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DISSIPATIVE HIGHER ORDER HYPERBOLIC EQUATIONS

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Abstract. In this paper, we describe a constructive method to find a dissipative term for any generic higher order, homogeneous, possibly weakly, hyperbolic operator $P(\partial_t, \partial_x)$, with $x \in \mathbb{R}^n$, $n \geq 1$. We derive long-time decay estimates for the solution to the related Cauchy problem. We provide an example of application to the theory of elastic waves.

1. Introduction

In this paper we consider a m-th order *hyperbolic* homogenous equation, $m \geq 2$,

$$P(\partial_t, \partial_x)u := \partial_t^m u + \sum_{1\leq|\alpha|\leq m} b_\alpha \, \partial_t^{m-|\alpha|} \partial_x^\alpha u = 0, \tag{1}$$

with $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, $n \geq 1$ and $b_\alpha \in \mathbb{R}$, and we find suitable lower order terms which, added to $P(\partial_t, \partial_x)u$, transform equation (1) into a dissipative one. Equation (1) is said to be *hyperbolic* if the polynomial

$$P(\lambda, i\xi) = \lambda^m + \sum_{1\leq|\alpha|\leq m} b_\alpha \, \lambda^{m-|\alpha|} (i\xi)^\alpha \tag{2}$$

has only pure imaginary roots $ia_j(\xi')$ for any $\xi' \in S^{n-1} = \{\xi \in \mathbb{R}^n \, : \, |\xi| = 1\}$; namely

$$P(\lambda, i\xi') = \prod_{j=1}^m (\lambda - ia_j(\xi')) \tag{3}$$

where $a_j(\xi') \in \mathbb{R}$.

If all roots are simple for any $\xi' \in S^{n-1}$, equation (1) and the polynomial P are said to be *strictly hyperbolic*, otherwise they are said to be *weakly hyperbolic*.

In order to present our result, we need the following

Definition 1. A polynomial

$$f(z) = \sum_{j=0}^\kappa c_j z^j, \tag{4}$$

with $\kappa \geq 1$, $c_j \in \mathbb{C}$, $c_\kappa \neq 0$, is said to be *strictly stable* (resp. *stable*) if $\mathrm{Re}\,(z_j) < 0$ (resp. $\mathrm{Re}\,(z_j) \leq 0$) for any $z_j : f(z_j) = 0$.

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Throughout the present paper, when we write a polynomial in the form (4), we always implicitly assume that its degree is κ, i.e. $c_\kappa \neq 0$. We also notice that $c_0 \neq 0$ in a strictly stable polynomial (4); hence, it is not restrictive to assume that $c_0 = 1$. Our main result is the following

Theorem 1. *Let P be a (possibly weakly) hyperbolic polynomial as in* (3)*, and let m_1 be the maximum multiplicity of the roots in* (3) *over $\xi' \in S^{n-1}$, i.e.*

$$m_1 := \max_{\xi' \in S^{n-1}} \left\{ k : \ \partial_\lambda^{k-1} P(a_j(\xi'), \xi') = 0, \ \text{for some } j = 1, \dots, m \right\}.$$

Let $\mathrm{r} \geq m_1$ and let $\psi(z) := \sum_{k=0}^{\mathrm{r}} c_k z^k$ be a strictly stable, real polynomial, according to Definition 1, with $c_0 = 1$. Moreover, let

$$Q(\lambda, i\xi) := \psi(\partial_\lambda) P(\lambda, i\xi) = P(\lambda, i\xi) + \sum_{k=1}^{\mathrm{r}} c_k \, \partial_\lambda^k P(\lambda, i\xi). \tag{5}$$

Then the equation $Q(\partial_t, \partial_x) u = 0$ is a dissipative equation. More precisely:

- *if $\mathrm{r} \leq m-1$, then the solution to*

$$\begin{cases} Q(\partial_t, \partial_x) u = 0, & t \geq 0, \quad x \in \mathbb{R}^n, \\ \partial_t^j u(0,x) = u_j(x), & j = 0, \dots, m-1, \end{cases} \tag{6}$$

  satisfies the following long-time decay estimate with polynomial speed:

$$\begin{aligned} \|\partial_x^\alpha \partial_t^k u(t,\cdot)\|_{L^2} \leq C & \sum_{j=0}^{m-\mathrm{r}-2} (1+t)^{-\frac{n}{4} - \frac{|\alpha|+k-j}{2}} \|u_j\|_{L^1} \\ & + C (1+t)^{-\frac{n}{4} - \frac{|\alpha|+k-(m-\mathrm{r}-1)}{2}} \sum_{j=m-\mathrm{r}-1}^{m-1} \|u_j\|_{L^1} \\ & + C\, e^{-\delta t} \sum_{j=0}^{m-1} \|u_j\|_{H^{|\alpha|+k-j+m_1-1}} \end{aligned} \tag{7}$$

  for any $\alpha \in \mathbb{N}^n$ and $k \in \mathbb{N}$ such that

$$\frac{n}{2} + |\alpha| + k > m - \mathrm{r} - 1, \tag{8}$$

  and for some $C > 0$, $\delta > 0$, not depending on the data, provided that $u_j \in L^1 \cap H^{|\alpha|+k-j+m_1-1}$ for $j = 0, \dots, m-1$;
- *if $\mathrm{r} \geq m$, then the solution to* (6) *satisfies the following long-time decay estimate with exponential speed:*

$$\|\partial_x^\alpha \partial_t^k u(t,\cdot)\|_{L^2} \leq C\, e^{-\delta t} \sum_{j=0}^{m-1} \|u_j\|_{H^{|\alpha|+k-j+m_1-1}}. \tag{9}$$

  for any $\alpha \in \mathbb{N}^n$ and $k \in \mathbb{N}$, for some $C > 0$, $\delta > 0$, not depending on the data, provided that $u_j \in H^{|\alpha|+k-j+m_1-1}$ for $j = 0, \dots, m-1$.

Theorem 1 has been proved in the case P strictly hyperbolic and $\mathrm{r} = 1$ by the authors in [13]. For $\mathrm{r} = 1$, the real polynomial $1 + c_1 z$ is strictly stable if, and only if, $c_1 > 0$. This led to the result that the operator $P(\partial_t, \partial_x) + c_1 \partial_\lambda P(\partial_t, \partial_x)$ was dissipative, for any $c_1 > 0$, in the sense of Theorem 1.

On the one hand, Theorem 1 shows how to extend this result for strictly hyperbolic operators when r ≥ 2; in particular, we notice that the decay rate for the solution improves as r grows. On the other hand, Theorem 1 allows to deal with weakly hyperbolic operators, if r is greater than or equal to the maximum multiplicity of the roots in (3).

Remark 1. As it is well known, in absence of lower order terms, i.e. if one only considers the Cauchy problem for the homogenous, hyperbolic equation $P(\partial_t, \partial_x)u = 0$, one may only prove a classical energy estimate in the form

$$\|\partial_x^\alpha \partial_t^k u(t,\cdot)\|_{L^2} \leq C \sum_{j=0}^{m-1} \|u_j\|_{H^{|\alpha|+k-j}},$$

which excludes blow-up of the energy as $t \to \infty$ (here we are also assuming that the equation is strictly hyperbolic). By invariance with respect to time reversal, dissipative effects may be excluded for homogeneous equations, in general. We remark that the presence of time-dependent, bounded, coefficients which include oscillations might invalidate this estimate and produce a blow-up in infinite time (see [8, 33]).

To get a dissipative effect on the energy, in general, it is necessary to include lower order terms. Otherwise, in absence of lower order terms, a decay rate may be produced, under suitable geometric assumptions on the characteristic roots, if dispersive estimates $L^q - L^{q'}$ are proved, where $q \in (2, \infty]$ and $q' = q/(q-1)$ is the Hölder conjugate of q (as for the non dissipative wave equation).

The study of long-time decay estimates for dissipative hyperbolic equations like (7) goes back to the study of the Cauchy problem for the damped wave equation (see [27]),

$$u_{tt} - \Delta u + 2c_1 u_t = 0, \qquad t \geq 0,\ x \in \mathbb{R}^n,$$

with $c_1 > 0$. Indeed, the full symbol $Q(\lambda, i\xi) = \lambda^2 + |\xi|^2 + 2c_1\lambda$ of the damped wave operator is obtained by taking the homogeneous symbol of the wave equation, i.e. $P(\lambda, i\xi) = \lambda^2 + |\xi|^2$, and setting r $= 1$ in Theorem 1. In particular,

$$\begin{aligned} \|u(t,\cdot)\|_{L^2} &\leq C(1+t)^{-\frac{n}{4}} \left(\|u_0\|_{L^1} + \|u_1\|_{L^1} + \|u_0\|_{L^2} + \|u_1\|_{L^2}\right), \\ \|(\nabla, \partial_t) u(t,\cdot)\|_{L^2} &\leq C(1+t)^{-\frac{n+2}{4}} \left(\|u_0\|_{L^1} + \|u_1\|_{L^1} + \|u_0\|_{H^1} + \|u_1\|_{L^2}\right). \end{aligned} \tag{10}$$

On the other hand, setting r $= 2$ in Theorem 1, one finds an exponential decay in (9), which is a well-known property for the damped Klein-Gordon equation:

$$u_{tt} - \Delta u + 2c_1 u_t + 2c_2 u = 0, \qquad t \geq 0,\ x \in \mathbb{R}^n,$$

with $c_1, c_2 > 0$. Indeed, the real polynomial $c_2 z^2 + c_1 z + 1$ is strictly stable if, and only if, $c_1, c_2 > 0$.

The decay rate in (10) comes from an asymptotic profile of the solution which is given by the solution to a Cauchy problem for the heat equation [19, 26, 27, 28, 29]. Long-time decay estimates like (7) are particularly helpful to study semilinear problems. In particular, estimates (10) have been applied to find the critical exponent for the global existence of small data solutions for the damped wave equation with power nonlinearity [21, 37] or nonlinear memory [9].

These results have been extended in recent years to linear and semilinear damped wave equations with time-dependent coefficients [3, 10, 15, 16, 25, 39] to wave equations with structural damping [4, 11, 12, 14, 24], and to more general operators (see, in particular, [6]). This list of results concerning decay estimates for damped wave equations and their applications is far from being complete.

