$\mathcal{W} := \left\{ v \text{ solution of (48)} \mid v_T \in D(\mathcal{A}^2) \right\},$$

where

$$D(\mathcal{A}^2) = \left\{ u \in D(\mathcal{A}) \mid \mathcal{A}u \in D(\mathcal{A}) \right\}.$$

Observe that  $D(\mathcal{A}^2)$  is densely defined in  $D(\mathcal{A})$  (see, for example, [12,  
 425 Lemma 7.2]) and hence in  $L^2_{\frac{1}{k}}(Q_{A,1})$ . Obviously,

$$\mathcal{W} = C^1([0, T]; D(\mathcal{A})) \subset \mathcal{V} := L^2(Q_{T,A}; \mathcal{H}^2_{\frac{1}{k}}(0, 1)) \cap H^1(Q_{T,A}; H^1_{\frac{1}{k}}(0, 1)) \subset \mathcal{U}.$$

**Proposition 4.2 (Caccioppoli's inequality).** *Assume Hypothesis 3.2 or 3.3.*

*Let  $\omega'$  and  $\omega$  two open subintervals of  $(0, 1)$  such that  $\omega' \subset \subset \omega \subset \subset (0, 1)$ . Let  $\psi(t, x) := \Theta(t, a)\Psi(x)$ , where  $\Theta$  is defined in (8) and  $\Psi \in C^1(0, 1)$  is a strictly negative function. Then, there exist two strictly positive constants  $C$  and  $s_0$*

430 *such that, for all  $s \geq s_0$ ,*

$$\begin{aligned} \int_0^T \int_0^A \int_{\omega'} v_x^2 e^{2s\psi} dx da dt &\leq C \left( \int_0^T \int_0^A \int_{\omega} v^2 dx da dt + \int_Q f^2 e^{2s\psi} dx da dt \right) \\ &\leq C \left( \int_0^T \int_0^A \int_{\omega} \frac{v^2}{k} dx da dt + \int_Q f^2 \frac{e^{2s\psi}}{k} dx da dt \right), \end{aligned} \quad (50)$$

for every solution  $v$  of (6).

PROOF. Let us consider a smooth function  $\xi : [0, 1] \rightarrow \mathbb{R}$  such that

$$\begin{cases} 0 \leq \xi(x) \leq 1, & \text{for all } x \in [0, 1], \\ \xi(x) = 1, & x \in \omega', \\ \xi(x) = 0, & x \in (0, 1) \setminus \omega. \end{cases}$$

Then, integrating by parts one has

$$\begin{aligned} 0 &= \int_0^T \frac{d}{dt} \left( \int_0^A \int_0^1 (\xi e^{s\psi})^2 v^2 dx da \right) dt \\ &= \int_Q 2s \psi_t (\xi e^{s\psi})^2 v^2 + 2(\xi e^{s\psi})^2 v (-v_a - k v_{xx} + \mu v + f) dx da dt \\ &= 2s \int_Q \psi_t (\xi e^{s\psi})^2 v^2 dx da dt + 2s \int_Q \psi_a (\xi e^{s\psi})^2 v^2 dx da dt + 2 \int_Q (\xi^2 e^{2s\psi} k)_x v v_x dx da dt \\ &\quad + 2 \int_Q (\xi^2 e^{2s\psi} k) v_x^2 dx da dt + 2 \int_Q \xi^2 e^{2s\psi} \mu v^2 dx da dt + 2 \int_Q \xi^2 e^{2s\psi} f v dx da dt. \end{aligned}$$

Hence, using Young's inequality

$$\begin{aligned}
2 \int_Q (\xi^2 e^{2s\psi} k) v_x^2 dx dt &= -2s \int_Q \psi_t (\xi e^{s\psi})^2 v^2 dx dt - 2s \int_Q \psi_a (\xi e^{s\psi})^2 v^2 dx dt \\
&\quad - 2 \int_Q (\xi^2 e^{2s\psi} k)_x \frac{\xi e^{s\psi} \sqrt{k}}{\xi e^{s\psi} \sqrt{k}} v v_x dx dt - 2 \int_Q \xi^2 e^{2s\psi} \mu v^2 dx dt \\
&\quad - 2 \int_Q \xi^2 e^{2s\psi} f v dx dt \\
&\leq -2s \int_Q \psi_t (\xi e^{s\psi})^2 v^2 dx dt - 2s \int_Q \psi_a (\xi e^{s\psi})^2 v^2 dx dt \\
&\quad + 4 \int_Q \left( \xi e^{s\psi} \sqrt{k} \right)_x^2 v^2 dx dt + \int_Q (\xi^2 e^{2s\psi} k) v_x^2 dx dt \\
&\quad + (2\|\mu\|_{L^\infty(Q)} + 1) \int_Q \xi^2 v^2 dx dt + \int_Q \xi^2 e^{2s\psi} f^2 dx dt.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\inf_{\omega'} \{k\} \int_0^T \int_0^A \int_{\omega'} e^{2s\psi} v_x^2 dx dt \\
&\leq \sup_{\omega \times (0,T)} \left\{ \left| 4 \left( \xi e^{s\psi} \sqrt{k} \right)_x^2 - 2s(\psi_t + \psi_a)(\xi e^{s\psi})^2 \right| \right\} \int_0^T \int_0^A \int_{\omega} v^2 dx dt + \int_Q f^2 e^{2s\psi} dx dt.
\end{aligned}$$

435 **Proposition 4.3 (Caccioppoli's inequality).** *Assume Hypothesis 3.4 and suppose that  $\frac{k'}{\sqrt{k}} \in L_{loc}^\infty([0,1] \setminus \{x_0\})$  if Hypothesis 2.2 holds. Let  $\omega'$  and  $\omega$  two open subintervals of  $(0,1)$  such that  $\omega' \subset\subset \omega \subset\subset (0,1)$  and  $x_0 \notin \bar{\omega}$ . Let  $\psi(t,x) := \Theta(t,a)\Psi(x)$ , where  $\Theta$  is defined in (8) and  $\Psi \in C^1(0,1)$  is a strictly negative function. Then the thesis of Proposition 4.2 holds.*

440 The proof of the previous result follows the one of Proposition 4.2. We underline only that, in this case,  $\left( \xi e^{s\psi} \sqrt{k} \right)_x$  can be estimated by

$$C \left( e^{2s\psi} + s^2 (\psi_x)^2 e^{2s\psi} + e^{2s\psi} \frac{(k')^2}{k} \right)$$

and  $\frac{(k')^2}{k}$  exists and is bounded in  $\omega$  thanks to the assumptions (recall that  $x_0 \notin \bar{\omega}$ ).

With the aid of Theorems 3.4, 3.5, 3.6 and Propositions 4.2, 4.3, we can now  
445 show  $\omega$ -local Carleman estimates for (6).

**Theorem 4.1.** Assume Hypothesis 3.2 and suppose that  $\omega$  satisfies (41). Then, there exist two strictly positive constants  $C$  and  $s_0$  such that every solution  $v$  of (6) in  $\mathcal{V}_1$  satisfies, for all  $s \geq s_0$ ,

$$\int_Q \left( s\Theta v_x^2 + s^3\Theta^3 \left( \frac{x}{k} \right)^2 v^2 \right) e^{2s\varphi} dx dadt \leq C \left( \int_Q f^2 \frac{e^{2s\Phi}}{k} dx dadt + \int_0^T \int_0^A \int_\omega \frac{v^2}{k} dx dadt \right).$$

PROOF. Let us consider a smooth function  $\xi : [0, 1] \rightarrow \mathbb{R}$  such that

$$\begin{cases} 0 \leq \xi(x) \leq 1, & \text{for all } x \in [0, 1], \\ \xi(x) = 1, & x \in [0, (2\alpha + \rho)/3], \\ \xi(x) = 0, & x \in [(\alpha + 2\rho)/3, 1]. \end{cases}$$

We define  $w(t, a, x) := \xi(x)v(t, a, x)$  where  $v \in \mathcal{V}_\infty$  satisfies (6). Then  $w$  satisfies

$$\begin{cases} w_t + w_a + kw_{xx} - \mu w = \xi f + k(\xi_{xx}v + 2\xi_x v_x) =: h, & (t, a, x) \in Q, \\ w(t, a, 0) = w(t, a, 1) = 0, & (t, a) \in Q_{T,A}. \end{cases}$$

450 Thus, applying Theorem 3.4 and Proposition 4.2,

$$\begin{aligned} & \int_0^T \int_0^A \int_0^{\frac{2\alpha+\rho}{3}} \left( s\Theta v_x^2 + s^3\Theta^3 \left( \frac{x}{k} \right)^2 v^2 \right) e^{2s\varphi} dx dadt \\ &= \int_0^T \int_0^A \int_0^{\frac{2\alpha+\rho}{3}} \left( s\Theta w_x^2 + s^3\Theta^3 \left( \frac{x}{k} \right)^2 w^2 \right) e^{2s\varphi} dx dadt \\ &\leq \int_Q \left( s\Theta w_x^2 + s^3\Theta^3 \left( \frac{x}{k} \right)^2 w^2 \right) e^{2s\varphi} dx dadt \leq C \int_Q h^2 \frac{e^{2s\varphi}}{k} dx dadt \\ &\leq C \left( \int_Q f^2 \frac{e^{2s\varphi}}{k} dx dadt + \int_0^T \int_0^A \int_{\omega'} \frac{v^2}{k} dx dadt + \int_0^T \int_0^A \int_{\omega'} v_x^2 e^{2s\varphi} dx dadt \right) \\ &\leq C \left( \int_Q f^2 \frac{e^{2s\varphi}}{k} dx dadt + \int_0^T \int_0^A \int_\omega \frac{v^2}{k} dx dadt \right), \end{aligned} \tag{51}$$

where  $\omega' := \left( \frac{2\alpha + \rho}{3}, \frac{\alpha + 2\rho}{3} \right)$ .

Now, consider  $z = \eta v$ , where  $\eta = 1 - \xi$  and take  $\bar{\alpha} \in (0, \alpha)$ . Then  $z$  satisfies

$$\begin{cases} z_t + z_a + kz_{xx} - \mu z = \eta f + k(\eta_{xx}v + 2\eta_x v_x) =: h, & (t, a, x) \in Q_{T,A} \times (\bar{\alpha}, 1) =: \bar{Q}, \\ z(t, a, \bar{\alpha}) = z(t, a, 1) = 0, & (t, a) \in Q_{T,A}. \end{cases} \tag{52}$$



Clearly the equation satisfied by  $z$  is not degenerate, thus applying Theorem 3.1 and Proposition 4.2, one has

$$\begin{aligned} \int_{\bar{Q}} (s^3 \phi^3 z^2 + s \phi z_x^2) e^{2s\Phi} dx dadt &\leq C \int_{\bar{Q}} h^2 e^{2s\Phi} dx dadt \\ &\leq C \left( \int_{\bar{Q}} f^2 e^{2s\Phi} dx dadt + \int_0^T \int_0^A \int_{\omega'} (v^2 + v_x^2) e^{2s\Phi} dx dadt \right) \\ &\leq C \left( \int_Q \frac{f^2}{k} e^{2s\Phi} dx dadt + \int_0^T \int_0^A \int_{\omega} \frac{v^2}{k} dx dadt \right). \end{aligned}$$

455 Hence

$$\begin{aligned} \int_0^T \int_0^A \int_{\frac{\alpha+2\rho}{3}}^1 (s^3 \phi^3 v^2 + s \phi v_x^2) e^{2s\Phi} dx dadt &= \int_0^T \int_0^A \int_{\frac{\alpha+2\rho}{3}}^1 (s^3 \phi^3 z^2 + s \phi z_x^2) e^{2s\Phi} dx dadt \\ &\leq C \left( \int_Q \frac{f^2}{k} e^{2s\Phi} dx dadt + \int_0^T \int_0^A \int_{\omega} \frac{v^2}{k} dx dadt \right), \end{aligned}$$

for a strictly positive constant  $C$ . Proceeding, for example, as in [31] one can prove the existence of  $\varsigma > 0$ , such that, for all  $(t, a, x) \in [0, T] \times [0, A] \times [\bar{\alpha}, 1]$ , we have

$$e^{2s\varphi} \leq \varsigma e^{2s\Phi}, \left( \frac{x}{k(x)} \right)^2 e^{2s\varphi} \leq \varsigma e^{2s\Phi}. \quad (53)$$