In the setting of long-time decay estimates for higher order inhomogeneous equations with constant coefficients, we address the interested reader to [34], where dispersive and Strichartz estimates are obtained. In particular, under different hypotheses of geometric type on the roots of the full symbol of the operator, the authors are able to derive $L^q - L^{q'}$ estimates, $q \in [2, \infty]$, where $q' = q/(q-1)$, for inhomogeneous hyperbolic equations. The decay rate in these estimates has the classical form $(1+t)^{-\kappa\left(\frac{1}{q}-\frac{1}{q'}\right)}$, where $\kappa > 0$ depends on the assumptions on the roots of the full symbol.

In Theorems 2.1.1 and 2.1.2 in [34], an exponential decay in time follows by the assumption that the roots of the full symbol are uniformly bounded by a negative constant (a property that we prove for our equation, when $\mathrm{r} = m$, in Theorem 1). Indeed, as it is well known, this assumption produces a strong dissipative effect, which leads to exponential decay in time, as in (9).

A huge literature exists for dissipative hyperbolic systems with constant coefficients, under suitable assumptions on the lower order term and its relations with the first-order term. We address the interested reader to [36], and to [2] and the references therein, being aware that this cannot be an exhaustive list. Recently, dissipative estimates for first-order hyperbolic systems with time-dependent coefficients have been obtained in [40].

This paper is intended to be a starting point to consider higher order operators, with properties and effects which are new if compared to the case of the wave operator, with a new approach. The question that we address is to find a constructive way to produce a dissipative effect for a given higher order hyperbolic operator, and to explicitly give the related decay estimate at the energy level, with no need to discuss the behavior or the roots of the full symbol, and assuming only the strict or weak hyperbolicity of the roots of the main symbol. We plan to consider $L^p - L^q$ estimates, with $1 \le p \le q \le \infty$, for higher order equations, in future works, keeping an explicit, constructive, approach.

We notice that our techniques, developed in Section 3, may be successfully applied to higher order operators which are more general than the ones considered in Theorem 1, as we show in Section 7.

Theorem 1 has the advantage that it allows to construct a dissipative term for a generic hyperbolic, homogeneous, equation $P(\partial_t, \partial_x)u = 0$ of any order $m \ge 2$, without any need to study the full symbol of the resulting equation. In other words, to find a dissipative structure, we do not have to compute the roots of a polynomial of order m, depending on a parameter ξ, with complex coefficients. Thanks to Theorem 1, it is sufficient to check the stability of a real polynomial in one variable, which degree is not smaller than the maximum multiplicity of the roots of the symbol $P(\lambda, i\xi)u = 0$.

In a forthcoming paper, we will provide applications of Theorem 1 to the study of semilinear problems, in particular linear estimates (7) can be successfully applied to study the global existence of small data solutions to problems with different type of power nonlinearities.

In Section 5, we provide a brief example of application of Theorem 1 to the theory of elastic waves. It is well known that a second order system of elastic waves may be reduced to the study of a scalar, fourth order, equation. Also, we will reduce the study of a system of elastic waves coupled with Maxwell equations to the study of a scalar, sixth order, equation.

To prove Theorem 1, we need some auxiliary results. For the ease of reading, we collect these results in separate sections. In Section 2 we collect some known results about strictly stable polynomials in one variable $z \in \mathbb{C}$; in Section 3 we give a standard way to perturb a hyperbolic polynomial obtaining a strictly stable polynomial, whereas in Section 4 we prove some lemmas about the behavior of the roots of $Q(\lambda, i\xi)$, as $\xi \to 0$ and as $|\xi| \to \infty$, from which the proof of Theorem 1 follows.

Notation.

$[x]$ — If $x \in \mathbb{R}$, we define the floor function $[x]$ as

$$[x] := \max\{k \in \mathbb{Z} : \ k \le x\}.$$

$g(D)f(z)$ — If $g(z)$ is a polynomial, $g(z) = \sum_{j=0}^{\kappa} a_j z^j$, and $f(z)$ is another polynomial, we define $g(D)f(z)$ as the polynomial

$$g(D)f(z) := \sum_{j=0}^{\kappa} a_j f^{(j)}(z).$$

$g(\partial_\lambda)P(\lambda, i\xi)$ — If $g(z)$ is a polynomial, $g(z) = \sum_{j=0}^{\kappa} a_j z^j$, and $P(\lambda, i\xi)$ is a polynomial in λ, depending on a parameter $i\xi$, we define $g(\partial_\lambda)P(\lambda, i\xi)$ as the polynomial

$$g(\partial_\lambda)P(\lambda, i\xi) := \sum_{j=0}^{\kappa} a_j \partial_\lambda^j P(\lambda, i\xi).$$

$g_e(z), g_o(z)$ — If $g(z)$ is a polynomial, $g(z) = \sum_{j=0}^{\kappa} a_j z^j$, we define $g_e(z), g_o(z)$ as the polynomials

$$g_e(z) = \sum_{j=0}^{[\kappa/2]} (-1)^j a_{2j} z^{2j}, \qquad g_o(z) = \sum_{j=0}^{[(\kappa-1)/2]} (-1)^j a_{2j+1} z^{2j+1}, \tag{11}$$

so that $g(iz) = g_e(z) + i g_o(z)$.

2. Preliminary results about strictly stable polynomials

We collect here some well-known results about strictly stable polynomials which will be useful in what follows. First of all, we state some immediate properties of strictly stable polynomials.

Lemma 1. *Let $f(z) = \sum_{j=0}^{\kappa} c_j z^j$ be a strictly stable polynomial. Then:*

(i) the polynomials

$$z^\kappa f(1/z) = \sum_{j=0}^{\kappa} c_j z^{\kappa-j} = \sum_{j=0}^{\kappa} c_{\kappa-j} z^j,$$

$$f(\gamma z) = \sum_{j=0}^{\kappa} (\gamma^j c_j)\, z^j, \qquad \textit{with } \gamma > 0,$$

are strictly stable;

(ii) if f is also real, then either $c_j > 0$ for all $j = 0, \dots, \kappa$, or $c_j < 0$ for all $j = 0, \dots, \kappa$.

Proof. Trivial. □

Definition 2 (see Definition 6.3.1 in [31])**.** We say that two non–constant real polynomials *weakly interlace* if they have both only real roots, their degrees differ at most by one and there exists an ordering such that

$$\alpha_1 \le \beta_1 \le \alpha_2 \le \beta_2 \le \cdots \le \alpha_\nu \le \beta_\nu \le \cdots \le \beta_{m-1} \le \alpha_m \le \beta_m \tag{12}$$

where the α_j are the zeros of one of the polynomials and the β_j are those of the other (the last term β_m must be struck in (12) if the degrees of the two polynomials differ by one). If no equality sign occurs in (12), then we say that the two polynomials *strictly interlace.*

Now let us recall two classical criteria for strictly interlacing polynomials.

Theorem 2 (Hermite–Kakeya)**.** *Let f, g be non–constant real polynomials. Then f, g strictly interlace if, and only if, for all $\lambda, \mu \in \mathbb{R} : (\lambda, \mu) \neq (0, 0)$ the polynomial $\lambda f(z) + \mu g(z)$ has real simple zeros.*

Proof. See for instance [31], Theorem 6.3.8. □

Theorem 3 (Hermite–Biehler)**.** *Let f, g be non–constant real polynomials. Then f, g strictly interlace if, and only if, the polynomial $h(z) := f(z) + ig(z)$ has all its zeros either in the half–plane $\{z : \mathrm{Im}(z) > 0\}$ or in the half–plane $\{z : \mathrm{Im}(z) < 0\}$.*

Proof. See for instance [31], Theorem 6.3.4. □

As an immediate corollary of Theorem 3 we get the following stability criterion:

Theorem 4 (Hermite–Biehler)**.** *Let $f(z) = \sum_{j=0}^{\kappa} c_j z^j$ be a real polynomial. Then one of the two polynomials $f(z), f(-z)$ is strictly stable if, and only if, $f_e(z), f_o(z)$ (see Notation) strictly interlace.*

Proof. Define $h(z) := f(iz) = f_e(z) + i f_o(z)$, and let Γ be the set of the roots of h. From Theorem 3 we get that f_e and f_o strictly interlace if, and only if, either $\Gamma \subset \{z : \mathrm{Im}(z) > 0\}$ or $\Gamma \subset \{z : \mathrm{Im}(z) < 0\}$. These two cases are, respectively, equivalent to the strict stability of either $f(z)$ or $f(-z)$. □

3. Perturbation of hyperbolic polynomials

We say that a monic polynomial

$$f(z) = z^n + \sum_{j=1}^{n} a_j\, z^{n-j}, \tag{13}$$

with $a_j \in \mathbb{C}$, is *hyperbolic* if it has only pure imaginary roots, i.e.

$$f(z) = \prod_{j=1}^{n} (z - ix_j), \qquad x_j \in \mathbb{R}.$$

The main goal of this section is to prove the following theorem, which provides a standard way to perturb a hyperbolic polynomial obtaining a strictly stable polynomial.

Theorem 5. *Let $p(z)$ be a monic hyperbolic polynomial of degree m, and let m_1 be the maximum multiplicity of its roots. Let $\mathrm{r} \geq m_1$, and let $\psi(z) := \sum_{k=0}^{\mathrm{r}} c_k z^k$ be a strictly stable real polynomial, according to Definition 1, with $c_0 = 1$. Then the polynomial (see Notation)*

$$q(z) := \psi(D)p(z) = p(z) + \sum_{k=1}^{\mathrm{r}} c_k\, p^{(k)}(z) \tag{14}$$

is strictly stable.

We notice that, if $\mathrm{r} > m$, then $p^{(k)}(z) \equiv 0$ in (14) for any $k = m+1, \ldots, \mathrm{r}$. Moreover, the assumption $\mathrm{r} \geq m_1$ is also necessary; indeed, if ix_j is a root of $p(z)$ with multiplicity $\mathrm{r}_1 \geq \mathrm{r}+1$, then $p^{(k)}(ix_j) = 0$ for $k = 1, \ldots, \mathrm{r}_1 - 1$; hence $q(ix_j) = 0$, so that $q(z)$ is not strictly stable.

Theorem 5 is the milestone of our construction of dissipative, higher order, weakly hyperbolic equations (Theorem 1). In order to prove Theorem 5, we need a preliminary result.

Proposition 1. *Let $g(z) = \sum_{k=0}^{\mathrm{r}} \beta_k\, z^k$, with $\beta_0 = 1$ and $\beta_{\mathrm{r}} \neq 0$, be a real polynomial with only real roots, and let $f(z)$ be another non–constant real polynomial with only real roots. Then, the polynomial*

$$g(D)f(z) = \sum_{k=0}^{\mathrm{r}} \beta_k f^{(k)}(z) \tag{15}$$

has only real roots. Moreover, x^* is a zero of $g(D)f(z)$ of order s, with $s \geq 2$, if, and only if, x^* is a zero of f of order $s + \mathrm{r}$.

Proof. This is a very particular case of the classical Hermite–Poulain–Jensen theorem, see for instance [31], Theorem 5.4.9. □

We are now ready to prove Theorem 5.

Proof of Theorem 5. The polynomial $\psi(z)$ is strictly stable; hence, by Theorem 4, $\psi_e(z), \psi_o(z)$ strictly interlace. Therefore, Theorem 2 implies that the polynomial

$$\psi_{\lambda,\mu}(z) := \lambda\psi_e(z) + \mu\psi_o(z) \tag{16}$$

has simple real roots for any $(\lambda,\mu)\neq(0,0)$. Moreover for any $(\lambda,\mu)\neq(0,0)$ the degree of $\psi_{\lambda,\mu}(z)$ is not less than $\mathrm{r}-1$.

The polynomial $p(z)$ is hyperbolic, i.e. $p(z)=\prod_{j=1}^{m}(z-ix_j)$, $x_j\in\mathbb{R}$. Hence, the polynomial

$$\varphi(z):=p(iz)/i^m=\prod_{j=1}^{m}(z-x_j), \tag{17}$$

is a monic real polynomial with only real roots whose multiplicity does not exceed m_1 and, by hypothesis, $m_1\leq r$; therefore, by Proposition 1, the polynomial $\psi_{\lambda,\mu}(D)\varphi(z)$ has only real simple roots for any $(\lambda,\mu)\neq(0,0)$.

But $\psi_{\lambda,\mu}(D)\varphi(z)=\lambda\psi_e(D)\varphi(z)+\mu\psi_o(D)\varphi(z)$; hence, by Theorem 2 we get that $\psi_e(D)\varphi(z),\psi_o(D)\varphi(z)$ strictly interlace. Now it is immediate to verify that, if $q(z)$ is defined by (14), then

$$q(iz)=i^m(\psi_e(D)\varphi(z)-i\psi_o(D)\varphi(z))\,; \tag{18}$$

therefore, by Theorem 3, either $q(iz)$ has all its zeros in the half–plane $\{z:\mathrm{Im}(z)>0\}$ or $q(iz)$ has all its zeros in the half–plane $\{z:\mathrm{Im}(z)<0\}$, i.e. one between $q(z),q(-z)$ is strictly stable. We claim that $q(z)$ is strictly stable. Indeed, let us denote by $i\sigma$ the sum of the roots of $p(z)$. Then σ is a real number, due to the hyperbolicity of $p(z)$, and we have

$$(-1)^m q(-z)=z^m-(mc_1-i\sigma)z^{m-1}+\dots\,;$$

hence the sum of the real parts of the roots of $q(-z)$ is equal to mc_1, which is a strictly positive number, as $c_1>0$ (see Lemma 1 (ii)). Therefore $q(-z)$ cannot be strictly stable. □

The hypothesis of strict stability on the polynomial $\psi(z)$ in Theorem 5 is highly reasonable; indeed, we can state a sort of converse of Theorem 5.

Theorem 6. *Let $\psi(z)=\sum_{k=0}^{\mathrm{r}}c_kz^k$ be a real polynomial, with $c_0=1$, and let us suppose that for any hyperbolic polynomial $p(z)$ of degree $m\geq\mathrm{r}$, with roots whose multiplicity does not exceed* r*, the complex polynomial*

$$q(z):=\psi(D)p(z)=p(z)+\sum_{k=1}^{\mathrm{r}}c_kp^{(k)}(z) \tag{19}$$

is stable. Then $\psi(z)$ is stable.

Proof. By contradiction. Let us suppose that $\psi(z)$ has (at least) a couple of complex conjugated roots with strictly positive real part. For any $m\in\mathbb{N}$, $m\geq\mathrm{r}$, define

$$q_m(z):=z^m+\sum_{k=1}^{\mathrm{r}}c_kD^k(z^m)=z^m+\sum_{k=1}^{\mathrm{r}}\frac{m!}{(m-k)!}c_kz^{m-k} \tag{20}$$

and

$$\psi_m(z):=\left(\frac{z}{m}\right)^m q_m\left(\frac{m}{z}\right)=1+\sum_{k=1}^{\mathrm{r}}\frac{m!}{m^k(m-k)!}c_kz^k\,. \tag{21}$$

Since $\psi_m(z)\to\psi(z)$ as $m\to\infty$, there exists a sufficiently large ν such that $\psi_m(z)$ has a couple of complex conjugated roots with strictly positive real part for $m\geq\nu$, and so the same happens for $q_m(z)$. Without loss of generality, we may suppose that $\nu = j\mathrm{r}$ for a suitable $j \in \mathbb{N}$. For any $\varepsilon > 0$, let

$$p_\varepsilon(z) = \prod_{h=0}^{j-1} (z - hi\varepsilon)^{\mathrm{r}}\,; \tag{22}$$

then $p_\varepsilon(z)$ is a hyperbolic polynomial with multiplicity of its roots equal to r. By continuity, when $\varepsilon \to 0$ the roots of the polynomial

$$p_\varepsilon(z) + \sum_{k=1}^{\mathrm{r}} c_k p_\varepsilon^{(k)}(z) \tag{23}$$

tend to the roots of $q_\nu(z)$; hence, for ε sufficiently small, the polynomial in (23) is not stable, an absurd. □

We conclude this section with two results which we will use in Section 4 to prove Theorem 1. The first is an immediate consequence of Theorem 5.

Corollary 1. *Let* $\psi(z) = \sum_{k=0}^{\mathrm{r}} c_k z^k$*, with* $c_0 = 1$*, be a real strictly stable polynomial and let* $\mathrm{r}_1 \leq \mathrm{r}$*. Then the real polynomial*

$$\psi(D)z^{\mathrm{r}_1} = \sum_{k=0}^{\mathrm{r}_1} \frac{\mathrm{r}_1!}{(\mathrm{r}_1 - k)!}\, c_k\, z^{\mathrm{r}_1 - k} \tag{24}$$

is strictly stable as well.

Proof. The proof follows by applying Theorem 5 to the polynomial z^{r_1}. □

The second result is a slight variation on the theme of Theorem 5.

Theorem 7. *Let* $\psi(z) = \sum_{k=0}^{\mathrm{r}} c_k z^k$*, with* $c_0 = 1$*, be a real strictly stable polynomial and let* $m \geq \mathrm{r}$*, and*

$$q(z) := \psi(D)z^m = z^{m-\mathrm{r}} \sum_{k=0}^{\mathrm{r}} \frac{m!}{(m-k)!}\, c_k\, z^{\mathrm{r}-k}\,. \tag{25}$$

Then the real polynomial

$$\frac{q(z)}{z^{m-\mathrm{r}}} = \sum_{k=0}^{\mathrm{r}} \frac{m!}{(m-k)!}\, c_k\, z^{\mathrm{r}-k}$$

is strictly stable, i.e. $q(z)$ *has* r *roots with strictly negative real parts, and the root* $z = 0$ *with multiplicity* $m - \mathrm{r}$*.*

Proof of Theorem 7. If $m = \mathrm{r}$, the proof follows from Corollary 1 with $\mathrm{r}_1 = m$. Let now $m \geq \mathrm{r} + 1$.

We define $\psi_1(z)$, $\psi_2(z)$ and $\psi_{\lambda,\mu}(z)$, which has simple roots for any $(\lambda, \mu) \neq (0, 0)$, as in the proof of Theorem 5, and

$$i^{-m} q(iz) = q_1(z) - i q_2(z)$$

with

$$q_1(z) = \psi_e(D)z^m = \sum_{k=0}^{[\mathrm{r}/2]} (-1)^k \frac{m!}{(m-2k)!}\, c_{2k} z^{m-2k},$$

$$q_2(z) = \psi_o(D) z^m = \sum_{k=0}^{[(\mathrm{r}-1)/2]} (-1)^k \frac{m!}{(m-(2k+1))!} c_{2k+1} z^{m-(2k+1)}.$$

By applying Proposition 1 with $f(z) = z^m$ and $g(z) = \psi_{\lambda,\mu}(z)$, then $\psi_{\lambda,\mu}(D)z^m$ has only real zeroes, the only possibly multiple zero is $z = 0$, and it has multiplicity $m-$ r. But

$$\psi_{\lambda,\mu}(D)z^m = \lambda\, q_1(z) + \mu\, q_2(z),$$

so that

$$\frac{\psi_{\lambda,\mu}(D)z^m}{z^{m-\mathrm{r}}} = \lambda\, \frac{q_1(z)}{z^{m-\mathrm{r}}} + \mu\, \frac{q_2(z)}{z^{m-\mathrm{r}}}$$

has only real, simple zeroes (clearly, different from $z = 0$) for any $(\lambda, \mu) \neq (0, 0)$. The rest of the proof is analogous to the proof of Theorem 5. □

4. Proof of Theorem 1

Thanks to the results obtained in Section 3, we obtain important information on the roots of $Q(\lambda, i\xi)$.