Thus, for a strictly positive constant  $C$ ,

$$\begin{aligned} \int_0^T \int_0^A \int_{\frac{\alpha+2\rho}{3}}^1 \left( s \Theta v_x^2 + s^3 \Theta^3 \left( \frac{x}{k} \right)^2 v^2 \right) e^{2s\varphi} dx dadt \\ \leq C \left( \int_0^T \int_0^A \int_{\frac{\alpha+2\rho}{3}}^1 (s^3 \phi^3 v^2 + s \phi v_x^2) e^{2s\Phi} dx dadt \right) \\ \leq C \left( \int_Q f^2 \frac{e^{2s\Phi}}{k} dx dadt + \int_0^T \int_0^A \int_{\omega} \frac{v^2}{k} dx dadt \right). \end{aligned} \quad (54)$$

Now, consider  $\tilde{\alpha} \in (\alpha, (2\alpha + \rho)/3)$ ,  $\tilde{\rho} \in ((\alpha + 2\rho)/3, \rho)$  and a smooth function  $\tau : [0, 1] \rightarrow \mathbb{R}$  such that

$$\begin{cases} 0 \leq \tau(x) \leq 1, & \text{for all } x \in [0, 1], \\ \tau(x) = 1, & x \in [(2\alpha + \rho)/3, (\alpha + 2\rho)/3], \\ \tau(x) = 0, & x \in [0, \tilde{\alpha}] \cup [\tilde{\rho}, 1], \end{cases}$$

460 and define  $\zeta(t, a, x) := \tau(x)v(t, a, x)$ . Clearly,  $\zeta$  satisfies (52) with  $h := \tau f + k(\tau_{xx}v + 2\tau_x v_x)$ . Observe that in this case  $\tau_x, \tau_{xx} \not\equiv 0$  in  $\bar{\omega} := \left(\tilde{\alpha}, \frac{2\alpha + \rho}{3}\right) \cup \left(\frac{\alpha + 2\rho}{3}, \tilde{\rho}\right)$ . As before, by Theorem 3.1, Proposition 4.2 and (53), we have

$$\begin{aligned}
& \int_0^T \int_0^A \int_{\frac{2\alpha+\rho}{3}}^{\frac{\alpha+2\rho}{3}} \left( s\Theta v_x^2 + s^3\Theta^3 \left(\frac{x}{k}\right)^2 v^2 \right) e^{2s\varphi} dx dadt \\
& \leq C \left( \int_0^T \int_0^A \int_{\frac{2\alpha+\rho}{3}}^{\frac{\alpha+2\rho}{3}} (s^3\phi^3 v^2 + s\phi v_x^2) e^{2s\Phi} dx dadt \right) \\
& = C \left( \int_0^T \int_0^A \int_{\frac{2\alpha+\rho}{3}}^{\frac{\alpha+2\rho}{3}} (s^3\phi^3 \zeta^2 + s\phi \zeta_x^2) e^{2s\Phi} dx dadt \right) \\
& \leq C \left( \int_Q f^2 \frac{e^{2s\Phi}}{k} dx dadt + \int_0^T \int_0^A \int_{\omega} \frac{v^2}{k} dx dadt \right).
\end{aligned} \tag{55}$$

Adding (51), (54) and (55), the thesis follows.

Proceeding as before one can prove

465 **Theorem 4.2.** *Assume Hypothesis 3.3 and suppose that  $\omega$  satisfies (41). Then, there exist two strictly positive constants  $C$  and  $s_0$  such that every solution  $v$  of (6) in  $\mathcal{V}_1$  satisfies, for all  $s \geq s_0$ ,*

$$\begin{aligned}
& \int_Q \left( s\Theta v_x^2 + s^3\Theta^3 \left(\frac{1-x}{k}\right)^2 v^2 \right) e^{2s\varphi} dx dadt \\
& \leq C \left( \int_Q f^2 \frac{e^{2s\Phi}}{k} dx dadt + \int_0^T \int_0^A \int_{\omega} \frac{v^2}{k} dx dadt \right).
\end{aligned}$$

The  $\omega$ -local Carleman estimates given in Theorems 4.1 and 4.2 hold also if  $k$  degenerates in the interior of the space domain:

470 **Theorem 4.3.** *Assume Hypothesis 3.4 and (45) if Hypothesis 2.2 holds. Suppose that  $\omega$  satisfies (42) or (43). Then, there exist two strictly positive constants  $C$  and  $s_0$  such that every solution  $v$  of (6) in  $\mathcal{V}_2$  satisfies, for all  $s \geq s_0$ ,*

$$\begin{aligned}
& \int_Q \left( s\Theta (v_x)^2 + s^3\Theta^3 \left(\frac{x-x_0}{k}\right)^2 v^2 \right) e^{2s\Gamma} dx dadt \\
& \leq C \left( \int_Q \frac{f^2}{k} dx dadt + \int_0^T \int_0^A \int_{\omega} \frac{v^2}{k} dx dadt \right).
\end{aligned}$$

PROOF. First of all assume that  $\omega = (\alpha, \rho) \subset (0, 1)$  is such that  $x_0 \in \omega$  and  
 475 take  $\omega_i$ ,  $i = 1, 2$ , as in Remark 4.1. Now, fix  $\bar{\lambda}_i, \bar{\rho}_i \in \omega_i = (\lambda_i, \rho_i)$ ,  $i = 1, 2$ , such  
 that  $\bar{\lambda}_i < \bar{\rho}_i$  and consider a smooth function  $\xi : [0, 1] \rightarrow [0, 1]$  such that

$$\xi(x) = \begin{cases} 0 & x \in [0, \bar{\lambda}_1], \\ 1 & x \in [\bar{\lambda}_1, \bar{\lambda}_2], \\ 0 & x \in [\bar{\rho}_2, 1], \end{cases}$$

where  $\tilde{\lambda}_i = (\bar{\lambda}_i + \bar{\rho}_i)/2$ ,  $i = 1, 2$ . Then, define  $w := \xi v$ , where  $v$  is any fixed  
 solution of (6). Hence, neglecting the final-time datum (of no interest in this  
 context),  $w$  satisfies

$$\begin{cases} w_t + w_a + kw_{xx} - \mu w = \xi f + k(\xi_{xx}v + 2\xi_x v_x) =: F, & (t, x) \in Q, \\ w(t, a, 0) = w(t, a, 1) = 0, & t \in Q_{T,A}. \end{cases}$$

480 Applying Theorem 3.6 and using the fact that  $w \equiv 0$  in a neighborhood of  $x = 0$   
 and  $x = 1$ , we have

$$\int_Q \left( s\Theta(w_x)^2 + s^3\Theta^3 \left( \frac{x-x_0}{k} \right)^2 w^2 \right) e^{2s\Gamma} dx dadt \leq C \int_Q \frac{e^{2s\Gamma}}{k} F^2 dx dadt, \quad (56)$$

for all  $s \geq s_0$ . Then, using the definition of  $\xi$  and in particular the fact that  $\xi_x$   
 and  $\xi_{xx}$  are supported in  $\hat{\omega}$ , where  $\hat{\omega} := (\bar{\lambda}_1, \bar{\lambda}_1) \cup (\bar{\lambda}_2, \bar{\rho}_2)$ , we can write

$$\frac{F^2}{k} \leq 2\frac{f^2}{k} + 2k(\xi_{xx}v + 2\xi_x v_x)^2 \leq 2\frac{f^2}{k} + C(v^2 + (v_x)^2)\chi_{\hat{\omega}}. \quad (57)$$

Hence, we find

$$\begin{aligned} & \int_0^T \int_0^A \int_{\bar{\lambda}_1}^{\bar{\lambda}_2} \left( s\Theta(v_x)^2 + s^3\Theta^3 \left( \frac{x-x_0}{k} \right)^2 v^2 \right) e^{2s\Gamma} dx dadt \\ &= \int_0^T \int_0^A \int_{\bar{\lambda}_1}^{\bar{\lambda}_2} \left( s\Theta(w_x)^2 + s^3\Theta^3 \left( \frac{x-x_0}{k} \right)^2 w^2 \right) e^{2s\Gamma} dx dadt \\ &\leq C \left( \int_0^T \int_0^A \int_{\hat{\omega}} e^{2s\Gamma} (v^2 + (v_x)^2) dx dadt + \int_Q f^2 \frac{e^{2s\Gamma}}{k} dx dadt \right). \end{aligned} \quad (58)$$

(by Proposition 4.3, since  $\hat{\omega} \subset \subset \omega$ )

$$\leq C \left( \int_0^T \int_0^A \int_{\omega} \frac{v^2}{k} dx dadt + \int_Q \frac{f^2}{k} dx dadt \right).$$

485 Now, consider a smooth function  $\eta : [0, 1] \rightarrow [0, 1]$  such that

$$\eta(x) = \begin{cases} 0 & x \in [0, \bar{\lambda}_2], \\ 1 & x \in [\tilde{\lambda}_2, 1], \end{cases}$$

and define  $z := \eta v$ . Then  $z$  satisfies

$$\begin{cases} z_t + z_a + k z_{xx} - \mu z = h, & (t, x) \in Q_{T,A} \times (\lambda_2, 1), \\ z(t, a, \lambda_2) = z(t, a, 1) = 0, & t \in Q_{T,A}, \end{cases} \quad (59)$$

with  $h := \eta f + k(\eta_{xx}v + 2\eta_x v_x) \in L^2((0, T) \times (\lambda_2, 1))$ .

Since the problem is *non degenerate* (observe that  $x \in (\lambda_2, 1)$ ), we can apply Theorem 3.3, with  $(0, 1)$  replaced by  $(\lambda_2, 1)$  and Proposition 4.3, obtaining that

490 there exist two strictly positive constants  $C$  and  $s_0$  such that, for all  $s \geq s_0$ ,

$$\begin{aligned} & \int_0^T \int_0^A \int_{\lambda_2}^1 (s\Theta(z_x)^2 + s^3\Theta^3 z^2) e^{2s\Phi} dx dadt \leq C \int_0^T \int_0^A \int_{\lambda_2}^1 h^2 e^{2s\Phi} dx dadt \\ & \leq C \left( \int_0^T \int_0^A \int_{\tilde{\omega}} e^{2s\Phi} (v^2 + (v_x)^2) dx dadt + \int_Q f^2 \frac{e^{2s\Phi}}{k} dx dadt \right) \\ & \leq C \left( \int_0^T \int_0^A \int_{\lambda_2}^{\beta_2} \frac{v^2}{k} dx dadt + \int_Q f^2 \frac{e^{2s\Phi}}{k} dx dadt \right) \\ & \leq C \left( \int_0^T \int_0^A \int_{\omega} \frac{v^2}{k} dx dadt + \int_Q f^2 \frac{e^{2s\Phi}}{k} dx dadt \right), \end{aligned} \quad (60)$$

where  $\tilde{\omega} = (\bar{\lambda}_2, \tilde{\lambda}_2)$  and  $\Phi$  is related to  $(\lambda_2, 1)$ . Observe that the boundary term which appears in the original estimate is nonpositive and thus is neglected.

Now, for a suitable choice of  $d_1$  (see, for example, [31]), there exists a strictly positive constant  $C$ , such that

$$e^{2s\Gamma(t,x)} \leq C e^{2s\Phi(t,x)} \quad (61)$$

495 and

$$\left( \frac{x - x_0}{k(x)} \right)^2 e^{2s\Gamma(t,x)} \leq C e^{2s\Phi(t,x)} \quad (62)$$

for all  $(t, x) \in Q_{T,A} \times [\lambda_2, 1]$ . Thus, by (61) and (62), via (60), we find

$$\begin{aligned} & \int_0^T \int_0^A \int_{\lambda_2}^1 \left( s\Theta(z_x)^2 + s^3\Theta^3 \left( \frac{x-x_0}{k} \right)^2 z^2 \right) e^{2s\Gamma} dx dadt \\ & \leq C \left( \int_0^T \int_0^A \int_{\omega} \frac{v^2}{k} dx dadt + \int_Q f^2 \frac{e^{2s\Phi}}{k} dx dadt \right), \end{aligned}$$

for a strictly positive constant  $C$  and  $s$  large enough. Hence, by definition of  $z$  and by the inequality above, we get

$$\begin{aligned} & \int_0^T \int_0^A \int_{\tilde{\lambda}_2}^1 \left( s\Theta(v_x)^2 + s^3\Theta^3 \left( \frac{x-x_0}{k} \right)^2 v^2 \right) e^{2s\Gamma} dx dadt \\ & \leq \int_0^T \int_0^A \int_{\lambda_2}^1 \left( s\Theta(z_x)^2 + s^3\Theta^3 \left( \frac{x-x_0}{k} \right)^2 z^2 \right) e^{2s\Gamma} dx dadt \quad (63) \\ & \leq C \left( \int_0^T \int_0^A \int_{\omega} \frac{v^2}{k} dx dadt + \int_Q f^2 \frac{e^{2s\Phi}}{k} dx dadt \right). \end{aligned}$$

Thus (58) and (63) imply

$$\begin{aligned} & \int_0^T \int_0^A \int_{\tilde{\lambda}_1}^1 \left( s\Theta(v_x)^2 + s^3\Theta^3 \left( \frac{x-x_0}{k} \right)^2 v^2 \right) e^{2s\Gamma} dx dadt \\ & \leq C \left( \int_0^T \int_0^A \int_{\omega} \frac{v^2}{k} dx dadt + \int_Q \frac{f^2}{k} dx dadt \right). \quad (64) \end{aligned}$$

500 To complete the proof it is sufficient to prove a similar inequality for  $x \in [0, \tilde{\lambda}_1]$ .

To this aim, we use the reflection procedure of [30] or [31], considering the functions

$$\begin{aligned} W(t, a, x) &:= \begin{cases} v(t, a, x), & x \in [0, 1], \\ -v(t, a, -x), & x \in [-1, 0], \end{cases} \\ \tilde{f}(t, a, x) &:= \begin{cases} f(t, a, x), & x \in [0, 1], \\ -f(t, a, -x), & x \in [-1, 0], \end{cases} \end{aligned}$$

and

$$\tilde{\mu}(t, a, x) := \begin{cases} \mu(t, a, x), & x \in [0, 1], \\ -\mu(t, a, -x), & x \in [-1, 0], \end{cases}$$

505 so that  $W$  satisfies the problem

$$\begin{cases} W_t + W_a + \tilde{k}W_{xx} - \tilde{\mu}W = \tilde{f}, & (t, x) \in Q_{T,A} \times (-1, 1), \\ W(t, a, -1) = W(t, a, 1) = 0, & t \in Q_{T,A}. \end{cases}$$

Here  $\tilde{k}$  is as in (46).

Now, consider a cut off function  $\zeta : [-1, 1] \rightarrow [0, 1]$  such that

$$\zeta(x) = \begin{cases} 0 & x \in [-1, -\bar{\rho}_1], \\ 1 & x \in [-\tilde{\lambda}_1, \tilde{\lambda}_1], \\ 0 & x \in [\bar{\rho}_1, 1], \end{cases}$$

and define  $Z := \zeta W$ . Then  $Z$  satisfies

$$\begin{cases} Z_t + Z_a + \tilde{k}Z_{xx} - \tilde{\mu}Z = \tilde{h}, & (t, x) \in Q_{T,A} \times (-\rho_1, \rho_1), \\ Z(t, a, -\rho_1) = Z(t, a, \rho_1) = 0, & t \in Q_{T,A}, \end{cases} \quad (65)$$

where  $\tilde{h} = \zeta \tilde{f} + \tilde{k}(\rho_{xx}W + 2\tilde{\rho}_x W_x)$ . Now, applying the analogue of Theorem 3.3 on  $(-\rho_1, \rho_1)$  in place of  $(0, 1)$ , using the definition of  $W$ , the fact that  $Z_x(t, a, -\rho_1) = Z_x(t, a, \rho_1) = 0$ , analogous estimates of (61) and (62) and since  $\zeta$  is supported in  $[-\bar{\rho}_1, -\tilde{\lambda}_1] \cup [\tilde{\lambda}_1, \bar{\rho}_1]$ , we get

$$\begin{aligned} & \int_0^T \int_0^A \int_0^{\tilde{\lambda}_1} \left( s\Theta(W_x)^2 + s^3\Theta^3 \left( \frac{x-x_0}{k} \right)^2 W^2 \right) e^{2s\Gamma} dx dadt \\ &= \int_0^T \int_0^A \int_0^{\tilde{\lambda}_1} \left( s\Theta(Z_x)^2 + s^3\Theta^3 \left( \frac{x-x_0}{k} \right)^2 Z^2 \right) e^{2s\Gamma} dx dadt \\ &\leq C \int_0^T \int_0^A \int_0^{\rho_1} (s\Theta(Z_x)^2 + s^3\Theta^3 Z^2) e^{2s\Phi} dx dadt \\ &\leq C \int_0^T \int_0^A \int_{-\rho_1}^{\rho_1} (s\Theta(Z_x)^2 + s^3\Theta^3 Z^2) e^{2s\Phi} dx dadt \\ &\leq C \int_0^T \int_0^A \int_{-\rho_1}^{\rho_1} \tilde{h}^2 \frac{e^{2s\Phi}}{\tilde{k}} dx dadt \leq C \int_0^T \int_0^A \int_{-\rho_1}^{\rho_1} \tilde{f}^2 \frac{e^{2s\Phi}}{\tilde{k}} dx dadt \\ &+ C \int_0^T \int_0^A \int_{-\bar{\rho}_1}^{-\tilde{\lambda}_1} (W^2 + (W_x)^2) e^{2s\Phi} dx dadt + C \int_0^T \int_0^A \int_{\tilde{\lambda}_1}^{\bar{\rho}_1} (W^2 + (W_x)^2) e^{2s\Phi} dx dadt \\ &\leq C \int_0^T \int_0^A \int_{-\rho_1}^{\rho_1} \frac{\tilde{f}^2}{\tilde{k}} dx dadt + C \int_0^T \int_0^A \int_{-\rho_1}^{-\tilde{\lambda}_1} W^2 dx dadt + C \int_0^T \int_0^A \int_{\tilde{\lambda}_1}^{\rho_1} W^2 dx dadt \\ &\text{(by Propositions 4.3 and since } \tilde{f}(t, a, x) = -f(t, a, -x), \text{ for } x < 0) \\ &\leq C \int_0^T \int_0^A \int_0^1 \frac{f^2}{k} dx dadt + C \int_0^T \int_0^A \int_\omega v^2 dx dadt, \end{aligned}$$

for some strictly positive constants  $C$  and  $s$  large enough. Here  $\Phi$  is related to  $(-\rho_1, \rho_1)$ .

515 Hence, by definitions of  $Z$ ,  $W$  and  $\zeta$ , and using the previous inequality one has

$$\begin{aligned}
& \int_0^T \int_0^A \int_0^{\tilde{\lambda}_1} \left( s\Theta(v_x)^2 + s^3\Theta^3\left(\frac{x-x_0}{k}\right)^2 v^2 \right) e^{2s\Gamma} dx dadt \\
&= \int_0^T \int_0^A \int_0^{\tilde{\lambda}_1} \left( s\Theta(W_x)^2 + s^3\Theta^3\left(\frac{x-x_0}{k}\right)^2 W^2 \right) e^{2s\Gamma} dx dadt \quad (66) \\
&\leq C \left( \int_Q \frac{f^2}{k} dx dadt + \int_0^T \int_0^A \int_\omega \frac{v^2}{k} dx dadt \right).
\end{aligned}$$

Moreover, by (64) and (66), the conclusion follows.

Nothing changes in the proof if  $\omega = \omega_1 \cup \omega_2$  and each of these intervals lye on different sides of  $x_0$ , as the assumption implies.

520 **Remark 4.3.** Observe that the results of Theorems 4.1, 4.2, 4.3 still hold true if we substitute the interval  $(0, T)$  with a general interval  $(T_1, T_2)$ , provided that  $\mu$  and  $\beta$  satisfy the required assumptions. In this case, in place of the function  $\Theta$  defined in (8), we have to consider the weight function

$$\tilde{\Theta}(t, a) := \frac{1}{(t - T_1)^4 (T_2 - t)^4 a^4}.$$

Using the previous local Carleman estimates one can prove the next observability inequalities.

**Theorem 4.4.** Assume Hypothesis 3.2 or 3.3 and Hypothesis 4.2 with  $T > \bar{a}$ . Suppose that  $\omega$  satisfies (41). Then, there exists a strictly positive constant  $C$  such that, for every  $\delta \in (0, A)$ , every solution  $v$  of (48) in  $\mathcal{V}_1$  satisfies

$$\begin{aligned}
\int_0^A \int_0^1 \frac{1}{k} v^2(T - \bar{a}, a, x) dx da &\leq C \int_0^T \int_0^\delta \int_0^1 \frac{1}{k} v^2(t, a, x) dx dadt \\
&+ C \left( \int_0^T \int_0^1 \frac{v_T^2(a, x)}{k} dx da + \int_0^T \int_0^A \int_\omega \frac{v^2}{k} dx dadt \right).
\end{aligned}$$

Moreover, if  $v_T(a, x) = 0$  for all  $(a, x) \in (0, T) \times (0, 1)$ , one has

$$\int_0^A \int_0^1 \frac{1}{k} v^2(T - \bar{a}, a, x) dx da \leq C \left( \int_0^T \int_0^\delta \int_0^1 \frac{1}{k} v^2(t, a, x) dx dadt + \int_0^T \int_0^A \int_\omega \frac{v^2}{k} dx dadt \right).$$

530 PROOF. Set

$$\tilde{T} := T - \bar{a}. \quad (67)$$

Using the method of characteristic lines, the assumption on  $\beta$  and the fact that  $v(t, A, x) = 0$  for all  $(t, x) \in Q_{T,1}$ , one can compute the following implicit formula for  $v$  solution of (48):

$$S(T - t)v_T(T + a - t, \cdot), \quad (68)$$

if  $t \geq \tilde{T} + a$  (observe that in this case  $T + a - t \leq \bar{a}$ ) and

$$v(t, a, \cdot) = \begin{cases} S(T - t)v_T(T + a - t, \cdot) + \int_a^{T+a-t} S(s - a)\beta(s, \cdot)v(s + t - a, 0, \cdot)ds, & \Gamma = \bar{a} \\ \int_a^A S(s - a)\beta(s, \cdot)v(s + t - a, 0, \cdot)ds, & \Gamma = \Gamma_{A,T}, \end{cases} \quad (69)$$

535 otherwise. Here  $(S(t))_{t \geq 0}$  is the semigroup generated by the operator  $\mathcal{A}_0 - \mu Id$  for all  $u \in D(\mathcal{A}_0)$  ( $Id$  is the identity operator),  $\Gamma_{A,T} := A - a + t - \tilde{T}$  and

$$\Gamma := \min\{\bar{a}, \Gamma_{A,T}\}. \quad (70)$$

In particular, it results

$$v(t, 0, \cdot) := S(T - t)v_T(T - t, \cdot). \quad (71)$$

Now, define, for  $\varsigma > 0$ , the function  $w = e^{\varsigma t}v$ , where  $v$  solves (48). Then  $w$  satisfies

$$\begin{cases} \frac{\partial w}{\partial t} + \frac{\partial w}{\partial a} + k(x)w_{xx} - (\mu(t, a, x) + \varsigma)w = -\beta(a, x)w(t, 0, x), & (t, x, a) \in \tilde{Q}, \\ w(t, a, 0) = w(t, a, 1) = 0, & (t, a) \in \tilde{Q}_{T,A}, \\ w(T, a, x) = e^{\varsigma T}v_T(a, x), & (a, x) \in Q_{A,1}, \\ w(t, A, x) = 0, & (t, x) \in \tilde{Q}_{T,1}, \end{cases} \quad (72)$$

540 where  $\tilde{Q} := (\tilde{T}, T) \times Q_{A,1}$ ,  $\tilde{Q}_{T,A} := (\tilde{T}, T) \times (0, A)$  and  $\tilde{Q}_{T,1} := (\tilde{T}, T) \times (0, 1)$ . Multiplying the equation of (72) by  $\frac{w}{k}$  and integrating by parts on  $Q_t := (\tilde{T}, t) \times$