Proposition 2. *Let P be a hyperbolic polynomial as in* (3)*, and let m_1 be the maximum multiplicity of the roots in* (3)*, $\mathrm{r} \geq m_1$ and $\psi(z) := \sum_{k=0}^{\mathrm{r}} c_k z^k$, be a strictly stable, real polynomial, with $c_0 = 1$. Then the roots $\lambda_j(\xi)$ of $Q(\lambda, i\xi)$, defined in* (5)*, satisfy $\operatorname{Re} \lambda_j(\xi) < 0$, for any $\xi \neq 0$. Moreover, $\max\{m - \mathrm{r}, 0\}$ roots λ_j vanish at $\xi = 0$, whereas the remaining roots satisfy $\operatorname{Re} \lambda_j(0) < 0$.*

Proof. The first part of the statement follows by applying, for any fixed $\xi \neq 0$, Theorem 5 to the hyperbolic polynomial

$$p(z) = P(z, i\xi) = z^m + \sum_{j=1}^{m} i^j\, b_j(\xi) z^{m-j}, \qquad b_j(\xi) := \sum_{|\alpha|=j} b_\alpha\, \xi^\alpha.$$

The second part of the statement follows by Theorem 7, since $P(z, 0) = z^m$. □

When $\mathrm{r} \leq m - 1$, we need to estimate $\operatorname{Re} \lambda_j(\xi)$, as $\xi \to 0$, for the $m - \mathrm{r}$ roots which vanish at $\xi = 0$. We have the following.

Lemma 2. *Let $\mathrm{r} \leq m - 1$ and $\lambda_j(\xi)$ be a root of* (5)*, satisfying $\lambda_j(0) = 0$. Then*

$$|\operatorname{Im} \lambda_j(\xi)| \leq K_1 |\xi|, \qquad -K_2\, \frac{c_{\mathrm{r}-1}}{c_{\mathrm{r}}}\, |\xi|^2 \leq \operatorname{Re} \lambda_j(\xi) \leq -K_3\, \frac{c_{\mathrm{r}-1}}{c_{\mathrm{r}}}\, |\xi|^2, \tag{26}$$

in a neighborhood of $\xi = 0$, for some $K_1 > 0$ and $K_2 > K_3 > 0$, which do not depend on ψ. Moreover, if $\mathrm{r} \leq m - 2$ and $\lambda_\ell(\xi)$ is another root of (5)*, i.e. $\ell \neq j$, satisfying $\lambda_\ell(0) = 0$, then*

$$|\operatorname{Im} \lambda_j(\xi) - \operatorname{Im} \lambda_\ell(\xi)| \geq K_4 |\xi|, \tag{27}$$

in a neighborhood of $\xi = 0$, for some $K_4 > 0$.

Proof. We fix $\xi' \in S^{n-1}$. Let $\rho > 0$ and $\xi = \rho\xi'$. We define $\eta = \lambda/\rho$, so that we may write

$$\partial_\lambda^k P(\lambda, i\xi) = (i\rho)^{m-k} Q_k(\eta), \quad \text{where}$$
$$Q_k(\eta) := \sum_{j=0}^{m-k} \frac{(m-j)!}{(m-j-k)!} b_j(\xi') \eta^{m-k-j}, \tag{28}$$

and $\eta_j(\rho) := \lambda_j(\rho\xi')/(i\rho)$.

We notice that Q_k is strictly hyperbolic for any $k \geq \mathrm{r}-1$, since $\mathrm{r} \geq m_1$, where m_1 is the maximum multiplicity of the roots of Q_0.

Now we may write $Q(\lambda, i\xi) = 0$ in the form:

$$\sum_{k=0}^{\mathrm{r}} c_k\,(i\rho)^{\mathrm{r}-k} Q_k(\eta) = 0.$$

It follows $Q_{\mathrm{r}}(\eta_j) \to 0$, as $\rho \to 0$, that is, η_j tends to a real, simple, root $\bar{\eta}$ of Q_{r}. We may write

$$Q_{\mathrm{r}}(\eta_j) = (\eta_j - \bar{\eta})\tilde{Q}_{\mathrm{r}}(\eta_j),$$

with $\tilde{Q}_{\mathrm{r}}(\bar{\eta}) \neq 0$, therefore,

$$\eta_j - \bar{\eta} = -\frac{1}{c_{\mathrm{r}}\,\tilde{Q}_{\mathrm{r}}(\eta_j)} \sum_{k=0}^{\mathrm{r}-1} c_k\,(i\rho)^{\mathrm{r}-k} Q_k(\eta_j);$$

so that

$$\lambda_j = i\rho\eta_j = i\rho\bar{\eta} + \rho^2\,\frac{c_{\mathrm{r}-1}\,Q_{\mathrm{r}-1}(\eta_j)}{c_{\mathrm{r}}\,\tilde{Q}_{\mathrm{r}}(\eta_j)} + O(\rho^3)\,.$$

We notice that $\bar{\eta}$ may be zero, in general. On the other hand, $\bar{\eta}$ cannot be a root of $Q_{\mathrm{r}-1}$, since this latter is strictly hyperbolic and $\bar{\eta}$ is a root of Q_{r}. Therefore, we get

$$\mathrm{Re}\left(\frac{Q_{\mathrm{r}-1}(\eta_j)}{\tilde{Q}_{\mathrm{r}}(\eta_j)}\right) \neq 0,$$

for sufficiently small ρ, thanks to $\eta_j \to \bar{\eta} \in \mathbb{R}$. By the compactness of S^{n-1} and by Proposition 2, it follows (26).

Now we prove (27). We define $\eta_\ell = \lambda_\ell/(i\rho)$ as we did for η_j. As $\rho \to 0$, η_j and η_ℓ tend to two different roots $\bar{\eta}_j$, $\bar{\eta}_\ell$ of Q_{r}. Being Q_{r} a strictly hyperbolic polynomial, it follows from the previous representation, that

$$|\mathrm{Im}\,\lambda_j - \mathrm{Im}\,\lambda_\ell| = \rho|\mathrm{Re}\,\eta_j - \mathrm{Re}\,\eta_\ell| = \rho|\bar{\eta}_j - \bar{\eta}_\ell| + O(\rho^2) \geq C'\rho.$$

Using again the compactness of S^{n-1}, we conclude the proof of (27). □

We recall that $c_{\mathrm{r}} > 0$, $c_{\mathrm{r}-1} > 0$ in (26), due to Lemma 1(ii).

Now we need to estimate the behavior of $\lambda_j(\xi)$ as $|\xi| \to \infty$.

Lemma 3. *For any fixed $\xi' \in S^{n-1}$, the roots $\lambda_j(\xi)$ of (5), where $\xi = |\xi|\,\xi'$, satisfy*

$$\exists \lim_{|\xi|\to\infty} \mathrm{Re}\,\lambda_j(\xi) < 0, \qquad \lim_{|\xi|\to\infty} \left(|\xi|^{-1}\mathrm{Im}\,\lambda_j(\xi) - a_j(\xi')\right) = 0, \quad j = 1, \ldots, m, \tag{29}$$

where $ia_j(\xi')$ are the roots of $P(\lambda, i\xi')$ (see (3)).

Proof. As in the proof of Lemma 2, we fix $\xi' \in S^{n-1}$ and we set $\xi = \xi'\rho$, for $\rho > 0$. Again, let $\eta = \lambda/(i\rho)$ and $\eta_j := \lambda_j/(i\rho)$. We write again the polynomials $\partial_\lambda^k P$ in the form (28), so that $Q(\lambda, i\xi) = 0$ may be written as

$$Q_0(\eta_j) + \sum_{h=1}^{\mathrm{r}} c_h\,(i\rho)^{-h} Q_h(\eta) = 0.$$

As $\rho \to \infty$, $Q_0(\eta_j) \to 0$, that is, $\eta_j \to a_j(\xi')$.

Let $ia_j(\xi')$ be a root of $P(\lambda, i\xi')$ with multiplicity r_1; we recall that $\mathrm{r}_1 \leq m_1 \leq \mathrm{r}$. We may write

$$Q_h(\eta_j) = (\eta_j - a_j)^{\mathrm{r}_1 - h} \tilde{Q}_h(\eta_j), \qquad h = 0, \ldots, \mathrm{r}_1,$$

where $\tilde{Q}_h(\eta_j) \neq 0$. In particular, $\tilde{Q}_{\mathrm{r}_1}(\eta_j) = Q_{\mathrm{r}_1}(\eta_j)$, and

$$\tilde{Q}_h(\eta_j) = \frac{\mathrm{r}_1!}{(\mathrm{r}_1 - h)!} \tilde{Q}_0(\eta_j), \qquad h = 0, \ldots, \mathrm{r}_1.$$

Therefore, recalling the expression in (24), it follows that

$$\begin{aligned}
0 &= \sum_{h=0}^{\mathrm{r}_1} c_h\,(i\rho)^{-h} (\eta - a_j)^{\mathrm{r}_1 - h} \tilde{Q}_h(\eta_j) + O(\rho^{-\mathrm{r}_1 - 1}) \\
&= \sum_{h=0}^{\mathrm{r}_1} \frac{\mathrm{r}_1!}{(\mathrm{r}_1 - h)!} c_h\ (i\rho)^{-h}\ (\eta_j - a_j)^{\mathrm{r}_1 - h} \tilde{Q}_0(\eta_j) + O(\rho^{-\mathrm{r}_1 - 1}) \\
&= (i\rho)^{-\mathrm{r}_1}\ \tilde{Q}_0(\eta_j) \sum_{h=0}^{\mathrm{r}_1} \frac{\mathrm{r}_1!}{(\mathrm{r}_1 - h)!} c_h\ ((\eta_j - a_j) i\rho)^{\mathrm{r}_1 - h} + O(\rho^{-\mathrm{r}_1 - 1}) \\
&= (i\rho)^{-\mathrm{r}_1}\ \tilde{Q}_0(\eta_j) \psi(D) z^{\mathrm{r}_1} + O(\rho^{-\mathrm{r}_1 - 1}),
\end{aligned}$$

with

$$z = (\eta_j - a_j) i\rho.$$

By Corollary 1, $\psi(D) z^{\mathrm{r}_1}$ is strictly stable. Therefore,

$$(\eta_j - a_j) i\rho = \tilde{z} + o(1), \qquad \text{as } \rho \to \infty,$$

where $\tilde{z}$ is a root of $\psi(D) z^{\mathrm{r}_1}$, in particular $\operatorname{Re} \tilde{z} < 0$; hence,

$$\lambda_j = i\rho\eta_j = i\rho a_j + \tilde{z} + o(1),$$

so that (29) follows. □

In particular, from Proposition 2 and Lemma 3, we derive that

$$\forall\, \varepsilon > 0\ \exists\, c_\varepsilon > 0\ :\ \operatorname{Re} \lambda_j(\xi) \leq -c_\varepsilon \quad \forall\, \xi : |\xi| \geq \varepsilon\,, \tag{30}$$

for any root verifying $\lambda_j(0) = 0$, whereas

$$\exists\, c > 0\ :\ \operatorname{Re} \lambda_j(\xi) \leq -c \quad \forall\, \xi \in \mathbb{R}^n\,, \tag{31}$$

for any root verifying $\operatorname{Re} \lambda_j(0) < 0$.