$(0, A) \times (0, 1)$ , it results

$$\begin{aligned}
& -\frac{1}{2} \int_{Q_{A,1}} \frac{1}{k} w^2(t, a, x) dx da + \frac{e^{\varsigma \tilde{T}}}{2} \int_{Q_{A,1}} \frac{1}{k} v^2(\tilde{T}, a, x) dx da + \frac{1}{2} \int_0^t \int_0^1 \frac{1}{k} w^2(\tau, 0, x) dx d\tau \\
& + \varsigma \int_{Q_t} \frac{1}{k} w^2(\tau, a, x) dx d\tau \leq \int_{Q_t} \frac{1}{k} \beta w(\tau, 0, x) w dx d\tau \\
& \leq \|\beta\|_{L^\infty(Q)} \frac{1}{\epsilon} \int_{Q_t} \frac{1}{k} w^2 dx d\tau + \epsilon A \|\beta\|_{L^\infty(Q)} \int_0^t \int_0^1 \frac{1}{k} w^2(\tau, 0, x) dx d\tau,
\end{aligned} \tag{73}$$

for  $\epsilon > 0$ . Choosing  $\epsilon = \frac{1}{2\|\beta\|_{L^\infty(Q)}A}$  and  $\varsigma = \frac{\|\beta\|_{L^\infty(Q)}}{\epsilon}$ , we have

$$\int_{Q_{A,1}} \frac{1}{k} v^2(\tilde{T}, a, x) dx da \leq C \int_{Q_{A,1}} \frac{1}{k} w^2(t, a, x) dx da \leq C \int_{Q_{A,1}} \frac{1}{k} v^2(t, a, x) dx da.$$

Now, take  $\delta \in (0, A)$ . Then, integrating over  $\left[\frac{T}{4}, \frac{3T}{4}\right]$ ,

$$\begin{aligned}
\int_{Q_{A,1}} \frac{1}{k} v^2(\tilde{T}, a, x) dx da & \leq C \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{Q_{A,1}} \frac{1}{k} v^2(t, a, x) dx d\tau \\
& = C \int_{\frac{T}{4}}^{\frac{3T}{4}} \left( \int_0^\delta + \int_\delta^A \right) \int_0^1 \frac{1}{k} v^2(t, a, x) dx d\tau.
\end{aligned} \tag{74}$$

545 Consider the term  $\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_\delta^A \int_0^1 \frac{1}{k} v^2(t, a, x) dx d\tau$ . By the Hardy - Poincaré inequality one has

$$\int_0^1 \frac{v^2}{k} dx \leq C \int_0^1 \frac{v^2}{x^2} dx \leq C \int_0^1 v_x^2 dx, \tag{75}$$

for a strictly positive constant  $C$ . Hence,

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_\delta^A \int_0^1 \frac{1}{k} v^2(t, a, x) dx d\tau \leq C \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_\delta^A \int_0^1 \Theta v_x^2 e^{2s\varphi} dx d\tau$$

and, by Theorem 4.1 or 4.2,

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_\delta^A \int_0^1 \Theta v_x^2 e^{2s\varphi} dx d\tau \leq C \left( \int_Q \frac{f^2}{k} dx d\tau + \int_0^T \int_0^A \int_\omega \frac{v^2}{k} dx d\tau \right),$$

where, in this case,  $f(t, a, x) := -\beta(a, x)v(t, 0, x)$ . Thus

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_\delta^A \int_0^1 \frac{1}{k} v^2(t, a, x) dx d\tau \leq C \|\beta\|_{L^\infty(Q)}^2 \left( \int_Q \frac{v^2(t, 0, x)}{k} dx d\tau + \int_0^T \int_0^A \int_\omega \frac{v^2}{k} dx d\tau \right), \tag{76}$$

550 for a strictly positive constant  $C$ . Now, using the fact that the semigroup generated by  $\mathcal{A}_0$  is a contraction semigroup and the hypothesis on  $\mu$ , we have that also the semigroup generated by  $\mathcal{A}_0 - \mu Id$  is bounded. Hence, by (71),

$$\int_Q \frac{v^2(t, 0, x)}{k} dx dadt \leq C \int_{Q_{T,1}} \frac{v_T^2(T-t, x)}{k} dx dt \leq C \int_{Q_{T,1}} \frac{v_T^2(a, x)}{k} dx da. \quad (77)$$

Hence, by (76) and (77), one has

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{\delta}^A \int_0^1 \frac{1}{k} v^2(t, a, x) dx dadt \leq C \|\beta\|_{L^\infty(Q)}^2 \left( \int_{Q_{T,1}} \frac{v_T^2(a, x)}{k} dx da + \int_0^T \int_0^A \int_{\omega} \frac{v^2}{k} dx dadt \right), \quad (78)$$

for a strictly positive constant  $C$ . From (74) and (78), it results

$$\begin{aligned} \int_{Q_{A,1}} \frac{1}{k} v^2(\tilde{T}, a, x) dx da &\leq C \int_0^T \int_0^{\delta} \int_0^1 \frac{1}{k} v^2(t, a, x) dx dadt \\ &+ C \left( \int_{Q_{T,1}} \frac{v_T^2(a, x)}{k} dx da + \int_0^T \int_0^A \int_{\omega} \frac{v^2}{k} dx dadt \right). \end{aligned} \quad (79)$$

555 The observability inequality proved in the previous theorem still holds when  $k$  degenerate at  $x_0$ :

**Theorem 4.5.** Assume Hypothesis 3.4, Hypothesis 4.2 and suppose that  $\frac{k'}{\sqrt{k}} \in L_{loc}^\infty([0, 1] \setminus \{x_0\})$  if Hypothesis 2.2 holds with  $T > \bar{a}$ . Suppose that  $\omega$  satisfies (42) or (43). Then, there exists a strictly positive constant  $C$  such that, for

560 every  $\delta \in (0, A)$ , every solution  $v$  of (48) in  $\mathcal{V}_2$  satisfies

$$\begin{aligned} \int_0^A \int_0^1 \frac{1}{k} v^2(T - \bar{a}, a, x) dx da &\leq C \int_0^T \int_0^{\delta} \int_0^1 \frac{1}{k} v^2(t, a, x) dx dadt \\ &+ C \left( \int_0^T \int_0^1 \frac{v_T^2(a, x)}{k} dx da + \int_0^T \int_0^A \int_{\omega} \frac{v^2}{k} dx dadt \right). \end{aligned}$$

Moreover, if  $v_T(a, x) = 0$  for all  $(a, x) \in (0, T) \times (0, 1)$ , one has

$$\int_0^A \int_0^1 \frac{1}{k} v^2(T - \bar{a}, a, x) dx da \leq C \left( \int_0^T \int_0^{\delta} \int_0^1 \frac{1}{k} v^2(t, a, x) dx dadt + \int_0^T \int_0^A \int_{\omega} \frac{v^2}{k} dx dadt \right).$$

The proof of the previous inequalities follows the one of Theorem 4.1 so we omit it. But we underline the fact that, in order to obtain (75) in this situation, we

distinguish the cases  $M < 1$  and  $M \geq 1$ . In the former case, define

$$p(x) = \frac{|x - x_0|^2}{k} \quad \text{and} \quad q = 2 - M.$$

565 Clearly, by [30, Lemma 2.1],  $p(x) \rightarrow 0$  as  $x \rightarrow x_0$  and  $x \mapsto \frac{p(x)}{|x - x_0|^q} = \frac{|x - x_0|^M}{k}$  is nonincreasing on the left of  $x = x_0$  and nondecreasing on the right of  $x = x_0$ . Moreover,  $q > 1$  since  $M < 1$ . Hence, by the Hardy-Poincaré inequality given in [30, Proposition 2.3], one has

$$\begin{aligned} \int_0^1 \frac{v^2}{k} dx &= \int_0^1 \frac{p(x)}{(x - x_0)^2} v^2 dx \leq C_{HP} \int_0^1 p(v_x)^2 dx \\ &\leq C \int_0^1 (v_x)^2 dx, \end{aligned}$$

for a strictly positive constant  $C$ .

570 If  $M \geq 1$ , we can apply [33, Lemma 3.7] obtaining again

$$\int_0^1 \frac{v^2}{k} dx \leq C \int_0^1 (v_x)^2 dx,$$

for a strictly positive constant  $C$ .

Hence, in both cases (75) holds also if the degeneracy is in the interior of the domain. So, proceeding as before, we obtain the thesis.

**Corollary 4.1.** *Assume  $\bar{a} = T$ , Hypotheses 4.1 and 4.2. Suppose that  $\omega$  satisfies (41), (42) or (43). Then, there exists a strictly positive constant  $C$  such*  
575 *that, for every  $\delta \in (0, A)$ , every solution  $v$  of (48) in  $\mathcal{V}_i$ ,  $i = 1, 2$ , satisfies*

$$\begin{aligned} \int_0^A \int_0^1 \frac{1}{k} v^2(0, a, x) dx da &\leq C \int_0^T \int_0^\delta \int_0^1 \frac{1}{k} v^2(t, a, x) dx da dt \\ &+ C \left( \int_0^T \int_0^1 \frac{v_T^2(a, x)}{k} dx da + \int_0^T \int_0^A \int_\omega \frac{v^2}{k} dx da dt \right). \end{aligned}$$

Moreover, if  $v_T(a, x) = 0$  for all  $(a, x) \in (0, T) \times (0, 1)$ , one has

$$\int_0^A \int_0^1 \frac{1}{k} v^2(0, a, x) dx da \leq C \left( \int_0^T \int_0^\delta \int_0^1 \frac{1}{k} v^2(t, a, x) dx da dt + \int_0^T \int_0^A \int_\omega \frac{v^2}{k} dx da dt \right).$$

Actually, we can improve the previous results in the following way:

**Theorem 4.6.** Assume Hypotheses 4.1 and 4.2. Suppose that  $\omega$  satisfies (41),

580 (42) or (43). Then, there exists a strictly positive constant  $C$  such that, for every  $\delta \in (T, A)$ , every solution  $v$  of (48) in  $\mathcal{V}_i$ ,  $i = 1, 2$ , satisfies

$$\int_0^A \int_0^1 \frac{1}{k} v^2(T - \bar{a}, a, x) dx da \leq C \left( \int_0^\delta \int_0^1 \frac{v_T^2(a, x)}{k} dx da + \int_0^T \int_0^A \int_\omega \frac{v^2}{k} dx dadt \right).$$

PROOF. We distinguish between the two cases  $T = \bar{a}$  and  $T > \bar{a}$ .

If  $T = \bar{a}$ : Taking  $\delta \in (T, A)$ , one has, as in (74),

$$\int_{Q_{A,1}} \frac{1}{k} v^2(0, a, x) dx da \leq C \int_{\frac{T}{4}}^{\frac{3T}{4}} \left( \int_0^{\delta - \frac{3T}{4}} + \int_{\delta - \frac{3T}{4}}^A \right) \int_0^1 \frac{1}{k} v^2(t, a, x) dx dadt. \quad (80)$$

As for (78),

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{\delta - \frac{3T}{4}}^A \int_0^1 \frac{1}{k} v^2(t, a, x) dx dadt \leq C \left( \int_{Q_{T,1}} \frac{v_T^2(a, x)}{k} dx da + \int_0^T \int_0^A \int_\omega \frac{v^2}{k} dx dadt \right). \quad (81)$$

585 Now, consider the term  $\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^{\delta - \frac{3T}{4}} \int_0^1 \frac{1}{k} v^2(t, a, x) dx dadt$  and let us prove that there exists  $C > 0$  such that

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^{\delta - \frac{3T}{4}} \int_0^1 \frac{1}{k} v^2(t, a, x) dx dadt \leq C \int_0^\delta \int_0^1 \frac{v_T^2(a, x)}{k} dx da. \quad (82)$$

In order to prove (82), we use (68) or (69). Observe, first of all, that  $\delta - \frac{3T}{4} > \frac{T}{4}$ ,

but we do not know if  $\delta - \frac{3T}{4} \geq \frac{3T}{4}$  or  $\delta - \frac{3T}{4} < \frac{3T}{4}$ . In the last case, if  $t \in \left[ \delta - \frac{3T}{4}, \frac{3T}{4} \right)$  and  $a \in \left( 0, \delta - \frac{3T}{4} \right)$ , we have easily that  $t \geq \tilde{T} + a = a$

590 (recall that we are in the case  $\tilde{T} = 0$ ); hence (68) holds. On the other hand,

if  $t < \delta - \frac{3T}{4}$  we do not know if  $t \geq \tilde{T} + a = a$  or  $t < \tilde{T} + a = a$ . Hence, we have to consider (68) or (69). Taking into account these considerations, using

the assumption on  $\beta$  and the boundedness of  $(S(t))_{t \geq 0}$ , one has:

If  $\delta - \frac{3T}{4} < \frac{3T}{4}$ :

$$\begin{aligned} \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^{\delta - \frac{3T}{4}} \int_0^1 \frac{1}{k} v^2(t, a, x) dx dadt &= \int_{\frac{T}{4}}^{\delta - \frac{3T}{4}} \int_0^{\delta - \frac{3T}{4}} \int_0^1 \frac{1}{k} v^2(t, a, x) dx dadt \\ &+ \int_{\delta - \frac{3T}{4}}^{\frac{3T}{4}} \int_0^{\delta - \frac{3T}{4}} \int_0^1 \frac{1}{k} v^2(t, a, x) dx dadt. \end{aligned} \quad (83)$$

595 By (68),

$$\begin{aligned}
& \int_{\delta-\frac{3T}{4}}^{\frac{3T}{4}} \int_0^{\delta-\frac{3T}{4}} \int_0^1 \frac{1}{k} v^2(t, a, x) dx dadt \leq C \int_{\delta-\frac{3T}{4}}^{\frac{3T}{4}} \int_0^{\delta-\frac{3T}{4}} \int_0^1 \frac{v_T^2(T+a-t, x)}{k} dx dadt \\
& \leq C \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^{\delta-\frac{3T}{4}} \int_0^1 \frac{v_T^2(T+a-t, x)}{k} dx dadt \\
& \leq C \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^{\delta-\frac{3T}{4}} \int_0^1 \frac{v_T^2(a+z, x)}{k} dx dadz \leq C \int_0^{\delta-\frac{3T}{4}} \int_{a+\frac{T}{4}}^{a+\frac{3T}{4}} \int_0^1 \frac{v_T^2(\sigma, x)}{k} dx d\sigma da \\
& \leq C \int_0^\delta \int_0^1 \frac{v_T^2(\sigma, x)}{k} dx d\sigma.
\end{aligned} \tag{84}$$

Consider now the integral

$$\int_{\frac{T}{4}}^{\delta-\frac{3T}{4}} \int_0^{\delta-\frac{3T}{4}} \int_0^1 \frac{1}{k} v^2(t, a, x) dx dadt$$

and divide it in the following way:

$$\begin{aligned}
\int_{\frac{T}{4}}^{\delta-\frac{3T}{4}} \int_0^{\delta-\frac{3T}{4}} \int_0^1 \frac{1}{k} v^2(t, a, x) dx dadt &= \int_{\frac{T}{4}}^{\delta-\frac{3T}{4}} \int_0^t \int_0^1 \frac{1}{k} v^2(t, a, x) dx dadt \\
&+ \int_{\frac{T}{4}}^{\delta-\frac{3T}{4}} \int_t^{\delta-\frac{3T}{4}} \int_0^1 \frac{1}{k} v^2(t, a, x) dx dadt.
\end{aligned} \tag{85}$$

Proceeding as before, one can prove

$$\int_{\frac{T}{4}}^{\delta-\frac{3T}{4}} \int_0^t \int_0^1 \frac{1}{k} v^2(t, a, x) dx dadt \leq C \int_0^\delta \int_0^1 \frac{v_T^2(\sigma, x)}{k} dx d\sigma. \tag{86}$$

Indeed, since  $\tilde{T} = 0$  and  $a \leq t$ , (68) holds, hence

$$\begin{aligned}
& \int_{\frac{T}{4}}^{\delta-\frac{3T}{4}} \int_0^t \int_0^1 \frac{1}{k} v^2(t, a, x) dx dadt \leq C \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^t \int_0^1 \frac{v_T^2(T+a-t, x)}{k} dx dadt \\
& \leq C \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^{T-z} \int_0^1 \frac{v_T^2(a+z, x)}{k} dx dadz \leq C \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^T \int_0^1 \frac{v_T^2(\sigma, x)}{k} dx d\sigma dz \\
& \leq C \int_0^\delta \int_0^1 \frac{v_T^2(\sigma, x)}{k} dx d\sigma.
\end{aligned}$$

600 Now, we estimate the second term in the right hand side of (85). First of all, assume that  $\Gamma = \bar{a}$  (we recall that  $\Gamma$  is defined in (70)). By (69) and (71), using

the assumption on  $\beta$  and the boundedness of  $(S(t))_{t \geq 0}$ , one has:

$$\begin{aligned}
& \int_{\frac{T}{4}}^{\delta - \frac{3T}{4}} \int_t^{\delta - \frac{3T}{4}} \int_0^1 \frac{1}{k} v^2(t, a, x) dx dadt \leq C \int_{\frac{T}{4}}^{\delta - \frac{3T}{4}} \int_t^{\delta - \frac{3T}{4}} \int_0^1 \frac{v_T^2(T + a - t, x)}{k} dx dadt \\
& + C \int_{\frac{T}{4}}^{\delta - \frac{3T}{4}} \int_t^{\delta - \frac{3T}{4}} \int_0^1 \left( \int_a^{T+a-t} \frac{v_T^2(T - s - t + a, x)}{k} ds \right) dx dadt \\
& \leq C \int_{\frac{T}{4}}^{\delta - \frac{3T}{4}} \int_0^{\delta - \frac{3T}{4}} \int_0^1 \frac{v_T^2(T + a - t, x)}{k} dx dadt \\
& + C \int_{\frac{T}{4}}^{\delta - \frac{3T}{4}} \int_t^{\delta - \frac{3T}{4}} \int_0^1 \left( \int_{-a}^{T-t-a} \frac{v_T^2(a + z, x)}{k} dz \right) dx dadt
\end{aligned}$$

(proceeding as in (83) for the first integral)

$$\leq C \int_0^\delta \int_0^1 \frac{v_T^2(\sigma, x)}{k} dx d\sigma + C \int_{\frac{T}{4}}^{\delta - \frac{3T}{4}} \int_t^{\delta - \frac{3T}{4}} \int_0^1 \left( \int_0^{T-t} \frac{v_T^2(\sigma, x)}{k} d\sigma \right) dx dadt$$

(since  $T < \delta$ )

$$\begin{aligned}
& \leq C \int_0^\delta \int_0^1 \frac{v_T^2(\sigma, x)}{k} dx d\sigma + C \int_{\frac{T}{4}}^{\delta - \frac{3T}{4}} \int_0^{\delta - \frac{3T}{4}} \int_0^1 \left( \int_0^\delta \frac{v_T^2(\sigma, x)}{k} d\sigma \right) dx dadt \\
& \leq C \int_0^\delta \int_0^1 \frac{v_T^2(\sigma, x)}{k} dx d\sigma.
\end{aligned}$$

(87)

Now, assume that  $\Gamma = A + \bar{a} - a + t - T$  (this implies that  $A - a \leq T - t$ ). By

(69) and (71), one has, as before:

$$\begin{aligned}
& \int_{\frac{T}{4}}^{\delta - \frac{3T}{4}} \int_t^{\delta - \frac{3T}{4}} \int_0^1 \frac{1}{k} v^2(t, a, x) dx dadt \\
& \leq C \int_{\frac{T}{4}}^{\delta - \frac{3T}{4}} \int_t^{\delta - \frac{3T}{4}} \int_0^1 \left( \int_a^A \frac{v_T^2(T - s - t + a, x)}{k} ds \right) dx dadt \\
& \leq C \int_{\frac{T}{4}}^{\delta - \frac{3T}{4}} \int_t^{\delta - \frac{3T}{4}} \int_0^1 \left( \int_{T-A-t}^{T-a-t} \frac{v_T^2(a + z, x)}{k} dz \right) dx dadt \\
& \leq C \int_{\frac{T}{4}}^{\delta - \frac{3T}{4}} \int_t^{\delta - \frac{3T}{4}} \int_0^1 \left( \int_{T-t-(A-a)}^{T-t} \frac{v_T^2(\sigma, x)}{k} d\sigma \right) dx dadt \quad (88) \\
& \text{(since } T < \delta \text{)} \\
& \leq C \int_{\frac{T}{4}}^{\delta - \frac{3T}{4}} \int_0^{\delta - \frac{3T}{4}} \int_0^1 \left( \int_0^\delta \frac{v_T^2(\sigma, x)}{k} d\sigma \right) dx dadt \\
& \leq C \int_0^\delta \int_0^1 \frac{v_T^2(\sigma, x)}{k} dx d\sigma.
\end{aligned}$$

605 Hence, in every case (82) holds.

By (80), (81) and (82), it follows that

$$\begin{aligned}
& \int_{Q_{A,1}} \frac{1}{k} v^2(0, a, x) dx da \leq C \int_{\frac{T}{4}}^{\delta - \frac{3T}{4}} \left( \int_0^{\delta - \frac{3T}{4}} + \int_{\delta - \frac{3T}{4}}^A \right) \int_0^1 \frac{1}{k} v^2(t, a, x) dx dadt \\
& \leq C \left( \int_0^\delta \int_0^1 \frac{v_T^2(a, x)}{k} dx da + \int_0^T \int_0^A \int_\omega \frac{v^2}{k} dx dadt \right).
\end{aligned}$$

If  $\delta - \frac{3T}{4} \geq \frac{3T}{4}$ : In order to obtain (82), we divide the integral

$$\int_{\frac{T}{4}}^{\delta - \frac{3T}{4}} \int_0^{\delta - \frac{3T}{4}} \int_0^1 \frac{1}{k} v^2(t, a, x) dx dadt$$

in the following way:

$$\begin{aligned}
& \int_{\frac{T}{4}}^{\delta - \frac{3T}{4}} \int_0^{\delta - \frac{3T}{4}} \int_0^1 \frac{1}{k} v^2(t, a, x) dx dadt = \int_{\frac{T}{4}}^{\delta - \frac{3T}{4}} \int_0^t \int_0^1 \frac{1}{k} v^2(t, a, x) dx dadt \\
& \quad + \int_{\frac{T}{4}}^{\delta - \frac{3T}{4}} \int_t^{\delta - \frac{3T}{4}} \int_0^1 \frac{1}{k} v^2(t, a, x) dx dadt. \quad (89)
\end{aligned}$$

Then, proceeding as before, the thesis follows.

610 If  $T > \bar{a}$ : We proceed as before substituting, for simplicity,  $\frac{T}{4}$  and  $\frac{3T}{4}$  with  $T - \bar{a}$  and  $T - \frac{\bar{a}}{4}$ , respectively. In particular, taking  $\delta \in (T, A)$ , we will consider, in place of (80), the following inequality:

$$\int_{Q_{A,1}} \frac{1}{k} v^2(T - \bar{a}, a, x) dx da \leq C \int_{T-\bar{a}}^{T-\frac{\bar{a}}{4}} \left( \int_0^{\delta-\bar{a}} + \int_{\delta-\bar{a}}^A \right) \int_0^1 \frac{1}{k} v^2(t, a, x) dx da dt. \quad (90)$$

Also in this case, since  $t \in \left(T - \bar{a}, T - \frac{\bar{a}}{4}\right)$  and  $a \in (0, \delta - \bar{a})$ , we do not know if  $t \geq \tilde{T} + a$  or  $t < \tilde{T} + a$ . Hence, to prove an estimate like (82), we have to  
615 consider different cases as before.