We are now ready to prove Theorem 1.

Proof of Theorem 1. Assume for a moment that the roots of $Q(\lambda, i\xi)$ are simple. Then, after performing the Fourier transform of the equation in (6), we may write

$$\hat{u}(t, \xi) = \sum_{j=1}^{m} e^{\lambda_j(\xi) t}\, \Delta_j(\xi) \sum_{h=0}^{m-1} \sigma_{m-1-h, j}(\xi)\, \hat{u_h}(\xi), \tag{32}$$

where

$$\Delta_j(\xi) = \prod_{k \neq j} \frac{1}{\lambda_j(\xi) - \lambda_k(\xi)},$$

$$\sigma_{0,j} = 1, \quad \sigma_{1,j} = -\sum_{k \neq j} \lambda_k, \qquad \sigma_{2,j} = \sum_{k<l} \lambda_k \lambda_l + \lambda_j \sigma_{1,j},$$

$$\sigma_{3,j} = -\sum_{k<l<p} \lambda_k\lambda_l\lambda_p + \lambda_j\sigma_{2,j}, \qquad \ldots$$

By Plancherel's theorem, we want to estimate the L^2 norm of

$$|\xi|^{|\alpha|}\partial_t^k\hat{u}(t,\xi).$$

Let $\varepsilon > 0$. By (30)–(31) we know that $\operatorname{Re}\lambda_j \leq -c_\varepsilon$ for any $\xi : |\xi| > \varepsilon$ and $j = 1,\ldots,m$.

Due to $|\lambda_j(\xi)| \lesssim |\xi|$ for $|\xi| > \varepsilon$, we get

$$|\Delta_j(\xi)| \lesssim |\xi|^{-(m-1)} \qquad |\sigma_{m-1-h,j}(\xi)| \lesssim |\xi|^{m-1-h},$$

and

$$|\partial_t^k\hat{u}(t,\xi)| \lesssim |\xi|^k \sum_{j=1}^m e^{\operatorname{Re}\lambda_j(\xi)t}\,|\Delta_j(\xi)| \sum_{h=0}^{m-1} |\sigma_{m-1-h,j}(\xi)|\,|\hat{u_h}(\xi)|.$$

Therefore

$$|\xi|^{|\alpha|}|\partial_t^k\hat{u}(t,\xi)| \lesssim e^{-c_\varepsilon t}\,|\xi|^{|\alpha|+k} \sum_{h=0}^{m-1} |\xi|^{-h}\,|\hat{u}_h(\xi)|,$$

for $|\xi| > \varepsilon$. Taking into account of the possible multiplicity of $\lambda_j(\xi)$, in a compact subset of $\{|\xi| > \varepsilon\}$, this estimate becomes

$$|\xi|^{|\alpha|}|\partial_t^k\hat{u}(t,\xi)| \lesssim (1+t)^m\,e^{-c_\varepsilon t}\,|\xi|^{|\alpha|+k} \sum_{h=0}^{m-1} |\xi|^{-h}\,|\hat{u}_h(\xi)|.$$

It is clear that $(1+t)^m\,e^{-c_\varepsilon t} \lesssim e^{-\delta t}$ for some $\delta \in (0, c_\varepsilon)$.

However, two or more roots $\lambda_j(\xi)$ tend to coincide as $|\xi| \to \infty$ along the direction $\xi' = \xi/|\xi|$, if $ia_j(\xi')$ is a multiple root of $P(\lambda, i\xi')$. In this case, if r_1 is the multiplicity of $ia_j(\xi')$, we may estimate

$$|\xi|^{|\alpha|}|\partial_t^k\hat{u}(t,\xi)| \lesssim (1+t|\xi|)^{\mathrm{r}_1-1}\,e^{-c_\varepsilon t}\,|\xi|^{|\alpha|+k} \sum_{h=0}^{m-1} |\xi|^{-h}\,|\hat{u}_h(\xi)|.$$

Therefore, if m_1 is the maximum multiplicity of the roots of $P(\lambda, i\xi')$, by applying Plancherel's theorem on the initial data, we obtain

$$\|\hat{u}(t,\cdot)\|_{L^2(|\xi|\geq\varepsilon)} \lesssim e^{-\delta t} \sum_{h=0}^{m-1} \|u_h\|_{H^{|\alpha|+k-h+m_1-1}},$$

for some $\delta \in (0, c_\varepsilon)$. That is, we have a loss of regularity in our estimate, due to the weak hyperbolicity of P.

For sufficiently small $\varepsilon > 0$, for any $\xi : |\xi| \leq \varepsilon$, we distinguish two cases.

We assume for a moment that the roots satisfying $\operatorname{Re}\lambda_j < 0$ are simple. Then they remain simple for $|\xi| \leq \varepsilon$, for sufficiently small ε and, for each root $\lambda_j(\xi)$ such that $\operatorname{Re}\lambda_j < 0$, we may estimate $\Delta_j(\xi)$ by a constant. On the other hand, we may also estimate $|\sigma_{m-1-h,j}(\xi)|$ by a constant. Taking into account of the possible multiplicity of $\lambda_j(\xi)$ and estimating again $(1+t)^m\,e^{-ct} \lesssim e^{-\delta t}$ for some $\delta \in (0, c)$, this leads to

$$|\lambda_j(\xi)|^k \sum_{j=1}^m e^{\operatorname{Re}\lambda_j(\xi)t}\,|\Delta_j(\xi)| \sum_{h=0}^{m-1} |\sigma_{m-1-h,j}(\xi)|\,|\hat{u_h}(\xi)| \lesssim e^{-ct} \sum_{h=0}^{m-1} |\hat{u_h}(\xi)|.$$

for any $|\xi|\leq\varepsilon$. In particular, by Plancherel theorem,

$$\Big\| |\xi|^{|\alpha|}\,|\lambda_j(\xi)|^k \sum_{j=1}^m e^{\operatorname{Re}\lambda_j(\xi)t}\,|\Delta_j(\xi)| \sum_{h=0}^{m-1} |\sigma_{m-1-h,j}(\xi)|\,|\hat{u_h}(\xi)|\Big\|_{L^2(|\xi|\leq\varepsilon)} \lesssim e^{-ct}\sum_{h=0}^{m-1}\|u_h\|_{H^{|\alpha|+k-h+m_1-1}}.$$

Indeed, the weighted norms $\|(1+|\xi|)^\kappa\hat{u}_h\|_{L^2(|\xi|\leq\varepsilon)}$ are all equivalent for $\kappa\in\mathbb{R}$.

If $\mathrm{r}\geq m$, estimate (9) immediately follows and we conclude the proof of Theorem 1.

Now let $\mathrm{r}\leq m-1$, so that we should also consider the roots $\lambda_j(\xi)$ which vanish at $\xi=0$.

Thanks to (27), there exists $\varepsilon>0$ such that each root $\lambda_j(\xi)$, which satisfies $\lambda_j(0)=0$, is a simple root of $Q(\lambda,i\xi)$, for any $0<|\xi|<\varepsilon$.

Then we may estimate $|\Delta_j(\xi)|\lesssim|\xi|^{-(m-\mathrm{r}-1)}$, thanks to (27). On the other hand, we may estimate $|\sigma_{m-1-h,j}(\xi)|\lesssim|\xi|^{m-\mathrm{r}-1-h}$, for any $h\leq m-\mathrm{r}-1$, thanks to (26), whereas we estimate $\sigma_{m-1-h,j}(\xi)$ by a constant for $h\geq m-\mathrm{r}-1$. We also remark that $|\lambda_j(\xi)|\lesssim|\xi|$ thanks to (26).

Therefore,

$$\Big\| |\xi|^{|\alpha|}(\lambda_j(\xi))^k\,e^{\lambda_j(\xi)t}\,\Delta_j(\xi)\sum_{h=0}^{m-1}\sigma_{m-1-h,j}(\xi)\,\hat{u_h}(\xi)\Big\|_{L^2(|\xi|\leq\varepsilon)} \lesssim \sum_{h=0}^{m-1}\|J_h\,e^{\operatorname{Re}\lambda_j(\xi)t}\|_{L^2(|\xi|\leq\varepsilon)}\,\|\hat{u_h}\|_{L^\infty}\,,$$

where

$$J_h:=\begin{cases}|\xi|^{|\alpha|+k-h} & \text{if } h=0,\dots,m-\mathrm{r}-1,\\ |\xi|^{|\alpha|+k-(m-\mathrm{r}-1)} & \text{if } h\geq m-\mathrm{r}-1.\end{cases}$$

We remark that $\|\hat{u}_h\|_{L^\infty}\lesssim\|u_h\|_{L^1}$.

In particular, J_h are in $L^2(|\xi|\leq\varepsilon)$ for any $h=0,\dots,m-1$, thanks to (8) (incidentally we remark that condition (8) may be relaxed if $u_{m-1}=u_{m-2}=\ldots=u_{m-\ell}=0$ for some $\ell\geq\mathrm{r}+1$ in (6)).