By Theorem 4.6 and using a density argument, one can prove Proposition 4.1. As a consequence one has the following null controllability results:

**Theorem 4.7.** *Assume Hypotheses 4.1 and 4.2. Then, given  $T > 0$  and  $y_0 \in L^2_{\frac{1}{k}}(Q_{A,1})$ , for every  $\delta \in (T, A)$  there exists a control  $f \in L^2_{\frac{1}{k}}(\tilde{Q})$  such that the  
620 solution  $y \in \mathcal{U}$  of*

$$\begin{cases} \frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} - k(x)y_{xx} + \mu(t, a, x)y = f(t, x, a)\chi_\omega & \text{in } \tilde{Q}, \\ y(t, a, 1) = y(t, a, 0) = 0 & \text{on } \tilde{Q}_{T,A}, \\ y(\tilde{T}, a, x) = y_0(a, x) & \text{in } Q_{A,1}, \\ y(t, 0, x) = \int_0^A \beta(a, x)y(t, a, x) da & \text{in } \tilde{Q}_{T,1}, \end{cases} \quad (91)$$

satisfies

$$y(T, a, x) = 0 \quad \text{a.e. } (a, x) \in (\delta, A) \times (0, 1).$$

Moreover, there exists  $C > 0$  such that

$$\|f\|_{L^2_{\frac{1}{k}}(\tilde{Q})} \leq C \|y_0\|_{L^2_{\frac{1}{k}}(Q_{A,1})}. \quad (92)$$

Here, we recall,  $\tilde{Q} = (\tilde{T}, T) \times (0, A) \times (0, 1)$ ,  $\tilde{Q}_{T,A} = (\tilde{T}, T) \times (0, A)$  and  $\tilde{Q}_{T,1} = (\tilde{T}, T) \times (0, 1)$ .

625 **PROOF.** Take  $g \in L^2_{\frac{1}{k}}(Q_{A,1})$  such that  $g(A, x) = 0$  in  $(0, 1)$  and fix  $\delta \in (T, A)$ .



Let  $v$  be the solution of

$$\begin{cases} \frac{\partial v}{\partial t} + \frac{\partial v}{\partial a} + k(x)v_{xx} - \mu(t, a, x)v = -\beta(t, x, a)v(t, 0, x), & (t, x, a) \in \tilde{Q}, \\ v(t, a, 0) = v(t, a, 1) = 0, & (t, a) \in \tilde{Q}_{T,A}, \\ v(T, a, x) = v_T(a, x) := \begin{cases} g(a, x), & (a, x) \in (\delta, A) \times (0, 1), \\ 0, & (a, x) \in (0, \delta) \times (0, 1), \end{cases} \\ v(t, A, x) = 0, & (t, x) \in \tilde{Q}_{T,1}. \end{cases} \quad (93)$$

Now, fixed  $y_0 \in L^2_{\frac{1}{k}}(Q_{A,1})$ , define as in [24],

$$J(g) = \frac{1}{2} \int_{\tilde{T}}^T \int_0^A \int_{\omega} \frac{v^2}{k} dx da dt + \int_0^A \int_0^1 \frac{1}{k} v(\tilde{T}, a, x) y_0(a, x) dx da.$$

The functional  $J$  is strictly convex, continuous and coercive over the Hilbert space  $\mathcal{H}$  defined by the completion of  $L^2((\delta, A) \times (0, 1))$  with respect to the norm  $\|v\|_{L^2(\tilde{Q}_{T,A} \times \omega)}$ . Thus, there exists a unique minimum,  $\hat{g}$ , of  $J$  and  $\hat{g}(A, x) = 0$  in  $(0, 1)$ . Let  $\hat{v}$  be the solution of (93) associated to  $\hat{g}$ . Define  $f := \hat{v}\chi_{\omega}$  and let  $y$  be the solution of (91) in  $\tilde{Q}$  associated to  $f$ . Since  $\hat{g}$  is the minimum of  $J$ , it results

$$0 = \left[ \frac{d}{dt} J(\hat{g} + tg) \right]_{t=0} = \int_{\tilde{T}}^T \int_0^A \int_{\omega} \frac{1}{k} v \hat{v} dx da dt + \int_0^A \int_0^1 \frac{1}{k} v(\tilde{T}, a, x) y_0(a, x) dx da, \quad (94)$$

for all  $g \in L^2(Q_{A,1})$  such that  $g(A, x) = 0$  in  $(0, 1)$ . In particular, for  $g = \hat{g}$ , one

has

$$0 = \int_{\tilde{T}}^T \int_0^A \int_{\omega} \frac{\hat{v}^2}{k} dx da dt + \int_0^A \int_0^1 \frac{1}{k} \hat{v}(\tilde{T}, a, x) y_0(a, x) dx da.$$

Hence

$$\int_{\tilde{T}}^T \int_0^A \int_{\omega} \frac{\hat{v}^2}{k} dx da dt = - \int_0^A \int_0^1 \frac{1}{k} \hat{v}(\tilde{T}, a, x) y_0(a, x) dx da, \quad (95)$$

and, by Hölder's inequality, by Proposition 4.1 applied to  $\hat{v}$  in  $\tilde{Q}$  and using the

fact that  $v_T(a, x) = 0$  for all  $(a, x) \in (0, \delta) \times (0, 1)$ , one has

$$\begin{aligned} \left| \int_0^A \int_0^1 \frac{1}{k} \hat{v}(\tilde{T}, a, x) y_0(a, x) dx da \right| &\leq \left( \int_0^A \int_0^1 \frac{1}{k} \hat{v}^2(\tilde{T}, a, x) dx da \right)^{\frac{1}{2}} \left( \int_0^A \int_0^1 \frac{1}{k} y_0^2(a, x) dx da \right)^{\frac{1}{2}} \\ &\leq C \left( \int_{\tilde{T}}^T \int_0^A \int_{\omega} \frac{\hat{v}^2}{k} dx dadt \right)^{\frac{1}{2}} \|y_0\|_{L^2_{\frac{1}{k}}(Q_{A,1})}. \end{aligned} \quad (96)$$

Thus, by (95) and (96),

$$\int_{\tilde{T}}^T \int_0^A \int_{\omega} \frac{1}{k} \hat{v}^2(t, a, x) dx dadt \leq C \left( \int_{\tilde{T}}^T \int_0^A \int_{\omega} \frac{\hat{v}^2}{k} dx dadt \right)^{\frac{1}{2}} \|y_0\|_{L^2_{\frac{1}{k}}(Q_{A,1})}. \quad (97)$$

640 Hence

$$\|f\|_{L^2_{\frac{1}{k}}(\tilde{Q})} = \left( \int_{\tilde{T}}^T \int_0^A \int_{\omega} \frac{\hat{v}^2}{k} dx dadt \right)^{\frac{1}{2}} \leq C \|y_0\|_{L^2_{\frac{1}{k}}(Q_{A,1})}.$$

Now, let  $y$  be the solution of (91) associated to  $f$  and  $y_0$ .

Multiplying the equation of (93) by  $\frac{y}{k}$  and integrating over  $\tilde{Q}$ , one has:

$$\begin{aligned} 0 &= \int_{\tilde{Q}} \left( \frac{\partial v}{\partial t} + \frac{\partial v}{\partial a} + k(x)v_{xx} - \mu(t, a, x)v + \beta(x, a)v(t, 0, x) \right) \frac{y}{k} dx dadt \iff \\ 0 &= \int_{\delta}^A \int_0^1 \frac{1}{k} y(T, a, x) g(a, x) dx da - \int_{Q_{A,1}} \frac{1}{k} y_0 v(\tilde{T}, a, x) dx da - \int_{\tilde{Q}_{T,1}} \frac{1}{k} y(t, 0, x) v(t, 0, x) dx dt \\ &\quad + \int_{\tilde{Q}} \frac{1}{k} \beta(a, x) v(t, 0, x) y(t, a, x) dx dadt - \int_{\tilde{Q}} \frac{v}{k} \left( \frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} - k(x)y_{xx} + \mu(t, a, x)y \right) dx dadt \end{aligned}$$

(recall that  $y(t, 0, x) = \int_0^A \beta(a, x) y(t, a, x) da$ ). But  $\frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} - k(x)y_{xx} + \mu(t, a, x)y = f\chi_{\omega}$ ; hence

$$0 = \int_{\delta}^A \int_0^1 \frac{1}{k} y(T, a, x) g(a, x) dx da - \int_{Q_{A,1}} \frac{1}{k} y_0 v(\tilde{T}, a, x) dx da - \int_{\tilde{T}}^T \int_0^A \int_{\omega} \frac{v \hat{v}}{k} dx dadt.$$

645 Thus, being by (94)

$$\int_{\tilde{T}}^T \int_0^A \int_{\omega} \frac{1}{k} v \hat{v} dx dadt = - \int_0^A \int_0^1 \frac{1}{k} v(\tilde{T}, a, x) y_0(a, x) dx da,$$

it follows

$$0 = \int_{\delta}^A \int_0^1 \frac{1}{k} y(T, a, x) g(a, x) dx da$$

for all  $g \in L^2_{\frac{1}{k}}(Q_{A,1})$  with  $g(A, x) = 0$  in  $(0, 1)$ . Hence  $y(T, a, x) = 0$  a.e.  $(a, x) \in (\delta, A) \times (0, 1)$ .

Observe that if  $T = \bar{a}$ , Theorem 4.7 is exactly the null controllability result  
 650 that we expect. Indeed, in this case (91) coincide with (1). On the other hand,  
 if  $T > \bar{a}$ , the null controllability for (1) is given in the next theorem and it is  
 based on the previous result:

**Theorem 4.8.** *Assume Hypotheses 4.1 and 4.2. Suppose that  $\omega$  satisfies (41),  
 (42) or (43). Then, given  $T > 0$  and  $y_0 \in L^2_{\frac{1}{k}}(Q_{A,1})$ , for every  $\delta \in (T, A)$ , there  
 655 exists a control  $f \in L^2_{\frac{1}{k}}(Q)$  such that the solution  $y$  of (1) satisfies*

$$y(T, a, x) = 0 \quad \text{a.e. } (a, x) \in (\delta, A) \times (0, 1).$$

Moreover, there exists  $C > 0$  such that

$$\|f\|_{L^2_{\frac{1}{k}}(Q)} \leq C \|y_0\|_{L^2_{\frac{1}{k}}(Q_{A,1})}. \quad (98)$$

PROOF. Fix  $\tilde{T} \in (0, T)$ . By Theorem 2.1, there exists a unique solution  $u$  of

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} - k(x)u_{xx} + \mu(t, a, x)u = 0 & \text{in } (0, \tilde{T}) \times (0, A) \times (0, 1), \\ u(t, a, 1) = u(t, a, 0) = 0 & \text{on } (0, \tilde{T}) \times (0, A), \\ u(0, a, x) = y_0(a, x) & \text{in } (0, A) \times (0, 1), \\ u(t, 0, x) = 0 & \text{in } (0, \tilde{T}) \times (0, 1). \end{cases} \quad (99)$$

Set  $\tilde{y}_0(a, x) := u(\tilde{T}, a, x)$ ; clearly  $\tilde{y}_0 \in L^2_{\frac{1}{k}}(Q_{A,1})$ . Now, consider

$$\begin{cases} \frac{\partial w}{\partial t} + \frac{\partial w}{\partial a} - k(x)w_{xx} + \mu(t, a, x)w = h(t, x, a)\chi_\omega & \text{in } \tilde{Q}, \\ w(t, a, 1) = w(t, a, 0) = 0 & \text{on } \tilde{Q}_{T,A}, \\ w(\tilde{T}, a, x) = \tilde{y}_0(a, x) & \text{in } Q_{A,1}, \\ w(t, 0, x) = \int_0^A \beta(a, x)w(t, a, x)da & \text{in } \tilde{Q}_{T,1}. \end{cases} \quad (100)$$

Again, by Theorem 2.1, there exists a unique solution  $w$  of (100) and, by the  
 660 previous Theorem, there exists a control  $h \in L^2_{\frac{1}{k}}(\tilde{Q})$  such that

$$w(T, a, x) = 0 \quad \text{a.e. } (a, x) \in (\delta, A) \times (0, 1).$$

Now, define  $y$  and  $f$  by

$$y := \begin{cases} u, & \text{in } [0, \tilde{T}], \\ w, & \text{in } [\tilde{T}, T] \end{cases} \quad \text{and} \quad f := \begin{cases} 0, & \text{in } [0, \tilde{T}], \\ h, & \text{in } [\tilde{T}, T]. \end{cases}$$

Then  $y$  satisfies (1) and  $f \in L^2_{\frac{1}{k}}(Q)$  is such that

$$y(T, a, x) = 0 \text{ a.e. } (a, x) \in (\delta, A) \times (0, 1).$$

Indeed  $y(T, a, x) = w(T, a, x) = 0$  a.e.  $(a, x) \in (\delta, A) \times (0, 1)$ .

It remains to prove (98). To this aim, observe that, by (92),

$$\|f\|_{L^2_v(Q)}^2 = \int_{\tilde{T}}^T \int_0^A \int_0^1 \frac{h^2}{k} dx da dt \leq C \|\tilde{y}_0\|_{L^2_{\frac{1}{k}}(Q_{A,1})}^2 = \int_0^A \int_0^1 \frac{u^2}{k}(\tilde{T}, a, x) dx da \quad (101)$$

665 for a strictly positive constant  $C$ . Thus, it is sufficient to estimate the last integral. To do this, we multiply the equation of (99) by  $\frac{u}{k}$  and we integrate over  $Q_{A,1}$ , obtaining:

$$\frac{1}{2} \frac{d}{dt} \int_0^A \int_0^1 \frac{u^2}{k} dx da + \frac{1}{2} \int_0^1 \frac{u^2(t, A, x)}{k} dx + \int_0^A \int_0^1 u_x^2 dx da + \int_0^A \int_0^1 \mu \frac{u^2}{k} = 0.$$

Hence

$$\frac{1}{2} \frac{d}{dt} \int_0^A \int_0^1 \frac{u^2}{k} dx da = -\frac{1}{2} \int_0^1 \frac{u^2(t, A, x)}{k} dx - \int_0^A \int_0^1 u_x^2 dx da - \int_0^A \int_0^1 \mu \frac{u^2}{k} \leq 0.$$

For all  $t \in (0, T)$ , integrating over  $(0, t)$ , we have

$$\int_0^A \int_0^1 \frac{u^2(t, a, x)}{k} dx da \leq \int_0^A \int_0^1 \frac{u^2(0, a, x)}{k} dx da = \int_0^A \int_0^1 \frac{y_0^2(a, x)}{k} dx da.$$

670 In particular,

$$\int_0^A \int_0^1 \frac{u^2(\tilde{T}, a, x)}{k} dx da \leq \int_0^A \int_0^1 \frac{y_0^2(a, x)}{k} dx da. \quad (102)$$

By (101) and (102), (98) follows.

Actually, in the (ISD) case, this result can be deduced directly by Theorem 4.8 in the (BD) case. Indeed, it holds also if we substitute the space interval  $(0, 1)$  with a general interval  $(\mathcal{A}, \mathcal{B})$  provided that  $k$  satisfies the required assumptions in this interval. Now, if we are in the (ISD) case, by [33, Proposition

3.6],  $y(t, a, x_0) = 0$  a.e.  $(t, a) \in Q_{T,A}$ ; hence, we can divide (1) into two problems stated in  $Q_{T,A} \times (0, x_0)$  and in  $Q_{T,A} \times (x_0, 1)$ , respectively, and we can apply Theorem 4.8 in the (BD) case, obtaining the thesis. This technique does not work in the weakly degenerate case since we are not able to divide the problem into two disjoint systems due the lack of the characterization of  $H_{\frac{1}{k}}^1(0, 1)$ .  
680 However, using observability inequalities and Carleman estimates, we are able to prove a null controllability result also in this case.

## 5. Appendix

*Proof of Theorem 3.2.* Let us proceed with the proof of Theorem 3.2. It is  
685 similar to the one of Theorem 3.4 (see also [31, Theorem 3.1]), so we sketch it.

As a first step assume that  $\mu \equiv 0$  and define, for  $s > 0$ , the function

$$w(t, a, x) := e^{s\Phi(t,a,x)}v(t, a, x)$$

where  $v$  is the solution of (6) in  $\mathcal{V}$ ; thus, since  $\Phi < 0$ ,  $w \in \mathcal{V}$ . Of course,  $w$  satisfies

$$\begin{cases} (e^{-s\Phi}w)_t + (e^{-s\Phi}w)_a + (k(x)(e^{-s\Phi}w)_x)_x = f(t, x, a), & (t, x) \in Q, \\ w(0, a, x) = w(T, a, x) = 0, & (a, x) \in Q_{A,1}, \\ w(t, A, x) = w(t, 0, x) = 0, & (t, x) \in Q_{T,1}, \\ w(t, a, 0) = w(t, a, 1) = 0, & (t, a) \in Q_{T,A}. \end{cases} \quad (103)$$

Defining  $Pw := w_t + w_a + (kw_x)_x$  and  $P_s w = e^{s\Phi}P(e^{-s\Phi}w)$ , the equation of  
690 (103) becomes

$$P_s w = P_s^+ w + P_s^- w = e^{s\Phi}f,$$

where

$$P_s^+ w := (kw_x)_x - s(\Phi_t + \Phi_a)w + s^2 k(\Phi_x)^2 w,$$

and

$$P_s^- w := w_t + w_a - 2sk\Phi_x w_x - s(k\Phi_x)_x w.$$

Moreover, setting  $\langle u, v \rangle_2 := \int_Q uv dx dadt$  and  $\|u\|_2 := \int_Q u^2 dx dadt$ , one has

$$\begin{aligned} 2\langle P_s^+ w, P_s^- w \rangle_2 &\leq 2\langle P_s^+ w, P_s^- w \rangle_2 + \|P_s^+ w\|_2^2 + \|P_s^- w\|_2^2 \\ &= \|P_s w\|_2^2 = \|f e^{s\Phi}\|_2^2. \end{aligned} \quad (104)$$

Proceeding as in the proof of Lemma 3.1, one can compute the scalar product

<sup>695</sup>  $\langle P_s^+ w, P_s^- w \rangle_2$ , which takes, in this case, the following form

**Lemma 5.1.** *The following identity holds:*

$$\left. \begin{aligned} &\langle P_s^+ w, P_s^- w \rangle_2 \\ &= \frac{s}{2} \int_Q (\Phi_{tt} + \Phi_{aa}) w^2 dx dadt + s^3 \int_Q (2k\Phi_{xx} + k'\Phi_x) k(\Phi_x)^2 w^2 dx dadt \\ &+ s \int_Q (2k\Phi_{xx} + k'\Phi_x) k(w_x)^2 dx dadt + s \int_Q k(k\Phi_x)_{xx} w w_x dx dadt \\ &- 2s^2 \int_Q k\Phi_x \Phi_{tx} w^2 dx dadt - 2s^2 \int_Q k\Phi_x \Phi_{ax} w^2 dx dadt + s \int_Q \Phi_{ta} w^2 dx dadt \end{aligned} \right\} \{D.T.\}_2 \quad (105)$$

$$\{B.T.\}_2 \left\{ \begin{aligned} &-\frac{1}{2} \int_0^A \int_0^1 [kw_x^2]_0^T dx da + \int_0^T \int_0^A [kw_x(w_t + w_a)]_0^1 dadt \\ &- s \int_0^T \int_0^A [k^2 \Phi_x w_x^2]_0^1 dadt \\ &- s \int_0^T \int_0^A [k(k\Phi_x)_x w w_x]_0^1 dadt + \frac{1}{2} \int_0^A \int_0^1 [(s^2 k \Phi_x^2 - s\Phi_t - s\Phi_a) w^2]_0^T dx da \\ &- s \int_0^T \int_0^A [(s^2 k^2 \Phi_x^3 - sk\Phi_x \Phi_t - sk\Phi_x \Phi_a) w^2]_0^1 dadt \\ &- \frac{1}{2} \int_0^T \int_0^1 [kw_x^2]_0^A dx dt + \frac{1}{2} \int_0^T \int_0^1 [(s^2 k \Phi_x^2 - s(\Phi_t + \Phi_a)) w^2]_0^A dx dt. \end{aligned} \right.$$

PROOF. Integrating by parts, one has

$$\langle P_s^+ w, P_s^- w \rangle_2 = I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 = \int_Q (kw_x)_x (w_t - 2sk\Phi_x w_x - s(k\Phi_x)_x w) dx dadt,$$

$$I_2 = \int_Q (-s\Phi_t w + s^2 k(\Phi_x)^2 w) (w_t - 2sk\Phi_x w_x - s(k\Phi_x)_x w) dx dadt,$$

$$I_3 = \int_Q -s\Phi_a w(w_t - 2sk\Phi_x w_x - s(k\Phi_x)_x w) dx dadt$$

and

$$I_4 = \int_Q w_a((kw_x)_x - s(\Phi_t + \Phi_a)w + s^2 k(\Phi_x)^2 w) dx dadt.$$

By, [31, Lemma 3.1],

$$\left. \begin{aligned} & I_1 + I_2 \\ &= \frac{s}{2} \int_Q \Phi_{tt} w^2 dx dadt + s^3 \int_Q (2k\Phi_{xx} + k'\Phi_x) k(\Phi_x)^2 w^2 dx dadt \\ &- 2s^2 \int_Q k\Phi_x \Phi_{tx} w^2 dx dadt + s \int_Q (2k\Phi_{xx} + k'\Phi_x) k(w_x)^2 dx dadt \\ &+ s \int_Q k(k\Phi_x)_{xx} w w_x dx dadt \end{aligned} \right\} \{\text{D.T.}\}_3 \quad (106)$$

$$\{\text{B.T.}\}_3 \left\{ \begin{aligned} &+ \int_0^T \int_0^A [kw_x w_t]_{x=0}^{x=1} dadt - \frac{s}{2} \int_0^A \int_0^1 [w^2 \Phi_t]_{t=0}^{t=T} dx da + \frac{s^2}{2} \int_0^A \int_0^1 [k(\Phi_x)^2 w^2]_{t=0}^{t=T} dx da \\ &- \frac{1}{2} \int_0^A \int_0^1 [k(w_x)^2]_{t=0}^{t=T} dx da + \int_0^T \int_0^A [-sk(k\Phi_x)_x w w_x]_{x=0}^{x=1} dadt \\ &+ \int_0^T \int_0^A [-s\Phi_x k^2(w_x)^2 + s^2 k\Phi_t \Phi_x w^2 - s^3 k^2(\Phi_x)^3 w^2]_{x=0}^{x=1} dadt. \end{aligned} \right. \quad (107)$$

705 Next, we compute  $I_3$  and  $I_4$ :

$$\begin{aligned} I_3 &= \frac{s}{2} \int_Q \Phi_{at} w^2 dx dadt - \frac{s}{2} \int_0^A \int_0^1 [\Phi_a w^2]_{t=0}^{t=T} dx da \\ &- s^2 \int_Q (k\Phi_x)_x \Phi_a w^2 dx dadt - s^2 \int_Q k\Phi_x \Phi_{ax} w^2 dx dadt + s^2 \int_0^T \int_0^A [k\Phi_a \Phi_x w^2]_{x=0}^{x=1} dadt \\ &+ s^2 \int_Q (k\Phi_x)_x \Phi_a w^2 dx dadt \\ &= \frac{s}{2} \int_Q \Phi_{at} w^2 dx dadt - \frac{s}{2} \int_0^A \int_0^1 [\Phi_a w^2]_{t=0}^{t=T} dx da \\ &- s^2 \int_Q k\Phi_x \Phi_{ax} w^2 dx dadt + s^2 \int_0^T \int_0^A [k\Phi_a \Phi_x w^2]_{x=0}^{x=1} dadt. \end{aligned} \quad (108)$$

On the other hand

$$\begin{aligned}
I_4 &= \int_0^T \int_0^A [kw_x w_a]_{x=0}^{x=1} dadt - \frac{1}{2} \int_0^T \int_0^1 [k(w_x)^2]_{a=0}^{a=A} dxdt \\
&+ \frac{1}{2} \int_0^T \int_0^1 [(s^2 k \Phi_x^2 - s(\Phi_t + \Phi_a))w^2]_{a=0}^{a=A} dxdt \\
&+ \frac{s}{2} \int_Q \Phi_{aa} w^2 dx dadt + \frac{s}{2} \int_Q \Phi_{ta} w^2 dx dadt - s^2 \int_Q k \Phi_x \Phi_{xa} w^2 dx dadt.
\end{aligned} \tag{109}$$

Adding (106), (108) and (109), we have the thesis.