If $t\leq1$, we simply estimate the $L^2(|\xi|\leq\varepsilon)$ norm of $J_he^{\operatorname{Re}\lambda_jt}$ by a constant. Let $t\geq1$. By the change of variable $\theta=\sqrt{t}\xi$, we derive

$$\int_{|\xi|\leq\varepsilon}|\xi|^{2(|\alpha|+k-h)}e^{-2K_3\frac{\mathrm{c}_{\mathrm{r}-1}}{\mathrm{c}_\mathrm{r}}|\xi|^2t}\,d\xi \lesssim t^{-\frac{n}{2}-|\alpha|-k+h}\int_{\mathbb{R}^n}|\theta|^{2(|\alpha|+k-h)}e^{-2K_3\frac{\mathrm{c}_{\mathrm{r}-1}}{\mathrm{c}_\mathrm{r}}|\theta|^2}\,d\theta.$$

Since the last integral is bounded, we obtain

$$\|J_he^{\lambda_jt}\|_{L^2(|\xi|\leq\varepsilon)}\lesssim\begin{cases}(1+t)^{-\frac{n}{4}-\frac{|\alpha|+k-h}{2}}, & h=0,\dots,m-\mathrm{r}-1\,;\\ (1+t)^{-\frac{n}{4}-\frac{|\alpha|+k-(m-\mathrm{r}-1)}{2}} & h=m-\mathrm{r}-1,\dots,m-1.\end{cases}$$

By gluing these estimates with the previous ones, we conclude the proof of (7). □

5. An application of Theorem 1 to the theory of elastic waves

There exists a large literature concerning the study of systems of elastic waves, which goes back to [18, 23]. We address the interested reader to [7] for a nice survey about the Lamé operator which drives the propagation of linear elastic waves.

In particular, in $\mathbb{R}^n$, dissipative systems of elastic waves have been studied in [5, 20, 32]. Strichartz estimates have been obtained in [1] for systems of elastic waves in $\mathbb{R}^3$.

Linear elastic waves are modeled by the hyperbolic system of equations

$$\rho\,\partial_t^2 u - \mu\Delta u - (\mu+\nu)\nabla\operatorname{div} u = 0, \qquad t\in\mathbb{R}, \quad x\in\mathbb{R}^n,$$

where $\rho > 0$ is the density, $\mu > 0$ and $\nu > -\mu$ are the Lamè constants, $u = (u_1,\dots,u_n)$ is the displacement n-dimensional vector, and

$$\nabla = \big(\partial_{x_1},\dots,\partial_{x_n}\big), \qquad \operatorname{div} u := \sum_{j=1}^n \partial_{x_j} u_j.$$

The scalar operator Δ is intended to be applied to each component of u, i.e. $\Delta u = (\Delta u_1,\dots,\Delta u_n)$. We do not distinguish between row and column vectors.

After re-scaling μ and ν, we may assume, without loss of generality, that $\rho = 1$, that is,

$$\partial_t^2 u - \mu\Delta u - (\mu+\nu)\nabla\operatorname{div} u = 0. \tag{33}$$

As a consequence of (33), $\operatorname{div} u$ verifies the scalar equation:

$$\partial_t^2 \operatorname{div} u - (2\mu+\nu)\Delta\operatorname{div} u = 0. \tag{34}$$

Therefore, as it is well-known, applying the scalar operator $\partial_t^2 - (2\mu+\nu)\Delta$ to (33), one finds that each component of u verifies the fourth-order scalar equation:

$$\partial_t^4 u - (3\mu+\nu)\Delta\partial_t^2 u + \mu(2\mu+\nu)\Delta^2 u = 0. \tag{35}$$

Equation (35) is strictly hyperbolic, with symbol:

$$P(\lambda, i\xi) = \lambda^4 + (3\mu+\nu)|\xi|^2\lambda^2 + \mu(2\mu+\nu)|\xi|^4,$$

and it is given by the factorization of two wave operators with different speeds:

$$\big(\partial_t^2 - (2\mu+\nu)\Delta\big)\,\big(\partial_t^2 - \mu\Delta\big)\,u = 0. \tag{36}$$

Therefore, Theorem 1 may applied to equation (35). Let r $= 1,2,3,4$. Then the equation $Q(\partial_t,\partial_x)u = 0$, with symbol defined in (5), reads as:

$$\begin{aligned} \partial_t^4 u - (3\mu+\nu)\Delta\partial_t^2 u + \mu(2\mu+\nu)\Delta^2 u + c_1\big(4\partial_t^3 u - 2(3\mu+\nu)\Delta\partial_t u\big)& \\ + c_2\big(12\partial_t^2 u - 2(3\mu+\nu)\Delta u\big) + 24c_3\partial_t u + 24 c_4 u = 0.& \end{aligned} \tag{37}$$

Remark 1. The hypotheses on the strict stability of the polynomial $\psi(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4$, in Theorem 1, are verified if, and only if, one of the following holds

- $c_1 > 0$ and $c_2 = c_3 = c_4 = 0$;
- $c_1 > 0$, $c_2 > 0$ and $c_3 = c_4 = 0$;
- $c_1 > 0$, $c_2 > 0$, $c_3 \in (0, c_1 c_2)$ and $c_4 = 0$;
- c_j are all positive, and they verify the conditions

$$4c_4 < c_2^2, \qquad c_1^2 c_4 + c_3^2 < c_1 c_2 c_3. \tag{38}$$

The proof follows, for instance, by applying Theorem 4 to $\psi(z)$.

Some interesting models from the theory of systems of electromagnetic elastic waves may also be studied by reducing them to higher order, scalar equations.

Let us consider the coupled system of linear elastic waves with Maxwell equations in $\mathbb{R}^3$, i.e.

$$\begin{cases} \partial_t^2 u - \mu\Delta u - (\mu+\nu)\nabla \operatorname{div} u + \gamma\nabla\times E = 0, \\ \partial_t E - \nabla\times H - \gamma\nabla\times\partial_t u = 0, \\ \partial_t H + \nabla\times E = 0, \end{cases} \tag{39}$$

where $\nabla\times$ denotes the curl operator. It is clear that $\operatorname{div} u$ still verifies (34). Model (39) can be considered a special case of the model studied in exterior domains in [17].

We may assume without restriction that the electric field E and the magnetic field H are divergence-free, since $\operatorname{div} E$ and $\operatorname{div} H$ are constant, as a consequence of the second and third equations in (39).

By the second and third equation in (39), we obtain:

$$\partial_t^2 E + \nabla\times(\nabla\times E) - \gamma\nabla\times\partial_t^2 u = 0$$

which, being $\operatorname{div} E = 0$, recalling that

$$\nabla\times(\nabla\times f) = -\Delta f + \nabla\operatorname{div} f, \tag{40}$$

gives the system

$$\begin{cases} \partial_t^2 u - \mu\Delta u - (\mu+\nu)\nabla \operatorname{div} u + \gamma\nabla\times E = 0, \\ \partial_t^2 E - \Delta E - \gamma\nabla\times\partial_t^2 u = 0. \end{cases} \tag{41}$$

Applying the operator $\partial_{tt} - \Delta$ to the first equation in (39), replacing the second equation in (41), recalling (40), we get

$$\partial_t^4 u - (1+\mu+\gamma^2)\Delta\partial_t^2 u + \mu\Delta^2 u - ((\mu+\nu)-\gamma^2)\nabla\operatorname{div}\partial_t^2 u + (\mu+\nu)\Delta\nabla\operatorname{div} u = 0. \tag{42}$$

Since $\operatorname{div} u$ verifies (34), in the special case$((\mu+\nu)-\gamma^2)(2\mu+\nu) = \mu+\nu$, equation (42) reduces to a fourth order scalar equation given by the factorization of

$$(\partial_t^2 - \alpha_+\Delta)(\partial_t^2 - \alpha_-\Delta)u = 0,$$

where

$$\alpha\pm = \frac{1+\mu+\gamma^2 \pm \sqrt{(1-\mu+\gamma^2)^2 + 4\mu\gamma^2}}{2}.$$

Otherwise, applying the operator $\partial_t^2 - (2\mu+\nu)\Delta$ to (42), we obtain the sixth-order scalar equation given by the factorization of

$$(\partial_t^2 - (2\mu+\nu)\Delta)(\partial_t^2 - \alpha_+\Delta)(\partial_t^2 - \alpha_-\Delta)u = 0.$$

Then one may construct lower order terms which make the equation above dissipative, by applying Theorem 1.

Remark 2. It is clear that there is no general way to solve a sixth order algebraic equation, depending on a parameter $\xi \in \mathbb{R}^n$, with complex-valued coefficients. Therefore, the task to explicitly compute the roots $\lambda_j(\xi)$ of the polynomial

$$Q(\lambda, i\xi) = (\lambda^2 + (2\mu+\nu)|\xi|^2)(\lambda^2 + \alpha_+|\xi|^2)(\lambda^2 + \alpha_-|\xi|^2) + \sum_{k+|\alpha|\leq 5} c_{k,\alpha}\lambda^k\,(i\xi)^\alpha,$$

with $c_{k,\alpha} \in \mathbb{R}$, seems to be not easy to be accomplished. Thus, it is not possible to apply some known result in literature, where assumptions are taken on the properties of the roots of the full symbol of an inhomogeneous operator. The application of Theorem 1 to the model discussed in the present section, gives at least a relatively general class of coefficients $c_{k,\alpha}$, for which a dissipative decay estimate is provided.

6. An example of dissipative equation not included in Theorem 1

It is clear that Theorem 1 only provides a class of lower order terms which, if added to a homogeneous hyperbolic operator, make it dissipative. Indeed, not all possible dissipative higher order inhomogeneous operators $R(\partial_t,\partial_x)$ with principal part $P(\lambda,\partial_\xi)$ may be written in the form (5). If this is possible, then the coefficients c_k may be easily and uniquely determined by the solving the polynomial identity:

$$R(\lambda,0)=P(\lambda,0)+\sum_{k=1}^{r} c_k\,\partial_\lambda^k P(\lambda,0)\equiv\lambda^m+\sum_{k=1}^{r} c_k\,\frac{m!}{(m-k)!}\,\lambda^{m-k}.$$

In this section, we give a complete characterization of third order, inhomogeneous operators, for which a dissipative effect appears, under the additional assumption that only terms which contain an even number of spatial derivatives appear. Thanks to this assumption, the full symbol of the inhomogeneous operator is real, and so Theorem 4 may be directly applied to this latter.

In the following of this section, we fix

$$P(\partial_t,\partial_x)=\partial_t^3-a(\partial_x)\,\partial_t, \tag{43}$$

$$Q(\partial_t,\partial_x)=P(\partial_t,\partial_x)+b_0\,\partial_t^2-b(\partial_x)+d_0\,\partial_t+e_0, \tag{44}$$

where $b_0,d_0,e_0\in\mathbb{R}$ and

$$a(\partial_x)=\sum_{j,k=1}^{n} a_{jk}\,\partial_{x_j}\,\partial_{x_k},$$

$$b(\partial_x)=\sum_{j,k=1}^{n} b_{jk}\,\partial_{x_j}\,\partial_{x_k},$$

with $a_{jk},b_{jk}\in\mathbb{R}$. The hyperbolicity condition for $P(\partial_t,\partial_x)$ holds if, and only if, $a(\xi)\geq 0$ for any $\xi\in S^{n-1}$; the roots of $P(\lambda,i\xi)=0$ are $0,\pm i\sqrt{a(\xi)}$.

Lemma 4. *Let $a(\xi)\geq 0$ and $b(\xi)$ be as above. Then the polynomial*

$$Q(\lambda,i\xi)=\lambda^3+b_0\lambda^2+\lambda(d_0+a(\xi))+e_0+b(\xi)$$

is strictly stable for any $\xi\in\mathbb{R}^n\setminus\{0\}$ if, and only if, $b_0>0$, and

- *either*

$$e_0=d_0=0,\qquad 0<b(\xi)<b_0\,a(\xi),\qquad \forall\xi\in S^{n-1} \tag{45}$$

  (in this case, the operator $P(\partial_t,\partial_x)$ must be be strictly hyperbolic);
- *or*

$$e_0=0,\quad d_0>0,\qquad 0<b(\xi)\leq b_0\,a(\xi),\qquad \forall\xi\in S^{n-1} \tag{46}$$

  (in this case, the operator $P(\partial_t,\partial_x)$ must be be strictly hyperbolic);
- *or*

$$0<e_0<b_0\,d_0,\qquad 0\leq b(\xi)\leq b_0\,a(\xi),\qquad \forall\xi\in S^{n-1}. \tag{47}$$

- *or*

$$0 < e_0 = b_0\, d_0, \qquad 0 \leq b(\xi) < b_0\, a(\xi), \qquad \forall \xi \in S^{n-1}. \tag{48}$$

Proof. Since the polynomial is real-valued for any $\xi \in \mathbb{R}^n \setminus \{0\}$, we may apply Theorem 4, so that $Q(\lambda, i\xi)$ is strictly stable if, and only if,

$$b_0 > 0, \qquad d_0 + a(\xi) > 0,$$

and

$$0 < e_0 + b(\xi) < b_0(d_0 + a(\xi)), \qquad \forall \xi \in \mathbb{R}^n \setminus \{0\}. \tag{49}$$

By homogeneity arguments, due to $a(\xi) \geq 0$, condition $d_0 + a(\xi) > 0$ for any $\xi \in \mathbb{R}^n \setminus \{0\}$ holds if, and only if, $d_0 \geq 0$ if $a(\xi) > 0$ for any $\xi \in S^{n-1}$, or $d_0 > 0$ if $a(\xi)$ vanishes for some $\xi \in S^{n-1}$. Again, by homogeneity arguments, condition (49) is equivalent to ask that

$$0 < b(\xi) < b_0\, a(\xi), \quad \forall \xi \in S^{n-1},$$

if $e_0 = d_0 = 0$, to

$$0 < b(\xi) \leq b_0\, a(\xi), \quad \forall \xi \in S^{n-1},$$

if $e_0 = 0$ and $d_0 > 0$, or to (47) or (48), if $e_0, d_0 > 0$. This concludes the proof. □

Remark 3. If $Q(\lambda, i\xi)$ is as in (5), then the assumptions of Theorem 8 are verified. Indeed, in this case, $b_0 = 3c_1$, $b(\xi) = c_1\, a(\xi)$, $d_0 = 6c_2$ and $e_0 = 6c_3$, and the polynomial $\psi(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3$ is strictly stable in one of the following three cases (see Remark 1):

- $c_1 > 0$ and $c_2 = c_3 = 0$;
- $c_1 > 0$, $c_2 > 0$, and $c_3 = 0$;
- $c_1 > 0$, $c_2 > 0$, and $0 < c_3 < c_1\, c_2$.

In the first case, (45) holds, in the second one (46) holds, whereas, in the last case, (47) is verified.

Remark 4. In Theorem 2.1 in [38], the authors prove that the stability of a polynomial in the form

$$Q(\lambda, i\xi) = P_m(\lambda, i\xi) + P_{m-1}(\lambda, i\xi) + P_{m-2}(\lambda, i\xi),$$

where $m \geq 3$ and P_k are homogeneous hyperbolic polynomials of order k, is equivalent to ask the following set of assumptions:

- P_{m-1} is strictly hyperbolic;
- there is no common zero of P_m, P_{m-1} and P_{m-2}, exception given for $\lambda = 0$ when $\xi = 0$;
- $P_m(1, 0)$, $P_{m-1}(1, 0)$ and $P_{m-2}(1, 0)$ are non-zero and they have the same sign;
- $P_m(\lambda, \xi)$ and $P_{m-1}(\lambda, \xi)$ weakly interlace for any $\xi \neq 0$;
- $P_{m-1}(\lambda, \xi)$ and $P_{m-2}(\lambda, \xi)$ weakly interlace for any $\xi \neq 0$.

It is easy to check that condition (46) is equivalent to the previous set of assumptions for the polynomial $Q(\lambda, i\xi)$ in Lemma 4 with $d_0 \neq 0$ and $e_0 = 0$.

The behavior of the roots of $Q(\lambda, i\xi)$ may then be checked as $\xi \to 0$ and as $|\xi| \to \infty$, as we did in the proofs of Lemmas 2 and 3.

We notice that if $a(\xi) = 0$, then the roots of $Q(\lambda, i\xi) = \lambda^3 + b_0\lambda^2 + d_0\lambda + e_0$ are constant, so that there is nothing to prove.

Let us first consider $\xi \to 0$. If (47) or (48) holds, then the three roots of $Q(\lambda, 0)$ have negative real parts. If (45) or (46) holds, then the roots of $Q(\lambda, i\xi)$ which vanish at $\xi = 0$ verify $|\mathrm{Im}\,\lambda_j(\xi)| \leq C|\xi|$ and $\mathrm{Re}\,\lambda_j(\xi) \leq -c|\xi|^2$, for some $C > 0$ and $c > 0$, as $\xi \to 0$. Moreover, if $d_0 = 0$, the two vanishing roots $\lambda_\pm(\xi)$ satisfy (27), i.e., $|\mathrm{Im}\,(\lambda_+ - \lambda_-)| \gtrsim |\xi|$.

Indeed, if $d_0 > 0$ and $e_0 = 0$, then there is one root $\lambda_j(\xi)$, vanishing at $\xi = 0$, since $Q(\lambda, 0) = \lambda^3 + b_0\lambda^2 + d_0\lambda$. Due to

$$d_0\lambda_j(\xi) = -\lambda_j^3(\xi) - \lambda_j(\xi)\,a(\xi) - b_0\,\lambda_j^2(\xi) - b(\xi),$$

this root verifies

$$\lambda_j(\xi) = -b(\xi)/d_0 + O(|\xi|^3).$$

On the other hand, if $d_0 = e_0 = 0$ then there are two roots $\lambda_\pm(\xi)$, vanishing at $\xi = 0$, since $Q(\lambda, 0) = \lambda^3 + b_0\lambda^2$. Due to

$$\big(\sqrt{b_0}\lambda_\pm(\xi) + i\sqrt{b(\xi)}\big)\big(\sqrt{b_0}\lambda_\pm(\xi) - i\sqrt{b(\xi)}\big) = -\lambda_\pm(\xi)^3 - \lambda_\pm(\xi)\,a(\xi),$$

these roots verify

$$\lambda_\pm(\xi) = \pm i\sqrt{b(\xi)/b_0} - \frac{a(\xi) - b(\xi)/b_0}{2} + O(|\xi|^3).$$

Similarly, we may follow the proof of Lemma 3 to prove that $\mathrm{Re}\,\lambda_j(\xi)$ remains away from zero, as $|\xi| \to \infty$, for any $j = 1, 2, 3$. Indeed, the three roots $\lambda_0(\xi)$, $\lambda_+(\xi)$ and $\lambda_-(\xi)$ verify:

$$\mathrm{Im}\,\lambda_0(\xi) = O(|\xi|^{-1}), \qquad \mathrm{Im}\,\lambda_\pm(\xi) = \pm i\sqrt{a(\xi)} + O(|\xi|^{-1}),$$
$$\mathrm{Re}\,\lambda_0(\xi) = -\frac{b(\xi)}{a(\xi)} + O(|\xi|^{-1}), \qquad \mathrm{Re}\,\lambda_\pm(\xi) = -\frac{a(\xi)b_0 - b(\xi)}{2a(\xi)} + O(|\xi|^{-1}),$$

if $a(\xi) > 0$.

Following the proof of Theorem 1, we are now in the position to prove the following.