The crucial step is to prove now the following estimate.

**Lemma 5.2.** *There exist two strictly positive constants  $s_0$  and  $C$  such that for*

all  $s \geq s_0$  the distributed terms of (105) satisfy the estimate

$$Cs \int_Q \Theta e^{\kappa\sigma} (w_x)^2 dx dadt + Cs^3 \int_Q \Theta^3 e^{3\kappa\sigma} w^2 dx dadt \leq \{D.T.\}_2.$$

PROOF. By [31, Lemma 3.2], there exist two strictly positive constants  $s_0$  and  $C$  such that for all  $s \geq s_0$ , the distributed terms given in (106)

$$\frac{C}{2} s \int_Q \Theta e^{\kappa\sigma} (w_x)^2 dx dadt + Cs^3 \int_Q \Theta^3 e^{3\kappa\sigma} w^2 dx dadt \leq \{D.T.\}_3. \tag{110}$$

Moreover, using the definition of  $\Phi$ , the other distributed terms of  $\langle P_s^+ w, P_s^- w \rangle_2$  take the form

$$-2s^2 \int_Q k \Theta \Theta_a \Psi_x^2 w^2 dx dadt + \frac{s}{2} \int_Q \Theta_{aa} \Psi w^2 dx dadt + s \int_Q \Theta_{ta} \Psi w^2 dx dadt. \tag{111}$$

Now, the first term in (111) can be estimated in the following way:

$$\begin{aligned}
\left| 2s^2 \mathfrak{d}^2 \kappa^2 \int_Q \Theta \Theta_a \frac{e^{2\kappa\sigma}}{k} w^2 dx dadt \right| &\leq \frac{2\mathfrak{d}^2 \kappa^2 c}{\min_{[0,1]} k \min_{[0,1]} e^{\kappa\sigma}} s^2 \int_Q \Theta^3 e^{3\kappa\sigma} w^2 dx dadt \\
&\leq \frac{C}{6} s^3 \int_Q \Theta^3 e^{3\kappa\sigma} w^2 dx dadt,
\end{aligned} \tag{112}$$

for some  $C > 0$  and  $s \geq \frac{12\mathfrak{d}^2 \kappa^2 c}{C \min_{[0,1]} k \min_{[0,1]} e^{2\kappa\sigma}}$ . Using again (27), we have

$$\begin{aligned}
\left| \frac{s}{2} \int_Q \Theta_{aa} \Psi w^2 dx dadt \right| &\leq s c \max_{[0,1]} |\Psi| \int_Q \Theta^3 w^2 dx dadt \leq \frac{cs \max_{[0,1]} |\Psi|}{\min e^{3\kappa\sigma}} \int_Q \Theta^3 e^{3\kappa\sigma} w^2 dx dadt \\
&\leq \frac{C}{6} s^3 \int_Q \Theta^3 e^{3\kappa\sigma} w^2 dx dadt
\end{aligned} \tag{113}$$



and

$$\left| s \int_Q \Theta_{ta} \Psi w^2 dx dadt \right| \leq \frac{C}{6} s^3 \int_Q \Theta^3 e^{3\kappa\sigma} w^2 dx dadt,$$

for  $s \geq \sqrt{\frac{6c \max_{[0,1]} |\Psi|}{C \min e^{3\kappa\sigma}}}$ . In conclusion, by the previous inequalities, we obtain

$$\{D.T.\}_2 \geq \frac{C}{2} s \int_Q \Theta e^{\kappa\sigma} (w_x)^2 dx dadt + C s^3 \int_Q \Theta^3 e^{3\kappa\sigma} w^2 dx dadt - \frac{C}{2} s^3 \int_Q \Theta^3 e^{3\kappa\sigma} w^2 dx dadt.$$

Hence, the thesis follows.

720 The next lemma holds.

**Lemma 5.3.** *The boundary terms in (105) become*

$$\{B.T.\}_2 = s\kappa \|k'\|_{L^\infty(0,1)} \int_0^T \int_0^A [k\Theta e^{\kappa\sigma} (w_x)^2]_{x=0}^{x=1} dadt. \quad (114)$$

PROOF. By [31, Lemma 3.4], the boundary terms given in (107) take the form

$$\{B.T.\}_3 = s\kappa \|k'\|_{L^\infty(0,1)} \int_0^T \int_0^A [k\Theta e^{\kappa\sigma} (w_x)^2]_{x=0}^{x=1} dadt.$$

Using the definition of  $\Phi$ , the other boundary terms of  $\langle P_s^+ w, P_s^- w \rangle_2$  become

$$\begin{aligned} & \int_0^T \int_0^A [kw_x w_a]_0^1 dadt - \frac{s}{2} \int_0^A \int_0^1 [\Phi_a w^2]_0^T dx da \\ & + s^2 \int_0^T \int_0^A [k\Phi_x \Phi_a w^2]_0^1 dadt - \frac{1}{2} \int_0^T \int_0^1 [kw_x^2]_0^A dx dt \\ & + \frac{1}{2} \int_0^T \int_0^1 [(s^2 k \Phi_x^2 - s(\Phi_t + \Phi_a)) w^2]_0^A dx dt \\ & = \int_0^T \int_0^A [kw_x w_a]_0^1 dadt - \frac{s}{2} \int_0^A \int_0^1 [\Theta_a \Psi w^2]_0^T dx da \\ & + s^2 \int_0^T \int_0^A [k\Theta \Theta_a \Psi \Psi_x w^2]_0^1 dadt - \frac{1}{2} \int_0^T \int_0^1 [kw_x^2]_0^A dx dt \\ & + \frac{1}{2} \int_0^T \int_0^1 [(s^2 k \Theta^2 \Psi_x^2 - s(\Theta_t + \Theta_a) \Psi) w^2]_0^A dx dt. \end{aligned}$$

725 As before, since  $w \in \mathcal{V}$ ,  $w(0, a, x)$ ,  $w(T, a, x)$ ,  $w_x(t, 0, x)$ ,  $w_x(t, A, x)$ ,  $w(t, A, x)$ ,  $w(t, 0, x)$ ,  $w(t, a, 0)$ ,  $w(t, a, 1)$ ,  $w_x(t, a, 0)$  and  $w_x(t, a, 1)$  are well defined. Moreover, we have that  $w_a(t, a, 0)$  and  $w_a(t, a, 1)$  make sense and are actually 0.

Thus, using the boundary conditions of  $w = e^{s\Phi}v$ , we get

$$\begin{aligned} & \int_0^T \int_0^A [kw_x w_a]_0^1 dadt - \frac{s}{2} \int_0^A \int_0^1 [\Theta_a \Psi w^2]_0^T dx da \\ & + s^2 \int_0^T \int_0^A [k\Theta \Theta_a \Psi \Psi_x w^2]_0^1 dadt - \frac{1}{2} \int_0^T \int_0^1 [kw_x^2]_0^A dx dt \\ & + \frac{1}{2} \int_0^T \int_0^1 [(s^2 k \Theta^2 \Psi_x^2 - s(\Theta_t + \Theta_a) \Psi) w^2]_0^A dx dt = 0. \end{aligned}$$

Hence the thesis.

730 By Lemmas 5.1, 5.2 and 5.3 the next estimate holds:

**Proposition 5.1.** *There exist two strictly positive constants  $C$  and  $s_0$  such that, for all  $s \geq s_0$ , all solutions  $w$  of (103) in  $\mathcal{V}$  satisfy*

$$\begin{aligned} & s \int_Q \Theta e^{\kappa\sigma} (w_x)^2 dx dadt + s^3 \int_Q \Theta^3 e^{3\kappa\sigma} w^2 dx dadt \\ & \leq C \left( \int_Q f^2 e^{2s\Phi} dx dadt - s\kappa \|k'\|_{L^\infty(0,1)} \int_0^T \int_0^A [k\Theta e^{\kappa\sigma} (w_x)^2]_{x=0}^{x=1} dadt \right). \end{aligned}$$

Recalling the definition of  $w$ , we have  $v = e^{-s\Phi}w$  and  $v_x = (w_x - s\Phi_x w)e^{-s\Phi}$ .

Thus,

$$\begin{aligned} (s\Theta e^{\kappa\sigma} (v_x)^2 + s^3 \Theta^3 e^{3\kappa\sigma} v^2) e^{2s\Phi} & \leq c [s\Theta e^{\kappa\sigma} (s^2 \Theta^2 w^2 + (w_x)^2) + s^3 \Theta^3 e^{3\kappa\sigma} w^2] \\ & \leq c [s\Theta e^{\kappa\sigma} (w_x)^2 + s^3 \Theta^3 e^{3\kappa\sigma} w^2], \end{aligned}$$

735 for a strictly positive constant  $c$ . Hence, Theorem 3.2 follows immediately by Proposition 3.1 when  $\mu \equiv 0$ .

Now, we assume that  $\mu \not\equiv 0$ .

To complete the proof of Theorem 3.2 we consider, as before, the function  $\bar{f} = f + \mu v$ . Hence, there are two strictly positive constants  $C$  and  $s_0$  such that,

740 for all  $s \geq s_0$ , the following inequality holds

$$\int_Q (s^3 \phi^3 v^2 + s\phi v_x^2) e^{2s\Phi} dx dadt \leq c \left( \int_Q \bar{f}^2 e^{2s\Phi} dx dadt - s\kappa \int_0^T \int_0^A [k e^{2s\Phi} \phi(v_x)^2]_{x=0}^{x=1} dadt \right). \quad (115)$$

On the other hand, we have

$$\int_Q \bar{f}^2 e^{2s\Phi} dx dadt \leq 2 \left( \int_Q |f|^2 e^{2s\Phi} dx dadt + \|\mu\|_{L^\infty(Q)}^2 \int_Q |v|^2 e^{2s\Phi} dx dadt \right). \quad (116)$$

Now, applying the Hardy-Poincaré inequality to the function  $\nu := e^{s\Phi}v$ , we obtain

$$\begin{aligned} \int_Q |v|^2 e^{2s\Phi} dx dadt &\leq \int_Q \frac{\nu^2}{x^2} dx dadt \leq C \int_Q (e^{s\Phi}v)_x^2 dx dadt \\ &\leq C \int_Q e^{2s\Phi} v_x^2 dx dadt + Cs^2 \int_Q \Theta^2 e^{2s\Phi} \Psi_x^2 v^2 dx dadt. \end{aligned}$$

Using this last inequality in (116), it follows

$$\begin{aligned} \int_Q |\bar{f}|^2 e^{2s\Phi} dx dadt &\leq 2 \int_Q |f|^2 e^{2s\Phi} dx dadt + C \int_Q e^{2s\Phi} v_x^2 dx dadt \\ &\quad + Cs^2 \int_Q \Theta^2 e^{2s\Phi} e^{2\kappa\sigma} v^2 dx dadt. \end{aligned} \tag{117}$$

745 Substituting in (115), one can conclude

$$\begin{aligned} \int_Q (s^3 \phi^3 v^2 + s \phi v_x^2) e^{2s\Phi} dx dadt &\leq C \left( \int_Q |f|^2 e^{2s\Phi} dx dadt \right. \\ &\quad \left. + \int_Q e^{2s\Phi} v_x^2 dx dadt + s^2 \int_Q \Theta^2 e^{2s\Phi} e^{2\kappa\sigma} v^2 dx dadt - s\kappa \int_0^T \int_0^A [k e^{2s\Phi} \phi(v_x)^2]_{x=0}^{x=1} dadt \right). \end{aligned}$$

This completes the proof of Theorem 3.2.

*Proof of Theorem 3.3.* As before, to prove Theorem 3.3, one can assume, first of all, that  $\mu \equiv 0$ . The case  $\mu \not\equiv 0$  follows as in the previous subsection.

If  $\mu \equiv 0$ , the proof in the divergence case is *formally* similar to the one of  
750 Theorem 3.2 (see also the proof of [31, Theorem 3.1]). Observe that, in this case, integrations by parts are not immediately justified since the function  $k$  is not  $C^1[0, 1]$ . However, proceeding as in [31], one can motivate them.

In the non divergence case one can proceed as in the proofs of [31, Theorems 3.1 and 3.2].

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 760 *non lineari.*

She dedicates this work to her father.

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