Theorem 8. *Let P and Q be as in* (43) *and* (44)*, with $b_0 > 0$. Assume one among* (45)-(46)-(47)-(48)*. Then the equation $Q(\partial_t, \partial_x)u = 0$ is a dissipative equation. More precisely:*

- *if* (45) *holds, then the solution to*

$$\begin{cases} Q(\partial_t, \partial_x)u = 0, & t \geq 0, \quad x \in \mathbb{R}^n, \\ \partial_t^j u(0, x) = u_j(x), & j = 0, 1, 2, \end{cases} \tag{50}$$

  satisfies the following long-time decay estimate with polynomial speed:

$$\|\partial_x^\alpha \partial_t^k u(t, \cdot)\|_{L^2} \leq C(1+t)^{-\frac{n}{4} - \frac{|\alpha|+k-1}{2}} \big((1+t)^{-\frac{1}{2}}\|u_0\|_{L^1} + \|u_1\|_{L^1} + \|u_2\|_{L^1}\big) + C\,e^{-\delta t} \sum_{j=0}^{2} \|u_j\|_{H^{|\alpha|+k-j}} \tag{51}$$

  for any $\alpha \in \mathbb{N}^n$ and $k \in \mathbb{N}$, provided that $n \geq 3$ if $|\alpha| = k = 0$;
- *if* (46) *holds, then the solution to* (50) *satisfies the following long-time decay estimate with polynomial speed:*

$$\|\partial_x^\alpha \partial_t^k u(t, \cdot)\|_{L^2} \leq C(1+t)^{-\frac{n}{4} - \frac{|\alpha|+k}{2}} \big(\|u_0\|_{L^1} + \|u_1\|_{L^1} + \|u_2\|_{L^1}\big)$$

$$+ C\, e^{-\delta t} \sum_{j=0}^{2} \|u_j\|_{H^{|\alpha|+k-j}} \tag{52}$$

for any $\alpha \in \mathbb{N}^n$ and $k \in \mathbb{N}$;

- *if (47) or (48) holds, then the solution to (50) satisfies the following long-time decay estimate with exponential speed:*

$$\|\partial_x^\alpha \partial_t^k u(t,\cdot)\|_{L^2} \leq C\, e^{-\delta t} \sum_{j=0}^{2} \|u_j\|_{H^{|\alpha|+k-j}}, \tag{53}$$

if $a(\xi) > 0$ for any $\xi \in S^{n-1}$, for any $\alpha \in \mathbb{N}^n$ and $k \in \mathbb{N}$, or

$$\|\partial_x^\alpha \partial_t^k u(t,\cdot)\|_{L^2} \leq C\, e^{-\delta t} \sum_{j=0}^{2} \|u_j\|_{H^{|\alpha|+k-j+2}}, \tag{54}$$

if $a(\xi)$ vanishes at some $\xi \in S^{n-1}$, for any $\alpha \in \mathbb{N}^n$ and $k \in \mathbb{N}$.

Proof. The same as the proof of Theorem 1. □

Lemma 4 guarantees the optimality of assumptions (45)-(46)-(47)-(48).

Remark 5. The chance to explicitly check the stability of the polynomial $Q(\lambda, i\xi)$, with Q as in (44) was strongly related to the assumption that it was a third order operator with real-valued coefficients, so that it was possible and easy to apply Theorem 4. This approach still works with fourth order polynomials with real-valued coefficients (for instance, for polynomials with principal part given by (35)), and other stability criteria may help in more complicated cases. However, in general, it seems to be a hard task to check the stability of a high order inhomogeneous polynomial in the general form. Theorem 1 provides a tool to construct at least a class of lower order terms, for a given high order homogeneous polynomial, for which the stability is guaranteed.

7. Further applications of Theorem 5

In this section, we show how Theorem 5 may be also applied to study equations, which are not covered by Theorem 1.

It is well-known (see, for instance, [4, 14, 22, 30, 35]) that the damped wave equation keeps its dissipative nature if the external damping term is replaced by a structural one, namely, if we consider the Cauchy problem for

$$u_{tt} - \Delta u + 2c(-\Delta)^\theta u_t = 0, \qquad t \geq 0,\ x \in \mathbb{R}^n, \tag{55}$$

with $\theta \in (0,1]$, $c > 0$, where the fractional Laplacian is described by its action

$$(-\Delta)^\theta f = \mathfrak{F}^{-1}(|\xi|^{2\theta} \mathfrak{F} f),$$

being $\mathfrak{F}$ the Fourier transform with respect to x. The full symbol of (55) may be written in the form

$$P(\lambda, i\xi) + c|\xi|^{2\theta} \partial_\lambda P(\lambda, i\xi),$$

where $P(\lambda, i\xi) = \lambda^2 + |\xi|^2$ is the symbol of the wave operator. In particular, the polynomial above is strictly stable for any $\xi \neq 0$, thanks to Theorem 5.

More in general, let $P(\partial_t, \partial_x)$ be a real, hyperbolic operator with maximum multiplicity of its roots m_1 over $\mathcal{S}^{n-1}$, and let $\mathrm{r} \geq m_1$, $\psi(z) = \sum_{j=0}^{\mathrm{r}} c_k z^k$ a strictly stable, real, polynomial, with $c_0 = 1$. Let $\theta \in (0,1]$. Then

$$Q_\theta(\lambda, i\xi) := \psi(|\xi|^{2\theta}\partial_\lambda)P(\lambda, i\xi) = P(\lambda, i\xi) + \sum_{k=1}^{\mathrm{r}} c_k\,|\xi|^{2k\theta}\,\partial_\lambda^k P(\lambda, i\xi), \tag{56}$$

is a real, strictly stable, polynomial, for any $\xi \neq 0$, thanks to Lemma 1(i) and Theorem 1. Therefore, we may deal with models like (55). However, to derive decay estimates for the higher order models, we need to estimate the behavior of the roots of (56) as $\xi \to 0$ and as $|\xi| \to \infty$, as we did in Section 4 for $Q(\lambda, i\xi)$.

However, in the special case $\theta = 1/2$, this analysis is not necessary, since $Q_{\frac12}(\lambda, i\xi)$ is homogeneous of degree m, and we have the following

Theorem 9. *Let P be a hyperbolic polynomial as in* (3)*, and let m_1 be the maximum multiplicity of the roots in* (3)*, over $\xi' \in S^{n-1}$. Let $\mathrm{r} \geq m_1$ and let $\psi(z) := \sum_{k=0}^{\mathrm{r}} c_k z^k$, be a strictly stable, real polynomial. Then a solution to*

$$\begin{cases} P(\partial_t, \partial_x)u + \sum_{k=1}^{\mathrm{r}} c_k\,(-\Delta)^{\frac{k}{2}}\,P^{(k)}(\partial_t, \partial_x)u = 0, & t \geq 0,\ x \in \mathbb{R}^n \\ \partial_t^j u(0,x) = u_j(x), & j = 0, \dots, m-1, \end{cases} \tag{57}$$

where $P^{(k)}(\lambda, i\xi) := \partial_\lambda^k P(\lambda, i\xi)$, satisfies the following long-time decay estimate with polynomial speed:

$$\|\partial_x^\alpha \partial_t^k u(t,\cdot)\|_{L^2} \leq C \sum_{j=0}^{m-1} (1+t)^{-\frac{n}{2}-|\alpha|-k+j}\,\|u_j\|_{L^1} + C\,e^{-\delta t} \sum_{j=0}^{m-1} \|u_j\|_{H^{|\alpha|+k-j}}, \tag{58}$$

for any $\alpha \in \mathbb{N}^n$ and $k \in \mathbb{N}$, such that

$$\frac{n}{2} + |\alpha| + k > m - 1, \tag{59}$$

for some $C > 0$, $\delta > 0$, which do not depend on the data.

Proof. For any $\xi \neq 0$, let $\lambda = |\xi|\tilde\lambda(\xi')$, with $\xi' = \xi/|\xi|$, so that

$$Q_{\frac12}(\lambda, i\xi) := |\xi|^m\,Q(\tilde\lambda, i\xi'),$$

where Q is defined as in (5). Since $\mathcal{S}^{n-1}$ is compact, by Theorem 5, there exist $C_1 > C_2 > 0$ such that

$$-C_1|\xi| \leq \operatorname{Re}\lambda_j(\xi) \leq -C_2|\xi|,$$

for any $\xi \neq 0$, $j = 1, \dots, m$. It is clear that $\lambda_j(0) = 0$, since $Q_{\frac12}(\lambda, 0) = \lambda^m$. We follow the proof of Theorem 1, setting $\varepsilon = 1$, and taking into account of the possible multiplicity of the roots of $Q(\tilde\lambda, i\xi')$ which, of course, is at most m. However, thanks to

$$e^{t\operatorname{Re}\lambda_j(\xi)} \leq e^{-C_2 t|\xi|} \leq e^{-C_2 t/2}\,e^{-C_2 t|\xi|/2}$$

for $|\xi| \geq 1$, we have now both exponential decay and *smoothing effect* for high frequencies. Therefore, for any $|\xi| \geq 1$, we may estimate

$$\begin{aligned} |\xi|^{|\alpha|}|\partial_t^k \hat{u}(t,\xi)| &\lesssim e^{-C_2t/2}\,(1+t|\xi|)^m\, e^{-C_2t|\xi|/2}\,|\xi|^{|\alpha|+k}\sum_{h=0}^{m-1}|\xi|^{-h}\,|\hat{u}_h(\xi)| \\ &\lesssim e^{-C_2t/2}\,|\xi|^{|\alpha|+k}\sum_{h=0}^{m-1}|\xi|^{-h}\,|\hat{u}_h(\xi)|, \end{aligned}$$

so that no loss of regularity appears, due to the possible multiplicity of the roots of $Q(\tilde{\lambda},\xi')$; indeed,

$$\||\xi|^{|\alpha|}\partial_t^k\hat{u}(t,\xi)\|_{L^2(|\xi|\geq 1)} \lesssim e^{-C_2t/2}\sum_{h=0}^{m-1}\||\xi|^{|\alpha|+k-h}\hat{u}_h\|_{L^2(|\xi|\geq 1)}.$$

Similarly, at low frequencies $|\xi| \leq 1$, we reduce to estimate

$$I_h := \int_{|\xi|\leq 1}|\xi|^{2(|\alpha|+k)}\,(t^2+|\xi|^{-2})^h e^{-2C_2\,|\xi|t}\,d\xi, \qquad h=0,\dots,m-1,$$

where again we took into account of the possible weak hyperbolicity of $Q(\tilde{\lambda},\xi')$. The integral is bounded, thanks to (59), and for $t \geq 1$ we apply the change of variable $\theta = t\xi$ to derive

$$I_h \lesssim t^{-n-2|\alpha|-2k+2h}\int_{\mathbb{R}^n}|\theta|^{2(|\alpha|+k-h)}e^{-2C_2|\theta|}\,d\theta,$$

so that the proof of (58) follows. □

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